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Generation of Time-Independent and Time-Dependent Harmonic Oscillator-Like Potentials Using Supersymmetric Quantum Mechanics

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ABSTRACT

The harmonic oscillator is a quantum mechanical system that represents one of the most basic potentials. In order to understand the behavior of a particle within this system, the time-independent Schrödinger equation was solved; in other words, its eigenfunctions and eigenvalues were found. The first goal of this study was to construct a family of single parameter potentials and corresponding eigenfunctions with a spectrum similar to that of the harmonic oscillator. This task was achieved by means of supersymmetric quantum mechanics, which utilizes an intertwining operator that relates a known Hamiltonian with another whose potential is to be built. Secondly, a generalization of the technique was used to work with the time-dependent Schrödinger equation to construct new potentials and corresponding solutions.

KEYWORDS: exact solutions, time-dependent Schrödinger equation, time-independent Schrödinger equation, supersymmetric quantum mechanics, intertwining operators

INTRODUCTION

Since the harmonic oscillator is an exactly solvable system (Flugge, 1971), it can be used to obtain other solvable anharmonicities using the supersymmetric quantum mechanics method (SUSY) (Mielnik, 1984). This technique utilizes intertwining operators to find new potentials and corresponding eigenfunctions from the time-independent Schrödinger equation (Fernandez and Fernandez-Garcia, 2004). This technique can be generalized in order to find new solutions of the time-dependent Schrödinger equation (Bagrov et. al., 1995; Contreras-Astorga, 2017; Zelaya and Rosas-Ortiz, 2017).

In this article, the basic ideas and principal formulas to perform a supersymmetric transformation of first and second order are presented in Section 1. In Section 2, the time-independent Schrödinger equation for the harmonic oscillator is solved, obtaining eigenfunctions and eigenvalues that are useful in the constructions of solvable systems. Then, in Section 3, time-independent SUSY partners will be constructed. The general theory of the time-dependent SUSY with a first order differential operator is first discussed, and then a a confluent time-dependent algorithm is developed in Section 4. Finally, in Section 5, our main results are presented: two sets of time-dependent potentials with their solutions generated using a coherent state, and also a generic Gaussian packet as transformation functions.

1. SUPERSYMMETRIC QUANTUM MECHANICS

SUSY is a convenient method used to find a new exactly solvable potential from a known Hamiltonian using an intertwining differential operator. Departing from a particular Schrödinger Hamiltonian $\mathbf{H}_0$, given by the expression

$$\mathbf{H}_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0, \tag{1}$$

where $V_0 = V_0(x)$ is the potential, and then solving the eigenvalue equation $H_0\Psi = E\Psi$, where E is the energy of the system, new Hamiltonians $\mathbf{H}_i$ with their solutions $\hat{\Psi}$ can be generated by proposing the existence of an intertwining operator $\mathbf{B}$ such that

$$\mathbf{H}_i\mathbf{B} = \mathbf{B}\mathbf{H}_0. \tag{2}$$

In the case where the operator B can be constructed, solutions of the new Schrödinger equation $\mathbf{H}_i\hat{\Psi} = E\hat{\Psi}$ will have the form $\hat{\Psi} \propto \mathbf{B}\Psi$. Specific expressions for $\mathbf{B}$ and the potential V_i corresponding to the Hamiltonian $\mathbf{H}_i$ will depend on the order of the transformation. The order of this differential operator determines the order of the transformation. In Subsection 1.1, an introduction of the method will be given for first order, while in Subsection 1.2, a special case of the second order will be studied. More details of these transformations can be found (Fernandez and Fernandez-Garcia, 2004; Khare and Sukhatme, 1995).

1.1 First-Order Transformation

The simplest form of this technique is called 1-SUSY. This method provides us with the capability to add a new energy level below the ground state energy, to eliminate the ground state energy, and to create new isospectral systems. If we know a solution u for the equation

$$-\frac{1}{2}u'' + V_0 u = \epsilon u, \tag{3}$$

where ε symbolizes factorization energy and u'' denotes the second order differentiation of u with respect to position, then the function

$$\hat{\Psi} = \mathbf{B}\hat{\Psi}, \qquad \text{where} \quad \mathbf{B} = -\frac{d}{dx} + \frac{u'}{u}, \tag{4}$$

is a solution of the Schrödinger equation:

$$\mathbf{H}_1\hat{\Psi} = E\hat{\Psi}, \qquad \text{where} \quad \mathbf{H}_1 = -\frac{1}{2}\frac{d^2}{dx^2} + V_1, \tag{5}$$

with

$$V_1(x) = V_0 - 2\frac{d^2}{dx^2}\ln(u) \tag{6}$$

being the transformed potential. In the case where $\Psi = u$, equation (4) gives us a trivial solution. It can be shown by direct substitution that under these circumstances, the function

$$\hat{\Psi} = \frac{1}{u} \tag{7}$$

is also solution of (5). The spectrum of $\mathbf{H}_0$ and $\mathbf{H}_1$ will be the same, or at most will differ in only the ground state energy, depending on properties of the selected transformation function u.

1.2 Second-Order Transformation by Means of the Confluent Case

In order to develop the confluent, second-order SUSY algorithm, a first-order transformation to equation (5) must be performed. This method requires a new transformation function $\hat{u}$ to generate the function

$$\bar{\Psi} = \left(-\frac{d}{dx} + \frac{\hat{u}'}{\hat{u}}\right)\hat{\Psi}. \tag{8}$$

The confluent case is a particular iteration where the factorization energy of $\hat{u}$ is the same as for u in (3), such that $\hat{u}$ will solve

$$-\frac{1}{2}\hat{u}'' + V_1\hat{u} = \epsilon\hat{u}. \tag{9}$$

Within this case where the factorization energy is kept constant, the function $\bar{\Psi}$ must be a solution of the corresponding Schrödinger equation:

$$\mathbf{H}_2\bar{\Psi} = E\bar{\Psi}, \qquad \text{where} \quad \mathbf{H}_2 = -\frac{1}{2}\frac{d^2}{dx^2} + V_2, \tag{10}$$

and

$$V_2(x) = V_1 - 2\frac{d^2}{dx^2}\ln(\hat{u}). \tag{11}$$

Recall that the particular solution of equation (9) is given by $\hat{u} = 1/u$; therefore, the potential V_2 can be constructed. Moreover, a general solution $\hat{u}$ can be obtained via the application of the reduction of order formula:

$$\hat{u} = c_1\hat{u} + c_2\hat{u}\int\frac{1}{\hat{u}^2}dx = \frac{c_1}{u} + \frac{c_2}{u}\int^x u^2dx, \tag{12}$$

where c_1 and c_2 are arbitrary constants. Substitution of V_1 from (6) and the explicit form of $\hat{u}$ can be used to determine the transformed potential V_2, where

$$\begin{aligned} V_2(x) &= V_1 - 2\frac{d^2}{dx^2}\ln(\hat{u}) \\ &= V_0 - 2\frac{d^2}{dx^2}\ln(u) - 2\frac{d^2}{dx^2}\ln(\hat{u}) \\ &= V_0 - 2\frac{d^2}{dx^2}\ln\left(\frac{c_1}{c_2} + \int^x u^2dt\right). \end{aligned} \tag{13}$$

The new eigenfuctions can be constructed as they were for the first transformation but with the new transformation function $\hat{u}$:

$$\bar{\Psi} = \hat{\mathbf{B}}\hat{\Psi}, \qquad \hat{\mathbf{B}} = -\frac{d}{dx} + \frac{\hat{u}'}{\hat{u}}. \tag{14}$$

An new energy level can also be generated:

$$\bar{\Psi} = \frac{1}{\hat{u}} = \frac{u}{\frac{c_1}{c_2} + \int^x u^2dt}\left[c_1 + c_2\int^x\frac{1}{u^2}\left(\frac{c_3}{c_4} + \int^t u^2ds\right)^2\right]. \tag{15}$$

2. HARMONIC OSCILLATOR

In order to generate a new potential using a first-order SUSY transformation of the harmonic oscillator, a solution u of (3) must be determined and substituted into equations (4, 6). A general expression of the time-independent Schrödinger equation will be considered:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V_0(x)\Psi = E\Psi. \tag{16}$$

By using the harmonic oscillator potential

$$V_0(x) = \frac{1}{2}m\omega^2x^2 \tag{17}$$

and defining the energy of the state E and the angular frequency ω as

$$E = \frac{k^2\hbar^2}{2m}, \quad \omega = \frac{\lambda\hbar}{m}, \tag{18}$$

the Schrödinger equation can be re-written as

$$\frac{d^2}{dx^2}\Psi + \left(k^2 - \lambda^2x^2\right)\Psi = 0. \tag{19}$$

With the substitution of

$$\Psi\left(x\right)=e^{-\frac{\lambda x^{2}}{2}}v\left(x\right) \tag{20}$$

and the introduction of the new variable, $y = \lambda x^2$, Kummer's Equation is generated. A general expression for v in terms of a confluent hypergeometric function (Abramowitz and Stegun, 1964) can be constructed

$$v(x)=A\ {}_1F_1\left(\frac{1}{4}-\frac{E}{2\hbar\omega},\frac{1}{2};\lambda x^2\right)+B\ \sqrt{\lambda x^2}{}_1F_1\left(\frac{1}{2}-\frac{E}{2\hbar\omega},\frac{3}{2};\lambda x^2\right), \tag{21}$$

where A and B are arbitrary constants. Note that $\mathbf{B}$ is an intertwining operator, whereas B is a constant. If boundary conditions are imposed on this solution, eigenfunctions and eigenvalues for the harmonic oscillator can be found.

Let the function u be equal to a particular solution,

$$u(x)=\left[A\ {}_1F_1\left(a,\frac{1}{2};\lambda x^2\right)+B\ x\sqrt{\lambda}\ {}_1F_1\left(a+\frac{1}{2},\frac{3}{2};\lambda x^2\right)\right]e^{-\frac{\lambda x^2}{2}}, \tag{22}$$

where

$$a=\frac{1}{4}-\frac{E}{2\hbar\omega}. \tag{23}$$

To obtain vanishing functions when $|x|\to\infty$, a must be zero or a negative integer in the hypergeometric function, in addition to the constants $A = 1$ and $B = 0$. Thus, the energy E must be $E_{2n+1} = \hbar\omega\,(2n + ½)$, and a solution can be written as

$$\Psi_{2n}\left(x\right)=\ {}_1F_1\left(-n,\frac{1}{2};\lambda x^2\right)e^{-\frac{\lambda x^2}{2}}. \tag{24}$$

A second condition can be obtained from the second hypergeometric function, when the first argument is zero or a negative integer, and the constants $A = 0$ and $B = 1$. As a result, E must be given by $E_{2n+1} = \hbar\omega\,(2n + 1 + ½)$, which produces the corresponding solution,

$$\Psi_{2n+1}\left(x\right)=x\sqrt{\lambda}\ {}_1F_1\left(-n,\frac{3}{2};\lambda x^2\right)e^{-\frac{\lambda x^2}{2}}. \tag{25}$$

Solutions Ψ_{2n} are all even eigenfunctions of the harmonic oscillator, while Ψ_{2n+1} are all odd eigenfuntions. The spectrum is then the union of the corresponding eigenvalues and can be written as $E_n = \hbar\omega\,(n + ½)$.

In Figure 1, the first three eigenfunctions and the potential of the harmonic oscillator are constructed and displayed. In the Appendix, the eigenfunctions and eigenvalues are obtained using creation and annihilation operators.

3. THE TIME-INDEPENDENT SUSY TRANSFORMATION

The SUSY transformation technique is a convenient method that can be used to construct a new potential and to find corresponding eigenstates of a particle. Two transformations of the harmonic oscillator are performed, first using the simplest version of the SUSY technique and secondly through the confluent version.

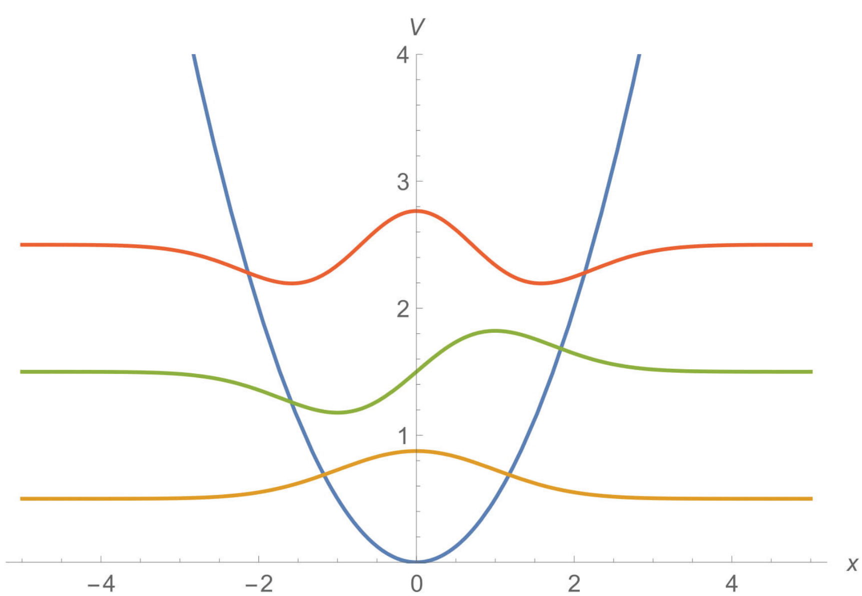

Figure 1.
The harmonic oscillator potential and its first three eigenfunctions.

3.1 First-Order Time-Independent SUSY Transformation

The mathematical solution u can be used to generate new exactly solvable potentials. Let us consider first the general solution

$$u(x) = \left[A\ {}_1F_1\left(\frac{1}{4} - \frac{E}{2\hbar\omega}, \frac{1}{2}; \lambda x^2\right) + B\sqrt{\lambda x^2}\ {}_1F_1\left(\frac{1}{2} - \frac{E}{2\hbar\omega}, \frac{3}{2}; \lambda x^2\right)\right] e^{-\frac{\lambda x^2}{2}}. \tag{26}$$

The first order transformation of the harmonic oscillator potential is constructed by substituting this solution into equation (6), where $V_0 = -\frac{x^2}{2}$. The corresponding eigenfunctions can then be solved by using the intertwining operator $\mathbf{B}$ as in (4):

$$\hat{\Psi}_n(x) = \mathbf{B}\Psi_n(x) = \left(\frac{d}{dx} + \frac{u'}{u}\right)\Psi_n(x). \tag{27}$$

In order to ensure regularity of the new potential and a well behavior of the new solutions, the constants A and B should be carefully chosen in order to avoid zeros in the transformation function u. If a regular potential is constructed, then all functions $\hat{\Psi}_n$ with different quantum numbers n can be shown to be orthogonal to each other. Additionally, functions $\hat{\Psi}$ will be square integrable. However, the range of the functions does not span the whole $L^2(R)$ space, resulting in a missing eigenfunction $\hat{\Psi}_\epsilon$ that can be constructed by substituting u into equation (7).

An example of the new potential with its first three lowest-level eigenfunctions and the missing eigenfunction are shown in Figure 2 with the parameters: $\varepsilon = -3/2$, $A = 1$, and $B = \frac{\Gamma(\frac{1}{2})}{\Gamma(1)}$.

3.2 Second-Order Time-Independent SUSY Transformation

In performing the second transformation from the first-order transformation of the harmonic oscillator, a new transformation function $\hat{u}$ must be obtained to generate the following set of solutions:

$$\bar{\Psi} = \left(-\frac{d}{dx} + \frac{\hat{u}'}{\hat{u}}\right)\hat{\Psi}. \tag{28}$$

This transformation corresponds to a confluent case, where the factorization energy of $\hat{u}$ must be chosen to be the same ε as for u, such that $\hat{u}$ solves the following:

$$-\frac{1}{2}\hat{u}'' + V_1\hat{u} = \epsilon\hat{u}. \tag{29}$$

Based on the assumption of the confluent case, the function $\bar{\Psi}$ must be a solution of the following Schrödinger equation:

$$\mathbf{H}_2\bar{\Psi} = E\bar{\Psi}, \quad \text{where} \quad \mathbf{H}_2 = -\frac{1}{2}\frac{d^2}{dx^2} + V_2, \quad V_2 = V_1 - 2\frac{d^2}{dx^2}\ln(\hat{u}). \tag{30}$$

By applying the reduction of order method, $\hat{u}$ is determined to be

$$\hat{u} = \frac{c_1}{u} + \frac{c_2}{u}\int u^2 dt \tag{31}$$

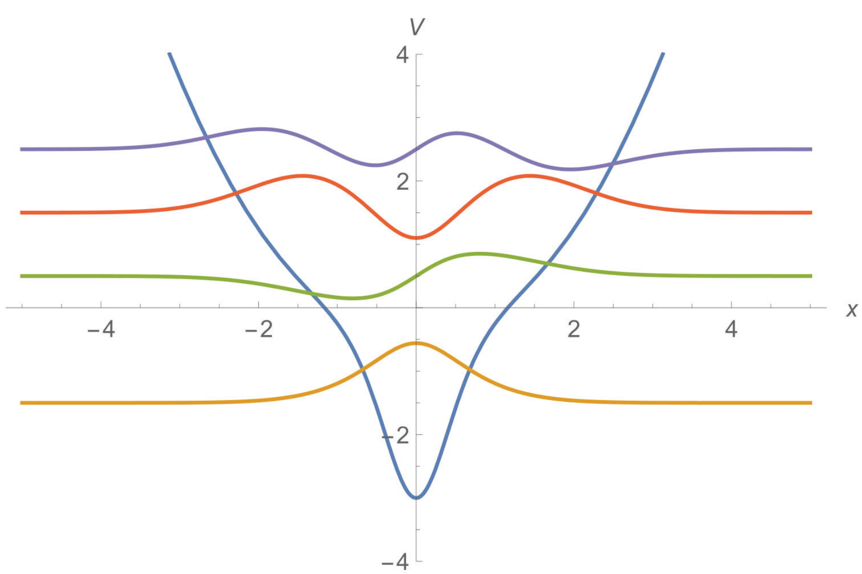

Figure 2.
A first-order SUSY partner of the harmonic oscillator potential with its first three eigenfunctions and an additional missing eigenstate.

with the transformed potential being

$$V_2 = V_0 - 2\frac{d^2}{dx^2}\log\left(\frac{c_1}{c_2} + \int u^2 dt\right). \tag{32}$$

The ratio $C = c_1/c_2$ plays the role of a parameter in this transformation. An alternative expression for this new potential can be generated in terms of Wronskians (Mielnik et al., 2000; Bermudez et al, 2012; Contreras-Astorga and Schulze-Halberg, 2015) and thus constructed as

$$V_2 = V_0 - \frac{(C-W)\,2\hat{u}\hat{u}' - \hat{u}^2\,(C-W)'}{(C-W)^2}, \quad \text{where} \quad W = u_1'u_2 - u_1u_2' \tag{33}$$

and

$$u_1 = u, \quad u_2(x) = Cu + \frac{\partial u}{\partial \epsilon}. \tag{34}$$

The eigenfunctions can then be solved using again a Wronskian representation as

$$\bar{\Psi}_n = \left(\frac{1}{1-2\epsilon}\right)\left(\Psi_n{}' - \frac{(C-W)'}{C-W}\Psi_n\right) + \left(\frac{1}{2}\frac{d}{dx}\left(\frac{(C-W)'}{C-W}\right) + \frac{1}{2}\left(\frac{(C-W)'}{C-W}\right)^2 - x^2 + 2\epsilon\right)\Psi_n.$$

The new potential with its three eigenfunctions and the missing eigenfunction are shown in Figure 3 with the parameters: $\varepsilon = -3/2$, $A = 1$, and $B = \frac{\Gamma(\frac{1}{2})}{\Gamma(1)}$.

It is important to note that (32) and (33) are equivalent expressions for the new potential. However, for some parameters, one may be easier than the other to compute.

4. TIME-DEPENDENT SUSY TRANSFORMATION

As in the stationary case, a time-dependent SUSY transformation of the harmonic oscillator can also be obtained with the use of intertwining operators on the Schrödinger operators. Both the first-order and second-order transformations are generated, showing an interesting perspective on the system with a new parameter.

4.1 First-Order Time-Dependent SUSY Transformation

The time-dependent Schrödinger equation is given by

$$\mathbf{S}_0\Psi = 0, \quad \text{where} \quad \mathbf{S}_0 = i\partial_t + \partial_{xx} - V_0. \tag{35}$$

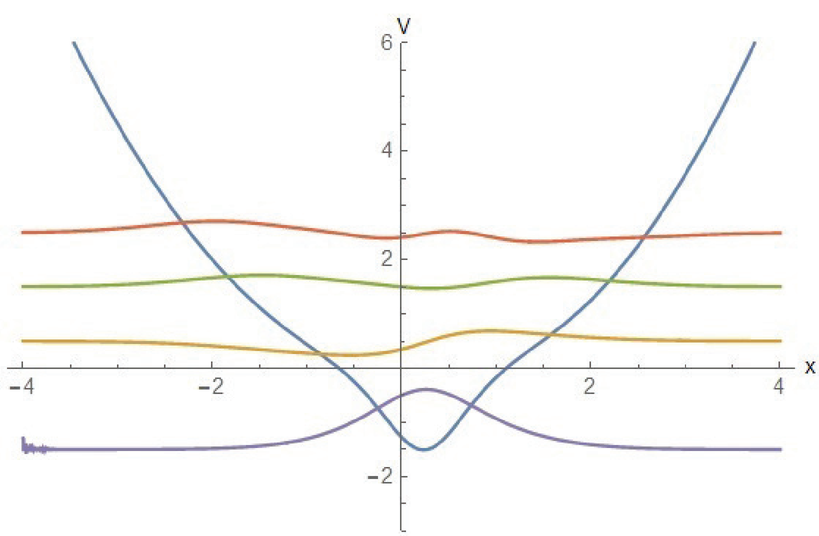

Figure 3.
A second-order SUSY partner of the harmonic oscillator potential with its first three eigenfunctions and a missing eigenfunction.

In order to generate a time-dependent transformation of the harmonic oscillator system, the following intertwining relationship between Schrödinger operators is used:

$$\mathbf{S}_1\mathbf{L}_1 = \mathbf{L}_1\mathbf{S}_0, \tag{36}$$

where

$$\mathbf{S}_1 = i\partial_t + \partial_{xx} + V_1$$
$$\mathbf{L}_1 = A_1\left[-\partial_x + \frac{u'(x,t)}{u(x,t)}\right],$$

where A_1 is a function of time and the function $u(x, t)$ is the transformation function. By solving the intertwining relationship shown in (36), the expression of a new potential V_1 and a relationship that u must fulfill are:

$$V_1 = V_0 + i\frac{A_{1t}}{A_1} - 2\partial_{xx}\ln(u), \quad i\partial_{xt}\ln(u) + \partial_x\left(\frac{u_{xx}}{u} - V_0\right) = 0. \tag{37}$$

With the integration of the (37) with respect to x, the function becomes

$$i\partial_t u + u\partial_{xx} - V_0 u = 0, \tag{38}$$

where A_1 is the real function

$$A_1 = e^{-i\int^t \partial_{xx}\ln\left[\frac{u(x,s)}{u^*(x,s)}\right]ds}. \tag{39}$$

In order for u to be an accepted function, it must satisfy a time-dependent Schrödinger equation. A substitution of the real function A_1 into the potential expression results in

$$V_1 = V_0 - 2\partial_{xx}\ln|u|. \tag{40}$$

The solutions $\hat{\Psi}$ of the new Schrödinger equation

$$(i\partial_t + \partial_{xx} - V_1)\,\hat{\Psi} = 0 \tag{41}$$

can be obtained by applying the intertwining operator $\mathbf{L}_1$ onto Ψ, in other words, $\hat{\Psi} = \mathbf{L}_1\Psi$. By referring to the relationship of the intertwining and Schrödinger operators of (36), the adjoint relationship states

$$\mathbf{L}_1^\dagger\mathbf{S}_1 = \mathbf{S}_0\mathbf{L}_1^\dagger, \quad \text{where} \quad \mathbf{L}_1^\dagger = A_1\left(\partial_x + \frac{u_x^*}{u^*}\right). \tag{42}$$

Thus, a function annihilated by the operator $\mathbf{L}_1^\dagger$ is also a solution of $\mathbf{S}_1\hat{\Psi} = 0$. This function, also known as the missing state, can be found by solving a first-order differential equation $\mathbf{L}_1^\dagger\hat{\Psi} = 0$, where

$$\hat{\Psi} = \frac{1}{A_1 u^*}. \tag{43}$$

As a result, the solutions of $\mathbf{S}_1\hat{\Psi} = 0$ can be constructed as $\hat{\Psi} = \mathbf{L}_1\Psi$ with the addition of the missing state.

4.2 Confluent Time-Dependent SUSY Transformation

Just as in the first SUSY transformation, an intertwining relationship with the Schrödinger operators must be considered:

$$\mathbf{S}_2\mathbf{L}_2 = \mathbf{L}_2\mathbf{S}_1, \quad \text{where} \quad \mathbf{S}_j = i\partial_t + \partial_{xx} - V_j, \quad j = 1, 2, \quad \mathbf{L}_2 = A_2\left(-\partial_x + \frac{v_x}{v}\right). \tag{44}$$

For this transformation, an intermediate transformation function $v(x, t)$ that satisfies $\mathbf{S}_1 v = 0$ must be used. If v is set equal to the missing state from (43), the original Schrödinger operator is obtained as a result of the second transformation. Therefore, a general expression for v is constructed as

$$v = \frac{1}{A_1 u^*}\left(p + \int |u(x,t)|^2 ds\right), \tag{45}$$

where p is a real constant. With this new transformation function, a new potential V_2 forms:

$$V_2(x) = V_1 - 2\partial_{xx}\ln|v| = V_0 - 2\partial_{xx}\ln\left|p + \int^x u(s,t)\,u^*(s,t)\,ds\right|. \tag{46}$$

In order to produce a regular potential for V_2, a transformation function is used that satisfies

$$p + \int^x |u(s,t)|^2 ds \neq 0. \tag{47}$$

Additionally, the transformation function must satisfy

$$\partial_{xxx}\ln\left(\frac{v}{v^*}\right) = 0 \quad \rightarrow \quad \partial_{xxx}\ln\left(\frac{u}{u^*}\right) = 0 \tag{48}$$

to ensure that V_2 is real, where the function $A_2(t)$ appearing in the intertwining operator $\mathbf{L}_2$ in (44) is defined as

$$A_2 = e^{i\int^2 \ln\left(\frac{v}{v^*}\right)ds} = e^{i\int^2 \ln\left(\frac{u}{u^*}\right)ds} = A_1. \tag{49}$$

The solutions $\bar{\Psi}$ of the equation $\mathbf{S}_2\bar{\Psi} = 0$ can be obtained from the solutions of the original equation

$$\bar{\Psi} = \mathbf{L}_2\mathbf{L}_1\Psi = A_1^2\left(\partial_{xx} - \partial_x \ln\left(uv\right)\partial_x + \left(\partial_x \ln v\right)\left(\partial_x \ln u\right) - \partial_{xx}\ln u\right)\Psi, \tag{50}$$

with the missing state

$$\bar{\Psi} = \frac{1}{A_1 v^*} = \frac{u\left(x,t\right)}{p + \int^x |u\left(s,t\right)|^2 ds}. \tag{51}$$

5. GENERATION OF THE TIME-DEPENDENT HARMONIC OSCILLATOR-LIKE POTENTIAL

In this section, two sets of time-dependent potentials are generated using SUSY departing from the harmonic oscillator utilizing a coherent state and a general Gaussian packet as transformation functions.

5.1 Coherent States

The time-dependent Schrödinger equation in one spatial dimension for the harmonic oscillator is of the form

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\partial_{xx}\Psi - \frac{1}{2}m\omega^2 x^2\Psi = 0. \tag{52}$$

For this particular case, we will let $\hbar = 1$, $m = 1/2$, and $\omega = 2$, for which the Schrödinger equation becomes

$$\left(i\partial_t + \partial_x x - x^2\right)\Psi = 0. \tag{53}$$

Given a Schrödinger equation with the set of solutions satisfying the boundary conditions, a transformation function $u(x, t)$ must fulfill three conditions: $i\partial_t u + \partial_{xx}u - V_0 u = 0$, $\partial_{xxx}\ln\left(\frac{u}{u^*}\right) = 0$, and $u(x, t) \neq 0$ for all $x \in (-\infty, \infty)$ and $t \in (-\infty, \infty)$. One such function that complies with all three conditions are the coherent states. These states are eigenstates of an annihilation operator characterized by a complex number. These states are important because they not only describe superfluids and super-conductors, but they also describe the quantum state of a laser in terms of the number of photons. One of these coherent states can be used as a transformation function to produce a SUSY partner of the harmonic oscillator [6]. The expression of a coherent state is

$$u\left(x,t\right) = \frac{1}{\pi^{1/4}} exp[-it - \frac{\left[x - r\cos\left(2t\right)\right]^2}{2} + i\frac{r^2\cos\left(2t\right)\sin\left(2t\right)}{2} - irx\sin\left(2t\right)]. \tag{54}$$

When a coherent state is used as a transformation function u to generate a first-order transformation of the harmonic oscillator potential, a new potential is generated as

$$V_1 = V_0 - 2\partial_{xx}\ln|u| = V_0 + 2. \tag{55}$$

It can be seen that this transformation generates a time-independent translation of the potential. The confluent SUSY transformation provides us with a more interesting potential

$$V_2 = x^2 - \partial_{xx}\ln\left(p + \frac{1}{\pi^{1/2}}\int_{-\infty}^{x} e^{-\left[s - r\cos(2t)\right]^2 ds}\right)^2, \tag{56}$$

where the value of the parameter p cannot be within $[-1, 0]$ in order to avoid singularities. To obtain solutions of the new Schrödinger equation with the potential V_2, the transformation function $u(x, t)$ given by the coherent state is used to form the v function (45). We then generate the solutions by using the intertwining operator $\mathbf{L}_2$ onto the original time-independent Schrödinger solutions. The transformed potential and wavefunctions are displayed in Figure 4 when $t = 0$, $t = \pi/4$, and $t = \pi/2$. The missing state was solved using (51):

$$\bar{\Psi} = \frac{e^{-it - \left[x - r\cos(2t)\right]^2 + i\frac{r^2\cos(2t)\sin(2t)}{2} - irx\sin(2t)}}{\pi^{\frac{1}{8}(x - r\cos(2t) + 1) + p}}. \tag{57}$$

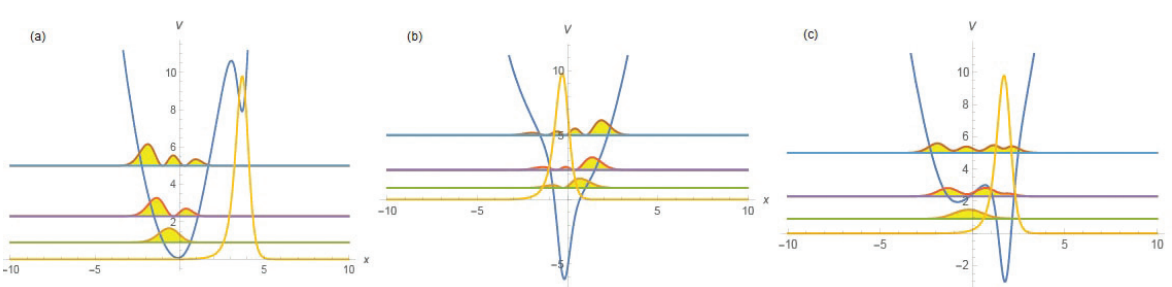

Figure 4.

A time-dependent second-order SUSY partner utilizing a coherent state as the transformation function with its first three eigenfunctions and a missing eigenstate at $t = 0$, $t = \pi/4$, and $t = \pi/2$. The coordinate system of the figure describes the potential; the y-axis for the wave functions would be $|\bar{\Psi}_n|^2$.

5.2 Simple Harmonic Oscillator Wave Packets

Given the time-dependent Schrödinger equation in equation (52) and the same parameter assignments for the coherent states example, a new system can be constructed using Gaussian wave packets. These wave packets can be described as a superposition of wave functions that travels as a localized wave (Robinett and Bassett, 2004). Those wave packets can also be used as transformation functions u with their mathematical expressions being

$$u\left(x,t\right)=e^{\frac{im\omega x^2\cos(\omega t)}{2\hbar\sin(\omega t)}\frac{1}{\sqrt{A(t)}\sqrt{\pi}}}e^{-\frac{im\omega\beta}{2\hbar\sin(\omega t)}\frac{(x-x_s(t))^2}{A(t)}}, \tag{58}$$

where $A\left(t\right)=\beta\cos\left(\omega t\right)+\left(\frac{\hbar}{m\omega\beta}\right)\sin\left(\omega t\right),\ x_s\left(t\right)=\frac{\sin(\omega t)}{m\omega}$, and $\beta = 1$. This transformation function obeys all three conditions demonstrating that it is an acceptable solution to the Schrödinger equation. By applying the first-order time-dependent SUSY transformation of the function as shown in Section 4.1, the new potential is an uninteresting translation. However, by applying the confluent second transformation as shown in Section 4.2, a unique potential is obtained. By utilizing the intertwining operator $\mathbf{L}_2$ on the wavefunctions, a series of wavefunctions, all following a pattern, is constructed. This can be explained from the idea that all energy levels of this wave packet move as a uniform system showing that all the energy levels depend on one another. From constructing this transformation in the confluent case, a missing state is obtained using (51). Figure 5 shows representations of this system at $t = \pi/8$, $t = 2\pi/3$, and $t = \pi$. Notice that the probability density of the missing wavefunction coincides with the anharmonic-like potential.

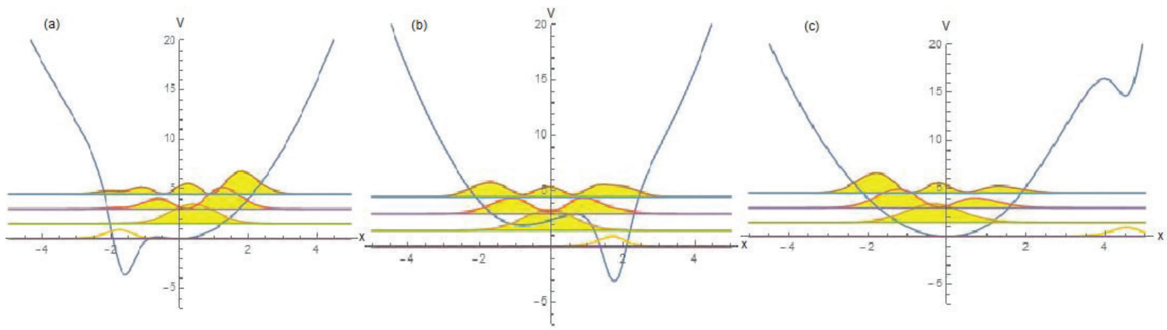

Figure 5.

A time-dependent second-order SUSY partner utilizing a Gaussian packet as the transformation function with its first three eigenfunctions and a missing wavefunction at $t = \pi/8$, $t = 2\pi/3$, and $t = \pi$. The coordinate system of the figure describes the potential; the y-axis for the wave functions would be $|\bar{\Psi}_n|^2$.

CONCLUSION

Both time-independent and time-dependent anharmonic oscillators were constructed using an intertwining SUSY transformation. In the time-dependent study, it was observed that an application of the first-order SUSY using both the coherent state and the Gaussian packet as transformation functions resulted in translation of the original potential. It required a confluent second-order SUSY to obtain a new anharmonic potential. By utilizing the intertwining operators $\mathbf{L}_1$ and $\mathbf{L}_2$, all solutions of the system became obtainable with the discovery of a missing state.

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APPENDIX: HARMONIC OSCILLATOR USING CREATION AND ANNIHILATION OPERATORS

The harmonic oscillator is one of the few quantum mechanical systems for which its exact analytical solutions are known. The factorization method in its most classical form was first used to obtain the spectrum of the Hamiltonian of this system in one dimension:

$$\mathbf{H}=-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2.$$

This method introduces the operators of "annihilation" and "creation," respectively:

$$\mathbf{a}=\frac{1}{\sqrt{2}}\left(\frac{d}{dx}+x\right)=\frac{1}{\sqrt{2}}e^{-\frac{x^2}{2}}\frac{d}{dx}e^{\frac{x^2}{2}}$$

$$\mathbf{a}^{\dagger}=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)=-\frac{1}{\sqrt{2}}e^{\frac{x^2}{2}}\frac{d}{dx}e^{-\frac{x^2}{2}}.$$

The commutation of the two operators is equal to one, which affords a relationship between them and the Hamiltonian

$$\mathbf{H}\mathbf{a}^{\dagger} = \mathbf{a}^{\dagger}\left(\mathbf{H}+1\right),$$
$$\mathbf{H}\mathbf{a} = \mathbf{a}\left(\mathbf{H}-1\right).$$

These relations allow for the generation of eigenfunctions and eigenvalues for the known Hamiltonian. With Ψ being an eigenfunction of $\mathbf{H}$ (i.e., a solution of the equation $\mathbf{H}\Psi = \lambda\Psi$), its corresponding eigenvalues arise:

$$\mathbf{H}\mathbf{a}^{\dagger}\Psi = (\lambda+1)\mathbf{a}^{\dagger}\Psi,$$
$$\mathbf{H}\mathbf{a}\Psi = (\lambda-1)\mathbf{a}\Psi.$$

With the assumption of an existing lowest eigenvalue and the former relations, the lowest eigenfunction can be constructed

$$\mathbf{a}\Psi = 0 \quad \Longrightarrow \quad 0 = \frac{d}{dx}e^{\frac{x^2}{2}}\Psi_0 \quad \Longrightarrow \quad \Psi_0\left(x\right) = C_0 e^{-\frac{x^2}{2}},$$

where C_0 is a normalization constant. Thus, the eigenvalue of the ground state is solved to be ½. The eigenfunction of the first excited state can be determined by applying the creation operator $\mathbf{a}^{\dagger}$ on Ψ_0. Hence, a general statement is made:

$$\Psi_n = \left(\mathbf{a}^{\dagger}\right)^n \Psi_0.$$

When $(\mathbf{a}^{\dagger})^n$ is acted upon $e^{-\frac{x^2}{2}}$, a hermite polynomial forms:

$$H_n = \left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

With this generalization, the Rodriguez formula can then be used to solve for any particular eigenfunction of any energy level within the harmonic oscillator system:

$$\Psi_n = C_n H_n\left(x\right)e^{-\frac{x^2}{2}},$$

where C_n are normalization constants.

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