NEW DIRECTIONS IN THE ENUMERATION OF TILINGS OF A CHESSBOARD

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Arthur L. Gershon

NEW DIRECTIONS IN THE ENUMERATION OF TILINGS OF A CHESSBOARD

We investigate the number of arrangements of $1 \times n$ rectangular strips of varying length on a chessboard, where each strip is endowed with a natural horizontal or vertical *direction*. Our principal aim is to study the number of *Restricted Oriented Strip Arrangements*, or *ROSAs*, on a chessboard with at most one horizontal strip in each row and at most one vertical strip in each column; these ROSAs have a natural interpretation as a variation on an important model in statistical physics. Using transfer matrices, we obtain exact formulas for the number $T(m,n)$ of ROSAs on an $m \times n$ rectangle for certain fixed positive integers m and any non-negative integer n. For general m, we deduce the asymptotic formula

$$T(m,n) \sim \frac{1}{(m!)^2} \prod_{j=1}^{m} \left[1 + \binom{j}{2} + \binom{m+1-j}{2}\right] n^m \left(1 + \binom{m+1}{2}\right)^n$$

as $n \to \infty$. We also apply the previous transfer matrix methods to compute generating functions for the numbers of strip arrangements in which the aforementioned restriction on the number of strips in each row and column is dropped. Finally, returning to the restricted case, we look at ROSAs on square chessboards; using convex analysis, we obtain for large L the logarithmic estimate

$$\log T(L,L) = 4L \log L - 2L \log 8 + O(L^{2/3})$$

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CHAPTER 1

INTRODUCTION & MOTIVATION

1.1 Background

Consider the two-dimensional square lattice Λ. A *polyomino* on Λ is defined to be a polygon that is congruent to some edge-connected collection of unit squares on Λ. Given a set τ of non-congruent *tiles*, or polyominoes, on Λ, the *enumerative tiling problem* for τ on a rectangular region R of Λ of finite area – which we will refer to colloquially as a *chessboard* – asks for the number of ways to partition the unit squares of R into polyominoes congruent to some element of τ. In this tiling problem, one typically requires that each and every unit square of the chessboard R be covered by some tile of τ. One can relax this condition, and consider the enumeration of ways to place tiles on R so that each unit square is covered by *at most one* tile; we call these tile placements *arrangements* to distinguish them from the previous complete tilings (although we consider complete tilings to be included in the set of arrangements).*

*One can generalize the notions of this paragraph, taking Λ to be any lattice – and thereby redefining polyominoes on Λ to be a collection of edge-connected *fundamental regions* – and taking R to be any region in Λ of finite area. For the most part, however, we will not have occasion to worry about such generalities.

One of the more classical tiling/arrangement enumeration problems studied is the *dimer (or domino) problem*, which asks for the number $D(m, n)$ of tilings of $m \times n$ chessboards by dimers, or 1×2 rectangles, when the area mn of the containing chessboard is even.[†] Originally using graph-theoretic techniques, this problem has been solved exactly:

Theorem 1.1 (Kasteleyn, Temperley-Fisher; see Aigner [2]). *If mn is even,*

$$D(m,n) = \prod_{k=1}^{m} \prod_{\ell=1}^{n} \left[4\cos^2\left(\frac{k\pi}{m+1}\right) + 4\cos^2\left(\frac{\ell\pi}{n+1}\right) \right]^{1/4}$$

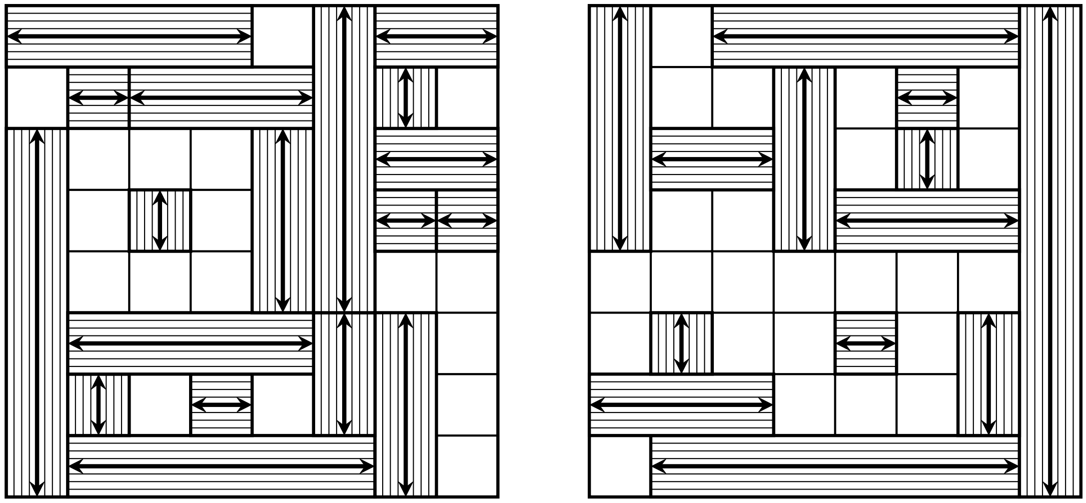
Another related classical problem is the *monomer-dimer problem*, which asks for the number of tilings of an $m \times n$ rectangle using 1×2 dimers as well as 1×1 monomers. Removing these monomers, this problem of enumerating tilings can be viewed as a problem of enumerating arrangements of dimers. Unfortunately, an exact formula for the number of monomer-dimer tilings, or dimer arrangements, is not known.

One can generalize the problems of enumerating tilings and arrangements of 1×2 dimers to problems using any single straight polyomino – that is, considering the tilings and arrangments of $1 \times k$ strips on a chessboard for a fixed positive integer k; Mathar [12] gives some results in these related directions. We aim to generalize the problem further, as we consider the number of arrangements that allow any number of strips of *any* length (provided, of course, that said strips fit on the chessboard). We give an example of the type of arrangement we have in mind in Figure 1.1(a).

[†]In the vein of the previous footnote, we note the related classical problem of enumerating the tilings of hexagonal regions in the triangular lattice by edge-connected pairs of triangles, typically called lozenges. This problem has also been completely solved: the number of lozenge tilings of a hexagon is zero if its opposite sides are not congruent; otherwise, if a, b, and c denote the lengths of three consecutive sides of the hexagon, the number of lozenge tilings is given by a plane partition formula of MacMahon [11]:

$$L(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)}$$

where, for any positive integer n, $H(n) = (1!)(2!)\dots(n!)$ denotes the *hyperfactorial* of n.



(a) *(b)*

Figure 1.1: Generic strip arrangements on an 8×8 chessboard. Figure 1.1(a) depicts an Un-Restricted Strip Arrangement (or URSA) that allows any number of strips in any row or column, whereas Figure 1.1(b) depicts a Restricted Oriented Strip Arrangement (or ROSA). We include arrows to easily indicate the direction of each strip; we note that such distinctions are required *for strips of unit length on a ROSA.*

In addition, we notice that each strip of length $\ell > 1$ has a natural horizontal or vertical orientation in the chessboard according to its longest side. We can extend this notion to 1×1 tiles by artificially imposing an orientation on each (so that in fact we have two possible monomers: one vertical, and one horizontal). We are therefore able to ask questions about strip arrangements that take these orientations into account; these orientations are the "directions" alluded to in the title of this dissertation. One interesting problem in this spirit is to determine the number of strip arrangements that have at most one horizontal strip in each row and at most one vertical strip in each column (like the arrangement illustrated in Figure 1.1(b)). The enumeration of such arrangements, which we call *Restricted Oriented Strip Arrangements*, or *ROSAs*, is the primary focus of this dissertation.

1.2 An Alternative Interpretation

The problems of enumerating various tilings and arrangements on a chessboard are of intrinsic interest to mathematicians as problems that can be simply stated but resist attempts at a complete solution by elementary means. However, these and other related combinatorial problems (e.g., the enumeration of graph colorings) are of a broader scientific interest. In physics, such arrangements often correspond to models of particles and their interactions; a statistical theory of particle interactions, therefore, has great interest in the enumeration of all possible configurations of the desired particles.

For example, tilings of a chessboard by dominoes that we discussed previously correspond to a two-dimensional Ising model illustrating bonds between magnetically charged particles arranged on a square lattice so that no two nearest neighbor particles have an identical charge (see Baxter [3] or Thompson [17]). Alternatively, Lieb realized the model by assigning up and down arrows to each unit square with the conditions that

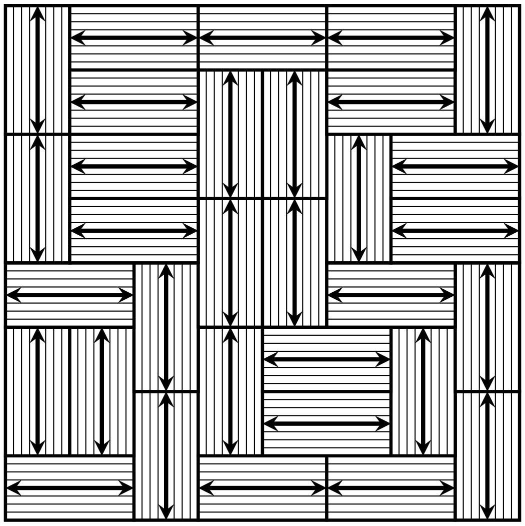
- There is no adjacent pair of up arrows in any column.
- In any row, the collection of down arrows that do *not* point to an up arrow can be partitioned into adjacent pairs.‡

It is this coloring of the unit squares (illustrated in Figure 1.2) that led Lieb to his novel algebraic exact solution of the dimer tiling problem [10], as opposed to the graph theoretic solutions previously mentioned. Lieb's methods can also be used (see Percus [14]) to determine an asymptotic formula

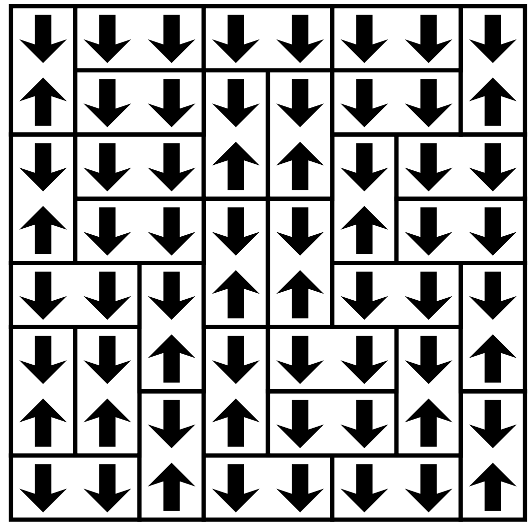
$$D(2m,2n) \sim \exp\left[\frac{(2m)(2n)}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\right] = \exp\left[\frac{(2m)(2n)}{\pi}\left\{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\right\}\right]$$

for the number of dimer coverings on a $2m \times 2n$ chessboard as $m, n \to \infty$.

‡In the corresponding arrangement enumeration problem, this condition may be dropped.



(a) A domino tiling of an 8×8 chessboard.



(b) Lieb's coloring of an 8×8 chessboard that corresponds to the tiling of Figure 1.2(a).

Figure 1.2

As another example, we consider the *Problem of Kings*, which asks for the number of ways to place a maximum number of non-attacking kings on a $2m \times 2n$ chessboard – that is, the number of ways to place objects on unit squares in a square lattice so that no two placed objects are adjacent horizontally, vertically, or diagonally. If one considers the square lattice with the upper-left to lower-right diagonals, then, as Baxter [3] illustrates, the Problem of Kings corresponds to the hard hexagon model of statistical mechanics (albeit without the usual periodic boundary conditions).[§]

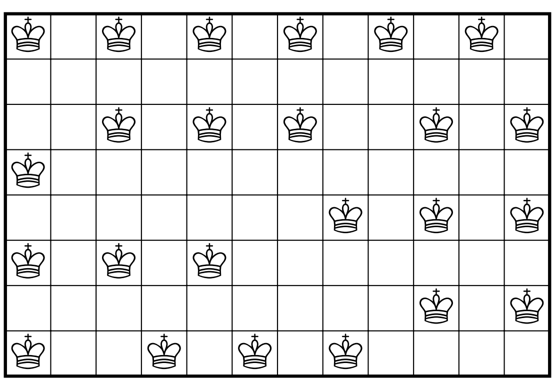
In the style of the previous example, we note an alternative realization of the model. In the case where each dimension of the ambient chessboard is even and each square of the chessboard either contains a king or is adjacent to a square containing a king, Wilf [20] assigns a four-coloring to each of the disjoint 2×2 square subregions that represents the

[§]It is worth noting that Baxter, in the same reference, is able to solve for the thermodynamic properties of the hard hexagon model exactly.

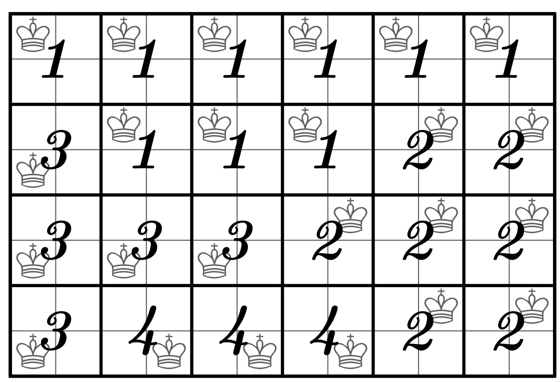
position of the king in that region, and he is thus able to correlate valid configurations of non-attacking kings on a $2m \times 2n$ chessboard to four-colorings of an $m \times n$ chessboard that respect a list of non-adjacency conditions (given by Wilf). We illustrate a configuration of non-attacking kings and Wilf's corresponding coloring in Figure 1.3.

Such models can be realized in various ways, and each viewpoint may be useful in its own right. To wit, we note that Larsen, in his independent study of the Problem of Kings [9], realizes the model using a pair of two-colorings – which is surely equivalent to a four-coloring. Similarly as in Lieb's analysis of the dimer problem described earlier, the first color is represented by a vertical arrow (either $\uparrow$ or $\downarrow$), and the second color is represented by a horizontal arrow (either $\leftarrow$ or $\rightarrow$). Then, in a manner again similar to Lieb's work, the non-adjacency conditions corresponding to valid placements of non-attacking kings on a chessboard admit a concise verbal description:

- In any column, a $\downarrow$ cannot be above a $\uparrow$.
- In any row, a $\rightarrow$ cannot be to the left of a $\leftarrow$.



(a)



(b)

Figure 1.3: Figure 1.3(a) depicts a placement of $24 = (8/2) * (12/2)$ *mutually non-attacking kings on an* 8×12 *chessboard that was originally used as an example by Larsen [9], whereas Figure 1.3(b) shows Wilf's coloring of a* 4×6 *chessboard corresponding to Figure 1.3(a).*

- $(\downarrow, \rightarrow)$ cannot be adjacent on its bottom-right corner to $(\uparrow, \leftarrow)$.
- $(\uparrow, \rightarrow)$ cannot be adjacent on its top-right corner to $(\downarrow, \leftarrow)$.

This coloring also has the benefit that the horizontal and vertical directions can be analyzed independently, and the corresponding results can be united by more careful reasoning.

Turning now to our case of enumerating ROSAs, we note that, in each row, the unit squares can be colored, or classified into states, as follows:

- State 0: A strip has not yet been seen in this row. Alternatively, the unit square is to the left of the horizontal strip in this row, or there is no horizontal strip in this row.
- State 1: The unit square is covered by a horizontal strip.
- State 2: A strip has already been seen in this row. Alternatively, the unit square is to the right of the horizontal strip in this row.

A similar coloring can be used for columns with respect to vertical strips. Hence, like Larsen's work above, we have a pair of three-colorings, or a nine-coloring, for each square of the chessboard. However, one of these colors is not compatible with the condition that strips are not to overlap, so we are left with eight. (Figure 1.4 gives a pictorial description of these colors; Figure 1.5 shows an eight-coloring for a given ROSA.) Such *eight-vertex models* are of particular interest in statistical mechanics as generalizations of the six-vertex model of water molecules in an ice lattice (see Baxter [3] for more details).

We find this alternative perspective to be of great service to our pure mathematical interests of enumerating strip arrangements. The view of ROSAs as models of statistical physics opens our problem to techniques used in this discipline, such as the method of *transfer matrices*, used by Lieb [10] in enumerating dimer tilings and by Wilf [20] in the Problem of Kings, and the method of *renormalization*, used by Larsen [9] in his results on the Problem of Kings. It is our hope that our findings may reciprocally be of use to other scientists, including those who study statistics of various molecular configurations.

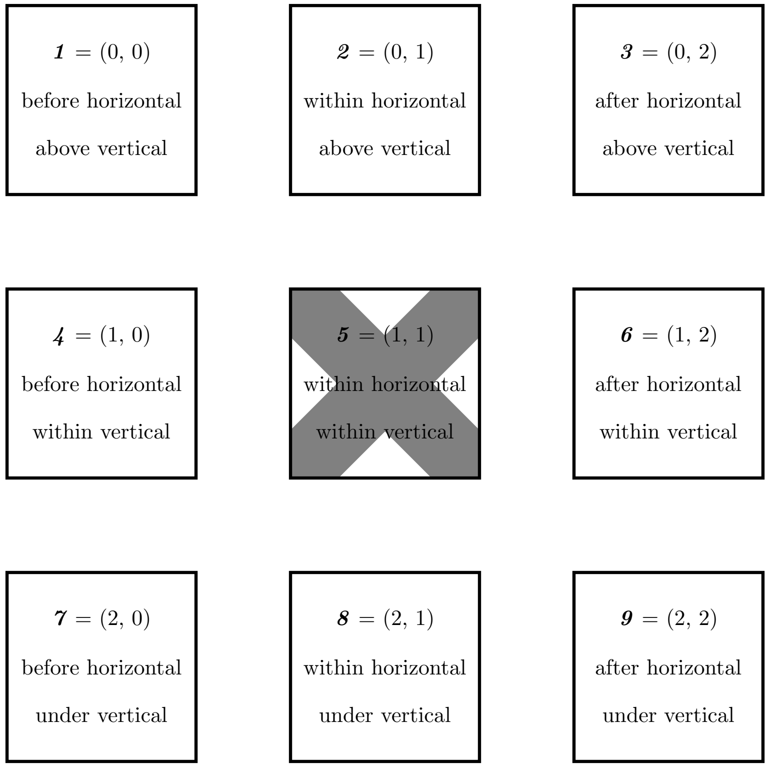


Figure 1.4: Translation of a ROSA into pairs of 3-colorings of unit squares of a chessboard with eight allowed colors and one disallowed color.

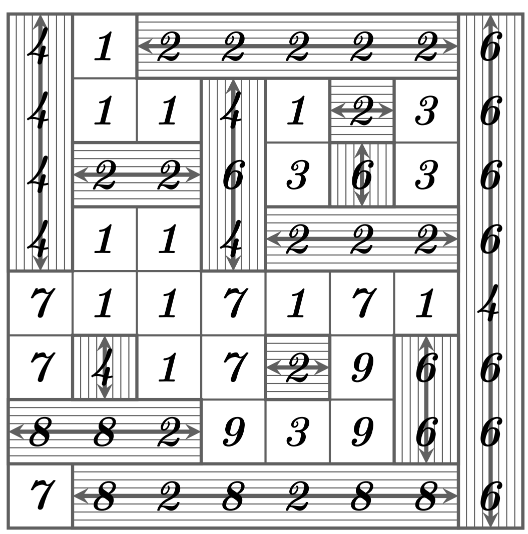


Figure 1.5: The eight-vertex model that corresponds to the ROSA given in Figure 1.1(b).

1.3 Overview of the Results

In Chapter 2, we investigate the number $T(m,n)$ of ROSAs on an $m \times n$ chessboard, where m is taken to be some fixed positive integer and n is allowed to be any non-negative integer. We are able to compute $T(m,n)$ exactly for small m by direct combinatorial reasoning. For larger m, we turn to the theory of *transfer matrices*, which illustrates how to build a ROSA one column at a time, and thereby computes for us the corresponding generating function $\mathcal{T}_m(x) = \sum_{n\geq 0} T(m,n)x^n$.

More specifically, for each fixed m, we show how the 3^m possible columns can be arranged as the vertices of a vertex-weighted directed graph G_m. There is a proper subset V_0 of vertices (i.e., columns) that can begin a ROSA; it then follows that $T(m,n)$ is equal to the number of weighted directed walks on G_m whose initial vertex belongs to V_0. From the particular construction of G_m, we see that there is a *unique* starting vertex $u \in V_0$ (namely the origin, or the m-tuple consisting solely of zeroes) for which there is an edge from u to every other vertex of V_0, and no edge from u to any other edge of G_m. We determine that, if $\mathcal{F}_v(x)$ is the generating function for which the coefficient of x^ℓ is the number of walks on G_m of length ℓ beginning at the vertex v, and $w(v)$ is the weight of v,

$$\mathcal{F}_u(x) = 1 + w(u)x \sum_{v\in V_0} \mathcal{F}_v(x) = 1 + w(u)x\mathcal{T}_m(x) \tag{1.2}$$

We remark that Equation 1.2 represents a general philosophy that applies not just to walks on the particular graph G_m, but to walks on any (vertex-weighted directed) graph whose desired start vertices are connected to a single start vertex (see p. 11).

We then construct a $3^m \times 3^m$ diagonal matrix D_m that captures the number of ways to place vertical strips in each column, and a $3^m \times 3^m$ matrix A_m that captures how we can *transfer* from one column to the next. With these matrices in hand, we are able to compute the left-hand side of Equation 1.2, and thereby obtain the generating function

Theorem 1.3 (Theorem 2.15, p. 25)**.**

$$\mathcal{T}_m(x) = \left(\frac{1}{\binom{m+1}{2}+1}\right) e_0 \cdot \left[(I_{3^m} - xD_mA_m)^{-1} \cdot D_m\right] \cdot (e_0 + \cdots + e_{3^m-1})^t$$

In other words, $\mathcal{T}_m(x)$ *is the sum of all of the entries in the zeroth row (under an indexing scheme given in Chapter 2) of the matrix* $(I_{3^m} - xD_mA_m)^{-1} \cdot D_m$*, where* I_{3^m} *denotes the* $3^m \times 3^m$ *identity matrix, divided by the constant* $1 + \binom{m+1}{2}$*.*

We then use Theorem 1.3 to actually compute generating functions $\mathcal{T}_m(x)$ for some new values of m that we could not obtain from the previous direct arguments. We thus compute, via partial fraction decomposition of $\mathcal{T}_m$, exact formulas for $T(m,n)$ for these values of m and for all integers $n \geq 0$.

The transfer matrix methods require exponentially more computations as m increases, and so we are unfortunately not able to provide an exact formula for all m. However, the exact formulas that we do compute provide clues toward an asymptotic formula; we close Chapter 2 by proving this asymptotic formula, namely

Theorem 1.4 (Theorem 2.29, p. 41)**.**

$$T(m,n) \sim \frac{1}{(m!)^2} \prod_{j=1}^{m} \left[1 + \binom{j}{2} + \binom{m+1-j}{2}\right] n^m \left(1 + \binom{m+1}{2}\right)^n$$

as $n \to \infty$ *for any fixed integer* m*.*

using other combinatorial arguments *not* involving transfer matrices; finally, we discuss how it may be possible to unite these results using the preceding transfer matrix theory.

In Chapter 3, we make a slight digression to study those strip arrangements on a chessboard for which the number of horizontal strips in each row and the number of vertical strips in each column is *not* restricted (to at most one each). We note that such strip arrangements also have an interpretation in statistical physics. Indeed, we may classify the unit squares in each row into states as follows:

- State $\tilde{0}$: The unit square is not covered by any horizontal strip.
- State $\tilde{1}$: A horizontal strip begins on this unit square. That is, the unit square is covered by a horizontal strip that does *not* cover the square's nearest left neighbor.
- State $\tilde{2}$: A horizontal strip is continued on this unit square. That is, the unit square is covered by a horizontal strip that covers the square's nearest left neighbor.

Using a similar characterization for vertical strips in columns, taking these pairwise gives again nine possible states for each unit square. However, since two of the three states in each coordinate indicate a unit square being covered by some strip, we must disallow *four* of these. Thus, we see that URSAs correspond to a *five-vertex model* of statistical mechanics, rather than an eight-vertex model that we saw for ROSAs. (Figure 1.6 displays the colors used (and not used) in this model, and Figure 1.7 shows such a coloring for a given URSA.)

Our approach, therefore, for determining the number $U(m,n)$ of such *Un-Restricted Strip Arrangements*, or *URSAs*, on an $m \times n$ chessboard, will be very similar to that of Chapter 2. We first determine generating functions $\mathcal{U}_m(x) = \sum_{n\geq 0} U(m,n)x^n$ for small fixed values of m. We then adapt the transfer matrix methods of Chapter 2 – that is, we show that the philosophy of Equation 1.2 applies – to obtain the generating function $\mathcal{U}_m(x)$ for any m similar to that of Theorem 1.3:

Theorem 1.5 (Theorem 3.12, p. 52). *With* $\tilde{w}_m(0) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2m+1}\right]$, *the generating function* $\mathcal{U}_m(x) = \sum_{n\geq 0} U(m,n)x^n$ *for which the coefficient of* x^n *is the number* $U(m,n)$ *of URSAs on an* $m \times n$ *chessboard is given by*

$$\mathcal{U}_m(x) = \left(\frac{1}{\tilde{w}_m(0)}\right) e_0 \cdot [(I_{3^m} - x\tilde{D}_m\tilde{A}_m)^{-1} \cdot \tilde{D}_m] \cdot (e_0 + \cdots + e_{3^m-1})^t$$

We then use Theorem 1.5 to compute $\mathcal{U}_m(x)$ for certain fixed values of m.

Finally, in Chapter 4, we return to restricted strip arrangements, as we consider the number of ROSAs on a square chessboard of side length L. While it is very natural to consider arrangements on such chessboards, enumerating them proves to be elusive in many cases. Here, we follow a general idea from Larsen's result on the Problem of Kings [9]. We begin by *renormalizing*, or rescaling the chessboard by dividing it into an $N \times N$ *template* of certain square cells of side length s (so that $L = Ns$). We characterize these cells as either H*-cells* or V*-cells* based on whether the cell can be crossed by a horizontal strip or a vertical strip respectively. This allows us to treat the enumeration of strips macroscopically – that is, in groups of rows, rather than each row individually. Specifically, for each row of a template, we determine

Theorem 1.6 (Theorem 4.1, p. 60)**.** *If a row of a template of a ROSA contains a number A of H-cells and a number B of V-cells, then the number of ways to place horizontal strips that lie within this row of cells in the template is given by*

$$\begin{cases} \left[1 + \binom{Ns+1}{2}\right]^s & B = 0 \\ \left[s + \sum \binom{a_i s + 1}{2} + \binom{s}{2}[3N + A - 2(a_0 + a_k + 1)]\right]^s & B \neq 0 \end{cases}$$

where k is the number of contiguous groups of V-cells in the row, and a_0 and a_k denote the lengths of the zeroth and k-th contiguous group of H-cells.

As the number of strip arrangements grows quite rapidly, we then resolve to estimate not $T(L, L)$ directly, but rather its natural logarithm; this has the added benefit of allowing us to use techniques of convex analysis. Unfortunately, we determine that the expression of Theorem 1.6 is not logarithmically concave everywhere, so we may not apply Jensen's Inequality directly, but we are able to develop a modification of Jensen's Inequality:

Theorem 1.7 (Theorem 4.11, p. 69)**.** *Let $f(x)$ be convex on $(0, I)$ and concave on (I, M). Let $x_1, \ldots, x_P \in [0, I)$, and let $x_{P+1}, \ldots, x_N \in [I, M]$. Then there exists $z \in [0, N)$ with*

$$f(x_1) + \cdots + f(x_N) \leq zf(0) + (N - z)f\left(\frac{x_1 + \cdots + x_N}{N - z}\right)$$

Refining this result with techniques of differential calculus (and algebraic geometry), we develop upper and lower bounds for $\log T(L, L)$, and thus conclude with the estimate

Theorem 1.8 (Theorem 4.38, p. 92)**.** *For all sufficiently large L,*

$$\log T(L, L) = 4L \log L - 2L \log 8 + O(L^{2/3})$$

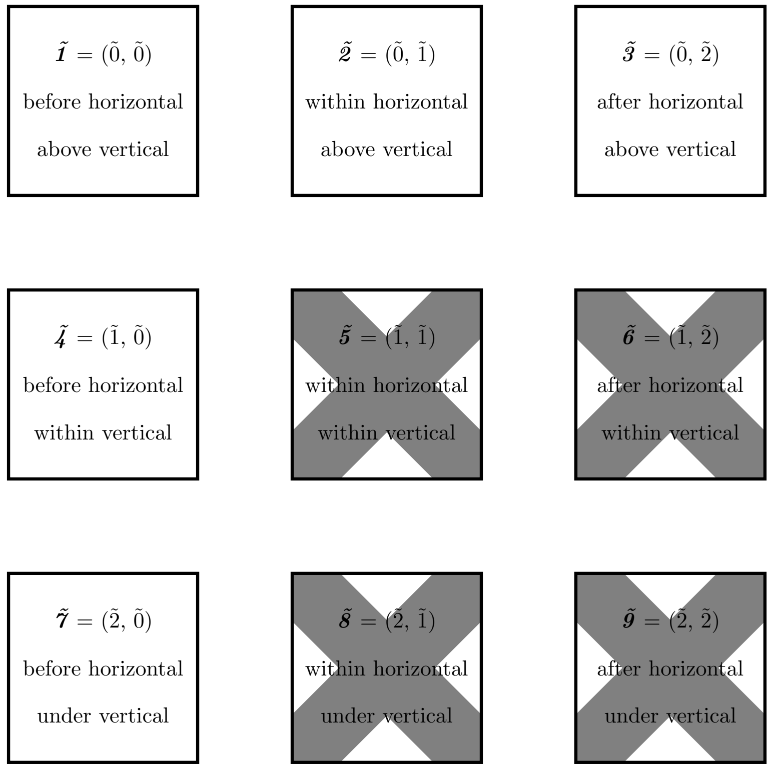


Figure 1.6: Translation of an URSA into pairs of 3-colorings of unit squares of a chessboard with five allowed colors and four disallowed colors.

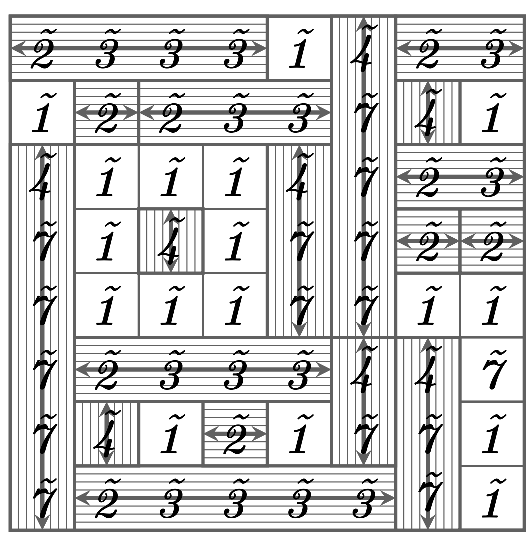


Figure 1.7: The five-vertex model that corresponds to the URSA given in Figure 1.1(a).

CHAPTER 2

RESTRICTED STRIP ARRANGEMENTS OF FIXED WIDTH

In this chapter, we investigate the number $T(m,n)$ of ROSAs on a $m \times n$ chessboard, where m is some fixed positive integer, and n is allowed to vary among all the positive integers. After computing some exact formulas for small m using elementary combinatorics, we turn to the algebraic method of *transfer matrices* to find an abstract matrix-theoretic formula for the corresponding generating function $\mathcal{T}_m(x) = \sum_{n\geq 0} T(m,n)x^n$. for any given value of m. We put this abstract formula into practice to find concrete generating functions for more values of m, and from these generating funtions we thereby determine exact formulas for $T(m,n)$ for all $m = 1, \ldots, 6$ and all $n \geq 0$. These exact formulas that we compute will provide some conjectures for an asymptotic formula for $T(m,n)$ as $n \to \infty$; we will conclude the chapter by proving these conjectures.

2.1 Elementary Results

We begin with an elementary counting of the number of ROSAs on a $1 \times n$ chessboard for any positive integer n:

Theorem 2.1. $T(1, n) = n2^n + 1$.

Proof. If the chessboard contains no horizontal tiles, then there are 2^n possible ways to tile the rectangle: each column is either filled with a vertical tile, or is empty. Otherwise, the row contains a horizontal tile of length j; there are $n + 1 - j$ ways to place this tile in the row, and the remainder of the row can be tiled in 2^{n-j} ways (see Figure 2.1). Thus,

$$T(1, n) = 2^n + n2^{n-1} + (n-1)2^{n-2} + \cdots + (2)2^1 + 1 = 2^n + \sum_{k=0}^{n-1}(k+1)2^k \tag{2.2}$$

Now note that (at least formally) we have

$$\begin{aligned} 1 + 2x + 3x^2 + \cdots + nx^{n-1} &= \frac{d}{dx}(1 + x + x^2 + \cdots + x^n) \\ &= \frac{d}{dx}\left(\frac{x^{n+1} - 1}{x - 1}\right) \\ &= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \end{aligned} \tag{2.3}$$

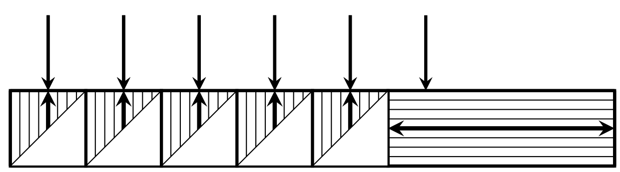


Figure 2.1: An example for Theorem 2.1. There are $6 = 8 + 1 - 3$ ways to place a horizontal strip of length 3 on a 1×8 chessboard, and there are two possibilities for each of the remaining $5 = 8 - 3$ squares. Thus, there are $192 = (8 + 1 - 3) \cdot 2^{(8-3)}$ ROSAs on a 1×8 chessboard that contain a horizontal strip of length 3. The total number of ROSAs on a 1×8 chessboard is therefore $2^8 + 8 \cdot 2^7 + 7 \cdot 2^6 + \cdots + 2 \cdot 2^1 + 1 = 2049 = 8 \cdot 2^8 + 1$.

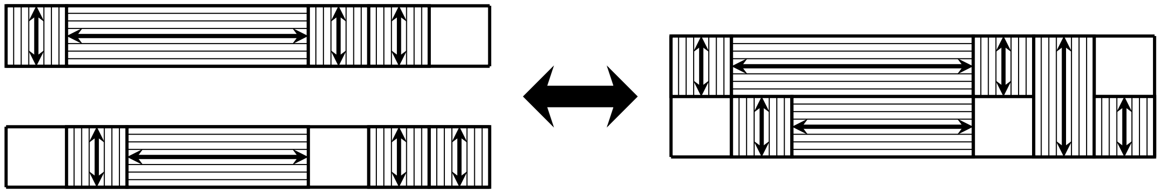


Figure 2.2: An example of the correspondence between pairs of ROSAs on $1 \times n$ boards and a single ROSA on a $2 \times n$ board. Note how the two vertical strips in the second-to-last column of each $1 \times n$ ROSA are transformed into a single vertical strip of length 2 in the $2 \times n$ ROSA.

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for all $x \neq 1$. Taking $x = 2$ in Equation 2.3*, we may conclusively rewrite Equation 2.2 as

$$T(1,n) = 2^n + n2^{n+1} - (n+1)2^n + 1 = (2n+1-n-1)2^n + 1 = n2^n + 1 \qquad \blacklozenge$$

Upon experimentally counting the number of ROSAs on $2 \times n$ chessboards, we find a direct relation between $T(2,n)$ and $T(1,n)$:

Theorem 2.4. $T(2,n) = [T(1,n)]^2 = [n2^n + 1]^2 = n^2 4^n + (2n)2^n + 1.$

This follows at once from the fact that there is a bijective map between ordered pairs (S,T) of ROSAs on the $1 \times n$ rectangle to ROSAs on the $2 \times n$ rectangle such that S is a tiling of the first row, T is a tiling of the second row, and, in any column that contains two vertical tiles, these two tiles are replaced with a single vertical tile of height 2 (see Figure 2.2).

---

*The technique of Equation 2.3 is inspired by the methods of Wilf [19]. The result can also be arrived at discretely, by rewriting the sum as a sum of geometric sums:

$$\begin{array}{ccccccccccc}
2^{n-1} & + & 2^{n-1} & + & \dots & + & 2^{n-1} & + & 2^{n-1} & = & n2^{n-1} \\
 & + & 2^{n-2} & + & \dots & + & 2^{n-2} & + & 2^{n-2} & = & (n-1)2^{n-2} \\
 & & & + & \ddots & + & \ddots & + & \vdots & = & \vdots \\
 & & & & & + & 2^1 & + & 2^1 & = & 2 \cdot 2^1 \\
 & & & & & & & + & 2^0 & = & 1 \cdot 2^0 \\
\hline
(2^n - 2^{n-1}) & + & (2^n - 2^{n-2}) & + & \dots & + & (2^n - 2^1) & + & (2^n - 2^0) & = & n2^n - (2^n - 1) = (n-1)2^n + 1
\end{array}$$

When the number m of rows of the ambient chessboard is 3 or greater, the analogous map from ordered m-tuples of ROSAs of the $1 \times n$ rectangle to strip arrangements of the $m \times n$ rectangle may introduce "gaps" between vertically oriented tiles (see Figure 2.3). Indeed, the range of this analogous map is no longer contained in the set of ROSAs; however, the inverse map that takes an ROSA of an $m \times n$ chessboard, breaks each vertical strip of length k into k individual vertical strips of unit length, and returns the corresponding m-tuple of $1 \times n$ ROSAs is well-defined, and is an injection. Hence, we have an upper bound $T(m,n) \leq [T(1,n)]^m = [n2^n + 1]^m$ for $m \leq n$, or $\log T(m,n) = O(mn \log 2 + m \log n)$ as $m, n \to \infty$ (where log denotes the natural logarithm).

We note, however, that we can immediately do better than this super-exponential estimate. Indeed, if we disregard the fact that no two strips can overlap, we can naively pick any two distinct boundaries to determine a single non-empty strip, or choose not to place a strip, in each row and column, so that

$$T(m,n) \leq \left(1 + \binom{m+1}{2}\right)^n \left(1 + \binom{n+1}{2}\right)^m \tag{2.5}$$

or $\log T(m,n) = O(2m \log n + 2n \log m)$. All of this gives a starting point for an asymptotic analysis of ROSAs, a subject to which we will return many times in the sequel.

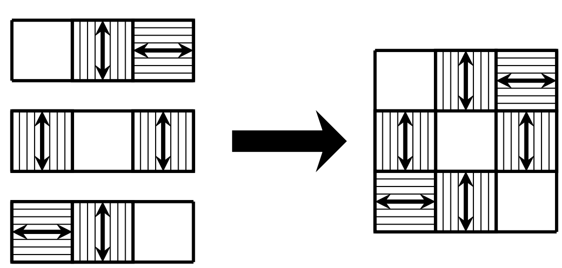


Figure 2.3: The left side of the diagram shows three proper ROSAs on 1×3 chessboard. Attempting to "fuse" these into a single strip arrangement on a 3×3 chessboard, we obtain two *non-adjacent vertical strips in the middle column, so the result is in fact* not *a ROSA.*

2.2 The Transfer Matrix Method

In any event, it would appear that determining $T(m,n)$ exactly for $m \geq 3$ using the previous methods involves rather delicate computations. There is, however, another general method that is often used for enumeration problems on chessboards such as ours. Looking at various strip arrangements and reading them from left to right, we notice that, in each row, each unit square must be in one of three states:

- State 0: A strip has not yet been seen in this row. Alternatively, the unit square is to the left of the horizontal strip in this row, or there is no horizontal strip in this row.
- State 1: The unit square is covered by a horizontal strip.
- State 2: A strip has already been seen in this row. Alternatively, the unit square is to the right of the horizontal strip in this row.

Furthermore, the states in any given row must occur in a proper sequence:

- A unit square in state 0 can be immediately followed by either a square in state 0 or a square in state 1 – we cannot instantaneously transition from not yet having seen a horizontal strip to having already seen the strip.
- A unit square in state 1 can be immediately followed by either a square in state 1 or a square in state 2 – the row of the ROSA cannot "forget" that a horizontal strip has already been seen.
- A unit square in state 2 can only be immediately followed by a square in state 2 – by transitivity, a square to the right of a square to the right of a horizontal strip must be to the right of the horizontal strip.

We may therefore construct a directed graph G_m whose vertices are $\{0,1,2\}^m$ (with one coordinate for each row) and whose edges are defined by the above rules. It then follows that the problem of counting the number of combinatorial objects (in this case, ROSAs on a chessboard with m rows) is equivalent to the problem of enumerating certain graph-theoretic walks on G_m (in this case, those walks that begin at those vertices in $\{0,1\}^m$, as no ROSA can have a row beginning in state 2), and the latter is determined by taking powers of the adjacency matrix of G_m – this is the method of *transfer matrices*. (Baxter [3] explains the transfer matrix method from the point of view of statistical mechanics, whereas Stanley [15] offers a perspective from the realm of pure mathematics, or combinatorics.)

We begin by constructing the unweighted adjacency matrix A_m for G_m; to this end, we first note that the unweighted adjacency matrix for the digraph G_1 that models ROSAs with a single row is given by

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

where the states are ordered $0, 1, 2$ as the columns of A_1 are read from left to right, the rows are read from top to bottom, and the edges of G_1 are directed according to the rules given previously.

We then notice that each row of a ROSA can progress according to the given rules independently of the other rows. This suggests that the digraph G_m for ROSAs with m rows can be constructed by taking the set of vertices to be the m-tuples of states (that is, $\{0,1,2\}^m$), and allowing an edge in G_m from a vertex $u = (u_1, \ldots, u_m)$ to another vertex $v = (v_1, \ldots, v_m)$ if and only if there are edges in G_1 from u_i to v_i for each $i = 1, \ldots, m$. This construction for G_m is known as the m-th graph-theoretic Kronecker power of G_1, or the graph-theoretic Kronecker product of G_1 taken with itself m times. It thus follows that the unweighted adjacency matrix A_m for G_m is given by the m-th matrix-theoretic Kronecker (or tensor) power of A_1, or the matrix-theoretic Kronecker (tensor) product of A_1 taken with itself m times. (More remarks about both the graph-theoretic and the matrix-theoretic Kronecker products, and their connection, can be found in the work of Cvetković, Doob, & Sachs [7].) For example, for the digraph G_2 that models ROSAs on chessboards with two rows, its unweighted adjacency matrix A_2 is given by

$$A_2 = A_1 \otimes A_1 = \begin{pmatrix} A_1 & A_1 & 0 \\ 0 & A_1 & A_1 \\ 0 & 0 & A_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

Next, we recall that each vertex of the digraph G_m should be thought of as an encoding of a column of the ROSA. We wish to consider the placement of vertical strips in these columns, and therefore we weight each vertex u with the number $w(u)$ of ways to place a vertical strip (including the empty strip) in the column encoded by u – or, equivalently, we weight each edge from u to v by $w_m(u)$. The weight of the vertex $u = (u_1, \ldots, u_m)$ is computed by the formula

$$w_m(u) = 1 + \sum_{i=1}^{k} \binom{\beta_i - \beta_{i-1}}{2} \tag{2.8}$$

where $\beta_0 = 0$, $\beta_k = m + 1$, and $\beta_1, \ldots, \beta_{k-1}$ denote the indices of each "1" in u in order – that is, $u_{\beta_i} = 1$ for all $1 \leq i < k \leq m$, $\beta_i < \beta_j$ if $i < j$, and $u_\alpha \neq 1$ if $\alpha \neq \beta_i$ for some $i = 1, \ldots, k-1$. From these weights, we are able to construct a diagonal matrix $D_m = (d_{i,j})_{0 \leq i,j \leq 3^m - 1}$ for which $d_{i,j} = 0$ if $i \neq j$, and $d_{i,i} = w_m(i)$, where we identify the integer $i = 0, 1, \ldots, 3^m - 1$ with the m-tuple $(u_1, \ldots, u_m) \in \{0, 1, 2\}^m$ given by the (unique) ternary representation $i = u_1 + 3u_2 + 3^2 u_3 + \cdots + 3^{m-1} u_m$.

We pause now to make some general remarks about generating functions obtained from transfer matrices. Let Γ be any (weighted) digraph, and let V be its set of vertices. If $W = (w_{u,v})_{u,v \in V}$ is the (weighted) adjacency matrix of Γ according to some total ordering on V, then, for any $u, v \in V$, the number of length ℓ walks on Γ from u to v is the (u, v)-entry of the product W^{ℓ}. It therefore follows that, for any indeterminate x, if I_{α} denotes the $\alpha \times \alpha$ identity matrix, then the power series

$$I_{|V|} + Wx + W^2x^2 + \cdots = (I_{|V|} - Wx)^{-1} \tag{2.9}$$

gives a matrix whose (u, v)-entry is a generating function such that the coefficient of x^{ℓ} is the number of walks on Γ from u to v of length ℓ. (These results can be found in Stanley [15].)

Thus, the generating function $\mathcal{F}_v$ for which the coefficient of x^{ℓ} is the number of walks on Γ of length ℓ that start at v and end at *any* other vertex of Γ is the sum of *all* the entries in the matrix $(I_{|V|} - Wx)^{-1}$ in the row corresponding to v – or, as a formula,

$$\mathcal{F}_v(x) = e_v \cdot (I_{|V|} - Wx)^{-1} \cdot \left(\sum_{u \in V} e_u \right)^t \tag{2.10}$$

where e_u denotes the row basis vector corresponding to $u \in V$ in the lattice $\mathbb{Z}^{|V|}$ (or in the group algebra $\mathbb{Z}[V]$), and $(\cdot)^t$ denotes the transpose operator that converts row vectors into column vectors and vice versa.

The foregoing matrix-theoretic discussion leads to an alternative way to compute $\mathcal{F}_v$:

Lemma 2.11. *For any (weighted) graph Γ, let V be its set of vertices, and let E be its set of edges. For any vertex $v \in V$, let $\mathcal{F}_v(x)$ denote the generating function for which the coefficient of x^n is the number of walks on Γ that begin at v. Then, if $W = (w_{\alpha,\beta})_{\alpha,\beta \in E}$ denotes the (weighted) adjacency matrix of Γ,*

$$\mathcal{F}_v(x) = 1 + \sum_{v' \in V} (w_{v,v'}x) \mathcal{F}_{v'}(x)$$

There is a nice intuition for this lemma: a walk in Γ of length ℓ beginning at v is determined by taking some first step from v to any other vertex w, and then constructing a walk from w of length $\ell - 1$. For a proof by formal calculation, if we write $\sigma = \sum_{u \in V} e_u$, then one has, by Equations 2.9 and 2.10,

$$\begin{aligned}
\mathcal{F}_v(x) &= e_v \cdot (I_{|V|} - Wx)^{-1} \cdot \sigma^t \\
&= e_v \cdot \left(I_{|V|} + Wx + W^2x^2 + \ldots\right) \cdot \sigma^t \\
&= e_v \cdot I_{|V|} \cdot \sigma^t + e_v \cdot \left(Wx + W^2x^2 + W^3x^3 + \ldots\right) \cdot \sigma^t \\
&= 1 + e_v \cdot Wx \cdot \left(I_{|V|} + Wx + W^2x^2 + \ldots\right) \cdot \sigma^t \\
&= 1 + \sum_{v' \in V} (w_{v,v'}x) \left[e_{v'} \cdot (I - Wx)^{-1} \cdot \sigma^t\right] \\
&= 1 + \sum_{v' \in V} (w_{v,v'}x) \mathcal{F}_{v'}(x)
\end{aligned}$$

In our situation, each edge of G_m that flows out of a vertex v is equally weighted by $w_m(v)$, and therefore we can rewrite the result of Lemma 2.11 as

$$\mathcal{F}_v(x) = 1 + xw_m(v) \sum_{(v,v') \in E_m} \mathcal{F}_{v'}(x) \tag{2.12}$$

where E_m is the set of directed edges of G_m. If we set $v = 0$ as the origin, whose nearest neighbors must have each of its coordinates equal to either 0 or 1, we obtain

$$\mathcal{F}_0(x) = 1 + xw_m(0) \sum_{v' \in \{0,1\}^m} \mathcal{F}_{v'}(x) = 1 + x\left(1 + \binom{m+1}{2}\right) \sum_{v' \in \{0,1\}^m} \mathcal{F}_{v'}(x)$$

The expression $\sum_{u \in \{0,1\}^m} \mathcal{F}_u(x)$ is the generating function that enumerates all walks that begin at any point in the unit square $\{0,1\}^m$. But we remarked earlier that these constitute all of the possible starting points for walks on G_m that correspond to ROSAs. Therefore,

Corollary 2.13. *The generating function $\mathcal{T}_m(x)$ for the sequence $\{T(m,n)\}_{n \geq 0}$ satisfies*

$$\mathcal{F}_0(x) = 1 + x\left(\binom{m+1}{2} + 1\right)\mathcal{T}_m(x)$$

where $\mathcal{F}_0(x)$ is the generating function for the sequence whose n-th term is the number of ROSAs on an $m \times n$ chessboard with no horizontal tiles crossing the first column.

The characterization of $\mathcal{F}_0(x)$ in Corollary 2.13 stems from the fact that the origin 0 encodes the only starting column (i.e., the only starting vertex in $\{0,1\}^m$) that does not have a horizontal strip crossing it (i.e., has no 1's among its coordinates). In addition, Corollary 2.13, like its predecessor Lemma 2.11, admits an intuitive explanation: for $n > 0$, the coefficient of x^n in $\mathcal{F}_0(x)$ is the number of ROSAs on an $m \times n$ chessboard that have no horizontal strips crossing the left-most row, whereas the coefficient of x^n in the power series $1 + x(1 + \binom{m+1}{2})\mathcal{T}_m(x)$ is equal to the number of ways to place a vertical strip anywhere in an empty column times the number of ROSAs on an $m \times (n-1)$ chessboard.

To apply the result of Corollary 2.13, we need to compute the generating function $\mathcal{F}_0(x)$ that enumerates ROSAs with m rows with no horizontal strips crossing the first (left-most) column. More generally, we would like to compute the generating function $\mathcal{F}_v(x)$ for ROSAs starting with any column v. Viewing G_m with weighted vertices, one way to do so is to start by evaluating the weight of v (using the weight matrix D_m), then transferring to a new column (using right-multiplication by the adjacency matrix A_m), taking the weight of this new column (using right-multiplication by D_m), and so on for any desired length. Alternatively, viewing G_m with weighted edges, we can use the product $D_m \cdot A_m$ as a single weighted adjacency matrix for G_m to account for the transitions between successive columns. However, transfer matrices, by their very nature, record transitions between vertices on the digraph – or, in our case, between columns of the ROSA. Thus, a ROSA with n columns corresponds to $n-1$ transitions, which is captured by the $(n-1)$-fold multiplication of the transfer matrix, but we still need to account for the weight of the last column. This can be done by taking one final transition from the last column to itself, and right-multiplying by the diagonal matrix D_m captures this phenomenon perfectly. [†]

[†]One could also take the matrix $A_m \cdot D_m$ and left-multiply by D_m, but this proves to be not as compatible with Theorem 2.11.

Therefore, the matrix-valued power series $\mathcal{P}(x)$ whose (u, v) entry is the generating function enumerating ROSAs beginning with some column u and ending with another column v is given by

$$\begin{aligned}\mathcal{P}(x) &= I_{3^m} + xD_m + x^2 D_m \cdot A_m \cdot D_m + x^3 (D_m \cdot A_m)^2 \cdot D_m \\ &\qquad + \cdots + x^n (D_m \cdot A_m)^{n-1} \cdot D_m + \ldots \\ &= I_{3^m} + (I_{3^m} - xD_m \cdot A_m)^{-1} \cdot (xD_m)\end{aligned}$$

In particular, $\mathcal{F}_0(x)$ is precisely the sum of the entries of the zeroth row of $\mathcal{P}(x)$, or, following the methods of Lemma 2.11 (and with σ denoting the sum of all the unit vectors in the standard basis on $\mathbb{Z}^{3^m}$),

$$\begin{aligned}\mathcal{F}_0(x) &= e_0 \cdot [I_{3^m} + (I_{3^m} - xD_m \cdot A_m)^{-1} \cdot (xD_m)] \cdot \sigma^t \\ &= e_0 \cdot I_{3^m} \cdot \sigma^t + e_0 \cdot [(I_{3^m} - xD_m \cdot A_m)^{-1} \cdot (xD_m)] \cdot \sigma^t \\ &= 1 + x \left\{ e_0 \cdot [(I_{3^m} - xD_m \cdot A_m)^{-1} \cdot D_m] \cdot \sigma^t \right\} \qquad (2.14)\end{aligned}$$

After applying Corollary 2.13, we conclude that

Theorem 2.15. *The generating function $\mathcal{T}_m(x)$ of the sequence $\{T(m, n)\}_{n \geq 0}$ is given by*

$$\mathcal{T}_m(x) = \left(\frac{1}{\binom{m+1}{2} + 1} \right) e_0 \cdot [(I_{3^m} - xD_m A_m)^{-1} \cdot D_m] \cdot (e_0 + \cdots + e_{3^m - 1})^t$$

In other words, $\mathcal{T}_m(x)$ is the sum of all of the entries in the zeroth row of the matrix $(I_{3^m} - xD_m A_m)^{-1} \cdot D_m$ divided by the constant $1 + \binom{m+1}{2}$.

2.3 Examples of the Transfer Matrix Method

Theorem 2.15 now gives us a procedure to compute generating functions for the number of ROSAs on $m \times n$ chessboards for any fixed number of rows m, from which we aim to compute exact formulas. Before we obtain new results in this direction, however, let us compare this with our previous results.

First, we take the case $m = 1$. We have already determined that the unweighted adjacency matrix A_1 for the corresponding digraph G_1 is given as in Equation 2.6. For the weights, we note that there is only one possible way to add a vertical strip to a column in state 1 (by adding nothing, since the column is already completely covered by a horizontal strip), whereas there are two possible ways to add a vertical strip to the other columns (either we add a strip or we do not). Therefore, $w_m(1) = 1$ and $w_m(0) = w_m(2) = 2$, so the weight matrix is given by

$$D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus, we can easily compute by hand that

$$\begin{aligned}
(I_3 - xD_1 \cdot A_1)^{-1} \cdot D_1 &= \begin{pmatrix} 1-2x & -2x & 0 \\ 0 & 1-x & -x \\ 0 & 0 & 1-2x \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{1-2x} & \frac{2x}{(1-x)(1-2x)} & \frac{2x^2}{(1-x)(1-2x)^2} \\ 0 & \frac{1}{1-x} & \frac{x}{(1-x)(1-2x)} \\ 0 & 0 & \frac{1}{1-2x} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{1-2x} & \frac{2x}{(1-x)(1-2x)} & \frac{4x^2}{(1-x)(1-2x)^2} \\ 0 & \frac{1}{1-x} & \frac{2x}{(1-x)(1-2x)} \\ 0 & 0 & \frac{2}{1-2x} \end{pmatrix}
\end{aligned}$$

By Theorem 2.15, the generating function that enumerates the ROSAs on a chessboard with one row is given by

$$\begin{aligned}
\mathcal{T}_1(x) &= \frac{1}{2}\left(\frac{2}{1-2x} + \frac{2x}{(1-x)(1-2x)} + \frac{4x^2}{(1-x)(1-2x)^2}\right) \\
&= \frac{1}{1-2x} + \left(\frac{1}{1-2x} - \frac{1}{1-x}\right) + \left(\frac{2}{1-x} - \frac{3}{1-2x} + \frac{1}{(1-2x)^2}\right) \\
&= \frac{1}{1-x} - \frac{1}{1-2x} + \frac{1}{(1-2x)^2} \qquad (2.16)
\end{aligned}$$

To recover the coefficient $T(1,n)$ of x^n in the power series for $\mathcal{T}_1(x)$, we recall some facts about generating functions from Wilf [19]. First, for any integer $k \geq 0$, the coefficient of y^n in the power series for $1/(1-y)^{k+1}$ is given by

$$[y^n]\left\{\frac{1}{(1-y)^{k+1}}\right\} = \binom{n+k}{k}$$

Secondly, making the substitution $y = Cx$ in the above and using a fundamental property of the coefficient extraction operator, we deduce that

$$\begin{aligned}
[x^n]\left\{\frac{1}{(1-Cx)^{k+1}}\right\} &= \left[\left(\frac{y}{C}\right)^n\right]\left\{\frac{1}{(1-y)^{k+1}}\right\} \\
&= C^n[y^n]\left\{\frac{1}{(1-y)^{k+1}}\right\} = C^n\binom{n+k}{k} \qquad (2.17)
\end{aligned}$$

Applying this to the partial fraction decomposition of $\mathcal{T}_1(x)$ given in Equation 2.16, we conclude that, for all positive integers n,

$$T(1,n) = \binom{n}{0} - 2^n\binom{n}{0} + 2^n\binom{n+1}{1} = 1 + 2^n(-1+(n+1)) = 1 + n2^n$$

which does indeed agree with the result of Theorem 2.1.

Let us also check the formula for the case $m = 2$. By the Kronecker product structure of the digraph $G_2 = G_1 \otimes G_1$, its unweighted adjacency matrix is given by $A_2 = A_1 \otimes A_1$, and is explicitly written as a matrix in Equation 2.7. By Equation 2.8, the weight matrix is given by

$$D_2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

We may therefore compute the matrix $I_9 - xD_2 \cdot A_2$ as

$$I_9 - xD_2 \cdot A_2 = \begin{pmatrix} 1-4x & -4x & 0 & -4x & -4x & 0 & 0 & 0 & 0 \\ 0 & 1-2x & -2x & 0 & -2x & -2x & 0 & 0 & 0 \\ 0 & 0 & 1-4x & 0 & 0 & -4x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2x & -2x & 0 & -2x & -2x & 0 \\ 0 & 0 & 0 & 0 & 1-x & -x & 0 & -x & -x \\ 0 & 0 & 0 & 0 & 0 & 1-2x & 0 & 0 & -2x \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-4x & -4x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-2x & -2x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-4x \end{pmatrix}$$

By Theorem 2.15, the generating function $\mathcal{T}_2(x)$ whose coefficients are the numbers $T(2, n)$ of ROSAs on a $2 \times n$ chessboard is given by the sum of the entries in the zeroth row of $(I_9 - xD_2 \cdot A_2)^{-1} \cdot D_2$. We can either compute the matrix inverse explicitly, or we can make use of the following formula which comes from the cofactor matrix:

Theorem 2.18 (Stanley [15], Theorem 4.7.2). *Let W denote the (weighted) adjacency matrix of the (weighted) digraph Γ with vertex set $\{v_1, \ldots, v_k\}$. Then, for any two vertices v_i, v_j of Γ, the generating function $P_{v,v'}(x)$ for which the coefficient of x^ℓ is the number of paths in Γ of length ℓ from v to v' is given by*

$$P_{v,v'}(x) = \frac{(-1)^{i+j} \det(I_k - xW; j, i)}{\det(I_k - xW)}$$

where $(B; \alpha, \beta)$ denotes the submatrix of the matrix B obtained by deleting the α-th row and the β-th column.

A direct corollary of Theorem 2.18 and Theorem 2.15 is

Corollary 2.19. *The generating function $\mathcal{T}_m(x)$ for which the coefficient of x^n is the number of ROSAs on an $m \times n$ chessboard is given by*

$$\mathcal{T}_m(x) = \sum_{j=0}^{3^m-1} \frac{(-1)^j w_m(j) \det(I_{3^m} - xD_m \cdot A_m; j, 0)}{w_m(0) \det(I_{3^m} - xD_m \cdot A_m)}$$

This result is all the more significant if we recall that the transfer matrix $D_m \cdot A_m$ is upper triangular, for we can easily compute

$$\det(I_{3^m} - xD_m \cdot A_m) = (1 - w_m(0)x)(1 - w_m(1)x) \ldots (1 - w_m(3^m - 1)x)$$

In addition, the submatrix $(I_{3^m} - xD_m \cdot A_m; j, 0)$ is diagonal below the deleted row, so that these diagonal elements in a numerator cancel with corresponding elements in a denominator. We can therefore rewrite the result of Corollary 2.19 as

$$\mathcal{T}_m(x) = \sum_{j=0}^{3^m-1} \frac{(-1)^j w_m(j) \det[I_{3^m} - xD_m \cdot A_m : (0, \ldots, j-1), (1, \ldots, j)]}{w_m(0)(1 - w_m(0)x) \ldots (1 - w_m(j)x)}$$

where $\det[M : (\alpha_1, \ldots, \alpha_z), (\beta_1, \ldots, \beta_z)]$ denotes the submatrix of the matrix M whose rows are the rows of M indexed by the tuple $(\alpha_1, \ldots, \alpha_z)$, and whose columns are the columns of M indexed by the tuple $(\beta_1, \ldots, \beta_z)$ (and, by convention, the determinant of the empty matrix $[M, \emptyset, \emptyset]$ is 1).

With $m = 2$, one has

$$\begin{aligned}
\mathcal{T}_2(x) &= \frac{1}{1-4x} + \frac{2x}{(1-4x)(1-2x)} + \frac{8x^2}{(1-4x)^2(1-2x)} + \frac{2x}{(1-4x)(1-2x)} \\
&\quad + \frac{x(1+2x)}{(1-4x)(1-2x)(1-x)} + \frac{2x^2(3-4x-8x^2)}{(1-4x)^2(1-2x)^2(1-x)} \\
&\quad + \frac{8x^2}{(1-4x)^2(1-2x)} + \frac{2x^2(3-4x-8x^2)}{(1-4x)^2(1-2x)^2(1-x)} \\
&\quad + \frac{4x^2(1+8x-20x^2-16x^3)}{(1-4x)^3(1-2x)^2(1-x)} \\
&= \frac{1}{1-4x} - 2\left(\frac{1}{1-2x} - \frac{1}{1-4x}\right) \\
&\quad + 2\left(\frac{2}{1-2x} - \frac{3}{1-4x} + \frac{1}{(1-4x)^2}\right) \\
&\quad + \left(\frac{1}{1-x} - \frac{2}{1-2x} + \frac{1}{1-4x}\right) \\
&\quad - 2\left(\frac{2}{1-x} - \frac{5}{1-2x} + \frac{1}{(1-2x)^2} + \frac{3}{1-4x} - \frac{1}{(1-4x)^2}\right) \\
&\quad + \left(\frac{4}{1-x} - \frac{12}{1-2x} + \frac{4}{(1-2x)^2} + \frac{9}{1-4x} - \frac{7}{(1-4x)^2} + \frac{2}{(1-4x)^3}\right) \\
&= \frac{1}{1-x} - \frac{2}{1-2x} + \frac{2}{(1-2x)^2} + \frac{1}{1-4x} - \frac{3}{(1-4x)^2} + \frac{2}{(1-4x)^3}
\end{aligned}$$

It therefore follows from Equation 2.17 that

$$\begin{aligned}
T(2,n) &= \binom{n}{0} + 2^n\left(-2\binom{n}{0} + 2\binom{n+1}{1}\right) + 4^n\left(\binom{n}{0} - 3\binom{n+1}{1} + 2\binom{n+2}{2}\right) \\
&= 1 + 2^n(2n) + 4^n(n^2)
\end{aligned}$$

which again agrees with our previous result of Theorem 2.4.

Exact formulas for $T(m,n)$ for increasing values of m can be found using the same method as above, only performing exponentially more computations; we list several of these results in Appendix A.

2.4 An Asymptotic Formula

Naturally, after computing these formulas, we find that some patterns begin to emerge. For example, the dominant term in $T(m,n)$ – i.e., the coefficient of x^n in $\mathcal{T}_m(x)$ – appears to be of the form $C(m)n^{P(m)}(\Lambda(m))^n$ for some functions C, P, Λ of m. This corresponds exactly to an observation of Flajolet and Sedgewick [8] that, for any rational generating function $f(x)$ with poles $\rho_1, \ldots, \rho_k$ that is analytic at $x = 0$, the coefficient of x^n in $f(x)$ must be of the form

$$[x^n]\{f(x)\} = \sum_{i=1}^{k} \Pi_i(n)(1/\rho_i)^n \tag{2.20}$$

for some polynomials $\Pi_1, \ldots, \Pi_k$. Stanley [15] tells us that any generating function that arises from the method of transfer matrices must be rational, and thus $f(x) = \mathcal{T}_m(x)$ must satisfy Equation 2.20 for all m.[‡] Furthermore, the dominant term in Equation 2.20 corresponds to a reciprocal of a pole of smallest (complex) magnitude – by construction, such a pole is itself a reciprocal of an eigenvalue of largest magnitude of the corresponding transfer matrix. In our situation, the transfer matrix $T_m = D_m \cdot A_m$ has a unique largest eigenvalue, which is the weight $1 + \binom{m+1}{2}$ of a column of a ROSA that is not crossed by any horizontal strip. Therefore,

$$T(m,n) \sim C(m)n^{P(m)} \left(1 + \binom{m+1}{2}\right)^n$$

We also know from Inequality 2.5 that the degree of the polynomial factor attached to this largest exponential term has degree at most $2m$ – that is, $P(m) \leq 2m$, using the above notation (for $1+\binom{n}{2}$ is a polynomial of degree 2, so its m-th power has degree $2m$). It seems, however, from the exact formulas in Appendix A that this degree is exactly m. Indeed, if

[‡]Indeed, Stanley goes further, showing that Flajolet and Sedgewick's necessary condition for a generating function to be rational is also sufficient – that is, any generating function $f(x)$ that satisfies Equation 2.20 for all n is a rational function (that is analytic at $x = 0$).

we restrict the set of ROSAs to those arrangements that have exactly one horizontal strip of unit length in each row and no more than one horizontal strip in each column, then, for $m \leq n$, we have the lower bound

$$T(m,n) \geq \binom{n}{m} m! \left(1 + \binom{m+1}{2}\right)^{n-m} \left\{ \prod_{j=1}^{m} \left[1 + \binom{j}{2} + \binom{m-j+1}{2} \right] \right\} \tag{2.21}$$

which stems from first choosing m of the n columns to contain a horizontal strip, then choosing distinct rows for the horizontal strips in each of the chosen columns (this is akin to selecting a placement of m non-attacking rooks on a $m \times m$ chessboard), and finally determining the number of ways to place a vertical strip in each column – for the $n - m$ columns without any horizontal strip, there are $w_m(0) = 1 + \binom{m+1}{2}$ possible placements of a vertical strip, whereas, for each $j = 1, \ldots, m$, the column that contains a horizontal strip in row j allows $w_m(3^j) = 1 + \binom{j}{2} + \binom{m+1-j}{2}$ possible placements of a vertical strip by Equation 2.8. From Inequality 2.21, we deduce that $P(m) \geq \deg \binom{n}{m} = m$.

In this same vein, we can refine our upper bound for $P(m)$. Define a *row signature* of a ROSA on an $m \times n$ chessboard as an m-tuple of pairs (a, x) with $a \in \{0, 1, \ldots, n\}$, and $x \in \{1, \ldots, \chi(a)\}$, where $\chi(a)$ is defined to be 1 if $a = 0$, and $\chi(a) = n + 1 - a$ if $a > 0$. For each $i = 1, \ldots, m$, the first element of the i-th pair is meant to signify the length of the horizontal strip in row i of the ROSA, while the second element is meant to indicate the first column that is covered by that strip (provided that there is indeed a horizontal strip in that row – or, equivalently, provided that the first element of the pair is non-zero). Each ROSA, therefore, can be obtained by first choosing a row signature, and then by choosing a placement of vertical strips that is compatible with the row signature. Hence, if $V(\rho)$ denotes the number of placements of vertical strips compatible with the row signature ρ,

$$T(m,n) = \sum_{\rho} V(\rho) \tag{2.22}$$

where the sum is taken over all row signatures ρ.

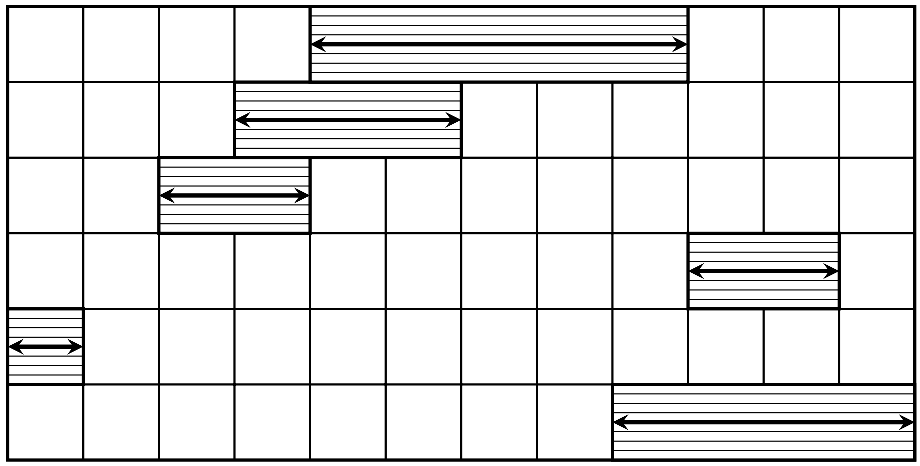
To obtain an upper bound for $V(\rho)$, we note that each horizontal strip obstructs the placement of a vertical strip in any given column; therefore, the number of compatible vertical strips increases as the number of columns crossed by a horizontal strip decreases. For a given row signature $\rho = ((a_1, x_1), \ldots (a_m, x_m))$, the fewest number of columns that must be crossed by a horizontal strip is $\mu = \max\{a_i : i = 1, \ldots, m\}$, and this minimum is attained when the corresponding largest strip acts as an umbrella for the others – see Figure 2.4. For those columns not crossed by a horizontal strip, the number of vertical strips that can be placed in the column is equal to the largest weight $w_m(0) = 1 + \binom{m+1}{2}$; for those columns crossed by at least one horizontal strip (i.e., with at least one obstruction to the placement of vertical strips), the number of compatible vertical strips is at most the next highest weight $w_m(1) = 1 + \binom{m}{2}$. It therefore follows that

$$\begin{aligned} V(\rho) &\leq \left(1 + \binom{m+1}{2}\right)^{n-\mu} \left(1 + \binom{m}{2}\right)^{\mu} \\ &\leq \left(1 + \binom{m+1}{2}\right)^{n-\frac{a_1+\cdots+a_m}{m}} \left(1 + \binom{m}{2}\right)^{\frac{a_1+\cdots+a_m}{m}} \qquad (2.23) \end{aligned}$$

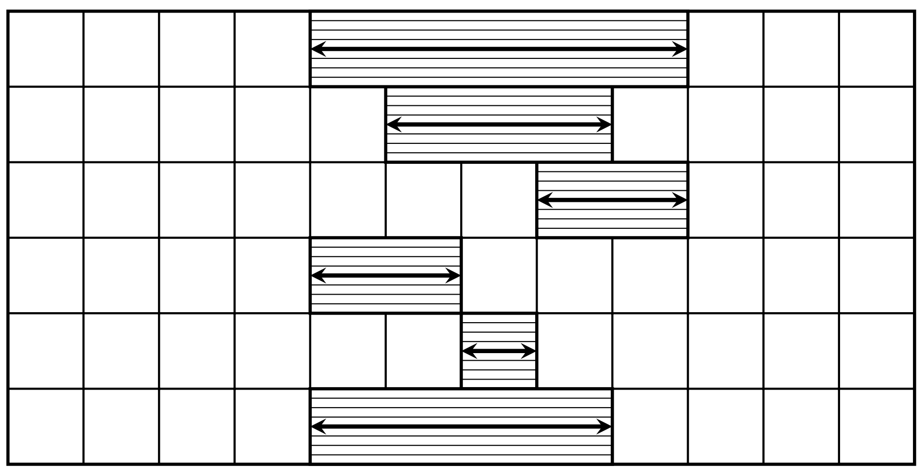
where Inequality 2.23 arises from the fact that $w_m(0) > w_m(1)$, and the fact that the average of a set of values cannot exceed its maximum.

We see that this upper bound for $V(\rho)$ does not depend fully on the row signature ρ of a ROSA, but rather on the corresponding *row prototype* – that is, the m-tuple of integers $(a_1, \ldots, a_m) \in \{0, \ldots, n\}^m$ that records only the lengths of the horizontal strips of the ROSA. One can easily see that the number $|\operatorname{Sig}(a_1, \ldots, a_n)|$ of row signatures corresponding to a row prototype $(a_1, \ldots, a_m)$ satisfies

$$|\operatorname{Sig}(a_1, \ldots, a_n)| = \chi(a_1)\chi(a_2)\ldots\chi(a_m) \leq [\chi(\max\{a_i : i = 1, \ldots, m\})]^m \leq n^m$$



(a)



(b)

Figure 2.4: Figure 2.4(a) shows an arrangement of horizontal strips in which 11 of the 12 available columns intersect at least one horizontal strip. Using strips of the same length in the same rows, Figure 2.4(b) shows an arrangement in which only 5 of the columns intersect at least one horizontal strip. With these given strips, we cannot intersect fewer columns, as the strip of length 5 must intersect 5 columns, and this minimum is achieved by "tucking" the remaining strips under the longest strip. The latter strip arrangement, which has fewer obstructed columns, allows a greater number of possible placements of vertical strips.

Therefore, if Φ is a function that takes row prototypes to integers, then a sum over row signatures has as an upper bound a corresponding sum over row prototypes:

$$\sum_{((a_1,x_1),\ldots,(a_m,x_m))} \Phi(a_1,\ldots,a_m) \le \sum_{(a_1,\ldots,a_m)} n^m \Phi(a_1,\ldots,a_m) = \sum_{a_1=0}^{n} \cdots \sum_{a_m=0}^{n} n^m \Phi(a_1,\ldots,a_m)$$

Applying this to Equation 2.22 and Inequality 2.23 gives

$$\begin{aligned}
T(m,n) &\le \sum_{a_1=0}^{n} \cdots \sum_{a_m=0}^{n} n^m \left(1+\binom{m+1}{2}\right)^{n-\frac{a_1+\cdots+a_m}{m}} \left(1+\binom{m}{2}\right)^{\frac{a_1+\cdots+a_m}{m}} \\
&\le n^m \left(1+\binom{m+1}{2}\right)^{n} \sum_{a_1=0}^{\infty} \cdots \sum_{a_m=0}^{\infty} \left[\frac{1+\binom{m}{2}}{1+\binom{m+1}{2}}\right]^{\frac{a_1+\cdots+a_m}{m}} && (2.24) \\
&\le n^m \left(1+\binom{m+1}{2}\right)^{n} \left\{\sum_{a=0}^{\infty} \left[\frac{1+\binom{m}{2}}{1+\binom{m+1}{2}}\right]^{\frac{a}{m}}\right\}^{m} \\
&\le n^m \left(1+\binom{m+1}{2}\right)^{n} \left\{1-\left[\frac{1+\binom{m}{2}}{1+\binom{m+1}{2}}\right]^{\frac{1}{m}}\right\}^{-m} && (2.25)
\end{aligned}$$

It follows from Inequality 2.24 that $P(m) \le m$, as we removed any dependence on n from the summation; therefore, with Inequality 2.21, we deduce that $P(m) = m$ as observed. However, continuing on to Inequality 2.25, we also obtain, again with Inequality 2.21, an estimate for the coefficient $C(m)$ of the summand of $T(m,n)$ with largest exponential base and largest polynomial degree in n:

$$\begin{aligned}
\left\{1-\left[\frac{1+\binom{m}{2}}{1+\binom{m+1}{2}}\right]^{\frac{1}{m}}\right\}^{-m} &\ge C(m) \\
&\ge \left(1+\binom{m+1}{2}\right)^{-m} \left\{\prod_{j=1}^{m}\left[1+\binom{j}{2}+\binom{m-j+1}{2}\right]\right\}
\end{aligned}$$

It remains only to determine the leading coefficient $C(m)$ exactly. We have been concentrating in the preceding arguments on the placements of the horizontal strips. Let us instead consider the placement of *gaps* between these horizontal strips. We formally define an *H-gap of size s* to be a contiguous set of s columns on the chessboard that are not crossed by any horizontal strip, and are such that the leftmost gridline of the gap is either the leftmost gridline of the chessboard or coincides with the rightmost boundary of some horizontal

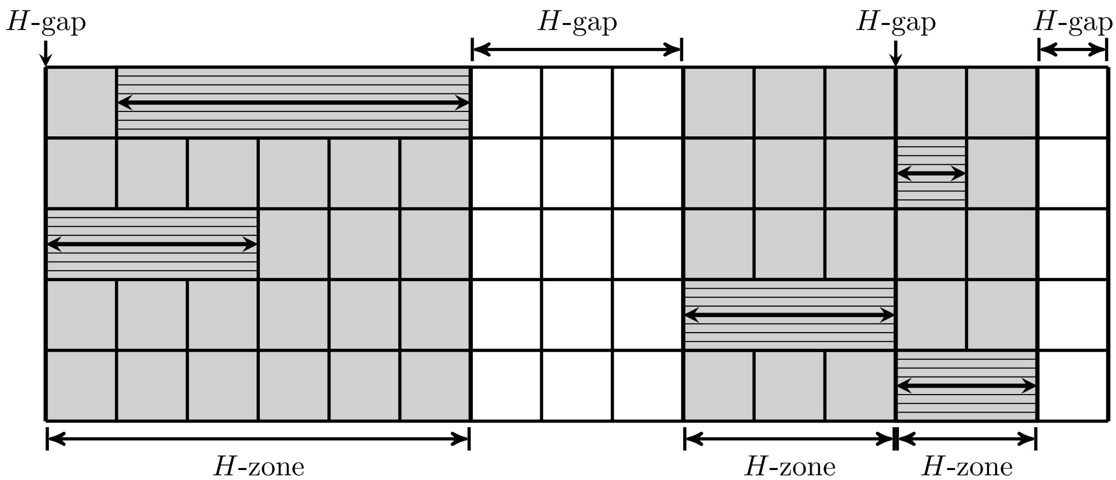


Figure 2.5: An arrangement of horizontal strips with H-gaps and H-zones labeled. Intuitively, H-gaps appear after one horizontal strip ends and another horizontal strip begins; any horizontal strips that intersect the same columns belong to the same H-zone. Note that there are always H-gaps at either end of the chessboard, and that it is possible for an H-gap to be of length 0.

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strip, and similarly the rightmost gridline of the gap is either the rightmost gridline of the chessboard or coincides with the leftmost boundary of some horizontal strip. Dually, we also define an *H-zone of size z* to be a set of contiguous columns of the chessboard that are covered by at least one horizontal strip, and are such that the leftmost gridline of the zone is the rightmost gridline of some H-gap, and the rightmost gridline of the zone coincides with the leftmost gridline of some other H-gap. We illustrate a configuration of H-gaps and H-zones in Figure 2.5.

We now list some remarks about these newly-defined H-gaps and H-zones. Firstly, we allow for H-gaps to have non-negative integer size of at most n, including those of size 0. An H-gap of size 0 indicates that either there is a horizontal strip that touches one of the ends of the chessboard, or there is a gridline of the chessboard that is touched by (at least) two distinct horizontal strips in distinct rows – one that touches from the left, and one from the right – and is not crossed by any other horizontal strip. However, since there are no horizontal strips of length 0, the H-zones must have strictly positive integer size.

Secondly, we note that there is at least one H-gap in any ROSA. A ROSA with exactly one H-gap, however, is one without any horizontal strips (i.e., with no H-zones), so, in the typical case, a ROSA will have at least two H-gaps (corresponding to the left and right edges), and consequently at least one H-zone. As there are at most m horizontal strips in a ROSA on a chessboard with m rows, there are at most m H-zones on such a ROSA, and consequently at most $m+1$ H-gaps. In fact, since each H-zone is bounded by H-gaps, the number of H-zones is always one less than the number of H-zones.

The great advantage of partitioning ROSAs into H-zones and H-gaps is that the placement of strips (both horizontal and vertical) in the H-gaps and the placement of strips in the H-zones are independent of one another. By definition, no horizontal strip can cross between H-zones and H-gaps (indeed, horizontal strips are forbidden in H-gaps); and, as both zones and gaps are collections of columns, no vertical strip can cross between them, since in fact no vertical strip can cross between any two columns. It therefore follows that

$$T(m,n) = Z(0,0)G(1,m) + \sum_{g=2}^{m+1} \sum_{\kappa=g-1}^{n} Z(g-1,\kappa)G(g,n-\kappa) \tag{2.26}$$

where Z and G are defined as follows:

$$\begin{aligned} Z(z,\kappa) &= \text{the number of ways to place strips in a number } z \text{ of } H\text{-zones of total size } \kappa \\ G(g,\lambda) &= \text{the number of ways to place strips in a number } g \text{ of } H\text{-gaps of total size } \lambda \end{aligned}$$

The sum of Equation 2.26 is indexed by the possible values κ of the number of columns in the ROSA that coincide with at least one horizontal strip and the possible values g of the number of H-gaps in the ROSA.

As we observed, the computation of Z and G are independent of one another, so we determine first the number $G(g, \lambda)$ of ways to place strips in a number g of H-gaps with total size λ. Note that there are by design no horizontal strips in the H-gaps, so that we only have to worry about the placement of vertical strips in each column. As each column is not obstructed by any horizontal strip, we have $w_m(0) = 1 + \binom{m+1}{2}$ possibilities for each column among the gaps; therefore, $G(g, \lambda)$ equals $w_m(0)^\lambda$ times the number of ways to arrange the columns into a collection of g gaps. But this number of ways of grouping the columns is equivalent to the number of non-negative integer solutions $y_1, \ldots, y_g$ to the equation $y_1 + \cdots + y_g = \lambda$, and it is a well-known combinatorial result (see, for example, Matoušek & Nešetřil [13]) that the number of such solutions to this equation is given by $\binom{\lambda+g-1}{g-1}$. Hence, we have

$$G(g, \lambda) = \binom{\lambda + g - 1}{g - 1} w_m(0)^\lambda = \binom{\lambda + g - 1}{g - 1} \left(1 + \binom{m+1}{2}\right)^\lambda$$

and so, with $Z(0, 0) = 1$, Equation 2.26 can be rewritten as

$$\begin{aligned} T(m, n) &= w_m(0)^n + \sum_{g=2}^{m+1} \sum_{\kappa=g-1}^{n} \binom{n - \kappa + g - 1}{g - 1} w_m(0)^{n-\kappa} Z(g - 1, \kappa) \\ &= w_m(0)^n \left[1 + \sum_{g^*=1}^{m} \sum_{\kappa^*=0}^{n-g^*} \binom{n - \kappa^*}{g^*} w_m(0)^{-(\kappa^* + \gamma^*)} Z(g^*, \kappa^* + g^*) \right] \end{aligned} \tag{2.27}$$

where $g^* = g - 1$ is the number of H-zones, and $\kappa^* = \kappa - g^*$ is the number of *other* columns that intersect a horizontal strip aside from the one required in each H-zone.

By a similar argument, using the fact that the number of ways to place a vertical strip in a column obstructed by some horizontal strip is at most $w_m(1) = 1 + \binom{m}{2}$, we can determine the upper bound

$$\frac{Z(g^*, \kappa)}{w_m(0)^\kappa} \leq \sum_{\substack{x_1 + \cdots + x_{g^*} = \kappa \\ x_i \geq 1}} \left[\frac{w_m(1)}{w_m(0)} \right]^\kappa = \sum_{\substack{x_1^* + \cdots + x_{g^*}^* = \kappa - g^* \\ x_i^* \geq 0}} \left[\frac{w_m(1)}{w_m(0)} \right]^\kappa = \binom{\kappa - 1}{g^* - 1} \left[\frac{w_m(1)}{w_m(0)} \right]^\kappa$$

We wish to determine the asymptotic behavior of $T(m,n)$ as $n \to \infty$, so let us regard κ as an expression that grows with n. If κ is sufficiently large compared to n – i.e., if $k > n^{\varepsilon}$ for some $\varepsilon > 0$ and all sufficiently large n – then, as $w_m(1) < w_m(0)$,

$$\frac{Z(g^*,\kappa)}{w_m(0)^{\kappa}} \leq \binom{\kappa-1}{g^*-1}\left[\frac{w_m(1)}{w_m(0)}\right]^{\kappa} < \binom{\kappa-1}{g^*-1}\left[\frac{w_m(1)}{w_m(0)}\right]^{n^{\varepsilon}} \to 0$$

as $n \to \infty$. Therefore, the exponentially dominant terms of Equation 2.27 correspond to ROSAs for which κ, the number of columns overlapped by any horizontal strip, is small enough so that $\kappa < n^{\varepsilon}$, or $\log\kappa < \varepsilon \log n$, for all $\varepsilon > 0$ and all sufficiently large n (i.e., in Landau notation, $\log\kappa = o(\log n)$), or equivalently have $\log(\kappa)/\log(n) \to 0$ as $n \to \infty$.

But $\log(\kappa)/\log(n) \to 0$ also implies that

$$\begin{aligned}\frac{\log\binom{\kappa-1}{g^*-1}}{\log n} &= \frac{\sum_{j=1}^{g^*-1}(\log(\kappa-j)-\log(j))}{\log n} \\ &< \frac{\log(\kappa-1)}{\log n} + \frac{\log(\kappa-2)}{\log n} + \cdots + \frac{\log(\kappa-1-g^*)}{\log n} \to 0\end{aligned}$$

and therefore the number $Z(g^*,\kappa)$ of strip placements in the H-zones does not contribute to the degree of the corresponding polynomial factor of n. Hence, the degree of the polynomial factor is equal to the degree of the polynomial factor $\binom{n-\kappa^*}{g^*}$ of $G(g^*+1, n-\kappa)$, which is exactly g^*. We wish to consider those terms whose polynomial factor has highest degree, and we know that this occurs when the number g^* of H-zones of the ROSA is equal to the number m of rows. In such ROSAs, we must have a horizontal strip in each and every row, and further there is one and exactly one horizontal strip in each of the H-zones.

Therefore,

$$\begin{aligned} C(m) &= \lim_{n\to\infty}\frac{T(m,n)}{n^m w_m(0)^n} \\ &= \lim_{n\to\infty}\frac{w_m(0)^n\sum_{\kappa^*=0}^{n-m}\binom{n-m}{m}w_m(0)^{-(\kappa^*+m)}Z(m,\kappa^*+m)}{n^m w_m(0)^n} \\ &= \frac{1}{m!}\sum_{\kappa^*=0}^{\infty} w_m(0)^{-(\kappa^*+m)}Z(m,\kappa^*+m) \end{aligned} \tag{2.28}$$

We need only evaluate $Z(m,\kappa)$, the number of ways to arrange horizontal strips of total length κ in a number m of H-zones. We remarked that each row must have a horizontal strip; for any $j = 1, \ldots, m$, let $x_j \geq 1$ denote the length of the horizontal strip in row j (as the rows are indexed from top to bottom). The number of the columns in the corresponding H-zone is exactly x_j, and the placement of vertical strips in each such column has a single obstruction in the j-th row, so, by Equation 2.8, there are $w_m(3^{j-1}) = 1 + \binom{j}{2} + \binom{m+1-j}{2}$ possible ways to place a vertical strip in each column of the H-zone. As the m zones can be permuted without affecting the number of placements, we have

$$\begin{aligned}
Z(m,\kappa) = m! \sum_{\substack{x_1+\cdots+x_m=\kappa \\ x_i \geq 1}} \left\{ \prod_{j=1}^{m} w_m(3^{j-1})^{x_j} \right\} \\
= m! \left(\prod_{j=1}^{m} w_m(3^{j-1}) \right) \sum_{\substack{x_1^*+\cdots+x_m^*=\kappa^* \\ x_i^* \geq 0}} \left\{ \prod_{j=1}^{m} w_m(3^{j-1})^{x_j^*} \right\}
\end{aligned}$$

Therefore, continuing Equation 2.28,

$$\begin{aligned}
C(m) &= \frac{1}{m!} \sum_{\kappa^*=0}^{\infty} w_m(0)^{-(\kappa^*+m)} \left[m! \left(\prod_{j=1}^{m} w_m(3^{j-1}) \right) \sum_{\substack{x_1^*+\cdots+x_m^*=\kappa^* \\ x_i^* \geq 0}} \left\{ \prod_{j=1}^{m} w_m(3^{j-1})^{x_j^*} \right\} \right] \\
&= \sum_{\kappa^*=0}^{\infty} \left(w_m(0)^{-m} \prod_{j=1}^{m} w_m(3^{j-1}) \right) \sum_{\substack{x_1^*+\cdots+x_m^*=\kappa^* \\ x_i^* \geq 0}} \left\{ w_m(0)^{-(x_1^*+\cdots+x_m^*)} \prod_{j=1}^{m} w_m(3^{j-1})^{x_j^*} \right\} \\
&= \sum_{\kappa^*=0}^{\infty} \left(\prod_{j=1}^{m} \frac{w_m(3^{j-1})}{w_m(0)} \right) \sum_{\substack{x_1^*+\cdots+x_m^*=\kappa^* \\ x_i^* \geq 0}} \left\{ \prod_{j=1}^{m} \left[\frac{w_m(3^{j-1})}{w_m(0)} \right]^{x_j^*} \right\} \\
&= \left(\prod_{j=1}^{m} \frac{w_m(3^{j-1})}{w_m(0)} \right) \prod_{j=1}^{m} \sum_{x_j^*=0}^{\infty} \left[\frac{w_m(3^{j-1})}{w_m(0)} \right]^{x_j^*} \\
&= \left(\prod_{j=1}^{m} \frac{w_m(3^{j-1})}{w_m(0)} \right) \prod_{j=1}^{m} \left(1 - \frac{w_m(3^{j-1})}{w_m(0)} \right)^{-1} \\
&= \prod_{j=1}^{m} \frac{w_m(3^{j-1})}{w_m(0) - w_m(3^{j-1})}
\end{aligned}$$

and thus we have a closed form for the coefficient $C(m)$. Carefully writing out the denominators of the factors, we also have

$$\begin{aligned} w_m(0) - w_m(3^{j-1}) &= \frac{1}{2}\{[2+m(m+1)] - [2+j(j-1)+(m+1-j)(m-j)]\} \\ &= \frac{1}{2}\{m(m+1) - j(j-1) - m(m+1) + mj + jm - j(j-1)\} \\ &= mj - j(j-1) = j(m+1-j) \end{aligned}$$

so that we may write[§]

$$C(m) = \prod_{j=1}^{m} \frac{1+\binom{j}{2}+\binom{m+1-j}{2}}{j(m+1-j)} = \frac{1}{(m!)^2}\prod_{j=1}^{m}\left[1+\binom{j}{2}+\binom{m+1-j}{2}\right]$$

and therefore deduce the asymptotic formula

Theorem 2.29.

$$T(m,n) \sim \frac{1}{(m!)^2}\prod_{j=1}^{m}\left[1+\binom{j}{2}+\binom{m+1-j}{2}\right] n^m \left(1+\binom{m+1}{2}\right)^n$$

as $n \to \infty$ for any fixed integer m.

2.5 From Transfer Matrices to the Asymptotic Formula

We conclude this chapter with some remarks about trying to understand the asymptotic nature of the numbers $T(m,n)$ of ROSAs from the algebraic point of view of the transfer matrix method. When one studies a transfer matrix that is diagonalizable, one can determine the number of (closed) walks of length ℓ on the associated digraph as a sum of ℓ-th powers of the eigenvalues of the transfer matrix, and one can thereby enumerate a desired class of combinatorial objects by considering an appropriate collection of such walks. In our case of enumerating ROSAs on an $m \times n$ chessboard for a fixed value of m, our transfer matrix $D_m \cdot A_m$ is unfortunately *not* diagonalizable, as A_1 is not diagonalizable (it is a

[§]Note that each factor in this expression for $C(m)$ is symmetric in j and $m+1-j$. It follows that $C(m)$ is the square of a rational number when m is even, and is very nearly so when m is odd.

Jordan block of size 3), and a Kronecker product is diagonalizable if and only if each factor is diagonalizable (this follows from elementary properties of the Kronecker product, as given in Steeb and Hardy [16]). In this case, the number of walks is given as a sum of products of powers of the eigenvalues and polynomial factors, very much like the observation of Flajolet and Sedgewick given in Equation 2.20; furthermore, the degree of the polynomial factor attached to a power of an eigenvalue λ is equal to one less than the size of a Jordan block corresponding to eigenvalue λ.

We compute the Jordan canonical form for the transfer matrices corresponding to the cases $m = 1$ and $m = 2$:

$$D_1 \cdot A_1 = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_2 \cdot A_2 = \begin{pmatrix} 4 & 4 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that we have

$$K_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

since $(K_2 - I_{2^2})^3 = 0$, so that the Jordan canonical form of K_2 contains a Jordan block of size 3, and hence, as K_2 is a 4×4 matrix, must also have a Jordan block of size 1.

It therefore seems – at least in these small cases – that the Jordan blocks corresponding to the largest eigenvalue arise from Kronecker powers of the Jordan block of size 2. We unfortunately see immediately that this cannot be the case, as the unweighted adjacency matrix A_m comes from Kronecker powers of the Jordan block A_1 of size 3, and Jordan blocks are irreducible in the sense that they cannot be represented as a direct sum of (i.e., a block diagonal matrix whose diagonals are) Jordan blocks of smaller size. Nevertheless, this ansatz accurately describes the behavior of the Jordan blocks with largest eigenvalue, and has been verified by Maple [1] for the cases $m = 3, 4, 5$ using the code given in Appendix B to construct the matrices. If this ansatz could be shown to hold true for general m, then a result of Brualdi [5] with an inductive argument would show that the largest eigenvalue $w_m(0) = 1 + \binom{m+1}{2}$ has a (unique) largest Jordan block of size $m + 1$, so that the degree of polynomial factor attached to $w_m(0)$ is exactly m. More research is required to understand if and why this ansatz holds in general, and also to understand how to recover the coefficient of this largest term (which we believe can be obtained by the similarity matrices).

CHAPTER 3

UNRESTRICTED STRIP ARRANGEMENTS OF FIXED WIDTH

In the study of arrangements of $1 \times k$ strips, we discussed that it is natural from the perspective of statistical physics to consider the class of strip arrangements that restricts the number of horizontal strips in each row and the number of vertical strips in each column to at most one each – these are indeed the ROSAs that we have studying so intently. It may, however, be more natural *a priori* to examine strip arrangements *without* these restrictions. We term these arrangements simply *Un-Restricted Strip Arrangements*, or URSAs, and we turn now to the problem of counting the number $U(m,n)$ of URSAs on an $m \times n$ chessboard. We will mainly follow the strategy and techniques of the previous chapter – that is, we will first fix one dimension of the chessboard (namely the number m of rows, just as in the previous chapter), and we will compute the corresponding generating function $\mathcal{U}_m(x) = \sum_{n \geq 0} U(m,n)x^n$ in the indeterminate x.

3.1 Elementary Results

Let us begin with a computation of the number $U(1,n)$ of URSAs on a $1 \times n$ chessboard for any integer $n \geq 0$:

Theorem 3.1. *The generating function $\mathcal{U}_1(x)$ whose coefficient is the number $U(1,n)$ of URSAs on a $1 \times n$ chessboard is given by*

$$\mathcal{U}_1(x) = \frac{1-x}{1-4x+2x^2} = 1 + 3x + 10x^2 + 34x^3 + \dots$$

Proof. By convention, we have $U(1,0) = 1$. For $n > 1$, we consider the left-most unit square of the chessboard. If it is not covered by a horizontal strip, then it is either empty or contains a vertical strip, and neither of these possibilities interact with the remainder of the chessboard. Thus, the rest of the arrangement is an URSA on a $1 \times (n-1)$ chessboard, and so we have $2U(1, n-1)$ possible URSAs in this case. If the leftmost square is indeed covered by a horizontal strip of some length $\ell \geq 1$, the remainder of the arrangement is an URSA on a $1 \times (n-\ell)$ chessboard, so each length ℓ corresponds to $U(1, n-\ell)$ arrangements. Therefore, for $n \geq 1$,

$$U(1,n) = 2U(1,n-1) + U(1,n-1) + U(1,n-2) + \cdots + U(1,0) \tag{3.2}$$

We form the generating function $\mathcal{U}_1(x)$ by multiplying both sides of Equation 3.2 by x^n and summing over all $n \geq 1$ (this is a common technique for obtaining a generating function from a recurrence relation, and is well-illustrated in Wilf [19]). We recall (also from Wilf) the fact that, given a generating function $A(x)$ for a sequence $\{a_n\}_{n\geq 0}$, the sequence of partial sums $\{\sum_{0\leq n\leq m} a_n\}_{m\geq 0}$ has generating function

$$A_\Sigma(x) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} a_n\right) x^m = \left(\sum_{j=0}^{\infty} a_j x^j\right)\left(\sum_{k=0}^{\infty} x^k\right) = \frac{A(x)}{1-x}$$

Thus, from the recurrence relation of Equation 3.2, we deduce the functional equation

$$\begin{aligned}
\sum_{n=1}^{\infty} U(1,n)x^n &= \sum_{n=1}^{\infty} 2U(1,n-1)x^n + \sum_{n=1}^{\infty}(U(1,n-1) + \cdots + U(1,0))x^n \\
\mathcal{U}_1(x) - 1 &= 2x\,\mathcal{U}_1(x) + \frac{x\,\mathcal{U}_1(x)}{1-x} \\
1 &= \mathcal{U}_1(x)\left(1 - 2x - \frac{x}{1-x}\right) \\
\mathcal{U}_1(x) &= \frac{1}{1-2x-\frac{x}{1-x}} = \frac{1-x}{1-4x+2x^2}
\end{aligned}$$

which completes the proof of the theorem. ♦

A couple of remarks are in order. First, we intended to mimic the style of the previous chapter, where we first obtained the number $T(1,n)$ of ROSAs on a $1 \times n$ chessboard by elementary means, but the technique here appears to be different. In fact, the same technique could be applied to Theorem 2.1, and we would have obtained

$$\mathcal{T}_1(x) - 1 = 2x\,\mathcal{T}_1(x) + \frac{x}{(1-x)(1-2x)}$$

as a functional equation for the generating function $\mathcal{T}_1(x)$ for the numbers $\{T(1,n)\}$ of ROSAs on a $1 \times n$ chessboard. It would follow from this that

$$\mathcal{T}_1(x) = \frac{1}{1-2x} + \frac{x}{(1-x)(1-2x)^2} = \frac{1}{1-x} - \frac{1}{1-2x} + \frac{1}{(1-2x)^2}$$

just as we deduced in Equation 2.16. Secondly, applying a partial fraction decomposition to $\mathcal{U}_1(x)$, we can compute an exact formula for $U(1,n)$ for all $n \geq 0$ as

$$U(1,n) = \frac{1}{4}\left[(2+\sqrt{2})^{n+1} + (2-\sqrt{2})^{n+1}\right] \tag{3.3}$$

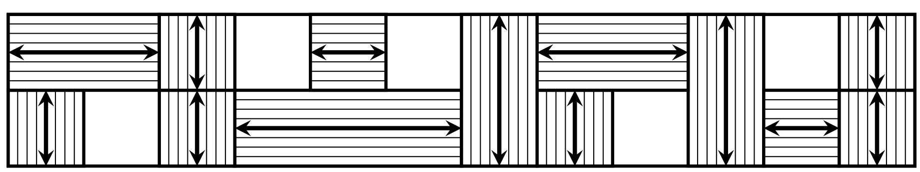
We recall now that we obtained the number $T(2,n)$ of ROSAs on a $2 \times n$ chessboard in Theorem 2.4 by gluing pairs of ROSAs on $1 \times n$ chessboards. Unfortunately, this phenomenon does not hold in general for URSAs on $2 \times n$ chessboards, as not all pairs of unit vertical strips need be glued together to form a vertical strip of length 2. However, let us consider the placement of the left-most vertical strip of length 2 (which we note is the only *new* strip for the $2 \times n$ chessboard). If there is such a long vertical strip located in column j for $j = 0, \ldots, n-1$, we see that the strip arrangement to the left of this vertical strip is indeed a pair of disconnected URSAs on $1 \times j$ chessboards, as by hypothesis no strip here crosses between the rows, and the arrangement to the right of the vertical strip can be considered as an URSA on a $2 \times (n-1-j)$ chessboard (see Figure 3.1). If there is no vertical strip of length 2 anywhere in the URSA, then again no strip crosses between the rows, so the arrangement is equivalent to a pair of disjoint URSAs on $1 \times n$ chessboards.

Therefore, for $n \geq 1$,

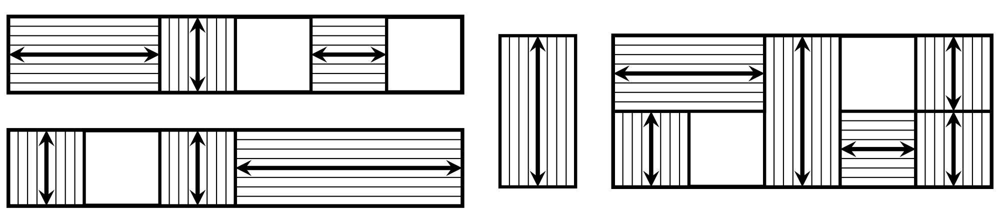
$$U(2,n) = [U(1,n)]^2 + \sum_{j=0}^{n-1} [U(1,j)]^2 U(2, n-1-j) \tag{3.4}$$

As in the proof of Theorem 3.1, we multiply both sides of Equation 3.4 by x^n and sum over all $n \geq 1$ to obtain the functional equation

$$\begin{aligned}
\mathcal{U}_2(x) - 1 &= \sum_{n=1}^{\infty} \left\{ [U(1,n)]^2 + \sum_{j=0}^{n-1} [U(1,j)]^2 U(2,n-1-j) \right\} x^n \\
&= \left(\sum_{n=1}^{\infty} [U(1,n)]^2 x^n \right) + x \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \left\{ [U(1,j)]^2 x^j \right\} \left\{ U(2,n-1-j) x^{n-1-j} \right\} \\
&= \left(\sum_{n=1}^{\infty} [U(1,n)]^2 x^n \right) + x \left\{ \sum_{j=0}^{\infty} [U(1,j)]^2 x^j \right\} \left\{ \sum_{k=0}^{\infty} U(2,k) x^k \right\} \\
&= \mathcal{U}_1^{\square}(x) - 1 + x\, \mathcal{U}_1^{\square}(x) \mathcal{U}_2(x)
\end{aligned}$$



(a)



(b)

Figure 3.1: For a given URSA on a $2 \times n$ chessboard, let $k \in \{0, \ldots, n-1\}$ denote the column containing the left-most vertical strip of length 2 (in this case $n = 12$ and $k = 6$). Then the strip arrangement to the left of the vertical strip comes from a pair of URSAs on $1 \times k$ boards, while the arrangement to the right of the vertical strip is an URSA on a smaller $2 \times (n - (k+1))$ chessboard.

or, equivalently,

$$\mathcal{U}_2(x) = \frac{\mathcal{U}_1^{\square}(x)}{1 - x\,\mathcal{U}_1^{\square}(x)} \tag{3.5}$$

where $\mathcal{U}_1^{\square}(x) = \sum_{n\geq 0}[U(1,n)]^2 x^n$. In general, there is no elementary way to pass from $A(x) = \sum_{n\geq 0} a_n x^n$ to $A^{\square}(x) = \sum_{n\geq 0} a_n^2 x^n$. However, since we determined $U(1,n)$ exactly in Equation 3.3, we can determine exactly

$$\begin{aligned} [U(1,n)]^2 &= \left\{\frac{1}{4}\left[(2+\sqrt{2})^{n+1} + (2-\sqrt{2})^{n+1}\right]\right\}^2 \\ &= \frac{1}{16}\left[(6+4\sqrt{2})^{n+1} + 2^{n+2} + (6-4\sqrt{2})^{n+1}\right] \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{U}_1^{\square}(x) &= \frac{6+4\sqrt{2}}{16}\left(\frac{1}{1-(6+4\sqrt{2})x}\right) + \frac{4}{16}\left(\frac{1}{1-2x}\right) + \frac{6-4\sqrt{2}}{16}\left(\frac{1}{1-(6-4\sqrt{2})x}\right) \\ &= \frac{1}{4}\left[\frac{3-2x}{1-12x+4x^2} + \frac{1}{1-2x}\right] = \frac{1-5x+2x^2}{1-14x+28x^2-8x^3} \end{aligned} \tag{3.6}$$

Substituting the result of Equation 3.6 into Equation 3.5 and simplifying, we conclude that

Theorem 3.7. *The generating function $\mathcal{U}_2(x)$ whose coefficient is the number $U(2,n)$ of URSAs on a $2 \times n$ chessboard is given by*

$$\mathcal{U}_2(x) = \frac{1-5x+2x^2}{1-15x+33x^2-10x^3} = 1 + 10x + 119x^2 + 1465x^3 + 18148x^4 + \ldots$$

We note that, unlike Theorem 3.1, it is unfortunately rather difficult to obtain an exact formula for $U(2,n)$ from Theorem 3.7. However, the rationality of the generating function $\mathcal{U}_2(x)$ gives us a recurrence relation, and therefore we can compute $U(2,n)$ from the formula

$$U(2,n) = 15U(2,n-1) - 33U(2,n-2) + 10U(2,n-3)$$

with $U(2,0) = 1$, $U(2,1) = 10$, and $U(2,2) = 119$.

3.2 The Transfer Matrix Method

As was the case with ROSAs, determining $T(m, n)$ exactly for $m \geq 3$ using the previous methods appears to be quite technical, so we would like to appeal to transfer matrix methods similar to those of Chapter 2. We began the transfer matrix argument for ROSAs by characterizing each unit square in each row as being in one of three states. We will do the same here, but using a slightly modified characterization:

- State $\tilde{0}$: The unit square is not covered by any horizontal strip.
- State $\tilde{1}$: A horizontal strip begins on this unit square. That is, the unit square is covered by a horizontal strip that does *not* cover the square's nearest left neighbor.
- State $\tilde{2}$: A horizontal strip is continued on this unit square. That is, the unit square is covered by a horizontal strip that covers the square's nearest left neighbor.

The transitions between the above states are as follows:

- A unit square in state $\tilde{0}$ can be immediately followed by either a square in state $\tilde{0}$ or a square in state $\tilde{1}$, but not by a square in state $\tilde{2}$. A horizontal strip cannot be continued when it has not yet begun.
- A unit square in state $\tilde{1}$ can be immediately followed by a square in any state.
- A unit square in state $\tilde{2}$ can be immediately followed by a square in any state.

From this, we construct (as before) a corresponding directed graph $\tilde{G}_m$ on the set of vertices $\{\tilde{0}, \tilde{1}, \tilde{2}\}^m$ whose edges are defined by the above rules. As the placement of horizontal strips in the rows are independent of one another, we again have that $\tilde{G}_m$ can be constructed as the m-th Kronecker power of the digraph $\tilde{G}_1$ corresponding to URSAs on a single row chessboard, and thus the unweighted adjacency matrix $\tilde{A}_m$ of $\tilde{G}_m$ is given by the m-th Kronecker power of the unweighted adjacency matrix $\tilde{A}_1$ of $\tilde{G}_1$.

According to the above rules, we may we may construct the unweighted adjacency matrix of $\tilde{G}_1$ as follows:

$$\tilde{A}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.8}$$

and from this we are able to determine the digraphs $\tilde{G}_m$ – and more importantly, the adjacency matrices $\tilde{A}_m$ – for any m. For example, the unweighted adjacency matrix $\tilde{A}_2$ for the digraph G_2 that models URSAs on chessboards with two rows is given by

$$\tilde{A}_2 = \tilde{A}_1 \otimes \tilde{A}_1 = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_1 & 0 \\ \tilde{A}_1 & \tilde{A}_1 & \tilde{A}_1 \\ \tilde{A}_1 & \tilde{A}_1 & \tilde{A}_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.9}$$

We next recall that each vertex $\tilde{u}$ of the digraph $\tilde{G}_m$ (or, equivalently, each edge flowing out of $\tilde{u}$) should be weighted with a value $\tilde{w}_m(\tilde{u})$ equal to the number of ways to place vertical strips in the corresponding column. To determine the weights, we see that we can only place such strips in regions between horizontal strips – that is, on groups of consecutive unit squares in state $\tilde{0}$ – and that the placement of strips in such a region is independent, but not disjoint, from any other region. Thus, the total weight $\tilde{w}_m(\tilde{u})$ is the product of the weights of these regions containing no horizontal strips.

Therefore, we need to compute the number of ways to place vertical strips on a $k \times 1$ chessboard column that contains no horizontal strips. This is surely equal to the number of ways $U^{\varepsilon}(1, k)$ to place horizontal strips on a $1 \times k$ chessboard row that contains no vertical strips. These numbers satisfy a recurrence similar to that of Equation 3.2, with the exception that we now have only one option if the left-most unit square is not covered by a strip. This gives us the recurrence

$$U^{\varepsilon}(1,k) = U^{\varepsilon}(1,k-1) + U^{\varepsilon}(1,k-1) + U(1,k-2) + \cdots + U(1,0)$$

Taking $U^{\varepsilon}(1,0) = 1$ and solving for the generating function $\mathcal{U}_1^{\varepsilon}(x) = \sum_{k\geq 0} U^{\varepsilon}(1,k)x^k$ as in Theorem 3.1, we have

$$\begin{aligned}\mathcal{U}_1^{\varepsilon}(x) = \frac{1}{1-x-\frac{x}{1-x}} &= \frac{1-x}{1-3x+x^2} \\ &= \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)^2 x}\right) - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)^2 x}\right)\end{aligned}$$

and therefore we have explicitly for each $k \geq 0$

$$U^{\varepsilon}(1,k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2k+1}\right]$$

The weight $\tilde{w}_m(\tilde{u})$ of the vertex $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ is thus computed by the formula

$$\begin{aligned}\tilde{w}_m(\tilde{u}) &= \prod_{i=1}^{r} U^{\varepsilon}(1, \tilde{\beta}_i - \tilde{\beta}_{i-1} - 1) \\ &= \prod_{i=1}^{r} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2(\tilde{\beta}_i-\tilde{\beta}_{i-1}-1)+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2(\tilde{\beta}_i-\tilde{\beta}_{i-1}-1)+1}\right] \qquad (3.10)\end{aligned}$$

where, as in Equation 2.8, $\tilde{\beta}_0 = 0$, $\tilde{\beta}_r = m+1$, and $\tilde{\beta}_1, \ldots, \beta_{r-1}$ denote the indices of each unit square not in state $\tilde{0}$ in $\tilde{u}$ in order – that is, $\tilde{u}_{\tilde{\beta}_i} \neq \tilde{0}$ for all $1 \leq i < r \leq m$, $\tilde{\beta}_i < \tilde{\beta}_j$ if $i < j$, and $\tilde{u}_{\tilde{\alpha}} = \tilde{0}$ if $\tilde{\alpha} \neq \tilde{\beta}_i$ for some $i = 1, \ldots, r-1$. With these weights, we construct the diagonal weight matrix $\tilde{D}_m = (\tilde{d}_{i,j})_{0\leq i,j\leq 3^m-1}$ for which $\tilde{d}_{i,j} = 0$ if $i \neq j$, and $\tilde{d}_{i,i} = \tilde{w}_m(i)$, again using the correspondence between integers $i = 0, \ldots, 3^m - 1$ and m-tuples in $\{\tilde{0}, \tilde{1}, \tilde{2}\}^m$ given by the representation of i in base 3.

In the same way we did for ROSAs, we now use the matrices $\tilde{A}_m$ and $\tilde{D}_m$ to construct the matrix-valued power series $\tilde{\mathcal{P}}(x)$ whose $(\tilde{u}, \tilde{v})$ entry enumerates URSAs with m rows that begin with column $\tilde{u}$ and end with column $\tilde{v}$:

$$\begin{aligned}\tilde{\mathcal{P}}(x) &= I_{3^m} + x\tilde{D}_m + x^2\tilde{D}_m \cdot \tilde{A}_m \cdot \tilde{D}_m + x^3(\tilde{D}_m \cdot \tilde{A}_m)^2 \cdot \tilde{D}_m \\ &\qquad + \cdots + x^n(\tilde{D}_m \cdot \tilde{A}_m)^{n-1} \cdot \tilde{D}_m + \ldots \\ &= I_{3^m} + (I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m)^{-1} \cdot (x\tilde{D}_m)\end{aligned}$$

We finally compute the generating functions $\mathcal{U}_m(x)$ enumerating all URSAs with m rows. For ROSAs, we noted that the possible initial columns were contained in the unit hypercube $\{0,1\}^m$, all of which were connected to the single vertex $0 = (0, \ldots, 0)$. Similarly, in the possible initial columns for URSAs, the unit squares must either not lie under a horizontal strip or must begin a horizontal strip, so they must lie in the unit hypercube $\{\tilde{0}, \tilde{1}\}^m$. The vertex $\tilde{0} = (\tilde{0}, \ldots, \tilde{0})$ connects to each of these vertices and *only* these vertices, so we may deduce from Theorem 2.11

$$\tilde{\mathcal{F}}_{\tilde{0}}(x) = 1 + \tilde{w}_m(\tilde{0})x \sum_{\tilde{v} \in \{\tilde{0}, \tilde{1}\}^m} \tilde{\mathcal{F}}_{\tilde{v}}(x) = 1 + \tilde{w}_m(\tilde{0})x\mathcal{U}(x) \tag{3.11}$$

where $\tilde{\mathcal{F}}_{\tilde{v}}(x)$ is the generating function enumerating URSAs with $\tilde{v}$ as an initial column. But we know that $\tilde{\mathcal{F}}_{\tilde{v}}(x)$ is equal to the sum of all entries of the matrix $\tilde{\mathcal{P}(x)}$ in the row corresponding to $\tilde{v}$; therefore,

$$\begin{aligned}\tilde{\mathcal{F}}_{\tilde{0}}(x) &= e_0 \cdot [I_{3^m} + (I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m)^{-1} \cdot (x\tilde{D}_m)] \cdot \sigma^t \\ &= e_0 \cdot I_{3^m} \cdot \sigma^t + e_0 \cdot [(I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m)^{-1} \cdot (x\tilde{D}_m)] \cdot \sigma^t \\ &= 1 + x\left\{e_0 \cdot [(I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m)^{-1} \cdot \tilde{D}_m] \cdot \sigma^t\right\}\end{aligned}$$

and so, by Equation 3.11, we have

Theorem 3.12. *If we define* $\tilde{w}_m(0) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2m+1}\right]$, *then*

$$\mathcal{U}_m(x) = \left(\frac{1}{\tilde{w}_m(0)}\right) e_0 \cdot [(I_{3^m} - x\tilde{D}_m\tilde{A}_m)^{-1} \cdot \tilde{D}_m] \cdot (e_0 + \cdots + e_{3^m-1})^t$$

3.3 Examples of the Transfer Matrix Method

We illustrate the application of Theorem 3.12 by recovering our previous results. First, we take the case $m = 1$. The unweighted adjacency matrix for the corresponding digraph G_1 is given as in Equation 3.8. For the corresponding weights, we see that there is only one possible way to add a vertical strip to a column in state $\tilde{1}$ or in state $\tilde{2}$ (by adding nothing, since the column is already completely covered by a horizontal strip), and there are two possible ways to add a vertical strip to a column in state $\tilde{0}$ (by either adding a strip or not). Therefore, $\tilde{w}_m(\tilde{0}) = 2$ and $\tilde{w}_m(\tilde{1}) = \tilde{w}_m(\tilde{2}) = 1$, so the weight matrix is given by

$$\tilde{D}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can compute by hand that

$$\begin{aligned}
(I_3 - x\tilde{D}_1 \cdot \tilde{A}_1)^{-1} \cdot \tilde{D}_1 &= \begin{pmatrix} 1-2x & -2x & 0 \\ -x & 1-x & -x \\ -x & -x & 1-x \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{1-4x+2x^2} \begin{pmatrix} 2(1-2x) & 2x(1-x) & 2x^2 \\ 2x & (1-x)(1-2x) & x(1-2x) \\ 2x & x & 1-3x \end{pmatrix}
\end{aligned}$$

By Theorem 3.12, the generating function that enumerates URSAs on a chessboard with one row is therefore given by

$$\mathcal{U}_1(x) = \frac{1}{2}\left(\frac{2(1-2x) + 2x(1-x) + 2x^2}{1-4x+2x^2}\right) = \frac{1-x}{1-4x+2x^2}$$

just as we determined in Theorem 3.1.

We now take the case $m = 2$. By the Kronecker product structure of the digraph $\tilde{G}_2 = \tilde{G}_1 \otimes \tilde{G}_1$, its unweighted adjacency matrix is given by $\tilde{A}_2 = \tilde{A}_1 \otimes \tilde{A}_1$, and is explicitly written as a matrix in Equation 3.9. By Equation 2.8, the weight matrix is given by

$$\tilde{D}_2 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We therefore have the matrix $I_9 - x\tilde{D}_2 \cdot \tilde{A}_2$ given by

$$\begin{pmatrix} 1-5x & -5x & 0 & -5x & -5x & 0 & 0 & 0 & 0 \\ -2x & 1-2x & -2x & -2x & -2x & -2x & 0 & 0 & 0 \\ -2x & -2x & 1-2x & -2x & -2x & -2x & 0 & 0 & 0 \\ -2x & -2x & 0 & 1-2x & -2x & 0 & -2x & -2x & 0 \\ -x & -x & -x & -x & 1-x & -x & -x & -x & -x \\ -x & -x & -x & -x & -x & 1-x & -x & -x & -x \\ -2x & -2x & 0 & -2x & -2x & 0 & 1-2x & -2x & 0 \\ -x & -x & -x & -x & -x & -x & -x & 1-x & -x \\ -x & -x & -x & -x & -x & -x & -x & -x & 1-x \end{pmatrix}$$

To compute the generating function $\mathcal{U}_2(x)$ using Theorem 3.12, we can explicitly compute the inverse of the matrix $I_9 - x\tilde{D}_m \cdot \tilde{A}_m$, multiply this inverse on the right by $\tilde{D}_m$, sum the entries of the zeroth row in the resulting matrix, and divide by $\tilde{w}_m(\tilde{0})$. Alternatively, we can make use of the a cofactor formula similar to that of Theorem 2.19 using URSA weights $\tilde{w}_m(j)$ instead of ROSA weights $w_m(j)$ – that is,

Theorem 3.13. *The generating function $\mathcal{U}_m(x)$ for which the coefficient of x^n is the number $U(m,n)$ of URSAs on an $m \times n$ chessboard is given by*

$$\mathcal{U}_m(x) = \sum_{j=0}^{3^m-1} \frac{(-1)^j \tilde{w}_m(j) \det(I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m; j, 0)}{\tilde{w}_m(0) \det(I_{3^m} - x\tilde{D}_m \cdot \tilde{A}_m)}$$

Taking $m = 2$ in Theorem 3.13, we see that the denominator of the generating function $\mathcal{U}_2(x)$ is given by $\det(I_9 - x\tilde{D}_2 \cdot \tilde{A}_2) = 1 - 15x + 33x^2 - 10x^3$, and the numerator is given by

$$\begin{aligned}
&\tfrac{1}{5}\big\{5(1-10x+8x^2) + 10x(1-5x+2x^2) + 10x^2(3-2x) + 10x(1-5x+2x^2) \\
&\qquad +5x(1-5x+2x^2) + 5x^2(3-2x) + 10x^2(2x-3) + 5x^2(3-2x) + 5x^2(1+2x)\big\} \\
&\quad = (1-10x+8x^2) + (5x-25x^2+10x^3) + (18x^2-12x^3) + (x^2+2x^3) \\
&\quad = 1 - 5x + 2x^2
\end{aligned}$$

Hence, $\mathcal{U}_2(x) = (1-5x+2x^2)/(1-15x+33x^2-10x^3)$, as we deduced in Theorem 3.7.

Using Theorem 3.12 in conjunction with Maple [1], (see Appendix B for the corresponding code), we are able to give generating functions for $U(m,n)$ for $m = 3$ and $m = 4$; we list these as follows.

Theorem 3.14. *The generating function $\mathcal{U}_3(x)$ whose coefficient is the number $U(3,n)$ of URSAs on a $3 \times n$ chessboard is given by*

$$\mathcal{U}_3(x) = \frac{1 - 30x + 233x^2 - 562x^3 + 456x^4 - 80x^5}{1 - 64x + 944x^2 - 5053x^3 + 9562x^4 - 6568x^5 + 1040x^6}$$
$$= 1 + 34x + 1465x^2 + 66155x^3 + 3013656x^4$$
$$+ 137507689x^5 + 6276095989x^6 + 286467901298x^7 + \dots$$

Theorem 3.15. *The generating function $\mathcal{U}_4(x)$ whose coefficient is the number $U(4,n)$ of URSAs on a $4 \times n$ chessboard is given by $\mathcal{U}_4(x) = \mathcal{P}_4(x)/\mathcal{Q}_4(x)$, where*

$$\begin{aligned}
\mathcal{P}_4(x) &= 1 - 132x + 5046x^2 - 78944x^3 + 610200x^4 \\
&\quad -2488088x^5 + 5257512x^6 - 5220480x^7 - 2116000x^8 + 208000x^9 \\
\mathcal{Q}_4(x) &= 1 - 248x + 15666x^2 - 409152x^3 + 5040016x^4 - 32793024x^5 \\
&\quad +116941744x^6 - 221424048x^7 + 200277120x^8 - 75272000x^9 + 7072000x^{10}
\end{aligned}$$

CHAPTER 4

RESTRICTED STRIP ARRANGEMENTS ON SQUARE CHESSBOARDS

We turn our attention now to the number of ROSAs on large chessboards. The methods of the previous chapter can theoretically be used for such cases, but the large matrices produced can become unwieldy rather quickly. We thus appeal to other more analytical methods, and we correspondingly adjust our aim slightly by attempting to give a *logarithmic estimate* for the number of ROSAs as the sides of the ambient chessboard grow arbitrarily large. More specifically, we note that, in problems of enumeration on a square lattice such as our problem with ROSAs and the classical monomer-dimer problem, statistical physicists have a particular interest in the case when the ambient region is itself square. This is in part because, as noted by Flajolet and Sedgewick [8], it is often not clear how to determine the behavior of the "diagonal" sequence in the corresponding multi-dimensional array (unless one already knows a general formula for all cases). Therefore, in this chapter, we will narrow our focus to counting the number of ROSAs on *square* chessboards whose sides have equal side length L, and we will analyze how these numbers grow as $L \to \infty$.

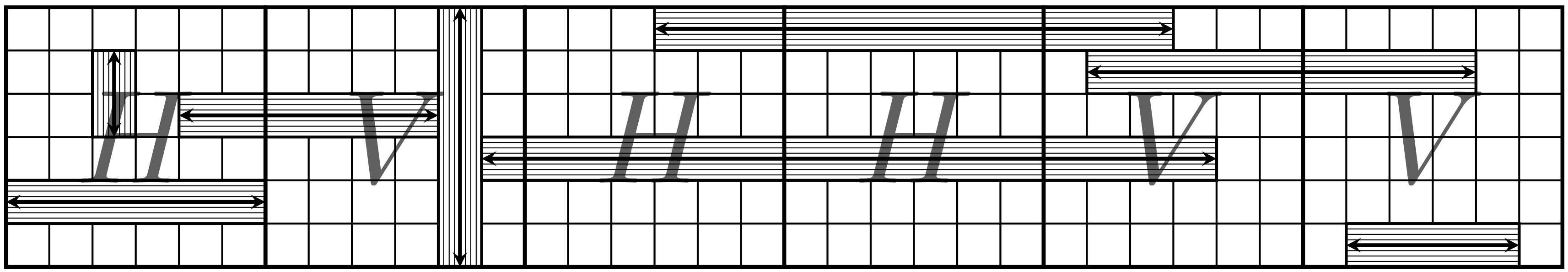
4.1 Renormalization and Templating

Viewing ROSAs as a model of statistical mechanics, there is a technique for dealing with models on a large surface by viewing them macroscopically and *renormalizing* the scale (such techniques are discussed in detail by Cardy [6]). In practical terms, we plan to subdivide our large chessboard of side length L into square cells of length $s < L$, ideally with s evenly dividing L; the idea is to reduce the problem for a large $L \times L$ chessboard of unit squares to a smaller $N \times N$ grid of cells, where $N = L/s$. This renormalization program was successfully carried out by Larsen [9], who was able to relate the placements of non-attacking kings on a chessboard to placements of non-interacting monomers in the established hard hexagon model of statistical physics.

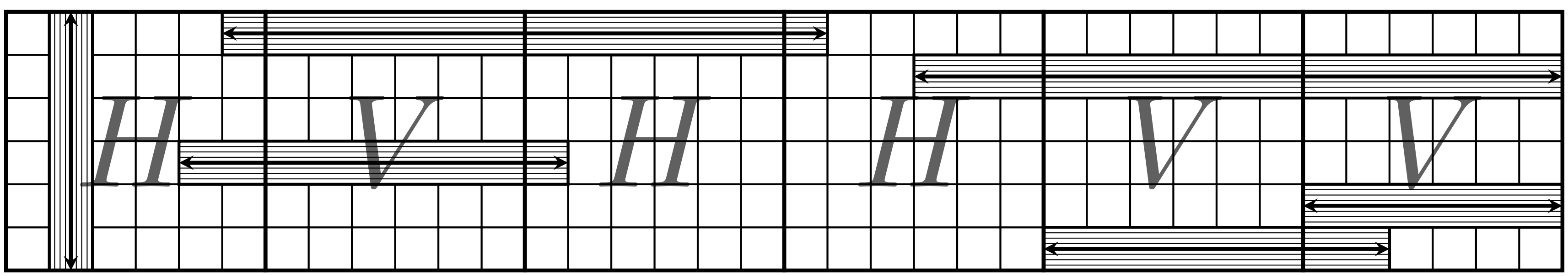
We label each of our N^2 cells as being of one of three types.

- An H-*cell* is a cell in which a horizontal strip can transit the cell – i.e., a horizontal strip is allowed to stretch across the width of the given H-cell and overlap with one or both of its horizontally neighboring cells.
- Similarly, a V-*cell* is a cell in which a vertical strip can transit the cell – i.e., a vertical strip is allowed to stretch across the height of the given V-cell and overlap with one or both of its vertically neighboring cells.
- An O-*cell* is a cell in which *neither* a horizontal strip nor a vertical strip can transit the cell – i.e., any strip that intersects an O-cell can intersect *at most one* of its horizontal or vertical nearest neighbors.

We note that, due to the condition that no two strips overlap, no H-cell can be a V-cell (since this would disallow any horizontal strip to transit the cell), and vice versa. In particular, no cell can be *both* an H-cell and a V-cell, so the above list is a complete characterization of cells, and we refer to a labeling of all $N \times N$ cells according to this list as a *template* of the chessboard. We also note that it is allowed for a horizontal strip in a V-cell to overlap with at most one of its horizontal neighbors, but it cannot overlap with both of them, for this would, as in the previous note, prohibit vertical strips from transiting the V-cell (and similar remarks hold for vertical strips in H-cells). In Figure 4.1(a), we illustrate some horizontal strips that are compatible with the given template, and we illustrate some incompatible strips in Figure 4.1(b).



(a) An illustration of allowed *strip placements on a given template.*



(b) An illustration of disallowed *strip placements on a given template.*

Figure 4.1

4.2 Placing Horizontal Strips on a Given Template

4.2.1 *A Single Row of the Template*

In the previous chapter, we analyzed ROSAs by constructing them one row (or column) of unit squares at a time; we will adopt a similar philosophy here by looking first at a single row of cells in a template, and thus, for the time being, we will focus on the placement of horizontal strips. In this respect, we note that a horizontal strip sees no difference between a V-cell and an O-cell – it cannot span the length of either. Therefore, as we analyze horizontal strips, we will simplify matters by labeling all O-cells as V-cells.

Let $\{a_0, a_1, \dots\}$ and $\{b_0, b_1, \dots\}$ be sequences of non-negative integers with the properties that finitely many terms of each sequence are non-zero, that $b_j = 0$ implies $a_{j+1} = 0$ for all j, and that $a_j = 0$ implies $b_j = 0$ for all $j \neq 0$. We say that this pair of sequences corresponds to a row of the template if that row consists of a_0 H-cells in the beginning (or the left-most position), followed by b_0 V-cells, followed by a_1 H-cells, followed by b_1 V-cells, etc. Define $A = \sum a_j$ to be the number of H-cells in the row of the template, and define $B = \sum b_j$ to be the number of V-cells (so that $A + B = N$ is the total number of cells in the row of the template).

Theorem 4.1. *If a row of a template of a ROSA corresponds to the sequences $\{a_0, a_1, \dots\}$ and $\{b_0, b_1, \dots\}$, then the number of ways to place a horizontal strip on a row of the ROSA that lies within this row of cells in the template is given by*

$$\begin{cases} 1 + \binom{Ns+1}{2} & B = 0 \\ s + \sum \binom{a_i s + 1}{2} + \binom{s}{2}[3N + A - 2(a_0 + a_k + 1)] & B \neq 0 \end{cases} \tag{4.2}$$

where k is the minimum value for which $b_k = 0$ (i.e., there are k contiguous groups of V-cells in the row).

Proof. A non-empty strip must exhibit one of the following behaviors:

1. *The strip is contained in a group of a_i consecutive H-cells.*

   In this case, the strip can transit any cell in this group. Thus, the number of strips in this case is the number of strips over $a_i s$ spaces – this is precisely $\binom{a_i s+1}{2}$.

2. It is contained in a group of b_j consecutive V-cells.

   In this case, the strip *cannot* transit over any cell in this group. Thus, either the strip is entirely contained in a single cell, and there are

   $$(s-1)+(s-1)+(s-2)+\cdots+2+1=(s-1)[1+s/2]$$

   possibilities in this case, or the strip straddles two consecutive cells (while avoiding the remaining vertical edge of each cell), and hence there are $(s-1)^2$ possibilities in this case. There are b_j single cells in this group and b_j-1 pairs of consecutive cells, which gives the total number of possible non-empty strips as

   $$b_j(s-1)(1+s/2)+(b_j-1)(s-1)^2=3b_j\binom{s}{2}-(s-1)^2$$

3. *The strip straddles a border between a group of a_i H-cells and a group of b_i V-cells.*

   In this case, the strip can transit through any H-cell in the first group, and one end of the cell must be contained in a single V-cell (but the strip must not transit the cell). Thus, we have sa_i choices for the end of the strip in the region of H-cells, and $s-1$ choices for the end of the strip in the region of V-cells, giving a total of $a_i s(s-1)$ possible strips in this case.

4. *The strip straddles a border between a group of b_i V-cells and a group of a_{i+1} H-cells.*

   This is the basically the same as the previous case, and we similarly have $a_{i+1}s(s-1)$ possible strips in this case.

5. *The strip stretches across an entire group of a_m H-cells and has its ends in a group of b_{m-1} V-cells and in a group of b_m V-cells.*

   The ends of this strip must be contained in a single V-cell to the left of the H-cells (for which there are $s-1$ choices) and in a single V-cell to the right of the H-cells (for which there are again $s-1$ choices). Thus, given any group of consecutive H-cells, there are $(s-1)^2$ non-empty strips that stretch across the group of H-cells and have their ends in the surrounding groups of V-cells.

If $B = 0$, then $a_1 = A = N$, so every strip exhibits the first behavior above, and hence (now counting the empty strip) there are $1 + \binom{As+1}{2} = 1 + \binom{Ns+1}{2}$ possible strips. Otherwise, we must sum over all possible cases. In what follows, let $k = \min\{x : b_x = 0\}$ be the number of contiguous groups of V-cells.

1. The contribution of those strips exhibiting the first behavior is simply $\sum \binom{a_i s+1}{2}$.
2. The contribution of those strips exhibiting the second behavior is

$$\sum_{j=0}^{k-1} \left[3b_j \binom{s}{2} - (s-1)^2 \right] = 3B\binom{s}{2} - k(s-1)^2$$

3 or 4. Each group of H-cells contributes exactly one term to the sum of the strips exhibiting the third behavior (except the last, or the k-th), and each group of H-cells contributes exactly one term to the sum of the strips exhibiting the fourth behavior (except the zeroth). As these behaviors are mutually exclusive, the total number of strips that exhibit either behavior is

$$2\sum(a_i s(s-1)) - a_0 s(s-1) - a_k s(s-1) = \binom{s}{2}[4A - 2a_0 - 2a_k]$$

5. Finally, for the number of strips that exhibit the last behavior, there is a summand of $(s-1)^2$ for each group of H-cells that is sandwiched between two groups of V-cells – this corresponds to the number of consecutive pairs of groups of consecutive V-cells, which is $k-1$ if $k > 0$ and 0 if $k = 0$. (Note that this is why we need to consider the case $k = 0$ separately.)

Therefore, the total number of strips (including the empty strip) that are compatible with the given template for the row is

$$
\begin{aligned}
1+\sum\binom{a_i s+1}{2}+3B\binom{s}{2}-k(s-1)^2+\binom{s}{2}[4A-2a_0-2a_k]+(k-1)(s-1)^2 \\
=1+\sum\binom{a_i s+1}{2}+\binom{s}{2}[4A+3B-2a_0-2a_k]-(s-1)^2 \\
=s+\sum\binom{a_i s+1}{2}+\binom{s}{2}[3N+A-2a_0-2a_k-2]
\end{aligned}
$$

This completes the proof of the theorem ♦

Theorem 4.1 gives exactly computable expressions for the number of ways to place a horizontal strip in a single row of a chessboard with a given template. However, it is made somewhat unwieldy by the presence of a variable number of parameters, so we attempt to simplify matters by taking an upper bound. The sum of binomial coefficents admits a nice simplification with the following observation.

Lemma 4.3. *For any finite collection $x_0, \ldots, x_r$ of non-negative integers,*

$$\binom{x_0+1}{2}+\cdots+\binom{x_r+1}{2}\leq\binom{x_1+\cdots+x_r+1}{2}$$

This can be proven algebraically by an inductive argument with the inequality

$$
\begin{aligned}
\binom{x_0+1}{2}+\binom{x_1+1}{2} &= \frac{x_0^2+x_0}{2}+\frac{x_1^2+x_1}{2} \\
&\leq \frac{x_0^2+2x_0x_1+x_1^2+x_0+x_1}{2} \\
&\leq \frac{(x_0+x_1)^2+(x_0+x_1)}{2} \\
&\leq \binom{x_0+x_1+1}{2}
\end{aligned}
$$

However, Lemma 4.3 also follows from a combinatorial argument that is particularly germane to our problem. Indeed, we may regard the right hand side of Lemma 4.3 as the number of unordered pairs of elements belonging to a totally ordered set X of size $x_1 + \cdots + x_r$, and we may regard the left-hand side as the number of such unordered pairs whose elements belong to the same equivalence class X_i of X of size x_i for some $i = 1, \ldots, r$.

In particular, if we take $x_i = a_i s$, Lemma 4.3 states algebraically that the sum of the binomial coefficients $\sum \binom{a_i s+1}{2}$ is dominated by the single binomial coefficient $\binom{As+1}{2}$. The combinatorial interpretation of this statement is interesting. Let X be the set of all vertical lattice lines that intersect an H-cell (even on the edge of the cell) in the desired row of the template, and order the elements of X lexicographically as they appear from left to right on the chessboard; then, for $i = 0, \ldots, k$, let X_i be the subset of X whose elements intersect the i-th contiguous group of H-cells (that appears before the i-th group of V-cells). Notice that selecting two elements of X is equivalent to selecting the endpoints of a horizontal strip on the chessboard; therefore, the combinatorial interpretation of Lemma 4.3 with $x_i = a_i s$ is that the number of ways to place a horizontal strip on a ROSA that is entirely contained in a group of H-cells on a template is maximized when there is a single group of contiguous H-cells in the corresponding row of the template.

With a single group of contiguous H-cells, it follows that this optimal row has at most two groups of contiguous V-cells. Thus, either we have $k = 0$, in which the group of H-cells stretches the width of the template; we have $k = 1$, in which the group of H-cells is on one side of the template; or we have $k = 2$, in which the group of H-cells occupies a place in the middle of the row away from the endpoints. From Theorem 4.1 and Lemma 4.3, we can compute the maximal number of ways to place a horizontal strip on a single row of a ROSA that is contained in a row of an $N \times N$ template that contains a given number A of H-cells in each case; we have organized the results in Table 4.2.

$k = 0$	$T_{full} = 1 + \binom{Ns+1}{2} = \left(\frac{s^2}{2}\right) N^2 + \left(\frac{s}{2}\right) N + 1$
$k = 1$	$T_{side}(A) = s + \binom{As+1}{2} + \binom{s}{2}[3N - A - 2]$ $= \frac{s^2}{2}\left(A^2 - \left(\frac{s-2}{s}\right) A + \frac{(s-1)(3N-2)+2}{s}\right)$
$k = 2$	$T_{mid}(A) = s + \binom{As+1}{2} + \binom{s}{2}[3N + A - 2]$ $= \frac{s^2}{2}\left(A^2 + A + \frac{(s-1)(3N-2)+2}{s}\right)$

Table 4.2: The maximum number of ways to place a horizontal strip in a row of a template with a given number A of H-cells and a given number k of contiguous groups of V-cells. Note that, in the case when $k = 0$, every cell in the row is an H-cell, and therefore A is not variable – it must *be equal to the total number N of cells in the row.*

Let $T^H_{max}(A)$ be the maximum number of ways to place a horizontal strip in a row of the ROSA whose corresponding row in a compatible template has a number A of H-cells. Directly comparing the expressions in the above table as polynomials in A (for fixed values of N and s), the expression that gives the largest value is the one corresponding to the case $k = 2$, and therefore we can confidently assert that $T_{mid}(A)$ is an upper bound for $T^H_{max}(A)$. It may, however, be appropriate in the sequel to replace T_{mid} with a sharper bound involving either the expression T_{full} or the polynomial T_{side}, and so we will keep them in mind as we obtain results for T_{mid}.

For a quick remark, we note that T_{mid} and T_{side}, considered as polynomials of A, have the same "constant term" – i.e., the same output value for the input value $A = 0$ – and the same leading coefficient of A, as abstract expressions of N and s. The ratio of this "constant term" to the leading coefficient comes up rather frequently in the sequel, so, to ease the discussion, we will henceforth define

$$K = \frac{s + \binom{s}{2}[3N-2]}{s^2/2} = \frac{(s-1)(3N-2)+2}{s} = 3N - 2 - \left(\frac{3N-4}{s}\right)$$

We also remark that K is "well-behaved" in the sense that it is bounded for a given N; indeed, as $s \geq 1$, we have $2 \leq K \leq 3N - 2 < 3N$, as K is a decreasing function of s.

4.2.2 *Multiple Rows of the Template: Inequalities of Convex Analysis*

Now let us consider all N rows of the template together. For a given template, let $\mathcal{T}^H$ be the maximum number of ways to place at most one horizontal strip in each row of the ROSA that is compatible with that template. Surely the placement of horizontal strips in the distinct rows of the template are independent of one another. Furthermore, each row of the template corresponds to s rows of the ROSA, so, if $A_1, A_2, \ldots, A_N$ denote the numbers of H-cells in each of the rows of the template, numbered in order from top to bottom, then

$$\begin{aligned} \mathcal{T}^H &= [T^H_{max}(A_1)]^s\,[T^H_{max}(A_2)]^s\;\ldots\;[T^H_{max}(A_N)]^s \\ &\leq [T_{mid}(A_1)]^s\,[T_{mid}(A_2)]^s\;\ldots\;[T_{mid}(A_n)]^s \end{aligned} \tag{4.4}$$

or equivalently $\log(\mathcal{T}^H) \leq s\log[T_{mid}(A_1)] + \cdots + s\log[T_{mid}(A_N)]$, where log here and throughout this chapter denotes the natural logarithm. We could easily find an upper bound for $\mathcal{T}^H$ if T_{mid} were log-concave (i.e., if the composition $\log \circ T_{mid}$ were concave), for we could maximize the right-hand side of Inequality 4.4 by replacing each A_i with their collective average. Thus, we attempt to determine if T_{mid} is indeed a log-concave function. We begin by establishing a simple criterion* for log-concavity that parallels the definition of log-concave sequences.

*Boyd and Vandenberghe [4] present a generalization of Lemma 4.5 to functions of multiple variables without proof. The proof of Lemma 4.5 that we have given can easily be adapted to this generalization.

Lemma 4.5. *A twice-differentiable function $f(x) : \mathbb{R} \to \mathbb{R}$ is log-concave if and only if*

$$f(x)f''(x) < (f'(x))^2$$

Proof. $f(x)$ is log-concave if and only if

$$0 > \frac{d^2(\log f(x))}{dx^2} = \frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = \frac{f(x)f''(x) - (f'(x))^2}{(f(x))^2}$$

As the denominator is always positive, this holds if and only if $0 > f(x)f''(x) - (f'(x))^2$. ♦

Applying Lemma 4.5 to quadratic functions gives

Theorem 4.6. *Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with $a > 0$. Then f is log-concave in a neighborhood of all x that satisfy*

$$x > \frac{-b + \sqrt{|b^2 - 4ac|}}{2a} \qquad \textit{or} \qquad x < \frac{-b - \sqrt{|b^2 - 4ac|}}{2a}$$

Proof. We have $f'(x) = 2ax + b$ and $f''(x) = 2a$; thus, by the previous lemma, f is log-concave if and only if

$$\begin{aligned} 2a(ax^2 + bx + c) &< (2ax + b)^2 \\ 0 &< 2a^2x^2 + 2abx + (b^2 - 2ac) \end{aligned} \tag{4.7}$$

The right-hand side of Inequality 4.7 is another quadratic polynomial with discriminant $D = (2ab)^2 - 4(2a^2)(b^2 - 2ac) = -4a^2(b^2 - 4ac)$. If $D < 0$ (which occurs if and only if $b^2 - 4ac > 0$), then Inequality 4.7 holds for all x in the domain of $\log f(x)$ – that is, for all x with $f(x) > 0$. By the quadratic formula applied to f and the fact that $a > 0$, we have $f(x) > 0$ if and only if

$$x > \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x < \frac{-b - \sqrt{b^2 - 4a}}{2a}$$

Otherwise, if $D \geq 0$ (or, equivalently, $b^2 - 4ac \leq 0$), it follows from the quadratic formula (and the fact that $2a^2 > 0$) that Inequality 4.7 is satisfied by all x such that

$$x > \frac{-2ab + \sqrt{4a^2b^2 - 4(2a^2)(b^2 - 2ac)}}{2(2a^2)} = \frac{-b + \sqrt{4ac - b^2}}{2a} \tag{4.8}$$

or by all x such that

$$x < \frac{-2ab - \sqrt{4a^2b^2 - 4(2a^2)(b^2 - 2ac)}}{2(2a^2)} = \frac{-b - \sqrt{4ac - b^2}}{2a} \tag{4.9}$$

This covers all cases, and thus completes the proof of the theorem. ♦

We are mainly interested in the behavior of the quadratic polynomials T_{side} and T_{mid} for non-negative input. With this in mind, we easily deduce from Theorem 4.6 the following:

Corollary 4.10.

1. *$T_{mid}(A)$ is log-concave for all $A > A_{mid} = \frac{1}{2}(-1 + \sqrt{4K - 1})$.*
2. *$T_{side}(A)$ is log-concave for all*

$$A > A_{side} = \frac{(s-2) + \sqrt{4Ks^2 - (s-2)^2}}{2s}$$

In particular, $T_{side}(A)$ is log-concave for all $A > \frac{1}{2}(1 + \sqrt{4K - 1})$.

Proof. Part (a) follows directly from Theorem 4.6. The first claim of part (b) also follows from Theorem 4.6 – we note that A_{side} is well-defined as a real number since

$$\left(\frac{s-2}{2s}\right)^2 < \left(\frac{1}{2}\right)^2 < 2 < K$$

The second claim in part (b) follows from the fact that the expression on the right hand side of Inequality 4.8 is a decreasing function of b for $b > -\sqrt{2ac}$. In the case of $T_{side}/(s^2/2)$, whose leading coefficient is 1 and whose constant term is K, we have

$$-\left(\frac{s-2}{s}\right) > -1 > -2 = -\sqrt{2(2)} > -\sqrt{2K}$$

and therefore the input value of the inflection point for $b = -1$ is greater than the corresponding value for $b = -(s-2)/2$. ♦

In particular, the quadratic polynomial T_{mid} is *not* log-concave everywhere (and indeed neither is T_{side}), so we cannot directly use Jensen's inequality. However, we do have a partial result, the intuition for which is illustrated in Figure 4.3.

Theorem 4.11. *Let $f(x)$ be convex on $(0, I)$ and concave on (I, M). Let $x_1, \ldots, x_P \in [0, I)$, and let $x_{P+1}, \ldots, x_N \in [I, M]$. Then there exists $z \in [0, N)$ such that*

$$f(x_1) + \cdots + f(x_N) \leq zf(0) + (N-z)f\left(\frac{x_1 + \cdots + x_N}{N-z}\right)$$

Proof. For each $j = 1, \ldots, P$, we have, by the definition of convexity,

$$f(x_j) = f\left(\left(\frac{x_i}{I}\right)I + \left(1 - \frac{x_i}{I}\right)0\right) \leq \left(\frac{x_i}{I}\right)f(I) + \left(1 - \frac{x_i}{I}\right)f(0)$$

Therefore, we have

$$\begin{aligned} f(x_1) + \cdots + f(x_P) &\leq \left(\frac{x_1 + \cdots + x_P}{I}\right)f(I) + \left(P - \frac{x_1 + \cdots + x_P}{I}\right)f(0) \\ &\leq zf(0) + (P-z)f(I) \end{aligned}$$

where $z = P - (x_1 + \cdots + x_P)/I$ is the coefficient of $f(0)$. Note that this definition of z gives $x_1 + \cdots + x_P = (P-z)I$.

Furthermore, by Jensen's inequality and the concavity of f on (I, M), we have

$$f(x_{P+1}) + \cdots + f(x_N) \leq (N-P)f\left(\frac{x_{P+1} + \cdots + x_N}{N-P}\right)$$

Combining these results with one more application of Jensen's inequality gives

$$\begin{aligned} f(x_1) + \cdots + f(x_N) &\leq zf(0) + (P-z)f(I) + (N-P)f\left(\frac{x_{P+1} + \cdots + x_N}{N-P}\right) \\ &\leq zf(0) \\ &\quad + ((P-z) + (N-P))f\left(\frac{(P-z)I + (N-P)\left(\frac{x_{P+1}+\cdots+x_N}{N-P}\right)}{(P-z)+(N-P)}\right) \\ &\leq zf(0) + (N-z)f\left(\frac{x_1 + \cdots + x_N}{N-z}\right) \end{aligned}$$

This completes the proof of the theorem. ♦

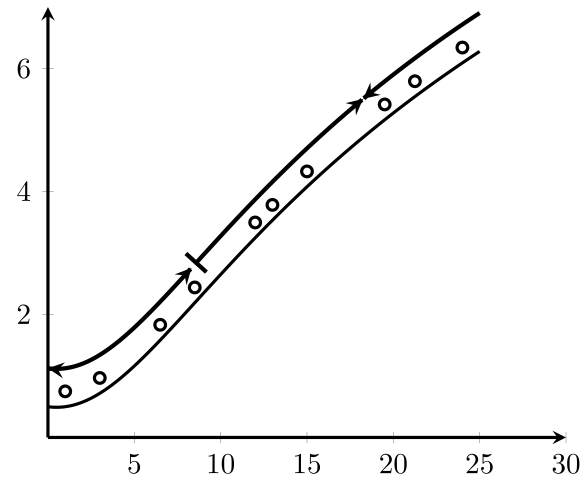


Figure 4.3: Intuition for Theorem 4.11. Informally, we think of the given values $x_1, \ldots, x_N$ as particles or balls on a hillside. Then, to obtain an upper bound, the balls in the concave region will tend toward either end (i.e., to 0 or the inflection point); afterward, the balls in the concave region (including those from the convex region that have migrated to the inflection point) will tend toward a common point.

We pause for a moment to note the following about Theorem 4.11 in comparison with Jensen's Inequality. In vague terms, Jensen's Inequality implies that, for a concave function $g(t)$ on a convex domain, a sum of output values $g(y_1) + \cdots + g(y_N)$ has an upper bound in the form of a multiple of an output value $g(\overline{y})$ whose corresponding input value $\overline{y}$ depends on the sum of input values $y_1 + \cdots + y_N$ and the number N of such values, and where g is concave in a neighborhood about $\overline{y}$ (indeed, it is concave near any point in its domain).

Similarly, for a function f that is part convex and part concave, Theorem 4.11 implies that a sum of output values $f(x_1) + \cdots + f(x_N)$ has an upper bound that in the form of a linear combination of output values $f(0)$ and $f(\overline{x})$. Like the value $\overline{y}$ previously discussed, this special argument $\overline{x}$ also depends on the sum of the input values $x_1 + \cdots + x_N$ and on the number N of input values, but $\overline{x}$ depends additionally on a new parameter z, which in turn depends on the sum $x_1 + \cdots + x_P$ of input values that are smaller than the input value I of the inflection point of f, as well as the number P of these values. The similarities between $\overline{x}$ and $\overline{y}$ do not end here: f must also be concave in a neighborhood about $\overline{x}$, since

$$x_1 + \cdots + x_N = (x_1 + \cdots + x_P) + (x_{P+1} + \cdots + x_N) \geq I(P - z) + I(N - P) = I(N - z)$$

Returning to our original problem, we apply Theorem 4.11 to the particular function $f(A) = s\log(T_{mid}(A))$ to obtain the following result:

Corollary 4.12. *Consider a given template whose rows have N blocks of size s, and let $\mathcal{H}$ be the total number of H-cells in the template. Then there exists some $0 \leq z \leq N$ such that the maximum number $\mathcal{T}^H$ of horizontal strips that can be placed in a board compatible with the given template satisfies*

$$\begin{aligned} \log \mathcal{T}^H &\leq zs\log\frac{s^2K}{2} + (N-z)s\log\left(\frac{s^2}{2}\left(\left(\frac{\mathcal{H}}{N-z}\right)^2 + \left(\frac{\mathcal{H}}{N-z}\right) + K\right)\right) \\ &\leq Ns\log\frac{s^2K}{2} + (N-z)s\log\left(\frac{1}{K}\left(\left(\frac{\mathcal{H}}{N-z}\right)^2 + \left(\frac{\mathcal{H}}{N-z}\right) + K\right)\right) \end{aligned}$$

4.2.3 *The Optimum Configuration of H-Cells*

This result in effect binarizes (and thus simplifies) our analysis. In the ideal event that z and $\mathcal{H}/(N-z)$ are integers, Corollary 4.12 has a very practical interpretation. Given a template with a set number $\mathcal{H}$ of H-cells, Corollary 4.12 allows us to replace this template with one that has the same number of H-cells but admits a greater number of horizontal strip placements. The corresponding value of z can be thought of as the number of rows in this optimized template that are devoid of H-cells, and the remaining $N-z$ rows of the optimized template have a uniform number $\mathcal{H}/(N-z)$ of H-cells. The question remains, however, as to whether it is better to have more rows that contain H-cells but where each row has a relatively small number of them, or to have fewer rows containing H-cells but where each such row has a comparatively large amount of them.

To answer this question, we set

$$M_{mid}^{\mathcal{H}}(z) = \begin{cases} Ns\log T_{mid}(0) + (N-z)s\log\left[T_{mid}\left(\dfrac{\mathcal{H}}{N-z}\right)\Big/ T_{mid}(0)\right] & 0 \le z < N \\ Ns\log T_{mid}(0) & z = N \end{cases}$$

That is, we regard the greater side of the inequality of Corollary 4.12 as a function of z (with an auxiliary integer parameter $\mathcal{H}$). We indicated above that we are most interested in the case where z (and $\mathcal{H}/(N-z)$) is an integer, but, given that we are looking for an upper bound and not necessarily for an exact answer, we may relax this restriction to the integers and thus allow z to take on *real* values – the benefit of this is that we may take full advantage of any analytic methods we may require. Therefore, for any integer $\mathcal{H} = 1, \ldots, N^2$, we view $M_{mid}^{\mathcal{H}}$ as a real valued function that takes an input value anywhere in the real interval $[0, N]$.

Then, for a given value of $\mathcal{H}$, let $\tilde{z} = \tilde{z}(\mathcal{H}) \in [0, N]$ be such that $M_{\mathcal{H}}(\tilde{z})$ is a maximum; we note that such a value must exist, as $M_{mid}^{\mathcal{H}}$ is a *continuous* function on a compact domain. Given how $\tilde{z}$ is defined, we know that it must satisfy

$$\begin{aligned} 0 &= \left.\frac{\partial M_{mid}^{\mathcal{H}}(z)}{\partial z}\right|_{z=\tilde{z}} \\ &= -s\log\left(\frac{1}{K}\left(\left(\frac{\mathcal{H}}{N-\tilde{z}}\right)^2 + \left(\frac{\mathcal{H}}{N-\tilde{z}}\right) + K\right)\right) \\ &\qquad + \left(\frac{\mathcal{H}s}{N-\tilde{z}}\right)\left(\frac{2\left(\frac{\mathcal{H}s}{N-\tilde{z}}\right)+1}{\left(\frac{\mathcal{H}}{N-\tilde{z}}\right)^2 + \left(\frac{\mathcal{H}}{N-\tilde{z}}\right) + K}\right) && (4.13) \\ &= -s\log\left(\frac{1}{K}\left(x^2+x+K\right)\right) + s\left(\frac{2x^2+x}{x^2+x+K}\right) && (4.14) \end{aligned}$$

where $x = \mathcal{H}/(N-\tilde{z})$. Equation 4.14 defines x implicitly as a positive function $g(K)$ of K, and, since x relates $\mathcal{H}$ and $\tilde{z}$ inversely, we see that $\tilde{z}$ is a decreasing function of $\mathcal{H}$; more explicitly, if $\mathcal{H}/(N-\tilde{z}) = x = g(K)$, then $\tilde{z} = N - \mathcal{H}/g(K)$, so that $\tilde{z}$ is a linear function of $\mathcal{H}$ with slope $-1/g(K) < 0$.

Now consider $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H}_1 < \mathcal{H}_2$. By the above argument, the corresponding z-values $\tilde{z}_1 = \tilde{z}(\mathcal{H}_1)$ and $\tilde{z}_2 = \tilde{z}(\mathcal{H}_2)$ satisfy $\tilde{z}_1 > \tilde{z}_2$. If $M_{mid}^{\mathcal{H}_1}(\tilde{z}_1) > M_{mid}^{\mathcal{H}_2}(\tilde{z}_2)$, then

$$\begin{aligned}\tilde{z}_1 s \log(T_{mid}(0)) + (N - \tilde{z}_1) s \log\left(T_{mid}\left(\frac{\mathcal{H}_1}{N-\tilde{z}_1}\right)\right)& \\ > \tilde{z}_2 s \log(T_{mid}(0)) + (N - \tilde{z}_2) s \log\left(T_{mid}\left(\frac{\mathcal{H}_2}{N-\tilde{z}_2}\right)\right)&\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&(\tilde{z}_1 - \tilde{z}_2) s \log(T_{mid}(0)) \\ &\qquad > (N-\tilde{z}_2) s \log\left(T_{mid}\left(\frac{\mathcal{H}_2}{N-\tilde{z}_2}\right)\right) - (N-\tilde{z}_1) s \log\left(T_{mid}\left(\frac{\mathcal{H}_1}{N-\tilde{z}_1}\right)\right)\end{aligned} \tag{4.15}$$

However, Jensen's Inequality implies that

$$\begin{aligned}&(N-\tilde{z}_2) s \log\left(T_{mid}\left(\frac{\mathcal{H}_2}{N-\tilde{z}_2}\right)\right) \\ &\qquad \geq (\tilde{z}_1 - \tilde{z}_2) s \log\left(T_{mid}\left(\frac{\mathcal{H}_2 - \mathcal{H}_1}{\tilde{z}_1 - \tilde{z}_2}\right)\right) + (N-\tilde{z}_1) s \log\left(T_{mid}\left(\frac{\mathcal{H}_1}{N-\tilde{z}_1}\right)\right)\end{aligned} \tag{4.16}$$

Note that Jensen's Inequality applies, as we have $\mathcal{H}_i > (N - \tilde{z}_i)A_0$ for $i = 1, 2$, both of which imply $\mathcal{H}_2 - \mathcal{H}_1 > A_0[(N - \tilde{z}_2) - (N - \tilde{z}_1)] = A_0[\tilde{z}_1 - \tilde{z}_2]$. Therefore, Inequalities 4.15 and 4.16 together imply

$$\begin{aligned}(\tilde{z}_1 - \tilde{z}_2) s \log(T_{mid}(0)) &> (\tilde{z}_1 - \tilde{z}_2) s \log\left(T_{mid}\left(\frac{\mathcal{H}_2 - \mathcal{H}_1}{\tilde{z}_1 - \tilde{z}_2}\right)\right) \\ T_{mid}(0) &> T_{mid}\left(\frac{\mathcal{H}_2 - \mathcal{H}_1}{\tilde{z}_1 - \tilde{z}_2}\right)\end{aligned}$$

But this contradicts the fact that T_{mid} is strictly increasing on $(0, \infty)$. Hence, if $\mathcal{H}_1 \leq \mathcal{H}_2$, then $M_{mid}^{\mathcal{H}_1}(\tilde{z}_1) \leq M_{mid}^{\mathcal{H}_2}(\tilde{z}_2)$.

We remark that Theorem 4.11 applies also to the function $h(A) = s \log(T_{side}(A))$. We may thus, in analogy with the above, define the function

$$M_{side}^{\mathcal{H}}(z) = \begin{cases} Ns \log T_{side}(0) + (N-z) s \log\left[T_{side}\left(\frac{\mathcal{H}}{N-z}\right) \Big/ T_{side}(0)\right] & 0 \leq z < N \\ Ns \log T_{side}(0) & z = N \end{cases}$$

and we may also define the corresponding function $\tilde{z}_{side}(\mathcal{H})$ that is decreasing, and has the additional property that $M_{side}^{\mathcal{H}}(z) < M_{side}^{\mathcal{H}}(\tilde{z}_{side}(\mathcal{H}))$ for all $\mathcal{H}$ and all z. We would like to apply to M_{side} the same (sort of) argument that we made for M_{mid}, but more care must be taken for $M_{side}^{\mathcal{H}}$, as T_{side} is *not* increasing on all of $(0, \infty)$. In spite of this, we still have

$$T_{side}(A) = \frac{s^2}{2}\left[\left(A - \frac{s-2}{2s}\right)^2 + \frac{4Ks^2 - (s-2)^2}{4s^2}\right]$$

from which it follows that T_{side} is symmetric about the axis $A - (s-2)/2s = 0$; thus, $T_{side}(A) < T_{side}(0)$ if and only if $0 < A < (s-2)/s$. Hence, for $\mathcal{H}_1 < \mathcal{H}_2$, we have $M_{side}^{\mathcal{H}_1}(\tilde{z}_{side}(\mathcal{H}_1)) > M_{side}^{\mathcal{H}_2}(\tilde{z}_{side}(\mathcal{H}_2))$ only if the special input value

$$\frac{\mathcal{H}_2 - \mathcal{H}_1}{\tilde{z}_{side}(\mathcal{H}_1) - \tilde{z}_{side}(\mathcal{H}_2)}$$

analogous to that of Inequality 4.16 is positive (which it is easily seen to be) and less than $(s-2)/s$. Recall, however, that we justified Inequality 4.16 using Jensen's Inequality, and we note that Jensen's Inequality may be applied to T_{side} only if

$$\begin{aligned}\frac{\mathcal{H}_2 - \mathcal{H}_1}{\tilde{z}_{side}(\mathcal{H}_1) - \tilde{z}_{side}(\mathcal{H}_2)} \geq A_{side} &= \frac{s-2}{2s} + \sqrt{K - \left(\frac{s-2}{2s}\right)^2} \\ &> \frac{s-2}{2s} + \sqrt{2-1} > \frac{s-2}{2s} + \frac{1}{2} > \frac{s-2}{2s} + \frac{s-2}{2s} = \frac{s-2}{s}\end{aligned}$$

Therefore, the output of T_{side} at this special input cannot be less than the output at 0, so we cannot have $M_{side}^{\mathcal{H}_1}(\tilde{z}(\mathcal{H}_1)) > M_{side}^{\mathcal{H}_2}(\tilde{z}(\mathcal{H}_2))$ if $\mathcal{H}_1 < \mathcal{H}_2$.

In all cases, then, we deduce that (the upper bound for) the maximum possible number of ways to place horizontal strips on a template increases as the number of H-cells of the template increases and, consequently, as the number z of rows without H-cells decreases. Indeed, the situation is optimized when $\tilde{z}$ is as small as possible, and we are now interested in determining the values of $\mathcal{H}$ corresponding to this minimum value of $\tilde{z}$. We can easily see that we must have $\tilde{z} = 0$ for $\mathcal{H} = N^2$, as the H-cells exhaust all of the available cells. In fact, this is (fortunately) not an isolated phenomenon – we must also have $\tilde{z} = 0$ for

$\mathcal{H} > N(N-1)$, as there are exactly $N(N-1)$ cells in any collection of $N-1$ rows, so having more than this number of H-cells forces us to have an H-cell in the one remaining row not in the aforementioned collection, and thus we have H-cells in every row. Since the optimum value of $\tilde{z}(\mathcal{H})$ is a decreasing function of $\mathcal{H}$, there must be a minimum value of $\mathcal{H}$ for which $\tilde{z}(H) = 0$. Using asymptotic analysis, we can give a non-effective result that nevertheless provides a good understanding of where we should expect to see $\tilde{z}(H) = 0$.

Theorem 4.17. *Suppose $\mathcal{H} = \Omega(N^{3/2+\varepsilon})$ – that is, $\mathcal{H} \geq CN^{3/2+\varepsilon}$ for some positive constants C and ε. Then, for sufficiently large N (depending on C and ε), $M^{\mathcal{H}}_{mid}(z)$ is maximized when $z = 0$ – that is,*

$$\log \mathcal{T}_{max} \leq M^{\mathcal{H}}_{mid}(0) = Ns\log\left(\frac{s^2}{2}\left[\left(\frac{\mathcal{H}}{N}\right)^2 + \left(\frac{\mathcal{H}}{N}\right) + K\right]\right) = Ns\log\left(T_{mid}\left(\frac{\mathcal{H}}{N}\right)\right)$$

Proof. For any $0 \leq z < N$, we have

$$x = \frac{\mathcal{H}}{N-z} \geq \frac{\mathcal{H}}{N} \geq CN^{1/2+\varepsilon} \qquad \Longleftrightarrow \qquad x^2 = \left(\frac{\mathcal{H}}{N-z}\right)^2 \geq \left(\frac{\mathcal{H}}{N}\right)^2 \geq C^2N^{1+2\varepsilon}$$

Then we have from Equation 4.14 that

$$\begin{aligned}
\frac{d(M^{\mathcal{H}}_{mid}(z))}{dz} &= s\left[-\log\left(\frac{1}{K}\left(x^2+x+K\right)\right) + \left(\frac{2x^2+x}{x^2+x+K}\right)\right] \\
&\leq s\left[\frac{2x^2+x}{x^2} - \log\left(\frac{x^2}{K}\right)\right] \\
&\leq s\left[2 + \frac{1}{x} - \log\left(\frac{x^2}{K}\right)\right] \\
&\leq s\left[2 + \frac{1}{CN^{1/2+\varepsilon}} - \log\left(\frac{CN^{1+2\varepsilon}}{K}\right)\right]
\end{aligned}$$

Note that $K \leq 3N - 2 < 3N$, and hence

$$\begin{aligned}
\frac{d(M^{\mathcal{H}}_{mid}(z))}{dz} &\leq s\left[2 + \frac{1}{CN^{1/2+\varepsilon}} - \log\left(\frac{C^2N^{1+2\varepsilon}}{3N}\right)\right] \\
&\leq s\left[2 - \log\left(\frac{C^2}{3}\right) + \frac{1}{CN^{1/2+\varepsilon}} - 2\varepsilon\log(N)\right]
\end{aligned}$$

$\log(N) \to \infty$ as $N \to \infty$, whereas $1/N^{1/2+\varepsilon} \to 0$; hence, for sufficiently large N, the logarithmic term must dominate the rational term. This implies that $M^{\mathcal{H}}_{mid}$ is decreasing in a neighborhood of all $z \in (0, N)$, and therefore $M^{\mathcal{H}}_{mid}$ achieves its maximum when $z = 0$. ♦

Theorem 4.17 seems to imply that the least number of H-cells for which we should not have any rows without H-cells is of the form $O(N^{3/2})$ as $N \to \infty$. For a more explicit and a more effective result, we have

Theorem 4.18. $\log \mathcal{T}^{H}_{max} \leq M^{\mathcal{H}}_{mid}(0) = Ns \log\left(T_{mid}\left(\mathcal{H}/N\right)\right)$ *if*

$$\frac{\mathcal{H}}{N} \geq C_{mid} = \frac{-1+\sqrt{16K+1}}{2}$$

Recall that $K = O(N)$, which implies that $C_{mid} = O(N^{1/2})$, and thus Theorem 4.18 holds for $\mathcal{H} \geq N[O(N^{1/2})] = O(N^{3/2})$, which is in agreement with our preceding remark.

Proof. With $x = \mathcal{H}/(N-z)$, which is positive for any $z < N$, Equation 4.14 gives

$$\begin{aligned}
\frac{d(M^{\mathcal{H}}_{mid}(z))}{dz} &= s\left[\frac{2x^2+x}{x^2+x+K} - \log\left(\frac{x^2+x+K}{K}\right)\right] \\
&= s\left[2 - \frac{x+2K}{x^2+x+K} - \log\left(\frac{x^2+x+K}{K}\right)\right] \\
&< s\left[2 - \frac{2}{\frac{x^2+x+K}{K}} - \log\left(\frac{x^2+x+K}{K}\right)\right] \\
&< s[2 - 2w^{-1} - \log(w)]
\end{aligned}$$

where $w = (x^2+x+K)/K$.

A simple analysis of the expression $2 - 2w^{-1} - \log(w)$ (see Figure 4.4) shows that it is negative for all $w \geq 5$, and hence $M^{\mathcal{H}}_{mid}$ is decreasing whenever

$$\frac{x^2+x+K}{K} = w \geq 5 \qquad \Longleftrightarrow \qquad x^2+x-4K \geq 0 \qquad \Longleftrightarrow \qquad x \geq \frac{-1+\sqrt{1+16K}}{2}$$

For $z \in [0, N)$, we have $x > \frac{H}{N}$, and so, in order to ensure that $M^{\mathcal{H}}_{mid}$ is decreasing in a neighborhood of all $z \in (0, N)$ (and hence ensure that $M^{\mathcal{H}}_{mid}$ is maximized when $z = 0$), we require that $\mathcal{H}/N \geq \frac{1}{2}(-1+\sqrt{1+16K})$. ♦

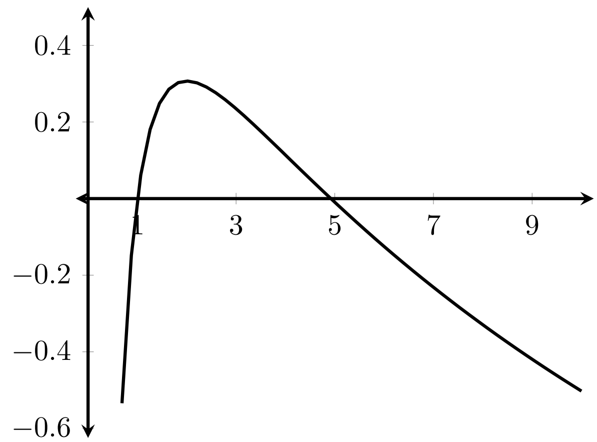


Figure 4.4: A graph of the function $f(w) = 2 - 2w^{-1} - \log(w)$ *showing that the function is indeed negative for* $w \geq 5$*. One may also argue by determining that the function* f *is decreasing for all* $w > 2$*, and then compute that* $f(5) < 0$*.*

4.3 Placing Horizontal and Vertical Strips: The Optimum Template

It should be noted that all of our results for horizontal strips and H-cells hold exactly the same for vertical strips and V-cells. Therefore, if we consider a template where $\mathcal{H}$ is the number of H-cells and $\mathcal{V}$ is the number of V-cells in the template, there exist (real) numbers $z_{\mathcal{H}}, z_{\mathcal{V}} \in [0, N]$ such that the number $\overline{\mathcal{T}}$ of ROSAs that are compatible with the template, that have no two strips overlapping, and are such that each row and column of the chessboard contains at most one strip satisfies

$$\begin{aligned}\log(\overline{\mathcal{T}}) \quad &\leq \quad \left\{Ns\log\left(T_{mid}(0)\right) + (N - z_{\mathcal{H}})s\log\left[T_{mid}\left(\frac{\mathcal{H}}{N - z_{\mathcal{H}}}\right)\Big/ T_{mid}(0)\right]\right\} \\ &\quad + \left\{Ns\log\left(T_{mid}(0)\right) + (N - z_{\mathcal{V}})s\log\left[T_{mid}\left(\frac{\mathcal{V}}{N - z_{V}}\right)\Big/ T_{mid}(0)\right]\right\}\end{aligned} \tag{4.19}$$

We recall that the expressions $\mathcal{H}/(N - z_{\mathcal{H}})$ and $\mathcal{V}/(N - z_{\mathcal{V}})$ arise from Theorem 4.11; immediately after proving this theorem, we noted that the value of such expressions must lie in the region of concavity for the function under consideration (see the associated discussion on p. 71). In particular, concerning the function $f(A) = s\log(T_{mid}(A))$, we find that both values $\mathcal{H}/(N-z_{\mathcal{H}})$ and $\mathcal{V}/(N-z_{\mathcal{V}})$ must be larger than the input value A_{mid} of the (positive) inflection point of f. We may therefore apply Jensen's Inequality to improve (and simplify) our upper bound from Inequality 4.19 with the result

$$\begin{aligned}\log(\overline{\mathcal{T}}) \leq{}& 2Ns\log\left(\frac{s^2K}{2}\right)\\ &+ ((N-z_H)+(N-z_V))s\log\left[T_{mid}\left(\frac{\mathcal{H}+\mathcal{V}}{(N-z_H)+(N-z_V)}\right)\Big/\, T_{mid}(0)\right] \quad (4.20)\end{aligned}$$

Defining $\zeta = z_{\mathcal{H}}+z_{\mathcal{V}}$ to be a real number in the interval $[0, 2N]$, and noting that $\mathcal{H}+\mathcal{V} = N^2$ must equal the total number of cells in the template, we can write our improved upper bound of Inequality 4.20 in the concise form

$$\log(\overline{\mathcal{T}}) \leq 2Ns\log\left(\frac{s^2K}{2}\right) + (2N-\zeta)s\log\left[\frac{1}{K}\left(\left(\frac{N^2}{2N-\zeta}\right)^2 + \left(\frac{N^2}{2N-\zeta}\right) + K\right)\right] \quad (4.21)$$

We recall that it was optimal for us in the previous arguments to have $z_{\mathcal{H}} = z_{\mathcal{V}} = 0$ – it should therefore be optimal for us in the current situation to have $\zeta = z_{\mathcal{H}} + z_{\mathcal{V}} = 0$. Theorem 4.18 gave us a sufficient condition under which we have $z_{\mathcal{H}} = 0$ and $z_{\mathcal{V}} = 0$; applying the very same methods of Theorem 4.18 to the expression in Inequality 4.21, we obtain the following corresponding result.

Theorem 4.22. *The number of ROSAs on a chessboard with a given $N \times N$ template is at most $2Ns\log\left(T_{mid}\left(\frac{N}{2}\right)\right)$ if $\frac{N}{2} \geq C_{mid} = \frac{-1+\sqrt{16K+1}}{2}$.*

Proof. Indeed, we may define the function $\mathcal{M}_{mid} : [0, 2N] \to [0, \infty)$ by

$$\mathcal{M}_{mid}(\zeta) = \begin{cases} 2Ns\log\left(T_{mid}(0)\right) + (2N-\zeta)s\log\left[T_{mid}\left(\frac{N^2}{2N-\zeta}\right)\Big/\, T_{mid}(0)\right] & 0 \leq \zeta < 2N \\ 2Ns\log\left(T_{mid}(0)\right) & \zeta = 2N \end{cases}$$

Then, setting $y = N^2/(2N - \zeta)$, which is positive for all $\zeta < 2N$, we have

$$\begin{aligned}
\frac{d(\mathcal{M}_{mid}(\zeta))}{d\zeta} &= s\left[\frac{2y^2+y}{y^2+y+K} - \log\left(\frac{y^2+y+K}{K}\right)\right] \\
&= s\left[2 - \frac{y+2K}{y^2+y+K} - \log\left(\frac{y^2+y+K}{K}\right)\right] \\
&< s\left[2 - \frac{2}{\frac{y^2+y+K}{K}} - \log\left(\frac{y^2+y+K}{K}\right)\right] \\
&< s[2 - 2v^{-1} - \log(v)]
\end{aligned}$$

where $v = (y^2+y+K)/K$. As in Theorem 4.18, the expression $2-2v^{-1}-\log(v)$ is negative for all $v \geq 5$ (visually, we refer again to Figure 4.4), and hence $\mathcal{M}_{mid}$ is decreasing whenever

$$\frac{y^2+y+K}{K} = v \geq 5 \qquad \Longleftrightarrow \qquad y^2+y-4K \geq 0 \qquad \Longleftrightarrow \qquad y \geq \frac{-1+\sqrt{1+16K}}{2}$$

For $\zeta \in [0, 2N)$, we have $y > N^2/2N = N/2$, and so, in order to ensure that $\mathcal{M}_{mid}$ is decreasing in a neighborhood of all $\zeta \in (0, 2N)$ (and hence ensure that $\mathcal{M}_{mid}$ is maximized when $\zeta = 0$), we require that $\frac{N}{2} \geq \frac{1}{2}(-1+\sqrt{1+16K})$. ♦

Notice that, as $K < 3N$, the result of Theorem 4.22 holds for all N with

$$\begin{aligned}
\frac{N}{2} &\geq \frac{-1+\sqrt{16(3N)+1}}{2} \\
N+1 &\geq \sqrt{48N+1} \\
N^2+2N+1 &\geq 48N+1 \\
N^2-46N &\geq 0 \\
N &\geq 46
\end{aligned}$$

Thus, if N is sufficiently large *and even*, the $N \times N$ template with the highest number of compatible strip arrangements contains an equal number ($N/2$) of H-cells in each of its N rows (for a total number $N^2/2$ of H-cells) and an equal number ($N/2$) of V-cells in each of its N columns (for a total number $N^2/2$ of V-cells). We determined earlier that the number of compatible strip arrangements is maximized if the H-cells in each row and the V-cells in each column are contiguous; this implies that the templates that admit the most strip placements are those that are divided into four square quadrants of size $N^2/4$ that consist solely of H-cells, V-cells, H-cells, and V-cells respectively as one traverses them in a cyclic order (see Figure 4.5). If N is odd, we are able to verify empirically that essentially the same picture holds, with some leeway with the arrangement of cells in the center row and in the center column. However, as we are interested in an upper bound for (the logarithm of) the number of ROSAs as the side length L of the chessboard (and consequently as N) tends to ∞, knowledge of what happens when N is even is enough – indeed, when N is odd, one may take the corresponding result for the even integer $N+1$ as an upper bound for $\log T(Ns, Ns)$, for it is easy to see that $\log T(L, L)$ increases as L increases.

Therefore, there is no better template than the "quartered" template. In such a template, we notice that each group of H-cells in any given row of the template lies on one side of that row, and each group of V-cells in any given column of the template is positioned on one end of that column. It therefore follows that we can obtain a tighter upper bound by replacing each use of the polynomial T_{mid} with T_{side} in Inequalities 4.19, 4.20, and 4.21, and thus we deduce that

$$\log(\overline{\mathcal{T}}) \leq 2Ns\log\left(\frac{s^2K}{2}\right) + \\ (2N-\zeta)s\log\left[\frac{1}{K}\left(\left(\frac{N^2}{2N-\zeta}\right)^2 - \left(\frac{s-2}{s}\right)\left(\frac{N^2}{2N-\zeta}\right) + K\right)\right] \quad (4.23)$$

in analogy to Inequality 4.21. Again, we wish to know when we have a maximum at $\zeta = 0$; we do this with a result analogous to Theorem 4.22.

Theorem 4.24. *The number of ROSAs on a chessboard with a given $N \times N$ template is at most $2Ns\log\left(T_{side}\left(\frac{N}{2}\right)\right)$ if $\frac{N}{2} \geq C_{side} = \frac{(s-2)+\sqrt{39Ks^2+(s-2)^2}}{2s}$. In particular, the claim holds if*

$$\frac{N}{2} \geq \frac{1+\sqrt{39K+1}}{2} \quad (4.25)$$

H	H	H	H	V	V	V	V

(a)

H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V

(b)

H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V
H	H	H	H	V	V	V	V
V	V	V	V	H	H	H	H
V	V	V	V	H	H	H	H
V	V	V	V	H	H	H	H
V	V	V	V	H	H	H	H

(c)

Figure 4.5: A visual explanation for why the quartering template is optimal. Suppose that the cell in the upper left corner is an H-cell. Then this cell begins a contiguous group of H-cells up until the midpoint of the side of the chessboard, and the remainder of the cells in this row must be V-cells, as in Figure 4.5(a). These V-cells must begin contiguous groups of V-cells in their respective columns whose length is half the side length of the chessboard, just as in Figure 4.5(b), and this forces the remainder of the template to be filled as in Figure 4.5(c). An analogous argument holds if the cell in the upper left is a V-cell.

Proof. Similarly as in the proof of Theorem 4.22, we define $\mathcal{M}_{side} : [0, 2N] \to [0, \infty)$ by

$$\mathcal{M}_{side}(\zeta) = \begin{cases} 2Ns\log\left(T_{side}(0)\right) + (2N-\zeta)s\log\left[T_{side}\left(\dfrac{N^2}{2N-\zeta}\right)\Big/ T_{mid}(0)\right] & 0 \le \zeta < 2N \\ 2Ns\log\left(T_{side}(0)\right) & \zeta = 2N \end{cases}$$

Then, setting $y = N^2/(2N-\zeta)$, which is positive for all $\zeta < 2N$, we have

$$\begin{aligned}
\frac{d(\mathcal{M}_{side}(\zeta))}{d\zeta} &= s\left[\frac{2y^2 - \left(\frac{s-2}{s}\right)y}{y^2 - \left(\frac{s-2}{s}\right)y + K} - \log\left(\frac{y^2 - \left(\frac{s-2}{s}\right)y + K}{K}\right)\right] \\
&= s\left[2 + \frac{\left(\frac{s-2}{s}\right)y - 2K}{y^2 - \left(\frac{s-2}{s}\right)y + K} - \log\left(\frac{y^2 - \left(\frac{s-2}{s}\right)y + K}{K}\right)\right] \\
&= s\left[2 + \frac{\left(\frac{s-2}{s}\right)y}{y^2 - \left(\frac{s-2}{s}\right)y + K} - \frac{2}{\frac{y^2+y+K}{K}} - \log\left(\frac{y^2 - \left(\frac{s-2}{s}\right)y + K}{K}\right)\right] \\
&= s\left[2 + \frac{\left(\frac{s-2}{s}\right)y}{y^2 - \left(\frac{s-2}{s}\right)y + K} - 2u^{-1} - \log(u)\right]
\end{aligned}$$

where $u = (y^2 - \left(\frac{s-2}{s}\right)y + K)/K$.

We again end up with a rational expression

$$F(y) = \frac{\left(\dfrac{s-2}{s}\right)y}{y^2 - \left(\dfrac{s-2}{s}\right)y + K}$$

in the computation of the derivative of $\mathcal{M}_{side}$ that is unlike the others; unfortunately, unlike the proof of Theorem 4.22, $F(y)$ is *positive* for positive y, and so we cannot simply bound $F(y)$ above by zero. Nevertheless, viewing $F(y)$ as a rational function of y, we see that the denominator of $F(y)$ has no (real) zeroes, and, as the degree of the numerator is less than that of the denominator, $F(y) \to 0$ as $y \to \infty$. It therefore follows that the image of $F(y)$ is bounded above by some constant $\alpha > 0$. Attempting to solve for the value of α, we have

$$\begin{aligned}
\frac{\left(\dfrac{s-2}{s}\right)y}{y^2 - \left(\dfrac{s-2}{s}\right)y + K} &\le \alpha \\
\frac{1}{\alpha}\left(\frac{s-2}{s}\right)y &\le y^2 - \left(\frac{s-2}{s}\right)y + K \\
0 &\le y^2 - \left(1 + \frac{1}{\alpha}\right)\left(\frac{s-2}{s}\right)y + K
\end{aligned}$$

This resulting quadratic is non-negative for all y if and only if its corresponding discriminant is *non-positive*, so we must have

$$\begin{aligned}
\left[\left(1+\frac{1}{\alpha}\right)\left(\frac{s-2}{s}\right)\right]^2 - 4K &\leq 0 \\
\left(1+\frac{1}{\alpha}\right)\left(\frac{s-2}{s}\right) &\leq 2\sqrt{K} \\
\left(1+\frac{1}{\alpha}\right) &\leq \frac{2s\sqrt{K}}{s-2} \\
\alpha &\geq \left(\frac{2s\sqrt{K}}{s-2} - 1\right)^{-1}
\end{aligned}$$

Thus, $F(y) \leq ((2s\sqrt{K}/(s-2)) - 1)^{-1}$ for all y. This upper bound is difficult to use in practice, so we note that, as $s/(s-2) > 1$ and $K > 2$, $2s\sqrt{K}/(s-2) > 2\sqrt{2}$, which implies $\alpha < 1/(2\sqrt{2}-1)$, and therefore

$$\frac{d\mathcal{M}_{side}(\zeta)}{d\zeta} \leq s\left[2 + (2\sqrt{2}-1)^{-1} - 2u^{-1} - \log(u)\right]$$

We may now continue as we did in the proof of Theorem 4.22: the bracketed expression $2 + (2\sqrt{2}-1)^{-1} - 2u^{-1} - \log(u)$ is negative for all $u \geq 10.75$ (see Figure 4.6), and hence $\mathcal{M}_{side}$ is decreasing whenever

$$\frac{y^2 - \left(\frac{s-2}{s}\right)y + K}{K} = u \geq 10.75 \qquad \Longleftrightarrow \qquad y^2 - \left(\frac{s-2}{s}\right)y - 9.75K \geq 0$$

which holds if and only if

$$\frac{1}{2}\left[\left(\frac{s-2}{s}\right) + \sqrt{\left(\frac{s-2}{s}\right)^2 + 39K}\right] < \frac{1+\sqrt{1+39K}}{2} \leq y$$

For $\zeta \in [0, 2N)$, we have $y > N^2/2N = N/2$, and so, in order to ensure that $\mathcal{M}_{side}$ is decreasing in a neighborhood of all $\zeta \in (0, 2N)$ (and hence ensure that $\mathcal{M}_{mid}$ is maximized when $\zeta = 0$), it is enough that $\frac{N}{2} \geq \frac{1}{2}(1 + \sqrt{1+39K})$. ♦

More concretely, we can determine, just as in our remarks following Theorem 4.22, that Theorem 4.24 holds for all N that satisfy

$$\frac{N}{2} \geq \frac{1+\sqrt{39(3N)+1}}{2} \quad \Longleftrightarrow \quad (N-1)^2 \geq 117N+1$$
$$\Longleftrightarrow \quad N^2 - 119N \geq 0 \quad \Longleftrightarrow \quad N \geq 119$$

We have thus far determined which single template of a given dimension $N \times N$ produces the highest number of compatible strip arrangements. However, if we are to consider *all* possible strip arrangements on a chessboard, we must consider *all* possible $N \times N$ templates that can be constructed over that chessboard. As there are two possible labels for each cell, there are clearly 2^{N^2} possible templates, and the number of compatible strip arrangements for each such template is bounded above by the expression in Theorem 4.22, assuming N is sufficiently large. Therefore, we are finally ready to give an upper bound for the number of ROSAs on a square chessboard of dimension $L \times L$:

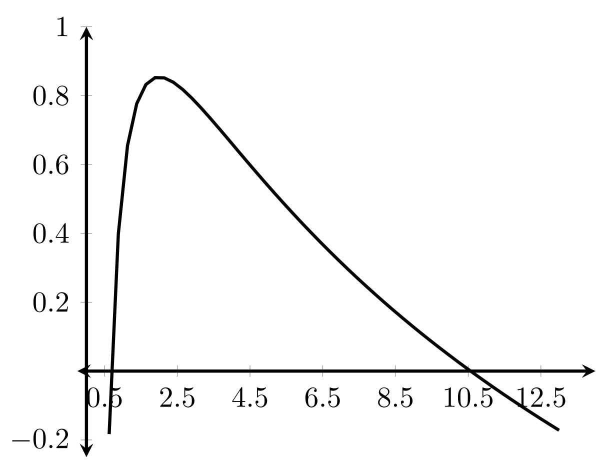


Figure 4.6: A graph of the function $g(w) = 2 + (2\sqrt{2}-1)^{-1} - 2w^{-1} - \log(w)$ showing that the function is indeed negative for $w \geq 10.75$. As we noted in Figure 4.4, one may also argue by showing that g is decreasing for all $w > 2$, and then compute that $g(10.75) < 0$.

Theorem 4.26. *Let L be a positive integer, and let s be an integer divisor of L.*

1. *If $N = L/s \geq 46$, then $T(L,L) \leq 2^{N^2}(T_{mid}(N/2))^{2Ns}$, or, equivalently,*

$$\begin{aligned}\log T(L,L) \;&\leq\; N^2 \log 2 &(4.27)\\ &\qquad +2Ns\log\left[\frac{s^2}{2}\left(\left(\frac{N}{2}\right)^2+\left(\frac{N}{2}\right)+3N-2-\frac{3N-4}{s}\right)\right]\\ &\leq\; \overline{\mathcal{M}_{mid}}(s)\\ &\qquad = \frac{L^2}{s^2}\log 2 + 2L\log\left[\frac{L^2-12L+(14L+16)s-8s^2}{8}\right] &(4.28)\end{aligned}$$

2. *If $N = L/s \geq 119$, then $T(L,L) \leq 2^{N^2}(T_{side}(N/2))^{2Ns}$, or, equivalently,*

$$\begin{aligned}\log T(L,L) \;&\leq\; N^2 \log 2 &(4.29)\\ &\qquad +2Ns\log\left[\frac{s^2}{2}\left(\left(\frac{N}{2}\right)^2-\left(\frac{s-2}{s}\right)\left(\frac{N}{2}\right)+3N-2-\frac{3N-4}{s}\right)\right]\\ &\leq\; \overline{\mathcal{M}_{side}}(s)\\ &\qquad = \frac{L^2}{s^2}\log 2 + 2L\log\left[\frac{L^2-8L+(10L+16)s-8s^2}{8}\right] &(4.30)\end{aligned}$$

4.4 A Logarithmic Upper Bound for All ROSAs: The Optimum Mesh for a Template

Theorem 4.26 holds for all pairs of integers L and s that satisfy the requisite condition for whichever bound one wishes to use. We would, of course, want to improve this result to give as small of an upper bound as possible, so, since we are initially given the length of the board L, we would like to know what corresponding cell size s minimizes the lower bound(s) of Theorem 4.26. Since we are again asking another optimization question that is best solved using analytic methods, we will again relax the integer conditions, and thereby allow s to be any real number.

Since Inequality 4.30 represents the tighter upper bound, we will apply our analytic methods to $\overline{\mathcal{M}_{side}}(s)$. It is easily seen that $\overline{\mathcal{M}_{side}}$ is a differentiable function of s in the closed interval $[1, L/119]^{\dagger}$, so, as s ranges continuously over this interval, we find that the optimal value of s (if it is not located at the endpoints of the target interval) must satisfy

$$\begin{aligned} 0 &= \frac{d\overline{\mathcal{M}_{side}}(s)}{ds} \\ 0 &= \frac{-2L^2}{s^3}\log 2 + 2L\left[\frac{(10L+16)-16s}{(L^2-8L)+(10L+16)s-8s^2}\right] \end{aligned} \tag{4.31}$$

Of course, we can simplify Equation 4.31 to obtain the polynomial equation

$$\begin{aligned} (10L+16)s^3 - 16s^4 = (L^3\log 2 - 8L^2\log 2) \\ + (10L^2\log 2 + 16L\log 2)s - (8L\log 2)s^2 \end{aligned} \tag{4.32}$$

We may solve Equation 4.32 for s as a Puiseux series in L by using the method of Newton polygons, as outlined by Walker [18]. However, there is one caveat: if we were apply the method of Newton polygons to Equation 4.32 directly, the Puiseux series so obtained would be valid only for those values of L in a neighborhood of 0. We require that the Puiseux series be valid as $L \to \infty$ – or as $\lambda = 1/L \to 0^+$ – and thus we substitute $\lambda = 1/L$ in Equation 4.32 to obtain

$$\begin{aligned} (10\lambda^{-1}+16)s^3 - 16s^4 = (\lambda^{-3}\log 2 - 8\lambda^{-2}\log 2) \\ + (10\lambda^{-2}\log 2 + 16\lambda^{-1}\log 2)s - (8\lambda^{-1}\log 2)s^2 \end{aligned}$$

†We set the lower endpoint of our target interval to 1 to avoid any unnecessary subdivisions of the unit cells. The upper endpoint of the target interval comes from the fact that Theorem 4.26 is guaranteed for $L/s \geq 119$, or, equivalently, $L/119 \geq s$.

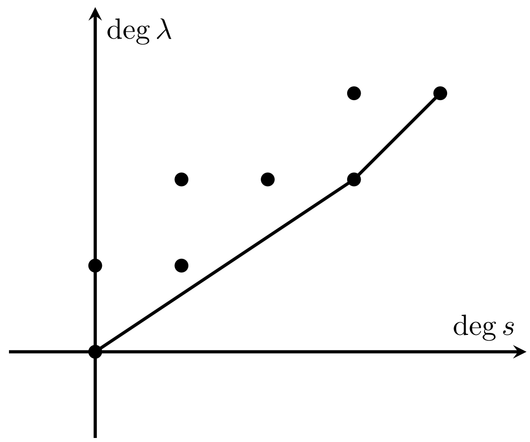


Figure 4.7: The Newton polygon corresponding to Equation 4.33 – that is, the convex hull of the set of points (a, b) for which the coefficient of $s^a\lambda^b$ in Equation 4.33 is non-zero.

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or, equivalently, after multiplying both sides by λ^3 to clear fractions,

$$(10\lambda^2 + 16\lambda^3)s^3 - 16\lambda^3 s^4 = (\log 2 - 8\lambda \log 2) + (10\lambda \log 2 + 16\lambda^2 \log 2)s - (8\lambda^2 \log 2)s^2 \quad (4.33)$$

and it is to Equation 4.33 that we will apply the technique of Newton polygons. Indeed, when we construct the Newton polygon corresponding to Equation 4.33 (see Figure 4.7), we see that its lower boundary contains a line segment with slope 2/3, which corresponds to the (triple) solution of Equation 4.33 given by $s = c\lambda^{-2/3}(1 + O(\lambda^{1/3}))$ as $\lambda \to 0^+$, or $s = cL^{2/3}(1 + O(L^{-1/3}))$ as $L \to \infty$[‡]; substituting this expression of λ into Equation 4.33 and equating the coefficients of the terms of least degree, we compute more specifically that

$$s = \sqrt[3]{\frac{\log 2}{10}}\lambda^{-2/3} + O(\lambda^{-1/3}) = \sqrt[3]{\frac{\log 2}{10}}L^{2/3} + O(L^{1/3})$$

[‡]The lower boundary of the Newton polygon also contains a line segment of slope 1 – this corresponds to the solution $s = (5/8)L - O(1)$ (of multiplicity 1), which we note is much larger than our upper bound $s \leq L/119$, and therefore this solution is extraneous to our purposes.

We would like to explain now why this value of s gives a minimum for $\overline{\mathcal{M}}_{side}$. Since $d(\overline{\mathcal{M}}_{side})/ds$ is continuous and has only the one zero $\sigma = ((\log 2)/10)^{1/3}L^{2/3} + O(L^{1/3})$ on the interval $[1, L/119]$, it follows from the Intermediate Value Theorem that the derivative must have the same sign for all s with $1 \leq s < \sigma$ – in particular, we need only the check the sign for $s = 1$. If we apply this substitution to Equation 4.32, we see that the left hand side is a polynomial in L of degree 1, whereas the right hand side is a polynomial in L of degree 3, and thus the right hand side is larger for sufficiently large L – this implies that $d(\overline{\mathcal{M}}_{side})/ds$ is negative for $s = 1$, and therefore it is negative for all $s \in [1, \sigma)$. Similarly, $d(\overline{\mathcal{M}}_{side})/ds$ must have the same sign for all s with $\sigma < s \leq L/119$, so we need only check the sign for $s = L/119$. Applying this substitution to Equation 4.32 gives the left hand side as a polynomial in L of degree 4 and the right hand side as a polynomial in L of degree 3, which implies that $d(\overline{\mathcal{M}}_{side})/ds$ is positive for $s = L/119$, and therefore it must be positive for all $s \in (\sigma, L/119]$. Putting these results together with the First Derivative Test of elementary calculus, we conclude that σ gives the absolute minimum value for $\overline{\mathcal{M}}_{side}$ on $[1, L/119]$.

Therefore, combining this result with Theorem 4.26 gives the following upper bound of the number $T(L, L)$ of ROSAs on the $L \times L$ chessboard that depends only on L and no other auxiliary parameters:

Theorem 4.34. *For sufficiently large L (namely, for $L > ((\log 2)/10)(119)^3$),*

$$\begin{aligned}
\log T(L,L) &\leq \frac{L^2 \log 2}{\left[\sqrt[3]{\frac{\log 2}{10}}L^{2/3} + O(L^{1/3})\right]^2} + 2L \log\left(\frac{L^2 + O(L^{5/3})}{8}\right) \\
&\leq (L^2 \log 2)\left(\sqrt[3]{\left(\frac{10}{\log 2}\right)^2} L^{-4/3} + O(L^{-5/3})\right) \\
&\qquad +2L \log\left(\frac{L^2}{8}\right) + 2L\log(1 + O(L^{-1/3})) \\
&\leq 4L \log L + \sqrt[3]{100 \log 2}L^{2/3} + O(L^{1/3}) - 2L \log 8 + LO(L^{-1/3}) \\
&\leq 4L \log L - 2L \log 8 + O(L^{2/3})
\end{aligned}$$

4.5 A Logarithmic Lower Bound

Let us take a moment to reflect on our progress so far. We obtained the upper bound of Theorem 4.34 for $T(L, L)$ from Theorem 4.1 by taking a maximum with respect to the number of H-cells in the rows of a template (and the number of V-cells in the columns) and taking the minimum with respect to the size s of the cells. By reversing this process, we can obtain a lower bound for $T(L, L)$.

We begin the search for a lower bound for $T(L, L)$ by noting that, in each case, Expression 4.2 of Theorem 4.1 is a quadratic polynomial in s with positive coefficients, and therefore the value of the expression is increasing for positive s. It therefore follows that Expression 4.2 is minimized for positive integers s when $s = 1$ – in other words, the number T of ways to place a horizontal row in a single row of a ROSA compatible with a template corresponding to the sequences $\{a_0, \dots\}$ and $\{b_0, \dots\}$ satisfies

$$T \geq 1 + \sum_{j \geq 0} \binom{a_j + 1}{2}$$

It is easily seen that the polynomial $g(x) = \binom{x+1}{2} = \frac{1}{2}x(x + 1)$ is a *convex* function (as are all quadratic functions that have a positive leading coefficient). It therefore follows from the convex form of Jensen's Inequality that, if J is the number of non-zero terms in the sequence $\{a_0, \dots\}$, then

$$T \geq 1 + \sum_{j \geq 0} \binom{a_j + 1}{2} \geq 1 + J\binom{(A/J) + 1}{2} = 1 + \tfrac{1}{2}A((A/J) + 1)$$

where $A = \sum a_j$ is the number of H-cells in the row (as above).

Considering all rows of the chessboard together, the previous result implies that the number $\mathcal{T}^H$ of ways to place horizontal strips anywhere in a ROSA satisfies

$$\mathcal{T}^H \geq \left(1 + J_1\binom{(A_1/J_1)+1}{2}\right)\left(1 + J_2\binom{(A_2/J_2)+1}{2}\right)\dots\left(1 + J_L\binom{(A_L/J_L)+1}{2}\right)$$

or, equivalently,

$$\log \mathcal{T}^H \geq \log\left(1 + \frac{1}{2}A_1\left(\frac{A_1}{J_1}+1\right)\right) + \log\left(1 + \frac{1}{2}A_2\left(\frac{A_2}{J_2}+1\right)\right) + \dots + \log\left(1 + \frac{1}{2}A_L\left(\frac{A_L}{J_L}+1\right)\right) \tag{4.35}$$

where A_j denotes the total number of H-cells, and J_j denotes the number of contiguous groups of H-cells, in row j of some compatible template (or, what amounts to the same since the cells have side length $s = 1$, A_j denotes the number of unit squares in row j that are covered by a horizontal strip in some ROSA).

Inequality 4.35 must hold for *all* templates (or *all* ROSAs), so we can improve our result by determining the maximum value for this lower bound. First, we note that, as in the argument for the upper bound, the number of ways to lay a horizontal strip is greatest when the H-cells are consolidated into a single contiguous group. Thus, we may set $J_j = 1$ for all j, which gives

$$\log \mathcal{T}^H \geq \log\left(1 + \binom{A_1+1}{2}\right) + \log\left(1 + \binom{A_2+1}{2}\right) + \dots + \log\left(1 + \binom{A_L+1}{2}\right)$$

Next, we recall Theorem 4.6, which implies that the polynomial $1 + \frac{1}{2}x(x+1)$ is logarithmically concave for $x > \frac{1}{2}(-1+\sqrt{7})$ – and in particular for $x \geq 1$. We may assume that $A_j \geq 1$ for all j – or, equivalently, that the number $\mathcal{H}$ of H-cells in the template is at least L – since having any row of a template be devoid of H-cells only serves to decrease the lower bound of Inequality 4.35, and therefore Theorem 4.6 allows us to apply Jensen's Inequality to deduce that $\log \mathcal{T}^H \geq L \log\left(1 + \binom{(\mathcal{H}/L)+1}{2}\right)$.

Surely, the above results also hold for vertical strips and V-cells. Taking the results for horizontal strips and vertical strips together, we see that

$$T(L,L) \geq L \log\left(1 + \binom{(\mathcal{H}/L)+1}{2}\right) + L \log\left(1 + \binom{(\mathcal{V}/L)+1}{2}\right) \qquad (4.36)$$

where $\mathcal{H}$ and $\mathcal{V}$ are the number of H-cells and V-cells respectively in some compatible template. Similarly to the previous argument, Inequality 4.36 must hold for *all* templates. We recall that Inequality 4.35 was maximized when $\mathcal{H} \geq L$ and (similarly) $\mathcal{V} \geq L$, or when we have $\mathcal{H}/L \geq 1$ and $\mathcal{V}/L \geq 1$. Thus, $\mathcal{H}/L$ and $\mathcal{V}/L$ lie in the interval of log-concavity, so we may invoke Jensen's Inequality once more to sharpen our lower bound as follows:

Theorem 4.37. *For any positive integer L,*

$$\begin{aligned}
T(L,L) &\geq 2L\log\left(1+\binom{(L^2/2L)+1}{2}\right)\\
&\geq 2L\log\left(1+\binom{(L/2)+1}{2}\right)\\
&\geq 2L\log\left(1+\frac{L}{4}+\frac{L^2}{8}\right)\\
&\geq 2L\log\left(\frac{L^2}{8}\right)+2L\log\left(\frac{8}{L^2}+\frac{2}{L}+1\right)\\
&\geq 4L\log L - 2L\log 8 + O(1)
\end{aligned}$$

One can also arrive at Theorem 4.37 in a more direct manner. We recall that we obtained the upper bound of Theorem 4.34 by considering all templates on the chessboard (with an optimum fixed cell side length) at once. It can be easily seen that no single template is compatible with all ROSAs on a given chessboard, so it is enough for a lower bound to consider a single template (of a minimum cell side length), and one can tighten this lower bound by determining which template is compatible with the maximum number of strip placements. But we already determined in the previous arguments that the template that admits the most strip arrangements is the quartered template (see Inequality 4.23 and the preceding discussion); using Theorem 4.1 to count the number of compatible ROSAs gives the exact same lower bound as we have in Theorem 4.37 (see Figure 4.8).

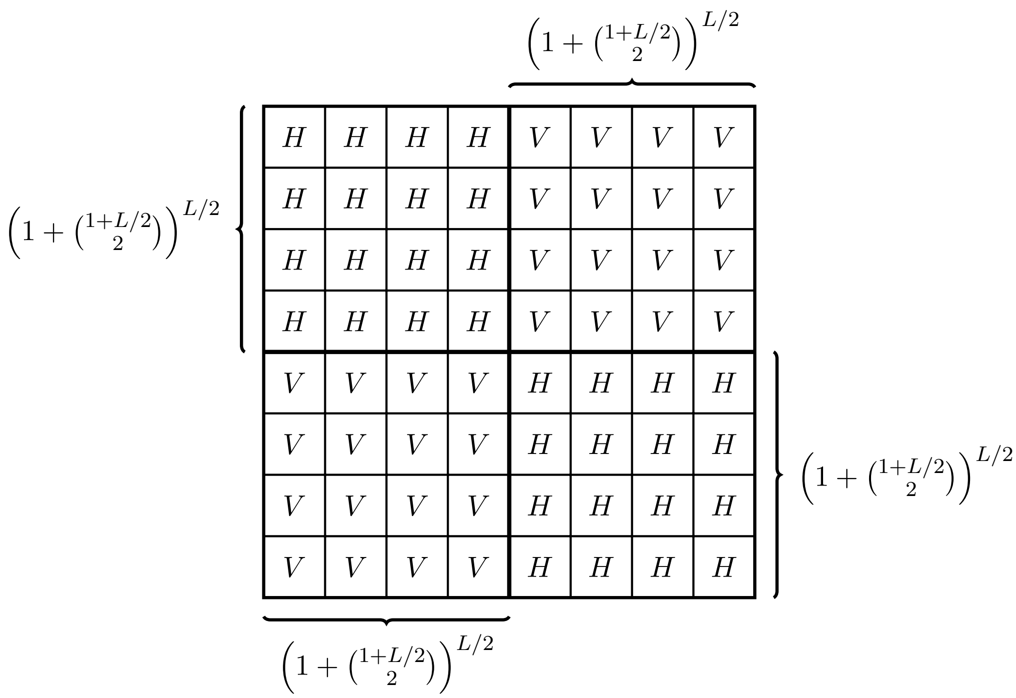

Figure 4.8: If each quadrant of the quartering pattern contains only horizontal or vertical strips respectively, there are $(1+\binom{1+L/2}{2})^{L/2}$ *ways to place strips in each quadrant, so that* $T(L,L) \geq [(1+\binom{1+L/2}{2})^{L/2}]^4 = (\frac{L^2}{8})^{2L}(1+\frac{2}{L}+\frac{8}{L^2})^{2L}$*, or* $\log T(L,L) \geq 2L\log(L^2/8) + O(1)$.

4.6 Conclusion

Taking the results of Theorem 4.34 and Theorem 4.37 together, we conclude with a concise estimate for $T(L,L)$ for sufficiently large L:

Corollary 4.38. *For all sufficiently large* L, $\log T(L,L) = 4L\log L - 2L\log 8 + O(L^{2/3})$.

It is natural to compare this result with the asymptotic formula of Theorem 2.29, which implies that $\log T(m,n) = 2n\log m + m\log n + O(n)$, and therefore seems to suggest that $\log T(n,n) = 3n\log n + O(n)$, which would contradict Theorem 4.38. However, the suggestion is false, as the two results apply to different situations. Theorem 4.38 applies to the regime in which both sides of the chessboard are nearly equal (or indeed are exactly equal), whereas Theorem 2.29 concerns the case where one side is (much) larger than another.

The above discussion also suggests that, as $n \to \infty$, the asymptotic value of $T(n,n)$ is greater than that of $T(m,n)$ for any fixed value of m. This suggestion is true, and follows from the facts that $T(m,n) = T(n,m)$ and $T(m,j) < T(m,k)$ if $j < k$. While both of these facts are evident from the definition of ROSAs, we note that the latter fact that $T(m,j) < T(m,k)$ if $j < k$ – which is equivalent to $T(m,k) < T(m,k+1)$ for all m and k – can also be proven by the arguments in Chapter 2, regarding the generating functions as analytic objects. Indeed, an application of Leibniz's differentiation formula to the result of Theorem 2.13 shows that

$$\mathcal{F}_0^{(k+1)}(x) = w_m(0)x\mathcal{T}_m^{(k+1)}(x) + (k+1)w_m(0)\mathcal{T}_m^{(k)}(x)$$

or, equivalently,

$$T(m,k) = \frac{\mathcal{T}_m^{(k)}(0)}{k!} = \left(\frac{1}{w_m(0)}\right)\frac{\mathcal{F}_0^{(k+1)}(0)}{(k+1)!}$$

The claim that the coefficients $T(m,k)$ of $\mathcal{T}_m(x)$ are increasing is thus equivalent to the claim that the coefficients of $\mathcal{F}_0(x)$ are increasing, which follows from Equation 2.14, as the eigenvalues of $D_m \cdot A_m$ are greater than 1 (or equal to 1).

Thus, we should not directly compare the asymptotic formula of Theorem 2.29 to that of Theorem 4.38, but rather we should try to understand how the former result transitions into the latter. We hope to pursue this in future research, and we hope that such understanding will take us closer to an exact formula for all ROSAs on any chessboard.

APPENDIX A

EXACT FORMULAS FOR $T(M,N)$ FOR SOME FIXED VALUES OF m

We list closed form expressions for the generating functions $\mathcal{T}_m(x) = \sum_{n\geq 0} T(m,n)x^n$ for $m = 1,2,3,4,5,6$, and from these we deduce exact formulas for the number $T(m,n)$ of ROSAs on an $m \times n$ chessboard for $m \in \{1,2,3,4,5,6\}$ and any integer $n \geq 0$.

The following results were obtained using the transfer matrix methods of Chapter 2, and were computed using Maple [1], making particular use of the native *LinearAlgebra* library. We will list all of the corresponding Maple code in Appendix B.

Theorem A.1. *The generating function $\mathcal{T}_1(x)$ for the sequence $\{T(1,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $1\times n$ chessboard is given by*

$$\mathcal{T}_1(x)=\frac{1}{1-x}-\frac{1}{1-2x}+\frac{1}{(1-2x)^2}$$

and hence $T(1,n)=1+n2^n$ for all non-negative integers n.

Theorem A.2. *The generating function $\mathcal{T}_2(x)$ for the sequence $\{T(2,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $2\times n$ chessboard is given by*

$$\mathcal{T}_2(x)=\frac{1}{1-x}-\frac{2}{1-2x}+\frac{2}{(1-2x)^2}+\frac{2}{1-4x}-\frac{3}{(1-4x)^2}+\frac{1}{(1-4x)^3}$$

and hence $T(2,n)=1+(2n)2^n+n^2 4^n$ for all non-negative integers n.

Theorem A.3. *The generating function $\mathcal{T}_3(x)$ for the sequence $\{T(3,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $3\times n$ chessboard is given by*

$$\begin{aligned}\mathcal{T}_3(x)=\frac{25}{16(1-x)}&-\frac{1012}{25(1-2x)}+\frac{36}{25(1-2x)^2}\\&+\frac{1881}{32(1-3x)}-\frac{783}{32(1-3x)^2}+\frac{9}{2(1-3x)^3}\\&-\frac{1760}{81(1-4x)}-\frac{224}{27(1-4x)^2}+\frac{128}{81(1-4x)^3}\\&+\frac{2113189}{64800(1-7x)}-\frac{1673}{7200(1-7x)^2}-\frac{2083}{135(1-7x)^3}+\frac{8}{(1-7x)^4}\end{aligned}$$

and hence $T(3,n)$ has the closed form

$$\begin{aligned}T(3,n)=\frac{25}{16}&+2^n\left(\frac{-1012}{25}+\frac{36(n+1)}{25}\right)\\&+3^n\left(\frac{1881}{32}-\frac{783(n+1)}{32}+\frac{9}{2}\binom{n+2}{2}\right)\\&+4^n\left(\frac{-1760}{81}-\frac{224(n+1)}{27}+\frac{128}{81}\binom{n+2}{2}\right)\\&+7^n\left(\frac{2113189}{64800}-\frac{1673(n+1)}{7200}-\frac{2083}{135}\binom{n+2}{2}+8\binom{n+3}{3}\right)\end{aligned}$$

Theorem A.4. *The generating function $\mathcal{T}_4(x)$ for the sequence $\{T(4,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $4 \times n$ chessboard is given by*

$$\begin{aligned}
\mathcal{T}_4(x) = \frac{361}{100(1-x)} &- \frac{889462264}{2460375(1-2x)} - \frac{315256}{18225(1-2x)^2} \\
&+ \frac{396241821}{524288(1-3x)} - \frac{10115667}{65536(1-3x)^2} + \frac{117009}{2048(1-3x)^3} \\
&+ \frac{9243898448}{175039(1-4x)} - \frac{25070512}{83349(1-4x)^2} + \frac{174080}{3969(1-4x)^3} \\
&- \frac{645442475}{93312(1-5x)} + \frac{3481425}{15552(1-5x)^2} - \frac{833225}{1296(1-5x)^3} + \frac{12325}{108(1-5x)^4} \\
&- \frac{471969631259}{2985984000(1-7x)} - \frac{476738297}{3317760(1-7x)^2} \\
&\qquad - \frac{28448567}{276480(1-7x)^3} + \frac{453691}{13824(1-7x)^4} \\
&+ \frac{74906009752781929}{619431631257600(1-11x)} + \frac{28362110652389}{12290310440(1-11x)^2} \\
&\qquad - \frac{1706381921}{97542144(1-11x)^3} - \frac{606371}{5376(1-11x)^4} + \frac{1225}{24(1-11x)^5}
\end{aligned}$$

and hence $T(4,n)$ has the closed form

$$\begin{aligned}
T(4,n) = \frac{361}{100} &- 2^n\left(\frac{889462264}{2460375} + \frac{315256(n+1)}{18225}\right) \\
&+ 3^n\left(\frac{396241821}{524288} - \frac{10115667(n+1)}{65536} + \frac{117009}{2048}\binom{n+2}{2}\right) \\
&+ 4^n\left(\frac{9243898448}{175039} - \frac{25070512(n+1)}{83349} + \frac{174080}{3969}\binom{n+2}{2}\right) \\
&+ 5^n\left(\frac{-645442475}{93312} + \frac{3481425(n+1)}{15552} - \frac{833225}{1296}\binom{n+2}{2} + \frac{12325}{108}\binom{n+3}{3}\right) \\
&+ 7^n\left(\frac{-471969631259}{2985984000} - \frac{476738297(n+1)}{3317760}\right. \\
&\qquad \left. -\frac{28448567}{276480}\binom{n+2}{2} + \frac{453691}{13824}\binom{n+3}{3}\right) \\
&+ 11^n\left(\frac{74906009752781929}{619431631257600} + \frac{28362110652389(n+1)}{12290310440}\right. \\
&\qquad \left. -\frac{1706381921}{97542144}\binom{n+2}{2} - \frac{606371}{5376}\binom{n+3}{3} + \frac{1225}{24}\binom{n+4}{4}\right)
\end{aligned}$$

Theorem A.5. *The generating function $\mathcal{T}_5(x)$ for the sequence $\{T(5,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $5 \times n$ chessboard is given by*

$$\begin{aligned}
\mathcal{T}_5(x) = {} & \frac{50794129}{4410000(1-x)} - \frac{137556123392}{40186125(1-2x)} - \frac{223857488}{893025(1-2x)^2} \\
& + \frac{148867530544920501}{9358868480000(1-3x)} - \frac{1475040687111}{899891200(1-3x)^2} + \frac{485636735}{1081600(1-3x)^3} \\
& - \frac{99055890145}{1750329(1-4x)} - \frac{88861373}{83349(1-4x)^2} - \frac{644416}{1323(1-4x)^3} + \frac{162}{(1-4x)^4} \\
& + \frac{715046863218175}{15027990912(1-5x)} + \frac{1205382331925}{683090496(1-5x)^2} \\
& \quad - \frac{2968531375}{1724976(1-5x)^3} - \frac{1643825}{8712(1-5x)^4} \\
& + \frac{5450984070174819449}{2176782336000(1-7x)} + \frac{495995672525759}{4031078400(1-7x)^2} \\
& \quad + \frac{403278349649}{22394880(1-7x)^3} - \frac{591001397}{373248(1-7x)^4} + \frac{30625}{54(1-7x)^5} \\
& - \frac{191132077579526}{66976875(1-8x)} + \frac{48096536294}{212625(1-8x)^2} \\
& \quad - \frac{1275364}{81(1-8x)^3} - \frac{770872}{405(1-8x)^4} + \frac{2800}{3(1-8x)^5} \\
& + \frac{376491725085404151707}{129048256512000000(1-11x)} - \frac{46677175268317}{348364800000(1-11x)^2} \\
& \quad - \frac{66157062419459}{33868800000(1-11x)^3} - \frac{418777788539}{362880000(1-11x)^4} + \frac{141933}{400(1-11x)^5} \\
& - \frac{772416754180034937152181}{18715782852344546718 75(1-16x)} + \frac{716240720294675465017}{290843556368990625(1-16x)^2} \\
& \quad + \frac{66713198326386532}{32283667040625(1-16x)^3} - \frac{23060586782}{79633125(1-16x)^4} \\
& \quad + \frac{3604456}{3375(1-16x)^5} + \frac{6776}{15(1-16x)^6}
\end{aligned}$$

and hence $T(5,n)$ has the closed form

$$
\begin{aligned}
T(5,n) = \frac{50794129}{4410000} &- 2^n\left(\frac{137556123392}{40186125} + \frac{223857488(n+1)}{893025}\right) \\
&+ 3^n\left(\frac{148867530544920501}{9358868480000} - \frac{1475040687111(n+1)}{899891200} + \frac{485636735}{1081600}\binom{n+2}{2}\right) \\
&+ 4^n\left(\frac{-99055890145}{1750329} - \frac{88861373(n+1)}{83349} - \frac{644416}{1323}\binom{n+2}{2} + 162\binom{n+3}{3}\right) \\
&- 5^n\left(\frac{-715046863218175}{15027990912} - \frac{1205382331925(n+1)}{683090496}\right. \\
&\qquad\left. + \frac{2968531375}{1724976}\binom{n+2}{2} + \frac{1643825}{8712}\binom{n+3}{3}\right) \\
&+ 7^n\left(\frac{5450984070174819449}{2176782336000} + \frac{495995672525759(n+1)}{4031078400}\right. \\
&\qquad\left. + \frac{403278349649}{22394880}\binom{n+2}{2} - \frac{591001397}{373248}\binom{n+3}{3} + \frac{30625}{54}\binom{n+4}{4}\right) \\
&+ 8^n\left(\frac{-191132077579526}{66976875} + \frac{48096536294(n+1)}{212625}\right. \\
&\qquad\left. - \frac{1275364}{81}\binom{n+2}{2} - \frac{770872}{405}\binom{n+3}{3} + \frac{2800}{3}\binom{n+4}{4}\right) \\
&+ 11^n\left(\frac{3764917250854041517 07}{129048256512000000} - \frac{46677175268317(n+1)}{348364800000}\right. \\
&\qquad - \frac{66157062419459}{33868800000}\binom{n+2}{2} - \frac{418777788539}{362880000}\binom{n+3}{3} \\
&\qquad\left. + \frac{141933}{400}\binom{n+4}{4}\right) \\
&+ 16^n\left(\frac{-77241675418003493715 2181}{18715782852344546718 75} + \frac{716240720294675465017(n+1)}{290843556368990625}\right. \\
&\qquad + \frac{66713198326386532}{32283667040625}\binom{n+2}{2} - \frac{23060586782}{79633125}\binom{n+3}{3} \\
&\qquad\left. + \frac{3604456}{3375}\binom{n+4}{4} + \frac{6776}{15}\binom{n+5}{5}\right)
\end{aligned}
$$

Theorem A.6. *The generating function $\mathcal{T}_6(x)$ for the sequence $\{T(6,n)\}_{n\geq 0}$ whose n-th term is the number of ROSAs on a $6\times n$ chessboard is given by*

$$
\begin{aligned}
\mathcal{T}_6(x) = {} & \frac{57336304661476}{1191317675625(1-x)} - \frac{49434478326603281111}{1350253800000000(1-2x)} - \frac{42920736964541}{14288400000(1-2x)^2} \\
& + \frac{2891485957085700003559691}{1083699434877696000(1-3x)} \\
& \quad - \frac{1413858896262753267 01}{5292793332736000(1-3x)^2} + \frac{13696900458281}{4783105600(1-3x)^3} \\
& - \frac{2645965612135793}{1640558367(1-4x)} \\
& \qquad - \frac{2977793528827}{121522842(1-4x)^2} - \frac{20405090828}{964467(1-4x)^3} + \frac{5305520}{729(1-4x)^4} \\
& - \frac{2847865871363641947846627475}{537931516706927082432(1-5x)} + \frac{806259529978787396848 25}{68491407780357408(1-5x)^2} \\
& \quad + \frac{1560964221378025}{198194920308(1-5x)^3} - \frac{341441519975}{16328466(1-5x)^4} \\
& + \frac{635053933560827133}{81920000000(1-6x)} - \frac{1557923106134457}{409600000(1-6x)^2} \\
& \quad + \frac{22073970059037}{20480000(1-6x)^3} - \frac{20325884319}{102400(1-6x)^4} + \frac{2573829}{128(1-6x)^5} \\
& + \frac{2006227081107167951244107}{183666009600000000(1-7x)} - \frac{4937860033653514023 07}{204073344000000(1-7x)^2} \\
& \quad - \frac{486381104228336633 47}{51018336000000(1-7x)^3} - \frac{994707046236211}{18895680000(1-7x)^4} + \frac{32238668}{2187(1-7x)^5} \\
& - \frac{1114140880200435616}{32162295375(1-8x)} + \frac{3947416877923936}{1531537875(1-8x)^2} \\
& \quad + \frac{4210309471808}{4862025(1-8x)^3} - \frac{31373341696}{138915(1-8x)^4} + \frac{49655552}{1323(1-8x)^5} \\
& - \frac{4993691901767511925}{72589644288(1-10x)} - \frac{1220019361300625}{61725888(1-10x)^2} - \frac{5461808845775}{1524096(1-10x)^3} \\
& \quad + \frac{667650265525}{979776(1-10x)^4} - \frac{943461575}{2916(1-10x)^5} + \frac{4217125}{108(1-10x)^6} \\
& + \frac{4214574363849930141746 3519}{864162432000000000(1-11x)} + \frac{146394969007535726786 29}{480090240000000(1-11x)^2} \\
& \quad - \frac{1952674788019889}{21168000000(1-11x)^3} - \frac{23303950898989}{226800000(1-11x)^4} + \frac{29796768}{240000(1-11x)^5}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2968054223097158473361}{7149540234375(1-12x)} + \frac{1192657008611987}{99041250(1-12x)^2} + \frac{7490951878742}{5359375(1-12x)^3} \\
&\quad - \frac{5880158106}{21875(1-12x)^4} - \frac{29796768}{4375(1-12x)^5} + \frac{1232352}{125(1-12x)^6} \\
&+ \frac{20234485472093696488426109236448}{88432073977327983246093750(1-16x)} + \frac{17156097582421877075033 6}{2617592007320915625(1-16x)^2} \\
&\quad - \frac{693940491150589 01248}{7844931090871875(1-16x)^3} - \frac{961903204820992}{32251415625(1-16x)^4} \\
&\quad - \frac{134972240896}{9021375(1-16x)^5} + \frac{1338681344}{273375(1-16x)^6} \\
&- \frac{39204838298637902282863397489288 99997419}{2244663082527760654173388800000000(1-22x)} \\
&\quad - \frac{9866017453013016777759145585 3151}{6549553812230860919040000000(1-22x)^2} \\
&\quad + \frac{5239836055374741998127487637}{12874448498168678400000 0(1-22x)^3} + \frac{1258960497238501 2251}{581509303200000(1-22x)^4} \\
&\quad - \frac{7325375745431}{1518142500(1-22x)^5} - \frac{58691576}{4725(1-22x)^6} + \frac{5120}{(1-22x)^7}
\end{aligned}
$$

and hence $T(6,n)$ *has the closed form*

$$
\begin{aligned}
T(6,n) = \frac{57336304661476}{1191317675625} &- 2^n\left[\frac{494344783266032811111}{1350253800000000} + \frac{429207369645 41(n+1)}{14288400000}\right] \\
&+ 3^n\left[\frac{28914859570857000355 9691}{1083699434877696000}\right. \\
&\quad \left. - \frac{14138588962627532670 1(n+1)}{5292793332736000} + \frac{13696900458281}{4783105600}\binom{n+2}{2}\right] \\
&+ 4^n\left[\frac{-2645965612135793}{1640558367} - \frac{2977793528827(n+1)}{121522842}\right. \\
&\quad \left. - \frac{20405090828}{964467}\binom{n+2}{2} + \frac{5305520}{729}\binom{n+3}{3}\right] \\
&- 5^n\left[\frac{28478658713636419478466 27475}{537931516706927082432} - \frac{806259529978787396848 25(n+1)}{68491407780357408}\right. \\
&\quad \left. - \frac{1560964221378025}{198194920308}\binom{n+2}{2} + \frac{341441519975}{16328466}\binom{n+3}{3}\right] \\
&+ 6^n\left[\frac{635053933560827133}{81920000000} - \frac{1557923106134457(n+1)}{409600000}\right. \\
&\quad \left. + \frac{22073970059037}{20480000}\binom{n+2}{2} - \frac{20325884319}{102400}\binom{n+3}{3} + \frac{2573829}{128}\binom{n+4}{4}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+7^n\Bigg[\frac{200622708110716795124410 7}{183666009600000000} \\
&\quad-\frac{4937860033653514023 07(n+1)}{204073344000000}-\frac{4863811042283366334 7}{51018336000000}\binom{n+2}{2} \\
&\quad-\frac{994707046236211}{18895680000}\binom{n+3}{3}+\frac{32238668}{2187}\binom{n+4}{4}\Bigg] \\
&+8^n\Bigg[\frac{-1114140880200435616}{32162295375}+\frac{3947416877923936(n+1)}{1531537875} \\
&\quad+\frac{4210309471808}{4862025}\binom{n+2}{2}-\frac{31373341696}{138915}\binom{n+3}{3}+\frac{49655552}{1323}\binom{n+4}{4}\Bigg] \\
&+10^n\Bigg[\frac{-4993691901767511925}{72589644288}-\frac{1220019361300625(n+1)}{61725888} \\
&\quad-\frac{5461808845775}{1524096}\binom{n+2}{2}+\frac{667650265525}{979776}\binom{n+3}{3} \\
&\quad-\frac{943461575}{2916}\binom{n+4}{4}+\frac{4217125}{108}\binom{n+5}{5}\Bigg] \\
&+11^n\Bigg[\frac{4214574363849930141746351 9}{86416243200000000} \\
&\quad+\frac{1463949690075357267862 9(n+1)}{480090240000000}-\frac{1952674788019889}{21168000000}\binom{n+2}{2} \\
&\quad-\frac{23303950898989}{226800000}\binom{n+3}{3}+\frac{29796768}{240000}\binom{n+4}{4}\Bigg] \\
&+12^n\Bigg[\frac{-2968054223097158473361}{7149540234375}+\frac{1192657008611987(n+1)}{99041250} \\
&\quad+\frac{7490951878742}{5359375}\binom{n+2}{2}-\frac{5880158106}{21875}\binom{n+3}{3} \\
&\quad-\frac{29796768}{4375}\binom{n+4}{4}+\frac{1232352}{125}\binom{n+5}{5}\Bigg] \\
&-16^n\Bigg[\frac{-202344854720936964884261092364 8}{884320739773279832460937 5} \\
&\quad-\frac{17156097582421877075033 6(n+1)}{2617592007320915625} \\
&\quad+\frac{693940491150589012 48}{7844931090871875}\binom{n+2}{2}+\frac{961903204820992}{32251415625}\binom{n+3}{3} \\
&\quad+\frac{134972240896}{9021375}\binom{n+4}{4}-\frac{1338681344}{273375}\binom{n+5}{5}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ 22^n \left[\frac{-3920483829863790228286339748928899974 19}{224466308252776065417338880000000} \right. \\
&\quad - \frac{98660174530130167777591455853151(n+1)}{65495538122308609190400000000} \\
&\quad + \frac{5239836055374741998127487637}{12874448498168678400000} \binom{n+2}{2} \\
&\quad + \frac{1258960497238501 2251}{581509303200000} \binom{n+3}{3} - \frac{7325375745431}{1518142500} \binom{n+4}{4} \\
&\quad \left. - \frac{58691576}{4725} \binom{n+5}{5} + 5120 \binom{n+6}{6} \right]
\end{aligned}
$$

APPENDIX B

MAPLE CODE FOR CONSTRUCTING TRANSFER MATRICES

In this appendix, we provide the Maple code used to construct the transfer matrices that were used to compute the results of Chapter 3 and Appendix A.

First, we consider the unweighted adjacency matrices A_m for ROSAs. These were computed using the Maple procedure

$AdjacencyMatrix := \mathbf{proc}(m)$
  if $m \leq 0$ **then**
    return $LinearAlgebra :- IdentityMatrix(1)$
  else
    return $LinearAlgebra :- KroneckerProduct(AdjacencyMatrix(m-1),$
      $LinearAlgebra :- Matrix([[1,1,0],[0,1,1],[0,0,1]]))$
  end if
end proc

Unweighted adjacency matrices $\tilde{A}_m$ for URSAs were computed similarly, only substituting the matrix $[[1,1,0],[0,1,1],[0,0,1]]$ with the matrix $[[1,1,0],[1,1,1],[1,1,1]]$.

To construct the diagonal weight matrices, we first construct the necessary weights. We do so by measuring the distances between each unit square in the column in state 1, and then sum the number of ways to place a horizontal strip between these squares in state 1, counting the empty square only once. The corresponding code is as follows.

$RosaWeight := \textbf{proc}(m, x)$
    local $j,\ r,\ y := x,\ sum := 1,\ count := 0;$
    for j **from** 0 **to** $m-1$ **do**
        $r := y$ **mod** $3;$
        if $r = 1$ **then**
            $sum := sum + (count \cdot (count + 1)/2);$
            $count := 0;$
        else
            $count := count + 1;$
        end if;
        $y := (y - r)/3;$
    end do;
    return $sum + (count \cdot (count + 1)/2);$
end proc;

The weight matrix D_m for ROSAs can be constructed from the previous procedure thusly:

$RosaWeightMatrix := \textbf{proc}(m)$
    local $i,\ v = [];$
    for i **from** 0 **to** $3^m - 1$ **do**
        $v := [op(v),\ op(RosaWeight(m,\ i))]$
    end do;
    return $LinearAlgebra :- DiagonalMatrix(v);$
end proc;

Alternatively, we can construct the weighted adjacency matrix $D_m \cdot A_m$ using a modular approach. We noted in Chapter 2 that the weight of a column (or a vertex of a digraph) is determined by the states in that column. The transitions are also determined by the states in the column; for example, the allowed transitions for a square in state 0 are given by the first row of the single row adjacency matrix A_1. Since a column corresponds to a independent m-tuple, the allowed transitions for the entire column is then given by Kronecker product of these single row transitions. We therefore compute $D_m \cdot A_m$ by computing the weight and the allowed transitions of each column simultaneously, and summing the products over all possible columns. To make this procedure more general, we take as additional matrices A, B, and C, which are intended to denote the single row transfers from a square in state 0, 1, and 2 respectively. Thus, if A is the matrix with zeroth row $[1, 1, 0]$, B is the matrix with first row $[0, 1, 1]$, and C is the matrix with second row $[0, 0, 1]$ (and zeroes in all other rows), we would obtain $D_m \cdot A_m$. However, with other values of A, B, and C, we may investigate certain other results, such as the ansatz given at the end of Chapter 2 for obtaining the asymptotic formula from the exact results. The procedure thus described is given on the following page.

```
RosaTransferMatrix := proc(m, A, B, C)
local i, j, x, r, sum, count, M, T;
    T := LinearAlgebra :- ZeroMatrix(3^m);
    for i from 0 to 3^m − 1 do
        sum := 1;
        count := 0;
        M := LinearAlgebra :- IdentityMatrix(1);
        x := i;
        for j from 0 to m − 1 do
            r := x mod 3;
            if r = 1 then
                sum := sum + (count · (count + 1)/2);
                count := 0;
                M := LinearAlgebra :- KroneckerProduct(M, B)
            else
                count := count + 1;
                if r = 0 then
                    M := LinearAlgebra :- KroneckerProduct(M, A)
                else
                    M := LinearAlgebra :- KroneckerProduct(M, C)
                end if;
            end if;
            x := (x − r)/3;
        end do;
        sum := sum + (count · (count + 1)/2);
        T := T + sum · M;
    end do;
end proc;
```

We now turn to procedures that construct matrices for URSAs. We noted that the procedure to compute the unweighted adjacency matrix $\tilde{A}_m$ is very much the same as that for ROSAs, where we merely replace the single row adjacency matrix A_1 for ROSAs with the single row adjacency matrix $\tilde{A}_1$ for URSAs.

For the weights of URSAs, we could use an exact formula for the number of ways to place vertical strips between horizontal strips in a column. However, here, we exploit the fact that these numbers are the coefficients of a certain rational generating function.

$UrsaWeight := \mathbf{proc}(m, x)$
$\mathbf{local}\ j,\ r,\ y := x,\ prod := 1,\ count := 0;$
  $\mathbf{for}\ j\ \mathbf{from}\ 0\ \mathbf{to}\ m-1\ \mathbf{do}$
    $r := y\ \mathbf{mod}\ 3;$
    $\mathbf{if}\ r \neq 0\ \mathbf{then}$
      $prod := prod \cdot eval\left(\frac{1}{(count)!} \cdot diff\left(\frac{1-z}{1-3z+z^2}, [z\$count]\right), z=0\right);$
      $count := 0;$
    $\mathbf{else}$
      $count := count + 1;$
    $\mathbf{end\ if};$
    $y := (y-r)/3;$
  $\mathbf{end\ do};$
  $\mathbf{return}\ prod \cdot eval\left(\frac{1}{(count)!} \cdot diff\left(\frac{1-z}{1-3z+z^2}, [z\$count]\right), z=0\right);$
$\mathbf{end\ proc};$

Thus, a procedure for constructing the matrix $\tilde{D}_m$ of weights for URSAs is

$UrsaWeightMatrix := \mathbf{proc}(m)$
$\mathbf{local}\ i,\ v := [];$
  $\mathbf{for}\ i\ \mathbf{from}\ 0\ \mathbf{to}\ 3^m - 1\ \mathbf{do}$
    $v := [op(v), op(UrsaWeight(m, i))]$
  $\mathbf{end\ do};$
  $\mathbf{return}\ LinearAlgebra :- DiagonalMatrix(v);$
$\mathbf{end\ proc};$

Similar to the case of ROSAs, we conclude with a procedure that computes the weighted adjacency matrix $\tilde{D}_m \cdot \tilde{A}_m$ by computing the weight of and the transfers from each column (or vertex of the digraph) simultaneously given single row transfer matrices A, B, and C from states $\tilde{0}$, $\tilde{1}$, and $\tilde{2}$ respectively.

$UrsaTransferMatrix :=$ **proc**(m, A, B, C)
local $i,\ j,\ x,\ r,\ prod,\ count,\ M,\ T := LinearAlgebra :- ZeroMatrix(3^m);$
    for i **from** 0 **to** $3^m - 1$ **do**
        $prod := 1;$
        $count := 0;$
        $M := LinearAlgebra :- IdentityMatrix(1);$
        $x := i$
        for j **from** 0 **to** $m - 1$ **do**
            $r := x$ **mod** $3;$
            if $r \neq 0$ **then**
                $prod := prod \cdot eval\left(\frac{1}{(count)!} \cdot diff\left(\frac{1-z}{1-3z+z^2}, [z\$count]\right), z = 0\right);$
                $count := 0;$
                if $r = 1$ **then**
                    $M := LinearAlgebra :- KroneckerProduct(M, B)$
                else
                    $M := LinearAlgebra :- KroneckerProduct(M, C)$
                end if;
            else
                $count := count + 1;$
                $M := LinearAlgebra :- KroneckerProduct(M, A)$
            end if;
            $x := (x - r)/3;$
        end do;
        $prod := prod \cdot eval\left(\frac{1}{(count)!} \cdot diff\left(\frac{1-z}{1-3z+z^2}, [z\$count]\right), z = 0\right);$
        $T := T + prod * M;$
    end do;
    return T;
end proc;

BIBLIOGRAPHY

[1] Maple 18.01. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario, Canada.

[2] Martin Aigner. *A Course in Enumeration.* Number 238 in Graduate Texts in Mathematics. Springer, Berlin, 2007.

[3] Rodney J. Baxter. *Exactly Solved Models in Statistical Mechanics.* Dover Publications, Mineola, NY, 2007.

[4] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization.* Cambridge University Press, New York, 2004.

[5] Richard A. Brualdi. Combinatorial verification of the elementary divisors of tensor products. *Linear Algebra Appl.*, 71, 1985.

[6] John Cardy. *Scaling and Renormalization in Statistical Physics.* Number 5 in Cambridge Lecture Notes in Physics. Cambridge University Press, New York, 1996.

[7] Dragos M. Cvetković, Michael Doob, and Norst Sachs. *Spectra of Graphs: Theory and Application.* Pure and Applied Mathematics. Academic Press, New York, 1979.

[8] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics.* Cambridge University Press, Cambridge, 2009.

[9] Michael J. Larsen. The problem of kings. *Elec. J. Comb.*, 2, 1995.

[10] Elliott H. Lieb. Solution of the dimer problem by the transfer matrix method. In B. Nachtergaele, J.P. Solovej, and J. Yngvason, editors, *Condensed Matter Physics and Exactly Soluble Models: Selecta of Elliott H. Lieb*. Springer, New York, 2004.

[11] Percy A. MacMahon. *Combinatory Analysis*. AMS Chelsea Publishing, Providence, RI, 1984.

[12] Richard J. Mathar. Tilings of rectangular regions by rectangular tiles: Counts derived from transfer matrices. *ArXiv e-prints*, Jun 2014.

[13] Jiří Matoušek and Jaroslav Nešetřil. *Invitation to Discrete Mathematics*. Oxford University Press, Oxford, 1998.

[14] Jerome K. Percus. *Combinatorial Methods*. Number 4 in Applied Mathematical Sciences. Springer-Verlag, New York, 1971.

[15] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. Number 49 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York, 2012.

[16] Willi-Hans Steeb and Yorick Hardy. *Matrix Calculus and Kronecker Product: A Practical Approach to Linear and Multilinear Algebra*. World Scientific Publishing Co., Hackensack, New Jersey, second edition, 2011.

[17] Colin J. Thompson. *Mathematical Statistical Mechanics*. Princeton University Press, Princeton, NJ, 1972.

[18] Robert J. Walker. *Algebraic Curves*. Princeton University Press, Princeton, NJ, 1950.

[19] Herbert S. Wilf. *generatingfunctionology*. Academic Press, New York, 1994.

[20] Herbert S. Wilf. The problem of the kings. *Elec. J. Comb.*, 2, 1995.

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Participated in and coordinated independent reading courses in Algebraic Combinatorics, Local Field Theory, and Representation Theory

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Teaching Experience

Associate Instructor: Indiana University, Bloomington, IN: 2007 – 2014

Taught my own standalone sections of College Algebra and Pre-Calculus Mathematics

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Led and graded for recitation sections and nightly help sessions in conjunction with Calculus I and Calculus II courses

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Tutored in various levels of mathematics ranging from primary school to college

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Wrote macros in Visual Basic and Java for image processing software in order to perform a quantitative analysis of angiogenesis in laboratory mice

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