Non-Stationarity in the Stochastic Distribution of Cryptocurrency Returns

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ABSTRACT

This paper uses a functional approach to analyze the distributions of weekly returns in Bitcoins on leading cryptocurrency exchanges. The results present strong evidence for non-stationarity, which suggests unpredictability and time-varying statistical properties. In addition, non-stationary fluctuations tend to be primarily concentrated in even moments, such as volatility and kurtosis—however their effect is significant and persistent in every moment, including higher moments. The analysis in this paper proposes that the Bitcoin market is maturing and tending towards stability, but retains a high degree of unpredictability in the case of random shocks due to underlying market dynamics.

KEYWORDS: cryptocurrencies, asset returns, functional time series

INTRODUCTION

The financial crisis of 2008 revealed some major flaws in the structure of the global financial system, and while regulators pushed for meaningful policy reforms and more effective regulatory oversight, others looked to alternative financial systems entirely. On January 3, 2009, a completely decentralized payment network operating on digital peer-to-peer technology was invented under the pseudonym, Satoshi Nakamoto. This network was known as Bitcoin.

Unlike the existing financial system which relies extensively on financial institutions to act as trusted third parties to verify, process, and record transactions, Bitcoin uses cryptography and algorithmic hashing\(^1\) to verify and process secure transactions directly between parties (Nakamoto, 2009). After verification, all transactions are then appended to an unalterable public ledger known as the Blockchain.

The primary innovation of the technology is a solution to the double-spending problem\(^2\) of digital payment networks, and addresses security vulnerabilities in the existing financial network such as those exploited in the recent Bangladesh Bank Heist of 2016 where hackers attempted to steal nearly one billion U.S. dollars (Das & Spicer, 2016).

Since the invention of Bitcoin, over 1000 alternative digital networks based on the technology have been created—collectively known as cryptocurrencies. The security, anonymity, and unregulated nature of cryptocurrencies have immense appeal to many in an increasingly digital era. In particular, recent years have seen increased applications of the technology such as in Initial Coin Offerings\(^3\), early-stage cryptocurrency start-up firms (Securities & Commission, 2018). In December 2017, leading derivatives exchanges CME Group and Chicago Board Options Exchange announced the offering of future contracts on Bitcoins (Reuters, 2017).

As the technology becomes increasingly adopted, we are forced to confront the potential economic impact of cryptocurrencies. The decentralized, unregulated, and completely novel nature of cryptocurrencies have landed them as some of the most controversial assets to date. The financial industry is split, with some prominent industry leaders and financial regulators actively voicing their opinions against cryptocurrencies while others express their support in the revolutionary potential of the technology. Many remain undecided. However, the impact that cryptocurrencies will have on the existing monetary system must be carefully considered and analyzed—regardless of one’s positional views on the technology.

This paper focuses on an analysis of the distributions of Bitcoin returns over time, defined by log-differences in prices. Unlike other more mature financial markets such as stocks and bonds, cryptocurrencies are vastly understudied and thus become particularly prone to price shocks and speculative movements without any solid justifications for their valuation. The highly volatile and unstable prices of cryptocurrencies in their current state poses a significant challenge to their intended usage as a currency, and may be a basis for hesitation towards adoption of the technology.\(^4\) Thus, an analysis of returns and better understanding of the valuation of Bitcoins may ultimately lead to more efficient cryptocurrency markets that are more mature, less speculative, and more stable which is necessary to serve their intended function as digital currencies.

Of the limited studies done on cryptocurrency returns, most analyze only specific characteristics of the distributions of returns over time. Previous studies such as Wang and Vergne (2017) use a measure of innovation potential to demonstrate that the most important factor in explaining weekly cryptocurrency returns is improvements to its underlying technology, but rely only on the mean\(^5\) of weekly returns in their model. Balcilar, Bouri, Gupta, and Roubaud (2017) demonstrate that trading volume is a significant predictor for returns, but only when restricted to specific quantiles and is insignificant for the variance or higher moments in the conditional distribution. Thus, existing approaches to understanding the cryptocurrency returns model only limited characteristics of the distributions of returns, and may even impose potentially misspecified parametric assumptions on the underlying distributions. This creates the risk of model misspecification and can be particularly severe when applied to long-term forecasting or pricing purposes. To account for these limitations in the current literature, the approach used in this paper closely follows the functional time series methodology proposed by Chang, Kim, and Park (2016).

\(^1\)This process is colloquially known as mining, where computers on the network would dedicate computing power to algorithmically verify pending transactions in exchange for a small reward.

\(^2\)The double-spending problem is the possibility of spending the same digital currency multiple times, due to the ease of replicating digital information. This is similar to the issue of counterfeit money in other currency systems, i.e. most modern paper currencies.

\(^3\)In an Initial Coin Offering, a percentage of the new cryptocurrency is sold to early backers of a venture similar to the established process of Initial Public Offerings.

\(^4\)For example, a vendor may be hesitant to accept payments in cryptocurrencies due to the inherent price risk of such a volatile asset.

\(^5\)This is common in standard regression models, where the full set of observations for a given time period is implicitly aggregated into a single data point such as the mean without considering the distribution of these observations. For example, a time series based on daily observations essentially aggregates more granular observations (hour, minute, second, etc) into a single data point for each day.
METHODOLOGY

Model Set-up

Consider a sequence of density functions \( f \) that are well-defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Let \( f_t \) be regarded as a functional random variable that takes values on a measurable Hilbert space \( H \), where \( f_t : \Omega \rightarrow H \) for \( t = 1, 2, \ldots \). The space \( H \) is formally defined as,

\[
H = \{ f : \int_C f(x) dx = 0, \int_C f^2 dx < \infty \}
\]

on some compact \( C \subseteq \mathbb{R} \), with inner product \( \langle f, g \rangle = \int f(x)g(x)dx \) for all \( f, g \in H \), and \( x \in \mathbb{R} \). Denote \( \| \cdot \| \) as the vector norm in \( H \), and \( A' \) as the Hermitian adjoint of any linear operator \( A \) on \( H \). Next, define \( (u_k) \) as the sequence of demeaned density functions, where \( u_k = f_k - \mathbb{E}[f] \).

The space \( H \) can then be decomposed into the direct sum \( H = H_s \oplus H_n \), where \( H_s \) and \( H_n \) are the stationary and non-stationary subspaces of \( H \), respectively. It is assumed that \( H_s \) is finite-dimensional, with \( \dim(H_s) = M \) for some \( 0 \leq M < \infty \). Let \( \Pi_s \) and \( \Pi_n \) be the projections onto \( H_s \) and \( H_n \), such that \( u_k^T = \Pi_s[u_k] \) and \( u_k^2 = \Pi_n[u_k] \).

Testing Procedure

The testing procedure to determine the dimension \( M \) of the non-stationary subspace \( H_n \) is based on functional principal component analysis.

First, define the unnormalized sample variance operator \( Q^T \) of \( u_k \) by,

\[
Q^T = \sum_{k=1}^T u_k \otimes u_k
\]

where \( T \) is the sample size and \( \otimes \) is the tensor product operator. Let the eigenvalues and corresponding eigenvectors of \( Q^T \) be denoted by \( \lambda_i(Q^T), v_i(Q^T) \) for \( i = 1, \ldots, T \). For ease of notation, the set is ordered such that \( \lambda_1(Q^T) \geq \lambda_2(Q^T) \geq \ldots \geq \lambda_T(Q^T) \).

Assuming the sample size \( T \) is sufficiently large, such that \( T > M \), consider the subspace spanned by the eigenvectors associated with the \( M \) largest eigenvalues of \( Q^T \). This subspace is defined as the sample unit root subspace and denoted \( H^N_{k_M} \), or in other words \( H^N_{k_M} = \text{span} \{ v_1(Q^T), \ldots, v_M(Q^T) \} \).

With these definitions in place, we may determine \( M \) by successively testing downwards from some initial value \( M > M \). More precisely, set the null and alternate hypotheses as follows,

\[
H_0 : |\dim(H_N) = M \|
H_A : |\dim(H_N) \leq M - 1 \|
\]

Let the smallest value of \( M \) such that the null hypothesis is rejected in favor of the alternative hypothesis be denoted by \( M^{\text{min}} \), and so \( M = M^{\text{min}} - 1 \).

Test Statistic

Consider the sequence \( (z_t) \) based on the coordinate processes of the \( M \)-leading eigenvectors that span \( H^N_{k_M} \), where

\[
z_t = (\langle v_1(Q^T), u_t \rangle, \ldots, \langle v_M(Q^T), u_t \rangle)'
\]

for \( t = 1, \ldots, T \). Let \( Z_T = (z_1, \ldots, z_T) \), and \( Q^T = Z_T'Z_T \). Define the autocovariance function of \( (z_t) \) by

\[
\Gamma_T(k) = \frac{1}{T} \sum_{t=1}^T \Delta z_t\Delta z_{t-k}^T.
\]

Then, the long run sample variance of \( (\Delta z_t) \) can be defined by \( \Omega^T_M \), where

\[
\Omega^T_M = \sum_{k=1}^j u_j(k)\Gamma_T(k)
\]

for some bounded weight function \( \omega_j \). Finally, the test statistic \( \tau^T_M \) can be introduced as follows,

\[
\tau^T_M = \frac{1}{T^2} \lambda_M(Q^T_M, \Omega^T_M).
\]

Critical values for the test statistic are presented in Table 1, where they were obtained by simulations in Chang et al. (2016).

Unit Root Proportions

In addition to determining the dimension of the unit root subspace, understanding the concentration of non-stationary movements in various moments of the distribution can be useful. First, define

\[
u_k(x) = x^k - \left( \frac{1}{|C|} \int_C x^k dx \right)
\]

for \( k = 1, 2, \ldots \), where \( |C| \) is the Lebesgue measure of \( C \subseteq \mathbb{R} \). The inner product \( \langle \mu_k, u_k \rangle \) is then a coordinate process representing the random movements in the \( k \)th moment of \( f \), where

\[
\langle u_k, u_l \rangle = \int_C x^k f(x)dx - \mathbb{E}[\int_C x^k f(x)dx].
\]

Next, decompose \( \mu_k \) into its projections onto \( H_s \) and \( H_n \) such that \( \mu_k = \Pi_s[\mu_k] + \Pi_n[\mu_k] \). By taking the norm of both sides, then

\[
|\mu_k|^2 = |\Pi_N[\mu_k]|^2 + |\Pi_S[\mu_k]|^2 = \sum_{i=1}^M \langle u_k, u_i \rangle^2 + \sum_{p=M+1}^T \langle u_k, u_p \rangle^2
\]

The unit root proportion of the \( k \)th moment, denoted by \( \pi_k \), may then be interpreted as the proportion of the coordinate process lying in the non-stationary subspace \( H_n \). Thus,

\[
\pi_k = \frac{|\Pi_N[\mu_k]|}{|\mu_k|} = \sqrt{\frac{\sum_{i=1}^M \langle u_k, u_i(Q^T) \rangle^2}{\sum_{i=1}^T \langle u_k, u_i(Q^T) \rangle^2}}.
\]

Data

The data used in this paper are obtained from the Bitcoincharts database, which provides historical trades books\(^6\) for leading cryptocurrency exchanges. It is important to note that due to the decentralized nature of cryptocurrencies, there exists hundreds of digital exchanges operating globally around the clock. However,

\(^6\) Due to ease of storage, most cryptocurrency exchanges elect to provide historical data in the form of a trades book, or a history of all trades executed on the exchange in a given time frame. This gives traders and analysts the flexibility to construct datasets for any timeframe and in any level of granularity desired, without the burden of additional server costs and maintenance.
any cross-exchange arbitrage assumptions are likely unrealistic, as persistent price differences of hundreds to even thousands of dollars can be seen in the trading price of Bitcoin across different exchanges.

Of the hundreds of active cryptocurrency exchanges, the majority of trading volume is concentrated in a few leading exchanges. This paper focuses on an analysis of the four largest exchanges by volume: Bitstamp, Coinbase, Kraken, and Coincheck. As cryptocurrencies trade on a global market, data was collected for XBT/USD (Bitcoin in USD denominations) on Bitstamp, Coinbase, and Kraken, and XBT/JPY on Coincheck. These exchanges capture the majority of the total market capitalization of Bitcoin, and recent months have seen an increased consolidation of trading activity towards these exchanges away from smaller ones.

After the full history of trades were obtained, time series datasets of any granularity can be constructed by resampling the trades data into desired time intervals. Two important considerations were accounted for when deciding on the time interval. First, the asymptotic theories establishing the consistency of the estimators in the model require sufficient time periods, with enough observations in each period to consistently estimate the density functions. On the other hand, overly granular time intervals within each period can result in the capture of excessive noise from sources such as market micro-structure. Thus, time intervals were set at 1-minute and time periods 1-week. In other words, the dataset is transformed into a time series with minutely observations and grouped by weeks. This will be used to analyze the distribution of returns for each week, which are defined as log-returns.

In all four datasets, there were issues of extreme outliers (for example, a spike of over 900% returns). These spikes occurred at different times for each exchange, and since the other exchanges demonstrate otherwise normal behavior at the time, the most likely cause is data integrity issues or isolated exchange-specific phenomenon. Thus, the datasets were restricted to the [0.5%, 99.5%] quantiles, or in other words the top and bottom 0.5% observations dropped.

In addition, the first few weeks of an exchange’s inception saw major issues with liquidity, with many time intervals during this period not containing any trades. While this issue is particularly severe towards the beginning of trade history, there are still a few isolated periods with no trades throughout the dataset. This may be attributed to causes such as server downtime, and so its effect on the distributions of returns should not be considered. Thus, any time intervals without trades are dropped. Table 2 shows a summary of the constructed returns dataset, with the amount of outliers dropped and number of time intervals with no trading activity for the four exchanges.

**RESULTS**

Since density functions are not observed in practice, they may be estimated given a set of observations by methods such as Kernel Density Estimation. Following the procedures in Chang et al. (2016), the Epanechnikov kernel is used with optimal bandwidth $h = 2.3440 \hat{\sigma} n^{-1/5}$, where $\hat{\sigma}$ and $n$ is the sample standard deviation and sample size at week $t$, respectively.

Next, the density functions are decomposed by applying a discrete wavelet transform, using the Daubechies db3 wavelet with 7 levels. The Daubechies wavelet is then implemented with 1053 basis functions, thus representing the density functions by a set of 1053-dimensional vectors.

The Bitstamp dataset spanned 335 weeks, with an average number of 4742 observations per week. The Kraken dataset spanned 211 weeks, with an average number of 2520 observations per week. The Coinbase dataset spanned 164 weeks, with an average of 7539 observations per week. Finally, the Coincheck dataset spanned 172 weeks, with an average of 4594 observations per week. Figures 1-4 show the evolution of the estimated density and demeaned density functions over time.

**Testing Procedure for Non-stationarity**

The test statistic $\tau_M^T$ is then computed for each exchange. The testing procedure, as described in the Methodology section, is carried out by comparing the test statistic against the critical values with a starting maximum unit root dimension of $M = 5$. The values of the test statistics are presented in Tables 3-6.

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8 Under a cross-exchange arbitrage assumption, if an asset is traded at different prices this would theoretically lead traders to capitalize on this riskless opportunity by buying at the lower valuation and shorting at the higher valuation—creating an equilibrium price. However in cryptocurrencies, a lack of regulation, the exchange’s reputation, and independent exchange policies may be preventing this mechanism from working efficiently.

9 Major exchanges that have recently had particularly negative news, such as ongoing regulatory investigation under suspicion of fraud, were excluded.

10 See Chang et al. (2016)
Figure 1.
Bitstamp: Estimated density functions (left) and demeaned densities (right) for XBT/USD.

Figure 2.
Kraken: Estimated density functions (left) and demeaned densities (right) for XBT/USD.

Figure 3.
Coinbase: Estimated density functions (left) and demeaned densities (right) for XBT/USD.

Figure 4.
Coincheck: Estimated density functions (left) and demeaned densities (right) for XBT/JPY.
Wu: Stochastic Distribution of Cryptocurrency

**Figure 5.**
Bitstamp: Scree plot of leading eigenvalues suggesting 1-dimensional unit root subspace, with non-stationary coordinate process.

**Figure 6.**
Kraken: Scree plot of leading eigenvalues suggesting 1-dimensional unit root subspace, with non-stationary coordinate process.

**Figure 7.**
Coinbase: Scree plot of leading eigenvalues suggesting 2-dimensional unit root subspace, with two non-stationary coordinate processes plotted.

**Figure 8.**
Coincheck: Scree plot of leading eigenvalues suggesting 1-dimensional unit root subspace, with non-stationary coordinate process.
Table 7. Unit Root Proportions for Bitstamp

<table>
<thead>
<tr>
<th>( \hat{\pi}_1^T )</th>
<th>( \hat{\pi}_2^T )</th>
<th>( \hat{\pi}_3^T )</th>
<th>( \hat{\pi}_4^T )</th>
<th>( \hat{\pi}_5^T )</th>
<th>( \hat{\pi}_6^T )</th>
<th>( \hat{\pi}_7^T )</th>
<th>( \hat{\pi}_8^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0844</td>
<td>0.3652</td>
<td>0.0973</td>
<td>0.2518</td>
<td>0.1022</td>
<td>0.2107</td>
<td>0.1047</td>
<td>0.1906</td>
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</table>

Table 8. Unit Root Proportions for Kraken

<table>
<thead>
<tr>
<th>( \hat{\pi}_1^T )</th>
<th>( \hat{\pi}_2^T )</th>
<th>( \hat{\pi}_3^T )</th>
<th>( \hat{\pi}_4^T )</th>
<th>( \hat{\pi}_5^T )</th>
<th>( \hat{\pi}_6^T )</th>
<th>( \hat{\pi}_7^T )</th>
<th>( \hat{\pi}_8^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1344</td>
<td>0.4455</td>
<td>0.1488</td>
<td>0.3236</td>
<td>0.1522</td>
<td>0.2775</td>
<td>0.1535</td>
<td>0.2546</td>
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</tbody>
</table>

Table 9. Unit Root Proportions for Coinbase

<table>
<thead>
<tr>
<th>( \hat{\pi}_1^T )</th>
<th>( \hat{\pi}_2^T )</th>
<th>( \hat{\pi}_3^T )</th>
<th>( \hat{\pi}_4^T )</th>
<th>( \hat{\pi}_5^T )</th>
<th>( \hat{\pi}_6^T )</th>
<th>( \hat{\pi}_7^T )</th>
<th>( \hat{\pi}_8^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1064</td>
<td>0.3240</td>
<td>0.1169</td>
<td>0.2276</td>
<td>0.1216</td>
<td>0.1935</td>
<td>0.1244</td>
<td>0.1770</td>
</tr>
</tbody>
</table>

Table 10. Unit Root Proportions for Coincheck

<table>
<thead>
<tr>
<th>( \hat{\pi}_1^T )</th>
<th>( \hat{\pi}_2^T )</th>
<th>( \hat{\pi}_3^T )</th>
<th>( \hat{\pi}_4^T )</th>
<th>( \hat{\pi}_5^T )</th>
<th>( \hat{\pi}_6^T )</th>
<th>( \hat{\pi}_7^T )</th>
<th>( \hat{\pi}_8^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0594</td>
<td>0.2287</td>
<td>0.0659</td>
<td>0.1498</td>
<td>0.0682</td>
<td>0.1252</td>
<td>0.0694</td>
<td>0.1139</td>
</tr>
</tbody>
</table>

At the 1% significance level, the testing procedure suggests a 1-dimensional unit root subspace for Bitstamp, Kraken, and Coincheck. Interestingly, the results found a 2-dimensional unit root subspace for Coinbase. A scree plot of the leading eigenvalues further supports these results, due to the significantly larger magnitude of the first (or first 2) eigenvalues. In addition, the non-stationary coordinate processes are plotted against the normalized mean in Figures 5-8.

**Unit Root Proportions**

The results from the previous section suggest a non-zero dimensional unit root subspace, or in other words, there is evidence of non-stationarity. Thus, the unit root proportions \( \hat{\pi}_k^T \) are computed and presented below in Tables 7-10 for the first 8 moments.

**DISCUSSION**

The estimated density functions in Figures 1-4 clearly show significant variations in the mean and volatility of the distributions of returns over time. In a stationary process, statistical properties are constant and non-varying with time so this supports the test’s results of non-stationarity. In addition, the demeaned densities show that while returns in Bitcoins are consistently growing over time, they are increasingly leptokurtic and closer distributed around 0. In other words, large movements in either direction are less likely as time passes which may suggest that the Bitcoin market is beginning to mature and become more stable.

On all four cryptocurrency exchanges analyzed, there is strong evidence of long-term non-stationarity in Bitcoin returns. However, when comparing the unit root proportion ratios across the different exchanges, a particularly interesting result is that the ratios on Kraken tend to be significantly higher than on other exchanges. This suggests that in the event of a random shock, its effect tends to much more persistent on Kraken compared with other exchanges—which may be indicative of underlying market dynamics such as a higher degree of speculative trades. On the other hand, the JPY-denominated Coincheck has the lowest unit root proportions, with many of its ratios less than half of those for Kraken.

**CONCLUSION**

The analysis in this paper exhibits four key findings, the first being that the distribution of intra-weekly returns over time tends to become increasingly concentrated around 0, with less likelihood of significant movements in either direction. This may suggest that momentum is slowing down, which may be signs of an increasingly mature, stable, and efficient market. Second, there is strong evidence for non-stationarity in the distributions of returns for all four exchanges, with a unit root subspace of between 1-2 dimensions. This finding should be carefully interpreted along with the first, as despite signs of stability random shocks will still cause a high degree of unpredictability. Third, the unit root proportions of certain exchanges are notably higher than on others, despite generally highly correlated prices due to cross-exchange arbitrage. This may be indicative of more underlying market dynamics, as exceptionally high unit root proportions on one exchange relative to another may be signs of a higher degree of speculation. Lastly, the unit root proportion statistic is persistent in all moments, including the higher moments. The significance of this final finding is most evident when compared to the equity markets. In the case of a price shock to a stock process, the non-stationary component tends to be concentrated in volatility and kurtosis with negligible effects on the mean and skewness. In other words, volatility and kurtosis become unpredictable while mean and skewness is relatively mean-reverting. For Bitcoin however, all moments up to the 8th are found to have significant persistent non-stationarity. This paper suggests that the Bitcoin market is maturing and tending towards stability and efficiency, but is also highly sensitive to random shocks and retains a high degree of unpredictability due to underlying market dynamics such as speculative behavior.

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REFERENCES


