



Selection games and the Vietoris space

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ABSTRACT

We explore the connections between selection games on Hausdorff spaces and their corresponding Vietoris space of compact subsets. These considerations offer a similar relationship as the well-known relationship between ω -covers of X and ordinary open covers of the finite powers of X . The primary utility of this method is to establish similar relationships with k -covers and the Vietoris space of compact subsets. Particularly, we show that some commonly studied selection principles are equivalent to a related hyperspace being Menger or Rothberger. We then apply these equivalences to correct a flawed argument in a previous paper which attempted to show that Hurewicz/Pawlikowski theorems are true for k -covers.

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1. Introduction

Relationships between selection principles on a ground space and the hyperspace of closed subsets with various topologies has been a growing area of investigation [3,4,7,11,13,14]. One of the common techniques employed is to translate certain cover types to families of closed sets via the complement operation. The resulting relationship is thus between covers of a certain type and dense sets in a related topology given to the space of closed sets. In [1, Theorem 3.22], a more direct topological relationship is suggested. By restricting our attention to compact sets, we bring to bear relationships between a space X and the space $\mathbb{K}(X)$ of compact subsets endowed with the Vietoris topology in terms of the cover types themselves. In particular, we establish relationships between ω -covers and open covers on the space of finite subsets of X viewed as a subspace of $\mathbb{K}(X)$ as well as relationships between k -covers and open covers of $\mathbb{K}(X)$.

We also point out an application of these methods, following existing results of [8,9,16], to prove strategic equivalence of some Rothberger- and Menger-like games on X with the corresponding games on the disjoint

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union $X^{<\omega}$ of finite powers of X . Particularly, these classical results establish a relationship between selection principles involving ω -covers on X and open covers on $X^{<\omega}$. This is natural since ω -covers are to cover all finite subsets of X and one can code finite subsets of X with tuples.

Finally, we also address the following. Steven Clontz pointed out that the without loss of generality claim in the beginning of the proof of [1, Proposition 3.25] (restated in a slightly generalized version in [2] as Lemma 7) is flawed. We then noticed a similar flaw at the end of the proof of [1, Proposition 3.27] (restated in a slightly generalized version in [2] as Lemma 8). In this note, we recover the conclusions of those results for k -covers. However, we remain with the following question. Are [2, Lemma 7] and [2, Lemma 8] true as stated?

Throughout, we assume that all spaces X considered are Hausdorff, infinite, and, when relevant, non-compact.

2. Preliminaries

Definition 2.1. For a topological space X , we let \mathcal{T}_X denote the collection of all proper, non-empty open subsets of X .

Definition 2.2. Generally, for an open cover \mathcal{U} of a topological space X , we say that \mathcal{U} is *non-trivial* provided that $X \notin \mathcal{U}$. We let \mathcal{O}_X denote the collection of all non-trivial open covers of X .

Definition 2.3. For a space X and a class \mathcal{A} of closed proper subsets of X , a non-trivial open cover \mathcal{U} is an \mathcal{A} -cover if, for every $A \in \mathcal{A}$, there exists $U \in \mathcal{U}$ so that $A \subseteq U$. We let $\mathcal{O}(X, \mathcal{A})$ denote the collection of all \mathcal{A} -covers of X .

Remark 2.4. Note that

- if \mathcal{A} consists of the finite subsets of X , then $\mathcal{O}(X, \mathcal{A})$ is the collection of all ω -covers of X , which will be denoted by Ω_X ;
- if \mathcal{A} consists of the compact (proper) subsets of X , then $\mathcal{O}(X, \mathcal{A})$ is the collection of all k -covers of X , which will be denoted by \mathcal{K}_X .

Definition 2.5. Given a set \mathcal{A} and another set \mathcal{B} , we define the *finite selection game* $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ for \mathcal{A} and \mathcal{B} as follows:

$$\begin{array}{c|cccc} \text{I} & A_0 & A_1 & A_2 & \dots \\ \hline \text{II} & \mathcal{F}_0 & \mathcal{F}_1 & \mathcal{F}_2 & \dots \end{array}$$

where $A_n \in \mathcal{A}$ and $\mathcal{F}_n \in [A_n]^{<\omega}$ for all $n \in \omega$. We declare Two the winner if $\bigcup \{\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Definition 2.6. Similarly, we define the *single selection game* $G_1(\mathcal{A}, \mathcal{B})$ as follows:

$$\begin{array}{c|cccc} \text{I} & A_0 & A_1 & A_2 & \dots \\ \hline \text{II} & x_0 & x_1 & x_2 & \dots \end{array}$$

where each $A_n \in \mathcal{A}$ and $x_n \in A_n$. We declare Two the winner if $\{x_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Definition 2.7. We define strategies of various strength below.

- A *strategy for player One* in $G_1(\mathcal{A}, \mathcal{B})$ is a function $\sigma : (\bigcup \mathcal{A})^{<\omega} \rightarrow \mathcal{A}$. A strategy σ for One is called *winning* if whenever $x_n \in \sigma \langle x_k : k < n \rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If player One has a winning strategy, we write $I \uparrow G_1(\mathcal{A}, \mathcal{B})$.
- A *strategy for player Two* in $G_1(\mathcal{A}, \mathcal{B})$ is a function $\tau : \mathcal{A}^{<\omega} \rightarrow \bigcup \mathcal{A}$. A strategy τ for Two is *winning* if whenever $A_n \in \mathcal{A}$ for all $n \in \omega$, $\{\tau(A_0, \dots, A_n) : n \in \omega\} \in \mathcal{B}$. If player Two has a winning strategy, we write $II \uparrow G_1(\mathcal{A}, \mathcal{B})$.
- A *predetermined strategy* for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves, they are just running through a pre-planned script. Formally it is a function $\sigma : \omega \rightarrow \mathcal{A}$. If One has a winning predetermined strategy, we write $I \uparrow_{\text{pre}} G_1(\mathcal{A}, \mathcal{B})$.
- A *Markov strategy* for Two is a strategy which only considers the most recent move of player One and the current turn number. Formally it is a function $\tau : \mathcal{A} \times \omega \rightarrow \bigcup \mathcal{A}$. If Two has a winning Markov strategy, we write $II \uparrow_{\text{mark}} G_1(\mathcal{A}, \mathcal{B})$.
- If there is a single element $x_0 \in \mathcal{A}$ so that the constant function with value x_0 is a winning strategy for One, we say that One has a *constant winning strategy*, denoted by $I \uparrow_{\text{cnst}} G_1(\mathcal{A}, \mathcal{B})$.

These definitions can be extended to $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ in the obvious way.

Definition 2.8. The reader may be more familiar with selection principles than selection games. Let \mathcal{A} and \mathcal{B} be collections. The selection principle $S_1(\mathcal{A}, \mathcal{B})$ for a space X is the following property: Given any sequence $\langle A_n : n \in \omega \rangle$ from \mathcal{A} , there exists $\{x_n : n \in \omega\}$ with $x_n \in A_n$ for each $n \in \omega$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$. $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ is similarly defined, but with finite selections instead of single selections. We will use the notation $X \models S_{\square}(\mathcal{A}, \mathcal{B})$ to denote that the selection principle $S_{\square}(\mathcal{A}, \mathcal{B})$ holds for X .

Note that

- $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ is the Menger property.
- $S_1(\mathcal{O}, \mathcal{O})$ is the Rothberger property.

Remark 2.9. In general, $S_{\square}(\mathcal{A}, \mathcal{B})$ holds if and only if $I \not\uparrow_{\text{pre}} G_{\square}(\mathcal{A}, \mathcal{B})$ where $\square \in \{1, \text{fin}\}$. See [5, Prop. 15].

Definition 2.10. An even more fundamental type of selection is inspired by the Lindelöf property. Let \mathcal{A} and \mathcal{B} be collections. Then $\binom{\mathcal{A}}{\mathcal{B}}$ means that, for every $A \in \mathcal{A}$, there exists $B \subseteq A$ so that $B \in \mathcal{B}$. Scheepers calls this a *Bar-Ilan selection principle* in [17].

Remark 2.11. Let \mathcal{A} and \mathcal{B} be collections. We let $\text{ctbl}(\mathcal{B}) = \{B \in \mathcal{B} : |B| \leq \omega\}$. Then One fails to have a constant strategy in $G_1(\mathcal{A}, \mathcal{B})$ if and only if $\binom{\mathcal{A}}{\text{ctbl}(\mathcal{B})}$ holds as shown in [5, Prop. 15].

In fact,

Lemma 2.12. For any space X ,

$$X \models \binom{\mathcal{A}}{\text{ctbl}(\mathcal{B})} \iff I \not\uparrow_{\text{cnst}} G_1(\mathcal{A}, \mathcal{B}) \iff I \not\uparrow_{\text{cnst}} G_{\text{fin}}(\mathcal{A}, \mathcal{B})$$

Proof. By Remark 2.11, the only thing to show is the equivalence of the non-existence of a constant strategy for One in the single selection and finite selection games. This equivalence can be seen by the fact that any

play by Two in the finite selection game can be translated to a play in the single selection game since a countable collection of finite sets is countable. \square

Note that, in the language of [10],

- $\left(\begin{smallmatrix} \mathcal{O} \\ \text{ctbl}(\mathcal{O}) \end{smallmatrix}\right)$ is the Lindelöf property.
- $\left(\begin{smallmatrix} \Omega \\ \text{ctbl}(\Omega) \end{smallmatrix}\right)$ is the ω -Lindelöf property, most commonly known as the ϵ -space property.
- $\left(\begin{smallmatrix} \mathcal{K} \\ \text{ctbl}(\mathcal{K}) \end{smallmatrix}\right)$ is the k -Lindelöf property.

Definition 2.13. We say that \mathcal{G} is a *selection game* if there exist classes \mathcal{A} , \mathcal{B} and $\square \in \{1, \text{fin}\}$ so that $\mathcal{G} = G_{\square}(\mathcal{A}, \mathcal{B})$.

2.1. Game-theoretic tools

Definition 2.14. We say that two selection games \mathcal{G} and \mathcal{H} are *equivalent*, denoted $\mathcal{G} \equiv \mathcal{H}$, if the following hold:

- $\text{II} \uparrow_{\text{mark}} \mathcal{G} \iff \text{II} \uparrow_{\text{mark}} \mathcal{H}$
- $\text{II} \uparrow \mathcal{G} \iff \text{II} \uparrow \mathcal{H}$
- $\text{I} \not\uparrow \mathcal{G} \iff \text{I} \not\uparrow \mathcal{H}$
- $\text{I} \not\uparrow_{\text{pre}} \mathcal{G} \iff \text{I} \not\uparrow_{\text{pre}} \mathcal{H}$

If, in addition, $\text{I} \not\uparrow_{\text{cnst}} \mathcal{G} \iff \text{I} \not\uparrow_{\text{cnst}} \mathcal{H}$, then we write that $\mathcal{G} \doteq \mathcal{H}$.

Definition 2.15. Given selection games \mathcal{G} and \mathcal{H} , we say that $\mathcal{G} \leq_{\text{II}} \mathcal{H}$ if the following implications hold:

- $\text{II} \uparrow_{\text{mark}} \mathcal{G} \implies \text{II} \uparrow_{\text{mark}} \mathcal{H}$
- $\text{II} \uparrow \mathcal{G} \implies \text{II} \uparrow \mathcal{H}$
- $\text{I} \not\uparrow \mathcal{G} \implies \text{I} \not\uparrow \mathcal{H}$
- $\text{I} \not\uparrow_{\text{pre}} \mathcal{G} \implies \text{I} \not\uparrow_{\text{pre}} \mathcal{H}$

If, in addition, $\text{I} \not\uparrow_{\text{cnst}} \mathcal{G} \implies \text{I} \not\uparrow_{\text{cnst}} \mathcal{H}$, then we write that $\mathcal{G} \leq_{\text{II}}^+ \mathcal{H}$.

Note that \leq_{II} and \leq_{II}^+ are transitive and that if $\mathcal{G} \leq_{\text{II}} \mathcal{H}$ and $\mathcal{H} \leq_{\text{II}} \mathcal{G}$, then $\mathcal{G} \equiv \mathcal{H}$. Moreover, if $\mathcal{G} \leq_{\text{II}}^+ \mathcal{H}$ and $\mathcal{H} \leq_{\text{II}}^+ \mathcal{G}$, then $\mathcal{G} \doteq \mathcal{H}$. We use the subscript of II since each implication is related to a transference of winning plays by Two. Also, for classes \mathcal{A} and \mathcal{B} ,

$$G_1(\mathcal{A}, \mathcal{B}) \leq_{\text{II}}^+ G_{\text{fin}}(\mathcal{A}, \mathcal{B}).$$

We now recall some of the Translation Theorems from [3] with slight additions that will be relevant in the sequel.

Theorem 2.16. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be collections. Suppose there are functions

- $\overleftarrow{T}_{I,n} : \mathcal{B} \rightarrow \mathcal{A}$ and
- $\overrightarrow{T}_{II,n} : [\bigcup \mathcal{A}]^{<\omega} \times \mathcal{B} \rightarrow [\bigcup \mathcal{B}]^{<\omega}$

for each $n \in \omega$ so that

(P1) If $\mathcal{F} \in [\overleftarrow{T}_{I,n}(B)]^{<\omega}$, then $\overrightarrow{T}_{II,n}(\mathcal{F}, B) \in [B]^{<\omega}$.

(P2) If $\mathcal{F}_n \in [\overleftarrow{T}_{I,n}(B_n)]^{<\omega}$ for each $n \in \omega$ and $\bigcup_{n \in \omega} \mathcal{F}_n \in \mathcal{C}$, then $\bigcup_{n \in \omega} \overrightarrow{T}_{II,n}(\mathcal{F}_n, B_n) \in \mathcal{D}$.

Then $G_{fin}(\mathcal{A}, \mathcal{C}) \leq_{II} G_{fin}(\mathcal{B}, \mathcal{D})$. If, in addition, $\overleftarrow{T}_{I,n} = \overleftarrow{T}_{I,m}$ for all $n, m \in \omega$, then $G_{fin}(\mathcal{A}, \mathcal{C}) \leq_{II}^+ G_{fin}(\mathcal{B}, \mathcal{D})$.

Proof. Most of the proof of this is in [3, Theorem 16]. The only thing that remains to be proved is the implication

$$I \not\Upsilon_{\text{cnst}} G_{fin}(\mathcal{A}, \mathcal{C}) \implies I \not\Upsilon_{\text{cnst}} G_{fin}(\mathcal{B}, \mathcal{D})$$

under the assumption that $\overleftarrow{T}_{I,n} = \overleftarrow{T}_{I,m}$ for all $n, m \in \omega$.

Suppose One does not have a constant winning strategy in $G_{fin}(\mathcal{A}, \mathcal{C})$ and let $B \in \mathcal{B}$ be arbitrary. Then $\overleftarrow{T}_{I,n}(B)$ represents a constant strategy for One in $G_{fin}(\mathcal{A}, \mathcal{C})$ since $\overleftarrow{T}_{I,n} = \overleftarrow{T}_{I,m}$ for all $n, m \in \omega$. As $\overleftarrow{T}_{I,n}(B) \in \mathcal{A}$, there exist $\mathcal{F}_n \in [\overleftarrow{T}_{I,n}(B)]^{<\omega}$ so that $\bigcup_{n \in \omega} \mathcal{F}_n \in \mathcal{C}$. Hence, $\bigcup_{n \in \omega} \overrightarrow{T}_{II,n}(\mathcal{F}_n, B) \in \mathcal{D}$. As $B \in \mathcal{B}$ was arbitrary, we see that One does not have a constant winning strategy in $G_{fin}(\mathcal{B}, \mathcal{D})$. \square

Corollary 2.17. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be collections. Suppose there are functions

- $\overleftarrow{T}_{I,n} : \mathcal{B} \rightarrow \mathcal{A}$ and
- $\overrightarrow{T}_{II,n} : (\bigcup \mathcal{A}) \times \mathcal{B} \rightarrow \bigcup \mathcal{B}$

for each $n \in \omega$ so that the following two properties hold.

(Ft1) If $x \in \overleftarrow{T}_{I,n}(B)$, then $\overrightarrow{T}_{II,n}(x, B) \in B$.

(Ft2) If $\mathcal{F}_n \in [\overleftarrow{T}_{I,n}(B_n)]^{<\omega}$ and $\bigcup_{n \in \omega} \mathcal{F}_n \in \mathcal{C}$, then $\bigcup_{n \in \omega} \left\{ \overrightarrow{T}_{II,n}(x, B_n) : x \in \mathcal{F}_n \right\} \in \mathcal{D}$.

Then, for $\square \in \{1, fin\}$, $G_{\square}(\mathcal{A}, \mathcal{C}) \leq_{II} G_{\square}(\mathcal{B}, \mathcal{D})$. If, in addition, $\overleftarrow{T}_{I,n} = \overleftarrow{T}_{I,m}$ for all $n, m \in \omega$, then $G_{\square}(\mathcal{A}, \mathcal{C}) \leq_{II}^+ G_{\square}(\mathcal{B}, \mathcal{D})$.

Corollary 2.18. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be collections. Suppose there is a map $\varphi : [\bigcup \mathcal{B}] \times \omega \rightarrow \bigcup \mathcal{A}$ so that the following two conditions hold.

- For all $B \in \mathcal{B}$ and all $n \in \omega$, $\{\varphi(y, n) : y \in B\} \in \mathcal{A}$.
- If $\mathcal{G}_n \in [B_n]^{<\omega}$ where $B_n \in \mathcal{B}$ for each $n \in \omega$ and $\bigcup_{n \in \omega} \varphi[\mathcal{G}_n \times \{n\}] \in \mathcal{C}$, then $\bigcup_{n \in \omega} \mathcal{G}_n \in \mathcal{D}$.

Then, for $\square \in \{1, fin\}$, $G_{\square}(\mathcal{A}, \mathcal{C}) \leq_{II}^+ G_{\square}(\mathcal{B}, \mathcal{D})$.

Definition 2.19. Consider a class \mathcal{C} and a collection $C \in \mathcal{C}$. We say that C' is an enlargement of C if $C' \subseteq \bigcup C$ and $(\forall x \in C)(\exists y \in C')[x \subseteq y]$.

Definition 2.20. We say that a class \mathcal{C} is *closed under enlargement* if the following property holds: if $C \in \mathcal{C}$ and C' is an enlargement of C , then $C' \in \mathcal{C}$.

Note that $\mathcal{O}(X, \mathcal{A})$ for any family of closed sets \mathcal{A} of a space X is closed under enlargement.

Definition 2.21. Let \mathcal{A} and \mathcal{B} be classes and $\varphi : \bigcup \mathcal{B} \rightarrow \bigcup \mathcal{A}$. For $A \in \mathcal{A}$, we define the φ -refinement of A to be

$$\left\{ y \in \bigcup \mathcal{B} : (\exists x \in A)[\varphi(y) \subseteq x] \right\}.$$

Corollary 2.22. Suppose \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} are classes so that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose there is a function $\varphi : \bigcup \mathcal{B} \rightarrow \bigcup \mathcal{A}$ so that, for any $B \in \mathcal{B}$, $\varphi[B] \in \mathcal{A}$ and, for any $E \subseteq \bigcup \mathcal{B}$ so that $\varphi[E] \in \mathcal{C}$, $E \in \mathcal{D}$. Then, for $\square \in \{1, \text{fin}\}$, $G_{\square}(\mathcal{A}, \mathcal{C}) \leq_{II}^+ G_{\square}(\mathcal{B}, \mathcal{D})$.

If, in addition,

- \mathcal{C} is closed under enlargement,
- for any $A \in \mathcal{A}$, the φ -refinement of A is an element of \mathcal{B} , and
- for any $E \subseteq \bigcup \mathcal{B}$, $E \in \mathcal{D}$ implies $\varphi[E] \in \mathcal{C}$,

then $G_{\square}(\mathcal{A}, \mathcal{C}) \rightleftharpoons G_{\square}(\mathcal{B}, \mathcal{D})$.

Proof. The first condition of Corollary 2.18 holds. Now, suppose $\mathcal{G}_n \in [B_n]^{<\omega}$ where $B_n \in \mathcal{B}$ for all $n \in \omega$ is so that $\bigcup_{n \in \omega} \varphi[\mathcal{G}_n] \in \mathcal{C}$. Then $\bigcup_{n \in \omega} \varphi[\mathcal{G}_n] = \varphi[\bigcup_{n \in \omega} \mathcal{G}_n] \in \mathcal{C}$ implies that $\bigcup_{n \in \omega} \mathcal{G}_n \in \mathcal{D}$. Hence, $G_{\square}(\mathcal{A}, \mathcal{C}) \leq_{II}^+ G_{\square}(\mathcal{B}, \mathcal{D})$.

For the remainder, we use Corollary 2.17. Define $\overleftarrow{T}_{I,n} : \mathcal{A} \rightarrow \mathcal{B}$ to be the φ -refinement of \mathcal{A} . Note that $\overleftarrow{T}_{I,n}$ does not depend on n . Then define $\overrightarrow{T}_{II,n} : (\bigcup \mathcal{B}) \times \mathcal{A} \rightarrow \bigcup \mathcal{A}$ in the following way. If $y \in \bigcup \mathcal{B}$ and $A \in \mathcal{A}$ are so that there exists $x \in A$ with $\varphi(y) \subseteq x$, let $\overrightarrow{T}_{II,n}(y, A) \in A$ be so that $\varphi(y) \subseteq \overrightarrow{T}_{II,n}(y, A)$. Otherwise, let $\overrightarrow{T}_{II,n}(y, A) = y$.

By our definitions, if $y \in \overleftarrow{T}_{I,n}(A)$, then $\overrightarrow{T}_{II,n}(y, A) \in A$. So suppose, for every $n \in \omega$, $y_{1,n}, \dots, y_{k_n,n} \in \overleftarrow{T}_{I,n}(A_n)$ are so that $\bigcup_{n \in \omega} \{y_{1,n}, \dots, y_{k_n,n}\} \in \mathcal{D}$. By the hypotheses, we have that $\bigcup_{n \in \omega} \{\varphi(y_{1,n}), \dots, \varphi(y_{k_n,n})\} \in \mathcal{C}$. Observe that $\varphi(y_{j,n}) \subseteq \overrightarrow{T}_{II,n}(y_{j,n}, A_n)$ for any $n \in \omega$ and $1 \leq j \leq k_n$ which provides that $\bigcup_{n \in \omega} \{\overrightarrow{T}_{II,n}(y_{1,n}, A_n), \dots, \overrightarrow{T}_{II,n}(y_{k_n,n}, A_n)\} \in \mathcal{C}$ since \mathcal{C} is closed under enlargement. \square

As we will see in the applications, Corollary 2.22 is capturing game equivalence under the condition that there is an adequate way to translate between cover types via some translation of open sets.

As an introductory application, the translation of winning plays for Two is monotone with respect to closed subspaces, just as one would expect.

Lemma 2.23. Let X be a space, \mathcal{A} be a family of closed subsets of X , and $Y \subseteq X$ be closed so that $Y \notin \mathcal{A}$. Then $\mathcal{B} := \{A \cap Y : A \in \mathcal{A}\}$ is a family of closed subsets of Y and, for $\square \in \{1, \text{fin}\}$,

$$G_{\square}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{A})) \leq_{II}^+ G_{\square}(\mathcal{O}(Y, \mathcal{B}), \mathcal{O}(Y, \mathcal{B})).$$

Proof. First, for any open $V \subseteq Y$, let W_V be open in X so that $W_V \cap Y = V$. Then define $\varphi : \mathcal{T}_Y \rightarrow \mathcal{T}_X$ by the rule $\varphi(V) = W_V \cup (X \setminus Y)$. If $\mathcal{V} \in \mathcal{O}(Y, \mathcal{B})$, we show that $\varphi[\mathcal{V}] \in \mathcal{O}(X, \mathcal{A})$. Let $A \in \mathcal{A}$ and find $V \in \mathcal{V}$ so that $A \cap Y \subseteq V$. Then $A \subseteq \varphi(V)$.

Now, suppose \mathcal{E} is a collection of open subsets of Y so that $\varphi[\mathcal{E}] \in \mathcal{O}(X, \mathcal{A})$. We show that $\mathcal{E} \in \mathcal{O}(Y, \mathcal{B})$. Let $B \in \mathcal{B}$ and $A \in \mathcal{A}$ be so that $B = A \cap Y$. There is some $E \in \mathcal{E}$ so that $A \subseteq \varphi(E)$ and so we see that $B = A \cap Y \subseteq \varphi(E) \cap Y = E$. So Corollary 2.22 applies. \square

Notice that Two has a winning Markov strategy in $G_1(\mathcal{K}_{\mathbb{R}}, \mathcal{K}_{\mathbb{R}})$ and since \mathbb{Q} is not hemicompact, by [1, Theorem 3.22], Two does not have a winning Markov strategy in $G_1(\mathcal{K}_{\mathbb{Q}}, \mathcal{K}_{\mathbb{Q}})$. Thus, $G_1(\mathcal{K}_{\mathbb{R}}, \mathcal{K}_{\mathbb{R}}) \not\leq_{II} G_1(\mathcal{K}_{\mathbb{Q}}, \mathcal{K}_{\mathbb{Q}})$ so the requirement that the subspace be closed is necessary.

Similarly, the inequality does not reverse as Two has a winning Markov strategy in $G_1(\mathcal{O}_{\mathbb{Z}}, \mathcal{O}_{\mathbb{Z}})$ but does not have a Markov winning strategy in $G_1(\mathcal{O}_{\mathbb{R}}, \mathcal{O}_{\mathbb{R}})$, proving $G_1(\mathcal{O}_{\mathbb{Z}}, \mathcal{O}_{\mathbb{Z}}) \not\leq_{II} G_1(\mathcal{O}_{\mathbb{R}}, \mathcal{O}_{\mathbb{R}})$.

3. Applications to finite powers

Let $X^{<\omega}$ be the disjoint union of all X^n for $n \geq 1$. Clearly, $X^{<\omega}$ is a coding space for all finite subsets of X so one may anticipate a relationship between open covers of $X^{<\omega}$ and ω -covers of X . Indeed, we revisit those well-known connections.

The following result concerning ω -covers and how they interact with finite powers can be seen as the real driving force behind the results of this section and, moreover, the inspiration behind Lemma 4.4.

Lemma 3.1 (Adapted from Lemmas 3.2 and 3.3 of [9]). *Let X be a space and $n \geq 1$. Then,*

- (a) *if \mathcal{U} is an ω -cover of X , then $\{U^n : U \in \mathcal{U}\}$ is an ω -cover of X^n ;*
- (b) *if \mathcal{U} is an ω -cover of X^n ,*

$$\mathcal{V} = \{V \in \mathcal{F}_X : (\exists U \in \mathcal{U}) [V^n \subseteq U]\}$$

is an ω -cover of X .

Lemma 3.2. *Let X be a space and $n \geq 1$. For any $A \subseteq X$, define $A^{\leq n}$ be the disjoint union of A, A^2, \dots, A^n . Then,*

- (a) *if \mathcal{U} is an ω -cover of X , then $\{U^{\leq n} : U \in \mathcal{U}\}$ is an ω -cover of $X^{\leq n}$;*
- (b) *if \mathcal{U} is an ω -cover of $X^{\leq n}$,*

$$\mathcal{V} = \{V \in \mathcal{F}_X : (\exists U \in \mathcal{U}) [V^{\leq n} \subseteq U]\}$$

is an ω -cover of X .

Proof. Though the proof here is similar to a proof of Lemma 3.1, we provide it in full for the convenience of the reader.

- (a) Suppose \mathcal{U} is an ω -cover of X and let \mathcal{F} be any finite subset of $X^{\leq n}$. Notice that

$$p(\mathcal{F}) := \{x \in X : (\exists \mathbf{x} \in \mathcal{F})(\exists j \in \omega)[\pi_j(\mathbf{x}) = x]\}$$

is a finite subset of X where π_j is the usual projection onto the j^{th} coordinate. Then we can find $U \in \mathcal{U}$ so that $p(\mathcal{F}) \subseteq U$. Notice that $\mathcal{F} \subseteq U^{\leq n}$.

(b) Now suppose \mathcal{U} is an ω -cover of $X^{\leq n}$ and let $F = \{x_1, x_2, \dots, x_m\} \subseteq X$. Certainly, $F^{\leq n}$ is a finite subset of $X^{\leq n}$ so there exists $U \in \mathcal{U}$ so that $F^{\leq n} \subseteq U$. For any $\vec{y} = (y_1, y_2, \dots, y_k) \in F^{\leq n}$, let $V_1(\vec{y}), \dots, V_k(\vec{y})$ be so that

$$(y_1, y_2, \dots, y_k) \in \prod_{j=1}^k V_j(\vec{y}) \subseteq U.$$

Observe that

$$\mathcal{V} = \{V_j(\vec{y}) : \vec{y} \in F^{\leq n}, 1 \leq j \leq \text{len}(\vec{y})\}$$

is a finite collection of open subsets. So, for $1 \leq \ell \leq m$, define $W_\ell = \bigcap \{V \in \mathcal{V} : x_\ell \in V\}$ and then $W = \bigcup_{\ell=1}^m W_\ell$. Clearly, W is an open subset of X and $F \subseteq W$.

The only thing that remains to be shown is that $W^{\leq n} \subseteq U$. For $1 \leq k \leq n$, consider $(y_1, \dots, y_k) \in W^k$. Let $1 \leq \ell_1, \ell_2, \dots, \ell_k \leq m$ be so that $y_j \in W_{\ell_j}$ for each $1 \leq j \leq k$. We can now note that

$$\vec{x} = (x_{\ell_1}, x_{\ell_2}, \dots, x_{\ell_k}) \in \prod_{j=1}^k V_j(\vec{x}) \subseteq U.$$

As $W_{\ell_j} \subseteq V_j(\vec{x})$, we see that

$$(y_1, \dots, y_k) \in \prod_{j=1}^k W_{\ell_j} \subseteq \prod_{j=1}^k V_j(\vec{x}) \subseteq U.$$

As k was chosen to be arbitrary, the proof is finished. \square

Lemma 3.3. For any space X , $n \geq 1$, and $\square \in \{1, \text{fin}\}$,

$$\mathsf{G}_\square(\Omega_X, \Omega_X) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^n}, \Omega_{X^n}) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^{\leq n}}, \Omega_{X^{\leq n}}) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}}).$$

Proof. For the equivalence $\mathsf{G}_\square(\Omega_X, \Omega_X) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^n}, \Omega_{X^n})$, we use the map $\varphi : \mathcal{T}_X \rightarrow \mathcal{T}_{X^n}$ defined by $\varphi(U) = U^n$. By Lemma 3.1, we know that $\varphi[\mathcal{W}] \in \Omega_{X^n}$ given $\mathcal{W} \in \Omega_X$. Moreover, if \mathcal{E} is any collection of open subsets of X so that $\varphi[\mathcal{E}]$ is an ω -cover of X^n , it is clear that \mathcal{E} must be an ω -cover of X . Just take $x \in X$ to the tuple of length n consisting of x in each coordinate.

Observe that Ω_{X^n} is closed under enlargement and that, given any ω -cover \mathcal{W} of X^n , by Lemma 3.1,

$$\{V \in \mathcal{T}_X : (\exists U \in \mathcal{W})[V^n \subseteq U]\} \in \Omega_X.$$

Hence, Corollary 2.22 applies.

The equivalence $\mathsf{G}_\square(\Omega_X, \Omega_X) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^{\leq n}}, \Omega_{X^{\leq n}})$ follows in a similar way, except by using Lemma 3.2.

For the equivalence $\mathsf{G}_\square(\Omega_X, \Omega_X) \rightleftharpoons \mathsf{G}_\square(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}})$, we first note that

$$\mathsf{G}_\square(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}}) \leq_{\text{II}}^+ \mathsf{G}_\square(\Omega_X, \Omega_X)$$

by Lemma 2.23. To obtain

$$\mathsf{G}_\square(\Omega_X, \Omega_X) \leq_{\text{II}}^+ \mathsf{G}_\square(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}}),$$

we will first need to fix a bijection $\beta : \omega^2 \rightarrow \omega$. Though the information transfer across the strategy types is uniform and thus, something similar to one of our translation theorems should apply, we will prove this without referring to them explicitly.

What we will do is describe how Two is to play the game assuming they have a winning play in $\mathsf{G}_\square(\Omega_X, \Omega_X)$. Since the statement we wish to prove involves a transferal of winning plays by Two, this

will prove what we want. Notice that, for $\{(n, m) : m \in \omega\}$, Two can play with their attention only on $X^{\leq n}$. In particular, for each $m \in \omega$, in the $\beta(n, m)^{\text{th}}$ inning of $G_{\square}(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}})$, given One's play $\mathcal{U}_{\beta(n, m)}$, let Two choose $V_{n, m} \subseteq X$ and $U_{n, m} \in \mathcal{U}_{\beta(n, m)}$ so that $V_{n, m}^{\leq n} \subseteq U_{n, m}$ in such a way that $\{V_{n, m} : m \in \omega\}$ is an ω -cover of X . This is possible by Lemma 3.2 and since $G_{\square}(\Omega_X, \Omega_X) \rightleftharpoons G_{\square}(\Omega_{X^{\leq n}}, \Omega_{X^{\leq n}})$. Now, the $U_{n, m}$ correspond to a play by Two in the game $G_{\square}(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}})$ and we need only check that it is a winning play. For any finite subset F of $X^{<\omega}$, there is a maximal length n of any tuple in F . Since $\{V_{n, m}^{\leq n} : m \in \omega\}$ is an ω -cover of $X^{\leq n}$ by Lemma 3.2, we see that there must be some $m \in \omega$ for which $F \subseteq V_{n, m}^{\leq n} \subseteq U_{n, m}$. This finishes the proof. \square

Lemma 3.4. For any space X and an ideal \mathcal{A} of compact sets so that $X = \bigcup \mathcal{A}$,

$$G_{fin}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{A})) \leq_{II}^+ G_{fin}(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. We use Theorem 2.16. Note that $\bigcup \mathcal{O}(X, \mathcal{A}) = \bigcup \mathcal{O}_X = \mathcal{T}_X$. Define $\overleftarrow{T}_{I, n} : \mathcal{O}_X \rightarrow \mathcal{O}(X, \mathcal{A})$ by the rule

$$\overleftarrow{T}_{I, n}(\mathcal{U}) = \left\{ \bigcup \mathcal{F} : \mathcal{F} \in [\mathcal{U}]^{<\omega} \right\}.$$

Observe that $\overleftarrow{T}_{I, n}(\mathcal{U}) \in \mathcal{O}(X, \mathcal{A})$ as \mathcal{A} is compact.

Now we define $\overrightarrow{T}_{II, n} : [\mathcal{T}_X]^{<\omega} \times \mathcal{O}_X \rightarrow [\mathcal{T}_X]^{<\omega}$ in the following way. If $V_1, \dots, V_n \in \mathcal{T}_X$ and $\mathcal{U} \in \mathcal{O}_X$ are so that $V_k = \bigcup \mathcal{F}_k$ for $\mathcal{F}_k \in [\mathcal{U}]^{<\omega}$, $1 \leq k \leq n$, choose $\mathcal{F}_{k, \mathcal{U}} \in [\mathcal{U}]^{<\omega}$ so that $V_k = \bigcup \mathcal{F}_{k, \mathcal{U}}$. Then we define

$$\overrightarrow{T}_{II, n}(\{V_1, V_2, \dots, V_n\}, \mathcal{U}) = \bigcup_{k=1}^n \mathcal{F}_{k, \mathcal{U}}.$$

Otherwise, let $\overrightarrow{T}_{II, n}(\{V_1, V_2, \dots, V_n\}, \mathcal{U}) = \{V_1, V_2, \dots, V_n\}$.

Suppose $V_1, \dots, V_n \in \overleftarrow{T}_{I, n}(\mathcal{U})$. Then notice that, for $1 \leq k \leq n$, $\mathcal{F}_{k, \mathcal{U}} \in [\mathcal{U}]^{<\omega}$ and thus

$$\overrightarrow{T}_{II, n}(\{V_1, V_2, \dots, V_n\}, \mathcal{U}) \in [\mathcal{U}]^{<\omega}.$$

Now, suppose $V_{n, 1}, \dots, V_{n, m_n} \in \overleftarrow{T}_{I, n}(\mathcal{U}_n)$ are so that $\bigcup_{n \in \omega} \{V_{n, j} : 1 \leq j \leq m_n\} \in \mathcal{O}(X, \mathcal{A})$. The last thing we need to show is that

$$\bigcup_{n \in \omega} \overrightarrow{T}_{II, n}(\{V_{n, 1}, \dots, V_{n, m_n}\}, \mathcal{U}_n) \in \mathcal{O}_X.$$

Suppose $x \in X$ and notice that $\mathcal{O}(X, \mathcal{A}) \subseteq \mathcal{O}_X$. Then there exists $n \in \omega$ and $1 \leq j \leq m_n$ so that

$$x \in V_{n, j} = \bigcup \mathcal{F}_{n, j, \mathcal{U}_n}.$$

Hence, there is an $W \in \mathcal{F}_{n, j, \mathcal{U}_n}$ so that $x \in W$. Finally, note that $W \in \overrightarrow{T}_{II, n}(\{V_{n, 1}, \dots, V_{n, m_n}\}, \mathcal{U}_n)$. \square

Lemma 3.4 can be strengthened to single selections when \mathcal{A} is the collection of finite subsets of X . However, we have not found a way to apply any of the Translation Theorems in this particular instance. Before stating it explicitly, we need some helpful facts.

Lemma 3.5 (Sakai [16]). If $X \models S_1(\Omega, \Omega)$, then X is Rothberger.

The proof of this relies on a bijection $\omega^2 \rightarrow \omega$ that ensures that, given a sequence \mathcal{U}_n of open covers, single selections from a sequence of ω -covers consisting of a particular kind of closure under finite unions of the \mathcal{U}_n form single selections from the \mathcal{U}_n . The primary reason this creates a problem for strategic transferal is because the way the open covers are translated to ω -covers requires the entire sequence up front. Hence, as Two only knows finitely many of One's moves at any stage in the game, they cannot bring this information to bear.

Also, we employ the Hurewicz and Pawlikowski strategy strengthenings for the Menger and Rothberger games.

Theorem 3.6 (Hurewicz, Pawlikowski [15]). *For any space X and $\square \in \{1, \text{fin}\}$,*

$$I \uparrow_{\text{pre}} G_{\square}(\mathcal{O}_X, \mathcal{O}_X) \iff I \uparrow G_{\square}(\mathcal{O}_X, \mathcal{O}_X).$$

Lemma 3.7. *For any space X ,*

$$G_1(\Omega_X, \Omega_X) \leq_{II}^+ G_1(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. The implication $I \not\uparrow G_1(\Omega_X, \Omega_X) \implies I \not\uparrow_{\text{cst}} G_1(\mathcal{O}_X, \mathcal{O}_X)$ follows from Lemmas 3.4 and 2.12 and $I \uparrow_{\text{pre}} G_1(\mathcal{O}_X, \mathcal{O}_X) \implies I \uparrow_{\text{pre}} G_1(\Omega_X, \Omega_X)$ is the content of Lemma 3.5. Now, using Theorem 3.6,

$$I \uparrow_{\text{pre}} G_1(\mathcal{O}_X, \mathcal{O}_X) \implies I \uparrow_{\text{pre}} G_1(\mathcal{O}_X, \mathcal{O}_X) \implies I \uparrow_{\text{pre}} G_1(\Omega_X, \Omega_X) \implies I \uparrow G_1(\Omega_X, \Omega_X).$$

To finish the proof, we note that [6, Theorem 15] states that

$$II \uparrow G_1(\Omega_X, \Omega_X) \iff II \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$$

and that [6, Theorem 17] states that

$$II \uparrow_{\text{mark}} G_1(\Omega_X, \Omega_X) \iff II \uparrow_{\text{mark}} G_1(\mathcal{O}_X, \mathcal{O}_X). \quad \square$$

Theorem 3.8. *For any space X , $n \geq 1$, and $\square \in \{1, \text{fin}\}$,*

$$G_{\square}(\Omega_X, \Omega_X) \iff G_{\square}(\Omega_{X^n}, \Omega_{X^n}) \iff G_{\square}(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}}) \iff G_{\square}(\mathcal{O}_{X^{<\omega}}, \mathcal{O}_{X^{<\omega}}).$$

Proof. Lemmas 3.4 and 3.7 obtain

$$G_{\square}(\Omega_{X^{<\omega}}, \Omega_{X^{<\omega}}) \leq_{II}^+ G_{\square}(\mathcal{O}_{X^{<\omega}}, \mathcal{O}_{X^{<\omega}}).$$

By Lemma 3.3, to finish the proof, we need only show that

$$G_{\square}(\mathcal{O}_{X^{<\omega}}, \mathcal{O}_{X^{<\omega}}) \leq_{II}^+ G_{\square}(\Omega_X, \Omega_X).$$

Define $\varphi: \mathcal{T}_X \rightarrow \mathcal{T}_{X^{<\omega}}$ by letting $\varphi(U)$ be the disjoint union of all the U^n , $n \geq 1$. If \mathcal{U} is an ω -cover of X , observe that $\varphi[\mathcal{U}]$ is an open cover of $X^{<\omega}$.

Now, suppose \mathcal{E} is a collection of open subsets of X so that $\varphi[\mathcal{E}]$ is an open cover of $X^{<\omega}$. To see that \mathcal{E} must indeed be an ω -cover of X , let $\{x_1, x_2, \dots, x_m\} \subseteq X$. Then notice that $(x_1, x_2, \dots, x_m) \in X^m$ so there must be some $E \in \mathcal{E}$ so that $(x_1, x_2, \dots, x_m) \in \varphi(E)$. By our definition of φ , this means that $\{x_1, x_2, \dots, x_n\} \subseteq E$, so we apply Corollary 2.22 to obtain what we claimed. \square

Corollary 3.9 (Gerlits & Nagy [8]). *Let X be a space. The following are equivalent:*

- (a) X is an ϵ -space,
- (b) every finite power of X is an ϵ -space,
- (c) every finite power of X is Lindelöf, and
- (d) $X^{<\omega}$ is Lindelöf.

Corollary 3.10 (Just, Miller, and Scheepers [9, Theorem 3.9]). *Let X be a space. The following are equivalent:*

- (a) $X \models S_{fin}(\Omega, \Omega)$,
- (b) $(\forall n \in \omega)[X^{n+1} \models S_{fin}(\Omega, \Omega)]$,
- (c) $(\forall n \in \omega)[X^{n+1} \models S_{fin}(\mathcal{O}, \mathcal{O})]$, and
- (d) $X^{<\omega}$ is Menger.

Corollary 3.11 (Sakai [16]). *Let X be a space. The following are equivalent:*

- (a) $X \models S_1(\Omega, \Omega)$,
- (b) $(\forall n \in \omega)[X^{n+1} \models S_1(\Omega, \Omega)]$,
- (c) $(\forall n \in \omega)[X^{n+1} \models S_1(\mathcal{O}, \mathcal{O})]$, and
- (d) $X^{<\omega}$ is Rothberger.

In the sequel, we will extend Theorem 3.6 and Corollaries 3.9, 3.10, and 3.11 as much as possible.

4. Applications to ideals of compact sets

Definition 4.1. For a space X , let $\mathbb{K}(X)$ be the collection of all non-empty compact subsets of X endowed with the Vietoris topology; that is, the topology generated by sets of the form $\{K \in \mathbb{K}(X) : K \subseteq U\}$ and $\{K \in \mathbb{K}(X) : K \cap U \neq \emptyset\}$ for $U \subseteq X$ open. For U_1, \dots, U_n open in X , define

$$[U_1, \dots, U_n] = \left\{ K \in \mathbb{K}(X) : K \subseteq \bigcup_{j=1}^n U_j \text{ and } (\forall j) [K \cap U_j \neq \emptyset] \right\}.$$

These sets form a basis for the topology on $\mathbb{K}(X)$.

For a detailed treatment of the Vietoris topology, see [12].

Definition 4.2. We say that an ideal of compact subsets \mathcal{A}^X of X is *closed under \mathcal{A} -unions* if there is an ideal $\mathcal{A}^{\mathbb{K}(X)}$ of compact subsets of $\mathbb{A} := \mathcal{A}^X$ as a subspace of $\mathbb{K}(X)$ so that

- $K_0 \in \mathcal{A}^X \implies \{K \in \mathbb{A} : K \subseteq K_0\} \in \mathcal{A}^{\mathbb{K}(X)}$ and
- $\mathbf{K} \in \mathcal{A}^{\mathbb{K}(X)} \implies \bigcup \mathbf{K} \in \mathcal{A}^X$.

Lemma 4.3. *Let X be a space.*

- If \mathcal{A}^X is the ideal of finite subsets of X , then \mathcal{A}^X is closed under \mathcal{A} -unions.
- If \mathcal{A}^X is the ideal of compact subsets of X , then \mathcal{A}^X is closed under \mathcal{A} -unions.

Proof. For the finite sets, notice that, for any finite set $F \subseteq X$, $\mathbf{F} := \{K \in \mathbb{K}(X) : K \subseteq F\}$ consists of finite sets. Moreover, \mathbf{F} is finite, thus compact. Similarly, if \mathbf{K} is a finite set consisting of finite subsets of X , then $\bigcup \mathbf{K}$ is a finite subset of X .

The second item follows from results of [12]. \square

Colloquially, one may say that *compact sets are closed under compact unions* when \mathcal{A}^X is the ideal of compact subsets, for example.

The following result finds inspiration from Lemma 3.1 and is the foundation for most of what follows.

Lemma 4.4. *Let X be a space and \mathcal{A}^X be an ideal of compact subsets that is closed under \mathcal{A} -unions.*

- (a) *If $\mathcal{U} \in \mathcal{O}(X, \mathcal{A}^X)$, then $\{[U] : U \in \mathcal{U}\} \in \mathcal{O}(\mathbb{A}, \mathcal{A}^{\mathbb{K}(X)})$.*
- (b) *If $\mathcal{U} \in \mathcal{O}(\mathbb{A}, \mathcal{A}^{\mathbb{K}(X)})$, then*

$$\mathcal{V} = \{V \in \mathcal{T}_X : \exists U \in \mathcal{U} ([V] \subseteq U)\} \in \mathcal{O}(X, \mathcal{A}^X).$$

Proof. (a) Suppose $\mathcal{U} \in \mathcal{O}(X, \mathcal{A}^X)$ and $\mathbf{K} \in \mathcal{A}^{\mathbb{K}(X)}$. Since $\bigcup \mathbf{K} \in \mathcal{A}^X$, there exists $U \in \mathcal{U}$ so that $\bigcup \mathbf{K} \subseteq U$. Hence, $\mathbf{K} \subseteq [U]$.

(b) Suppose $\mathcal{U} \in \mathcal{O}(\mathbb{A}, \mathcal{A}^{\mathbb{K}(X)})$ and let $K_0 \in \mathcal{A}^X$ be arbitrary. Observe that

$$K_0^* := \{K \in \mathbb{A} : K \subseteq K_0\} \in \mathcal{A}^{\mathbb{K}(X)}.$$

Let $U \in \mathcal{U}$ be so that $K_0^* \subseteq U$. Now, for each $K \in K_0^*$, we can find a basic neighborhood \mathcal{B}_K so that $K \in \mathcal{B}_K \subseteq U$. By compactness, there are K_1, \dots, K_n so that $K_0^* \subseteq \bigcup_{j=1}^n \mathcal{B}_{K_j}$. Let $\mathcal{B}_{K_j} = [W_{j,1}, \dots, W_{j,m_j}]$ for $1 \leq j \leq n$. For $x \in K_0$, set $N_x = \bigcap \{W_{j,k} : x \in W_{j,k}\}$ and define $V = \bigcup_{x \in K_0} N_x$. Clearly, $K_0 \subseteq V$ and thus $K_0 \in [V]$, so it suffices to show that $[V] \subseteq U$.

So let $K \in [V]$; i.e. $K \subseteq V$. Then we can find $x_1, \dots, x_p \in K_0$ so that $K \subseteq \bigcup_{\ell=1}^p N_{x_\ell}$ and $K \cap N_{x_\ell} \neq \emptyset$ for each $1 \leq \ell \leq p$. Since $\{x_1, \dots, x_p\}$ is a compact subset of K_0 , it must be an element of some $[W_{j,1}, \dots, W_{j,m_j}]$.

Now, for each $1 \leq \ell \leq p$, there is a $q_\ell \leq m_j$ so that $x_\ell \in W_{j,q_\ell}$; thus $N_{x_\ell} \subseteq W_{j,q_\ell} \subseteq \bigcup_{q=1}^{m_j} W_{j,q}$. So $K \subseteq \bigcup_{\ell=1}^p N_{x_\ell} \subseteq \bigcup_{q=1}^{m_j} W_{j,q}$.

For each $1 \leq q \leq m_j$, let $1 \leq \ell_q \leq p$ be so that $x_{\ell_q} \in W_{j,q}$. As $K \cap N_{x_{\ell_q}} \neq \emptyset$ and $N_{x_{\ell_q}} \subseteq W_{j,q}$, we see that $K \cap W_{j,q} \neq \emptyset$.

Hence, we see that $K \in [W_{j,1}, \dots, W_{j,m_j}] \subseteq U$. Therefore $[V] \subseteq U$. \square

Lemma 4.5. *Let X be a space and $\mathcal{A} = \mathcal{A}^X$ be an ideal of compact subsets that is closed under \mathcal{A} -unions where $\mathbb{A} = \mathcal{A}^X$ is viewed as a subspace of $\mathbb{K}(X)$. Then, for $\square \in \{1, \text{fin}\}$,*

$$G_\square(\mathcal{O}_\mathbb{A}, \mathcal{O}_\mathbb{A}) \leq_{II}^+ G_\square(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{A})).$$

Proof. Define $\varphi : \mathcal{T}_X \rightarrow \mathcal{T}_\mathbb{A}$ by the rule $\varphi(U) = [U]$. Certainly, if $\mathcal{U} \in \mathcal{O}(X, \mathcal{A})$, then $\varphi[\mathcal{U}] \in \mathcal{O}_\mathbb{A}$. Moreover, suppose \mathcal{E} is a collection of open subsets of X so that $\varphi[\mathcal{E}] \in \mathcal{O}_\mathbb{A}$. It is clear that $\mathcal{E} \in \mathcal{O}(X, \mathcal{A})$. Hence, Corollary 2.22 applies. \square

Theorem 4.6. *Let X be a space and $\mathcal{A} = \mathcal{A}^X$ be an ideal of compact subsets that is closed under \mathcal{A} -unions where $\mathbb{A} = \mathcal{A}^X$ is viewed as a subspace of $\mathbb{K}(X)$ and $\mathcal{B} = \mathcal{A}^{\mathbb{K}(X)}$ is the corresponding ideal of \mathbb{A} . Then, for $\square \in \{1, \text{fin}\}$,*

$$G_\square(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{A})) \rightleftharpoons G_\square(\mathcal{O}(\mathbb{A}, \mathcal{B}), \mathcal{O}(\mathbb{A}, \mathcal{B})).$$

Proof. Define $\varphi : \mathcal{T}_X \rightarrow \mathcal{T}_\mathbb{A}$ by the rule $\varphi(U) = [U]$. If $\mathcal{U} \in \mathcal{O}(X, \mathcal{A})$, then, by Lemma 4.4, $\varphi[\mathcal{U}] \in \mathcal{O}(\mathbb{A}, \mathcal{B})$. If \mathcal{E} is a collection of open subsets of X so that $\varphi[\mathcal{E}] \in \mathcal{O}(\mathbb{A}, \mathcal{B})$, then $\mathcal{E} \in \mathcal{O}(X, \mathcal{A})$. This is seen by considering any $K \in \mathcal{A}$ and noticing that $K \in \{F \in \mathbb{A} : F \subseteq K\} \in \mathcal{B}$.

Certainly, $\mathcal{O}(\mathbb{A}, \mathcal{B})$ is closed under enlargement and, by Lemma 4.4, we see that the φ -refinement of $\mathcal{U} \in \mathcal{O}(\mathbb{A}, \mathcal{B})$ is an element of $\mathcal{O}(X, \mathcal{A})$. Therefore, Corollary 2.22 applies and the proof is complete. \square

4.1. The space of finite subsets

Let $\mathcal{P}_{\text{fin}}(X)$ be the subspace of $\mathbb{K}(X)$ consisting of the finite subsets of X . As in Section 3, one would anticipate some relationship between open covers of $\mathcal{P}_{\text{fin}}(X)$ and ω -covers of X . To some degree, $\mathcal{P}_{\text{fin}}(X)$ also avoids all of the unnecessary information the finite powers offer such as repetition and order. To begin, we offer an analog to Lemma 3.1 which follows immediately from Lemmas 4.3 and 4.4.

Corollary 4.7. *Let X be a space.*

- (a) *If \mathcal{U} is an ω -cover of X , then $\{[U] : U \in \mathcal{U}\}$ is an ω -cover of $\mathcal{P}_{\text{fin}}(X)$.*
- (b) *If \mathcal{U} is an ω -cover of $\mathcal{P}_{\text{fin}}(X)$, then*

$$\mathcal{V} = \{V \in \mathcal{T}_X : \exists U \in \mathcal{U} ([V] \subseteq U)\}$$

is an ω -cover of X .

Theorem 4.8. *For any space X and $\square \in \{1, \text{fin}\}$,*

$$G_{\square}(\Omega_X, \Omega_X) \rightleftarrows G_{\square}(\Omega_{\mathcal{P}_{\text{fin}}(X)}, \Omega_{\mathcal{P}_{\text{fin}}(X)}) \rightleftarrows G_{\square}(\mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}, \mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}).$$

Proof. The fact that

$$G_{\square}(\Omega_X, \Omega_X) \rightleftarrows G_{\square}(\Omega_{\mathcal{P}_{\text{fin}}(X)}, \Omega_{\mathcal{P}_{\text{fin}}(X)})$$

follows from Theorem 4.6. By Lemmas 3.4 and 3.7, we see that

$$G_{\square}(\Omega_{\mathcal{P}_{\text{fin}}(X)}, \Omega_{\mathcal{P}_{\text{fin}}(X)}) \leq_{\text{II}}^+ G_{\square}(\mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}, \mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}).$$

Finally,

$$G_{\square}(\mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}, \mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}) \leq_{\text{II}}^+ G_{\square}(\Omega_X, \Omega_X),$$

follows from Lemma 4.5. This finishes the proof. \square

Corollary 4.9. *For any space X , the following are equivalent:*

- (a) *X is an ϵ -space*
- (b) *$\mathcal{P}_{\text{fin}}(X)$ is an ϵ -space*
- (c) *$\mathcal{P}_{\text{fin}}(X)$ is Lindelöf.*

Corollary 4.10. *For any space X , the following are equivalent:*

- (a) *$X \models S_{\text{fin}}(\Omega, \Omega)$*

- (b) $\mathcal{P}_{fin}(X) \models \mathcal{S}_{fin}(\Omega, \Omega)$
 (c) $\mathcal{P}_{fin}(X)$ is Menger.

Corollary 4.11. For any space X , the following are equivalent:

- (a) $X \models \mathcal{S}_1(\Omega, \Omega)$
 (b) $\mathcal{P}_{fin}(X) \models \mathcal{S}_1(\Omega, \Omega)$
 (c) $\mathcal{P}_{fin}(X)$ is Rothberger.

Corollary 4.12 (Scheepers [18]). For any space X and $\square \in \{1, fin\}$,

$$I \uparrow_{pre} G_{\square}(\Omega_X, \Omega_X) \iff I \uparrow G_{\square}(\Omega_X, \Omega_X).$$

Proof. These follow immediately from Theorems 3.6 and 4.8. \square

4.2. The space of compact subsets

When we move to compact sets, one might expect that all of the analogous theorems from Sections 3 and 4.1 hold between k -covers of X and open covers of $\mathbb{K}(X)$. Though we cannot obtain the full scope of those results, we are able to recover a significant fragment; namely, everything about finite selection games goes through, and we are able to recover a version of Pawlikowski's theorem.

In a similar spirit to the results of Lemma 3.1 and Corollary 4.7, we establish a way to transfer k -cover information between X and $\mathbb{K}(X)$. It follows immediately from Lemmas 4.3 and 4.4.

Corollary 4.13. Let X be a space.

- (a) If \mathcal{U} is a k -cover of X , then $\{\{U\} : U \in \mathcal{U}\}$ is a k -cover of $\mathbb{K}(X)$.
 (b) If \mathcal{U} is a k -cover of $\mathbb{K}(X)$, then

$$\mathcal{V} = \{V \in \mathcal{T}_X : \exists U \in \mathcal{U} ([V] \subseteq U)\}$$

is a k -cover of X .

Corollary 4.14. For any space X and $\square \in \{1, fin\}$,

$$G_{\square}(\mathcal{O}_{\mathbb{K}(X)}, \mathcal{O}_{\mathbb{K}(X)}) \leq_{II}^+ G_{\square}(\mathcal{K}_X, \mathcal{K}_X) \iff G_{\square}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}).$$

Proof. This follows immediately from Lemma 4.5 and Theorem 4.6. \square

Theorem 4.15. For any space X ,

$$G_{fin}(\mathcal{K}_X, \mathcal{K}_X) \iff G_{fin}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}) \iff G_{fin}(\mathcal{O}_{\mathbb{K}(X)}, \mathcal{O}_{\mathbb{K}(X)}).$$

Proof. This follows immediately from Lemma 3.4 and Corollary 4.14. \square

Corollary 4.16. For any space X , the following are equivalent:

- (a) X is k -Lindelöf
 (b) $\mathbb{K}(X)$ is k -Lindelöf

(c) $\mathbb{K}(X)$ is Lindelöf.

Corollary 4.17. For any space X , the following are equivalent:

- (a) $X \models \mathcal{S}_{fin}(\mathcal{K}, \mathcal{K})$
- (b) $\mathbb{K}(X) \models \mathcal{S}_{fin}(\mathcal{K}, \mathcal{K})$
- (c) $\mathbb{K}(X)$ is Menger.

Corollary 4.18. For any space X ,

$$I \uparrow_{pre} \mathcal{G}_{fin}(\mathcal{K}_X, \mathcal{K}_X) \iff I \uparrow \mathcal{G}_{fin}(\mathcal{K}_X, \mathcal{K}_X).$$

Proof. This follows from Theorems 3.6 and 4.15. \square

Like before, single selections present an obstacle. In the context of k -covers, we don't even obtain an analog to Corollary 4.11.

Example 4.19. In general, $\mathbb{S}_1(\mathcal{K}, \mathcal{K}) \not\equiv \mathbb{S}_1(\mathcal{O}, \mathcal{O})$. Observe that $\mathbb{R} \models \mathbb{S}_1(\mathcal{K}, \mathcal{K})$ but $\mathbb{R} \not\models \mathbb{S}_1(\mathcal{O}, \mathcal{O})$. If $\{\mathcal{U}_n : n \in \omega\}$ consists of k -covers of \mathbb{R} , simply choose $U_n \in \mathcal{U}_n$ to be so that $[-n, n] \subseteq U_n$. Then $\{U_n : n \in \omega\}$ is a k -cover of \mathbb{R} . On the other hand, consider $\mathcal{V}_n = \{B(q, 2^{-n}) : q \in \mathbb{Q}\}$ and any sequence of selections $V_n \in \mathcal{V}_n$. Notice that the union of the V_n has finite Lebesgue measure so they cannot cover \mathbb{R} .

Because of this non-example, we cannot obtain a version of Pawlikowski's result for k -covers as easily as we did for ω -covers. In the next two results we nevertheless prove that there is a Pawlikowski style strategy reduction for the k -Rothberger game. The basic idea is to take the game up to the hyperspace and play with the right kind of open sets to guarantee that Two's play results in a k -cover.

Lemma 4.20. Suppose \mathcal{A} is any ideal of closed sets that contains all singletons of a space X . Also suppose $\bigcup\{\mathcal{F}_n : n \in \omega\} \in \mathcal{O}(X, \mathcal{A})$ where \mathcal{F}_n is a finite collection of open sets for each $n \in \omega$. Then, for any $A \in \mathcal{A}$, there exists an increasing sequence $\{\alpha_n : n \in \omega\}$ so that

$$(\forall n \in \omega)(\exists U \in \mathcal{F}_{\alpha_n}) [A \subseteq U].$$

Proof. Let $A = A_0 \in \mathcal{A}$ be arbitrary and, for $n \geq 0$, suppose we have $A_n \in \mathcal{A}$ and $\alpha_n \in \omega$ defined so that

$$\alpha_n = \min \{m \in \omega : (\exists U \in \mathcal{F}_m) [A_n \subseteq U]\}.$$

As U is a proper open set for each $U \in \mathcal{F}_{\alpha_n}$, we can find $x_U \in X \setminus U$. Notice that

$$A_{n+1} := A_n \cup \{x_U : U \in \mathcal{F}_{\alpha_n}\} \in \mathcal{A}$$

since \mathcal{A} is an ideal containing singletons and \mathcal{F}_{α_n} is finite. So then we can set

$$\alpha_{n+1} = \min \{m \in \omega : (\exists U \in \mathcal{F}_m) [A_{n+1} \subseteq U]\}.$$

Observe that $\alpha_{n+1} > \alpha_n$. This finishes the proof. \square

Theorem 4.21. For any space X , $I \uparrow_{pre} \mathcal{G}_1(\mathcal{K}_X, \mathcal{K}_X) \iff I \uparrow \mathcal{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

Proof. We need only show

$$I \not\Upsilon_{\text{pre}} G_1(\mathcal{K}_X, \mathcal{K}_X) \implies I \not\Upsilon G_1(\mathcal{K}_X, \mathcal{K}_X).$$

Suppose $X \models S_1(\mathcal{K}_X, \mathcal{K}_X)$ and that One is playing according to some fixed strategy in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. Any k -cover can be made into a countable k -cover by the selection principle. Hence, we can code One’s strategy with $\{U_s : s \in \omega^{<\omega}\}$ with the property that $\{U_{s \smallfrown k} : k \in \omega\}$ is a k -cover for any $s \in \omega^{<\omega}$. Using this strategy for One in $G_1(\mathcal{K}_X, \mathcal{K}_X)$, we will define a strategy for One in $G_{\text{fin}}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)})$ which will produce a winning counter-play by Two as

$$\begin{aligned} X \models S_1(\mathcal{K}_X, \mathcal{K}_X) &\implies \mathbb{K}(X) \models S_1(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}) \\ &\implies \mathbb{K}(X) \models S_{\text{fin}}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}) \\ &\implies I \not\Upsilon G_{\text{fin}}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}). \end{aligned}$$

Moreover, we will show the counter-play produced actually corresponds to a counter-play to One’s strategy in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. We will do this through a sequence of useful claims.

The first claim is that

$$(\forall m \geq 0)(\forall j > 0)(\forall \mathbf{K} \in \mathbb{K}(\mathbb{K}(X)))(\exists s \in \omega^{|j^m|})(\forall t \in j^m)(\forall L \in \mathbf{K})(\exists k < |j^m|) [L \subseteq U_{t \smallfrown (s \upharpoonright_{k+1})}].$$

Fix $m \geq 0, j > 0$, let $\mathbf{K} \subseteq \mathbb{K}(X)$ be compact, and consider $K := \bigcup \mathbf{K}$, which forms a compact subset of X . Enumerate j^m as $\{t_\ell : \ell < |j^m|\}$. Let $s(0) \in \omega$ so that $K \subseteq U_{t_0 \smallfrown s(0)}$. Then, for $n \geq 0$, suppose we have $s(0), \dots, s(n) \in \omega$ defined so that $K \subseteq U_{t_\ell \smallfrown s(0) \smallfrown \dots \smallfrown s(\ell)}$ for each $\ell \leq n$. Let $s(n+1) \in \omega$ be so that

$$K \subseteq U_{t_{n+1} \smallfrown s(0) \smallfrown \dots \smallfrown s(n+1)}.$$

This defines $s : |j^m| \rightarrow \omega$.

Next, fix some $t \in j^m$, let $L \in \mathbf{K}$ be arbitrary, and find $k < |j^m|$ so that $t = t_k$. Then,

$$L \subseteq K \subseteq U_{t_k \smallfrown s(0) \smallfrown \dots \smallfrown s(k)} = U_{t \smallfrown s \upharpoonright_{k+1}}.$$

This establishes the claim.

The second claim involves defining, for $m \geq 0, j > 0$, and $s : |j^m| \rightarrow \omega$,

$$\mathbf{V}_s(m, j) = \bigcap_{t \in j^m} \bigcup_{k=1}^{|j^m|} [U_{t \smallfrown (s \upharpoonright_k)}].$$

The second claim is that, for fixed $m \geq 0$ and $j > 0$, $\{\mathbf{V}_s(m, j) : s \in \omega^{|j^m|}\}$ is a k -cover of $\mathbb{K}(X)$. So let $\mathbf{K} \subseteq \mathbb{K}(X)$ be compact and choose $s : |j^m| \rightarrow \omega$ so that

$$(\forall t \in j^m)(\forall L \in \mathbf{K})(\exists k < |j^m|) [L \subseteq U_{t \smallfrown (s \upharpoonright_{k+1})}],$$

which is guaranteed by the first claim. Fix $t \in j^m$, let $L \in \mathbf{K}$, and observe that, for some $k < |j^m|$,

$$L \in [U_{t \smallfrown (s \upharpoonright_{k+1})}] \subseteq \bigcup_{\ell=1}^{|j^m|} [U_{t \smallfrown (s \upharpoonright_\ell)}].$$

Since this is true for any $L \in \mathbf{K}$, we see that

$$\mathbf{K} \subseteq \bigcup_{\ell=1}^{|j^m|} [U_{t \smallfrown (s \upharpoonright \ell)}].$$

Since $t \in j^m$ was also taken to be arbitrary, we see that

$$\mathbf{K} \subseteq \bigcap_{t \in j^m} \bigcup_{k=1}^{|j^m|} [U_{t \smallfrown (s \upharpoonright k)}] = \mathbf{V}_s(m, j).$$

The third claim is that there are increasing functions $g, h : \omega \rightarrow \omega$ so that

$$(\forall \mathbf{K} \in \mathbb{K}(\mathbb{K}(X))) (\exists^\infty n \in \omega) (\exists s : (g(n+1) - g(n)) \rightarrow h(n+1)) [\mathbf{K} \subseteq \mathbf{V}_s(g(n), h(n))].$$

To accomplish this, we define a particular strategy σ for One in $\mathbf{G}_{\text{fin}}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)})$. First, for $m \geq 0$ and $j > 0$, define

$$\mathcal{V}_{m,j} = \left\{ \mathbf{V}_s(m, j) : s \in \omega^{|j^m|} \right\},$$

which is a k -cover of $\mathbb{K}(X)$ by the second claim. Also, for $m \geq 0, j > 0$, and $p > 0$, define

$$\mathcal{F}_{m,j,p} = \left\{ \mathbf{V}_s(m, j) : s \in p^{|j^m|} \right\},$$

a finite subset of $\mathcal{V}_{m,j}$. Observe that, for $m \geq 0$ and $j > 0$, if $0 < p \leq q$, then $\mathcal{F}_{m,j,p} \subseteq \mathcal{F}_{m,j,q}$. In fact,

$$(\forall \mathcal{E} \in [\mathcal{V}_{m,j}]^{<\omega}) (\exists p > 0) [\mathcal{E} \subseteq \mathcal{F}_{m,j,p}].$$

To see this, let $\mathcal{E} \in [\mathcal{V}_{m,j}]^{<\omega}$ and $A \in [\omega^{|j^m|}]^{<\omega}$ be so that $\mathcal{E} = \{\mathbf{V}_s(m, j) : s \in A\}$. Then set

$$p = 1 + \max\{s(\ell) : \ell < |j^m|, s \in A\}.$$

It follows that $A \subseteq p^{|j^m|}$ which further implies that $\mathcal{E} \subseteq \mathcal{F}_{m,j,p}$. Now define $p_{m,j} : [\mathcal{V}_{m,j}]^{<\omega} \rightarrow \omega$ to be

$$p_{m,j}(\mathcal{E}) = \min\{p : \mathcal{E} \subseteq \mathcal{F}_{m,j,p}\}.$$

We next define the strategy σ . Set $m_0 = 0, j_0 = 1$, and $\sigma(\emptyset) = \mathcal{V}_{m_0, j_0}$. For $n \geq 0$, suppose $\{\mathcal{E}_\ell : \ell < n\}, \{m_\ell : \ell \leq n\}$, and $\{j_\ell : \ell \leq n\}$ have been defined. Let $m_{n+1} = m_n + |j_n^{m_n}|$. Then for $\mathcal{E}_n \in [\mathcal{V}_{m_n, j_n}]^{<\omega}$ set $j_{n+1} = \max\{j_n, p_{m_n, j_n}(\mathcal{E}_n)\}$ and define

$$\sigma(\mathcal{V}_{m_0, j_0}, \mathcal{E}_0, \dots, \mathcal{V}_{m_n, j_n}, \mathcal{E}_n) = \mathcal{V}_{m_{n+1}, j_{n+1}}$$

This finishes the definition of the strategy σ .

As One does not have a winning strategy in $\mathbf{G}_{\text{fin}}(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)})$, Two can produce a counter-play $\{\mathcal{E}_n : n \in \omega\}$ so that

$$\bigcup \{\mathcal{E}_n : n \in \omega\}$$

is a k -cover of $\mathbb{K}(X)$. Notice that this provides increasing sequences $\langle m_n : n \in \omega \rangle$ and $\langle j_n : n \in \omega \rangle$. Moreover, as $\mathcal{E}_n \in [\mathcal{V}_{m_n, j_n}]^{<\omega}$ and $j_{n+1} \geq p_{m_n, j_n}(\mathcal{E}_n)$, we have that $\mathcal{E}_n \subseteq \mathcal{F}_{m_n, j_n, j_{n+1}}$. That is,

$$\bigcup \{ \mathcal{F}_{m_n, j_n, j_{n+1}} : n \in \omega \}$$

is a k -cover of $\mathbb{K}(X)$.

Define $g, h : \omega \rightarrow \omega$ by the rules $g(n) = m_n$ and $h(n) = j_n$ and notice that they are increasing functions. To verify they are as desired, we first find one $n \in \omega$ that meets the requisite criterion. Let $\mathbf{K} \subseteq \mathbb{K}(X)$ be compact. Since $\bigcup \{ \mathcal{F}_{m_n, j_n, j_{n+1}} : n \in \omega \}$ is a k -cover of $\mathbb{K}(X)$, there must be some $n \in \omega$ and $s : |j_n^{m_n}| \rightarrow j_{n+1}$ so that $\mathbf{K} \subseteq \mathbf{V}_s(m_n, j_n)$. Behold that $m_n = g(n)$, $j_n = h(n)$, $j_{n+1} = h(n+1)$, and $|j_n^{m_n}| = m_{n+1} - m_n = g(n+1) - g(n)$. The fact that infinitely many such n exist follows from Lemma 4.20.

The final thing to show is that we can actually construct a counter-play against One’s strategy in $G_1(\mathcal{K}_X, \mathcal{K}_X)$ with the help of the defined g and h . For $n \geq 1$, $k_1 < k_2 < \dots < k_n$, and $s_i : (g(k_i+1) - g(k_i)) \rightarrow h(k_i+1)$, $1 \leq i \leq n$, we define

$$\mathbf{W}_n(k_1, \dots, k_n; s_1, \dots, s_n) = \bigcap_{i=1}^n \mathbf{V}_{s_i}(g(k_i), h(k_i)).$$

To assist with notation, we let $F_{k_i} = h(k_i+1)^{(g(k_i+1) - g(k_i))}$. By the third claim,

$$\mathcal{W}_n := \{ \mathbf{W}_n(k_1, \dots, k_n; s_1, \dots, s_n) : (k_1 < \dots < k_n) \text{ and } (\forall 1 \leq i \leq n) [s_i \in F_{k_i}] \}$$

is a k -cover of $\mathbb{K}(X)$. Since we are assume $X \models S_1(\mathcal{K}_X, \mathcal{K}_X)$ and we know that

$$X \models S_1(\mathcal{K}_X, \mathcal{K}_X) \implies \mathbb{K}(X) \models S_1(\mathcal{K}_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}),$$

for each $n \geq 1$, we can select $k_{n,1} < \dots < k_{n,n}$ and $s_{n,i} \in F_{k_{n,i}}$ for $1 \leq i \leq n$, so that

$$\{ \mathbf{W}_n(k_{n,1}, \dots, k_{n,n}; s_{n,1}, \dots, s_{n,n}) : n \in \omega \}$$

is a k -cover of $\mathbb{K}(X)$.

For each $n \geq 1$, choose $k_{n,\alpha_n} \in \{k_{n,1}, \dots, k_{n,n}\} \setminus \{k_{\ell,\alpha_\ell} : 1 \leq \ell < n\}$ and consider

$$B_n := \{g(k_{n,\alpha_n}) + i : i < g(k_{n,\alpha_n} + 1) - g(k_{n,\alpha_n})\}.$$

We argue that the sets $\{B_n : n \geq 1\}$ are pair-wise disjoint. Suppose $m < n$ and notice that $k_{m,\alpha_m} \neq k_{n,\alpha_n}$ by definition. Without loss of generality, suppose $k_{m,\alpha_m} < k_{n,\alpha_n}$. Since it follows that $g(k_{m,\alpha_m}) < g(k_{n,\alpha_n})$, to establish that B_m and B_n are disjoint, it suffices to check that $g(k_{m,\alpha_m} + 1) \leq g(k_{n,\alpha_n})$. Indeed,

$$k_{m,\alpha_m} + 1 \leq k_{n,\alpha_n} \implies g(k_{m,\alpha_m} + 1) \leq g(k_{n,\alpha_n}).$$

Moreover, for any $\ell \in B_n$, $\ell - g(k_{n,\alpha_n}) < g(k_{n,\alpha_n} + 1) - g(k_{n,\alpha_n})$. So $s_{n,\alpha_n}(\ell - g(k_{n,\alpha_n}))$ is defined. This allows us to define $f : \omega \rightarrow \omega$ by the rule

$$f(\ell) = \begin{cases} s_{n,\alpha_n}(\ell - g(k_{n,\alpha_n})), & \ell \in B_n \\ 0, & \text{otherwise} \end{cases}$$

We claim that this f is Two’s desired play. Let $K \subseteq X$ be compact, and notice that $\{K\}$ is a compact subset of $\mathbb{K}(X)$. So there exists some $n \geq 1$ so that

$$\{K\} \subseteq \mathbf{W}_n(k_{n,1}, \dots, k_{n,n}; s_{n,1}, \dots, s_{n,n}) = \bigcap_{i=1}^n \mathbf{V}_{s_{n,i}}(g(k_{n,i}), h(k_{n,i})).$$

For ease of notation, let $E_n = h(k_{n,\alpha_n})^{g(k_{n,\alpha_n})}$ and notice that

$$\{K\} \subseteq \mathbf{V}_{s_n,\alpha_n}(g(k_{n,\alpha_n}), h(k_{n,\alpha_n})) = \bigcap_{t \in E_n} \bigcup_{\ell=1}^{|E_n|} [U_{t \smallfrown (s_n,\alpha_n \upharpoonright \ell)}].$$

We wish to show that $f \upharpoonright_{g(k_{n,\alpha_n})}: g(k_{n,\alpha_n}) \rightarrow h(k_{n,\alpha_n})$. So let $\ell < g(k_{n,\alpha_n})$ be arbitrary. If $\ell \notin B_m$ for any $m \geq 1$, $f(\ell) = 0 < h(k_{n,\alpha_n})$. Otherwise, $\ell \in B_m$ for some $m \geq 1$. Then there is some $i < g(k_{m,\alpha_m} + 1) - g(k_{m,\alpha_m})$ with $\ell = g(k_{m,\alpha_m}) + i$. Hence,

$$f(\ell) = s_{m,\alpha_m}(\ell - g(k_{m,\alpha_m})) = s_{m,\alpha_m}(i) < h(k_{m,\alpha_m}).$$

Also, as

$$g(k_{m,\alpha_m}) \leq g(k_{m,\alpha_m}) + i = \ell < g(k_{n,\alpha_n}),$$

we see that $k_{m,\alpha_m} < k_{n,\alpha_n}$ which provides

$$f(\ell) < h(k_{m,\alpha_m}) \leq h(k_{n,\alpha_n}).$$

Thus,

$$\{K\} \subseteq \bigcup_{\ell=1}^{|E_n|} [U_{(f \upharpoonright_{g(k_{n,\alpha_n})}) \smallfrown (s_n,\alpha_n \upharpoonright \ell)}] \implies K \in \bigcup_{\ell=1}^{|E_n|} [U_{(f \upharpoonright_{g(k_{n,\alpha_n})}) \smallfrown (s_n,\alpha_n \upharpoonright \ell)}]$$

which means that for some $1 \leq \ell \leq |E_n|$,

$$K \in [U_{(f \upharpoonright_{g(k_{n,\alpha_n})}) \smallfrown (s_n,\alpha_n \upharpoonright \ell)}] \implies K \subseteq U_{(f \upharpoonright_{g(k_{n,\alpha_n})}) \smallfrown (s_n,\alpha_n \upharpoonright \ell)}.$$

Finally, by our definition of f , we note that

$$(f \upharpoonright_{g(k_{n,\alpha_n})}) \smallfrown (s_n,\alpha_n \upharpoonright \ell) = f \upharpoonright_{g(k_{n,\alpha_n} + 1)}$$

which means

$$K \subseteq U_{f \upharpoonright_{g(k_{n,\alpha_n} + 1)}}.$$

Therefore, if Two plays according to f , Two produces a k -cover of X , finishing the proof. \square

5. Final remarks

We end with a couple of other applications of these techniques which relate to the interplay between cover types.

Theorem 5.1. For any space X and $\square \in \{1, fin\}$,

$$G_{\square}(\mathcal{K}_{\mathbb{K}(X)}, \Omega_{\mathbb{K}(X)}) \leq_{II}^+ G_{\square}(\mathcal{K}_X, \Omega_X).$$

Proof. Define $\varphi : \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{K}(X)}$ by the rule $\varphi(U) = [U]$. By Corollary 4.13, we know that $\varphi[\mathcal{U}] \in \mathcal{K}_{\mathbb{K}(X)}$ when $\mathcal{U} \in \mathcal{K}_X$. Now, suppose $\mathcal{E} \subseteq \mathcal{T}_X$ is so that $\varphi[\mathcal{E}] \in \Omega_{\mathbb{K}(X)}$. Observe that $\varphi[\mathcal{E}]$ is also an ω -cover of $\mathcal{P}_{\text{fin}}(X)$. By Corollary 4.7, we know that

$$\{V \in \mathcal{T}_X : \exists U \in \varphi[\mathcal{E}] ([V] \subseteq U)\} \in \Omega_X$$

which demonstrates that \mathcal{E} is an ω -cover of X . Thus, Corollary 2.22 applies. \square

Theorem 5.2. For any space X and $\square \in \{1, \text{fin}\}$,

$$G_{\square}(\Omega_X, \mathcal{K}_X) \leq_{II}^+ G_{\square}(\Omega_{\mathbb{K}(X)}, \mathcal{K}_{\mathbb{K}(X)}).$$

Proof. In this case, we apply Corollary 2.17. Define $\overleftarrow{T}_{I,n} : \Omega_{\mathbb{K}(X)} \rightarrow \Omega_X$ by the rule

$$\overleftarrow{T}_{I,n}(\mathcal{U}) = \{V \in \mathcal{T}_X : \exists U \in \mathcal{U} ([V] \subseteq U)\}.$$

Since every ω -cover of $\mathbb{K}(X)$ is an ω -cover of $\mathcal{P}_{\text{fin}}(X)$, $\overleftarrow{T}_{I,n}$ is defined by Corollary 4.7.

Now we define $\overrightarrow{T}_{II,n} : \mathcal{T}_X \times \Omega_{\mathbb{K}(X)} \rightarrow \mathcal{T}_{\mathbb{K}(X)}$ in the following way. Let $\mathcal{U} \in \Omega_{\mathbb{K}(X)}$ and if $V \in \overleftarrow{T}_{I,n}(\mathcal{U})$, let $\overrightarrow{T}_{II,n}(V, \mathcal{U})$ be so that $[V] \subseteq \overrightarrow{T}_{II,n}(V, \mathcal{U})$. Otherwise, let $\overrightarrow{T}_{II,n}(V, \mathcal{U}) = V$.

Suppose $\mathcal{F}_n \in \left[\overleftarrow{T}_{I,n}(\mathcal{U}_n) \right]^{<\omega}$ are so that $\bigcup_{n \in \omega} \mathcal{F}_n \in \mathbb{K}(X)$. By Corollary 4.13, we know that

$$\bigcup_{n \in \omega} \{[V] : V \in \mathcal{F}_n\} \in \mathcal{K}_{\mathbb{K}(X)}$$

and, as $[V] \subseteq \overrightarrow{T}_{II,n}(V, \mathcal{U}_n)$ for each $V \in \mathcal{F}_n$, we see that

$$\bigcup_{n \in \omega} \left\{ \overrightarrow{T}_{II,n}(V, \mathcal{U}_n) : V \in \mathcal{F}_n \right\} \in \mathcal{K}_{\mathbb{K}(X)}.$$

This finishes the proof. \square

For further work,

- are [2, Lemma 7] and [2, Lemma 8] true as stated?
- Can Theorem 5.1 be used to establish a Pawlikowski style strategy reduction for $G_{\square}(\mathcal{K}, \Omega)$?
- Are there any other interesting ideals \mathcal{A} of compact sets that are closed under \mathcal{A} -unions?

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