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SIMCALC: DEMOCRATIZING ACCESS TO ADVANCED MATHEMATICS

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Historically, *what* people can learn is co-determined by the *representational infrastructure* for knowledge building. When Latin was the required medium of knowledge building, few could engage in scholarly activities; without the change to the vernacular, nearly universal access to higher education would not be possible. The highly compact, abstract, and opaque symbolism of mathematics presents similar barriers to the necessary *democratization of access* to important mathematics.

Over the course of a program of research lasting more than 20 years and involving contributors from institutions throughout the United States and worldwide (Hegedus & Roschelle, 2013), the representationally innovative design of SimCalc Mathworlds® has provided affordances for novel and effective approaches to teaching important algebraic and calculus-related ideas. When integrated with appropriate curricular workbooks, teacher professional development, and other instructional factors, *dynamic representation* has enabled diverse populations to learn more advanced mathematics. Research has included both design research as well as large-scale experiments involving hundreds of teachers and thousands of students; overall, the approach also has an unusually strong base of empirical support. We focus on lasting, essential design contributions of this body of work with a special emphasis on the dialectic relationship between affordances of technology and curricular progressions.

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Jeremy Roschelle is co-director of the Center for Technology in Learning at SRI International. In his research, Roschelle examines the design and classroom use of innovations that enhance learning of complex and conceptually difficult ideas in mathematics and science. Through cognitive science-based research on the Envisioning Machine and later SimCalc, he has explored how computer-based representations can make the mathematics of change and the related physics of motion accessible to many more students. Two running themes in his work are the study of collaboration in learning and the appropriate use of advanced or emerging technologies (such as component software and wireless handhelds) in education. His B.S. degree in computer science is from the Massachusetts Institute of Technology. His Ph.D. in learning sciences is from the University of California, Berkeley.

Stephen Hegedus is Dean of the School of Education at Southern Connecticut State University. Previously, he was the founding director of the Kaput Center for Research and Innovation in Science, Technology, Engineering and Mathematics at University of Massachusetts Dartmouth where he was also professor of mathematics and mathematics education. He also served as department chair in mathematics education from 2010-13. His present work involves the study and development of dynamic software environments, with associated curriculum focusing on classroom connectivity and haptic (force-feedback) technology. This project includes the professional development of pre- and in-service teachers and the large-scale integration of innovative technologies into K-12 curriculum. Previously, he held appointments as research fellow, educational consultant and lecturer at the University of Oxford in England. Dr. Hegedus earned his Ph.D. in mathematics education from the University of Southampton, England, where he earlier received a B.Sc. with honors in mathematics and economics.

INTELLECTUAL BACKGROUND

Backgrounds of the SimCalc Team and Present Authors

The founder and long-time leader of the SimCalc project was James J. Kaput, a mathematician and mathematics educator with a strong vision for the appropriate role of technology in mathematics education. Kaput often shared his vision via an analogy:

> *When knowledge was expressed in Latin, very few people could learn by reading; however, as knowledge was re-expressed in the vernacular, many more people could learn by reading. Likewise, when math is expressed in an arcane symbolism, very few people can understand it well; however, as technology allows us to find new ways to represent mathematical ideas, there is the possibility of enabling many more people to deeply learn mathematics.* (Kaput, Personnel communication)

Kaput was gifted not only in developing a vision and themes for his projects, but also in pulling together a diverse, multi-disciplinary team to execute them. Many kinds of expertise were represented on the team over time (Roschelle, Tatar & Kaput, 2008). At the onset, the team included Kaput, a mathematician; Nemirovsky, an expert in child development and mathematics; and Roschelle, one of the co-authors and a computer scientist and learning scientist. Over time, the team expanded and contracted. At various times the team included: mathematicians, scientists, teachers, experts in teacher professional development, experts in assessment, computer programmers, experimental psychologists, curriculum designers, and industry experts.

Two of the three authors of this retrospective analysis started their careers working with Logo and Boxer, an important rethinking of Logo that integrated programming, specific microworlds and hypertext to create a multi-purpose computational medium (diSessa, 2000, 1991; diSessa & Abelson, 1986). Based on this preparation, they recognized the great opportunity inherent in SimCalc. The first author, Deborah Tatar, was involved in several projects, primarily from 2000 until 2008 and is currently a professor of computer science with a focus on the design of systems that restructure knowing. The second author, Jeremy Roschelle, brings expertise both in computation and the learning sciences and has the longest history with SimCalc, running from 1994 until the present time and encompassing every aspect of the project from implementation to scaling. The third, Stephen Hegedus, brought a background in mathematics and mathematics education to bear on the project starting in 2000 and took on the running of the overarching project after Kaput's sudden and untimely death in 2005.

Strands of Research and Development for Technology in Mathematics Education

Approaches to the use of technology in mathematics education can be understood as drawing on three different predominant approaches (Drijvers, 2012). One approach emphasizes productivity, and emphasizes tools that are useful in everyday life, such as rulers, slide-rules, calculators, graphing calculators, spreadsheets, and the like. Another approach emphasizes support for structured practice with feedback and tutorials, and includes self-paced workbooks, drill and practice software, and game-like wrappers that motivate practice of mathematical skills. Today, the orientation to optimal practice of mathematical skills continues in the form of intelligent tutors and adaptive learning systems. A third approach is the subject of this paper, and concerns students' development of the ability to make sense of mathematics and to develop conceptual understanding. Predominant tools in this category emphasize simulation, visualization, and representation -—and often involve tools that are constructed to support deep learning, even if those tools are not currently used professionally in the way that a spreadsheet is.

Each of these approaches has a past, a present, and future. Each approach has existed across a range of underlying platforms from timeshare systems through to today's cloud and tablet solutions. Thus, it is not the case the one approach is more "modern" than the others. Further, each approach has applications and evidence of effectiveness. Thus, is not the case that one approach is "better" than the others. The different approaches have different goals, corresponding to different goals that co-exist in mathematics education. Designs, of course, are successful relative to goals. Consequently, will not attempt to compare designs across these different approaches. However, we draw on examples from different approaches to highlight the contrasting approaches with this specific design and its features.

LOGO AND THE CONCEPT OF MICROWORLDS

One of the great promises of computer technology in education has been transformation. Some projects try to change learning by, for example, liberating learning from the classroom; others by liberating learning from teachers; yet others, as in the SimCalc project1, have sought to use technology to liberate mathematics learning from arcane, esoteric symbol systems and render it more readily approachable and understandable.

Historical roots of the Simcalc Mathworlds® approach built upon an early great educational movement that was based on the Logo computer language for children; Logo was

[1] We use "SimCalc" to refer to the project and SimCalc Mathworlds® to refer to the implementation.

promoted and memorialized in Seymour Papert's 1980 book, Mindstorms (Papert, 1980). Authors Tatar and Roschelle were both deeply involved with the Logo movement. This approach contrasts with a "CAI" (Computer-Assisted Instruction) approach, for example as represented contemporaneously by PLATO (Hammon, 1972). Logo was built to be learnable along principles influenced by the great Swiss psychologist Jean Piaget; Papert offered the evocative analogy from how he explored the concept of ratio as a child using physical gears to how children could now explore a broader range of mathematical concepts using Logo as "gears for the mind." PLATO, in contrast, automated a traditional instructional approach consisting of providing the student with information, practice tasks, and feedback. Whereas PLATO offered teachers an approach to authoring instruction, Logo sought to offer children opportunities to construct their own computer programs. But Logo was more than a computer language.

The expressive form of *turtle geometry* allowed children to explore a rich panoply of outcomes related to the details of their programs. Programs could control the actions of a physical or virtual turtle, by asking it to, for example, move forward. A physical or virtual pen left a trail, thus allowing the children to at once draw and picture and have a trace of whether the commands had been executed as imagined.

As an educational community developed around Logo, its use moved beyond programming towards the development of constrained, playful environments in which students could explore powerful ideas of mathematics and science. These environments were termed "microworlds," and like Einstein's famous *gedanken* (thought) experiments, rendered technical ideas in a form conducive to playful engagement with fundamental ideas. Important principles (Hoyles & Noss, 1993) included: putting learning into children's hands, that is, treating them as *bricoleurs* (tinkerers) and letting them create; seeking newly accessible ways to render powerful ideas in a experience students could interact with; and "no threshold, no ceiling" environments which were initially simple but allowed engagement, over time, with complex endeavors (Abelson & DiSessa, 1986).

Whereas the design target in CAI systems was usually a course of study, the design of microworlds often began by identifying a foundational concept of science or mathematics which students were not reliably learning in a traditional course. Design work included the identification of foundational concepts and thinking out exactly how to invite and encourage engagement with those ideas, creating a kind of playground in which the learner would be brought back to them time-and-again. With these playgrounds, modeling was often a fundamental activity: students were invited to use scientific or mathematical constructs to reproduce a familiar phenomena or experience. For example, students might use the ability of a turtle to move forward and turn in small increments to model a circle as the limit of a regular polygon with increasingly short sides and small turns. Elements that are now often brought into discussion of learning strategies were taken as foundational. In particular, embodied learning and the use of virtual—and physical—manipulatives.

Over the years, there have been hundreds of implementations of micro-worlds in different areas of endeavor, ranging from music (Bamberger, 1974, 1976) through to chemistry (Schank & Kozma, 2002) and physics (White, 1993). The direction continues in projects such as those reported in diSessa's Changing Minds (2000), which focuses on bringing children into contact with powerful ideas, and has found new life in a variety of intellectual homes: via the Scratch language (http://scratch.mit.edu/), in Media Computation (http://coweb.cc.gatech.edu/mediaComp-teach, Guzdial, 2003), and Storytelling Alice (Kelleher & Pausch, 2007). Other areas of focus include: manifestations of computationally controllable objects (c.f. Hendrix & Eisenberg, 2006; Weller, Do & Gross, 2008); complex programming environments such as the parallel, distributed environment of NetLogo (Wilensky & Stroup, 2000) and AgentSheets (Repenning & Sumner, 1995); and game design (c.f. Nemirovsky, 1994); however, some of the most profound, long-lasting and widespread have been in the area of mathematics education.

The Context of Mathematical Instruction

In the 1980s, the possibility of new ways of engaging students with mathematical ideas began to intersect with a movement towards reform of mathematical curricula. Just as the launch of Sputnik in the 1960s gave rise to "new math," the influential report "A nation at risk: The imperative for educational reform" (Gardner, 1983) incited a wave of thinking about the future of mathematics education. Whereas in the early years of the 20th century, educators sought to enable all students to master shopkeeper arithmetic, now the focus began to shift to algebra for all—a dramatic increase in instructional challenge. Simultaneously, mathematics educators began to question whether educational goals should be limited to computational and symbol manipulation skill and pushed for mathematical attainment to include conceptual understanding and mathematical practices (such as expressing generalities). Presently, this shift continues with newer curriculum standards that emphasize not just skillful and accurate execution of mathematical calculations and procedures, but also focusing on conceptual development and enculturation into mathematical practices (e.g., the Common Core State Standards for Mathematics).

This shift in educational goals was supported by emerging mathematics education research, which was grounded in developmental and cognitive science approaches. Unlike instructional research, which tends to ask: "does this or that teaching strategy produce greater test score gains?", the newly emerging body of mathematics education research studied how individual learners build the next stage of mathematical thinking upon ideas and competencies they already had. This research was represented, for example, by the scholarly society "Psychology in Mathematics Education" and interlinked with the policy prerogatives noted above through the agency of an association of mathematics teachers, the National Council of Teachers of Mathematics. In some sense, this scholarship starts from the ur-question of why it is so difficult for so many people to learn mathematical concepts that are quite plain to those who already know them. The kinds of answers provided have to do with uncovering the detailed hidden entailments of mathematical thinking and the aspects of human psychology that make representations work or, sometimes, not work for particular learners as particular developmental moments.

This is the perspective most strongly represented by the third author of this account. Hegedus had completed his doctoral work investigating the metacognitive behavior of mathematics undergraduates solving single and multi-variable integrals when he joined the project in 2000, and championed continued thought about representational elements.

Dynamic Representations

SimCalc Mathworlds® (http://www.kaputcenter.umassd.edu/products/software/) constitutes one of a number of technologies for learning that dovetailed with and elaborated the opportunities for reform of mathematics within the context of a pre-existing body of scholarly thought about mathematics education. Other similar approaches which emerged at roughly the same time include Geometer Sketchpad (Jackiw, 1987-2007) and Cabri Geometre (LaBoarde, 1984-2007). This class of technology eventually became known for its "dynamic representation" approach. Like Logo, dynamic representations enabled learners to be active, playful, constructive, and expressive in a computer-based medium. But unlike Logo, dynamic representations do not focus on programming. Like microworlds, dynamic representations provide an invented, pedagogical environment that is meant to engage students with fundamental ideas of mathematics, rendered in an interactive and dynamic form. Relevant to the emergent development and cognitive psychology of the time, both microworlds and dynamic representations intend to activate students' prior knowledge, and through the activities of exploring and constructing, allow students to build new knowledge. However, whereas microworlds have somewhat more focus on a fanciful context for mathematical ideas, dynamic representations have more focus on providing interactive mathematical notions and representations.

Democratizing Access to Calculus: The Mathematics of Change and Variation

The overall educational purpose of SimCalc was, in Jim Kaput's, its progenitor's, words, to *democratize access to Calculus*. In 1992, when the project that would be SimCalc started, it was clear that the rate of change, co-variation, accumulation, approximation, continuity, and limits were arguably some of the topics that would be most important to children moving forward. Kaput was fond of arguing that whereas "algebra for all" was a necessary advance in educational goals for society in the 20th century, "calculus for all" would be a necessary advance in the 21st century due to the importance of mathematics in understanding and regulating processes of change. Importantly, Kaput conceptualized Calculus not as a course of study taken at the end of a long sequence of mathematical prerequisites, but rather as a strand of mathematical thinking that could develop beginning as early as elementary school and which could enrich classic middle school topics, such are proportionality. Thus Kaput used the phrase "mathematics of change and variation" (MCV) to break the mindset of Calculus as a specific course, and to instead focus on how the underlying ideas could develop over a decade or more of a students' mathematical development (Kaput & Roschelle, 1998).

At the heart of the SimCalc approach to MCV is the idea of considering rate as the relative change of two quantities (for example, position and time) which could be represented as the slope of a graph or a parameter in an algebraic expression or a motion or a set of values in a table. Technology provided a technical affordance for realizing these representations in a dynamic interactive form. Pedagogical and curricular research sought to exploit technology to allow a potential restructuring of when mathematical ideas could be explored by young students as well as upper high school. Introducing a dynamic, technological medium also allowed young children easy access to touch and manipulate mathematical objects, including moving pieces of graphs and watching the resulting changes to the movement of one to linked actors in a simulation. Later on in the evolution of the SimCalc program of research and development, the affordances of classroom networks were incorporated into the integrated software/curriculum suite of resources to enable students to make personal mathematical constructions that could be shared within the classroom and publically displayed by the teacher in many different configurations. This allowed some researchers to not only investigate the cognitive dimensions of learning the MCV with diverse populations of students but also affective dimensions of engagement and motivation as the participatory nature of the classroom changed (Dalton & Hegedus, 2013).

The aim of this program of design and research, "democratizing access," diverged from the contemporary emphasis on raising test scores, because Kaput sought to introduce students to concepts which were not commonly measured on tests—and to focus on conceptual understanding, whereas most assessments measure procedural skill. It also diverged from an emphasis on preparing students to use modern workplace tools, such as spreadsheets, by focusing more on mathematical insight than on mathematical applications. "Access" did not mean availability or affordability of technologies or textbooks, but rather access to meaningful opportunities to learn. Operationally, "democratizing" meant an emphasis on design and development of activities for students who would ordinarily be excluded from reaching a traditional Calculus course by deciding "I'm no good at math" or by not achieving suitable grades in prerequisites.

To achieve democratization of access, Kaput was always committed to the idea that technology and curriculum should be, indeed, *had to be*, co-developed to better build on *learner strengths*. Additionally, he was always committed to classroom-based education; classrooms are places where all students can have an opportunity to learn (overcoming, for instance, limitations of the resources available in their homes) and where socialization into a mathematical culture can occur. Emphasizing classroom-based education has consequences. First, it means that design must address the situation of having a teacher together with a group of students as well as the situation of the individual learner. Second, it means that (truly) no learner can be left behind.

These braids of thought, stemming from the potential of the computer, the detailed examination of the cognitive bases of mathematical knowledge, and a commitment to classroom-based education, led to a formulation of the SimCalc research project as one which *restructured knowing* through finding points of possible design action where learners' strengths, representational affordances, and a reorganized curriculum provided the opportunity to understand MVC in a new way. Some of this history has been reported, particularly in Roschelle, Tatar and Kaput (2008), which focuses on the research (rather than the design) trajectory of the project and itself draws on and summarizes diverse earlier sources, including a number of different studies (Kaput, 2001; Kaput, Noss & Hoyles, 2001; Nemirovsky, Tierney & Wright, 1998; Nickerson, Nydam & Bowers, 2001; Nemirovsky, Kaput, & Roschelle, 1998; Roschelle, Kaput & Stroup, 2000).

THREE DESCRIPTIONS OF SIMCALC DESIGNS

The issue of how the design of SimCalc technologies is described has depended on the context of the description and on the unfolding of projects that have themselves depended on opportunistic factors such as the particulars of novel technologies, shifts in policy concerns, alignments with school districts, teachers and curricula, funding opportunities, and the developments of thought about pedagogical leverage. Indeed, we prefer to think of it as a representational infrastructure or set of design principles that are, and could be used, in other mathematics software (Hegedus & Morena-Armella, 2009; Noss & Hoyles, 1996). Under this rubric, a wide range of functions have been investigated (including new curricular materials available at: http://www.kaputcenter.umassd.edu/products/curriculum_new/).

Nonetheless, in a major 2010 paper reporting the use of SimCalc in three large-scale randomized trials, the technology is described as follows, with five components:

1. Anchoring students' efforts to make sense of conceptually rich mathematics in their experience of familiar motions, which are portrayed as computer animations;
2. Engaging students in activities to make and analyze graphs that control animations;
3. Introducing piecewise linear functions as models of everyday situations with changing rates;
4. Connecting students' mathematical understanding of rate and proportionality across key mathematical representations (algebraic expressions, tables, graphs) and familiar representations (narrative stories and animations of motion); and
5. Structuring pedagogy around a cycle that asks students to make predictions, compare their predictions with mathematical reality, and explain any differences.

   (Shechtman et al., 2010 p. 839)

These components are explained as follows:

> *The SimCalc MathWorlds software provides a "representational infrastructure" (Kaput et al. 2007; Kaput & Roschelle, 1998) that is central to enabling this approach. Most distinctively, the software presents animations of motion (Figure 1). Students can control the motions of animated characters by building and editing mathematical functions in either graphical or algebraic forms. After editing the functions, students can press a play button to see the corresponding animation. Functions can be displayed in algebraic, graphical, and tabular form, and students are often asked to tell stories that correspond to the functions (and animations)…. In addition to proportional and linear functions, students and teachers can make piecewise linear functions, which can be used to model familiar situations.* (Shechtman et al., 2010, p. 839-840)

and were accompanied by a picture (Figure 1).

These five definitional elements were largely present in Kaput's 1994 description of what would become SimCalc:

> *"Imagine a pair of 12-year old students driving a computer-simulated vehicle that provides a windshield view and a carefully linked user- or system-configurable collection of data displays for the dashboard; one set of displays for time, another for velocity, and a third for position. These include sounds for each set (metronome for time, engine pitch for*

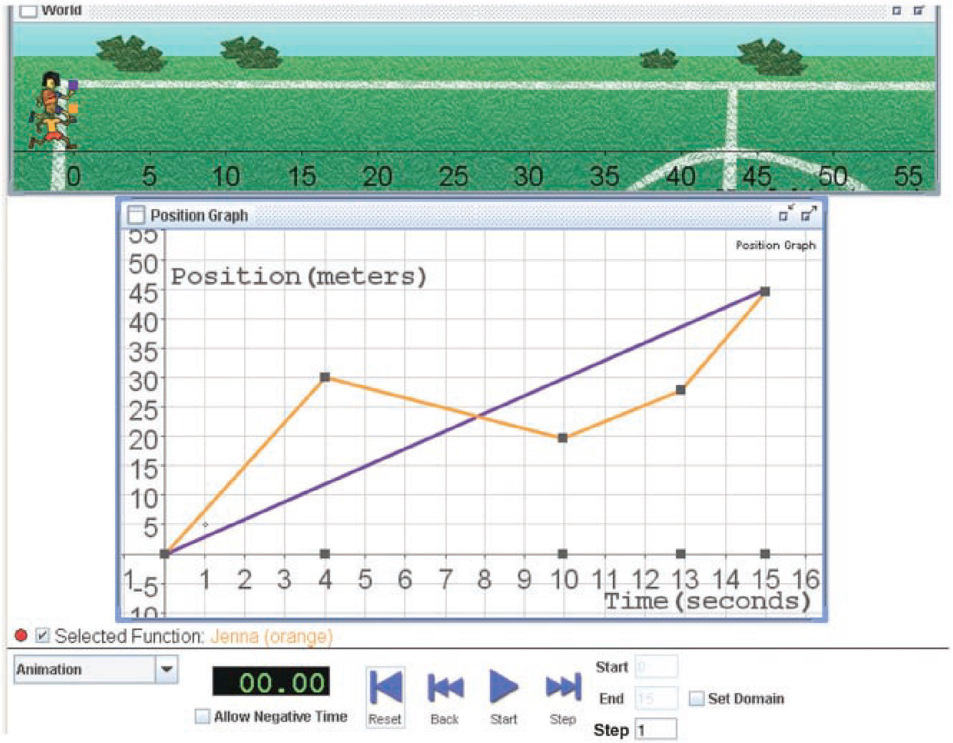

FIGURE 1. A picture used to explain a SimCalc Mathworlds microworld in the 2010 article. A position graph is shown related to the simulated situation shown in the "world" portion of the screen. The manipulation and animation functions are set in a window below. Playing the animation causes both the sweeping out of time on the position graph and the animation of the characters in the world. The motion of the character with the orange shirt (and rectangle) is described by the orange line while the motion of the character with the purple shirt (and rectangle) is described by the purple line. (Used with Permission.)

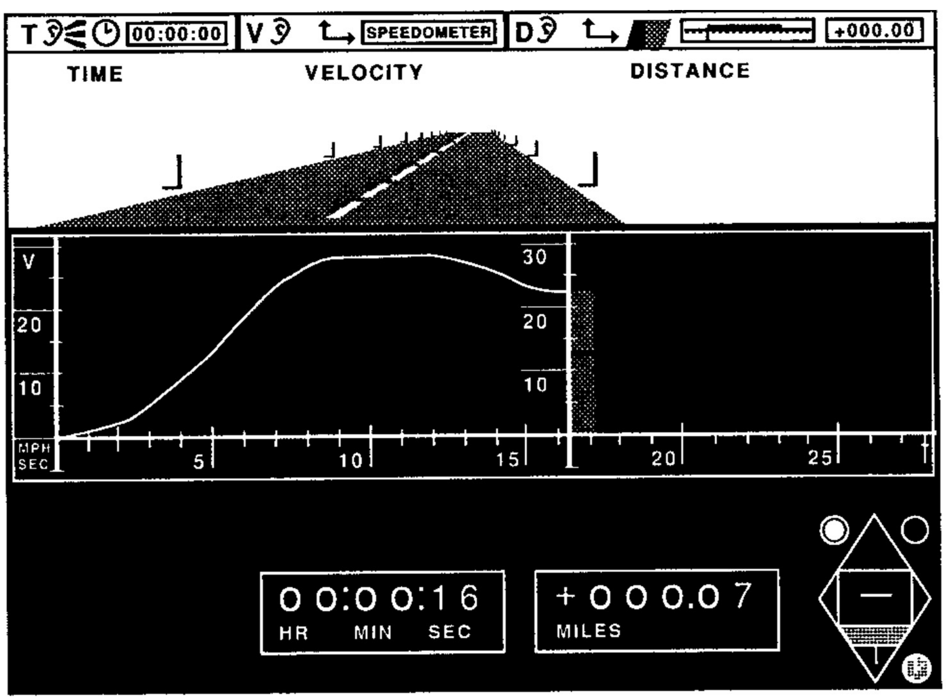

FIGURE 2. An Early Envisionment of SimCalc Mathworlds, circa 1992. The "experience" is shown as if through a windshield and the graph shows velocity as it sweeps out over time. (Used with Permission, quality as in the original.)

velocity and "echo" when passing roadside objects for position). The dashboard display can include velocity and/or position versus time graphs generated in "real-time" as well as clocks, odometers, tables and so forth. This "MathCars" system is designed to help link the phenomenologically rich everyday experience of motion in a vehicle to more structured and formal representations and to provide exciting and intensely experienced contexts for reasoning about change, accumulation and relations between them.

After some unstructured driving trips, they are now planning to follow a school bus whose (highly variable) velocity has been specified beforehand based on (one-dimensional) velocity data they collected on their own bus trip home the day before. . . ." (Kaput, 1994, p. 391)

Aspects of this vision for the design appear in an even earlier picture as reproduced in Figure 2 (Kaput, 1992, p. 540).

The major focus in this view is on authenticity of the motion phenomenon. However, the design focus in the 2010 paper is also mentioned:

"They will also set up and run simulated "ToyCars" on parallel tracks to study relative motion more systematically, describing the motion of each algebraically, confronting such questions as how to describe a later start versus describing a simultaneous start but from different locations. . . . " (Kaput, 1992, p.392)

These two descriptions represent both views of the opportunity space, and perspectives on what constitutes research on learning. In fact, Kaput's 1994 article reads like a mathematical proof. It makes arguments for a set of apparently disconnected beliefs and circumstances, taking particular care to unpack the relationship between the child's physical interactions with the world, the child's experience of physical interactions and the mathematician's formalisms. It then assembles the findings into the vision quoted above. In contrast, the 2010 description is a starting place for exploration of how a particular implementation of "technology + curriculum" fares as it faces the world.

An Important Sidetrack

Despite the impressive similarities between the 1994 and 2010 presentations, the five definitional elements were *not* obvious at the beginning. Instead, the focus on these five elements evolved and emerged throughout the life of the effort. With hindsight, we can see that the principles are consistent with the earliest designs even if they were not articulated in 1992 or even by 1994.

Kaput was very enthusiastic, in the beginning, about designing features that would engage students' kinesthetic sense, such as the visual sense of the world "zooming by" both sides of the perceptual field. Over time, this design feature was de-emphasized. Likewise, Kaput was interested in supporting motion detectors to measure motion in the physical world and sensors to measure a student's physical motion; these capabilities continued to be supported in the software, but became less important to the classroom experience of most users over time and played a smaller role in design thinking.

On inspection, it might appear that the design process to go from Figure 2, the initial vision, to Figure 1, the eventual software, was a straightforward elaboration of design principles specified at the onset. This was far from the case. Indeed, as is the case with many educational design processes, the team was distracted by "red herring" design principles which have nearly ubiquitous presence in public and scholarly discourse about learning technology. Then, as today, "games" were hot, and the team spent much time trying to translate the initial design concept into an educational game. Likewise, there was a belief that students would not pay attention to educational software unless it had the highest quality artwork and animations. In addition, advisors advocated for a rich "narrative" context as necessary to motivate students. Many of these concerns still loom large today as developers design modern software for mathematics learning.

Figure 3 illustrates an important path the design team followed for about a year and which turned out to be a dead end. Figure 3 is a screen shot of Alien Elevators, which was the first software designed by the SimCalc project, in 1993 and early 1994. Looking back, Roschelle, who is one of our authors, said "What were we thinking?" The two other authors were puzzled by the image: "what is this?" And indeed, the reason this was design was abandoned is that students were similarly lost and *not* engaged in productive mathematics. Yet, this design was purposeful, and ultimately led to key insights.

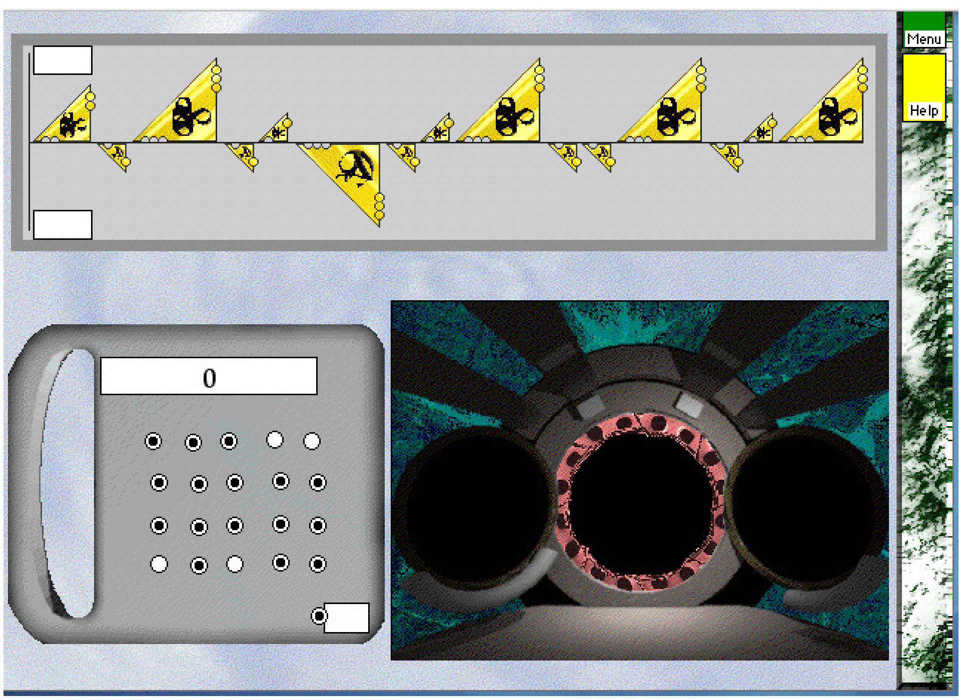

FIGURE 3. The "Alien Elevators" 1994 version of SimCalc created an interface that was gamelike, based on a narrative, and involved high-quality graphics but which submerged the mathematics. Students were asked to set velocity in 1 second increments (bottom left). They could see the alien elevator experience (bottom right). Velocity is shown by the glyphs at the top.

Much work went into an exciting narrative, about a journey to a planet where a lost alien society had disappeared, but the elevators were still running, and a mystery had to be resolved. Students would have to learn some mathematics to solve the mystery: on this planet, elevators were controlled not by final destination but rather by setting their velocity in 1-second increments. Further, velocity was indicated by a glyph—the strange triangles in the upper portion of the image. Students could experience being "on" the elevator as they travelled the lost alien world to solve the mystery—and the lower right section showed an animation of an elevator door opening and closing like an camera's iris on different floors. The lower left image was a controller which

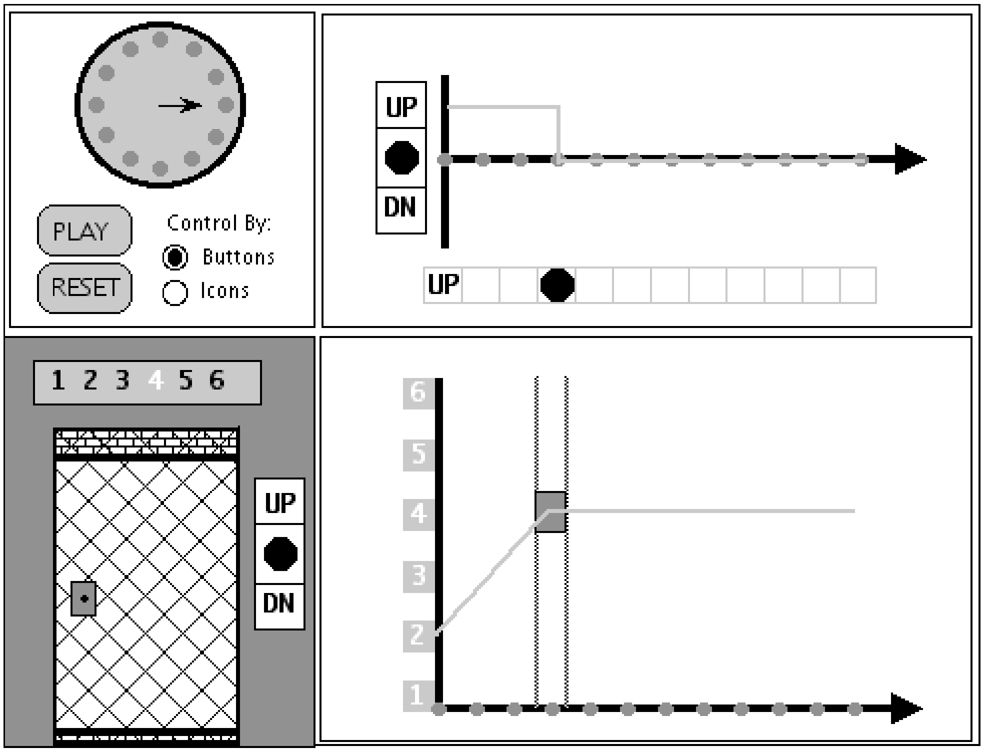

FIGURE 4. Controls for the 1994 "Alien Elevator" version of SimCalc included a stylized position graph (bottom right), a velocity graph (top right), a clock and an elevator. The mathematics was difficult for students to perceive.

gave access to tools—such as more traditional graphs, by which the strange glyphs and the motion of the elevator might be analyzed. Figure 4 shows some of these tools in the form of an early design section—they included a velocity graph, a stylized position graph which included an "elevator" shaft which moved to the right on the horizontal axis with time and a clock.

On reflection, the biggest failure in this design is that the mathematics was obscured by the narrative, game elements, and graphic art. When the SimCalc team turned away from this design thinking, the turn was towards an approach that was deeply mathematical and squarely focused on supports for students to conceptualize mathematics—with a much lighter touch on narrative, gaming, and graphic art.

Enduring Features

Despite the side-track of alien elevators, some key design differences between SimCalc as Kaput first imagined it in 1992 were largely finished by 1997. These changes SimCalc Mathworlds were realized across five years through a process of considerable design research and included:

- The representation of motion shifted from a 1st person ("point of view") perspective to a 3rd person, flattened perspective. Although the 1st person view is experientially compelling, it was hard for students to make connections between distance in a graph and distance in a windshield view.
- The students' opportunity for control shifted from controlling via a gas pedal and brake to control by changing the graph itself (as indicated by the square control points on the graph in Figure 1). This followed the realization that by giving students the ability to construct the more mathematical representation (rather than just see it as an output) they could better come to understand what it meant. The output became the movement of the soccer players.
- The nature of the mathematical function changed from a curve to a piecewise linear function. This reflected important growth in understanding about mathematics education. In particular, the project learned that curves were cognitively difficult objects for students to make sense of and that the learning progression could eventually get to curves from piecewise functions by showing how functions made of smaller and smaller pieces could come to approximate curves.
- There is also a noticeable simplification in the number of display elements in the eventual SimCalc Mathworlds design, reflecting the insight that it was essential to focus the learner's attention on a few representations at a time.

There is one contrast between Figures 1 and 2 that is *not* indicative of a design change: one image shows a velocity graph and the other shows a position graph. SimCalc MathWorlds has always had activities with both velocity and position graphs.

Figures 1 and 2 present a snapshot of changes; however, the contrast does not adequately explicate the nature of the design. Hence, we now move to a broader overview of the set of SimCalc projects and the design thinking that emerged in them.

Importantly, the *design* principles that constitute the focus of this article were complemented by a larger *implementation* principle when SimCalc was introduced into a large number of classrooms between 1997 and 2005. The larger implementation principle is to present teachers with an integrated system of the software, curricular workbooks, and teacher

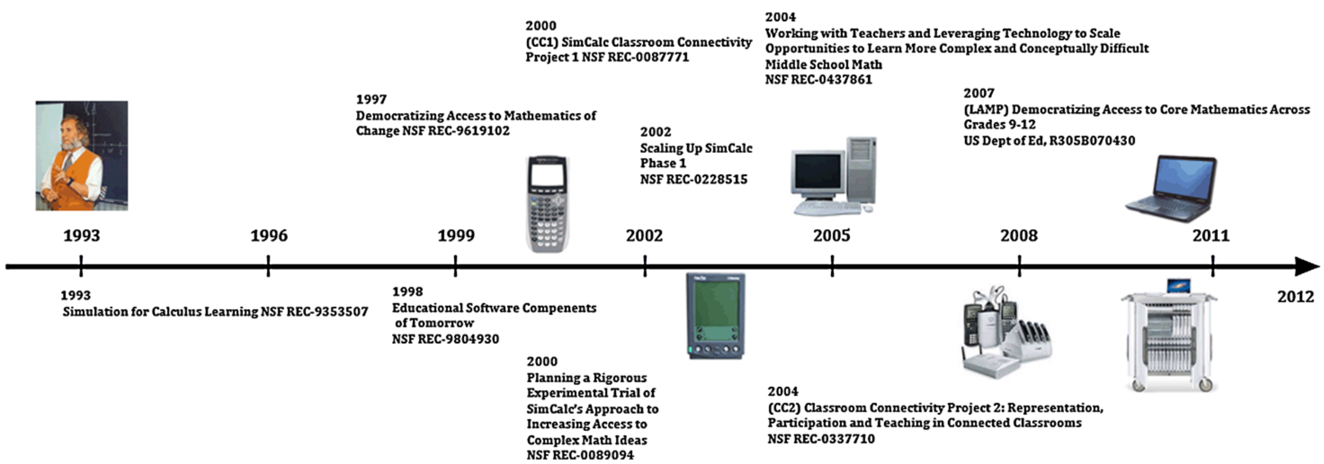

FIGURE 5. Timeline of the SimCalc project grants (Hegedus & Roschelle, 2013. Used with permission).

professional development. The achievement of a stable learning effect when SimCalc is introduced in hundreds of classrooms is importantly *not* due only to software features. Rather, it also is the consequence of carefully designed workbooks that lead teachers and students through a curricular learning progression with the software (and including exercises and discussions without the software) and is a consequence of teacher professional development that encultures teachers into appropriate classroom use of the software.

An Overview of SimCalc Projects

The larger SimCalc project starts from an ideal of improving mathematical teaching and learning, a mechanism, the computer, and a series of perceptions about learning and learners. It developed into a family of projects, each of which explored a facet or aspect of the whole. One history of the effort is given in Roschelle, Tatar and Kaput (2008), with a focus on the processing of going from small design studies to larger classroom tests. A graphical timeline appears in Figure 5.

1992-1994: Planning

An initial planning period was concerned with the examination of curriculum, the history of mathematical thought and a review of the learning sciences. Also during this time, several years were spent conducting the microanalysis of very small numbers of students working with different designs for the representations.

1994-1997: First Iteration

The first software design was implemented and then abandoned after less than a year. This design had an overarching narrative concept called "Alien Elevators" and was an extended game in which students would infer rules by which elevators were controlled on an alien planet, where the elevator buttons controlled velocity, not the target floor. This was abandoned because it was found in user testing that the story distracted students from the mathematics and the interface did not yield mathematical insights for students. However, one component of the interface was very productive for students and the project moved forward focused on this element. The element that was retained was a representation of velocity on a graph as a step function, where each step specified a constant velocity for a duration of time.

The work of this time consisted of experiments with a small number of students in a lab, or short teaching experiments in a classroom, each examining how the emerging SimCalc Mathworlds dynamic representations could enable students to develop particular target mathematical understandings.

1997-1999: Working with Teachers and Classrooms

This was followed by a second phase, three years spent on curriculum, that involved different educational settings and partners---in Newark, NJ, Syracuse, NY and San Diego, California. Notably, sites were chosen to include diverse students who would not ordinarily go on to study calculus. In addition, tests were conducted with students at different grades, include middle school, high school, and early undergraduate years. At this time—while many of the ideas were beginning to gel but not yet set—there was enough stability to involve teachers and classrooms full of students. However, even after nine years of work and development, measurement of learning outcomes only used research-designed pre-test/post-test assessments, consisting of items as created in response to the special purposes of the particular innovations. These projects resulted in an important diversity of curricular materials, variations of the software, and test questions.

2000 and after: Three Parallel Investigations

At this point, nine years into the project, it split. This split was did not reflect a lack of sympathy between project participants, but rather the need to pursue parallel avenues of exploration and therefore to involve people with expertise in those particular areas. There was good reason to believe that the core ideas were solid but it was not clear how they could become widely used. Three avenues were explored: technological, political and scientific.

Technological explorations

The technological exploration started from the observation that, although most students had theoretical access to computers, only graphing calculators received widespread, frequent use. Therefore, the research turned to how smaller, less-expensive devices could be used to make the key affordances available. The small size of the devices pushed the research to explore distributed, social, networked activities (Hegedus & Kaput, 2004a, 2004b, 2003; Hegedus, Kaput & Lesh, 2007; Hegedus & Penuel, 2008; Hegedus & Roschelle, 2012; Dalton & Hegedus, 2013; Roschelle et al. 2003; Tatar, Roschelle, Vahey & Penuel, 2003; Vahey, Tatar, & Roschelle, 2004, 2006). Some of this work was supported by Texas Instruments, building on a network infrastructure that they were developing and teacher professional development facilities that they supported. Work with the graphing calculator was awkward because the screen was small with low resolution and calculator keys had to be repurposed to implement SimCalc Mathworlds functions. Therefore another avenue was also explored: the then novel (and now defunct!) Personal Data Assistant, in particular, Palm Pilot handheld computers. These devices provided infra-red beaming, a low-overhead technology nicely suited to classroom communication (Tatar, Roschelle, Vahey, & Penuel, 2003; Vahey, Tatar & Roschelle, 2006; Vahey, Tatar & Roschelle, 2004). These projects led to the design of distributed activities that were social and fun but that always drew the student's attention back to important and difficult mathematics.

Political explorations

The political development had to do with influencing the key state mathematics examinations in Massachusetts. In particular, Kaput's influence over the construction of the high-stakes examinations resulted in a more rational and principled framework. Kaput and Hegedus additionally worked on the construction of SimCalc-based curricula, which they conceived of as a progression throughout middle and high school (http://www.kaputcenter.umassd.edu/products/curriculum_new/).

Scientific explorations

The third avenue was scientific demonstration. From 2000 until 2008, culminating in the 2010 paper, the project planned and then conducted a series of large-scale experiments, including randomized trials (Roschelle, Tatar, Shechtman, & Knudsen, 2008; Shectman et al., 2010; Tatar, Roschelle, Knudsen, Shechtman, Kaput & Hopkins, 2008). More than two thousand students, and 150 schools were involved. A pilot plus three different experiments with 7th and 8th graders in Texas demonstrated and replicated that SimCalc Mathworlds could produce significant learning gains in important mathematical concepts across a wide-range of teaching circumstances.

This level of demonstration was a triumph and should be seen not just as a confirmation SimCalc Mathworlds itself but also of the design-based research methods used at different scales throughout the early phases of the project. Such methods are necessarily complex, require intense scholarship, and can lead to substantial setbacks, as in the initial implementation; however, they can report real and important learning changes. The success of SimCalc at scale confirms the importance of support for the slow accretion of knowledge about learning and educational change.

Creating the assessments used in these experiments was difficult. It required over $1,000,000 in funding to develop, assessments that were altogether sensitive to the intervention, spoke to teacher and administrator concerns about curriculum, had the right reading and cultural properties, and could be administered within a classroom period. However, none of this development could have happened without pre-existing theories of learning and the roughly 700 test items gleaned from the classroom work over the years (as well as other scholarly studies of algebra learning also primarily supported by the National Science Foundation). The assessments used in these experiments are one example of an element that only worked because of diversity and persistence in the prior work.

The Changing Landscape

In theory, the technological, political and scientific elements of the project could have been more substantially supplemented by a fourth element, an economic strategy. Indeed, work with Texas Instruments moved in that direction and resulting, in part, in the TI nSpire handheld device. This device does incorporate dynamic representations, particularly for geometry, graphing, and data; however, it stopped short of including SimCalc representations such as motion and editable piecewise graphs. Further, other dynamic representation-based projects, such as The Geometer's Sketchpad, did pursue and succeed as business ventures (for a time); eventually, The Geometer's Sketchpad was undermined by changes in the market and by the availability of a free, open-source clone. Furthermore, at the very time that these projects were attempting to improve and widen instruction in mathematics, policies such as No Child Left Behind (http://www2.ed.gov/nclb/landing.jhtml) were in essence causing teachers and districts to become more risk-averse (Dickey-Kurdziolek & Tatar, 2012). It may well be that economically self-sustaining models of adoption are an unrealistic burden on an intervention aimed at changing so many elements of existing practice at the intimate level of the learning invisible from outside the classroom.

DESIGN RATIONALE

With this overview of the construction and development of the family of projects until 2008, we consider the rationale behind, and the implementation of, each element of the 2010 definition. These are the elements that, as the larger project has developed, have become assumptions in research papers. Yet the pedagogic opportunity lies in the details of how these elements are supported by the technology, and understood and utilized in the classroom. These are the elements that each teacher, curriculum designer, assessmentcreator, and technology designer needs to grapple with.

Anchoring students' efforts to make sense of conceptually rich mathematics in their experience of familiar motions, which are portrayed as computer animations

Kaput approached the ideas that would become SimCalc though historical, curricular and literature analysis. By the end of the 1980's, some kinds of computational environments for learning allowed learners to connect formal algebraic expressions with graphical representations, so that the learner could "follow along." Building on prior work examining mathematical representations, and starting as early as a 1992 publication, Kaput identified the *context of motion* as missing from instruction in Calculus (and algebra).

Although the development of calculus was historically motivated by the desire to describe motion phenomena, instruction had virtually no relationship to authentic contexts that motivated the work itself. Indeed, often teachers explicitly reject the idea of using motion to introduce algebraic or calculus on the grounds that motion is not mathematics. It is, instead, physics. But motion is not only an academic topic that must be described in formal terms. It is also a universal human experience.

Kaput perceived that integrating familiar aspects of motion into mathematics instruction could benefit students by allowing the redistribution of "sources of structure and action from the mental to the physical realm." (1994, p. 394). However, Kaput's initial thought about how to implement this experience evolved through small group work with children. The elements of realism that were featured so vividly in the original description were refined into more strategic and abstracted representations of motion seen in the "world" graphics. Although the earliest work resembled current approaches to games and game-like environments, this was soon dropped. While capturing student interest and engagement is important, and using existing student strengths, such as their experience of the natural world, is crucial, the experiences must not overwhelm or downplay the mathematics to be learned.

The decision to implement motion as animations or depictions in an artificial "world" allowed the scope of inquiry to be simplified to the representation of movement along a line (or, better yet, a number line!). This simplification created a congruence between the portion of motion depicted in the system and what was actually modeled in high school algebra.

Other aspects of the early vision did not make it into the branch of exploration expressed in the 2010 description, albeit for pragmatic rather than pedagogical reasons.[2]

Engaging students in activities to make and analyze graphs that control animations

Using proper notation is a metric of understanding of algebra and calculus; therefore, use of that notation is usually prioritized in instruction. Yet, arguably, the roots of student understanding lie not in algebra, but in the depiction of what is important about the motion phenomena. Graphical representations are less compact than algebraic ones; however, graphs are a more common, everyday representation. For example, one often sees graphs in the newspaper, but hardly ever sees algebra in the newspaper.

In particular, like the motion itself, graphs can be animated over time. Contrast between the depiction of the motion in the "world" and the depiction of the graph help students learn *how* a graph represents. The graph is an abstraction, that, by its nature draws our attention to certain aspects of motion (change in position over time, change in velocity over time) which we suggest are important.

The context of motion implicitly suggests that time might be an important thing to think about. The everyday description of motion utilizes time. The graphical representation of motion makes time an explicit element. But when we make time an explicit element, we change what we are showing about position, compared to an actual depiction of that motion. Thus, a graph, such as shown in Figure 1, does not show a picture of a person going up and down hills but rather a metric of the person's displacement in two-dimensions.

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[2] For example, a branch of pedagogical exploration considered collecting real data using motion detectors (ultrasound sensors from cameras), but did not become part of the mainstream project, because it would have introduced another object for schools to purchase. Yet, the exploration of physically-embodied phenomena and varieties of mathematical notations, and the use of hybrid physical/cybernetic devices embodying dynamical systems continued and continue (Brady, C. (2013). Perspectives in Motion (Unpublished doctoral dissertation). University of Massachusetts, Dartmouth, MA).

This point often seems simple to those who already know algebra, but it is hard to overestimate its importance to large categories of learners. In recent years, research has suggested that people differ in their ability to interpret different kinds of information. Some people, those who prefer linguistic kind of information, do well with current teaching and learning practices. Others already bring to the project of learning mathematics, a tendency to interpret depictions as symbolic. But a subset of students tend to see graphs as pictures (Hegarty & Kozhevnikov, 1999; Kozhevnikov, Kosslyn & Shephard, 2005). Even those who are inclined to see graphs as symbolic representations may become confused about the nature of a particular representation.

Engagement with *how a graph represents* was enabled by a core innovation in the SimCalc approach: to *make* and *edit* graphs without having to edit the algebraic notation, and this gave students an easy way into manipulating mathematical representations of motion. Indeed, in the SimCalc approach, students often learn about graphs and tables *before* they encounter algebraic notation. Rather than starting with formal symbolic notations, students' mathematical experiences are gradually formalized.

Animating graphs interacted with design decisions in ways that were not central to the mathematics but that were central to the HCI and pedagogical usability of the system. Drag-and-drop facilities meant that many graphs could be made and their motion consequences easily explored. One key interface element in enabling easy exploration was the implementation of snap-to-grid "hot spots" that allowed students to easily explore integer end-points. This was controversial because it compromised continuity, which is an important mathematical concept. However, pragmatically, trying to make lines do exactly the right thing can be a time consuming distraction. The downloadable version of SimCalc Mathworlds (http://www.kaputcenter.umassd.edu/products/software/) allows users to turn off snap-to-grid facilities. Another set of difficult usability issues had to do with the relationship between grabbing and pulling function lines as compared to changing axes or labels in the world's ruler or the Cartesian coordinate plane.

These two elements, animation and tying animation to easily manipulated graphical representations, can lead to subtle but important curricular changes. In particular, one place that we lose active cognition amongst students is in introducing the idea of slope. Most students learn the slope of a line as a calculation of "rise over run", often fixating on the identification of points that make the actual calculation easy. The slope then becomes one calculation among many, a calculation that, for mysterious reasons, is sometimes negative. SimCalc allows the teacher to ground an understanding of slope in a far more sophisticated context, a context in which rate is demonstrated to be instantaneous as it sweeps out, connecting the characters' motions with their positions at a given time. Conceptualizing slope as a description of the relationship between time and position leads towards calculus without demanding the mastery of algebra and grounds the concept of negative slope as "going backwards".

Another kind of curricular change permitted by SimCalc representations is the more coherent presentation of proportion as a reduced case of rate in which the line just happens to go through 0. Proportion is a major middle school topic, but often is presented merely as a "calculate the missing quantity" problem, where three numbers are given and the fourth must be calculated using the formula "a/b = c/d." Of course, this formula can be useful to permit calculations to figure out how much 5 pounds of potatoes will cost if potatoes are $3/2 pounds. However, it is also a mathematical dead end—it doesn't lead anywhere in further mathematics. SimCalc represents proportion instead as a constant of proportionality, k, in $y=kx$, which is the slope of a line. The analysis of slope as a ratio, $k = y/k$, and a proportional function as a simple case of a linear function, allows a trajectory of mathematical development that continues from middle school through calculus.

Introducing piecewise linear functions as models of everyday situations with changing rates

Traditional instruction in algebra and calculus emphasized the definition of a function and the importance of continuity in the definition of a function. The continuity assumption is key to the ability to calculate inherent in Calculus. But those students who went on to become engineers would go on to use piecewise functions extensively, because many physical systems are best modeled not as one continuous curve but as discontinuous segments that may each be represented as linear (at least well enough for their engineering purposes).

All motion of an object over time is continuous. However, people's experience of motion is not continuous. One of Kaput's major insights from the first iteration was that, by introducing piecewise linear graphs earlier and delaying the introduction of the idea of continuity, many important ideas could be introduced earlier and more effectively into the curriculum to a wider range of students. This notion utilizes the principle of building on existing student strengths, but it is legitimized by engineering practices.

This representation was easy for students to control, by adjusting the height and width of rectangle "chunks" of velocity (where the height was speed, the width was time, and the area represented change-in-position). It also turned out that students could easily understand the area as position change, and this led to interesting mathematical challenges, such as finding different ways to move 6 meters (see Figure 6).

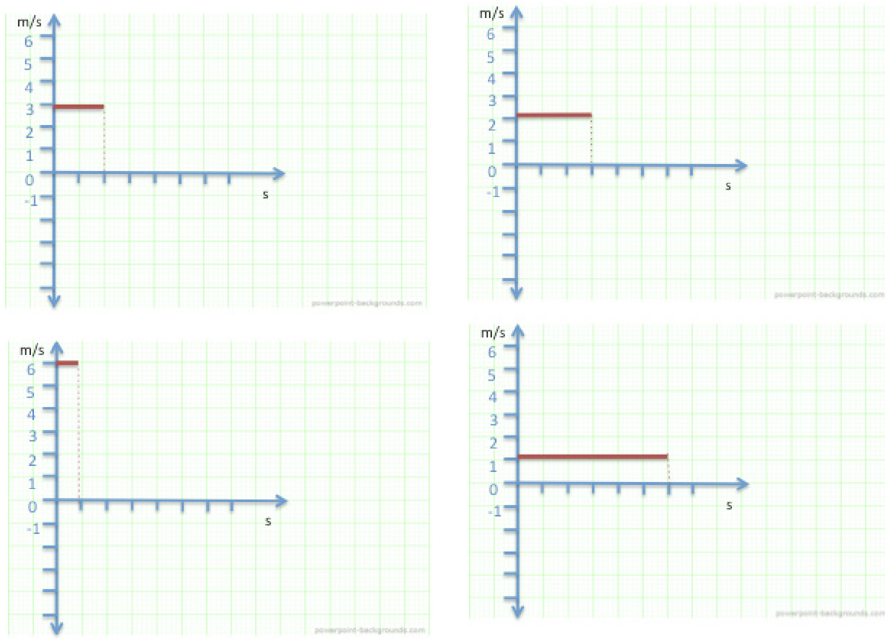

FIGURE 6. Changes in velocity: different ways to move six meters.

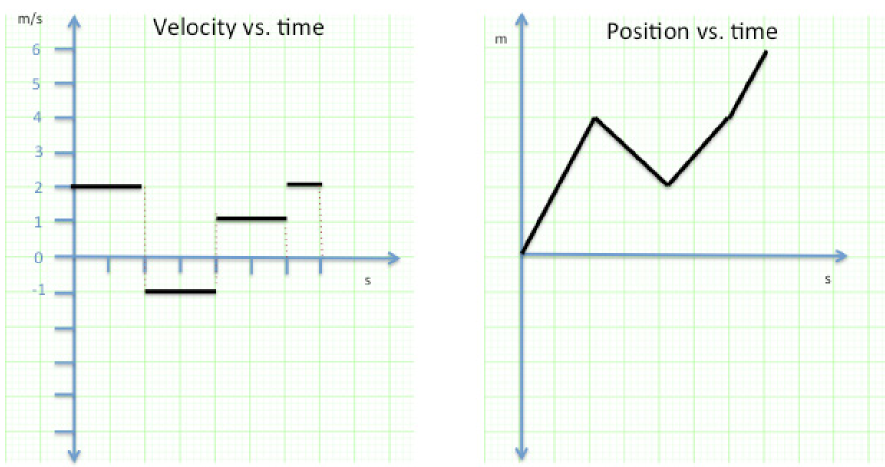

FIGURE 7. Piecewise graphs are easier for children to understand than continuous ones.

Further, the velocity graphs could be related to piecewise position graphs, which were also found to be productive in terms of student insight. Figure 7, for example, represents a complex way to get to a final position of 6 meters, but with changing speed and backwards motion.

The introduction and prioritization of graphical experience with piece-wise linear models is the cross-cutting computational, pedagogic, and conceptual insight that democratizes access to the math of change and variation. By enabling students to work with piecewise linear functions, that is, functions over a limited domain, we can let them explore the descriptive properties of the mathematical language we are introducing first, before showing them that the physical experiences of motion that arise as a consequence of being human are not precisely what Newton was modeling in creating that mathematical language.

A key activity used in conjunction with piecewise linear functions is the "exciting sack race" lesson. This is the more developed form of "driving behind a school bus" from the original conception. The students are given or create one function representing a person who runs a race at a constant speed over some domain (the straight line in Figure 1). They then have to create another line, representing a "crazy" race---like a sack race---putting together functions piecewise on the graph. The only rules are the race must start at time=0 and end in a tie. Subsequently, students are asked to write (in words) the story of the race. Often these stories are on the order of "Jane started really fast, but then she realized that she had forgotten her sunglasses so she ran back to the starting line to get them, but by then she was so exhausted that she couldn't run as fast, so she moved along, but she started staggering and being confused and sometimes went backwards until she finally stopped for a while. That gave her a rest, so she finished really quickly." Some or all students read their stories aloud, while the class looks at their graphs. Sometimes, teachers ask students to exchange these stories, and as the new person to draw the function line from just the story. Then the students compare the original and the new lines. This is usually fun, but also motivates the future use of more precise, specialized mathematical language.

In particular, by starting with graphs as compared to algebraic expressions, by tying those graphs to motion phenomenon and finally by allowing the graphs and the motion to model complex motion phenomenon, student learning can be grounded in the desire to represent tractable and interesting problems. This enables complex material to be taught significantly earlier in the curriculum, and indeed elements of the core idea in SimCalc have been taught as early as 5th grade (9-10 years old).

Both the inclusion of motion phenomenon in mathematics and the emphasis on animating graphs allows changes to the curriculum and changes in emphasis within existing curricula. However, the introduction of piecewise linear functions requires a shift in thought about the core material taught in Calculus. The graphical, piecewise approach *motivates* the more succinct expressions found in Algebra and Calculus.

Connecting students' mathematical understanding of rate and proportionality across key mathematical representations (algebraic expressions, tables, graphs) and familiar representations (narrative stories and animations of motion)

SimCalc turns algebra upside down by introducing piecewise functions early and also by introducing graphical interpretations of rate and proportionality, and allowing grounded explorations of slope. Graphical understandings are important in their own right, but they are also important in two other ways: as a pathway to other (algebraic) understandings and as a pathway towards understanding the system that compromises the mathematics of change and variation. Each kind of representation---computer models of the "world", graphs, tables and algebraic expressions, even word-based stories---emphasizes different aspects of the system, some of which are more usefully mathematical. Exercises that ask students to move across different representations develop fluidity and familiarity.

One could say that SimCalc projects a different image of what it means to "know" algebra. In a traditional symbolic approach, knowing algebra is often tantamount to knowing the grammatical transformation rules that correctly re-write one expression into another form. SimCalc still honors this as important, but aligns with an image of "knowing" which has to do with connections among representations. In this view, "knowing" a concept like rate means being able to coherent trace the connections of the concept in different forms—to be able to see rate as an experienced speed, a slope of a graph, covariation in a table of number pairs, and in a symbolic form. For students to build this connected sense of "knowing" algebra, they need tools which help them make the connections. SimCalc is squarely aimed at this connected epistemology.

One of the design challenges associated with this epistemology is that students cannot reasonably make all connections, all at once. Thus connections among representations must be introduced gradually, which corresponds to giving students access to different visual representations only as the master prior representations. In the early versions of SimCalc Mathworlds, this was handled by developing generic and powerful software which could show all possible representational forms, but configuring the software in saved documents. Teachers and students could then load documents in a sequence corresponding to the learning progression in a curricular workbook. More recently, the application/documents approach has been superceded with cloud-based solutions which deliver variant representations to students through activities arranged in a playlist.

Structuring pedagogy around a cycle that asks students to make predictions, compare their predictions with mathematical reality, and explain any differences.

This concept was always inherent in SimCalc Mathworlds in that the point draws student attention to aspects of the world and models that they might otherwise overlook. However, the idea of an explicit cycle of comparative prediction centered on the problem at hand developed slowly over time, and in conjunction with other related theorizing. One highly related pedagogical move is articulated by Schwartz and Bransford (Bransford & Schwartz, 1999; Schwartz & Bransford, 1998) as contrasting cases. The chief idea is that the designer or teacher creates a situation that makes the problem that will eventually be solved in the lesson clear before offering the solution.

The original conception of SimCalc was that the children would explore. Enabling exploration is still a key principle. However, exploration by itself does not necessarily lead to learning—for example, students may get to "solution states" for a particular challenge by exploration, but may not know how they got there. The predict-compare-explain cycle is meant to engage students in overt planning and reflection, with an eye towards developing stable explanations of the mathematical representations they are using. Further, the cycle plays into conventional classroom structures, where teachers lead discourse and ask students to make predictions and give explanations, as a way to check for and cultivate desired understandings. The commitment to classroom-based instruction means that the technology and related curriculum must respond to the teacher's need to ensure that certain material is encountered.

The predict-compare-explain cycle may be used in whole class activities. Additionally, often SimCalc is used with worksheets that ask students, as individuals or in small groups, to engage in specific activities and record the history of their interaction with the system and the lesson. The cycle is a generally beneficial practice that particularly helps ensure that the student thinks about and processes that she or he is experiencing.

The development and exploration of this principle is a design response of the SimCalc project to the problem of enabling both structured progress and exploration. It is a design response implemented in curriculum and use-practices rather than in the technology itself. It is thus aligned with some research on classroom orchestration (Dillenbourg & Jermann, 2010; Dillenbourg, Zuffrey, Alavi, Jermann, Do-Lenh, Bonnard, & Kaplan, 2011), but differs from responses that implement process in the technology itself, via scripts and successive disclosure of information (Diziol, Rummel, Spada, & McLaren, 2007; Fischer, Kollar, Mandl, & Haake, 2007).

FUTURE EXPLORATION

The five elements of the 2010 description and the associated practices are key and enduring pedagogical contributions of the SimCalc project. However, a number of issues have fallen by the wayside, not through any lack of merit.

1. Physically embodied algebra learning. SimCalc long anticipated the importance of embodiment in learning, but with the arrival of the DYI movement including Raspberry PI's, Arduino's and really inexpensive sensors and actuators, new opportunities are cropping up that have a chance of having impact in real K-12 classrooms.
2. Encompassing curriculum. When Jim Kaput passed away suddenly in 2005, he and Stephen Hegedus were in the process of creating an ambitious curriculum that reconceptualized the mathematics of change and variation from sixth to twelfth grade. This project was pursued by Hegedus and has to some extent been picked up and continued in the context projects housed at Roschelle's Center for Technology and Learning, but suffers from the lack of Kaput's single-minded focus.
3. Restriction of curricular scope. In order to encourage teachers and districts to use SimCalc, exploration has focused primarily on a high-impact setting, that of algebra learning. Algebra is indeed very important. It is the gateway course into four-year colleges in the United States. Enabling children to learn algebra and learn it well is a therefore a social justice issue that transcends other aspects of education. But Kaput's vision started with the phrase "democratizing access to the mathematics of change and variation." The project of creating a development sequence focused on this mathematics that would grow from middle school through college is as of yet, not complete.

CONCLUSION: DYNAMIC REPRESENTATIONS AND THE PROBLEM OF WICKED PROBLEMS

The fundamental advance in the SimCalc line of work has been to develop a principled design of a dynamic representation system for learning an important and difficult area of mathematics, conduct research deeply interconnecting that design with cognitive, developmental and pedagogical knowledge bases, and further expand the work to educational evaluations that show the learning gains that are achievable in diverse populations at the scale of hundreds of schools and thousands of students. Further, throughout the course of doing this work, the team has been reflective about refining their account of the key principles in the design.

The design story is thus a story of progress, but also a story of how complex the realization of the deep, transformative potential of technology in mathematics learning is. Technology is not a singular, causal factor in promoting learning and design of successful learning experiences but involves interweaving multiple concerns and levels of design.

In his one and only paper, Berkeley architecture professor Horst Rittel (Rittel & Webber, 1973) advanced the idea of the *wicked problem* in design. Wicked problems exist in contrast to *tame problems*. Tame problems (1) have single-valenced solutions and (2) require only that a person figure the solution out. Wicked problems do not necessarily have solutions. Furthermore, wicked problems are such that the exact formulation of the problem is tied to the kinds and ranges of solutions we consider.

One lesson from the SimCalc project is that important problems in education are wicked problems (Tatar, 2007). They must simultaneously determine the utility of technology, whose own properties are constitute wicked problems, in relationship to curriculum that may be designed in many different ways for promoting learning—which itself remains ultimately mysterious—in the tremendously complex environment of classrooms and schools. To begin to do this, and to keep the difficult exploration going, requires not just devotion and wide-ranging expertise, but the garnering of funding from sources with different requirements, expertise in the management of teams, and the ability to focus on the whole and the parts at the same time.

In the end, diffusion of innovation is not simple either. Deeply accepting Kaput's premise of representation change means sometimes *not* addressing today's curricular expectations and end-of-the-year examinations directly—but can result in changing the sequence of learning, so that learning accomplishments occur in different years and time frames than what is conventionally expected. It may also include changing the expected outcomes, which, after curricular changes, may not be fully measured by existing examinations. Kaput aimed his designs at addressing long-term societal change, which can mean that the designs do not tackle short-term desires to increase today's test scores fully—and can result in slow adoption.

And yet, historically we are clearly in the midst of transformation in what people need to know and be able to do to fully participate in an information age economy. Designs that democratize access to ways of thinking and reasoning that have long-term societal value, such as the ability to reason mathematically about change, have a likelihood of long-term societal impact. The opportunity to design effective dynamic representations—representations which express mathematical meaning through interactive, linked, time-based properties and give a wider range of people the opportunity learn and master corresponding ways of reasoning mathematically—is a wicked design problem worth solving.

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REFERENCES

Abelson, H. & DiSessa, A. (1986) *Turtle geometry*. Cambridge, MA: MIT Press.

Bamberger, J. (1974). *Progress report: Logo music project*. Cambridge, MA: Massachusetts Institute of Technology, AI Lab.

Bamberger, J. (1986). *Music logo*. Cambridge, MA: Terrapin Inc.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, *17*(1), 25-66.

Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. *Review of Research in Education*, *24*(1), 61-100.

Dalton, S., & Hegedus, S. (2013). Learning and participation in high school classrooms. In S. Hegedus and J. Roschelle (Eds.), *The SimCalc Vision and Contributions: Democratizing Access to Important Mathematics* (pp. 245-166). Dordrecht, NL: Springer.

Dickey-Kurdziolek, M. and Tatar, D. (2012). "They need to be solid in standard skills first": How standards can become the upper bound. In S. Hegedus and J. Roschelle (Eds.), *The SimCalc Vision and Contributions: Democratizing Access to Important Mathematics* (pp. 299-318). Dordrecht, NL: Springer.

Dillenbourg, P., & Jermann, P. (2010). Technology for classroom orchestration. In Khine, M.S. and Saleh, I. (Eds.), *New Science of Learning* (pp. 525-552). New York, NY: Springer.

Dillenbourg, P., Zufferey, G., Alavi, H., Jermann, P., Do-Lenh, S., Bonnard, Q., ... & Kaplan, F. (2011). Classroom orchestration: The third circle of usability. In *Proceedings of Computer-Supported Collaborative Learning* (pp. 510-517). International Society of the Learning Sciences.

DiSessa, A. A. (1991). An overview of Boxer. *Journal of Mathematical Behavior*, *10(1)*, 3-15.

DiSessa, A. (2000). *Changing Minds*. Cambridge, MA: MIT Press.

DiSessa, A. A., & Abelson, H. (1986). Boxer: A reconstructible computational medium. *Communications of the ACM*, *29*(9), 859-868.

Diziol, D., Rummel, N., Spada, H., & McLaren, B. M. (2007). Promoting learning in mathematics: Script support for collaborative problem solving with the Cognitive Tutor Algebra. In *Proceedings of the Conference on Computer-Supported Collaborative Learning.* Retrieved from http://repository.cmu.edu/cgi/viewcontent.cgi?article=1139&context=hcii

Doerr, H. M., & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics*, *41*(2), 143-163.

Drijvers, P. (2012, July). Digital technology in mathematics education: Why it works (or doesn't). Paper presented at the *12th International Congress on Mathematics Education*, Seoul, Korea.

Fischer, F., Kollar, I., Mandl, H., & Haake, J. M. (2007). *Scripting computer-supported collaborative learning: Cognitive, computational and educational perspectives* (Vol. 6). Dordrecht, NL: Springer.

Gardner, D. P. (1983). *A nation at risk*. Washington, DC: The National Commission on Excellence in Education, US Department of Education.

Guzdial, M. (2003, June). A media computation course for non-majors. *ACM SIGCSE Bulletin, 35*(3), 104-108.

Hammon, A.L. (1972). Computer-assisted instruction: Two major demonstrations. *Science*, *176 (4039)*, 1110-1112. Retrieved from http://eric.ed.gov/?id=EJ060645

Hegedus, S., & Kaput, J. (2003). Exciting new opportunities to make mathematics an expressive classroom activity using newly emerging connectivity technology. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 293). Honolulu, Hawaii: College of Education, University of Hawaii.

Hegedus, S., & Kaput, J. (2004a). An introduction to the profound potential of connected algebra activities: Issues of representation, engagement and pedagogy. Proceedings of the *28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Bergen, NO.

Hegedus, S., & Kaput, J. (2004b). An introduction to the profound potential of connected algebra activities: Issues of representation, engagement and pedagogy. Proceedings of the *28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Bergen, NO.

Hegedus, S., Kaput, J., & Lesh, R. (2007). Technology becoming infrastructural in mathematics education. In R. Lesh, E. Hamilton & J. Kaput (Eds.), *Foundations for the future in mathematics and science* (pp. 172-192). Mahwah, NJ: Lawrence Erlbaum Associates.

Hegedus, S., & Moreno-Armella, L. (2009). Intersecting representation and communication infrastructures. *ZDM: The International Journal on Mathematics Education: Transforming Mathematics Education through the Use of Dynamic Mathematics Technologies, 41*(4), 399-412.

Hegedus, S., & Penuel, W. (2008). Studying new forms of participation and classroom identity in mathematics classrooms with integrated communication and representational infrastructures. *Educational Studies in Mathematics*, *68*(2), 171-184.

Hegedus, S. & Roschelle, J. (Eds) (2013). *Democratizing access to important mathematics through dynamic representations: Contributions and visions from the SimCalc research program*. New York, NY: Springer.

Hegedus, S.J. & Roschelle, J. (2012). Highly Adaptive, Interactive Instruction: Insights for the Networked Classroom. In C. Dede & J. Richards (Eds.), *Digital teaching platforms* (pp. 103-115). New York: Teachers College Press.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual–spatial representations and mathematical problem solving. *Journal of Educational Psychology*, *91*(4), 684.

Hendrix, S. L., & Eisenberg, M. (2006). Computer-assisted pop-up design for children: Computationally enriched paper engineering. *Advanced Technology for Learning*, *3*(2), 119-127.

Hoyles, C. & Noss, R. (1993). Deconstructing microworlds. In D. L. Ferguson (Ed.), *Advanced Educational Technologies for Mathematics and Science*, *NATO ASI Series* (Vol. 107, pp. 415-438). New York, NY: Springer.

Jackiw, N. (1987-2007). *Geometer's Sketchpad®* [Software]. Emeryville, CA: Key Curriculum Press.

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.) *Handbook on research in mathematics teaching and learning* (pp. 515–556). New York, NY: Macmillan.

Kaput, J. (1994). Democratizing access to Calculus: New routes using old routes. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (2001). Changing representational infrastructures changes most everything: The case of SimCalc algebra, and calculus. In *Proceedings of the NAS Symposium on "Improving Learning with Informational Technology.* Washington, DC.

Kaput, J., Noss, R., & Hoyles, C. (2002). Developing new notations for a learnable mathematics in the computational era. In L. D. Church (Ed), *Handbook of international research in mathematics education*, (pp. 51-75). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. & Roschelle, J. (1998). The mathematics of change and variation from a millennial perspective: New content, new context. In Hoyles, C., Morgan, C., & Woodhouse, G. (Eds.) *Rethinking the mathematics curriculum* (pp. 155-170). London, UK: Falmer Press.

Kaput, J., & Schorr, R. (2008). The case of SimCalc, algebra, and calculus. In Blume, G. W., & Heid, M. K. (Eds.), *Research on Technology and the Teaching and Learning of Mathematics: Cases and Perspectives* (Vol. 2). Charlotte, NC: Information Age Publishing.

Kelleher, C., & Pausch, R. (2007). Using storytelling to motivate programming. *Communications of the ACM*, 50(7), 58-64.

Kozhevnikov, M., Kosslyn, S., & Shephard, J. (2005). Spatial versus object visualizers: A new characterization of visual cognitive style. *Memory & cognition*, *33*(4), 710-726.

Laborde, J-M. (1984-2007). *Cabri-Geometry™* [Software]. Grenoble, FR: CabriLog.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the Velocity sign. *The Journal of Mathematical Behavior*, *13*(4), 389-422.

Nemirovsky, R., Borba, M., Dimattia, C., Arzarello, F., Robutti, O., Schnepp, M., & Scheffer, N. F. (2004). PME Special Issue: Bodily activity and imagination in mathematics learning. *Educational Studies in Mathematics*, *57*(3), 303-321.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction*, *16*(2), 119-172.

Nemirovsky, R., Kaput, J. & Roschelle, J. (1998). Enlarging mathematical activity from modeling phenomena to generating phenomena. In *Proceedings of the 22nd Psychology of Mathematics Education Conference*.

Nickerson, S. D., Nydam, C. & Bowers, J. S. (2001). Linking algebraic concepts and contexts: Every picture tells a story. *Mathematics Teaching in the Middle School*, 6(2), 92-98.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht, NL: Kluwer Academic Publishers.

Papert, S. (1980). *Mindstorms*. New York, NY: Basic Books.

Repenning, A., & Sumner, T. (1995). Agentsheets: A medium for creating domain-oriented visual languages. *Computer*, *28*(3), 17-25.

Rittel, H. W., & Webber, M. M. (1973). Dilemmas in a general theory of planning. *Policy sciences*, *4*(2), 155-169.

Roschelle, J., Kaput, J, & Stroup, W. (2000) SimCalc: Accelerating students' engagement with the mathematics of change. In M. Jacobson & R. Kozma (Eds.) *Educational Technology and Mathematics and Science for the 21st Century* (pp. 47-75). Mahwah, NJ: Lawrence Erlbaum Associates.

Roschelle, J., Tatar, D., & Kaput, J. (2008). Getting to scale with innovations that deeply restructure how students come to know mathematics. In A. E. Kelly, R. Lesh & J.Y. Baek (Eds.), *Handbook of Design Research Methods in Education* (pp. 369-395). New York, NY: Routledge.

Roschelle, J., Tatar, D., Shechtman, N., & Knudsen, J. (2008). The role of scaling up research in designing for and evaluating robustness. *Educational Studies in Mathematics*, Special Issue on Democratizing Access to Mathematics through Technology: Issues of Design, Theory and Implementation— In Memory of James Kaput (S. Hegedus and R. Lesh, Eds.). 68(2), 149-170.

Roschelle, J., Vahey, P., Tatar, D., Kaput, J., & Hegedus, S. J. (2003). Five key considerations for networking in a handheld-based mathematics classroom. In N. A. Pateman & B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA* (Vol. 4, pp. 71-78). Honolulu, HI: University of Hawaii.

Schank, P., & Kozma, R. (2002). Learning chemistry through the use of a representation-based knowledge building environment. *Journal of Computers in Mathematics and Science Teaching*, *21*(3), 253-279.

Shechtman, N., Roschelle, J., Tatar, D., Hegedus, S., Hopkins, B., Empson, S., Knudsen, J., and Gallagher, L. (2010). Integration of teacher training, curriculum and technology for advancing middle school mathematics: Three large-scale studies. *American Educational Research Journal*, *47*(4), 833-878.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and instruction*, *16*(4), 475-522.

Tatar, D. (2007) The design tensions framework. *Journal of Human-Computer Interaction*, *22*(4), 413-451.

Tatar, D., Roschelle, J., Knudsen, J., Shechtman, N., Kaput, J. & Hopkins, B. (2008) Scaling up innovative technology-based math. *Journal of the Learning Science*s, 17(2), 248-286.

Tatar, D., Roschelle, J., Vahey, P. & Penuel, W. (2003) Handhelds go to school. *IEEE Computer*, *36*(9), 30-37

Vahey, P., Tatar, D. & Roschelle, J. (2006) Using handheld technology to move between the private and public in the classroom. In van't Hooft, M. A., & Swan, K. (Eds.) *Ubiquitous computing: Invisible technology, visible impact* (pp. 187-210). Mahwah, NJ: Lawrence Erlbaum Associates.

Vahey, P., Tatar, D., & Roschelle, J. (2004) Leveraging Handhelds to increase student learning: Engaging middle school students with the mathematics of change. In *Proceedings of the Sixth International Conference of the Learning Sciences* (pp. 553-560). Los Angeles, CA.

Weller, M.P., Do, E. Y-L., & Gross, M.D. (2008) Posey: Instrumenting a poseable hub and strut construction toy. In *Proceedings of Tangible and Embedded Interaction* (pp. 39-46). Bonn, DE.

White, B. Y. (1993). ThinkerTools: Causal models, conceptual change, and science education. *Cognition and Instruction*, *10*(1), 1-100.

Wilensky, U. & Stroup, W. (2000) Networked gridlock: Students enacting complex dynamic phenomena with the Hubnet architecture. In *Proceedings of The Fourth Annual International Conference of the Learning Sciences*. Ann Arbor, MI.