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PARTICIPANT RESEARCH ESSAY FOR DIME RESEARCH TEAM

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My research has two broad foci: (1) understanding how middle and high school students construct quantitative reasoning (especially reasoning that is multiplicative in nature) as a basis for algebraic reasoning; and (2) understanding how students and teachers form relationships supportive of

integrating attention to both reasoning and emotions in mathematics learning and teaching. This second focus includes understanding how students and teachers learn to express and manage emotions, as well as changes in energy levels, that accompany interactions aimed toward mathematical learning. I explored both foci in my dissertation research; I have explored the first focus in an interview study conducted during the 2009-2010 academic year; and I have “grand plans” for studying the second focus in future research (I submitted a grant application toward that end this past July 2010). I believe the second focus fits well with what I understand about the DIME research group, although I see the first focus as a critical support to developing the second.

Theoretical Frame for Reasoning and Emotions

Because I am interested in how reasoning and emotions are intertwined in mathematics learning and teaching, I begin with brief theoretical comments.

Following Piaget (1970, 1971) and von Glasersfeld (1995), I view mathematical *learning* in the context of making reorganizations, or accommodations, in schemes and operations in on-going interaction with one’s experiential world. *Operations* are mental actions such as dividing a unit into parts (partitioning), or repeatedly instantiating a unit to create a plurality of units (iterating). Operations are the components of *schemes*: goal-directed ways of operating that consist of a perceived situation, an activity, and a result (Piaget, 1971; von Glasersfeld, 1995). *Reasoning* is the functioning of a person’s schemes and operations in on-going interaction. An accommodation may occur when a person’s current schemes and operations produce an unexpected result: The person does not achieve the intended goal. This “disturbed” state of affairs is one example of a *perturbation*, a state of disequilibrium, and is often accompanied by a sense of disappointment or surprise. That is, a perturbation often yields an emotion.

In psychology and in education, an *emotion* is often characterized as one outcome of a person’s appraisal of the extent to which a person’s goal is being met (e.g., Lazarus, 1991; Mandler, 1989; McLeod, 1992; Pekrun, 2006; Schutz & DeCuir, 2002; Schutz & Pekrun, 2007). This view of emotion provides a solid link between the process of learning and the experience of emotions, because not meeting a goal is one way a perturbation may occur. However, this view often leaves out neurological and physiological sources of emotions (Buck, 1999; Damasio, 1996). Buck has proposed that emotions have three levels: adaptive homeostatic arousal responses such as chemical changes (not accessible to the individual without special measuring devices); expressive displays such as facial expressions (primarily accessible to others); and subjective experience (accessible only to the individual). Damasio has described how emotions are first and foremost about changes in body states. In my work I take emotions to be rooted biologically as Buck and Damasio describe. Yet I am usually concerned with higher-level emotions (Damasio’s secondary emotions and background emotions) that derive from these biological bases, in which cognitive appraisal is paramount. Thus far I have focused on eliciting expressive displays and subjective experience as the two main ways to assess research participants’ emotions.

Theoretical Frame for Students’ Multiplicative Reasoning

In addition to being based in the broad theoretical frame sketched above, my research has focused on using the units’ coordinations that students make in reasoning about numbers and quantities as a frame for understanding their mathematical activity. I present a brief overview of these ideas here so that what I say about the results of my research below may be sensible.¹ In particular, I have worked with students who have constructed the first, second, and third

¹ For some of you, these ideas will be very familiar, and you may accuse me of oversimplifying. I am likely guilty as charged, but I am trying to present a “quick” frame for those who are not familiar.

multiplicative concepts described by Hackenberg and Tillema (2009), based on the research of Steffe (1992, 1994).

Students with the first multiplicative concept (MC1)² operate as if they have two number sequences, which means they can coordinate two levels of units in solving multiplication problems. For example, these students can determine the number of days a person has been on vacation if she has been gone for three weeks, and they can learn to coordinate and keep track of the number of weeks and number of days (two levels of units) in a “story” of a person whose length of vacation keeps changing. In other words, MC1 students make and track composite units (units of units) *in activity*.

Students with the second multiplicative concept (MC2) can imagine one number sequence as embedded within another. This concept allows them to disembed and break apart numbers strategically, use tens and ones in reasoning, and use composite units iteratively in solving multiplication problems. For example, to determine the number of cookies in 4 bags if there are 8 cookies in each bag, a MC2 student might reason as follows: “8 plus 8 is 16, and then 16 plus 16 is... 16 plus 10 is 26. 26 and 4 more is 30, plus 2 more is 32.” In this solution the student keeps track of four 8s by uniting two 8s (16) and two more 8s (another 16). The student also breaks apart both 16 and 6 strategically in order to reason with tens and ones. In short, MC2 students can take composite units as given (i.e., 8 is a composite unit of 8 ones) and operate further with them. Another way to state this idea is that for MC2 students, 1 is an *iterable* unit. So a number like 8 implies a unit of 1 iterated 8 times.

Students with the third multiplicative concept (MC3) can do everything the MC2 students can do, but they can take a unit of composite units as given (i.e., 32 is a unit of 4 units, each of which contains 8 units). Another way to state this idea is that for MC3 students, composite units are iterable units. So a number like 32 implies a unit of 4 units iterated 8 times. One outcome of the third multiplicative concept is strategic reasoning with multiplication, i.e., using smaller multiplication facts to solve multiplication and division problems. For example, to solve the problem of 104 divided by 8, a 6th grade MC3 student in my dissertation research reasoned as follows: “8 times 10 is 80, and then you have 24 left. 24 is 8 times 3, so the answer must be 13.”

Dissertation Research

In my dissertation research I conducted an 8-month teaching experiment in which I taught two pairs of 6th grade students at a rural middle school in north Georgia. One pair was supposed to be MC2 students and the other to be MC3 students, as assessed in short selection interviews. During the study I determined that each of the pairs contained one MC2 student and one MC3 student. The pairs and I met twice per week in 30-minute sessions for two to three weeks, followed by a week off, which resulted in approximately 34 sessions for each pair of students. All sessions were videotaped with two cameras and digitally processed.

One purpose of the study was to investigate how the students constructed quantitative reasoning with fractions, and what aspects of their reasoning were foundational for algebraic reasoning. Toward that end, we worked on a range of topics, including making and comparing improper fractions, making fraction compositions (i.e., taking fractions of fractions), and solving problems involving reversible multiplicative relationships. An example of the latter topic, which was featured in the study, is the following:

2 When I use phrases like “students with the first multiplicative concept,” I mean only that my model of the student’s mathematics includes this idea—and that it is a researcher’s idea, attributed to a student based on interactions and experiences with him or her. I do not mean to imply anything about the “actual” nature of the student’s mathematics, which I cannot know.

The Peppermint Stick Problem. A 7-inch peppermint stick is three times the length of another stick. Can you draw a picture of this situation? How long is the other stick in inches?

More complicated versions of this problem occur when the known quantity and/or known relationship are fractions (see Hackenberg, 2010a).

Findings from this part of the dissertation study include:

- The third multiplicative concept is required to construct improper fractions as numbers that can be taken as given and used in further reasoning (Hackenberg, 2007);
- Students' whole number multiplicative concepts constrain and open possibilities for students' construction of fraction compositions as a basis for fraction multiplication, and these ideas can be considered in the province of mathematical knowledge for teaching (Hackenberg & Tillema, 2009);
- MC2 and MC3 students solve problems involving reversible multiplicative relationships in qualitatively different ways; notably MC3 students can construct anticipatory schemes to solve such problems because they can flexibly switch between viewing a quantity as two different three-levels-of-units structures (Hackenberg, 2010a);
- Even the third multiplicative concept may not be sufficient to construct fractions as operators on known and unknown quantities (Hackenberg, 2010a).

The other purpose of the dissertation study was to establish and maintain what I call *mathematical caring relations* (MCRs) with the students (Hackenberg, 2005, 2010b, in press). A mathematical caring relation (MCR) is a quality of interaction between a student and a teacher that conjoins attention to affect and cognition. Teachers who establish MCRs seek to help students and themselves manage fluctuations in emotions and energy levels that may accompany mathematical learning.

MCRs are derived from Noddings's (2002, 2005) caring relations, which are established only if there is evidence that the student receives the teacher's care. Such evidence includes renewed interest, an increase in energy, or a "glow of well-being" (2002, p. 28). In turn, this reception of care is what a teacher needs to feel cared for—to also experience increased energy or positive emotions. To specify what "energy" means, I use Ryan & Frederick's (1997) construct of *subjective vitality* as the positive energy one experiences as available to the self. Thus far I have taken assessments of the subjective vitality of students and teachers in interaction as indicators of the establishment of MCRs (Hackenberg, 2010b, in press).

Key tools for the teacher in establishing MCRs are to pose situations that *harmonize* with students' current schemes, emotions, and energetic responses, as well as situations that *challenge* students—that open opportunities for them to make accommodations and thereby develop their mathematical ways of operating. Throughout this process, teachers track students' reasoning, emotions, and energetic responses; reinstate harmonizing and challenging; and monitor their own emotions and subjective vitality.

Findings from this part of the study include:

- Harmonizing with students' mathematical ways of operating and challenging students to expand their ways of operating are closely connected (Hackenberg, 2010, in press);
- Student perturbations can provoke teacher perturbations, which can in turn provoke further student perturbations; teachers who aim to establish MCRs endeavor to influence this linked chain of perturbations so that it tends toward perturbations that are bearable (Tzur, 1995) for both teachers and students (Hackenberg, 2010b);

- MCRs are mathematical because to establish MCRs teachers must decenter from their own mathematical thinking and construct mathematical ways of operating that fit with their experience of the students (cf. Steffe & Thompson, 2000), and because students' mathematical activity is a significant part of what the teacher receives (Hackenberg, 2010b);
- The establishment of MCRs may influence students' mathematical self-concepts and teachers' personal teaching efficacy (Hackenberg, 2010b, in press);
- Establishing MCRs with students who consider themselves (to be) mathematically talented may be particularly demanding because doing so can challenge these students' ideas about what is entailed in being a strong mathematics student, and because teachers may assume more than is warranted about these students' schemes and operations (Hackenberg, in press).

Interview Study (analysis in progress)

Since my dissertation research I have conducted an interview study with MC1, MC2, and MC3 7th and 8th grade students on the relationship between their quantitative reasoning with fractions and their algebraic reasoning in the area of writing equations for quantitative situations (Hackenberg, 2009, reports on a pilot for this study with 9th grade students). Data collection occurred during the 2009-2010 academic year. Each of 18 students (6 with each multiplicative concept) participated in two 45-minute, semi-structured, task-based interviews, one focused on fractions and one on algebra problems; each also completed a fractions written assessment (Norton & Wilkins, 2009).

In contrast with my dissertation study, in the interview study I investigated more purposefully students' use of notation to represent unknown quantities. The interview protocols were designed so that the reasoning involved in the fractions interview was a foundation for solving problems in the algebra interview. For example, in the fractions interview students were posed this situation: A 65-cm stack of CDs is 5 times the height of another stack. Students were asked to make a drawing of the situation and determine the height of the other stack. In the algebra interview, students were posed a similar situation but both heights were unknown. Students were asked to make a drawing and write equations to represent the situation.

Data analysis is currently in progress, but these findings have emerged so far:

- MC1 and MC2 students had not constructed a scheme for improper fractions, while all MC3 students had. This result confirms prior findings that MC3 is necessary to construct improper fractions (Hackenberg, 2007; Steffe & Olive, 2010);
- MC1 students did not use letters to represent unknown quantities, except with guidance from the interviewer. Consequently, there was no evidence that they conceived of whole numbers or fractions as multiplicative operators on unknowns.
- MC2 students used letters to represent unknown quantities, often only at the interviewer's suggestion. In writing equations, four of the six MC2 students did not use whole numbers as multiplicative operators on unknowns, and none conceived of fractions as operators.
- MC3 students used whole numbers and fractions as operators on unknown quantities and reasoned reciprocally with unknowns. Reciprocal reasoning was not "automatic" but was elicited during the interviews.

My current explanation for these findings is that the coordination of composite units required for the construction of improper fractions facilitates the construction of quantitative unknowns as multiplicative in nature—i.e., of unknowns that can be operated on multiplicatively with whole numbers and fractions. Space does not permit me to say more about this here, but I am excited about the idea!

Future Plans

This summer I wrote a proposal for a 5-year project to study MCRs as one aspect of students' and teachers' social and emotional learning (SEL) in mathematics learning and teaching. In the past 20 years, SEL (Goleman, 1995; Zins, et al., 2004) has emerged as a new field of inquiry that aims to help students and teachers develop emotional literacy and trusting relationships. SEL has been linked to improved academic learning (Zins et al., 2004), but little research on SEL has occurred in mathematics education. The research goals of the proposed project are to understand (1) how MCRs are established with groups of 7th and 8th grade students; (2) how practicing middle grades mathematics teachers understand, value, and implement MCRs as one aspect of SEL in mathematics teaching and learning; and (3) what SEL looks like in middle grades mathematics classrooms.

My research plan is organized in three phases: two phases of iterative design experiments with groups of 7th and 8th grade students, and a year-long observational and interview study with practicing middle grades mathematics teachers. In the first two years of the project (Phase 1), graduate assistants and I plan to teach six 10-week design experiments. The purpose of this phase is to understand how MCRs are established with students who have different multiplicative concepts (MC1, MC2, and MC3), as well as different affective orientations to mathematics. In the third and fourth years of the project (Phase 2), practicing middle grades mathematics teachers will teach four 10-week design experiments. During this phase I will study how the teachers understand, value, and implement MCRs, and I will also refine analysis and products from Phase 1. In the fifth year (Phase 3), I plan to conduct an observational and interview study with the teachers focused on understanding their perspectives on SEL in mathematics classrooms, as well as how studying MCRs has informed their classroom practice. The design experiments and the interview study will be supported in the summers by data analysis and workshops for the teachers and interested prospective teachers on SEL in mathematics teaching and learning, as well as on quantitative reasoning as a basis for mathematics teaching and learning.

I am quite eager for feedback on how to develop this agenda, and to determine in what ways these plans intersect with potential collaborative work through DIME.

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