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Modular Forms as Clues to the Emergence of Spacetime in String Theory

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Abstract

In order to be consistent as a quantum theory, string theory requires our universe to have extra dimensions. The extra dimensions form a particular type of geometry, called Calabi-Yau manifolds. A fundamental open question in string theory over the past two decades has been the problem of establishing a direct relation between the physics on the worldsheet and the structure of spacetime. The strategy used in our work is to use methods from arithmetic geometry to find a link between the geometry of spacetime and the physical theory on the string worldsheet. The approach involves identifying modular forms that arise from the Omega motives with modular forms derived from the underlying conformal field theory. The aim of this article is to first give a motivation and explanation of string theory to a general audience. Next, we briefly provide some of the mathematics needed up to modular forms, so that one has the tools to find a string theoretic interpretation of modular forms derived from K3 surfaces of Brieskorn-Pham type. The problem involves the calculation of number theoretic constructions called Jacobi sums, and the use these to construct the Omega-motivic L-function and q-expansion. Then, one must determine whether the q-expansion can be factored into Hecke-indefinite modular forms that arise from the worldsheet.

1 Introduction

1.1 The Deep Questions in Life

We live in a wonderfully complex, but elegant universe. Time and time again humans have asked questions like, What is space and time, and what is the universe made of? We are lucky to live in a time when much progress has been made toward finding some of these answers. String theory is an attempt to answer questions like these unify all of physics as we know it. But before we start on string theory, let’s briefly review the developments in theoretical physics over the last one hundred years that have led so many theoretical physicists have taken an interest in string theory.

1.2 Relativity

Toward the end of the 19th century, it was thought by many physics professors that the task of physics was more or less complete. The main theories were thought to already in place, through the technological triumphs of Newton's laws, Maxwell's electromagnetic theory, and thermodynamics. They believed that only a few details needed to be filled here and there by the future generations, and the study of physics would be complete. This view could not have been any more wrong.

In 1905, Albert Einstein submitted a paper that would soon completely overturn the prior notions of space and time, and replace them with something one would never have guessed from common experience. It was known from Maxwell's formulation of
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electromagnetism that light always traveled at the same speed, roughly $3 \cdot 10^8$ m/s. As a teenager, Einstein wondered what would happen if one were to chase after light at nearly the speed of light. Intuition and Newtonian mechanics tells us that we could catch up to it so that light would appear stationary. On the other hand, Maxwell tells us that there is no such thing as stationary light. Einstein resolved this paradox with what is called the Special Theory of Relativity. In order for light to travel at the same speed for two observers that are in relative motion to one another, each will have different perceptions of distance and time. In other words, two identical atomic clocks with cameras attached in relative motion to each other will tick at different rates, disagree on the time that elapsed between two recorded events, and will disagree on measurements taken during these two events. Einstein's theory showed that concepts such as space and time are not separate, but deeply interwoven. From now on, space and time will be called spacetime. He also was able to show that two other physical concepts were not separate: He discovered a deep connection between mass and energy,  

$$E = mc^2,$$

where $E$ is the rest energy of a particle, $m$ is the mass of the particle, and $c$ is the speed of light.

With the success of special relativity, Einstein set out to conquer two new problems. One was to generalize his idea of relativity to accelerating reference frames, and the other to formulate gravity into this framework. In Newton's successful theory of gravity, two masses seem to instantly know about each other, which goes against the notion that no information can travel faster than the speed of light. Einstein came to the realization that being in accelerated motion is indistinguishable from being in a gravitational field that produces the same force, and called this the equivalence principle. With this, he was able to declare that all observers, regardless of whether they are in an accelerated state of motion or not, share the same laws of nature. In 1916, he fully developed these thoughts into a mathematical framework that is called the general theory of relativity (GR). This formulation led to a second link in unifying gravity and accelerated motion, the curvature of space and time. He found that since accelerated motion warps spacetime, gravity also warps spacetime. Further, strength of gravity, or this spacetime warp, is proportion to the produce of the two products. As an example of what this means, consider a bowling ball placed on a trampoline. Think of the bowling ball as the sun and the trampoline fabric to be a 2-dimensional version of our 3-dimensional space. When the bowling ball is absent, any other smaller ball on the trampoline will just move in a straight line. But with the presence of the bowling ball, other smaller balls will be forced to move in a curved path. In the right conditions, the smaller ball will go into an orbit around the bowling ball. This idea is analogous to why the planets orbit the sun, except the 2-dimensional trampoline fabric is replaced with our three dimensional space, and the warping of space is what a gravitational field is (due to a mass). Even the path of light will bend due to the presence of a large mass. Later that year, a physicist Karl Schwarzschild use the equations of GR to predict the existence of black holes. Also with this framework physicists were able to make accurate models on the past, present, and future of our universe, giving birth to modern cosmology.

1.3 Quantum Mechanics and Quantum Field Theory

We will now look at how our understanding of the world has matured. The story starts in 1900, when physicist Max Planck was studying the black body problem, which is when energy is quantized. In order to derive a formula from the experiment, the idea that energy is quantized. In other words, energy can only be in discrete units. Einstein explained this by stating that the energy of light is quanta of energy, and that light behaves as both a wave and a particle. Bohr then went on to develop a theory that explained the spectral lines of a hydrogen atom, which was a major breakthrough in the field of physics. Schrödinger also showed that the principles of quantum mechanics applied to all forms of matter, not just light. The key idea was that all quantum objects, such as position and momentum, are subject to uncertainty. This means that we can never know both the position and momentum of an object simultaneously. The equation that describes this is the uncertainty principle, which states that the product of the uncertainties in position and momentum is always greater than or equal to Planck's constant, $\hbar$. This principle has far-reaching implications for our understanding of the physical world.

In 1927, British physicist Paul Dirac took Planck's quantum theory of the electron to develop a new theory of quantum mechanics. In 1928, he developed the Dirac equation for the electron. This new formulation also included the concept of intrinsic spin. Also, it predicted the existence of anti-matter, which includes particles that have opposite charges and spins. In the 1930’s, some physicists began to realize that the quantum world and the world of matter are deeply intertwined. The equations that describe the behavior of particles at the subatomic level, such as electrons, are now known as quantum electrodynamics. By the 1930s, it was clear that the laws of physics are not the same at all energy scales. The strong and weak nuclear forces, which are responsible for the binding of protons and neutrons in atomic nuclei, are much stronger at the microscopic level, and are not described by the same rules as the electromagnetic field. It is this difference in the strength of these forces that makes it possible to construct the strong force known as the strong nuclear force.
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completed around 1975. Also in the 1960's, a few physicists began to realize that the weak force looks very similar to the electromagnetic force when the energies being used to prove it are above what is called the "unification energy" of roughly 100 GeV. This development is now called the electroweak theory, which describes what is now known as the standard model of particle physics.

The standard model describes three messenger particles for the electromagnetic, strong and weak forces that are called the photon, gluon, and the weak gauge bosons respectively. These are the smallest possible units of energy each force can transmit. In this framework, the particles are not thought to have any internal structure; instead they are elementary and can be thought of as points. The standard model also consists of 6 quarks, 6 leptons, their associated antiparticles, and the Higgs Boson. After the success of the electroweak theory, people have worked to unify the electroweak theory and QCD into a QCD-electroweak interaction. These similar theories have come to be known as grand unification theories (GUT)s. It predicts that at extremely high energies of about $10^{14}$ GeV, the electromagnetic, weak, and strong forces are fused into a single unified QCD-electroweak field. Up until now, energies can’t be made high enough to confirm these GUTs. By applying other asymmetric principles, physicists have come up with extensions to the standard model such as supersymmetry, which proposes a whole new list of particles that have never been directly observed due to the fact that they are to massive to be produced in a particle accelerator. The standard model falls short of being a complete theory of fundamental interactions due to the fact that gravity is not included and a host of other problems. Also, in many places constants are put into place to match data from experiments, and are not derived from the theory itself.

1.4 The Incompatibility of General Relativity and Quantum Field Theory

So far we have not talked about gravity in quantum mechanics. The equations of GR are applied mainly to distances of astronomical lengths, although there are some exceptions. GR says nothing about what the microscopic properties of space would look like, making it a classical theory. The strength of the gravitation force is some $10^{-30}$ times as strong as the strength of the electromagnetic field. In almost all cases, this implies that the gravitational force in quantum systems is negligible, which is why the standard model was able to get away with totally ignoring the gravitational force. GR demands that space is smooth, and that it stays smooth no matter what lengths we are looking at. However, quantum mechanics says that everything is subjected to quantum fluctuations due to the uncertainty principle. The smaller the region of space we probe into, the more the gravitational field will fluctuate. These quantum fluctuations cause violent distortions to the surrounding space, sometimes called quantum foam. When one tries to incorporate the equations of GR into those of quantum mechanics, probabilities that are supposed to be between 0 and 1 become infinity. Getting infinity in a probability is nature's way of telling us that something is very wrong in our framework. In order for the effects of gravity to become apparent on the quantum scale, one would have to look at distances as small as the plank length, which is roughly $10^{-34}$ m. Physicists have tried again and again to incorporate gravity into the framework of the standard model with no luck. This incompatibility between the theories of the big and small points to some kind of flaw in the understanding of our universe. Due to the successful unification of the weak and electromagnetic force, many physicists feel that if we really understood how the universe works, then the universe can be described by a logical, consistent and describes all four forces on earth. Modifying either GR or QM have met with little success, a few proposed quantum theories of gravity are string theory.

2 String Theory

2.1 History of String Theory

The story of string theory can be traced back to the 1960s, when physicists were trying to understand the nature of the strong nuclear force. He discovered a theory called the Euler beta-function, which seemed to describe the forces between particles. In the year 1970, physicists Yoichiro Nambu showed that if one modeled elementary "objects" composed of the strong nuclear force, then the nuclear force could be described by a mathematical expression. The 1970's showed that the string model made more sense, and a point particle theory called Quantum Chromodynamics (QCD) was born. The string theory was successful at describing the nuclear force, and the string model was not dead. The electroweak force was established, and high enough energies, the electroweak would come one force. Through the success of quantum mechanics, physicists began to wonder if the string model was right. At the time, additional messenger-like particles that did not exist in the standard model were found. String theory was not just a theory of gravity! String theory was able to explain missing from the standard model, the early pioneers thought string theory must be right. At the time, additional messenger-like particles that did not exist in the standard model were found. String theory was not just a theory of gravity! String theory was able to explain missing from the standard model, the early pioneers thought string theory must be right. At the time, additional messenger-like particles that did not exist in the standard model were found. String theory was not just a theory of gravity! String theory was able to explain missing from the standard model, the early pioneers thought string theory must be right. At the time, additional messenger-like particles that did not exist in the standard model were found. String theory was not just a theory of gravity! String theory was able to explain missing from the standard model, the early pioneers thought string theory must be right. At the time, additional messenger-like particles that did not exist in the standard model were found. String theory was not just a theory of gravity! String theory was able to explain missing from the standard model, the early pioneers thought string theory must be right. At the time, additional messenger-like particles that did not exist in the standard model were found.
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The story of string theory can be traced back to the late 1960's. In 1968, a physicist named Gabriel Veneziano was trying understand the experimentally observed properties of the strong nuclear force. He discovered that a two hundred year old equation, the Euler beta-function, seemed to describe various properties of the nuclear force. In the year 1970, physicists Yoichiro Nambu, Holger Nielsen, and Leonard Susskind showed that if one modeled elementary "objects" as vibrating, one-dimensional strings, then the nuclear force could be described by Euler's function. Unfortunately, it was not long before the string description was rejected. High energy experiments in the 1970's showed that the string model made a number of predictions that were wrong, and a point particle theory called Quantum Chromodynamics (QCD) was much more successful at describing the nuclear force. At that point, it looked like string theory was dead. The electroweak force was established and physicists believed that given high enough energies, the electroweak would have united with the strong force to become one force. Through the success of quantum field theories and the accumulation of data from high energy experiments, physicists were able develop a QFT called the standard model. Most people had lost interest in string theory because of the success of the standard model, but the early pioneers of string theory were still convinced that string theory must be right. At the time, their string theory has problems with additional messenger-like particles that did not appear to have any relevance to current experiments. A predicted particle called the tachyon that traveled faster than the speed of light. It seemed that the more they studied string theory, the more problems were found.

In 1970s, physicists John Schwarz and Joel Scherk tried all sorts of ways to get rid of these additional particles, but kept failing. For four years, they tried to tame those equations. Then one day while brainstorming, they made the startling discovery concerning one of these messenger-like particles. While studying the patterns of a strings vibration, they came to the realization that if they were to make the strings smaller, than this massless hypothetical particle had the same properties that a graviton would have. String theory was not just a theory of the nuclear force, but also a quantum theory of gravity! String theory was able to produce a piece of the puzzle that was missing from the standard model. Schwarz quickly submitted his research for publication. However, it was shown during this time that string theory still had fatal flaws, and their discovery was given almost no attention from the physics community. After some time, Michael Greene joined Schwarz on his quest to conquer these anomalies for over a decade. In 1984, Schwarz and Green finally made a breakthrough and their work was culminated in the summer of 1984. Amazingly, everything boiled down to a single calculation. They succeeded and the reactions to their success was heard

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throughout the worldwide physics community. This new string theory had the mathematical depth to encompass all four forces, and all known matter. In less than a year, hundreds of particle physicists dropped their projects to either join in the quest to either develop string theory or show that it can't possible be true. This period from 1984 to around 1986 is now known as the "first superstring revolution". [1]. Thousands of research papers were published during this time on string theory. They showed that numerous features of the standard model that took decades of experimental research to understand seemed to emerge naturally from the structure of string theory. Many of the features of the standard model had a more satisfying explanation in string theory than what could be found in the standard model. It also became apparent that in order for string theory to work, it needed to have six extra dimensions, where the six dimensions are thought to be very small. In order to produce the four dimensional spacetime we know, these extra dimensions must be of a particular geometry, known as Calabi-Yau manifolds, which are small and compact. One of the few examples of Calabi-Yau manifolds that can be visualized is a torus.

However, time and time again, string theorists found new anomalies that had to be tamed, only to slowly figure out a way around them. By the 1990's it was shown that there was not just one, but five different string theories that explained the universe as we know it. This was frustrating, since theorists felt as if there should only be one theory of everything. In 1995, theoretical physicist Edward Witten gave a talk where he suggested the existence of M-theory which has one more dimension to explain a number of observed dualities between the different string theories. This marked the birth of what is now called the "second superstring revolution". Here some of the versions of string theory were unified by new equivalences known as S-duality, T-duality, U-duality and mirror symmetry.

2.2 Basics of String Theory

As complicated as the mathematical formalism of string theory is, it can be understood in a very elegant and simple way conceptually. Consider the strings on a violin. When plucked, each string can go through a number of different vibrational patterns. Just as our ears can hear the different musical notes different resonant vibrations, the different vibrational patterns of a fundamental string can give rise to a host of different masses and forces charges. So one vibrational pattern can describe an electron while another vibrational pattern may describe a photon. The amplitude of a string vibration correspond to a larger energy with a shorter wavelength. Due to special relativity, we can confer that the mass of an elementary particle is determined by the energy of the vibrational pattern of its internal string. Similarly although a bit more complicated, the electric, weak, and strong charge carried by a particular string is determined precisely by the way the string vibrates. Instead of the notion in the standard model that different particles are made of different "stuff", string theory states that all matter and all the forces really come from the same constituent, a string. The universe is nothing but a cosmic symphony. Notice that in the violin example, the string has tension, which causes it to be able to plucked. The fundamental string's tension that would produce a graviton-like particle has been computed to be about $8.9 \times 10^{33}N$. Very detailed calculations have showed that with this tension, the string length is about the same as the Planck length,

$$l_p = \sqrt{\frac{\hbar G}{c^3}} = 1.61 \times 10^{-35}m.$$
This new string theory had the mathematical structure to explain all known matter. In less than a year, theorists were already projecting time on string theory. They showed that it could be understood in terms of the physics of a particular geometry, known as a torus.

One consequence of extremely large tension is that a string can be in an infinite number of different vibrational patterns. This at first may seem to be in conflict with experimental data. However, the large tension means that only a few of the vibrational patterns will correspond to particles at masses of the particles we observe. It is likely that the heavier particles that string theory predicts would decay into the lighter particles.

The answer to the apparent contradiction between general relativity and string theory is now one that is conceptually understandable. As a reminder, the main conflict between QM and GR is that general relativity demands the properties of the spatial fabric to be smooth while QM's uncertainty principle gives a picture of spatial fabric that is more violent distortions. Due to the large tension on a string, the string has a lot of energy and a very short length. With this property, it simply smears out this microscopic discreteness and fluctuations of the gravitational field just enough so that GR and QM can work together. These fluctuations then, in some sense, do not arise in string theory. They only exist in a formulation that thinks of fundamental particles as point particles.

### 3 String Theory Research at IUSB

#### 3.1 Motivation

Here at IUSB, the two string theorists Rolf Schimmrigk and Monika Lynker are working to understand the structure of spacetime from first principles in terms of the physics on the worldsheet [2]. The aim here is to use methods from arithmetic geometry to understand the problem in the context of exactly solvable Calabi-Yau varieties. They are formulating a framework that combines methods from algebraic number theory and arithmetic geometry to Calabi-Yau hypersurfaces of many dimensions. The goal is to show that the basic building blocks of underlying string partition functions can be derived from the geometry of these Calabi-Yau hypersurfaces. It has been shown that the modular forms derived in elliptic curves can be expressed in terms of modular forms derived from the underlying superconformal field theory. The generalization of this result to higher genus curves is difficult since no analog of the elliptic modularity theorem is known, even conjecturally [3]. It then suffices to look at special classes of higher genus surfaces to determine whether it has modular forms and then whether they can be written in modular forms derived from the conformal field theory. It has already been shown in [4] that some of these surfaces have modular forms that can be written in terms of string partition functions. To help in this work, I will have to write various computer programs that can compute very large computations in less than a day.

#### 4 Relevant Mathematics and Data Collected

This next section assumes that the reader has had enough prior knowledge in mathematics to read definitions and fill in some of the spaces between the blanks.
4.1 Calabi-Yau Hypersurfaces

Definition 4.1.1. Projective Space
A complex projective space $\mathbb{P}^n = \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \}$ with an equivalence relation $\sim$ defined by:

\[(z_1, \ldots, z_{n+1}) \sim (Az_1, \ldots, Az_{n+1}), \quad A \in \mathbb{C}^*, \quad \mathbb{C}^* = \mathbb{C} - \{0\} \]

Note if $A = 1/z_1$, where $z_1 \neq 0$, then

\[(z_1, \ldots, z_{n+1}) \sim (1/z_1, \ldots, 1/z_{n+1}) \]

In other words,

\[\dim \mathbb{P}^n = n \quad (3)\]

Definition 4.1.2. We define the degree of a polynomial $p(x) \in \mathbb{P}^n$ to be the largest exponent in which a variable is raised to.

Definition 4.1.3. Weighted Projective Space
A projective space is called a weighted projective space if:

\[\mathbb{P}_{(k_1, \ldots, k_{n+1})} = (\mathbb{C}^{n+1} - \{0\}) / \sim \]

with $\sim$ being an equivalence relation defined by:

\[(z_1, \ldots, z_{n+1}) \sim (\lambda^{k_1}z_1, \ldots, \lambda^{k_{n+1}}z_{n+1}), \quad \lambda \in \mathbb{C}^* \]

Definition 4.1.4. Let $x \in \mathbb{P}_{(k_1, \ldots, k_{n+1})}$. Then $x$ has degree $d$ if:

\[p(\lambda^{k_i}z_i) = \sum_{i} (\lambda^{k_i}z_i)^{d_i} = \sum_{i} \lambda^{k_i}z_i^{d_i} = 0 \quad (5)\]

Then

\[\forall i, \quad p(\lambda^{k_i}z_i) = \lambda^{d_i}p(z_i) \quad \text{or} \quad d_i = k_i \quad \forall i \quad (6)\]

Definition 4.1.5. Hypersurfaces in Weighted Projected Spaces
A polynomial $x \in \mathbb{P}_{(k_1, \ldots, k_{n+1})}$ is called a hypersurface in a weighted projective space if $\deg x = d$ for some $d \in \mathbb{Z}^+$

Let $\mathbb{P}^d_{(k_1, \ldots, k_{n+1})}$ denote the set of hypersurfaces with degree $d$ in $\mathbb{P}_{(k_1, \ldots, k_{n+1})}$

Definition 4.1.6. A Calabi-Yau manifold is a Kähler manifold with a vanishing first Chern class.

Example 1. In one complex dimension, a Calabi-Yau (CY) manifold is a complex elliptic curve. The only compact example is a torus. A K3 surface is an example of a Calabi-Yau manifold with complex dimension two.

Definition 4.1.7. Let $x \in \mathbb{P}^d_{(k_1, \ldots, k_{n+1})}$. Then $x$ is a Calabi-Yau Hypersurface if:

\[d = \sum_{i=1}^{n+1} k_i \]

4.2 Congruence Zeta Function

Definition 4.2.1. $\mathbb{F}_p$ is the field of integers modulo $p$.

One way to find properties of some mathematical objects (e.g., in number theory or algebraic geometry) that may reveal some information about a function made for this use is known as the Artin zeta function. Grothendieck showed in [5] that this is true for the congruence zeta function. Let

\[Z(X/\mathbb{F}_p, t) = \exp \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \]

with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The congruence zeta function is a generating function. Grothendieck showed in [5] that this is related to the Artin zeta function as,

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Let $z_1, z_2, \ldots, z_{n+1}$ be complex lines in $\mathbb{C}^{n+1}$ through the origin.

\[ \{ (z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \} \]  

\[ (\lambda z_1, \lambda z_2, \ldots, \lambda z_{n+1}), \lambda \in \mathbb{C}^*, \mathbb{C}^* = \mathbb{C} \setminus \{0\} \]  

\[ z_1, z_2, \ldots, z_{n+1} \sim 1/z_1(z_1, \ldots, z_{n+1}) \]  

Let $n = n$.

A polynomial $p(x) \in \mathbb{P}_n$ is to be the largest effective space if:

\[ \mathbb{P}^{n+1} - \{0\} / \sim \]  

and by:

\[ \{ (\lambda z_1, \ldots, \lambda z_{n+1}), \lambda \in \mathbb{C}^* \} \]  

Then $x$ has degree $d$ if:

\[ \sum_{i} \lambda^{k_i} = 0 \]  

or $d = k_i d_i \in \mathbb{Z}^+ n_i$.

Projected Spaces

A hypersurface in a weighted projective space is a Kähler manifold with a vanishing first Betti number. A Calabi-Yau (CY) manifold is a complex torus. A K3 surface is an example of a CY hypersurface.

Then $z$ is a Calabi-Yau Hypersurface if CY hypersurfaces with $n = 3$ are a special type of K3 surfaces that are algebraic. I am to investigate this class of CY hypersurfaces. Here is a list of all possible CY hypersurfaces with $n = 3$:

\[
\begin{align*}
\mathbb{P}^4 &\sim S^4, \\
\mathbb{P}^6 &\sim S^6, \\
\mathbb{P}^8 &\sim S^8, \\
\mathbb{P}^{10} &\sim S^{10}, \\
\mathbb{P}^{12} &\sim S^{12}, \\
\mathbb{P}^{18} &\sim S^{18}, \\
\mathbb{P}^{20} &\sim S^{20}, \\
\mathbb{P}^{24} &\sim S^{24}.
\end{align*}
\]

4.2 Congruence Zeta Function

Definition 4.2.1. $\mathbb{F}_p$ is the field of integers $\{0, 1, \ldots, p - 1\}$ modulo $p$.

Let

\[ N_{r,p}(X) = \#(X/\mathbb{F}_p^r) \]  

denote the number of solutions manifold $x$ over the field $\mathbb{F}_p^r$.

One way to find properties of some mathematical object is to define a function (usually arithmetic) that may reveal some information about the object itself. A generating function made for this use is known as the congruence zeta function defined by Emil Artin as,

\[ Z(X/\mathbb{F}_p, t) = \exp \left( \sum_{r=1}^{\infty} N_{r,p}(X) t^r \right) \]  

with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The congruence zeta function uses $N_{r,p}(X)$ to define a generating function. Grothendieck showed in [5] that the congruence zeta function can be written as

\[ Z(X/\mathbb{F}_p, t) = \prod_{r=0}^{\infty} P_{p[r]}^{2r+1}(t) \]  

w/ $\deg P_{p[r]}(t) = \dim H^1(x) = b^1(x)$ given by the $i$th Betti number.

Next is a theorem conjectured by Andre Weil and proven by Bernard Dwork in 1960.

Theorem 4.1. The congruence zeta function can be written as a ratio of polynomials,

\[ Z(X/\mathbb{F}_p, t) = \frac{P(t)}{Q(t)} \]  

The definition of $Z$ needs apparently an infinite # of $N_{r,p}(z)$. But since $P$ and $Q$ are polynomials, it must be the case that only a finite # of $N_{r,p}(z)$ will determine it. This is important because it means we are more likely to extract out of an object a finite amount of information as opposed to infinite amount of information. For the case of K3 surfaces can be shown that

\[ Z(X/\mathbb{F}_p, t) = \frac{1}{(1 - t)P(t)(1 - P(t))} \]  

which leads us to the Hasse Weil $L$-function.
PHYSICS

4.3 L-functions

Definition 4.3.1. A Dirichlet L-series is an infinite series in the form

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},
\]

where \( \chi \) is the Dirichlet character and \( s \) is a complex variable with \( \text{Re}(s) > 1 \). When the function is extended to a meromorphic function on the whole complex plane by analytic continuation, it is called a Dirichlet L-function and denoted \( L(X, s) \).

The factorization of (3) leads to the definition of the cohomological L-functions

\[
L^{(i)}(X/\mathbb{F}_p, s) = \prod_{p \text{ prime}} \frac{1}{P_k(p^{-s})}
\]

associated to the \( i \)-th cohomology group \( H^i(X) \). Sometimes the L-functions of a variety \( X \) can lead to a nice complex function that has lots of symmetries. This was proven to be true by Andrew Wiles for elliptic curves. For higher dimensional surfaces, a program called the Langlands program has made a series of conjectures. The hope is that the L-functions can be used to find modular forms, and that some of these modular forms are the same as the string modular forms derived from conformal field theories. This paper will focus on the computation of a special part of an L-function that has lots of nice symmetries known as the \( \Omega \)-motivic L-function, which will be defined later.

4.4 Jacobi Sums

For weighted Fermat varieties there is a second method for obtaining both the cardinalities and the L-function.

Theorem 4.2. For a smooth weighted projective surface with degree vector \( \mathbf{n} = (n_0, \cdots, n_s) \),

\[
X^n = \{z_0^{n_0} + z_1^{n_1} + \cdots + z_s^{n_s} = 0\} \subset \mathbb{P}(k_0, k_1, \cdots, k_s)
\]

defined over the finite field \( \mathbb{F}_q \), define the set \( A^{q,n}_s \) of rational vectors \( \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_s) \) as

\[
A^{q,n}_s = \left\{ \alpha \in \mathbb{Q}^{s+1} \mid 0 < \alpha_i < 1, d_i = \text{gcd}(n_i, q - 1), d_i \alpha_i = 0 \mod 1, \sum_{i=0}^{s} \alpha_i = 0 \mod 1 \right\}
\]

For each \((s+1)\)-tuple \( \alpha \) define the Jacobi sum

\[
j_q(\alpha_0, \alpha_1, \cdots, \alpha_s) = \frac{1}{q - 1} \sum_{u_i \in \mathbb{F}_q, \sum_{u_0 + u_1 + \cdots + u_s = 0 \mod p}} \chi_{\alpha_0}(u_0) \chi_{\alpha_1}(u_1) \cdots \chi_{\alpha_s}(u_s),
\]

where \( \chi_{\alpha_i}(u_i) = e^{2\pi i \alpha_i u_i} \) with integers \( m_i \) determined via \( u_i = g^{m_i} \), where \( g \in \mathbb{F}_q \) is a generator. Then the cardinality \( X^n \) is given by

\[
\#(X^n/\mathbb{F}_q) = N_{1,q}(X^n) = 1 + q + \cdots + q^{s-1} + \sum_{\alpha \in A^{q,n}_s} j_q(\alpha).
\]
is an infinite series in the form
\[ f(x) = \sum_{n=1}^{\infty} \frac{x(n)}{n^s}, \quad (12) \]

where \( x(n) \) is a complex variable with \( \text{Re}(s) > 1 \). When \( s = -1 \), \( f \) is a Dirichlet L-function on the whole complex plane by definition.

The L-function \( L(X, s) \) is defined as the cohomological L-functions in the form
\[ L(X, s) = \prod_{p \text{ prime}} \frac{1}{\mathcal{P}^{s}(p^{-s})} \quad (13) \]

where \( \mathcal{P}^{s} \) is a prime number. Sometimes the L-functions of a variety on the whole complex plane by definition.

The hope is to find modular forms, and that some of these modular forms derived from conformal field theories. The computation of a special part of a L-function that is \( \Omega \)-motive L-function, which will be defined later.

The second method for obtaining both the cardioid and projective surface with degree vector \( n = (k_0, k_1, \ldots, k_s) \) is determined via
\[ \mathcal{P}^{s}(p^{-s}) = \prod_{p \text{ prime}} \frac{1}{\mathcal{P}^{s}(p^{-s})} \quad (14) \]

Let \( \mathbb{A}^{n} \) of rational vectors \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_s) \)
\[ \{ \alpha \in \mathbb{Q}^{n}, 0 < \alpha_i < 1, n_i \alpha_i = 0 \mod 1, \sum_{i=0}^{s} \alpha_i = 0 \mod 1 \} \quad (15) \]

This is the Jacobi sum
\[ \sum_{\alpha \in \mathbb{A}^{n}} \chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1) \cdots \chi_{\alpha_s}(u_s), \quad (16) \]

where \( u_i \) is determined via \( u_i = g^{m_i} \), where \( g \in \mathbb{F}_q \) is given by
\[ u = q + \cdots + q^{m-1} + \sum_{\alpha \in \mathbb{A}^{n}} j_{q}(\alpha). \quad (17) \]

With these ingredients, one can solve for the polynomials \( P_{\mathcal{P}^{s}}^{n}(t) \) in terms of the Jacobi sum variables,
\[ P_{\mathcal{P}^{s}}^{n}(t) = \prod_{\alpha \in \mathbb{A}^{n}} \left( 1 - (-1)^{s-1}t_{\mathcal{P}^{s}}(\alpha)^{f} \right)^{1/f} \quad (18) \]

where
\[ A_{\mathcal{P}^{s}}^{n} = \left\{ \alpha \in \mathbb{Q}^{n}, 0 < \alpha_i < 1, n_i \alpha_i = 0 \mod 1, \sum_{i=0}^{s} \alpha_i = 0 \mod 1 \right\} \quad (19) \]
and \( f \) is determined via
\[ (p^{f} - 1) = 0 \mod 1, \forall i. \quad (20) \]

For my purposes, \( s = 3 \) and \( f = 1 \). Here are some of the Jacobi sum results for some of the Calabi-Yau hypersurfaces mentioned above. The following will describe some of the Jacobi sum results that have been collected.
\begin{align*}
\mathbb{P}^{[8]}_{(1,1,2,4)} \\
\begin{array}{|c|c|}
\hline
p & 17 \quad 41 \\
\hline
\mathcal{J}_p(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & 3 + 8\sqrt{2} + (-12 + 2\sqrt{2})i \quad 15 + 16\sqrt{2} + (-12 + 20\sqrt{2})i \\
\mathcal{J}_p(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & 3 - 8\sqrt{2} + (12 + 2\sqrt{2})i \quad 15 - 16\sqrt{2} + (12 + 20\sqrt{2})i \\
\mathcal{J}_p(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & 3 - 8\sqrt{2} - (12 + 2\sqrt{2})i \quad 15 - 16\sqrt{2} - (12 + 20\sqrt{2})i \\
\mathcal{J}_p(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & 3 + 8\sqrt{2} - (-12 + 2\sqrt{2})i \quad 15 + 16\sqrt{2} - (-12 + 20\sqrt{2})i \\
\hline
\end{array}
\end{align*}

\begin{align*}
\mathbb{P}^{[12]}_{(1,1,4,6)} \\
\begin{array}{|c|c|c|c|}
\hline
p & 7 & 13 & 19 & 31 & 37 \\
\hline
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{3}{2} - 7\sqrt{2}i & -\frac{13}{2} - 13\sqrt{2}i & \frac{19}{2} + 19\sqrt{2}i & -31 & -\frac{37}{2} - 37\sqrt{2}i \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{3}{2} + 7\sqrt{2}i & -\frac{13}{2} + 13\sqrt{2}i & \frac{19}{2} - 19\sqrt{2}i & -31 & -\frac{37}{2} + 37\sqrt{2}i \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & 13i & 13i & 37 & 37 \\
\hline
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{15}{2} + 3\sqrt{3} + i(-5 + 9\sqrt{3}) & \frac{15}{2} - 9\sqrt{3} + i(-33 - 3\sqrt{3}) & \frac{15}{2} - 9\sqrt{3} + i(-33 + 3\sqrt{3}) & 37 & 37 \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{15}{2} - 3\sqrt{3} + i(-5 - 9\sqrt{3}) & \frac{15}{2} + 9\sqrt{3} + i(-33 + 3\sqrt{3}) & \frac{15}{2} + 9\sqrt{3} - i(-33 - 3\sqrt{3}) & 37 & 37 \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{15}{2} - 3\sqrt{3} - i(-5 - 9\sqrt{3}) & \frac{15}{2} + 9\sqrt{3} - i(-33 + 3\sqrt{3}) & \frac{15}{2} - 9\sqrt{3} - i(-33 + 3\sqrt{3}) & 37 & 37 \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{15}{2} + 3\sqrt{3} - i(-5 + 9\sqrt{3}) & \frac{15}{2} - 9\sqrt{3} - i(-33 - 3\sqrt{3}) & \frac{15}{2} + 9\sqrt{3} - i(-33 - 3\sqrt{3}) & 37 & 37 \\
\mathcal{J}_p(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}) & \frac{15}{2} + 3\sqrt{3} - i(-5 + 9\sqrt{3}) & \frac{15}{2} - 9\sqrt{3} - i(-33 - 3\sqrt{3}) & \frac{15}{2} + 9\sqrt{3} - i(-33 - 3\sqrt{3}) & 37 & 37 \\
\hline
\end{array}
\end{align*}

\begin{align*}
\mathbb{P}^{[12]}_{(1,2,3,6)} \\
\begin{array}{|c|c|c|}
\hline
p & 13 & 37 \\
\hline
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 13i & 37 \\
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 2 + 6\sqrt{3} + (3 - 4\sqrt{3})i & 5 + 12\sqrt{3} + (-30 + 2\sqrt{3})i \\
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 2 + 6\sqrt{3} - (3 + 4\sqrt{3})i & 5 + 12\sqrt{3} - (-30 - 2\sqrt{3})i \\
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 2 - 6\sqrt{3} - (3 - 4\sqrt{3})i & 5 - 12\sqrt{3} - (-30 - 2\sqrt{3})i \\
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 2 - 6\sqrt{3} + (3 - 4\sqrt{3})i & 5 - 12\sqrt{3} + (-30 + 2\sqrt{3})i \\
\mathcal{J}_p(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{2}) & 2 - 6\sqrt{3} - (3 + 4\sqrt{3})i & 5 - 12\sqrt{3} - (-30 + 2\sqrt{3})i \\
\hline
\end{array}
\end{align*}
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<tr>
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<th>( 13 )</th>
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<td>( \frac{11}{2} + 3\sqrt{3} + i(-5 + 9\sqrt{3}) )</td>
<td>( \frac{13}{2} + 3\sqrt{3} - i(-5 + 9\sqrt{3}) )</td>
<td>( \frac{15}{2} + 3\sqrt{3} - i(-5 + 9\sqrt{3}) )</td>
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\( [12] \)

\( (1,2,3,6) \)

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<tr>
<td>( j_p\left(\frac{7}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2}\right) )</td>
<td>( 2 - 6\sqrt{3} + (3 + 4\sqrt{3}) )</td>
<td>( 5 - 12\sqrt{3} + (-30 - 2\sqrt{3})i )</td>
</tr>
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<td>( 5 + 12\sqrt{3} (-30 - 2\sqrt{3})i )</td>
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\( [12] \)

\( (2,3,3,4) \)

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</tr>
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<td>( \frac{5}{2} + \frac{7\sqrt{3}i}{2} )</td>
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<td>( j_p\left(\frac{7}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2}\right) )</td>
<td>( 5 - \frac{9\sqrt{3}}{2} + i(\frac{13}{2} + 3\sqrt{3}) )</td>
<td>( -\frac{11}{2} + 9\sqrt{3} + i(33 + 3\sqrt{3}) )</td>
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<td>( j_p\left(\frac{9}{2},\frac{9}{2},\frac{9}{2},\frac{9}{2}\right) )</td>
<td>( 5 + \frac{9\sqrt{3}}{2} + i(\frac{13}{2} - 3\sqrt{3}) )</td>
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4.5 Modular Forms

Definition 4.5.1. The modular group \( \Gamma_0(N) \) is defined as

\[
\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}
\]

(21)

Definition 4.5.2. A modular form of weight \( w \), level \( N \), and character \( \chi \) with respect to \( \Gamma_0(N) \) is a map \( f : \mathcal{H} \to \mathbb{C} \) on the upper half-plane such that for any \( \tau \in \mathcal{H} \) and \( \gamma \in \Gamma_0(N) \) Also, \( f \) is said to be meromorphic at the cusp

\[
f(\gamma \tau) = \chi(d)(cr + d)^w f(\tau)
\]

(22)

Eigenforms of these operators have been shown to be important in a geometric context.

One modular form of importance in this context is the Dedekind eta function with weight \( 1/2 \) and

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)
\]

(23)

4.6 Conformal Field Theory

Definition 4.6.1. Conformal symmetry is a symmetry that is invariant under scaling and under the special conformal transformations.

Conformal field theories provide toy models for genuinely interacting QFT's, and play a crucial role in string theory.

Definition 4.6.2. Conformal field theory

A conformal field theory (CFT) is a two dimensional quantum field theory that is invariant under conformal transformations.

A quantity that has proven usefull in the string modularity analysis of elliptic Brieskorn-Pham curves, [6] [7] are the theta functions

\[
\theta_{l,m}^{(k)}(\tau) = \sum_{(x,y) \in \mathbb{Z}^2 + \left( \frac{1}{2(k+2)}, \frac{1}{2} \right)} \text{sign}(x)e^{2\pi i((k+2)x^2 - ky^2)}
\]

(24)

defined by Kac and Peterson. These are related to the string function of the \( A_1^{(1)} \) at level \( k \) as

\[
c_{l,m}^{(k)}(\tau) = \frac{\theta_{l,m}^{(k)}(\tau)}{\eta^{(k)}(\tau)}
\]

(25)

It has been found that the following modular forms explain the modularity of extremal K3 surfaces of Brieskorn-Pham type in a string theoretic way:

\[
\theta_{1,1}^{(1)}(q) = q^{1/12} (1 - 2q - 2q^3 + q^4 + 2q^5 - 2q^6 - 2q^8 - 2q^9 + q^{10} + \ldots)
\]

(26)

\[
\theta_{1,1}^{(2)}(q) = q^{1/8} (1 - q - 2q^2 + q^3 + 2q^5 - 2q^6 + q^8 - 2q^9 - 2q^{10} + \ldots)
\]

(27)

4.7 Galois representations and \( \Omega \)-motives

This section will define the notion of \( \Omega \)-hypersurfaces. Before we do this, we need to define

Definition 4.7.1. Cyclotomic field

A cyclotomic field is a number field obtained by a primitive \( d^{th} \) root of unity.

The Jacobi sums \( j_p \) act as algebraic numbers by considering the orbit of the multiplicative group \( (\mathbb{Z}/d\mathbb{Z})^\times \), with \( d \) lead to integral coefficients in the L-function and can be interpreted as Galois groups of cyclotomic fields (see [9]).

We obtain a L-series with rationally integrable coefficients in the \( \Omega \)-function of the \( \Omega \)-motive of hypersurfaces

\[
\Omega_{\alpha} = \{ \beta \in \mathbb{A}_\mathbb{C}^d | \beta = (t, \alpha) \}
\]

where \( t \cdot \alpha = (t_0, \ldots, t_\alpha) \). Now for any

\[
L_{\alpha}(X^d, s) = \prod \frac{1}{1 - \frac{1}{(t, \alpha)}}
\]

where \( O_{\alpha} \) denotes the orbit of the element and our aim is to check whether the \( \Omega \)-motives of elliptic Brieskorn-Pham surfaces have a modular interpretation.

4.8 Calculation of \( L_{\alpha}(S^8, s) \)

Here we will need the example of how to compute for \( \Omega \)- Nepal values given the the Jacobi set above,

\[
L_{\alpha}(S^8, s) = \prod_{p \in \mathbb{Z}_p} 1 + j_p
\]

For our purposes here, we will not look at the higher term.

\[
L_{\alpha}(S^8, s) = (1 + (3 + 8 \sqrt{2} + i(-12 + 2 \sqrt{2})) \times (1 + (3 - 8 \sqrt{2} - (12 + 2 \sqrt{2})) \times (1 + (15 + 16 \sqrt{2} + (12 + 20 \sqrt{2})) \times (1 + (15 - 16 \sqrt{2} - (12 + 20 \sqrt{2})))
\]

(28)
4.7 Galois representations and \( \Omega \)-motives of CY varieties

This section will define the notion of \( \Omega \)-motives for the general class of Calabi-Yau hypersurfaces. Before we do this, we need a definition.

**Definition 4.7.1. Cyclotomic field**

A cyclotomic field is a number field obtained by adjoining a complex root of unity to \( \mathbb{Q} \). The \( \eta^{th} \) cyclotomic field \( \mathbb{Q}(\mu_d) \) is obtained by adjoining a primitive \( d^{th} \) root of unity \( \mu_d \) to \( \mathbb{Q} \).

The Jacobi sums \( j_{\tau} \) act as algebraic numbers in the cyclotomic field \( \mathbb{Q}(\mu_d) \). The idea here is to achieve the necessary integral-ness of the coefficients by considering the orbits of Jacobi sums defined by the action of the multiplicative group \((\mathbb{Z}/d\mathbb{Z})^\times\) with \( d \) the degree of the hypersurface. These orbits lead to integral coefficients in the \( L \)-function since the multiplicative groups \((\mathbb{Z}/d\mathbb{Z})^\times\) can be interpreted as Galois groups of cyclotomic fields \( \mathbb{Q}(\mu_d) \) (relevant concepts in [8], [9]). Jacobi sum orbits for the Galois group \( \text{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}) = (\mathbb{Z}/d\mathbb{Z})^\times \) are obtained by defining

\[
(\tau, \alpha) = t \cdot \alpha \pmod{1},
\]

where \( t \cdot \alpha = (t \alpha_0, \ldots, t \alpha_s) \) Now for any \( \alpha \), one can use the orbit

\[
O_{\alpha} = \{ \beta \in \mathbb{A}_1^d | \beta = t \cdot \alpha, \ t \in (\mathbb{Z}/d\mathbb{Z})^\times \},
\]

to obtain a \( L \)-series with rationally integral coefficients [2]. Further, this leads to the \( L \)-function of the \( \Omega \)-motive of hypersurfaces \( X^d \)

\[
L_{\Omega}(X^d, s) = \prod_{p} \prod_{\alpha \in O_{\alpha}} \frac{1}{1 - (-1)^{s-1} j_{\tau}(\alpha)p^{-s}f^{1/f}},
\]

where \( O_{\alpha} \) denotes the orbit of the element \( \alpha \) that corresponds to the \( \Omega \)-form. The aim is to check whether the \( \Omega \)-motives of extremal Brieskorn-Pham K3 surfaces admit a modular interpretation.

4.8 Calculation of \( L_{\Omega}(S^6, s) \)

Here will be an example of how to compute \( L_{\Omega}(X^6, s) \). Note \( f = 1 \), and using the values given the the Jacobi section above, we can rewrite the equation above as

\[
L_{\Omega}(S^6, s) = \prod_{p} \prod_{\alpha \in O_{\alpha}} \frac{1 + j_{\tau}(\alpha)p^{-s} + (j_{\tau}(\alpha)p^{-s})^2 + \cdots}{1 + (1 + j_{\tau}(\alpha)p^{-s})^2 + \cdots}
\]

For our purposes here, we will not look at the squared or any other higher orderd term.

\[
L_{\Omega}(S^6, s) = (1 + (3 - 8\sqrt{2} + i(-12 + 2\sqrt{2}))17^{-s})(1 + (3 - 8\sqrt{2} + i(12 + 2\sqrt{2}))17^{-s})
\]
\[
\times (1 + (3 - 8\sqrt{2} - (12 + 2\sqrt{2}i))17^{-s})(1 + (3 + 8\sqrt{2} - (12 + 2\sqrt{2}i))17^{-s})
\]
\[
\times (1 + (15 + 16\sqrt{2} + (12 + 20\sqrt{2}i))41^{-s})(1 + (15 - 16\sqrt{2} + (12 + 20\sqrt{2}i))41^{-s})
\]
\[
\times (1 + (15 - 16\sqrt{2} - (12 + 20\sqrt{2}i))41^{-s})(1 + (15 + 16\sqrt{2} - (12 + 20\sqrt{2}i))41^{-s})\ldots
\]
and similar orbit products for the next prime fields that have $\Omega$ orbits, such as $p = 73, 89, 97,$ and 113. The Jacobi sums for $p=100$ have taken to long for the current computer program I wrote to compute in a reasonable amount of time. Fortunately, knowing the values for primes less than 100 is enough information to find a modular form. Also, the $p^2$ fields of interest have been shown to usually have Jacobi sums equal to $p^2$. Multiplying the terms out above gives

$$L_\Omega(S^8, s) = 1 + \frac{a_9}{9^s} + \frac{12}{17^s} + \frac{a_{25}}{25^s} + \frac{60}{41^s} \frac{a_{49}}{49^s} - \frac{12}{73^s} + \frac{a_{81}}{81^s} + \frac{180}{89^s} - \frac{180}{97^s} + \cdots \quad (32)$$

where $\doteq$ means that the $L$ function is known up to the terms given. This gives the $q$-expansion

$$f_\Omega(S^8, q) \doteq q + a_9 q^9 + 9q^{17} + a_{25} q^{25} + 60 q^{41} + a_{49} q^{49} - 12q^{73} + \cdots \quad (33)$$

What’s left next is to find a modular form associated to this $q$-expansion. It is expected that the modular form of a K3 surface to be of weight 2. It is speculated that exactly solvable models admit complex multiplication (CM) in the classical sense [10]. For example, $S^4$ admits CM with respect to the field $\mathbb{Q}(i)$ while $S^6A$ $S^8B$ with respect to the field $\mathbb{Q}(i\sqrt{3})$, and $S^8$ with respect to the field $\mathbb{Q}(i\sqrt{2})$. Further, it is expected that the level of the geometrically derived modular form is divisible by the what is known as bad primes, or primes that divide $n$. For $S^8$, the level should be some power of 2. For $S^6$, it should be of the form $2^a5^b$ for some non-negative integers $a$, $b$.

So far, the Jacobi sums of each surface up to $p = 43$ have been computed, which puts certain pieces of the $\Omega$-motive $L$-function together. Modular forms for the CY hypersurfaces with $n = 4, 6$ have already been computed, and it has been shown [2] that they can be written in terms of modular forms constructed from the worldsheet.

5 Future Goals

The most general goal that is hoped to be accomplished by the end of the summer is to find a modular form associated to the $\Omega$-motive $L$-function of the surfaces mentioned earlier, and see if these can be written in terms of the modular forms derived from the underlying conformal field theory. A more efficient program for computing Jacobi sums may have to be written in order to compute the Jacobi sums of larger primes. Also, more computer programs will have to be written to test different aspects of the $q$-expansions.

References


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prime fields that have $\Omega$ orbits, such as $\text{Gal}(J_{p_{\leq 100}})$ have taken to long for the current amount of time. Fortunately, $\text{Gal}(J_{p_{\leq 100}})$ is enough information to find a modular form shown to usually have Jacobi sums equal to

$$\frac{19^8}{73^8} + \frac{180}{81^8} + \frac{180}{89^8} - \frac{97^8}{97^8} + \cdots \quad (32)$$

as the terms given. This gives the

$$q^{25} + 60q^{41} + a_{49}q^{49} - 12q^{73} + \cdots \quad (33)$$

associated to this $q$-expansion. It is expected of weight 2. It is speculated that exactly complex multiplication (CM) in the classical sense [10]. For the field $\mathbb{Q}(i)$ while $S^6 \cong S^6$ with respect to the field $\mathbb{Q}(i\sqrt{2})$. Further, it is expected that the modular form is divisible by the what is known $\mathbb{Q}$, the level should be some power of 2. For non-negative integers $a, b, c$, up to $p = 43$ have been computed, which is done to the level. Modular forms for the CY have been computed, and it has been shown [2] that these modular forms constructed from the worldsheet.

The goal accomplished by the end of the summer is to find an efficient program for computing Jacobi sums of larger primes. This will be written to test different aspects of the


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