Conceptualism, Ramified Logic, and Nominalized Predicates

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Abstract

Conceptualism differs from intuitionism in being a theory about the construction of concepts and not about the construction of proofs. Constructive conceptualism is similar to nominalism in excluding an impredicative comprehension principle, but differs from nominalism in the kind of ramified predicative logic each validates. Ramified constructive conceptualism leads in a natural way to holistic conceptualism, and, unlike nominalism, both can extended to a type of realism in which some nominalized predicates denote abstract objects. Intermediate positions of conceptual realism are distinguished regarding which concepts can be projected to have abstract objects corresponding to their nominalizations.

The problem of universals as the problem of what predicates stand for in meaningful assertions is discussed in contemporary philosophy mainly in terms of the opposing theories of nominalism and logical realism. Conceptualism, when it is mentioned, is usually identified with intuitionism, which is not a theory of predication but a theory of the activity of constructing proofs in mathematics. Both intuitionism and conceptualism are concerned with the notion of a mental construction, to be sure, and both maintain that there can be only a potentially infinite number of such constructions. But whereas the focus of concern in intuitionism is with the construction of proofs, in conceptualism our concern is with the construction of concepts. This difference sets the two frameworks apart and in pursuit of different goals, and in fact it is not at all clear how the notion of a mental construction in the one framework is related to that in the other. This is especially true insofar as mathematical objects, according to intuitionism, are nothing but mental constructions, whereas in conceptualism concepts are anything but objects.

In any case, whatever the relation between the two, our concern in this paper is with conceptualism as a philosophical theory of predication and not with intuitionism as a philosophy of mathematics.

Now conceptualism differs from nominalism insofar as it posits universals, namely, concepts, as the semantic grounds for the correct or incorrect application of predicate expressions. Conceptualism differs from logical realism, on the other hand, insofar as the universals it posits are not assumed to exist independently of the human capacity for thought and representation. Concepts, in other words, are neither predicate expressions nor independently real properties and relations. But then, at least for the kind of conceptualism we have in mind here, neither are they mental images or ideas in the sense of particular mental occurrences. That is, concepts are not objects (saturated individuals) but are rather cognitive capacities, or cognitive structures otherwise based upon such capacities, to identify and classify or characterize and relate objects in various ways. Concepts, in other words, are intersubjectively realizable cognitive abilities which may be exercised by different persons at the same time as well as by the same person at different times. It is for this reason that we speak of concepts as *objective universals*, even though they are not independently real properties and relations.

As cognitive structures, concepts in the sense intended here are not Fregean concepts (which for Frege are independently real unsaturated functions from objects to truth values). But they may be modeled by the latter (assuming that there are Fregean concepts to begin with) especially since as cognitive capacities which need not be exercised at any given time (or even ever for that matter), concepts in the sense intended here also have an unsaturated nature corresponding to, albeit different from, the unsaturated nature of Fregean concepts. Thus, in particular, the saturation (or exercise) of a concept in the sense intended here results not in a truth value but a mental act, and, if overtly expressed, a speech act as well. The unsaturatedness of a concept consists in this regard in its non-occurrent or purely dispositional status as a cognitive capacity, and it is the exercise (or saturation) of this capacity as a cognitive structure which informs particular mental acts with a predicable nature (or with a referential nature in the case of concepts corresponding to quantifier expressions).

In regard to the construction of concepts, or what is usually called concept-formation, we distinguish two major types or forms of conceptualism, which we shall call *constructive* and *holistic conceptualism,* respectively. Constructive conceptualism, as we shall see, has affinities with nominalism with which it is sometimes confused, especially because both validate only a predicative logic with a restricted form of comprehension principle. Holistic conceptualism, on the other hand, has affinities with logical realism with which it in turn is sometimes confused insofar as both similarly validate an unrestricted or impredicative comprehension principle. Holistic conceptualism is not opposed to constructive conceptualism, it should be noted, even though a strict form of the latter will reject even the possibility of impredicative concept-formation. Such a strict form of constructive conceptualism will maintain that all concept-formation must be in accord with the so-called vicious-circle principle that concepts cannot be constructed or formed in a manner that involves or presupposes a totality to which they belong or form a part. Holistic conceptualism does violate this principle, but in doing so it presupposes the construction of predicative concepts that are in accord with the principle and that form the basis of the process which leads to impredicative concept-formation. (We shall hereafter refer to the concepts which can be formed in constructive conceptualism as *predicative* concepts.) For this reason we shall be primarily concerned in this paper with the construction of a logic for constructive conceptualism and only secondarily with how this logic is to be extended to the impredicative logic of holistic conceptualism without nullifying the constructive nature of the predicative concepts it posits.

The Poincare–Russell vicious-circle principle is interpreted in conceptualism as a principle about concept-formation, it should be emphasized, and not about the introduction of new objects. This is because concepts are not objects to begin with, and a pattern of concept-formation that is in accord with the vicious-circle principle will involve only the construction of concepts and not the introduction of new objects. There is a realist counterpart to the vicious-circle principle, however, and this counterpart does involve conceptualism insofar as the objects in question are introduced only in a manner that correlates them with concepts—the way classes, for example, are correlated with the concepts whose extensions they are. Such objects might not be classes, of course, but whatever they are we shall associate the correlation in question with the nominalization transformation of predicates into abstract singular terms. It is in its account of nominalized predicates, in other words, that constructive conceptualism also provides an account of the realist counterpart of the vicious-circle principle. We shall refer to the assumption that nominalized predicates actually denote as *conceptual realism* (or even *conceptual Platonism* if the objects in question are assumed to be the properties and relations we purport to denote with the nominalized predicates of natural language). Following Frege, we shall refer to the objects denoted by nominalized predicates as *concept-correlates*. As part of our goal of constructing a logic for constructive conceptualism, accordingly, we shall also be concerned with developing within this framework a logic of nominalized predicates.

1 Nominalism versus Conceptualism as Predicative Second-Order Logics

The basic principle of standard predicative second-order logic is the restricted comprehension principle:

(CP!) $$(\exists F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi],$$

where F is an n-place predicate variable that does not occur in φ, and φ is a formula in which no predicate variable has a bound occurrence, and $x_{1,...,}x_n$ are pairwise distinct individual variables. Though this principle appears to posit a universal corresponding to any first-order formula, it need in fact involve no universals beyond first-order formulas themselves. That is, by interpreting the predicate quantifier substitutionally, (CP!) will not involve us in any ontological commitments beyond those we are already committed to in our use of first-order formulas. It is in this sense that nominalism can be said to validate (CP!) and in fact provide a semantics with respect to which standard predicative second-order logic is complete.[1]

In constructive conceptualism, however, predicate quantifiers are interpreted referentially and not substitutionally, and this difference is reflected in the kind of predicative logic that is validated. Consider, for example, an applied theory with E as a 2-place predicate constant and the following as an axiom:

(A) $$(\forall F)(\exists y)(\forall x)[E(x, y) \leftrightarrow F(x)].$$

In standard predicative second-order logic, (A) leads directly to Russell's paradox by simply substituting $\neg E(x, x)$ for F. Yet in constructive conceptualism (A) is not inconsistent and may be interpreted as simply stipulating that every predicative concept has an extension (a basic thesis of conceptual realism).

[1]See Cocchiarella [2] for a semantics and proof of completeness.

It is not the principle (CP!) that is validated in constructive conceptualism, in other words, but the somewhat more restricted principle:

(CCP!) $$(\forall G_1)...(\forall G_k)(\exists F)(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi],$$

where (1) φ is a pure second-order formula (i.e., one in which no non-logical constants occur), (2) F is an n-place predicate variable such that neither it nor the identity sign occur in φ, (3) φ is predicative in nominalism's purely grammatical sense (i.e., no predicate variable has a bound occurrence in φ), (4) $G_1,...,G_k$ are all of the distinct predicate variables occurring (free) in φ, and (5) $x_1,...,x_n$ are pairwise distinct individual variables. Otherwise than replacing (CP!) by (CCP!), we may assume that the (unramified) predicative second-order logic of constructive conceptualism consists of the same axioms and rules as standard predicative second-order logic. We will use '$\vdash_c$', however, to signify the derivability relation of this nonstandard predicative second-order logic.[2]

Now it is clear of course that by clause (3) every instance of (CCP!) is also an instance of (CP!) (because universal generalizations of instances of axiom schemes are also assumed to be instances of those schemes). The converse, however, does not hold; i.e., not every instance of (CP!) is also an instance of (CCP!). The important differences are that in (CP!) the initial quantifiers regarding all of the predicate variables occurring free in the comprehending formula φ are not required as a necessary condition and that φ need not be a *pure* second-order formula in which the identity sign does not occur. That is, in (CP!), but not (CCP!), φ may contain occurrences of the identity sign as well as of any of the predicate constants of the language or applied theory in question. This is allowed in nominalism because in that framework predicate quantifiers are interpreted only substitutionally, and in particular as having first-order formulas as their substituends. It is not acceptable in constructive conceptualism where predicate quantifiers are interpreted referentially and where it is not assumed that any first-order formula, and especially not an identity formula, will stand for a predicative concept merely because it is a first-order formula. Thus, in particular, what follows in constructive conceptualism regarding the predicate constant or first-order formula involved in (A) above is that as posited in (A) such an

[2]See Cocchiarella [3], Chapter 2, for a more detailed discussion of this nonstandard predicative second-order logic.

expression cannot stand for a predicative (relational) concept. That is,

$$(A) \vdash_c \neg(\exists R)(\forall x)(\forall y)[R(x,y) \leftrightarrow E(x,y)].$$

Constructive conceptualism, in other words, is 'free of existential presuppositions' regarding predicate constants (and variables), and in that regard it is free of the presupposition that any formula that is predicative in nominalism's purely grammatical sense must for that reason stand for a value of the bound predicate variables.

Of course if $\varphi(x_1, ...x_n)$ is a first-order formula of an applied theory Σ such that (1) the identity sign does not occur in φ and (2) for each k-place predicate constant P that occurs in φ, for all $k \in \omega$,

$$\Sigma \vdash_c (\exists G)(\forall x_1)..., (\forall x_k)[G(x_1, ..., x_k) \leftrightarrow P(x_1, ..., x_k)],$$

then it is provable in constructive conceptualism that $\varphi(x_1, ..., x_n)$ stands for a predicative concept in the context represented by Σ, i.e.,

$$\Sigma \vdash_c (\exists F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi].$$

What determines which of the predicate constants of Σ stand for predicative concepts and which, if any, do not will of course depend on Σ's domain of discourse and on how that domain is to be conceptually represented. The primitive predicate constants of Σ that are understood to stand for predicative concepts will then be stipulated as doing so in terms of the 'meaning postulates' of Σ.

Another way of seeing the difference between the predicative logics of nominalism and constructive conceptualism is in the role that free predicate variables have in each. In nominalism, of course, free predicate variables are construed as dummy schema letters standing for arbitrary first-order formulas. This means that the substitution rule (where G is a free n-place predicate variable),

$$if \ \vdash \varphi, \ then \ \vdash \varphi[\psi/G(x_1, ..., x_n)],$$

is valid in standard predicative second-order logic only when ψ is predicative in nominalism's purely grammatical sense, i.e., only when no predicate variable has a bound occurrence in ψ. The rule is not derivable in standard predicative second-order logic when ψ contains bound predicate variables, and in fact if such an extension of the rule were allowed in this logic, then—because

$$(\exists F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow G(x_1, ..., x_n)]$$

is an instance of (CP!)—we would be able to derive the full unrestricted impredicative comprehension principle. In the predicative logic of constructive conceptualism, however, the extended substitution rule is not only acceptable but is even derivable (by a simple induction on the proof of φ), and without having the impredicative comprehension principle as a consequence (because the above instance of (CP!) is not also an instance of (CCP!)). This means that the notion of a possible (explicit) definition of a predicate constant is broader in constructive conceptualism than it is in nominalism where only first-order formulas are allowed as definiens. That is, the validity of the extended rule of substitution in constructive conceptualism shows us that definitions in this framework whose definiens contain bound predicate variables will still be noncreative and will still allow for the provable eliminability of defined predicate constants. Thus, it is only by confusing nominalism with constructive conceptualism that one can maintain that concepts are admissible in constructive conceptualism insofar as they are representable by definable predicates.

One important consequence of these differences that should be especially noted here, incidentally, is the irreducibility of identity in the predicative logic that is validated by constructive conceptualism as opposed to its reducibility in principle in the predicative logic that is validated by nominalism. In particular, while both will agree that any applied theory will contain at most finitely many predicate constants, only nominalism can maintain that the formula representing indiscernibility with respect to these finitely many predicate constants is itself a predicative formula and that such indiscernibility suffices in principle for full substitutivity. That is, as far as nominalism is concerned, the identity sign can be eliminated in any applied context in favor of the indiscernibility formula regarding the finitely many predicate constants of the theory describing that context (cp. Quine [7], p. 63f). It is in fact for this reason that nominalism can validate (CP!) even when the identity sign occurs in the comprehending formula.

None of this holds in constructive conceptualism, however, because in general indiscernibility with respect to all of the finitely many predicate constants of a given theory will not amount to a predicative formula; i.e., in general such a formula will not stand for a predicative concept. Moreover, even if it were stipulated in meaning postulates that all of the predicate constants of an applied theory stand for predicative concepts, the indiscernibility formula with respect to those constants would still not suffice for full substitutivity in complex formulas that do not stand for predicative concepts. The identity sign, in other words, is not eliminable or otherwise reducible in constructive conceptualism; and because identity allows for the full substitutivity of singular terms even in impredicative contexts, then an identity formula will not in general stand for a predicative concept, which is why the identity sign is not allowed to occur in instances of (CCP!). Thus, whereas

$$x = y \leftrightarrow (\forall F)[F(x) \leftrightarrow F(y)]$$

is provable in the standard predicative second-order logic that is validated by nominalism, the right-to-left direction of this same formula is not provable in constructive conceptualism's nonstandard predicative second-order logic.

2 Ramified Constructive Conceptualism

The differences between nominalism and constructive conceptualism noted so far are differences of logistic and not of notation. This is no longer so when we turn to ramified second-order logic. Thus the sort of ramification validated in nominalism is one that begins with a language for standard predicative second-order logic and that proceeds from there through a potential infinity of languages $L_1, ..., L_n, ...(n \in \omega)$ where the predicate variables of L_{j+1} have the formulas of L_j as their substituends and where the predicate quantifiers binding those variables are interpreted substitutionally with respect to their substituends. These languages can be cumulative, but since the formulas of L_j may contain bound occurrences of predicate variables, the predicate variables (and constants) of L_{j+1} will have to be distinguished from those of L_j.

This need for distinct predicate variables for each language L_j does not arise in constructive conceptualism, on the other hand, and this is because predicate quantifiers are interpreted referentially and not substitutionally in this framework. The sort of ramification validated in constructive conceptualism, in other words, is not based on a sequence of languages in which predicate quantifiers are interpreted substitutionally but on a potentially infinite sequence of stages of concept-formation where the concepts constructed or formed at one stage become the contents of the concepts constructed or formed at the next. These stages are cumulative and what is added at each stage $j+1$ beyond the initial stage of the predicative concepts of an applied theory are concepts whose construction involves a reference to the totality of concepts constructible at the jth stage. These new concepts are themselves predicative, according to constructive conceptualism, insofar as they do not involve or presuppose any reference to their own totality, i.e., insofar as they are formed in accordance with the vicious-circle principle.

But now since free predicate variables will have at any given stage arbitrary formulas as their substituends, including formulas that contain predicate quantifiers that refer to concepts formed at stages presupposing that stage, what is required is not a different type of predicate variable for each stage but a different type of predicate quantifier. In other words, instead of a sequence of different types of predicate variable for each of nominalism's languages $L_1, ..., L_j, ...(j \in \omega - \{0\})$, what is required in constructive conceptualism is a sequence of different quantifier signs $\forall^1, ..., \forall^j, ...(j \in \omega - \{0\})$ that can all be affixed to the same predicate variables. (We understand $(\exists^j F)$ to abbreviate $\neg(\forall^j F)\neg$.) The predicate quantifier $(\forall^j F)$, where F is an arbitrary n-place predicate variable, will then be understood to refer in a given applied theory to all of the n-ary predicative concepts that can be formed in that theory at the jth stage of the potentially infinite sequence of stages of concept-formation in question.

Thus, as applied at the jth stage, the pattern of concept-formation that in constructive conceptualism is to be in accord with the vicious-circle principle may be described as the following *ramified conceptualist comprehension principle*:

(RCCP!) $$(\forall^j G_1)...(\forall^j G_k)(\exists^j F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi],$$

where (1) φ is a *pure* ramified formula (i.e., one in which no nonlogical constants occur) in which the identity sign does not occur, (2) F is an n-place predicate variable not occurring free in φ, (3) no predicate variable is bound in φ by a quantifier of a stage $\geq j$ (i.e., for all $i \geq j$, $\forall^i$ does not occur in φ), (4) $G_1, ..., G_k$ are all of the predicate variables occurring free in φ, and (5) $x_1, ..., x_n$ pairwise distinct individual variables. Note that because the stages of concept-formation are cumulative, ramified constructive conceptualism validates the following axiom schema where $i \leq j$:

$$(\forall^j F)\varphi \rightarrow (\forall^i F)\varphi.$$

3 From Constructive to Holistic Conceptualism

The nonstandard ramified second-order logic described above differs from standard ramified second-order logic in essentially the same way that the nonstandard predicative second-order logic described earlier differs from standard predicative second-order logic.[3] This difference arises, of course, not only because predicate quantifiers are interpreted referentially in constructive conceptualism but also because the referential logic validated by this framework is 'free of existential presuppositions' regarding predicate expressions. In other words, the *grammatical predicativity* of the definiens for a new predicate constant of an applied context will not always coincide in constructive conceptualism with *semantical predicativity* in the sense of actually standing for a predicative concept of a given stage of concept-formation regarding that context. Such a situation cannot arise in nominalism, of course, precisely because, given nominalism's substitutional interpretation of predicate quantifiers, semantical predicativity is none other than grammatical predicativity.

Note by way of contrast that because first-order quantifiers are interpreted referentially in nominalism as well as in conceptualism, we may assume that the first-order logic that is validated in either or both of these frameworks is 'free of existential presuppositions' regarding singular terms. Indeed, such an assumption is particularly appropriate in any extension of the ramified logic of either of these frameworks that allows nominalized predicates to occur as abstract singular terms. In this way, either or both may assume what we call *the Abelardian thesis* that nominalized predicates in fact denote nothing.

Independently of the existential presuppositions regarding singular terms, the difference between grammatical and semantical predicativity that results from the freedom of such presuppositions for predicate expressions corresponds in constructive conceptualism to the idea that concept-formation is essentially an open process. Indeed, this openness, as represented by a 'gap' between semantical and grammatical predicativity, constitutes in conceptualism in general a kind of conceptual tension or disequilibrium in the structure of concepts that is the motivating or driving force of the pattern of reflective abstraction in which concepts constructed or formed at one stage become the basis of the concepts constructed or formed at the next. Thus, formulas

[3] See Church [1], § 58, for a formulation of standard ramified second order logic.

that are grammatically but not semantically predicative at a given stage of concept-formation may become semantically predicative at the next stage, thereby closing the 'gap' as it were between semantical and grammatical predicativity at the preceding stage. But because that closure will involve introducing new predicate quantifiers corresponding to the idea of taking all of the concepts of the preceding stage as referents, there will then be new formulas at the next stage that are grammatically but not semantically predicative. In this way, the conceptual tension or disequilibrium between grammatical and semantical predicativity will continue on indefinitely through all of the potentially infinite number of stages of concept-formation in question. On the other hand, because there is no such difference between grammatical and semantical predicativity in nominalism, one can only wonder at what motivates a nominalistic construction of standard ramified second-order logic.

Whatever the motivation in nominalism, it is clear that what moves us from one stage of concept-formation to the next in constructive conceptualism is a drive for a closure in which grammatical predicativity finally coincides with semantical predicativity. Such a closure cannot be realized in constructive conceptualism, of course, because it amounts in effect to a violation of the vicious-circle principle, and being in accord with the vicious-circle principle is the one constraint that characterizes the construction of concepts in constructive conceptualism. In holistic conceptualism, however, this constraint is violated, or rather transcended, by the allowance for an idealized transition to a limit where impredicative concept-formation becomes possible. Such an idealized transition is conceptually similar to, but ontologically different from, an actual transition to a limit at an infinite or ω-stage of concept-formation that is the summation of all of the finite stages. Of course, there cannot really be an infinite stage of concept-formation for concepts as cognitive capacities, which is why an idealized transition to a limit should not be confused with an actual transition. This is not to say, on the other hand, that there cannot be an actual transition to a limit regarding all of the classes that are the extensions of the predicative concepts constructed at each finite stage. But such an actual transition is a realist posit regarding classes as objects and should not be confused with an idealized transition regarding the process of concept-formation. In fact, the realist posit may even be in accord with the realist counterpart of the vicious-circle principle, in which case classes might also be posited corresponding to an $\omega + 1$ stage, and then corresponding to an $\omega + 2$ stage, and so on for each recursive ordinal α.[4] An idealized transition regarding the process of concept-formation, on the other hand, is a conceptual transition based on some form of closure, or equilibration of cognitive structure, which in the present case is a closure in which grammatical predicativity coincides with semantical predicativity.[5] Such a closure is opposed to the whole idea of further stages of concept-formation that are themselves based on the drive for closure, and in that regard an idealized transition transcends rather than rejects the vicious-circle principle. Of course, such a transcendence amounts to the validation of an impredicative comprehension principle as well; but such a validation should not be confused with the different validation of that principle in logical realism where it is assumed that properties and relations exist independently of all concept-formation. Nor should it be confused with the realist posit that classes exist corresponding to all concepts, predicative or otherwise.

Finally, it should be noted that impredicative concept-formation is based not only on the finite stages of concept-formation described in ramified constructive conceptualism but also on the capacity humans have for language as well. This is because the closure where grammatical predicativity coincides with semantical predicativity, and which is the basis of the idealized transition to a limit leading to impredicative concept-formation, is a closure regarding the interconnectedness of language and thought. In this regard, impredicative concept-formation is a mediated process, and language and the linguistic ability to use predicate expressions is the means used to master and direct such a process. This does not amount to a compromise between holistic conceptualism and nominalism, however, for even though grammatical predicativity coincides with semantical predicativity in both frameworks, in nominalism semantical predicativity is none other than grammatical predicativity. Also, unlike holistic conceptualism, nominalism cannot validate an impredicative comprehension principle. Nominalism also differs from constructive conceptualism in this regard, because unlike the latter it does not

[4]Wang's hierarchy of systems Σ in [8], chapter 23, is precisely such a framework based on the realist counterpart of the vicious-circle principle. Wang does not describe his hierarchy this way, it should be noted, but that is because he conflates the vicious-circle principle as a principle about concept-formation with its realist counterpart. Strictly speaking, for each recursive ordinal α, Σ_α is actually a type-theoretical set theory based on the realist counterpart of the vicious-circle principle.

[5]See Piaget [6] for a general discussion of the equilibration of cognitive structures by means of a closure involving an idealized transition to a limit.

validate a ramified logic that has a natural extension to an impredicative logic that is validated by an interpretation that presupposes the way the ramified logic that it contains is validated.

4 Ramified Constructive Conceptualism with Nominalized Predicates

There are two sorts of ways in which 'higher-order' logic is involved in conceptualism. The first is in the order of the stages of concept-formation that are generated in ramified constructive conceptualism where the concepts that are constructed or formed at one stage become the contents of the concepts constructed or formed at the next. This notion of 'higher-order' is properly formulated as a ramified second-order logic. The second is the distinction between the occurrence of predicates as predicates and their occurrence as abstract singular terms. Strictly speaking, of course, predicates are not themselves singular terms but can be formally transformed into such by deletion of the argument or subject positions that come with them in their role as predicates. Traditionally, this transformation is marked by a deletion of the parentheses (and commas in the case of a relational predicate) that precede and succeed (or separate in the case of commas) the singular terms to which the predicates can be applied. Thus whereas $F(x)$ and $R(x, y)$ are formulas in which F and R occur as predicates, $G(F)$ and $G(R)$ are formulas in which F and R occur as singular terms. In $F(F)$ and $R(F, R)$, of course, F and R occur both as predicates and as singular terms—though no single occurrence can be both as a predicate and as a singular term. We shall retain this traditional practice here when marking the transformation of a predicate into a singular term, or what we also call its nominalization.

Now in this second notion of 'higher-order' logic, predicates were traditionally allowed to occur as singular terms only of predicates that were assigned a 'higher order' than the predicates themselves. This led to a grammatically stratified hierarchy of predicates and singular terms in which formulas such as $F(F)$ and $R(F, R)$ were said to be meaningless. Such a grammatical stratification is really unnecessary, however, and the reason why it was originally incorporated was based on a confusion between the concepts predicates stand for in their role as predicates and the objects that their nominalized forms denote as singular terms.[6] Keeping this distinction in mind, we need only extend the ramified second-order logic already described so as to allow for the occurrence of nominalized predicates as singular terms on a par with individual variables.

Actually, we also need something like Church's λ-operator for the formation of complex predicates as well. Thus where φ is a formula, we also have $[\lambda x\varphi](\ )$ as a complex predicate (note the accompanying pair of parentheses) and $[\lambda x\varphi]$ as a complex singular term. For convenience, however, we shall usually drop the accompanying parentheses (and commas) when referring to predicates, though always, whenever a predicate occurs in a formula as a predicate, it will have an accompanying pair of parentheses (and commas as well if it is a relational predicate). We also adopt Frege's terminology here and refer to the objects (if any) denoted by nominalized predicates as *concept-correlates*.

In describing our logical grammar, we shall for convenience identify the different types of meaningful expressions by identifying them with different natural numbers. Thus, we shall understand 0 to represent the type of all *singular terms*, or what for brevity we will simply call *terms*. We then take 1 to be the type of all well-formed formulas (wffs) and $n+1$, for $n>0$, to be the type of all n-place predicate expressions. We assume the availability of denumerably many individual variables, and, for each natural number n, denumerably many n-place predicate variables. (We identify propositional variables with 0-place predicate variables.) We shall use 'x', 'y', 'z', with or without numerical subscripts, to refer (in the metalanguage) to individual variables, and similarly we use 'F^n', 'G^n','R^n', to refer to n-place predicate variables. (We shall usually drop the superscript when the context makes clear the degree of a predicate variable or when it otherwise does not matter what degree it is.) We shall also use 'u' to refer to variables in general, and 'a' and 'b' to refer to singular terms in general. We take $\rightarrow$, $\neg$, $=$, $\forall$, λ, and, for each positive integer j, $\forall^j$ as logical constants, and we assume the others to be defined (as abbreviations of the metalanguage) in the usual way. We recursively define, for each natural number n, the set of meaningful expressions of type n, in symbols ME_n, as follows:

1. every individual variable (or constant) is in ME_0, and every n-place predicate variable (or constant) is in both $ME_{n+1}$and ME_0;

[6] For a more detailed discussion involved in this confusion, see Cocchiarella [4] and [5].

2. if $a, b \in ME_0$, then $(a = b) \in ME_1$;

3. if $\pi \in ME_{n+1}$, and $a_1, ..., a_n \in ME_0$, then $\pi(a_1, ..., a_n) \in ME_1$;

4. if $\varphi \in ME_1$, and $x_1, ..., x_n$ are pairwise distinct individual variables, then $[\lambda x_1 ... x_n \varphi] \in ME_{n+1}$;

5. if $\varphi \in ME_1$, then $\neg\varphi \in ME_1$;

6. if $\varphi, \psi \in ME_1$, then $(\varphi \rightarrow \psi) \in ME_1$;

7. if $\varphi \in ME_1$, x is an individual variable, F is a predicate variable, and j is a positive integer, then $(\forall x)\varphi$, $(\forall^j x)\varphi$, $(\forall^j F)\varphi \in ME_1$;

8. if $\varphi \in ME_1$, then $[\lambda\varphi] \in ME_0$; and

9. if $n > 1$, then $ME_n \subseteq ME_0$.

As indicated, ME_0 is the set of terms, and for $n > 0$, ME_{n+1} is the set of n-place predicate expressions. Wffs or propositional forms are of course the members of ME_1. Note that whereas by clause (9) every predicate expression is (or rather can be transformed into) a term, not every wff is a term. Of course, by clause (4), where $n = 0$, $[\lambda\varphi]$ is a wff if φ is a wff, and by clause (8) $[\lambda\varphi]$ is a term. In other words, besides 0-place predicate variables (and constants), wffs are terms only when prefixed by the λ-operator. (We read $[\lambda\varphi]$ as 'that φ' when it occurs as a term.)

Note that we apply the quantifier $\forall^j$ to individual as well as predicate variables. Where a is a singular term, the wff $(\exists^j x)(a = x)$ is understood to express the idea that the object denoted by a is the correlate of a predicative concept that can be constructed or formed (at least) at the jth stage of concept-formation. (In the monadic fragment of the logic described below, $(\forall^j x)\varphi$ can be defined as $(\forall x)[(\exists^j F)(x = F) \rightarrow \varphi]$, where F is a 1-place predicate variable. This definition is not available here because relational concepts also have concept-correlates.)

In describing the axioms and rules of the 'higher-order' logic validated by constructive conceptualism, we assume that the notions of bondage and freedom and proper substitution are defined in the usual way. Following our practice in previous papers, we shall add a superscripted $*$-label to all theses that are understood to include wffs having nominalized predicates as singular terms among their instances. By Leibniz's law, (LL*), understood as extended in this way, we note that the ramified conceptualist comprehension principle (RCCP!) described earlier (but now prefixed by $(\forall^i y_1)...(\forall^i y_m)$ as well) can be derived from the following somewhat simpler form:

(RCCP!$^*_\lambda$) $$(\forall^i y_1)...(\forall^i y_m)(\forall^j G_1)...(\forall^j G_k)(\exists^j F)([\lambda x_1...x_n\varphi] = F),$$

where (1) φ is a wff in which no nonlogical constants occur and in which the identity sign does not occur, (2) F is an n-place predicate variable not occurring free in φ, (3) for all $i \geq j$, $\forall^i$ does not occur in φ, and (4) $G_1, ..., G_k$ are all of the pairwise distinct predicate variables occurring free in φ, and $y_1, ..., y_m, x_1, ..., x_n$ are all of the pairwise distinct individual variables occurring free in φ.

We will call the system consisting of (RCCP!$^*_\lambda$) and the following axioms and inference rules RC* (for the $*$-version of *ramified constructive conceptualism*). Where u is a predicate or an individual variable, these other axioms are as follows:

(A0*) all tautologous wffs,

(A1*) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$,

(A2*) $(\forall^j u)(\varphi \to \psi) \to ((\forall^j u)\varphi \to (\forall^j u)\psi)$,

(A3*) $\varphi \to (\forall x)\varphi$, where x is not free in φ,

(A4*) $\varphi \to (\forall^j u)\varphi$, where u is not free in φ,

(A5*) $(\forall^j u)\varphi \to (\forall^i u)\varphi$, where $i \leq j$,

(A6*) $(a = a)$, where a is a term,

(A7*) $(\forall x)(\exists y)(x = y)$,

(A8*) $(\forall^j x)(\exists y)(x = y)$,

(A9*) $(\forall^j x)(\exists^j y)(x = y)$,

(LL*) $(a = b) \to (\varphi \to \psi)$, where a, b are terms and ψ comes from φ by replacing one or more free occurrences of b by free occurrences of a,

(Id$^*_\lambda$) $[\lambda x_1...x_n R(x_1, ..., x_n)] = R$, where R is an n-place predicate variable (or constant),

($\exists/\lambda$-Conv*) $[\lambda x_1...x_n\varphi](a_1, ...a_n) \leftrightarrow (\exists x_1)...(\exists x_n)(a_1 = x_1 \wedge ... \wedge a_n = x_n \wedge \varphi)$,
where no x_i is free in any a_j, for $1 \leq i \leq j$,

(RW*) $[\lambda x_1...x_n\varphi] = [\lambda y_1...y_n\varphi(y_1/x_1, ..., y_n/x_n)]$, where no y_i occurs in

φ, for $1 \leq i \leq n$.

As inference rules we take *modus ponens*,

(MP) if $\vdash_{rc} \varphi$ and $\vdash_{rc} (\varphi \to \psi)$, then $\vdash_{rc} \psi$,

and *universal generalization* with respect to an individual or predicate variable,

(UG) if $\vdash_{rc} \varphi$, then $\vdash (\forall x)\varphi$, $\vdash (\forall^j x)\varphi$ and $\vdash_{rc} (\forall^j F)\varphi$.

We take '$\vdash_{rc}$' to stand for the derivability relation of RC*.

Note that by (LL*), (UG), (MP) and other axioms, the laws of universal instantiation are qualified as follows (where x is not free in a, F is not free in φ, and the substitutions are proper):

($\exists$/UI$_0^*$) $\vdash (\exists x)(a = x) \to [(\forall x)\psi \to \psi(a/x)]$,

($\exists$/UI$_{0/j}^*$) $\vdash_{rc} (\exists^j x)(a = x) \to [(\forall^j x)\psi \to \psi(a/x)]$,

($\exists$/Ul$_j^*$) $(\exists^j F)([\lambda x_1...x_n\varphi] = F) \to ((\forall^j F)\psi \to \psi([\lambda x_1...x_n\varphi]/F))$,

Also, by (RCCP!$_\lambda^*$), (A5*), (Id$_\lambda^*$) and (LL*), the cumulative nature of the concepts constructed at each stage is seen in the following theorem of RC* (where F and G are both n-place predicate variables):

$$\vdash_{rc} (\forall^i G)(\exists^j F)(G = F), \qquad \text{where } i \leq j.$$

That concept-correlates are also cumulative is seen in the following consequence of (A5*) and (A9*):

$$\vdash_{rc} (\forall^i x)(\exists^j y)(x = y), \qquad \text{where } i \leq j.$$

5 Aberlardian Conceptualism versus Conceptual Realism

Predicative concepts, it must be emphasized, are unsaturated cognitive capacities, and in that regard they cannot be the objects (if any) denoted by nominalized predicates. Nevertheless, by a development of the interplay between language and thought, predicative concepts as cognitive capacities can be transformed into secondary or derived abilities that enable us to apply these same concepts in a denotative manner corresponding to the use of nominalized predicates in natural language. It is by means of such a secondary or derived application, in particular, that we purport to refer to independently real properties and relations (or Platonic forms) as the denotata of nominalized predicates. Thus, for example, not only do we predicate of a shape that it is triangular and of a person that he is wise by applying a predicative concept in each case, but, in addition, we also purport to denote the properties of triangularity and wisdom, respectively, by applying these same concepts denotatively.

Now purporting to denote and actually denoting do not as always coincide, and in fact, despite all our purportings, there may be no independently real properties or relations at all that are actually denoted by any nominalized predicates. Such a view was apparently maintained by the medieval philosopher Peter Abelard who is usually described as a nominalist but who in fact seems really to have been a conceptualist of the sort we have in mind here. Thus, in particular, Abelard apparently agrees that individuals can fall under the same universal concept, but he refuses to grant that such a concept is a 'thing'; that is, that it too is an individual. In any case, whether nominalist or conceptualist, Abelard's view of nominalized predicates can be represented by means of the following schema:

$$(\text{Abelard}^*) \qquad \neg(\exists y)([\lambda x_1 ... x_n \varphi] = y).$$

Our point here is that (Abelard*) can be consistently added to RC*.

There is a realist alternative to the Abelardian view, however, and this alternative corresponds in its way to a realist counterpart of the vicious-circle principle. In particular, by assuming that each predicative concept that is constructed in accordance with (RCCP$!^*_\lambda$) has a concept-correlate, we in effect transform the vicious-circle principle as a principle regarding the process of concept-formation into its realist counterpart regarding the introduction of new objects. The realist counterpart of the vicious-circle principle, in other words, may be represented in RC* by the introduction of the following axiom schema:

$$(\text{A10}^*) \qquad (\forall^j F)(\exists^j x)(F = x), \qquad \text{for all } j \in \omega - \{0\}.$$

We shall refer to the result of adding (A10*) to the axioms of RC* as *realist ramified conceptualism*, or simply RRC*.

Note that with (A10*) we can now prove that the Russellian concept of being an object correlated with a concept formed at the jth stage and under which that object does not fall is not a concept that can itself be formed until after the jth stage. That is, for all $i \leq j$,

$$\neg(\exists^j F)([\lambda x(\exists^j G)(x = G \wedge \neg G(x))] = F)$$

is provable in RCC*

Note that where membership is defined at the jth stage as follows,

$$\in_j =_{df} [\lambda xy(\exists^j F)(y = F \wedge F(x))]$$

the principle (A) cited in section one to the effect that every predicative concept (of what was at that point the first stage) has a corresponding extension is now provable. That is,

$$(\forall^j F)(\exists y)(\forall x)[x \in_j y \leftrightarrow F(x)]$$

is provable in RRC*. Speaking of membership here of course suggests that concept-correlates are none other than the extensions of concepts, which in fact is precisely Frege's own interpretation of nominalized predicates (cf. [4], §§4–5). Such an interpretation of course also suggests assuming the following principle of extensionality (for all $n \in \omega$):

(Ext*) $$(\forall x_1)...(\forall x_n)(\varphi \leftrightarrow \psi) \rightarrow [\lambda x_1...x_n\varphi] = [\lambda x_1...x_n\psi].$$

We will not make this assumption here ourselves, however, but leave it as but one of several alternative ways to extend RRC*. Another alternative of course is to take the objects correlated with concepts to be intensional entities, and in fact to be the properties and relations that we purport to denote by means of the nominalized predicates of natural language. Classes as the extensions of concepts can then be reduced to intensional entities in one or another of the usual ways.

6 The Axiom of Reducibility

In formulating his original theory of ramified types, Russell found that in order to develop a theory of classes within that theory he needed an axiom of reducibility. This axiom has essentially the same form in RRC* that it has in standard ramified second-order logic:

(Red) $$(\forall^j G)(\exists^1 F)(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow G(x_1,...,x_n)]$$

In other words, according to (Red), any predicative concept that can be constructed at the jth stage of concept-formation is co-extensive with some concept that can be constructed at the initial or first stage of concept-formation. Note that without assuming the principle of extensionality, (Ext*), the axiom of reducibility does not nullify the vicious-circle principle as it applies to the process of concept-formation, since the predicative concepts in question are only assumed to be co-extensive and not also identical. In other words, without (Ext*) the axiom of reducibility does not imply

$$(\forall^j G)(\exists^1 F)(G = F)$$

in RCC*.

Adding (Red) to RRC* does result in an extensional impredicative comprehension principle, however, and so the question naturally arises as to whether Russell's paradox is derivable in RRC* on the basis of (Red). The answer is that it is not, and in fact RRC* + (Red) can be proved to be consistent in a finitistic proof-theoretic manner.

In regard first to Russell's paradox, note that

$$(\exists^1 F)(\forall x)[F(x) \leftrightarrow \neg x(x)]$$

is not well-formed in RCC*. For whereas predicate variables can be nominalized and allowed to occur as singular terms, individual variables cannot be 'predicativized' and allowed to occur as predicates. Of course we do have

$$(\exists^1 F)(\forall x)[F(x) \leftrightarrow x \notin_1 x]$$

as well-formed, and one might perhaps think that a singular term's occurring to the right of $\in_1$is an indirect way to 'predicativize' that singular term. Nevertheless, the above wff is not derivable in RRC* + (Red), because by definition $x \notin_1 x$ is really $\neg(\exists^1 F)[x = F \wedge F(x)]$, and, because of the presence of the identity sign, this wff is not allowed to occur in any instance of (RCCP!$^*_\lambda$). In other words, here with $\in_1$we have an example of a definable predicate constant that does not stand for a predicative concept.

Besides obtaining an extensional impredicative comprehension principle for the development of a theory of classes, Russell also used his reducibility axiom to justify his definition of identity as indiscernibility with respect to the initial predicative concepts. Thus, given the following definition,

$$x \equiv y =_{df} (\forall^1 F)[F(x) \leftrightarrow F(y)],$$

Russell's view can be reformulated here as the claim that = can be eliminated in RRC* in favor of ≡ on the basis of (Red). Such a reduction fails in RRC* + (Red), however. For although

$$(\exists^1 F)(\forall x)(F(x) \leftrightarrow (\exists^1 G)[x \equiv G \wedge \neg G(x)])$$

is derivable is derivable in RRC* + (Red), what follows is not Russell's paradox but only that some predicative concepts have indiscernible correlates even though the concepts themselves are not co-extensive with respect to those correlates. That is,

$$(\exists^1 F)(\exists^1 G)[F \equiv G \wedge F(F) \wedge \neg G(F)],$$

and therefore

$$(\exists^1 F)(\exists^1 G)(F \equiv G \wedge \neg(\forall x)[F(x) \leftrightarrow G(x)])$$

as well are provable in RRC* + (Red). But, by (LL*), identity suffices for co-extensivity, and therefore it is provable in RRC* + (Red) that some predicative concepts have correlates that are indiscernible but not identical. More specifically,

$$(\exists x)(\exists y)(x \equiv y \wedge x \neq y)$$

is provable in RRC* + (Red). Thus, instead of providing a basis by which identity might be reduced to indiscernibility with respect to the predicative concepts, (Red) actually leads to the refutation of such a reduction.

Now because it is refutable on the basis of (Red) that identity is reducible to indiscernibility with respect to the predicative concepts, it is also refutable on the basis of (Red) that some predicative relational concept corresponds to or otherwise stands for identity. That is,

$$\neg(\exists^j R)(\forall x)[R(x,y) \leftrightarrow x = y]$$

is provable in RRC* + (Red), for all positive integers j. Such a result is not counter-intuitive, it should be noted, or at least not in constructive conceptualism where in fact, for reasons already indicated, it seems to be a rather accurate assessment of the role of identity in this framework.

Nor should the result that some non-coextensive predicative concepts have indiscernible correlates be thought counter-intuitive either. For if $F \equiv G$ and $G(x)$ are true, then the idea that $F(x)$ should also be true depends on the assumption that an object's falling under a predicative concept is equivalent to (the correlate of) that concept itself falling under a predicative concept. What is disprovable in RRC* is the somewhat different thesis:

$$(\forall^1 x)(\exists^1 F)(\forall^1 G)[F(G) \leftrightarrow G(x)]$$

as well therefore as

$$(\forall x)(\exists^1 F))(\forall^1 G)[F(G) \leftrightarrow G(x)].$$

That is,the negations of these wffs are provable in RRC* + (Red), which is as it should be, because it is counter-intuitive to demand that the construction of a predicative concept corresponding to the condition of being a predicative concept of a given object should both be in accordance with the vicious-circle principle and yet occur at the same stage of concept-formation as the predicative concept in question.

A related consequence of the reducibility axiom, incidentally, is the fact that no relational predicative concept corresponds to, or otherwise stands for, predication. That is,

$$\neg(\exists^1 R)(\forall^1 F)(\forall x)[R(F, x) \leftrightarrow F(x)]$$

is provable in RRC* + (Red). This does not conflict with the fact that at any stage succeeding a given stage of concept-formation that is in accord with the vicious-circle principle one can assume the construction of a relational predicative concept that corresponds to predication at that stage; that is, it does not conflict with the fact that

$$(\exists^{j+1} R)(\forall^1 F)(\forall x)[R(F, x) \leftrightarrow F(x)]$$

is consistent in RRC*. For what is refuted is the construction of a relational concept of predication at the same stage of concept-formation as the stage in question.

Now it is noteworthy that all of these consequences of (Red) have been discussed in somewhat different terms in the related context of the impredicative theory T^*_λ described in Cocchiarella [3], chapter V, §4. For if we replace each predicate quantifier $\forall^i$, for all positive integers i, by $\forall$ in the wffs of RC*, then it is easily seen that every theorem of RRC* + (Red) is a theorem of T^*_λ. Moreover, in [3], chapter IV, §6, a strictly finitistic proof-theoretic consistency proof is given for the system T* that is easily extended to the λ-abstracts of T^*_λ as well. And therefore, by translating RRC* + (Red) into T^*_λ in the way indicated, it follows that we also have a finitistic proof-theoretic consistency proof for RRC* + (Red) as well.

Theorem: RRC* + (Red) is consistent and can be shown to be so in a finitistic proof-theoretic way.

7 Holistic Conceptualism

None of the consequences of (Red) described above, it should be emphasized, are in conflict with ramified constructive conceptualism. In fact, some of them might even be assumed in this framework independently of any assumption regarding (Red) itself. That is, aside from (Red) itself, we can for each $j \geq i$ replace $\forall^1$ and $\exists^1$ by $\forall^j$ and $\exists^j$ in several of the above consequences of (Red) and take the results as valid theses of ramified constructive conceptualism.

(Red) itself, on the other hand, as a principle about predicative concepts, is a very implausible hypothesis. (Red) is, after all, really about predicative concepts and only derivatively about the classes that are the extensions of these concepts. In other words, that (Red) has among its consequences an extensional impredicative comprehension principle that is useful in the development of a theory of classes as the extensions of concepts does not make (Red) itself any less implausible as a principle about predicative concepts. In fact, given its impredicative content, (Red) really amounts to a poor compromise between holistic conceptualism and a strict form of constructive conceptualism which refuses to acknowledge any concepts beyond those that can be constructed in accordance with the vicious-circle principle. If an impredicative comprehension principle is to be validated at all, then it is conceptually clearer that it be validated on the basis of an idealized transition to a limit where impredicative concept-formation becomes possible than to try to base it on predicative concepts alone. It is conceptually clearer, in other words, to extend RC*, with or without its realist assumptions, to the logic of holistic conceptualism than to extend it by adding (Red).

This is not the only reason for extending RC* to a logic for holistic conceptualism, needless to say; and, in doing so, we must again take care not to confuse transcending the vicious-circle principle as a principle about the process of concept-formation with violating its realist counterpart as a principle about the positing of new objects. In fact, the logic of holistic conceptualism, HC*, described below as an extension RRC*, is actually in full conformity with the realist counterpart of the vicious-circle principle.

The logic of holistic conceptualism, HC*, that we have in mind here is really a simple extension of RC* in which the unrestricted quantifier $\forall$ is allowed to be affixed to predicate as well as individual variables. We assume, accordingly, that clause (7) of the definition of a meaningful expression of type 1 is extended so that $(\forall F)\varphi \in ME_1$ whenever $\varphi \in ME_1$ and F is a predicate variable. In addition, besides allowing for wffs in the newly defined sense in all of the axiom schemes of RC* other than (RCCP!$^*_\lambda$), we also assume that axioms (A1*) and (A3*) and the inference rule (UG) are extended to apply to affixing the unrestricted quantifier $\forall$ to predicate variables as well as individual variables. The system HC* is then obtained by finally adding the following impredicative comprehension principle:

(CP$^*_\lambda$) $$(\exists F)([\lambda x_1 ... x_n \varphi] = F),$$

where φ is any wff (as newly defined) and F is an n-place predicate variable not occurring free in φ. Of course it is understood that the ramified comprehension principle (RCCP!$^*_\lambda$) remains restricted to the wffs of RC*, i.e., to wffs in which no predicate variable is bound by the unrestricted quantifier $\forall$.

Note that by (Id$^*_\lambda$), (LL*) and (UG), all predicative concepts are concepts *simpliciter*; that is,

$$(\forall^j G)(\exists F)(G = F)$$

is provable in HC*. Accordingly, where

$$x \in y =_{df} (\exists F)[y = F \wedge F(x)],$$

then we also have

$$x \in_j y \rightarrow x \in y$$

as provable in HC*.

In regard to Russell's paradox, note that although we do have

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F)$$

as provable in HC*, what follows by Russell's argument is not a contradiction but only that the predicate $[\lambda x(\exists G)(x = G \wedge \neg G(x))]$ is denotationless in its nominalized occurrences as an abstract singular term. That is, instead of a contradiction,

$$\neg(\exists y)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = y)$$

is provable in HC*. Thus what Russell's argument really shows is not that there can be no concept corresponding to $[\lambda x(\exists G)(x = G \wedge \neg G(x))]$ as a predicate, but rather only that any denotative application of such a concept will simply fail to denote a correlate of that concept.

Of course, since HC*, like RC*, is devoid of any realist assumptions at all about the existence of objects, we can consistently add the Abelardian thesis,

$$(\forall F)\neg(\exists x)(F = x)$$

to HC* just as we can add its schematic counterpart to RC*. All nominalized predicates, in other words, can be assumed to be denotationless in HC*. Thus, even though the vicious-circle principle is violated or transcended here as a principle regarding the process of concept-formation, HC* need not also thereby be committed to violating the realist counterpart of that principle. Indeed, where RHC* is the result of adding

(A10*) $\qquad (\forall^j F)(\exists^j x)(F = x), \qquad$ for all $j \in \omega - \{0\}$.

to HC*, it is clear that RHC* represents a minimal form of realist holistic conceptualism that actually is in full conformity with the realist counterpart of the vicious-circle principle. Impredicative characterizations are not objected to as such in RHC*, in other words, but only when used as a means for positing new objects.

Now it is clear that without further realist assumptions as to which concepts have correlates, such a minimal realist framework as RHC* is too weak to serve as a foundation for mathematics. Nevertheless, this does not affect its status as a general conceptual framework for pursuing studies in the foundations of mathematics, including in particular the question as to what additional constructive principles might be justified in such a foundation. Thus, for example, where

$$0 =_{df} [\lambda x(\exists^j F)(x = F \wedge \neg(\exists y)F(y))],$$

$$S =_{df} [\lambda yz(\exists w)(w \in_j z \wedge [\lambda x(x \in_j z \wedge x \neq w)] \in_j y)],$$

$$N =_{df} [\lambda x(\forall F)(F(0) \wedge (\forall y)(\forall z)[F(y) \wedge S(y,z) \rightarrow F(z)] \rightarrow F(x))],$$

the question arises as to what constructive principles other than the vicious-circle principle might justify assuming

$$(\text{N}^*) \qquad (\exists x)(N = x)$$

for some positive integer j. Similarly, assuming (N*) and an analysis of the real numbers based on the natural numbers, the question will also arise as to what constructive principles (if any) will justify assuming that the concept of a real number (as so analyzed) has a concept-correlate. (Such principles, incidentally, might in fact be based on Wang's type-theoretical set theory Σ_ω. That is, by identifying the values of Wang's variables of type ω as the correlates of concepts posited in a suitable extension of RHC*, we might in fact be able to provide a foundation for classical mathematics that is constructive at least in its realist assumptions.)

Finally, although we shall not go into the details of this alternative here, it might well be argued that we should simply drop the realist counterpart of the vicious-circle principle as an ontological principle about the positing of new objects and replace it instead by an epistemological principle about the constructive knowability of the objects correlated with predicative concepts. Thus, for example, where $\Box$ is a modal operator for '*it is constructively knowable that*', we would on this proposal revise the ramified comprehension principle (RCCP!^*_λ) and its realist counterpart (A10*) by simply inserting $\Box$ after the existential posits ($\exists^j F$) and ($\exists x$), respectively, thereby enabling us to assert instead that the predicative concept and concept-correlate being posited in (RCCP!^*_λ) and (A10*) are constructively knowable in terms of the conditions of their specification. No such revision will apply to the impredicative comprehension principle (CP^*_λ), on the other hand, nor to any realist assumption (such as the principle ($\exists/\text{HSCP}^*_\lambda$) described in [4], §15, and [5], §7) that posits correlates even for concepts specified impredicatively. Thus, rather than suggest that we are dealing with the construction of objects as well as of concepts, the epistemological version of the vicious-circle principle will make it clear that even though impredicative concepts can have concept-correlates, it is only the correlates of predicative concepts that are constructively knowable. In this way, neither impredicative concept-formation nor a realist assumption regarding the correlates of impredicative concepts need actually be in conflict with the vicious-circle principle. In addition, such an epistemological version of the vicious-circle principle may also provide the basis of an interpretation of intuitionism within conceptualism. For one of the theses validated in conceptualism regarding constructive knowledge is that an object will fall under a predicative concept only if it is constructively knowable that it falls under that concept. That is, in general, where F is an n-place predicate variable,

$$(\text{K!}) \qquad (\forall^j F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \to \Box F(x_1, ..., x_n)]$$

is valid in conceptualism when $\Box$ is interpreted as constructive knowability. But then, because the complement of a predicative concept is also a predicative concept, it follows that

$$(\forall^j F)(\forall x_1)...(\forall x_n)[\neg F(x_1, ..., x_n) \to \Box\neg F(x_1, ..., x_n)]$$

is also valid in conceptualism; and therefore so is

$$(\text{PR!}) \qquad (\forall^j F)(\forall x_1)...(\forall x_n)[\Box F(x_1, ..., x_n) \vee \Box\neg F(x_1, ..., x_n)].$$

None of these principles are valid for concepts in general, needless to say; and, according to conceptualism, it is the fact that (PR!) cannot be extended to concepts in general that explains why the principle of excluded middle fails in intuitionism. Constructive knowability, in other words, is the counterpart in conceptualism of the notion of constructive provability in intuitionism.

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