INITIAL AND BOUNDARY VALUE PROBLEMS FOR THE DETERMINISTIC AND STOCHASTIC ZAKHAROV-KUZNETSOV EQUATION IN A BOUNDED DOMAIN

Chuntian Wang

Submitted to the faculty of the University Graduate School
in partial fulfillment of the requirements
for the degree
Doctor of Philosophy
in the Department of Mathematics,
Indiana University
May 2015

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Doctoral Committee

Roger Temam, PhD

Nathan Glatt-Holtz, PhD

Shouhong Wang, PhD

Michael Jolly, PhD

Richard Bradley, PhD

April, 23 2014

ACKNOWLEDGMENTS

The years in the Department of Mathematics at Indiana University, Bloomington, has been the most joyful, meaningful and eventful. It is in this place that I have become motivated to pursue a career in mathematical research.

I am most grateful to my advisers Professors Roger Temam and Nathan Glatt-Holtz. Their devotion towards research has lead me to the same road, narrow and difficult it may be, but also adorned with roses alongside. When I fall, they come to my side; when I gain, they grant me praises but also advice about humbleness. I can never thank them enough for their kindness, wisdom and guidance.

I would like to pay special thanks to my initial adviser, Professor Shouhong Wang, and my undergraduate adviser, Professor Tian Ma, who have always been helping me along, especially when I first came to the United States.

I would also like to thank sincerely other committee members of my defense, Professors Michael Jolly and Richard Bradley. I am grateful as well for Professors Kevin Zumbrun, Alberto Torchinsky, Russell Lyons, Madan Puri, Hari Bercovici, Jean Claude-Saut, Arnaud Debussche, Franco Flandoli, Bruce Solomon, Chris Judge, Matthias Weber and Randall Bramley, for their kind help in various ways. Special thanks also go to Forrest Kate, Teresa Bunge, Jeff Taylor, Chawnn Blake and Connie Wright.

Friends are one of the most important assets that I have found at IU. I would like to especially thank Aseel Farhat, Ko-Shin Chen, Yinwei Li, Will Tune, Anne Carter, Prasit Bhattacharya, Tuyen Truong, and all my academic brothers and sisters.

Finally I would like to thank my family, especially my parents, for their whole-hearted support. A warm thanks is dedicated to my mother, who always wishes me happiness over achievements, and I would like to tell her now that I have found the theme of my life that I feel happy with.

Chuntian Wang

INITIAL AND BOUNDARY VALUE PROBLEMS FOR THE DETERMINISTIC AND STOCHASTIC ZAKHAROV-KUZNETSOV EQUATION IN A BOUNDED DOMAIN

We study in this thesis the well-posedness and regularity of the Zakharov-Kuznetsov (ZK) equation in the deterministic and stochastic cases, subjected to a rectangular domain in space dimensions 2 and 3. Mainly we have established the existence, in $3D$, and uniqueness, in $2D$, of the weak solutions, and the local and global existence of strong solutions in $3D$. Then we extend the results to the stochastic case and obtain in $3D$ the existence of martingale solutions, and in $2D$ the pathwise uniqueness and existence of pathwise solutions. The main focus is on the mixed features of the partial hyperbolicity, nonlinearity, nonconventional boundary conditions, anisotropicity and stochasticity, which requires methods quite different than those of the classical models of fluid dynamics, such as the Navier-Stokes equation, Primitive Equation and related equations.

Roger Temam, PhD

Nathan Glatt-Holtz, PhD

Shouhong Wang, PhD

Michael Jolly, PhD

Richard Bradley, PhD

Contents

1 Introduction **1**

1.1 Existence and Uniqueness of the Weak Solution to the Zakharov-Kuznetsov (ZK) Equation . . . 3

1.2 Local and Global Existence of Strong Solutions in $3D$ . . . 4

1.3 Martingale and Pathwise Solutions to the Stochastic ZK Equation . . . 5

2 Functional Spaces and the Linearized ZK Equation **8**

2.1 Density and Trace Results for the Space $\mathcal{X}(\mathcal{M})$. . . . 9

2.2 Density and Trace Results for the Space $\mathcal{X}_1(\mathcal{M})$. . . . 13

2.3 Operator A in the Dirichlet Case. . . . 19

2.3.1 Density and Regularity Results for $D(A)$. . . . 19

2.3.2 Characterization of A^* and $D(A^*)$. . . . 20

2.4 Operator A_{per} in the Space Periodic Case. . . . 22

2.4.1 Density and Regularity Results for $D(A_{per})$. . . . 22

2.4.2 Characterization of A^*_{per} and $D(A^*_{per})$. . . . 23

2.5 Linear Evolution ZK Equation . . . 24

2.5.1 Well-posedness of the Evolution Equation . . . 24

3 Nonlinear ZK Equation: Existence and Uniqueness of the Weak Solution **27**

3.1 Existence of Weak Solutions in Dimension 3 . . . 28

3.2 Uniqueness of Weak Solutions in Dimension 2 34

4 Local Existence of Strong Solutions in $3D$ **41**

4.1 Parabolic Regularization 41

4.1.1 Global Bounds Independent of ϵ 42

4.1.2 Local Bounds Independent of ϵ 42

4.1.3 A Singular Perturbation Argument 52

4.2 Passage to the Limit 56

5 Global Existence of Strong Solutions in $3D$ **58**

5.1 Parabolic Regularization 59

5.2 Estimates Independent of ϵ 59

5.2.1 L^2 Estimate Independent of ϵ 59

5.2.2 H^1 Estimate Independent of ϵ 62

5.2.3 Estimates Independent of ϵ for u^ϵ_{xxx} and $u^\epsilon u^\epsilon_x$ 67

5.3 The Main Result 69

5.4 Discussions about the Uniqueness of Solutions 70

6 Stochastic ZK Equation **72**

6.1 Stochastic Framework 72

6.2 Regularized Stochastic ZK Equation 76

6.2.1 Definition of Solutions 77

6.2.2 Pathwise Solutions in Dimensions 2 and 3 80

6.3 Passage to the Limit as $\epsilon \to 0$ 89

6.3.1 Martingale Solutions in Dimensions 2 and 3 91

6.3.2 Pathwise Solutions in Dimension 2 103

7 Appendix **115**

7.1 Space $\Sigma(\mathcal{M})$ and its Dual $\Sigma'(\mathcal{M})$ 115

7.2 Trace Results 117

7.3 Compact Embedding Theorems 121

7.4 Convergence Theorem for the Noise Term 121

7.5 Some Probability Tools 122

7.6 The Jakubowski-Skorokhod Representation Theorem 123

7.7 An Adapted Stochastic Gronwall Lemma 124

Bibliography **126**

Curriculum Vita

Chapter 1

Introduction

The Zakharov-Kuznetsov (ZK) equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \Delta\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = f, \tag{1.1}$$

where $u = u(x, x^{\perp}, t)$, $x^{\perp} = y$ or $x^{\perp} = (y, z)$, describes the propagation of nonlinear ionic-sonic waves in a plasma submitted to a magnetic field directed along the x-axis. It has been derived formally in a long wave, weakly nonlinear regime from the Euler-Poisson system in [45], [27] and [29] (see also [4] and [5] for more general physical backgrounds). When u depends only on x and t, the ZK equation reduces to the classical Korteweg-de Vries (KdV) equation. But it is quite different from the Kadomtsev-Petviashivili equation, where e.g. the ZK equation is not completely integrable but has a hamiltonian structure (see [29]).

Recently the ZK equation has caught considerable attention (see e.g. [15], [16], [28], [13], [1], [40]), not only because it is closely related with the physical phenomena but also the start to explore more general problems that are partially hyperbolic (such as the inviscid primitive equations). Moreover, the study of the stochastic hyperbolic equations is meaningful as general stochastic random waves have gathered interests nowadays to capture phenomena similar to those of more realistic fluid systems.

More specifically, on the one hand, for the Cauchy-problem for the ZK equation, it has been proven to be globally well posed in the two-dimensional case ($x^{\perp} = y$) for data in $H^1(\mathbb{R}^2)$ (see [14]), and locally well-posed in the three-dimensional case for data in $H^s(\mathbb{R}^3)$,

$s > \frac{3}{2}$ (see [30]), and recently in $H^s(\mathbb{R}^3)$, $s > 1$ (see [38]). Then for both numerical purposes and problems of controllability, people have turned to the initial-boundary value problem (IBVP) for the ZK equation. The well-posedness have been studied in the half space $\{(x, y) : x > 0\}$ (see [15]), on a strip like $\{(x, y) : x \in \mathbb{R}, y \in (0, L)\}$ (see [1]) or $\{(x, x^\perp) : x \in (0, 1), x^\perp \in \mathbb{R}^d, d = 1, 2\}$ (see [16] and [40]). On the other hand, as for the regularity of solutions, the global existence of strong solutions in space dimension 2 has been proven in a half strip $\{(x, y) : x > 0, y \in (0, L)\}$ in [28]. The existence and exponential decay of regular solutions to the linearized ZK equation in a rectangle $\{(x, y) : x \in (0, L), y \in (0, B)\}$ has been studied in [13]. Finally, for the stochastic case, the stochastic KdV equation has been studied extensively (see [10], [20] and [12])), where the main focus are on Wick-type SPDEs (see [46] and [33]) and exact solutions under the additive noise (see [24]).

To conclude, the problem of the well-posedness of the (deterministic and stochastic) ZK equation in a bounded domain and regularity of solutions (in space dimension 3) have not yet been thoroughly explored. Hence here we aim to systematically analyse these problems in the rectangular domain

$$\mathcal{M} := \{(x, x^\perp), \, x \in (0, 1), \, x^\perp \in (-\pi/2, \pi/2)^d, \, d = 1, 2\}. \tag{1.2}$$

Firstly for the deterministic case, we establish, for arbitrary large initial data, the existence of global weak solutions in space dimensions 2 and 3 ($d = 1$ and 2 respectively) and a result concerning the uniqueness of such solutions in the two-dimensional case (Chapter 3). Then we prove the short and then global time existence of strong solutions in space dimension 3 (Chapters 4 and 5). Finally we extend the results of existence and uniqueness of weak solutions to the stochastic case (Chapter 6).

The proof adopts new ideas other than those of the classical models in fluid dynamics, such as the Navier-Stokes equation, Primitive Equation and related equations. This is mainly because as a partial differential equation with mixed features (partial hyperbolicity, nonlinearity, nonconventional boundary conditions, anisotropicity and stochasticity), the ZK equation sits at the interface between probability theory, mathematical analysis and theory of parabolic and hyperbolic partial differential equations, and hence requires novel methods, which will be elaborated below.

1.1 Existence and Uniqueness of the Weak Solution to the Zakharov-Kuznetsov (ZK) Equation

We consider a limited domain $\mathcal{M}$ as in (1.2). We assume the boundary conditions on $x = 0, 1$ to be

$$u(0,\, x^{\perp},\, t) = u(1,\, x^{\perp},\, t) = u_x(1,\, x^{\perp},\, t) = 0,$$

In the y and z directions, we will choose either the Dirichlet boundary conditions or the periodic boundary conditions.

We first establish in Chapter 2 various results concerning basic functional spaces and their traces and regularity results. Proceeding in steps we first introduce a functional space $\mathcal{X}(\mathcal{M}) = \{u \in L^2(\mathcal{M}) : \Delta u_x + cu_x \in L^2(\mathcal{M})\}$, for which traces on the y, z boundaries can not be defined (Section 2.1), then a subspace $\mathcal{X}_1(\mathcal{M}) = \{u \in \mathcal{X}(\mathcal{M}),\, u = 0$ at $x = 0, 1,$ and $u_x = 0$ at $x = 1\}$, for which such traces are defined (Section 2.2), and finally the spaces $D(A) = \{u \in \mathcal{X}_1(\mathcal{M}), u = 0\ at\ y = \pm\frac{\pi}{2}\ (and\ z = \pm\frac{\pi}{2})\}$, and $D(A_{per})$=$\{u\in \mathcal{X}_1(\mathcal{M})$, u is periodic with the first order derivatives periodic in y, z, with period $\pi\}$, covering all the boundary conditions in the Dirichlet and space periodic cases (Sections 2.3, 2.4).

We then study in Section 2.5 the linear ZK equation, which will be applied to prove for the uniqueness of solutions for the nonlinear equation.

In Chapter 3 we establish for the full nonlinear problem the following well-posedness result:

Theorem 1.1.1. *We are given the initial data* $u_0 \in L^2(\mathcal{M})$ *and* $f \in L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{M}))$, $\mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d$, $d = 1, 2$. *Then:*

(*i*) *The initial and boundary-value problem for the ZK equation has a weak solution* u,

$$u \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H^1_0(\mathcal{M})),$$

$$\int_0^T \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} |u_x(0,x^\perp,t)|^2\, dx^\perp\, dt < \infty,$$

for all $T > 0$.

(*ii*) *If* $d = 1$, *then the solution* u *is unique. Moreover,* $u \in \mathcal{C}([0,T];\, L^2(\mathcal{M}))$ *and the flow map* $u_0 \to u(\cdot,t)$ *is continuous from* $L^2(\mathcal{M})$ *to* $L^2(\mathcal{M})$.

The proof is carried out mainly using the following parabolic regularization and a compactness argument:

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \Delta \dfrac{\partial u^\epsilon}{\partial x} + c\dfrac{\partial u^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon\,(\dfrac{\partial^4 u^\epsilon}{\partial x^4} + \dfrac{\partial^4 u^\epsilon}{\partial y^4} + \dfrac{\partial^4 u^\epsilon}{\partial z^4}) = f, \\ u^\epsilon(0) = u_0. \end{cases} \tag{1.3}$$

1.2 Local and Global Existence of Strong Solutions in $3D$

In chapter 4 we establish the short time existence of strong solutions in a $3D$ rectangular domain.

Theorem 1.2.1. *We assume suitable regularity on the data and compatibility conditions. Then there exists a local strong solution to the ZK equation on some time interval* $[0, T_*)$, $T_* > 0$ *depending only on the data, such that*

$$\nabla u, u_{yy}, u_{zz}, u_t \in L^\infty(0, T_*; L^2(\mathcal{M})),$$

$$u \in L^2(0, T_*; D(A) \cap \Xi \cap H^3(I_x; L^2(I_{x^\perp})) \cap H^3(I_{x^\perp}; L^2(I_x))),$$

$$u_t \in L^2(0, T_*; H^1(\mathcal{M})).$$

Moreover, we have for every $t \in (0, T_*)$,

$$u_{yy}(t)\big|_{y=\pm\frac{\pi}{2}} = u_{zz}(t)\big|_{z=\pm\frac{\pi}{2}} = 0.$$

The proof is different from e.g. that of the Navier-Stokes equation, due to the partial hyperbolicity of the model. We again use the parabolic regularization (1.3) and the key observation of the comparison between $|u_x^\epsilon(t)|_{L^2(\mathcal{M})}$ and $|u_t^\epsilon(t)|_{L^2(\mathcal{M})}$ (see Lemma 4.1.2).

Next in Chapter 5, we prove, with different assumptions on the boundary conditions on $x = 0$ and 1, the global existence of strong solutions in $3D$. That is, we assume periodic boundary conditions on u, u_x and u_{xx} on $x = 0$ and 1. Then the main result is the following:

Theorem 1.2.2. *We assume suitably regular data. Then the initial and boundary value problem for the ZK equation possesses at least a solution u:*

$$u \in \mathcal{C}([0,T]; H^1(\mathcal{M})) \cap W^{3,3/2}(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp}))).$$

For the proof we again use the parabolic regularization (1.3).

The main obstacle is the estimates of the nonlinear term as in the case of $3D$ Navier-Stokes equation. To overcome this difficulty we utilize the anisotropic resonance of the term u_{xxx} and the nonlinear term uu_x to cancel uu_x, which leads to a bound for the H^1 norm over $(0,T)$ of u. Also with the assumption of periodic boundary conditions for u and u_{x^j} at $x = 0,\ 1,\ j = 1, 2$, the trace $u_{xx}\big|_{x=0}^{x=1}$ now vanishes for our convenience in estimation.

However the uniqueness of solutions is still open in both $2D$ and $3D$, even with such a regularity and all the periodic boundary conditions satisfied.

1.3 Martingale and Pathwise Solutions to the Stochastic ZK Equation

In Chapter 6, we consider the stochastic Zakharov-Kuznetsov equation subject to multiplicative random noise

$$du + (\Delta u_x + cu_x + uu_x)\, dt = f\, dt + \sigma(u)\, dW(t), \tag{1.4}$$

posed in the same rectangular domain $\mathcal{M}$ as in (1.2). Then we extend the results of Theorem 1.1.1 to the stochastic case. Note that here we have different notions of solutions, namely, the martingale and pathwise solutions. In the former notion, the stochastic basis is not specified in the beginning and is viewed as part of the unknown, while in the latter case, the stochastic basis is fixed in advance as part of the assumptions.

The main results are as follows:

Theorem 1.3.1. *In dimensions* 2 *or* 3*, we suppose suitable conditions on the measure of the initial data,* σ *and* f*. Then there exists a global martingale solution* $(\tilde{\mathcal{S}}, \tilde{u})$ *to the stochastic ZK equation.*

Theorem 1.3.2. *In dimension* 2*, we assume, relative to a fixed stochastic basis* $\mathcal{S}$*, suitable conditions on the initial data,* σ *and* f*. Then there exists a unique global pathwise solution* u *to the stochastic ZK equation.*

One of the main novelties of the proof is the treatment of the boundary conditions, which are more complicated than the usual Dirichlet or periodic ones. Firstly, it is not clear whether all the boundary conditions are still preserved after the application of the Skorokhod embedding theorem (Theorem 2.4 in [9]) since the underlying stochastic basis has been changed. To solve this problem, a measurability result concerning Hilbert spaces is developed (Lemma 7.3.1). Secondly, we have extended the trace results in the deterministic to stochastic setting by establishing the trace properties of the linearized ZK equation depending on the probabilistic parameters (Lemma 7.2.2 and Lemma 7.2.3).

A further novelty is contained in the proof of the pathwise uniqueness (Section 6.3.2). Difficulties arise with the derivation of the energy inequality for the difference of the solutions due to the lack of regularity. Moreover, the method in the deterministic case can not be adapted to the stochastic case by the application of the stochastic version of the Gronwall lemma established in [21] (see also [34]), as issues would arise in passage to the limit on the terms involving stopping times. We overcome this difficulty by establishing a variant of the stochastic Gronwall lemma (Lemma 7.7.2), where we find that in certain situations we can weaken the hypotheses so as to avoid the stopping times.

Chapter 2

Functional Spaces and the Linearized ZK Equation

We consider the linearized Zakharov-Kuznetsov equation in a rectangular or parallelepiped domain, namely,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \Delta\frac{\partial u}{\partial x} = f, \tag{2.1}$$

considered in $\mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d$, with $d = 1, 2$, $\Delta u = u_{xx} + \Delta^{\perp} u$, $\Delta^{\perp} u = u_{yy}$ or $u_{yy} + u_{zz}$. We will use the notations $I_x = (0,1)_x$, $I_y = (-\pi/2, \pi/2)_y$ and $I_z = (-\pi/2, \pi/2)_z$ in the sequel. As in the previous work [40], we assume the boundary conditions on $x = 0, 1$ to be

$$u(0,\, x^{\perp},\, t) = u(1,\, x^{\perp},\, t) = u_x(1,\, x^{\perp},\, t) = 0, \tag{2.2}$$

and the initial condition reads:

$$u(x,\, x^{\perp},\, 0) = u_0(x,\, x^{\perp}). \tag{2.3}$$

We also need suitable boundary conditions in the y and z directions. As mentioned before, we will choose either the Dirichlet boundary conditions

$$u = 0 \ at\ y = \pm\frac{\pi}{2} \ \ (and\ z = \pm\frac{\pi}{2}), \tag{2.4}$$

or the periodic boundary conditions

$$u\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = u_y\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = 0 \ \ (and\ u\Big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = u_z\Big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = 0). \tag{2.5}$$

We will study in each case the initial and boundary value problem (2.1)-(2.3) supplemented with the boundary conditions (2.4) or (2.5). We will recast the linear system in the form of an abstract evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au = f, \\ \\ u(0) = u_0, \end{cases} \tag{2.6}$$

and prove that $-A$ is the infinitesimal generator of a contraction semigroup which will provide existence and uniqueness of solutions to (2.6) using the linear semi-group theory. In order to define and characterize A and its domain $D(A)$, we first introduce the auxiliary spaces $\mathcal{X}(\mathcal{M})$ and $\mathcal{X}_1(\mathcal{M})$ in Sections 2.1 and 2.2, and then proceed in Section 2.3 with the definition of $D(A)$.

2.1 Density and Trace Results for the Space $\mathcal{X}(\mathcal{M})$.

We consider the space $\mathcal{X} = \mathcal{X}(\mathcal{M}) = \{u \in L^2(\mathcal{M}) : \Delta u_x + cu_x \in L^2(\mathcal{M})\}$, endowed with its natural Hilbert norm: $|u|_{\mathcal{X}(\mathcal{M})} = (|u|^2_{L^2(\mathcal{M})} + |\Delta u_x + cu_x|^2_{L^2(\mathcal{M})})^{1/2}$.

As a preparation for the trace theorem, Theorem 2.1.1, we first prove the following density result.

Lemma 2.1.1. *$H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$ is dense in $\mathcal{X}(\mathcal{M})$.*

Proof. Fix $u \in \mathcal{X}(\mathcal{M})$, and assume that $d = 2$ (similar proof for $d = 1$). We aim to approximate u by a sequence $\{u_N\} \subseteq H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$, such that $u_N \to u$ in $\mathcal{X}(\mathcal{M})$.

Step 1. We construct a sequence of approximations u_N. By Fourier series expansion in $x^\perp$, we write

$$u = \sum_k \widehat{u}_k(x)\omega_{k_1}(y)\,\omega_{k_2}(z), \quad k = (k_1, k_2), \tag{2.7}$$

where the ω_{k_1} are the eigenfunctions of the operators d^2/dy^2 on I_y (sine and cosine functions) which form an orthonormal basis of $L^2(I_y)$, and the ω_{k_2} are the eigenfunctions of the operators d^2/dz^2 on I_z which form an orthonormal basis of $L^2(I_z)$.[1] Since $u \in L^2(\mathcal{M})$, these expansions converge in $L^2(\mathcal{M})$, and $|u|^2_{L^2(\mathcal{M})} = C_d \Sigma_k |\widehat{u}_k|^2_{L^2(I_x)}$, with C_d a constant depending only on d.

We set

$$\Delta u_x + c u_x = g \in L^2(\mathcal{M}). \tag{2.8}$$

According to Lemma 7.1.1 in the Appendix, $\Delta u_x + c u_x$ being in $L^2(\mathcal{M})$ makes sense in the space of distributions $\Sigma'(\mathcal{M})$ defined in that lemma. Hence by Corollary 7.1.1,

$$\widehat{u}_k''' - (\lambda_k - c)\widehat{u}_k' = \widehat{g}_k, \tag{2.9}$$

holds in $L^2(I_x)$, with $\widehat{g}_k(x)$ the Fourier coefficients of g as in (2.7).

Now we define the sequence of approximations

$$u_N = \sum_{|k| \leqslant N} \widehat{u}_k(x) \omega_{k_1}(y)\, \omega_{k_2}(z), \quad N \in \mathbb{N}. \tag{2.10}$$

By (7.7), we have $|u_{Nxxx}|^2_{L^2(\mathcal{M})} = \sum\limits_{|k| \leqslant N} |\widehat{u}_k'''|^2_{L^2(I_x)} < \infty$. Hence

$$u_N \in H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d). \tag{2.11}$$

Step 2. We show that $u_N \to u$ in $\mathcal{X}(\mathcal{M})$ as $N \to \infty$, that is

$$\begin{cases} u_N \to u \ in\ L^2(\mathcal{M}), \\ \Delta u_{Nx} + c u_{Nx} \to \Delta u_x + c u_x \ in\ L^2(\mathcal{M}). \end{cases} \tag{2.12}$$

It is easy to obtain $(2.12)_1$.

Multiplying (2.9) by $\omega_{k_1}(y)\, \omega_{k_2}(z)$ and summing up in k for $|k| \leq N$, we find that

$$\Delta u_{Nx} + c u_{Nx} = g_N, \tag{2.13}$$

[1]We will only use that ω_{k_i} form an orthonormal basis of $L^2(I_y)$ (and $L^2(I_z)$), and satisfy $\omega''_{k_i} + \lambda_{k_i}\omega_{k_i} = 0$, for some λ_{k_i}, $i = 1, 2$.

holds in $L^2(\mathcal{M})$, where

$$g_N = \sum_{|k|\leqslant N} \widehat{g}_k(x)\omega_{k_1}(y)\,\omega_{k_2}(z). \tag{2.14}$$

Since $g_N \to g$ in $L^2(\mathcal{M})$, (2.13) implies that $\Delta u_{Nx} + cu_{Nx} \to g$ in $L^2(\mathcal{M})$. Hence with (2.8), we obtain $(2.12)_2$.

The conclusions of Lemma 2.1.1 follow. □

Now we are in position to prove the following trace theorem.

Theorem 2.1.1. *For $u \in \mathcal{X}(\mathcal{M})$, we can define the traces, on the lines $x = 0$ and $x = 1$, of u, u_x, u_{xx}, and of their y and z derivatives:*

$$\gamma_i u,\ \gamma_i u_x \text{ and } \gamma_i u_{xx} \text{ belong to } H^{-3}((-\frac{\pi}{2}, \frac{\pi}{2})^d), \tag{2.15}$$

$$\gamma_i u_{yy},\ \gamma_i u_{zz},\ \text{and } \gamma_i \Delta^{\perp} u \text{ belong to } H^{-5}((-\frac{\pi}{2}, \frac{\pi}{2})^d), \tag{2.16}$$

where γ_i is the trace on $x = i$, $i = 0, 1$. Furthermore, the trace operators are continuous from $\mathcal{X}(\mathcal{M})$ into the corresponding spaces.

Remark 2.1.1. *i) The spaces for the traces may not be "optimal", but, for the sequel, it is sufficient to know that the traces are well defined and that they depend continuously on u in $\mathcal{X}(\mathcal{M})$.*

ii) Since the derivatives $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$ are continuous from $H^{-k}((-\frac{\pi}{2}, \frac{\pi}{2})^d)$ into $H^{-k-1}((-\frac{\pi}{2}, \frac{\pi}{2})^d)$, the traces of the y and z derivatives of arbitrary orders of the functions in (2.15) can also be well defined, with continuity in the natural spaces; (2.16) provides some examples of such cases.

Proof. Let $u \in \mathcal{X}(\mathcal{M})$. Then by Fourier series expansion as in (2.7), we find that (2.9) implies

$$|\widehat{u}_k'''|_{L^2(I_x)} \leq |\widehat{g}_k|_{L^2(I_x)} + |\lambda_k - c||\widehat{u}_k'|_{L^2(I_x)}. \tag{2.17}$$

Applying the interpolation theorem, $|\widehat{u}'_k|_{L^2(I_x)} \leq C|\widehat{u}_k|^{2/3}_{L^2(I_x)}|\widehat{u}'''_k|^{1/3}_{L^2(I_x)}$, with C a constant independent of k, we obtain

$$|\widehat{u}'''_k|_{L^2(I_x)} \leq |\widehat{g}_k|_{L^2(I_x)} + \frac{2}{3}C^{3/2}|\lambda_k - c|^{3/2}|\widehat{u}_k|_{L^2(I_x)} + \frac{1}{3}|\widehat{u}'''_k|_{L^2(I_x)}.$$

Hence

$$|\widehat{u}'''_k|_{L^2(I_x)} \lesssim |\widehat{g}_k|_{L^2(I_x)} + |\lambda_k - c|^{3/2}|\widehat{u}_k|_{L^2(I_x)}, \tag{2.18}$$

where $\lesssim$ means $\leq$ up to a multiplicative constant independent of k. Summing up (2.18) in k for $|k| \leq N$, we obtain

$$\sum_k \lambda_k^{-3}|\widehat{u}'''_k|^2_{L^2(I_x)} \lesssim \sum_k \lambda_k^{-3}|\widehat{g}_k|^2_{L^2(I_x)} + \sum_k |\widehat{u}_k|^2_{L^2(I_x)}.$$

Thus

$$u_{xxx} \in L^2(I_x;\, H^{-3}(-\pi/2, \pi/2)^d). \tag{2.19}$$

Since $u \in L^2(\mathcal{M}) = L^2(I_x;\, L^2(-\pi/2, \pi/2)^d)$, by the intermediate derivative theory, u_x and u_{xx} belong to at least the same space as in (2.19). Hence we obtain (2.15). □

Remark 2.1.2. *Note that one cannot define the traces on $y = \pm\pi/2$ or $z = \pm\pi/2$ yet. A counterexample is the following:*

When $d = 1$, let $u(x, y) = 1/(y + \pi/2)^{1/4}$, $(x, y) \in I_x \times I_y$. Then $u \in \mathcal{X}(\mathcal{M})$, but $u \to \infty$ as $y \to -\pi/2$.

However, certain subspaces of $\mathcal{X}(\mathcal{M})$ may have traces defined on the y and z boundaries. As an example, Theorem 2.2 below gives such a result for the space

$$\mathcal{X}_1(\mathcal{M}) = \{u \in \mathcal{X}(\mathcal{M}),\ u = 0 \text{ at } x = 0, 1, \text{ and } u_x = 0 \text{ at } x = 1\}. \tag{2.20}$$

Note that this space can be defined thanks to Theorem 2.1.1.

2.2 Density and Trace Results for the Space $\mathcal{X}_1(\mathcal{M})$.

Before we state and prove the trace theorem, Theorem 2.2.1, we establish the following preliminary results for $\mathcal{X}_1(\mathcal{M})$.

Firstly, we have a density result for $\mathcal{X}_1(\mathcal{M})$ similar to Lemma 2.1.1.

Lemma 2.2.1. $\mathcal{X}_1(\mathcal{M}) \cap H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$ *is dense in* $\mathcal{X}_1(\mathcal{M})$.

Proof. Fix $u \in \mathcal{X}_1(\mathcal{M})$, and define u_N as in (2.10). By a similar proof as for Lemma 2.1.1, we have

$$u_N \to u \textit{ in } \mathcal{X}(\mathcal{M}), \tag{2.21}$$

and

$$u_N \in H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d). \tag{2.22}$$

Now we show that $u_N \in \mathcal{X}_1(\mathcal{M})$. By (2.20), $u(0, x^\perp) = 0$. Hence

$$\widehat{u}_k(0) = \langle u(0, x^\perp),\, \omega_{k_1}(y)\,\omega_{k_2}(z)\rangle = 0.$$

Similarly we have

$$\widehat{u}_k(0) = \widehat{u}_k(1) = \widehat{u}'_k(1) = 0. \tag{2.23}$$

Multiplying (2.23) by $\omega_{k_1}(y)\,\omega_{k_2}(z)$ and summing up in k for $|k| \le N$, we obtain

$$u_N = 0 \text{ at } x = 0, 1, \text{ and } u_{Nx} = 0 \text{ at } x = 1. \tag{2.24}$$

Thus $u_N \in \mathcal{X}_1(\mathcal{M})$ as desired and (2.21) now holds in $\mathcal{X}_1(\mathcal{M})$. □

Secondly we introduce two results concerning the regularity of functions in $\mathcal{X}_1(\mathcal{M})$.

Lemma 2.2.2. *For* $u \in \mathcal{X}_1(\mathcal{M})$, *the traces*

$$u_x\big|_{x=0} \textit{ and } u_{xx}\big|_{x=1} \textit{ belong to } L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d), \tag{2.25}$$

and the trace operators are continuous from $\mathcal{X}_1(\mathcal{M})$ *into the corresponding spaces.*

Proof. Let $u \in \mathcal{X}_1(\mathcal{M})$. Assume first that $u \in \mathcal{X}_1(\mathcal{M}) \cap H^3_x(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$; then (2.9) and (2.23) hold in $L^2(I_x)$. We successively multiply (2.9) by $\widehat{u}_k$, $x\widehat{u}_k$ and x, integrate between 0 and 1 and integrate by parts.

Multiplying (2.9) by $\widehat{u}_k$, we obtain with (2.23)

$$-\int_0^1 \widehat{u}_k''\widehat{u}_k'\,dx + \widehat{u}_k''\widehat{u}_k\big|_0^1 - \frac{1}{2}(\lambda_k - c)\widehat{u}_k^2\big|_0^1 = \int_0^1 \widehat{g}_k\widehat{u}_k\,dx;$$

$$\frac{1}{2}(\widehat{u}_k'(0))^2 \leq \frac{1}{2}|\widehat{g}_k|^2_{L^2(I_x)} + \frac{1}{2}|\widehat{u}_k|^2_{L^2(I_x)}. \tag{2.26}$$

Summing (2.26) up in k,

$$\big|u_x\big|_{x=0}\big|^2_{L^2(\mathcal{M})} \lesssim |g|^2_{L^2(\mathcal{M})} + |u|^2_{L^2(\mathcal{M})}, \tag{2.27}$$

which implies (2.25).

Multiplying (2.9) by $x\widehat{u}_k$, we obtain

$$-\int_0^1 \widehat{u}_k''\widehat{u}_k'x\,dx - \int_0^1 \widehat{u}_k''\widehat{u}_k\,dx + \frac{1}{2}(\lambda_k - c)|\widehat{u}_k|^2_{L^2(I_x)} = \int_0^1 \widehat{g}_k\widehat{u}_k x\,dx;$$

$$\frac{1}{2}\int_0^1 (\widehat{u}_k')^2 dx + \int_0^1 (\widehat{u}_k')^2 dx + \frac{1}{2}(\lambda_k - c)|\widehat{u}_k|^2_{L^2(I_x)} = \int_0^1 \widehat{g}_k\widehat{u}_k x dx;$$

$$\frac{3}{2}|\widehat{u}_k'|^2_{L^2(I_x)} + \frac{\lambda_k}{2}|\widehat{u}_k|^2_{L^2(I_x)} = \frac{c}{2}|\widehat{u}_k|^2_{L^2(I_x)} + \int_0^1 \widehat{g}_k\widehat{u}_k x\,dx; \tag{2.28}$$

$$\frac{\lambda_k}{2}|\widehat{u}_k|^2_{L^2(I_x)} \leq \frac{c}{2}|\widehat{u}_k|^2_{L^2(I_x)} + |\widehat{g}_k|_{L^2(I_x)}|\widehat{u}_k|_{L^2(I_x)}.$$

Dividing both sides by $|\widehat{u}_k|_{L^2(I_x)}$, we find

$$\frac{\lambda_k}{2}|\widehat{u}_k|_{L^2(I_x)} \leq \frac{c}{2}|\widehat{u}_k|_{L^2(I_x)} + |\widehat{g}_k|_{L^2(I_x)};$$

$$\frac{\lambda_k^2}{4}|\widehat{u}_k|^2_{L^2(I_x)} \leq \frac{c^2}{2}|\widehat{u}_k|^2_{L^2(I_x)} + 2|\widehat{g}_k|^2_{L^2(I_x)}. \tag{2.29}$$

Multiplying (2.9) by x, we see that

$$-\int_0^1 \widehat{u}_k'' dx + \widehat{u}_k'' x\big|_0^1 - (\lambda_k - c)\int_0^1 \widehat{u}_k' x dx = \int_0^1 \widehat{g}_k x dx;$$

$$\widehat{u}_k''(1) = -\widehat{u}_k'(0) - \lambda_k \int_0^1 \widehat{u}_k\,dx + c\int_0^1 \widehat{u}_k\,dx + \int_0^1 \widehat{g}_k x\,dx.$$

Thus

$$(\widehat{u}_k''(1))^2 \lesssim (\widehat{u}_k'(0))^2 + (\lambda_k^2 + c^2)|\widehat{u}_k|^2_{L^2(I_x)} + |\widehat{g}_k|^2_{L^2(I_x)}. \tag{2.30}$$

From (2.30), (2.29) and (2.26) we infer that

$$(\widehat{u}_k''(1))^2 \lesssim c^2|\widehat{u}_k|^2_{L^2(I_x)} + |\widehat{g}_k|^2_{L^2(I_x)}. \tag{2.31}$$

Summing (2.31) up in k, we obtain

$$|u_{xx}\big|_{x=1}|^2_{L^2(\mathcal{M})} \lesssim c^2|u|^2_{L^2(\mathcal{M})} + |g|^2_{L^2(\mathcal{M})}, \tag{2.32}$$

which implies (2.25).

Hence the conclusions of Lemma 2.2.2 follow for functions smooth enough. For u non-smooth we proceed by approximation using Lemma 2.2.1. □

Proposition 2.2.1. *When $d = 1, 2$, if $u \in \mathcal{X}_1(\mathcal{M})$, we have*

$$\Delta^{\perp} u \text{ and } \Delta u \text{ belong to } L^2(\mathcal{M}). \tag{2.33}$$

Moreover,

$$\mathcal{X}_1(\mathcal{M}) \subset H_0^1 \cap H^2(I_x;\, L^2((-\pi/2, \pi/2)^d))), \tag{2.34}$$

with a continuous embedding. Furthermore, when $d = 1$, we actually have

$$\mathcal{X}_1(\mathcal{M}) \subset H_0^1 \cap H^2(I_x;\, H^2(I_y)). \tag{2.35}$$

Proof. Integrating (2.8) in x from x to 1, we find

$$\Delta u = u_{xx}\big|_{x=1} - cu - \int_x^1 g(\tilde{x}, x^{\perp})\, d\tilde{x}. \tag{2.36}$$

Together with (2.25), we find that $\Delta u \in L^2(\mathcal{M})$.

For u smooth enough, integrating (2.9) from x to 1, we obtain with (2.23)

$$\widehat{u}_k''(x) = -\lambda_k \widehat{u}_k(x) + c\widehat{u}_k(x) + \widehat{u}_k''(1) + \int_x^1 \widehat{g}_k(\tilde{x})\, d\tilde{x};$$

$$|\widehat{u}_k''|^2_{L^2(I_x)} \lesssim \lambda_k^2|\widehat{u}_k|^2_{L^2(I_x)} + c^2|\widehat{u}_k|^2_{L^2(I_x)} + (\widehat{u}_k''(1))^2 + |\widehat{g}_k|^2_{L^2(I_x)}. \tag{2.37}$$

From (2.37), (2.29) and (2.30) we obtain

$$|\widehat{u}_k''|^2_{L^2(I_x)} \lesssim c^2|\widehat{u}_k|^2_{L^2(I_x)} + |\widehat{g}_k|^2_{L^2(I_x)}. \tag{2.38}$$

Summing (2.38) up in k,

$$|u_{xx}|^2_{L^2(\mathcal{M})} \lesssim c^2|u|^2_{L^2(\mathcal{M})} + |g|^2_{L^2(\mathcal{M})}.$$

Thus

$$u_{xx} \in L^2(\mathcal{M}). \tag{2.39}$$

By the intermediate derivatives theorem, $u_x \in L^2(\mathcal{M})$, and hence we obtain (2.34). Since $\Delta u \in L^2(\mathcal{M})$ as is proven, we infer (2.33) from (2.39).

When $d = 1$, (2.33) implies that $u_{yy} \in L^2(\mathcal{M})$. By the intermediate derivatives theorem, $u_y \in L^2(\mathcal{M})$. Hence with (2.34) we obtain (2.35).

The conclusions of Proposition 2.2.1 follow for functions smooth enough. For u non-smooth, we proceed by approximation using Lemma 2.2.1. □

Remark 2.2.1. *Integrating (2.8) in x from 0 to x, we also find (compare with (2.25)),*

$$u_{xx}\big|_{x=0} \text{ belong to } L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d). \tag{2.40}$$

Remark 2.2.2. *In (2.35), we do not know if $u \in H^2(I_x \times I_y)$, because u_{xy} does not necessarily belong to $L^2(\mathcal{M})$ due to the lack of information at the y boundary.*

Now we are in position to prove the following trace theorem.

Theorem 2.2.1. *For $u \in \mathcal{X}_1(\mathcal{M})$, we can define more traces, in particular, at $y = \pm\pi/2$ and $z = \pm\pi/2$, namely,*

$$u\big|_{y=\pm\frac{\pi}{2}} \text{ and } u_y\big|_{y=\pm\frac{\pi}{2}} \text{ belong to } H^{-2}(I_x \times I_z);$$

$$u\big|_{z=\pm\frac{\pi}{2}} \text{ and } u_z\big|_{z=\pm\frac{\pi}{2}} \text{ belong to } H^{-2}(I_x \times I_y).$$

Furthermore, the trace operators are continuous from $\mathcal{X}_1(\mathcal{M})$ into the corresponding spaces.

Remark 2.2.3. *As mentioned in Remark 2.1.1, we can similarly define the traces on the y and z boundaries of the x derivatives of arbitrary orders of these functions. For example,*

$$u_x\big|_{y=\pm\frac{\pi}{2}} \text{ and } u_{xy}\big|_{y=\pm\frac{\pi}{2}} \text{ belong to } H^{-3}(I_x \times I_z). \tag{2.41}$$

Proof. Assume first u is smooth. Then by Fourier series expansion in $y^{\perp}$, we write

$$u = \sum_k \widehat{u}_k(y) e_{k_1}(x)\, w_{k_2}(z), \quad k = (k_1, k_2), \tag{2.42}$$

with $\{e_{k_1}\} \subseteq \mathcal{D}(I_x)$ an orthonormal basis of $L^2(I_x)$; setting $h = \Delta^{\perp} u \in L^2(\mathcal{M})$, (2.33) implies

$$-\lambda_{k_2}\widehat{u}_k(y) + \widehat{u}_k''(y) = \widehat{h}_k(y), \tag{2.43}$$

with $\widehat{h}_k(y)$ the Fourier coefficients of h as in (2.42). Hence

$$\lambda_{k_2}^{-2}|\widehat{u}_k''|^2_{L^2(I_y)} \lesssim \lambda_{k_2}^{-2}|\widehat{h}_k|^2_{L^2(I_y)} + |\widehat{u}_k|^2_{L^2(I_y)}.$$

Summing it up in k, we obtain

$$\sum_k \lambda_{k_2}^{-2}|\widehat{u}_k''|^2_{L^2(I_y)} \lesssim \sum_k \lambda_{k_2}^{-2}|\widehat{h}_k|^2_{L^2(I_y)} + \sum_k |\widehat{u}_k|^2_{L^2(I_y)}.$$

Hence $u_{yy} \in L^2(I_y;\, H^{-2}(I_x \times I_z))$. By the intermediate derivative theorem, u_y belongs to at least the same space as u_{yy}, and we can define the traces of u and u_y at the y boundary. Similarly, we can prove the trace results for the z boundary. Hence the conclusions follow for u smooth.

For u non-smooth, since $u \in L^2(\mathcal{M})$ and $\Delta u \in L^2(\mathcal{M})$ by (2.33), we can use Theorem 5.1 in [32], which says that with $\partial\mathcal{M}$ satisfying the cone condition, $\mathcal{C}^{\infty}(\bar{\mathcal{M}})$ is dense in the space $\{u \in L^2(\mathcal{M});\ \Delta u \in L^2(\mathcal{M})\}$. Hence we can proceed by approximation. □

Remark 2.2.4. *In Proposition 2.2.1, because of the lack of information about the trace of u on the y and z boundaries, (2.33) does not imply that $u_{yy} \in L^2(\mathcal{M})$ when $d = 2$. Here is a counterexample entailing this reasoning: Let $u = w(x)v(y,z)$, where $w \in \mathcal{C}_c^\infty(I_x)$, and $v = \ln[(y+\pi/2)^2 + (z+\pi/2)^2]$. We can show that*

$$v \in L^2(I_y \times I_z), \quad v_{yy} = -v_{zz} \notin L^2(I_y \times I_z). \tag{2.44}$$

Indeed by the change of variables $\tilde{y} = y + \pi/2$ and $\tilde{z} = z + \pi/2$, we are led to consider the function $\tilde{v}(\tilde{y},\tilde{z}) = \ln(\tilde{y}^2 + \tilde{z}^2)$ in $\tilde{I}_{\tilde{y}} \times \tilde{I}_{\tilde{z}}$, where $\tilde{I}_{\tilde{y}} = (0,\pi)_{\tilde{y}}$, $\tilde{I}_{\tilde{z}} = (0,\pi)_{\tilde{z}}$. The analogue of (2.44) is easy to check by direct calculation. Then (2.44) implies

$$u \in L^2(\mathcal{M}), \quad u_x = w'v \in L^2(\mathcal{M}); \tag{2.45}$$

and

$$\Delta u_x = u_{xxx} + u_{xyy} + u_{xzz} = w'''v + w'(v_{yy} + v_{zz}) = w'''v \in L^2(\mathcal{M}).$$

Hence $u \in \mathcal{X}(\mathcal{M})$.

Since $w \in \mathcal{C}_c^\infty(I_x)$, it is easy to see that

$$u = 0 \text{ at } x = 0, 1, \text{ and } u_x = 0 \text{ at } x = 1.$$

Finally, $u \in \mathcal{X}_1(\mathcal{M})$. But (2.44) also implies that neither $u_{yy} = wv_{yy}$ nor $u_{zz} = wv_{zz}$ belongs to $L^2(\mathcal{M})$, and hence $u \notin H^2(\mathcal{M})$.

In fact, the regularity of u in the y and z directions is even worse. We can see that $\tilde{v}_{\tilde{y}} \notin L^2(\tilde{I}_{\tilde{y}} \times \tilde{I}_{\tilde{z}})$. Hence $v_y \notin L^2(I_y \times I_z)$. Thus $u_y = wv_y \notin L^2(\mathcal{M})$, which implies that u does not even belong to $H^1(\mathcal{M})$. To conclude, while having H^2-regularity in the x-direction, u may be much less regular in the y, z directions.

However, certain subspaces of $\mathcal{X}_1(\mathcal{M})$ may have better regularity properties. For instance, adding suitable boundary conditions on y and z boundaries can lead to stronger regularity in these directions; we show this in the next section. □

2.3 Operator A in the Dirichlet Case.

As mentioned before, we can assume either the Dirichlet or the periodic boundary condition on the y, z boundaries. We first study the Dirichlet case, that is we assume (2.4). We rewrite (2.7) in the form

$$u = \sum_{k} \widehat{u}_k(x) \cos k_1 y \, \cos k_2 z, \tag{2.46}$$

where $k_1, k_2 \in 2\mathbb{N}+1$, and $\lambda_k = k_1^2 + k_2^2 = k^2$. We also rewrite (2.10) as

$$u_N = \sum_{|k| \leqslant N} \widehat{u}_k(x) \cos k_1 y \, \cos k_2 z, \quad N \in \mathbb{N}. \tag{2.47}$$

2.3.1 Density and Regularity Results for $D(A)$.

Thanks to Theorems 2.1.1, 2.2.1, we can define the operator A and its domain $D(A)$:

$$D(A) = \{u \in \mathcal{X}_1(\mathcal{M}), u = 0 \; at \; y = \pm\frac{\pi}{2} \; (and \; z = \pm\frac{\pi}{2})\}, \tag{2.48}$$

and for $u \in D(A)$,

$$Au = \Delta u_x + c u_x.$$

Firstly, as mentioned before, we have a regularity result for $D(A)$:

Proposition 2.3.1. *$D(A) \subset H_0^1(\mathcal{M}) \cap H^2(\mathcal{M})$ with a continuous imbedding, $d = 1, 2$.*

Proof. Since $u \in D(A)$, then $u \in L^2(\mathcal{M})$ and $u = 0$ on $\partial\mathcal{M}$; furthermore, by (2.33), $\Delta u \in L^2(\mathcal{M})$; from $\partial\mathcal{M}$ being Lipschitz, we infer that $u \in H^2(\mathcal{M})$, thanks to [22]. □

Secondly, we have a density result for $D(A)$ similar to Lemma 2.1.1:

Lemma 2.3.1. *$D(A) \cap H_x^3(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$ is dense in $D(A)$.*

Proof. Fix $u \in D(A)$, and define u_N as in (2.47). Since $u \in \mathcal{X}_1(\mathcal{M})$, Lemma 2.2.1 implies

$$u_N \to u \; in \; \mathcal{X}_1(\mathcal{M}),$$

and

$$u_N \in \mathcal{X}_1(\mathcal{M}) \cap H_x^3(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d). \tag{2.49}$$

Moreover, from (2.47) we obtain

$$u_N = 0 \; at \; y = \pm\frac{\pi}{2} \; (and \; z = \pm\frac{\pi}{2}). \tag{2.50}$$

By (2.50) and (2.49), $u_N \in D(A) \cap H_x^3(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$ as desired. Thus the conclusions of Lemma 2.3.1 follow. □

Remark 2.3.1. *Note that by Proposition 2.3.1 and Lemma 2.3.1, (2.28) implies*

$$(Au, xu) = \int_{\mathcal{M}} \left(\frac{3}{2}u_x^2 + \frac{1}{2}|\nabla^\perp u|^2 - \frac{c}{2}u^2\right) d\mathcal{M}, \quad u \in D(A). \tag{2.51}$$

2.3.2 Characterization of A^* and $D(A^*)$.

In order to study the linearized Z-K equation (2.1)-(2.4), as mentioned before, we need to prove that $-A$ in (2.6) is the infinitesimal generator of a contraction semigroup in view of applying the Hille-Yoshida theorem. For that purpose we first need to define and characterize the adjoint A^* and its domain $D(A^*)$.

Let $H = L^2(\mathcal{M})$. Assume that $u \in D(A)$ and $\tilde{u} \in H$ are smooth functions; then

$$(Au, \tilde{u})_H = \int_{\mathcal{M}} (\Delta u_x + cu_x)\tilde{u} \, d\mathcal{M} = I_1 + I_2 + I_3, \tag{2.52}$$

where

$$I_1 = -\int_{\mathcal{M}} u(\Delta \tilde{u}_x + c\tilde{u}_x) \, d\mathcal{M},$$

and

$$I_2 = \int_{(-\frac{\pi}{2}, \frac{\pi}{2})^d} u_{xx}\tilde{u}\Big|_{x=0}^{x=1} + u_x\tilde{u}_x\Big|_{x=0} \, dx^\perp;$$

$$I_3 = \int_{I_x \times I_z} u_{xy}\tilde{u}\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} \, dy^\perp + \int_{I_x \times I_y} u_{xz}\tilde{u}\Big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} \, dz^\perp.$$

According to [39], $D(A^*)$ consists of the $\tilde{u}$ in H such that the mapping $u \to (Au, \tilde{u})_H$ is continuous on $D(A)$ for the norm of H.

Step 1. Fix $\tilde{u}$ in H and consider the u's in $D(A) \cap \mathcal{C}_c^\infty(\mathcal{M})$. For such u's,

$$(Au, \tilde{u})_H = I_1 = - < u, \Delta\tilde{u}_x + c\tilde{u}_x >,$$

and if $u \to (Au, \tilde{u})_H$ is continuous for the norm of H for such u's, then necessarily we have $\Delta\tilde{u}_x + c\tilde{u}_x \in L^2(\mathcal{M})$. Hence

$$\tilde{u} \in \mathcal{X}(\mathcal{M}). \tag{2.53}$$

Step 2. Now consider the u's in $D(A) \cap H_x^3(I_x; \mathcal{C}_c^\infty((-\frac{\pi}{2}, \frac{\pi}{2})^d))$. For such u's, since (2.53) implies that the traces of $\tilde{u}$ at $x = 0$ and 1 are well defined as in Theorem 2.1.1, the following calculations are valid:

$$(Au, \tilde{u})_H = I_1 + I_2.$$

Hence the mapping $u \to (Au, \tilde{u})_H$ can only be continuous for the the norm of H if

$$\tilde{u} = 0 \ \textit{at} \ x = 0, \, 1, \ \textit{and} \ \tilde{u}_x = 0 \ \textit{at} \ x = 0. \tag{2.54}$$

From (2.54) and (2.53) we deduce that $\tilde{u} \in \tilde{\mathcal{X}}_1(\mathcal{M})$, where

$$\tilde{\mathcal{X}}_1(\mathcal{M}) = \{\tilde{u} \in \mathcal{X}(\mathcal{M}), \, \tilde{u} = 0 \ \textit{at} \ x = 0, \, 1, \ \tilde{u}_x = 0 \ \textit{at} \ x = 0\}. \tag{2.55}$$

Step 3. Now consider the u's in $D(A) \cap H_x^3(I_x; \mathcal{C}^\infty[-\pi/2, \pi/2]^d)$. By (2.55), the traces of $\tilde{u}$ can be defined at both the x boundary and the y, z boundaries (same reasoning as in Theorems 2.1.1 and 2.2.1 for $\mathcal{X}_1(\mathcal{M})$). Hence the following calculations are valid:

$$(Au, \tilde{u})_H = I_1 + I_3, \tag{2.56}$$

where I_2 vanishes because of (2.54). Thus the mapping $u \to (Au, \tilde{u})_H$ can only be continuous for the norm of H if

$$\tilde{u} = 0 \ \textit{at} \ y = \pm\frac{\pi}{2} \ \textit{and} \ z = \pm\frac{\pi}{2}. \tag{2.57}$$

From (2.57) and (2.55), we infer that $\tilde{u} \in \tilde{D}(A)$, where

$$\tilde{D}(A) = \{\tilde{u} \in \mathcal{X}(\mathcal{M}), \, \tilde{u} = 0 \ \textit{on} \ \partial\mathcal{M}, \, \tilde{u}_x = 0 \ \textit{at} \ x = 0\}. \tag{2.58}$$

Finally, if $\tilde{u} \in \tilde{D}(A)$, then it can be approximated in $\tilde{D}(A)$ by functions belonging to $\tilde{D}(A) \cap H^3_x(I_x; \mathcal{C}^\infty[-\pi/2, \pi/2]^d)$ (same reasoning as for $D(A)$ in Lemma 2.3.1). Also any function $u \in D(A)$ can be approximated in $D(A)$ by functions belonging to $D(A) \cap H^3_x(I_x; \mathcal{C}^\infty[-\pi/2, \pi/2]^d)$ by Lemma 2.3.1. The integration by parts of (2.56) are then justified by approximation, and by (2.57), $I_3 = 0$. This shows that $\tilde{u} \in D(A^*)$ and, finally,

$$D(A^*) = \tilde{D}(A). \tag{2.59}$$

For $\tilde{u} \in D(A^*)$,

$$A^*\tilde{u} = -(\Delta\tilde{u}_x + c\tilde{u}_x). \tag{2.60}$$

2.4 Operator A_{per} in the Space Periodic Case.

As mentioned at the end of Section 2.2, we can also consider periodic boundary conditions on the y and z boundaries, all the well defined derivatives being periodic with period π. We rewrite (2.7) in the form

$$u = \sum_k \widehat{u}_k(x) e^{ik_1 y} e^{ik_2 z}, \tag{2.61}$$

where $k_1, k_2 \in 2\mathbb{N}$, and $\lambda_k = k_1^2 + k_2^2 := k^2$. We also rewrite (2.10) as

$$u_N = \sum_{|k| \leqslant N} \widehat{u}_k(x) e^{ik_1 y} e^{ik_2 z}, \quad N \in \mathbb{N}. \tag{2.62}$$

2.4.1 Density and Regularity Results for $D(A_{per})$.

Thanks to Theorems 2.1.1, 2.2.1, (2.5) is legitimate, and hence we can define A_{per} and its domain $D(A_{per})$:

$$D(A_{per}) = \{u \in \mathcal{X}_1(\mathcal{M}), u \text{ is periodic with the first order derivatives periodic in } y \text{ and } z, \text{ with period } \pi\},$$

and equivalently

$$D(A_{per}) = \{u \in \mathcal{X}(\mathcal{M}), u = 0 \; at \, x = 0, 1, \; u_x = 0 \; at \; x = 1;$$
$$u\big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = u_y\big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = 0, \; u\big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = u_z\big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = 0\}.$$

For $u \in D(A_{per})$,

$$A_{per}u = \Delta u_x + cu_x.$$

It is worth noting that when $d = 1$, we can define $v := u - u\big|_{y=\frac{\pi}{2}} = u - u\big|_{y=-\frac{\pi}{2}}$. Then $v \in D(A)$. Hence each function in $D(A_{per})$ corresponds to a function in $D(A)$ up to a constant, and they should share the same properties.

Thus the more interesting case will be $d = 2$. Firstly, we have the same regularity result as in Proposition 2.3.1.

Proposition 2.4.1. $D(A_{per}) \subset H_0^1(\mathcal{M}) \cap H^2(\mathcal{M}),\; d = 1, 2,$ *with a continuous imbedding.*

Proof. Let $u \in D(A_{per})$. Since $u \in D(A_{per})$, then $u \in L^2(\mathcal{M})$ and $u = 0$ on the x boundary, and is periodic on the y and z boundaries. Hence we can prove Proposition 2.4.1 in the same way as Proposition 2.3.1. □

Secondly, we also have the same density result as Lemma 2.1.1:

Lemma 2.4.1. $D(A_{per}) \cap H_x^3(I_x; \mathcal{C}^\infty[-\frac{\pi}{2}, \frac{\pi}{2}]^d)$ *is dense in* $D(A_{per})$.

Proof. The proof is similar with Lemma 2.1.1, noting that (2.62) implies that u_N and its first order derivatives are periodic in y and z with period π. □

2.4.2 Characterization of A_{per}^* and $D(A_{per}^*)$.

Let $H = L^2(\mathcal{M})$. Assume that $u \in D(A_{per})$ and $\tilde{u} \in H$ are smooth functions. We calculate $(A_{per}u, \tilde{u})_H$, and the result is the same as that of (2.52). The reason is that when integrating by parts in y and z, whenever the boundary terms get canceled by the Dirichlet boundary condition, they also get canceled by the periodic boundary condition. Hence by a similar reasoning, we obtain

$$\tilde{D}(A^*_{per}) = \{\tilde{u} \in \mathcal{X}(\mathcal{M}), u = 0 \ at\, x = 0, 1, \ \tilde{u}_x = 0 \ at \ x = 0;$$
$$\tilde{u}\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = \tilde{u}_y\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = 0, \ \tilde{u}\Big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = \tilde{u}_z\Big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = 0\},$$

and for $\tilde{u} \in D(A^*_{per})$,

$$A^*_{per}\tilde{u} = -(\Delta\tilde{u}_x + c\tilde{u}_x). \tag{2.63}$$

□

2.5 Linear Evolution ZK Equation

We are now in position to study the linear evolution equation in the Dirichlet case, that is (2.1)-(2.4). The results for the periodic case, which we will not present, are very similar.

2.5.1 Well-posedness of the Evolution Equation

We want to apply the Phillips version of the Hille-Yoshida theorem to deduce the existence and uniqueness of strong solutions to (2.6). As indicated before, we only need to show that $-A$ is the infinitesimal generator of a semigroup of contractions in H. According to [25] and [36], it suffices to show that

i) A and A^* are closed operators and their domains $D(A)$ and $D(A^*)$ are dense in H, and

ii) A and A^* are positive:

$$\begin{cases} (Au,\, u)_H \geq 0, \quad \forall u \in D(A), \\ (A^*\tilde{u},\, \tilde{u})_H \geq 0, \quad \forall \tilde{u} \in D(A^*). \end{cases} \tag{2.64}$$

The proof of i) is direct.

To show $(2.64)_1$, we can assume that $u \in D(A) \cap H^3_x(I_x; \mathcal{C}^\infty[-\pi/2, \pi/2]^d)$. Replacing $\tilde{u}$ by u in (2.52), we see that all the integration by parts are legitimate for such a u and we obtain (2.52) with

$$I_1 = -(Au,\, u)_H; \quad I_2 = \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} u_x^2(0, x^\perp)\, dx^\perp; \quad I_3 = 0.$$

Hence

$$(Au,\, u)_H = \frac{1}{2} \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} u_x^2(0, x^\perp)\, dx^\perp \geq 0, \tag{2.65}$$

for such a u and by continuity for all $u \in D(A)$. Similarly, we have

$$(A^*\tilde{u},\, \tilde{u})_H = \frac{1}{2} \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} \tilde{u}_x^2(1, x^\perp)\, dx^\perp \geq 0, \quad \forall \tilde{u} \in D(A^*). \tag{2.66}$$

Thus $-A$ is the infinitesimal generator of a semigroup of contractions in H denoted by $\{S(t)\}_{t\geq 0}$. By the Hille-Phillips-Yoshida theorem, we then obtain the existence of strong solutions to the initial-value problem (2.6).

Theorem 2.5.1. *Assume that u_0 is given in $D(A)$ and f is given in $L^1_{loc}(\mathbb{R}_+; H)$, with $f' = df/dt$ in $L^1_{loc}(\mathbb{R}_+; H)$. Then the initial and boundary value problem (2.6) possesses a unique solution u such that*

$$u \in \mathcal{C}([0,T]; D(A)), \;\; u_t \in L^\infty(0,T;H) \textit{ for any } T > 0. \tag{2.67}$$

Furthermore, If u_0 is given in H and $f \in L^1_{loc}(\mathbb{R}_+; H)$, then (2.6) possesses a unique mild solution $u \in \mathcal{C}(\mathbb{R}_+; H)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds. \tag{2.68}$$

Thanks to the corresponding regularity results for the stationary equation, we have the following regularity properties for the evolution equation:

Corollary 2.5.1. *Under the hypotheses of Theorem 2.5.1, the solution u of (2.67) satisfies*

$$u \in L^\infty((0,T);\, H^1_0(\mathcal{M}) \cap H^2(\mathcal{M})), \;\; d = 1, 2, \tag{2.69}$$

and

$$u_x\big|_{x=0} \text{ and } u_{xx}\big|_{x=0,1} \text{ belong to } L^\infty((0,T);\, L^2((-\frac{\pi}{2},\frac{\pi}{2})^d)). \tag{2.70}$$

Proof. By (2.67), $u \in L^\infty(0,T;\, D(A))$ and $u_t \in L^\infty(0,T;H)$. Then we obtain (2.69) by Proposition 2.3.1, and obtain (2.70) by (2.25), (2.40). □

Chapter 3

Nonlinear ZK Equation: Existence and Uniqueness of the Weak Solution

We now proceed and consider the initial and boundary-value problem for the full (nonlinear) ZK equation in the limited domain $\mathcal{M}$ in the Dirichlet case, that is, (1.1) and (2.2)-(2.4). Our main result in this chapter is as follows.

Theorem 3.0.2. *We are given* $u_0 \in L^2(\mathcal{M})$ *and* $f \in L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{M}))$, $\mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d$, $d = 1, 2$. *Then:*

(*i*) *The initial and boundary-value problem for the ZK equation, that is, (1.1) and (2.2)-(2.4), possesses a weak solution* u,

$$u \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H^1_0(\mathcal{M})), \tag{3.1}$$

$$\int_0^T \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} |u_x(0,x^\perp,t)|^2 \, dx^\perp \, dt < \infty, \tag{3.2}$$

for all $T > 0$.

(*ii*) *If* $d = 1$, *then the solution* u *is unique. Moreover,* $u \in \mathcal{C}([0,T];\, L^2(\mathcal{M}))$ *and the flow map* $u_0 \to u(\cdot,t)$ *is continuous from* $L^2(\mathcal{M})$ *to* $L^2(\mathcal{M})$.

Remark 3.0.1. *Note that the boundary condition* $u = 0$ *on* $\partial\mathcal{M}$ *makes sense since* $u \in L^2(0,T;\, H^1_0(\mathcal{M})^d))$. *The boundary condition* $u_x\big|_{x=1} = 0$ *makes sense in* $H^{-2}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d)$ *as explained in Lemma 3.1.1 below.*

3.1 Existence of Weak Solutions in Dimension 3

Proof of Theorem 3.0.2. i) We start with the proof of existence. The existence is proven by parabolic regularization as in [40]; that is, for $\epsilon > 0$ "small", we consider the parabolic equation

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \Delta \dfrac{\partial u^\epsilon}{\partial x} + c \dfrac{\partial u^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon \, L u^\epsilon = f, \\ u^\epsilon(0) = u_0, \end{cases} \tag{3.3}$$

where

$$L u^\epsilon := \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4},$$

supplemented with the boundary conditions (2.2), (2.4) and the additional boundary conditions

$$u^\epsilon_y\big|_{y=\pm\frac{\pi}{2}} = u^\epsilon_z\big|_{z=\pm\frac{\pi}{2}} = 0, \tag{3.4}$$

$$u^\epsilon_{xx}\big|_{x=0} = 0. \tag{3.5}$$

Note that since $u^\epsilon_{yy} = u^\epsilon_{zz} = 0$ at $x = 0$ (and $x = 1$), (3.5) is equivalent to

$$\Delta u^\epsilon\big|_{x=0} = 0.$$

The following a priori estimates classically guarantee the existence of a solution u^ϵ for $\epsilon > 0$ fixed and then allow us to pass to the limit $\epsilon \to 0$, thus providing the existence of a solution for the ZK equation.

As for the linear ZK equation these a priori estimates are obtained by multiplying (3.3) by u^ϵ, then by $x u^\epsilon$, integrating and integrating by parts in each case. The contributions of the linear terms Δu^ϵ_x and $c u^\epsilon_x$ have already been studied, and we thus concentrate on the contributions of the other terms. Note that the solutions u^ϵ to the parabolic problem are sufficiently regular for the following calculations to be fully legitimate without any need of further regularization.

We drop the super index ϵ and start by multiplying $(3.3)_1$ with $u = u^\epsilon$. We find

- $\displaystyle\int_{\mathcal{M}} \frac{\partial u}{\partial t} u \, d\mathcal{M} = \frac{1}{2}\frac{d}{dt}|u|^2_{L^2(\mathcal{M})}$,
- $\displaystyle\int_{\mathcal{M}} uu_x \, u \, d\mathcal{M} = 0$,
- $\displaystyle\int_{\mathcal{M}} fu \, d\mathcal{M} \le \frac{1}{2}|f|^2_{L^2(\mathcal{M})} + \frac{1}{2}|u|^2_{L^2(\mathcal{M})}$,
- $\displaystyle\epsilon\int_{\mathcal{M}} (\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4}) \, u \, d\mathcal{M}$
$\displaystyle= \epsilon\int_{\mathcal{M}} (\left|\frac{\partial^2 u}{\partial x^2}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2 + \left|\frac{\partial^2 u}{\partial z^2}\right|^2) \, d\mathcal{M} =: \epsilon[u]_2^2.$

Hence with (2.65), we find

$$\frac{d}{dt}|u^\epsilon|^2_{L^2(\mathcal{M})} + \int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} |u_x^\epsilon(0, x^\perp, t)|^2 \, dx^\perp + 2\epsilon[u^\epsilon]_2^2 \le |f|^2_{L^2(\mathcal{M})} + |u^\epsilon|^2_{L^2(\mathcal{M})}. \tag{3.6}$$

Using the Gronwall lemma we classically infer from (3.6) (and the fact that $u_0 \in L^2(\mathcal{M})$) the following bounds *independent* of ϵ:

$$\begin{cases} u^\epsilon \text{ is bounded in } L^\infty(0, T; \, L^2(\mathcal{M})), \\ \sqrt{\epsilon} u^\epsilon \text{ is bounded in } L^2(0, T; \, H_0^1 \cap H^2(\mathcal{M})), \\ u_x^\epsilon(0, \cdot, \cdot) \text{ is bounded in } L^2(0, T; \, L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d)). \end{cases} \tag{3.7}$$

For $(3.7)_2$, note that $[u]_2$ is a norm on $H_0^1 \cap H^2(\mathcal{M})$ equivalent to the H^2-norm, since $|\Delta u|_{L^2(\mathcal{M})}$ is equivalent to the H^2-norm and

$$|\Delta u|_{L^2(\mathcal{M})} \le 2(|u_{xx}|_{L^2(\mathcal{M})} + |u_{yy}|_{L^2(\mathcal{M})} + |u_{zz}|_{L^2(\mathcal{M})}).$$

We now multiply $(3.3)_1$ by xu $(= xu^\epsilon)$, integrate and integrate by parts:

- $\displaystyle\int_{\mathcal{M}} \frac{\partial u}{\partial t} xu \, d\mathcal{M} = \frac{1}{2}\frac{d}{dt}|\sqrt{x}u|^2_{L^2(\Omega)}$,
- $\displaystyle\left|\int_{\mathcal{M}} uu_x xu \, d\mathcal{M}\right| \le \frac{1}{3}\int_{\mathcal{M}} |u|^3 \, d\mathcal{M}$

  $\le$ (by interpolation, $H^{1/2}(\mathcal{M}) \subset L^3(\mathcal{M})$ in dimension 3)

$$\leq c_1|u|^{3/2}_{L^2(\mathcal{M})}|\nabla u|^{3/2}_{L^2(\mathcal{M})} \leq \frac{1}{4}|\nabla u|^2_{L^2(\mathcal{M})} + c_2|u|^6_{L^2(\mathcal{M})}, \tag{3.8}$$

- $\displaystyle\int_{\mathcal{M}} fxu\, d\,\mathcal{M} \leq \frac{1}{2}|f|^2_{L^2(\mathcal{M})} + \frac{1}{2}|\sqrt{x}u|^2_{L^2(\mathcal{M})},$
- $\displaystyle\epsilon\int_{\mathcal{M}} (\frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial y^4})xu\, d\,\mathcal{M} = \epsilon\left|\sqrt{x}\frac{\partial^2 u}{\partial y^2}\right|^2_{L^2(\mathcal{M})} + \epsilon\left|\sqrt{x}\frac{\partial^2 u}{\partial z^2}\right|^2_{L^2(\mathcal{M})},$
- $\displaystyle\int_0^1 \frac{\partial^4 u}{\partial x^4}xu\, dx = -\int_0^1 \frac{\partial^3 u}{\partial x^3}(u + x\frac{\partial u}{\partial x})\, dx$
  $\displaystyle= \int_0^1 (2u_{xx}u_x + xu^2_{xx})\, dx = -|u_x(0,\cdot,\cdot)|^2 + \int_0^1 xu^2_{xx}dx,$
- $\displaystyle\epsilon\int_{\mathcal{M}} \frac{\partial^4 u}{\partial x^4}xu\, dx\, dx^\perp = \epsilon|\sqrt{x}u_{xx}|^2_{L^2(\mathcal{M})} - \epsilon\int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} |u_x(0,x^\perp,t)|^2\, dx^\perp.$

Hence, together with (2.51), we arrive at

$$\begin{aligned}
&\frac{d}{dt}|\sqrt{x}u^\epsilon|^2_{L^2(\mathcal{M})} + \frac{1}{2}|\nabla u^\epsilon|^2_{L^2(\mathcal{M})} + 2|u^\epsilon_x|^2_{L^2(\mathcal{M})}\\
&\qquad + 2\epsilon|\sqrt{x}u^\epsilon_{xx}|^2_{L^2(\mathcal{M})} + 2\epsilon|\sqrt{x}u^\epsilon_{yy}|^2_{L^2(\mathcal{M})} + 2\epsilon|\sqrt{x}u^\epsilon_{zz}|^2_{L^2(\mathcal{M})}\\
&\leq |f|^2_{L^2(\mathcal{M})} + |\sqrt{x}u^\epsilon|^2_{L^2(\mathcal{M})} + c|u^\epsilon|^2_{L^2(\mathcal{M})}\\
&\qquad +2c_2|u^\epsilon|^6_{L^2(\mathcal{M})} + 2\epsilon\int_{(-\frac{\pi}{2},\frac{\pi}{2})^d} |u^\epsilon_x(0,x^\perp,t)|^2\, dx^\perp.
\end{aligned} \tag{3.9}$$

Taking into account the previous estimates (3.7) we obtain the following estimates, also *independent* of ϵ:

$$\nabla u^\epsilon, \sqrt{\epsilon x}u^\epsilon_{xx}, \sqrt{\epsilon x}u^\epsilon_{yy}, \sqrt{\epsilon x}u^\epsilon_{zz} \text{ are bounded in } L^2(0,T;\, L^2(\mathcal{M})). \tag{3.10}$$

From (3.7) and (3.10) we infer a bound on $u^\epsilon u^\epsilon_x$ on which we now elaborate because of our needs below. We write (dropping momentarily the ϵ):

$$\begin{aligned}
&\int_{\mathcal{M}} |uu_x|^{9/8}\, d\,\mathcal{M} = \int_{\mathcal{M}} (u^2)^{\frac{3}{8}}(u^6)^{\frac{1}{16}}|u_x|^{\frac{9}{8}}\, d\,\mathcal{M}\\
&\leq (\text{with } H^1(\mathcal{M}) \subset L^6(\mathcal{M}) \text{ in dimension 3}) \leq |u|^{\frac{3}{4}}_{L^2(\mathcal{M})}|\nabla u|^{\frac{24}{16}}_{L^2(\mathcal{M})};
\end{aligned}$$

that is,

$$|u^\epsilon u^\epsilon_x|_{L^{9/8}(\mathcal{M})} \leq |u^\epsilon|^{\frac{2}{3}}_{L^2(\mathcal{M})}|\nabla u^\epsilon|^{\frac{4}{3}}_{L^2(\mathcal{M})}. \tag{3.11}$$

The function on the right-hand-side of (3.11) is bounded in $L^{3/2}(0,T)$; that is,

$$u^\epsilon u_x^\epsilon \text{ is bounded in } L^{3/2}(0,T;\, L^{9/8}(\mathcal{M})) \text{ in dimension 3,} \tag{3.12}$$

and hence it is bounded in $L^{9/8}(0,T;\, L^{9/8}(\mathcal{M}))$. Since L^9 is the dual of $L^{9/8}$ and $H^{\frac{7}{6}}(\mathcal{M}) \subset L^9(\mathcal{M})$ in space dimension 3, we also have

$$u^\epsilon u_x^\epsilon \text{ is bounded in } L^{9/8}(0,T;\, H^{-\frac{7}{6}}(\mathcal{M})) \text{ in dimension 3.} \tag{3.13}$$

Thanks to (3.7), (3.10), (3.13), equation (3.3) now implies

$$\frac{\partial u^\epsilon}{\partial t} \text{ is bounded (independently of } \epsilon) \text{ in } L^{9/8}(0,T;\, H^{-3}(\mathcal{M})). \tag{3.14}$$

Although the estimate (3.14) is a very poor one, it allows us to show that the family u^ϵ is relatively compact in $L^2(0,T;\, L^2(\mathcal{M}))$. As we have said, the estimates above are then sufficient to obtain the existence of u^ϵ for $\epsilon > 0$ fixed, and they permit also, in a second step, to pass to the limit $\epsilon \to 0$, using a compactness argument for the nonlinear term.

ii) Having shown that the limit u of (a subsequence extracted from) u^ϵ is a solution of (1.1), we want now to address the question of the boundary and initial conditions. The initial condition $u(x, x^\perp, 0) = u_0(x, x^\perp)$ is satisfied because, due in particular to (3.14), u^ϵ converges to u in $\mathcal{C}([0,T];\, H_w^{-3}(\Omega))$, where H_w^{-3} is H^{-3} equipped with the weak topology.

Similarly the Dirichlet boundary condition $u = 0$ on $\partial\mathcal{M}$ is satisfied since it is satisfied by u^ϵ and since u^ϵ converges to u weakly in $L^2(0,T;\, H_0^1(\mathcal{M}))$.

Hence for the existence, there remains to show that the boundary condition

$$u_x(1, x^\perp, t) = 0, \tag{3.15}$$

is satisfied. This boundary condition is the object of Lemma 3.1.1 below where we show that $u_x(1,\cdot,\cdot)$ is defined when $u \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ and u satisfies an equation like (1.1) and furthermore this trace depends continuously on u in a suitable topology, so that $u_x^\epsilon(1,\cdot,\cdot) = 0$ gives, at the limit, $u_x(1,\cdot,\cdot) = 0$.

We now pursue the proof of (3.15) and start with the following lemma which shows that whenever $u \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ satisfies equation (1.1) then $u_x(1,\cdot,\cdot)$ makes sense (as well as other traces).

Lemma 3.1.1. *If $u \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ satisfies equation (1.1), then*

$$u_x, u_{xx} \in \mathcal{C}_x(I_x;\, Y),\ Y = H^{-2}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d)), \tag{3.16}$$

and, in particular,

$$u_x\big|_{x=0,1},\ u_{xx}\big|_{x=0,1} \tag{3.17}$$

are well defined in Y. Furthermore these traces depend continuously on u in a sense made precise in the proof.

Proof. We write equation (1.1) in the form

$$u_{xxx} = f - cu_x - \Delta^\perp u_x - uu_x - u_t, \tag{3.18}$$

and we observe that, since

$$u, u_x \in L^2(0,T;\, L^2(\mathcal{M})) = L_x^2(I_x;\, L^2(0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d)), \tag{3.19}$$

it follows that

$$\begin{aligned} u_t &\in L_x^2(I_x;\, H^{-1}(0,T;\, L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d))), \\ \Delta^\perp u_x &\in L_x^2(I_x;\, L^2(0,T;\, H^{-2}((-\frac{\pi}{2}, \frac{\pi}{2})^d))). \end{aligned}$$

Also according to (3.12),

$$uu_x \in L_x^{9/8}(I_x;\, L_x^{9/8}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d))) \subset L_x^{9/8}(I_x;\, H^{-2}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d))). \tag{3.20}$$

Thus u_{xxx} belongs (at least) to the largest of these spaces, that is

$$u_{xxx} \in L_x^{9/8}(I_x;\, H^{-2}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d))). \tag{3.21}$$

Then (3.16), (3.17) follow.

Furthermore, if a sequence of functions u_m satisfies (1.1) with $f = f_m$, and $u_m \to u$ in $L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ and $f_m \to f$ in $L^2(0,T;\, L^2(\mathcal{M}))$ (strongly), then $u_{mx}\big|_{x=0,1}$, $u_{mxx}\big|_{x=0,1}$ converge respectively to $u_x\big|_{x=0,1}$, $u_{xx}\big|_{x=0,1}$ in Y. If the convergence of u_m and f_m are weak (weak-star for L^∞), then the convergences hold in $C_x(I_x;\, Y_w)$ and Y_w. These convergences use a compactness argument based on the analog of (3.14) , which is used to show that $u_m u_{mx}$ converges to uu_x. □

iii) We now need to show that the boundary condition $u_x(1,\cdot,\cdot) = 0$, which is satisfied in a strong sense for $\epsilon > 0$, "passes to the limit" to imply (3.15). It suffices here to use Lemma 7.2.1 in the appendix. Let u^ϵ be a solution of (3.3), (2.2)-(2.4), (3.4) and (3.5), with $p = \frac{9}{8}$, $Y = H^{-2}((0,T) \times (-\frac{\pi}{2}, \frac{\pi}{2})^d)$, as in Lemma 3.1.1 and

$$g^\epsilon := u^\epsilon_{xxx} + \epsilon u^\epsilon_{xxxx} = f - u^\epsilon_t - c u^\epsilon_x - \Delta^\perp u^\epsilon_x - u^\epsilon u^\epsilon_x - \epsilon u^\epsilon_{yyyy} - \epsilon u^\epsilon_{zzzz}. \tag{3.22}$$

We observe that, since u^ϵ remains bounded in $L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ as $\epsilon \to 0$, the following functions remain bounded in the indicated spaces:

$$\begin{aligned}
u^\epsilon_x \quad & in \quad L^2_x(I_x;\, L^2_t(0,T;\, L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d))),\\
u^\epsilon_t \quad & in \quad L^2_x(I_x;\, H^{-1}_t(0,T;\, L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d))),\\
\Delta^\perp u^\epsilon_x \quad & in \quad L^2_x(I_x;\, L^2_t(0,T;\, H^{-2}((-\frac{\pi}{2}, \frac{\pi}{2})^d))),\\
u^\epsilon_{yyyy} + u^\epsilon_{xxxx} \quad & in \quad L^2_x(I_x;\, L^2_t(0,T;\, H^{-4}((-\frac{\pi}{2}, \frac{\pi}{2})^d))).
\end{aligned}$$

We have thus shown that $u^\epsilon u^\epsilon_x$ remains bounded in $L^{9/8}(0,T;\, L^{9/8}(\mathcal{M})) = L^{9/8}_x(I_x;\, L^{9/8}((0,T)\times (-\frac{\pi}{2}, \frac{\pi}{2})^d))$.

Finally g^ϵ remains bounded in the reflexive Banach space $L^{9/8}_x(I_x;\, Y)$, where

$$Y = H^{-1}_t(0,T;\, L^2((-\frac{\pi}{2}, \frac{\pi}{2})^d)) + L^2_t(0,T;\, H^{-4}((-\frac{\pi}{2}, \frac{\pi}{2})^d)) + L^{9/8}_t(0,T;\, L^{9/8}((-\frac{\pi}{2}, \frac{\pi}{2})^d)).$$

Then we can apply Lemma 7.2.1 with this space Y.

Remark 3.1.1. *As in the stationary and the linear cases, the traces in (3.17) are defined in a weak sense (in a large space). We did not try to refine these trace results, as we only need to know that these traces exist and depend continuously on* u*.*

Remark 3.1.2. *Another way to estimate the term* $u^\epsilon u_x^\epsilon$ *was pointed out to one of us by Laure Saint-Raymond, namely, we have*

$$\int_0^T |u^\epsilon u_x^\epsilon|_{L^1(\mathcal{M})}^2\, dt \le \int_0^T |u^\epsilon|_{L^2(\mathcal{M})}^2 |u_x^\epsilon|_{L^2(\mathcal{M})}^2\, dt \le \left(\sup_{t\in(0,T)} |u^\epsilon(t)|_{L^2(\mathcal{M})}\right)^2 \int_0^T |u^\epsilon(t)|_{H_0^1(\mathcal{M})}^2\, dt,$$

and we infer from (3.7) and (3.10) that

$$u^\epsilon u_x^\epsilon \text{ is bounded in } L^2(0,T;\, L^1(\mathcal{M})) \text{ in dimension } 3. \tag{3.23}$$

Since $H_0^2(\mathcal{M}) \subset L^\infty(\mathcal{M})$ *in space dimension* 3, $L^1(\mathcal{M})$ *is included in* $H^{-2}(\mathcal{M})$*, and hence we also have*

$$u^\epsilon u_x^\epsilon \text{ is bounded in } L^2(0,T;\, H^{-2}(\mathcal{M})) \text{ in dimension } 3. \tag{3.24}$$

Thanks to (3.7), (3.10), (3.24), equation (3.3) now implies

$$\frac{\partial u^\epsilon}{\partial t} \text{ is bounded (independently of } \epsilon\text{) in } L^2(0,T;\, H^{-3}(\mathcal{M})). \tag{3.25}$$

With these bounds, we can also infer the weak convergence of $u_x^\epsilon(1,\cdot,\cdot) = 0$ *by applying Lemma 7.2.1 in a smaller space* Y*, observing that* g^ϵ *remains bounded in the Banach space* $L_x^1(I_x; Y)$*, where* $Y = H_t^{-1}(0,T;\, L^2((-\frac{\pi}{2},\frac{\pi}{2})^d)) + L_t^2(0,T;\, H^{-4}((-\frac{\pi}{2},\frac{\pi}{2})^d)) + L_t^2(0,T;\, L^1((-\frac{\pi}{2},\frac{\pi}{2})^d))$*.*

3.2 Uniqueness of Weak Solutions in Dimension 2

Finally we conclude the proof of Theorem 3.0.2 by proving the uniqueness and the strong continuity properties, when $d = 1$. This will follow from the following lemma.

Lemma 3.2.1. *Assume that* $w_0 \in H := L^2(\mathcal{M})$ *and* $g \in L^p(0,T;\, L^q(\mathcal{M}))$ *are given, such that* $1 \le p, q \le \infty$, *and*

$$L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M})) \subset L^{p'}(0,T;\, L^{q'}(\mathcal{M})), \tag{3.26}$$

with $1/p + 1/p' = 1/q + 1/q' = 1$, *the last injection (inclusion) being continuous. Then when* $d = 1$, *there exists a unique function* $w \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ *satisfying*

$$\begin{cases} \dfrac{\partial w}{\partial t} + \Delta \dfrac{\partial w}{\partial x} + c\dfrac{\partial w}{\partial x} = g, \\ w(0) = w_0, \end{cases} \tag{3.27}$$

and the boundary conditions (2.2), (2.4). This function w *satisfies also the energy inequality*

$$\frac{d}{dt}|\sqrt{1+x}\, w|^2_{L^2(\mathcal{M})} + |\nabla w|^2_{L^2(\mathcal{M})} - c|w|^2_{L^2(\mathcal{M})} \le 2\int_{\mathcal{M}} g(1+x)w\, d\mathcal{M}. \tag{3.28}$$

Proof. By parabolic regularization as in the nonlinear case, we can construct a solution of (3.27), (2.2) and (2.4) satisfying the energy type inequality (3.28). Indeed we consider the parabolic equation

$$\begin{cases} \dfrac{\partial w^\epsilon}{\partial t} + \Delta \dfrac{\partial w^\epsilon}{\partial x} + c\dfrac{\partial w^\epsilon}{\partial x} + \epsilon\,(\dfrac{\partial^4 w^\epsilon}{\partial x^4} + \dfrac{\partial^4 w^\epsilon}{\partial y^4} + \dfrac{\partial^4 w^\epsilon}{\partial z^4}) = g, \\ w^\epsilon(0) = w_0, \end{cases} \tag{3.29}$$

supplemented with the boundary conditions (2.2), (2.4) and the additional boundary conditions (3.4), (3.5).

Multiplying $(3.29)_1$ by w^ϵ and then xw^ϵ, integrating over $\mathcal{M}$ and integrating by parts, we can apply the same calculations as in the nonlinear case and obtain the analogues of (3.6) and (3.9) without the terms due to the nonlinearity. We can deduce the same uniform bounds as in (3.7) and (3.10), from which we infer that (compare to (3.14))

$$\frac{\partial w^\epsilon}{\partial t} \text{ is bounded (independently of } \epsilon\text{) in } L^2(0,T;\, H^{-3}(\mathcal{M})). \tag{3.30}$$

Hence w^ϵ converges up to a subsequence to w weakly in $L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T; H_0^1(\mathcal{M}))$ and strongly in $L^2(0,T; L^2(\mathcal{M}))$. By a similar argument as in the nonlinear case, we obtain the existence of a solution to the linear equation (3.27) with the boundary conditions (2.2), (2.4).

Summing up the analogues of (3.6) and (3.9), dropping the nonnegative terms, we obtain that when say $\epsilon < 1/2$,

$$\frac{d}{dt}|\sqrt{1+x}\,w^\epsilon|^2_{L^2(\mathcal{M})} + |\nabla w^\epsilon|^2_{L^2(\mathcal{M})} \leq c|w^\epsilon|^2_{L^2(\mathcal{M})} + 2\int_{\mathcal{M}} g(1+x)w^\epsilon\, d\mathcal{M}. \tag{3.31}$$

Since w^ϵ converges weakly to w in $L^\infty(0,T;L^2(\mathcal{M})) \cap L^2(0,T;H^1_0(\mathcal{M}))$ and strongly in $L^2(0,T;L^2(\mathcal{M}))$, we can pass to the lower limit on the left-hand-side of (3.31) and pass to the limit on the term $|w^\epsilon|^2_{L^2(\mathcal{M})}$. By (3.26), we obtain that w^ϵ converges weakly in $L^{p'}(0,T;\, L^{q'}(\mathcal{M}))$, which is the dual of $L^p(0,T;\, L^q(\mathcal{M}))$. Hence we can pass to the limit on the term $\int_{\mathcal{M}} g(1+x)w^\epsilon\, d\mathcal{M}$. To conclude, we can pass to the limit in (3.31) and obtain (3.28).

We finally observe that for such a function g, the solution w belonging to the space $L^2(0,T;\, H^1_0(\mathcal{M}))\cap L^\infty(0,T;\, L^2(\mathcal{M}))$ to the linear equation (3.27) with boundary conditions (2.2), (2.4) is necessarily unique. Indeed, if w_1, w_2 are two such solutions, setting $w = w_1 - w_2$ and $W(t) = \int_0^t w(s)ds$, we see that

$$\begin{cases} \dfrac{\partial W}{\partial t} + \Delta \dfrac{\partial W}{\partial x} + c\dfrac{\partial W}{\partial x} = 0, \\ W(0) = 0, \end{cases} \tag{3.32}$$

and W satisfies the same boundary conditions (2.2), (2.4). Since $W \in L^2(0,T;\, H^1_0(\mathcal{M}))$, and $W_t = w \in L^2(0,T;L^2(\mathcal{M}))$, we see from $(3.32)_1$ that $W(t) \in D(A)$ for almost every t and more precisely $W \in L^2(0,T;\, D(A))$, and (3.32) reads

$$\begin{cases} \dfrac{\partial W}{\partial t} + AW = 0, \\ W(0) = 0. \end{cases}$$

By the uniqueness in Theorem 2.5.1 (linear case), $W(t) = 0$ for all $t \geq 0$, and then $W_t = w = 0$ for all $t \geq 0$. $\square$

Now let u and v be two weak solutions of (1.1), (2.2)-(2.4), Then $u-v$ satisfies

$$\frac{\partial w}{\partial t} + \Delta \frac{\partial w}{\partial x} + c\frac{\partial w}{\partial x} = -uu_x + vv_x, \tag{3.33}$$

together with the boundary and initial conditions (2.2)-(2.4) (with initial data $u_0 - v_0 = 0$). Hence we can regard $u-v$ as a solution in the space $L^\infty(0,T;\ L^2(\mathcal{M})) \cap L^2(0,T;\ H_0^1(\mathcal{M}))$ of (3.33) written as (3.27), with $g = -uu_x + vv_x$.

Of course, u satisfies the boundary conditions (2.2), (2.4). We can prove that

$$|uu_x|_{L^{4/3}(\mathcal{M})} \le C''|u|_{L^4(\mathcal{M})}|u_x|_{L^2(\mathcal{M})} \le C''|u|^{1/2}_{L^2(\mathcal{M})}|\nabla u|^{1/2}_{L^2(\mathcal{M})}|u_x|_{L^2(\mathcal{M})}.$$

Hence we find (compare to (3.12))

$$uu_x \in L^{4/3}(0,T;L^{4/3}) \text{ in dimension 2}, \tag{3.34}$$

so that $g \in L^{4/3}(0,T;L^{4/3})$.

When $d = 1$, we see that $L^2(0,T;\ H_0^1(\mathcal{M})) \cap L^\infty(0,T;\ L^2(\mathcal{M})) \subset L^4(0,T;L^4(\mathcal{M}))$, which is the dual of $L^{4/3}(0,T;L^{4/3})$. Hence we can apply Lemma 3.2.1 and by the uniqueness of solutions to (3.27) with the boundary and initial conditions (2.2)-(2.4) (with initial data $u_0 - v_0 = 0$), we deduce that $u - v = w$. Thus from (3.28) we can derive another energy estimate:

$$\begin{aligned}
&\frac{d}{dt}(|w|^2_{L^2(\mathcal{M})} + |\sqrt{x}w|^2_{L^2(\mathcal{M})}) + |\nabla w|^2_{L^2(\mathcal{M})} \\
&\le -\int_{\mathcal{M}} w^2 v_x\, d - 2xw^2u + xw^2v\, d\mathcal{M} + vw^2 + cw^2\, d\mathcal{M} \\
&\le (\text{with } \sigma(t) = |v_x(t)|_{L^2(\mathcal{M})} + |u_x(t)|_{L^2(\mathcal{M})} + |v(t)|_{L^2(\mathcal{M})}) \\
&\le C'\sigma(t)|w|^2_{L^4(\mathcal{M})} + c\int_{\mathcal{M}} w^2\, d\mathcal{M} \\
&\le (\text{by interpolation } H^{1/2} \subset L^4 \text{ in dimension 2}) \\
&\le C'\sigma(t)|w|_{L^2(\mathcal{M})}|\nabla w|_{L^2(\mathcal{M})} + c\int_{\mathcal{M}} w^2\, d\mathcal{M} \\
&\le \frac{1}{2}|\nabla w|^2_{L^2(\mathcal{M})} + C'(\sigma^2(t)+1)|w|^2_{L^2(\mathcal{M})}.
\end{aligned}$$

Hence,

$$\frac{d}{dt}(|\sqrt{1+x}\,w|^2_{L^2(\mathcal{M})}) \le C'(\sigma^2(t)+1)|w|^2_{L^2(\mathcal{M})}. \tag{3.35}$$

Since $w(0) = 0$ and σ^2 is an integrable function ($\in L^1(0,T)$), (3.35) implies, using the Gronwall lemma, that $w(t) = 0$ for every $t \in (0,T)$. The uniqueness follows in dimension 2.

Now to prove the strong continuity in time and the continuity of the flow map, we use the classical Bona-Smith technique (see [6]) which we now adapt to this context.

Let u be the solution of (1.1) and (2.2)-(2.4). Then we have $u \in L^\infty(0,T;\,H)$ and by (3.14), we see that $u \in \mathcal{C}([0,T];\,H_w)$, where H_w is H equipped with the weak topology (see [42]). Hence for every $0 < s < T$, we can define $v_0 = u(s) \in H$. Then given v_0 as the initial data, we see that $u(t+s)$ serves as a solution of (1.1), (2.2)-(2.4) with the forcing term being replaced by $F(t) := f(t+s)$, which belongs to $L^2(0,T-s;\,L^2(\mathcal{M}))$. By the uniqueness of solution as proved before, $v(t) = u(t+s)$. Using (3.35) and the Gronwall lemma, we find

$$\begin{aligned}
&|\sqrt{1+x}\,(u(t)-v(t))|^2_{L^2(\mathcal{M})}\\
&\le \exp\,(C'\int_0^t(\sigma^2(r)+1)\,dr)\,|\sqrt{1+x}\,(u_0-v_0)|^2_{L^2(\mathcal{M})} + \int_0^t |F(r)-f(r)|^2_{L^2(\mathcal{M})}\,dr;\\
&|\sqrt{1+x}\,(u(t+s)-u(t))|^2_{L^2(\mathcal{M})}\\
&\le \exp\,(C'\int_0^t(\sigma^2(r)+1)\,dr)\,|\sqrt{1+x}\,(u(s)-u_0)|^2_{L^2(\mathcal{M})} + \int_0^t |f(r+s)-f(r)|^2_{L^2(\mathcal{M})}\,dr,
\end{aligned}$$

where $\sigma(t) = |v_x(t)|_{L^2(\mathcal{M})} + |u_x(t)|_{L^2(\mathcal{M})} + |v(t)|_{L^2(\mathcal{M})}$, $t \le T-s$. Since σ^2 is an integrable function ($\in L^1(0,T-s)$), and $f \in L^2(0,T;L^2(\mathcal{M}))$, in order to show that u is continuous in H at time $t \ge 0$, it now suffices to prove that u is strongly continuous in H at $t = 0$, namely, that

$$u(s) \to u_0 \text{ in } L^2(\mathcal{M}) \text{ as } s \to 0. \tag{3.36}$$

We have already observed that u is weakly continuous in H on $[0,T]$ and hence at $t = 0$, that is,

$$u(s) \rightharpoonup u_0 \text{ weakly in } L^2(\mathcal{M}) \text{ as } s \to 0. \tag{3.37}$$

Having proved (3.37), in order to show (3.36), it now suffices to prove the following

$$|u(s)|_{L^2(\mathcal{M})} \to |u_0|_{L^2(\mathcal{M})} \quad \text{as } s \to 0. \tag{3.38}$$

By uniqueness of solution, u must be the limit of u^ϵ constructed in (3.3). Hence passing to the limit in the sum of (3.6) and (3.9), we find (in both cases $d = 1, 2$)

$$\frac{d}{dt}|\sqrt{1+x}\,u|^2_{L^2(\mathcal{M})} \le |f|^2_{L^2(\mathcal{M})} + |\sqrt{1+x}\,u|^2_{L^2(\mathcal{M})} + c|u|^2_{L^2(\mathcal{M})} + 2c_2|u|^6_{L^2(\mathcal{M})}. \tag{3.39}$$

Integrating from 0 to s, we find

$$\begin{aligned} &|\sqrt{1+x}\,u(s)|^2_{L^2(\mathcal{M})} - |\sqrt{1+x}\,u(0)|^2_{L^2(\mathcal{M})} \\ &\qquad \le \int_0^s |f|^2_{L^2(\mathcal{M})}\,dt + (1+c)\int_0^s |u|^2_{L^2(\mathcal{M})}\,dt + 2c_2 s|u|^6_{L^\infty(0,s;\,L^2(\mathcal{M}))}. \end{aligned} \tag{3.40}$$

Since $f \in L^2(0,T;\, L^2(\mathcal{M}))$, $\int_0^s |f|^2_{L^2(\mathcal{M})}\,dt$ is absolutely continuous with respect to s. Hence it converges to 0 as $s \to 0$. Similarly we can deduce that each term on the right-hand-side of (3.40) converges to 0 as $s \to 0$. Thus

$$\limsup_{s\to 0} |\sqrt{1+x}\,u(s)|^2_{L^2(\mathcal{M})} \le |\sqrt{1+x}\,u_0|^2_{L^2(\mathcal{M})}.$$

By (3.37) and lower semicontinuity of the L^2-norm, we also have

$$\liminf_{s\to 0} |\sqrt{1+x}\,u(s)|^2_{L^2(\mathcal{M})} \ge |\sqrt{1+x}\,u_0|^2_{L^2(\mathcal{M})}.$$

Hence $\sqrt{1+x}\,u(s) \to \sqrt{1+x}\,u_0$ *in* $L^2(\mathcal{M})$ as $s \to 0$ and this is equivalent to (3.36).

This concludes the proof of Theorem 3.0.2. □

Remark 3.2.1. *Note that although (3.39) is true for both $d = 1, 2$, (3.36) is only true for $d = 1$. This is because we need the uniqueness of solution to deduce (3.39).*

Remark 3.2.2. *The results in Theorem 3.0.2 display the remarkable properties of the ZK equation posed in a limited domain* $I_x \times (-\frac{\pi}{2}, \frac{\pi}{2})^d$, $d = 1, 2$, *in contrast with the Cauchy problem posed in the whole space (see [14], [30]). In particular it is not known whether the Cauchy problem is well-posed for initial data in* $L^2(\mathbb{R}^{d+1})$, $d = 1, 2$.

Remark 3.2.3. *Theorem 3.0.2 implies obviously a global well-posedness of the initial-boundary-value problem for the ZK equation on a "rectangle"* $(0, L_1)_x \times (0, L_2)_y \times (0, L_3)_z$, *the estimates on the solution depending of course on the* L_i. *It would be interesting (for instance for numerical purposes) to obtain estimates independent of some* L_i, *allowing thus to pass to the limit as* $L_i \to +\infty$ *in order to obtain a solution of an initial-boundary-value problem in the infinite "rectangle", say* $(0, +\infty)_x \times (0, L_2)_y \times (0, L_3)_z$. *Such a result has been obtained in [8] for the KdV equation, under the boundary conditions* $u(0, t) = 0$, $u_x(L, t) = u_{xx}(L, t) = 0$.

Remark 3.2.4. *As for the periodic case, that is, (1.1) and the boundary and initial conditions (2.2), (2.3), (2.5), the results will be the same with the Dirichlet case as discussed above. The reasoning will be similar.*

Remark 3.2.5. *We have only treated the case of homogeneous boundary conditions. The case of nonhomogeneous boundary conditions will be treated in a separate work. Indeed there are two possibilities for the treatment of nonhomogeneous boundary conditions: one possibility is that the boundary data (*u *and* u_x *at* $x = 1$ *and* u *at* $x = 0$, u *at* y, $z = \pm\pi/2$ *in the case of Dirichlet boundary conditions in* y, z*) are given as traces of a sufficiently regular function* Φ. *In this case setting classically* $v = u - \Phi$, *we obtain a homogeneous problem for* v *which is very similar to the problem that we studied and which can be treated in a similar way if* Φ *is sufficiently regular. However if we want to consider less regular boundary values, then we need to study more carefully the properties of the traces of the functions that we consider.*

Chapter 4

Local Existence of Strong Solutions in $3D$

Now we turn to establish the local existence of strong solutions in $3D$.

For the purpose, we also consider the space

$$\Xi = \left\{ u \in H^2(\mathcal{M}) \cap H_0^1(\mathcal{M}),\ u_x\big|_{x=1} = 0 \right\}, \tag{4.1}$$

and endow this space, as in Section 3.1 with the scalar product and norm $[\cdot,\cdot]_2$ and $[\cdot]_2$,

$$[u,v]_2 = (u_{xx}, v_{xx}) + (u_{yy}, v_{yy}) + (u_{zz}, v_{zz}),$$

$$[u]_2^2 = |u_{xx}|^2 + |u_{yy}|^2 + |u_{zz}|^2, \tag{4.2}$$

which make it a Hilbert space. Furthermore by Proposition 2.3.1, we have

$$D(A) \subset \Xi. \tag{4.3}$$

4.1 Parabolic Regularization

For the sake of simplicity we will only treat the more complicated case when $d = 2$; the case when $d = 1$ is easier. To begin with, we recall the parabolic regularization introduced in Section 3.1, that is, for $\epsilon > 0$ "small", we consider the parabolic equation,

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \Delta \dfrac{\partial u^\epsilon}{\partial x} + c\dfrac{\partial u^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon\, L u^\epsilon = f, \\ u^\epsilon(0) = u_0, \end{cases} \tag{4.4}$$

where

$$Lu^\epsilon := \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4},$$

supplemented with the boundary conditions (2.2), (2.4) and the additional boundary conditions (3.4), (3.5).

4.1.1 Global Bounds Independent of ϵ

Firstly, we recall the following global bounds derived from (3.7) and (3.10) previously:

Lemma 4.1.1. *We assume that*

$$u_0 \in L^2(\mathcal{M}), \tag{4.5}$$

$$f \in L^2(0,T; L^2(\mathcal{M})), \tag{4.6}$$

then, for every $T > 0$ the following estimates independent of ϵ hold:

$$\begin{cases} u^\epsilon \textit{ is bounded in } L^\infty(0,T;\, L^2(\mathcal{M})), \\ u^\epsilon \textit{ is bounded in } L^2(0,T;\, H_0^1(\mathcal{M})), \\ u_x^\epsilon(0,\cdot,\cdot) \textit{ is bounded in } L^2(0,T;\, L^2(I_{x^\perp})). \end{cases} \tag{4.7}$$

4.1.2 Local Bounds Independent of ϵ

We first introduce a useful result:

Lemma 4.1.2. *Under the same assumptions as in Lemma 4.1.1, if we further suppose that*

$$f \in L^\infty(0,T; L^2(\mathcal{M})), \tag{4.8}$$

then we have

$$|u_x^\epsilon(t)|^2 \le |u_t^\epsilon(t)|^2 + \kappa, \qquad 0 \le t \le T, \tag{4.9}$$

where κ is a constant depending only on $|u_0|$, $|f|_{L^\infty(0,T;L^2(\mathcal{M}))}$ and T.

Proof. We rewrite $(4.4)_1$ as

$$\Delta \frac{\partial u^\epsilon}{\partial x} + c\frac{\partial u^\epsilon}{\partial x} + u^\epsilon \frac{\partial u^\epsilon}{\partial x} + \epsilon\, L u^\epsilon = -\frac{\partial u^\epsilon}{\partial t} + f. \tag{4.10}$$

We multiply (4.10) by $(1+x)u^\epsilon$, integrate over $\mathcal{M}$, integrate by parts, and follow the same calculations as in Section 3.1; we find when $\epsilon \le \frac{1}{4}$,

$$\begin{aligned} |u_x^\epsilon|^2 &\le -\int_{\mathcal{M}} u_t^\epsilon\,(1+x)u^\epsilon\, d\mathcal{M} + |f|^2 + |(1+x)u^\epsilon|^2 + c|u^\epsilon|^2 + c'|u^\epsilon|^6 \\ &\le |u_t^\epsilon|^2 + |f|^2 + c'|u^\epsilon|^2 + c'|u^\epsilon|^6, \end{aligned} \tag{4.11}$$

Here and below c' indicates an absolute constant which may be different at each occurrence. Hence if we call ν a bound of $|u^\epsilon|_{L^\infty(0,T;L^2(\mathcal{M}))}$ as in $(4.7)_1$, we can set

$$\kappa = |f|^2_{L^\infty(0,T;L^2(\mathcal{M}))} + c'\nu^2 + c'\nu^6 + |u_{0x}|^2; \tag{4.12}$$

and by (4.8) we obtain (4.9). Thus we have completed the proof of Lemma 4.1.2. □

Now we are ready to prove the following result giving the local bounds on u^ϵ independent of ϵ:

Proposition 4.1.1. *Under the same assumptions as in Lemma 4.1.2, if we further suppose that*

$$f_t \in L^\infty(0,T;L^2(\mathcal{M})), \tag{4.13}$$

$$f \in L^2(0,T;L^2(I_x;H^2(I_{x^\perp}))), \tag{4.14}$$

$$Lu_0 \in L^2(\mathcal{M}), \tag{4.15}$$

$$\Delta u_{0x} + u_0 u_{0x} + c u_{0x} - f(0) \in L^2(\mathcal{M}), \tag{4.16}$$

$$\nabla^\perp u_0, u_{0yy}, u_{0zz} \in L^2(\mathcal{M}), \tag{4.17}$$

then there exists $T_* = \min(T, T_1)$,

$$T_1 = \frac{c_3}{\mu^4}, \tag{4.18}$$

$$\mu = \mu(\kappa, |f_t|_{L^\infty(0,T;L^2(\mathcal{M}))}, |Lu_0|, |\Delta u_{0x} + u_0 u_{0x} + c u_{0x} - f(0)|),$$

such that for every t, $0 \le t \le T_*$,

$$|u_t^\epsilon(t)| \lesssim \mu, \tag{4.19}$$

$$\int_0^{T_*} |\nabla u_t^\epsilon(s)|^2\, ds \lesssim \mu, \tag{4.20}$$

$$|\nabla u^\epsilon(t)| \le C(\mu), \tag{4.21}$$

$$\int_0^{T_*} |\nabla u_y^\epsilon(s)|^2\, ds \le C(\mu), \quad \int_0^{T_*} |\nabla u_z^\epsilon(s)|^2\, ds \le C(\mu), \tag{4.22}$$

$$|u_{yy}^\epsilon(t)| \le C(\mu), \qquad |u_{zz}^\epsilon(t)| \le C(\mu), \tag{4.23}$$

$$\int_0^{T_*} |\nabla u_{yy}^\epsilon(s)|^2\, ds \le C(\mu), \quad \int_0^{T_*} |\nabla u_{zz}^\epsilon(s)|^2\, ds \le C(\mu), \tag{4.24}$$

$$\epsilon \int_0^{T_*} [u_{yy}^\epsilon]_2^2\, ds \le C(\mu), \qquad \epsilon \int_0^{T_*} [u_{zz}^\epsilon]_2^2\, ds \le C(\mu), \tag{4.25}$$

where $\lesssim$ means $\le$ up to a multiplicative constant independent of ϵ, the constant c_3 depends only on the data, and the constant $C(\mu)$ depends only on μ and the data and may be different at each occurrence.

Proof. We differentiate (4.4) in t, write $u_t^\epsilon = v^\epsilon$ and we find:

$$\begin{cases} \dfrac{\partial v^\epsilon}{\partial t} + \Delta \dfrac{\partial v^\epsilon}{\partial x} + c\dfrac{\partial v^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial v^\epsilon}{\partial x} + v^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon\, L v^\epsilon = f_t, \\ v^\epsilon(0) = u_{t0}^\epsilon = -\epsilon L u_0 - \Delta u_{0x} - u_0 u_{0x} - c u_{0x} + f(0). \end{cases} \tag{4.26}$$

Thus when $\epsilon \le 1$,

$$|u_{t0}^\epsilon| \le |Lu_0| + |\Delta u_{0x} + u_0 u_{0x} + c u_{0x} - f(0)|. \tag{4.27}$$

From (4.15) and (4.16), we obtain

$$u_{t0}^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(\mathcal{M}). \tag{4.28}$$

Multiplying (4.26) by $(1+x)v^\epsilon$, integrating over $\mathcal{M}$ and integrating by parts, dropping ϵ

for the moment we find

- $\displaystyle\int_{\mathcal{M}} \frac{\partial v}{\partial t}(1+x)v\, d\mathcal{M} = \frac{1}{2}\frac{d}{dt}|\sqrt{1+x}\, v|^2,$
- $\displaystyle\int_{\mathcal{M}} \Delta\frac{\partial v}{\partial x}\, v\, d\mathcal{M} = \frac{1}{2}\int_{I_{x^\perp}} (v_x\big|_{x=0})^2\, dx^\perp,$
- $\displaystyle\int_{\mathcal{M}} \Delta\frac{\partial v}{\partial x}\, xv\, d\,\mathcal{M} = \frac{3}{2}|v_x|^2 + \frac{1}{2}|\nabla^\perp v|^2,$
- $\displaystyle\int_{\mathcal{M}} cv_x(1+x)v\, d\mathcal{M} = -\frac{c}{2}\int_{\mathcal{M}} v^2\, d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} uv_x\,(1+x)v\, d\mathcal{M} = -\frac{1}{2}\int_{\mathcal{M}} (1+x)u_x v^2\, d\mathcal{M} - \frac{1}{2}\int_{\mathcal{M}} uv^2\, d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} vu_x\,(1+x)v\, d\mathcal{M} = \int_{\mathcal{M}} (1+x)u_x v^2\, d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} f_t(1+x)v\, d\mathcal{M} \le \frac{1}{2}|f_t|^2 + \frac{1}{2}|(1+x)v|^2 \le \frac{1}{2}|f_t|^2 + |v|^2,$
- $\displaystyle\epsilon\int_{\mathcal{M}} \frac{\partial^4 v}{\partial x^4}(1+x)v\, dx\, dx^\perp = \epsilon|\sqrt{1+x}\, v_{xx}|^2 - \epsilon\int_{I_{x^\perp}} (v_x\big|_{x=0})^2\, dx^\perp,$
- $\displaystyle\epsilon\int_{\mathcal{M}} (\frac{\partial^4 v}{\partial y^4} + \frac{\partial^4 v}{\partial z^4})\,(1+x)v\, d\mathcal{M} = \epsilon\left(\left|\sqrt{1+x}\, v_{yy}\right|^2 + \left|\sqrt{1+x}\, v_{zz}\right|^2\right).$

Hence we arrive, when $\epsilon \le \frac{1}{4}$, at

$$\begin{aligned}\frac{d}{dt}|\sqrt{1+x}\, u_t^\epsilon|^2 + |\nabla u_t^\epsilon|^2 + \frac{1}{4}|u_{tx}^\epsilon|_{x=0}|^2_{L^2(I_{x^\perp})}&\\ + 2\epsilon\left(|\sqrt{1+x}\, u_{txx}^\epsilon|^2 + |\sqrt{1+x}\, u_{tyy}^\epsilon|^2 + |\sqrt{1+x}\, u_{tzz}^\epsilon|^2\right)&\\ \le \left|\int_{\mathcal{M}} ((1+x)u_x^\epsilon - u^\epsilon)\,(u_t^\epsilon)^2\, d\mathcal{M}\right| + (c+2)|u_t^\epsilon|^2 + |f_t|^2.&\end{aligned} \tag{4.29}$$

For the first term on the right-hand-side of (4.29), we have

$$\left|\int_{\mathcal{M}} ((1+x)u_x^\epsilon - u^\epsilon)\,(u_t^\epsilon)^2\, d\mathcal{M}\right| \lesssim \sigma^\epsilon(t)|u_t^\epsilon|^2_{L^4(\mathcal{M})},$$

where $\sigma^\epsilon(t) := |u_x^\epsilon| + |u^\epsilon|$. Then with $H^{3/4} \subset L^4$ in $3D$ we have

$$\begin{aligned}\left|\int_{\mathcal{M}} ((1+x)u_x^\epsilon - u^\epsilon)\,(u_t^\epsilon)^2\, d\mathcal{M}\right| &\lesssim \sigma^\epsilon(t)|u_t^\epsilon|^{1/2}|\nabla u_t^\epsilon|^{3/2}\\ &\le c'(\sigma^\epsilon(t))^4|u_t^\epsilon|^2 + \frac{1}{8}|\nabla u_t^\epsilon|^2\\ &\le \text{(by (4.9))}\\ &\le c'\left(|u_t^\epsilon|^6 + \kappa^2|u_t^\epsilon|^2 + |u^\epsilon|^4|u_t^\epsilon|^2\right) + \frac{1}{8}|\nabla u_t^\epsilon|^2.\end{aligned} \tag{4.30}$$

Applying (4.30) to (4.29), we obtain

$$
\begin{aligned}
&\frac{d}{dt}|\sqrt{1+x}\,u_t^\epsilon|^2 + \frac{7}{8}|\nabla u_t^\epsilon|^2 + \frac{1}{4}|u_{tx}^\epsilon|_{x=0}|^2_{L^2(I_{x^\perp})} \\
&\qquad\qquad + 2\epsilon\left(|\sqrt{1+x}\,u_{yxx}^\epsilon|^2 + |\sqrt{1+x}\,u_{yyy}^\epsilon|^2 + |\sqrt{1+x}\,u_{yzz}^\epsilon|^2\right) \\
&\qquad\qquad\qquad \le c_1(|u_t^\epsilon|^2+1)^3 + |f_t|^2 \\
&\qquad\qquad\qquad \le c_1(|\sqrt{1+x}\,u_t^\epsilon|^2+1)^3 + |f_t|^2.
\end{aligned}
\tag{4.31}
$$

where c_1 depends only on κ. Setting $|\sqrt{1+x}\,u_t^\epsilon|^2 + 1 := \mathcal{Y}^\epsilon$, then (4.31) implies that

$$
\frac{d}{dt}\mathcal{Y}^\epsilon \le c_2(\mathcal{Y}^\epsilon)^3, \tag{4.32}
$$

with $c_2 := c_1 + |f_t|^2_{L^\infty(0,T;L^2(\mathcal{M}))}$. Thus

$$
\mathcal{Y}^\epsilon(t) \le 2\mu_0^2, \quad 0 \le t \le \frac{3}{8c_2\mu_0^4}, \tag{4.33}
$$

where μ_0 is a bound of $\sqrt{\mathcal{Y}^\epsilon(0)}$ independent of ϵ as provided by (4.28). Now (4.33) implies that

$$
|u_t^\epsilon(t)| \lesssim \mu_0, \quad 0 \le t \le \frac{3}{8c_2\mu_0^4}. \tag{4.34}
$$

Then by (4.34) and (4.9) we deduce that

$$
|u_x^\epsilon(t)| \lesssim \mu, \quad 0 \le t \le T_*. \tag{4.35}
$$

with $\mu := \mu_0 + \sqrt{\kappa}$, and $T_* = \min(T, T_1)$,

$$
T_1 = \frac{c_3}{\mu^4} \le \frac{3}{8c_2\mu_0^4}. \tag{4.36}
$$

By (4.34), (4.36) and (4.31) we obtain (4.20).

We multiply (4.4) by $(1+x)u_{yy}^\epsilon$, integrate over $\mathcal{M}$ and integrate by parts, dropping ϵ

for the moment, we find

- $\displaystyle\int_{\mathcal{M}} u_t\,(1+x)u_{yy}\,d\mathcal{M} = -\frac{1}{2}\frac{d}{dt}|\sqrt{1+x}\,u_y|^2,$
- $\displaystyle\int_{\mathcal{M}} u_{xxx}\,(1+x)u_{yy}\,d\mathcal{M} = -\frac{3}{2}|u_{xy}^2| - \frac{1}{2}\left|u_{xy}^2\right|_{x=0}\Big|_{L^2(I_y)},$
- $\displaystyle\int_{\mathcal{M}} u_{xyy}\,(1+x)u_{yy}\,d\mathcal{M} = -\frac{1}{2}|u_{yy}^2|,$
- $\displaystyle\int_{\mathcal{M}} u_{xzz}\,(1+x)u_{yy}\,d\mathcal{M} = -\frac{1}{2}|u_{zy}^2|,$
- $\displaystyle\int_{\mathcal{M}} cu_x(1+x)u_{yy}\,d\mathcal{M} = \frac{c}{2}\int_{\mathcal{M}} u_y^2\,d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} uu_x\,(1+x)u_{yy}\,d\mathcal{M} = \frac{1}{2}\int_{\mathcal{M}} u_y^2\,(u-(1+x)u_x)\,d\mathcal{M},$
- $$\begin{aligned}\epsilon\int_{\mathcal{M}} u_{xxxx}\,(1+x)u_{yy}\,d\mathcal{M} &= -\epsilon\int_{\mathcal{M}} u_{xxxxy}(1+x)u_y\,d\mathcal{M}\\ &= \epsilon\int_{\mathcal{M}} u_{xxxy}u_y\,d\mathcal{M} + \epsilon\int_{d\mathcal{M}} u_{xxxy}(1+x)u_{xy}\,d\mathcal{M}\\ &= -2\epsilon\int_{\mathcal{M}} u_{xxy}u_{xy}d\mathcal{M} - \epsilon\int_{\mathcal{M}}(1+x)u_{xxy}^2 d\mathcal{M}\\ &= \epsilon\int_{\mathcal{M}} u_{xy}^2\big|_{x=0}\,d\mathcal{M} - \epsilon\int_{\mathcal{M}}(1+x)u_{xxy}^2\,d\mathcal{M},\end{aligned}$$
- $\displaystyle\epsilon\int_{\mathcal{M}} u_{zzzz}\,(1+x)u_{yy}\,d\mathcal{M} = -\epsilon\int_{\mathcal{M}}(1+x)u_{zzy}^2\,d\mathcal{M},$
- $\displaystyle\epsilon\int_{\mathcal{M}} u_{yyyy}\,(1+x)u_{yy}\,d\mathcal{M} = -\epsilon\int_{\mathcal{M}}(1+x)u_{yyy}^{\epsilon}\,d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} f(1+x)\,u_{yy}\,d\mathcal{M} = -\int_{\mathcal{M}} f_y\,(1+x)u_y\,d\mathcal{M} \le \frac{1}{2}|(1+x)u_y|^2 + \frac{1}{2}|f_y|^2.$

Hence when $\epsilon \le \frac{1}{4}$, we have

$$\begin{aligned}&\frac{d}{dt}|\sqrt{1+x}\,u_y^{\epsilon}|^2 + |\nabla u_y^{\epsilon}|^2 + \frac{1}{4}\left|u_{xy}^{\epsilon}\big|_{x=0}\right|_{L^2(I_y)}^2\\ &\qquad + 2\epsilon\left(|\sqrt{1+x}\,u_{yxx}^{\epsilon}|^2 + |\sqrt{1+x}\,u_{yyy}^{\epsilon}|^2 + |\sqrt{1+x}\,u_{yzz}^{\epsilon}|^2\right)\\ &\qquad\le \left|\int_{\mathcal{M}}(u_y^{\epsilon})^2\left((1+x)u_x^{\epsilon} - u^{\epsilon}\right)\,d\mathcal{M}\right| + (c+2)|u_y^{\epsilon}|^2 + |f_y|^2.\end{aligned} \tag{4.37}$$

For the first term on the right-hand-side of (4.37), we find

$$
\begin{aligned}
\left|\int_{\mathcal{M}} (u_y^\epsilon)^2 \left((1+x)u_x^\epsilon - u^\epsilon\right)\, d\mathcal{M}\right| &\lesssim (\text{with } \sigma^\epsilon(t) := |u_x^\epsilon| + |u^\epsilon|) \\
&\lesssim \sigma^\epsilon(t) |u_y^\epsilon|^2_{L^4(\mathcal{M})} \\
&\leq c'(\sigma^\epsilon(t))^4 |u_y^\epsilon|^2 + \frac{1}{8}|\nabla u_y^\epsilon|^2 \\
&\leq (\text{by } (4.35)) \\
&\leq c'\left(\mu^4 + |u^\epsilon|^4\right)|u_y^\epsilon|^2 + \frac{1}{8}|\nabla u_y^\epsilon|^2, \quad 0 \leq t \leq T_*.
\end{aligned} \tag{4.38}
$$

Applying (4.38) to (4.37), we find

$$
\begin{aligned}
\frac{d}{dt}|\sqrt{1+x}\, u_y^\epsilon|^2 &+ \frac{7}{8}|\nabla u_y^\epsilon|^2 + \frac{1}{4}\left|u_{xy}^\epsilon\big|_{x=0}\right|^2_{L^2(I_y)} \\
&+ 2\epsilon\left(|\sqrt{1+x}\, u_{yxx}^\epsilon|^2 + |\sqrt{1+x}\, u_{yyy}^\epsilon|^2 + |\sqrt{1+x}\, u_{yzz}^\epsilon|^2\right) \\
&\leq c'\mu^4 |u_y^\epsilon|^2 + |f_y|^2 \\
&\leq c'\mu^4 |\sqrt{1+x}\, u_y^\epsilon|^2 + |f_y|^2, \qquad 0 \leq t \leq T_*.
\end{aligned} \tag{4.39}
$$

We can then close the Gronwall inequality on the time interval $(0, T_*)$, and obtain

$$
\begin{aligned}
|\sqrt{1+x}\, u_y^\epsilon(t)|^2 &\leq C(\mu)\left(|\sqrt{1+x}\, u_{0y}|^2 + \int_0^{T_*} |f_y(s)|^2\, ds\right) \\
&\leq C(\mu, |u_{0y}|, |f_y|_{L^2(0,T;L^2(\mathcal{M}))}), \qquad 0 \leq t \leq T_*,
\end{aligned}
$$

which implies

$$
|u_y^\epsilon(t)| \leq C(\mu), \quad 0 \leq t \leq T_*. \tag{4.40}
$$

By (4.40) and (4.39) we obtain

$$
\int_0^{T_*} |\nabla u_y^\epsilon(s)|^2\, ds \leq C(\mu). \tag{4.41}
$$

Similarly, we can obtain the same kind of estimates for u_z^ϵ, ∇u_z^ϵ, that is

$$
|u_z^\epsilon(t)| \leq C(\mu), \quad 0 \leq t \leq T_*, \tag{4.42}
$$

$$
\int_0^{T_*} |\nabla u_z^\epsilon(s)|^2\, ds \leq C(\mu). \tag{4.43}
$$

From (4.35), (4.40) and (4.42) we obtain (4.21).

We then multiply (4.4) by $(1+x)u^{\epsilon}_{yyyy}$, integrate over $\mathcal{M}$ and integrate by parts, to find

- $\displaystyle\int_{\mathcal{M}} u_t\,(1+x)u_{yyyy}\,d\mathcal{M} = \frac{1}{2}\frac{d}{dt}|\sqrt{1+x}\,u_{yy}|^2,$
- $\displaystyle\int_{\mathcal{M}} u_{xxx}\,(1+x)u_{yyyy}\,d\mathcal{M} = \frac{3}{2}|u_{xyy}|^2 + \frac{1}{2}\left|u_{xyy}\big|_{x=0}\right|^2_{L^2(I_y)},$
- $\displaystyle\int_{\mathcal{M}} u_{xyy}\,(1+x)u_{yyyy}\,d\mathcal{M} = \frac{1}{2}|u^2_{yyy}|^2,$
- $\displaystyle\int_{\mathcal{M}} u_{xzz}\,(1+x)u_{yyyy}\,d\mathcal{M} = \frac{1}{2}|u^2_{zyy}|^2,$
- $\displaystyle\int_{\mathcal{M}} cu_x\,(1+x)u_{yyyy}\,d\mathcal{M} = -\frac{c}{2}|u_{yy}|^2,$
- $\displaystyle\int_{\mathcal{M}} uu_x\,(1+x)u_{yyyy}\,d\mathcal{M}$

$$= -\int_{\mathcal{M}} u_y u_x(1+x)u_{yyy}\,d\mathcal{M} - \int_{\mathcal{M}} uu_{xy}(1+x)u_{yyy}\,d\mathcal{M}$$
$$= \int_{\mathcal{M}} u^2_{yy}u_x(1+x)\,d\mathcal{M} + 2\int_{\mathcal{M}} u_y u_{xy}(1+x)u_{yy}\,d\mathcal{M} + \int_{\mathcal{M}} uu_{xyy}(1+x)u_{yy}\,d\mathcal{M}$$
$$= \frac{1}{2}\int_{\mathcal{M}} (u_x(1+x)-u)\,u^2_{yy}\,d\mathcal{M} - \int_{\mathcal{M}} u^2_y u_{yy}\,d\mathcal{M} - \int_{\mathcal{M}} u^2_y(1+x)u_{xyy}\,d\mathcal{M}$$

- $\displaystyle\epsilon\int_{\mathcal{M}} u_{xxxx}\,(1+x)u_{yyyy}\,d\mathcal{M} = -\epsilon\int_{\mathcal{M}} u_{xxxxy}(1+x)u_{yyy}\,d\mathcal{M}$

$$= -\epsilon\int_{\mathcal{M}} u_{xxxy}u_{yyy}\,d\mathcal{M} + \epsilon\int_{\mathcal{M}} u_{xxxy}(1+x)u_{xyyy}\,d\mathcal{M}$$
$$= -\epsilon\int_{I_{x^\perp}} u^2_{xyy}\big|_{x=0}\,d\,I_{x^\perp} + \epsilon\int_{\mathcal{M}} (1+x)u^2_{xxyy}\,d\mathcal{M},$$

- $\displaystyle\epsilon\int_{\mathcal{M}} u_{yyyy}\,(1+x)u_{yyyy}\,d\mathcal{M} = \epsilon|\sqrt{1+x}\,u_{yyyy}|^2,$
- $\displaystyle\epsilon\int_{\mathcal{M}} u_{zzzz}\,(1+x)u_{yyyy}\,d\mathcal{M} = \epsilon|\sqrt{1+x}\,u_{zzyy}|^2,$
- $\displaystyle\int_{\mathcal{M}} f(1+x)u_{yyyy}\,d\mathcal{M} = -\int_{\mathcal{M}} f_{yy}\,(1+x)u_{yy}\,d\mathcal{M} \le \frac{1}{2}|(1+x)u_{yy}|^2 + \frac{1}{2}|f_{yy}|^2.$

Hence when $\epsilon \le \frac{1}{4}$,

$$\begin{aligned}
\frac{d}{dt}|\sqrt{1+x}\,u^\epsilon_{yy}|^2 + |\nabla u^\epsilon_{yy}|^2 + \frac{1}{4}|u^\epsilon_{xyy}|^2_{x=0}|_{L^2(I_y)} \\
+ 2\epsilon\left(|\sqrt{1+x}\,u^\epsilon_{yyxx}|^2 + |\sqrt{1+x}\,u^\epsilon_{yyyy}|^2 + |\sqrt{1+x}\,u^\epsilon_{yyzz}|^2\right) \\
\le \left|\int_{\mathcal{M}} \left((1+x)u^\epsilon_x - u^\epsilon\right)(u^\epsilon_{yy})^2\, d\mathcal{M}\right| + 2\left|\int_{\mathcal{M}} u_y^2 u_{yy}\, d\mathcal{M}\right| \\
+ 2\left|\int_{\mathcal{M}} u_y^2(1+x)u_{xyy}\, d\mathcal{M}\right| + (c+1)|u^\epsilon_{yy}|^2 + |f_{yy}|^2 \\
:= I^\epsilon_1 + I^\epsilon_2 + I^\epsilon_3 + (c+1)|u^\epsilon_{yy}|^2 + |f_{yy}|^2.
\end{aligned} \tag{4.44}$$

For I^ϵ_1, by the similar calculations in (4.38) we deduce

$$I^\epsilon_1 \le c'\left(\mu^4 + |u^\epsilon|^4\right)|u^\epsilon_{yy}|^2 + \frac{1}{8}|\nabla u^\epsilon_{yy}|^2, \quad 0 \le t \le T_*. \tag{4.45}$$

For I^ϵ_2 we have

$$\begin{aligned}
I^\epsilon_2 &\le 2|u^\epsilon_y|^2_{L^4(\mathcal{M})}|u^\epsilon_{yy}| \\
&\lesssim |u^\epsilon_y|^{1/2}|\nabla u^\epsilon_y|^{3/2}|u^\epsilon_{yy}| \\
&\le (\text{by } (4.40)) \\
&\le C(\mu)|\nabla u^\epsilon_y|^{3/2}|u^\epsilon_{yy}|^2 + C(\mu)|\nabla u^\epsilon_y|^{3/2}, \quad 0 \le t \le T_*.
\end{aligned} \tag{4.46}$$

For I^ϵ_3 we have

$$\begin{aligned}
I^\epsilon_3 &\le 2|u^\epsilon_y|^2_{L^4(\mathcal{M})}|u^\epsilon_{xyy}| \\
&\lesssim |u^\epsilon_y|^{1/2}|\nabla u^\epsilon_y|^{3/2}|u^\epsilon_{xyy}| \\
&\le \frac{1}{8}|u^\epsilon_{xyy}|^2 + c'|u^\epsilon_y||\nabla u^\epsilon_y|^3 \\
&\le \frac{1}{8}|u^\epsilon_{xyy}|^2 + c'|u^\epsilon_y||u^\epsilon_{xy}|^3 + c'|u^\epsilon_y||u^\epsilon_{yy}|^3 + c'|u^\epsilon_y||u^\epsilon_{yz}|^3 \\
&:= \frac{1}{8}|u^\epsilon_{xyy}|^2 + J^\epsilon_4 + J^\epsilon_5 + J^\epsilon_6.
\end{aligned} \tag{4.47}$$

We now estimate J^ϵ_4. We observe that since $u^\epsilon_x = 0$ at $y = \pm\frac{\pi}{2}$,

$$|u^\epsilon_{xy}|^2 = \int_{\mathcal{M}} \left(u^\epsilon_{xy}\right)^2\, d\mathcal{M} = -\int_{\mathcal{M}} u^\epsilon_x u^\epsilon_{xyy}\, d\mathcal{M} \le c'|u^\epsilon_x||u^\epsilon_{xyy}|. \tag{4.48}$$

Thus we have

$$
\begin{aligned}
J_4^\epsilon &\le c'|u_y^\epsilon||u_x^\epsilon|^{3/2}|u_{xyy}^\epsilon|^{3/2}\\
&\le c'|\nabla u^\epsilon|^{5/2}|u_{xyy}^\epsilon|^{3/2}\\
&\le \text{(by (4.21) already proven))}\\
&\le C(\mu)^{5/2}|u_{xyy}^\epsilon|^{3/2}\\
&\le C(\mu)^{10}+\frac{1}{8}|u_{xyy}^\epsilon|^2, \quad 0\le t\le T_*.
\end{aligned} \tag{4.49}
$$

Similarly for J_6^ϵ, since $u_z^\epsilon = 0$ at $y=\pm\frac{\pi}{2}$, we can apply the intermediate derivative theorem to u_z^ϵ, and deduce that $|u_{zy}^\epsilon|^2 \le c'|u_z^\epsilon||u_{zyy}^\epsilon|$. Hence by estimates similar as in (4.49) we have

$$
J_6^\epsilon \le C(\mu)^{10}+\frac{1}{8}|u_{zyy}^\epsilon|^2, \quad 0\le t\le T_*. \tag{4.50}
$$

To estimate J_5^ϵ, by (4.40) we have

$$
J_5^\epsilon \le C(\mu)|u_{yy}^\epsilon|^3, \qquad 0\le t\le T_*. \tag{4.51}
$$

Collecting the estimates in (4.49), (4.51) and (4.50), along with (4.47) we obtain

$$
I_3^\epsilon \le \frac{3}{8}|\nabla u_{yy}^\epsilon|^2 + C(\mu) + C(\mu)|u_{yy}^\epsilon|^3, \qquad 0\le t\le T_*. \tag{4.52}
$$

Collecting the estimates in (4.45), (4.46) and (4.52), along with (4.44) we obtain

$$
\begin{aligned}
\frac{d}{dt}|\sqrt{1+x}\,u_{yy}^\epsilon|^2 &+ \frac{1}{2}|\nabla u_{yy}^\epsilon|^2 + 2\epsilon\left(|\sqrt{1+x}\,u_{yyxx}^\epsilon|^2+|\sqrt{1+x}\,u_{yyyy}^\epsilon|^2+|\sqrt{1+x}\,u_{yyzz}^\epsilon|^2\right)\\
&\le c'\left(\mu^4+|u^\epsilon|^4+C(\mu)|\nabla u_y^\epsilon|^{3/2}+C(\mu)|u_{yy}^\epsilon|+c+1\right)|u_{yy}^\epsilon|^2\\
&\quad + C(\mu)|\nabla u_y^\epsilon|^{3/2}+C(\mu)+|f_{yy}|^2, \qquad 0\le t\le T_*.
\end{aligned} \tag{4.53}
$$

In particular, setting $\eta^\epsilon(t) = c'\left(\mu^4+|u^\epsilon|^4+C(\mu)|\nabla u_y^\epsilon|^{3/2}+C(\mu)|u_{yy}^\epsilon|+c+1\right)$, from (4.53) we infer that

$$
\frac{d}{dt}|\sqrt{1+x}\,u_{yy}^\epsilon|^2 \le \eta^\epsilon(t)|\sqrt{1+x}\,u_{yy}^\epsilon|^2 + C(\mu)|\nabla u_y^\epsilon|^{3/2}+C(\mu)+|f_{yy}|^2, \quad 0\le t\le T_*. \tag{4.54}
$$

Since $|\nabla u_y^\epsilon|^{3/2} \leq |\nabla u_y^\epsilon|^2 + c'$, along with (4.41) we deduce

$$\int_0^{T_*} \eta^\epsilon(s)\, ds \leq C(\mu).$$

We can then close the Gronwall inequality on the time interval $(0, T_*)$ in (4.53), and obtain

$$|\sqrt{1+x}\, u_{yy}^\epsilon(t)| \leq C(\mu, |u_{0yy}|, |f_{yy}|_{L^2(0,T;L^2(\mathcal{M}))}), \quad 0 \leq t \leq T_*,$$

which implies

$$|u_{yy}^\epsilon(t)| \leq C(\mu), \quad 0 \leq t \leq T_*. \tag{4.55}$$

By (4.55) and (4.44) we obtain

$$\int_0^{T_*} |\nabla u_{yy}^\epsilon(s)|^2\, ds \leq C(\mu), \tag{4.56}$$

$$\epsilon \int_0^{T_*} [u_{yy}^\epsilon]_2^2\; ds \leq C(\mu), \quad 0 \leq t \leq T_*. \tag{4.57}$$

Similarly we can obtain the same kind of estimates for u_{zz}^ϵ, ∇u_{zz}^ϵ and $\epsilon [u_{zz}^\epsilon]_2^2$.

Combining all the previous local bounds, we obtain (4.19)-(4.25). Hence we have completed the proof of Proposition 4.1.1. □

4.1.3 A Singular Perturbation Argument

We are now ready to show the local estimates for u_{xx}^ϵ and $u^\epsilon\, u_x^\epsilon$ by singular perturbation.

Proposition 4.1.2. *Under the same assumptions as in Proposition 4.1.1, we have*

$$u_{xx}^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(0, T_*,\, L^2(\mathcal{M})), \tag{4.58}$$

$$u^\epsilon u_x^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(0, T_*; L^2(\mathcal{M})). \tag{4.59}$$

Remark 4.1.1. *Note that by (5.35) and (4.23) we deduce that*

$$u^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(0, T_*,\, \Xi). \tag{4.60}$$

Remark 4.1.2. *We know that*

$$\int_{\mathcal{M}} (u^\epsilon u_x^\epsilon)^{3/2}\, d\mathcal{M} \le \left(\int_{\mathcal{M}} (u^\epsilon)^6\, d\mathcal{M}\right)^{1/4} \left(\int_{\mathcal{M}} (u_x^\epsilon)^2\, d\mathcal{M}\right)^{3/4}$$
$$= |u^\epsilon|^{3/2}_{L^6(\mathcal{M})} |u_x^\epsilon|^{3/2}$$
$$\le (\textit{by } H^1(\mathcal{M}) \subset L^6(\mathcal{M}) \textit{ in 3D})$$
$$\lesssim |\nabla u^\epsilon|^3.$$

Hence

$$\sup_{t\in(0,T_*)} |u^\epsilon u_x^\epsilon(t)|_{L^{3/2}(\mathcal{M})} \lesssim \sup_{t\in(0,T_*)} |\nabla u^\epsilon(t)|^2 \lesssim (\textit{by (4.21)}) \lesssim C(\mu)^2, \ \textit{a.e.t},$$

which implies that

$$u^\epsilon u_x^\epsilon \textit{ is bounded independently of } \epsilon \textit{ in } L^\infty(0,T_*; L^{3/2}(\mathcal{M})),$$
$$\textit{and hence in } L^{3/2}(I_x; L^{3/2}((0,T_*) \times I_{x^\perp})).$$

Thus we can apply Lemma 7.2.1 in the Appendix with $p = 3/2$ *and* $Y = L^{3/2}((0,T_*) \times I_{x^\perp})$*, and obtain*

$$u_{xx}^\epsilon \textit{ is bounded independently of } \epsilon \textit{ in } L^\infty(I_x; L^{3/2}((0,T_*) \times I_{x^\perp})). \tag{4.61}$$

However, to obtain more useful estimates as in (5.35) and (4.59), we need to use the following proof which provides a stronger result.

Proof of Proposition 5.2.2. We rewrite the regularized equation (4.4) as follows:

$$\begin{cases} u_{xxx}^\epsilon + u^\epsilon u_x^\epsilon + \epsilon u_{xxxx}^\epsilon = g^\epsilon, \\ u^\epsilon(0) = u^\epsilon(1) = u_x^\epsilon(1) = u_{xx}^\epsilon(0) = 0, \end{cases} \tag{4.62}$$

where $g^\epsilon := -u_t^\epsilon - \Delta^\perp u_x^\epsilon - c u_x^\epsilon - \epsilon u_{yyyy}^\epsilon - \epsilon u_{zzzz}^\epsilon + f$. Hence by (4.19), (4.24) and (4.25), we know that each term in g^ϵ is bounded independently of ϵ in $L^2(0, T_*, L^2(\mathcal{M}))$, and thus

$$g^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(0,T_*, L^2(\mathcal{M})). \tag{4.63}$$

Multiplying (4.62) by x and integrating in x from 0 to 1, we find

- $\int_0^1 x u_{xxx}\, dx = -\int_0^1 u_{xx}\, dx + u_{xx} x\big|_{x=0}^{x=1} = u_x\big|_{x=0} + u_{xx}\big|_{x=1}$,
- $\int_0^1 x u u_x\, dx = \int_0^1 \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) dx = -\frac{1}{2}\int_0^1 u^2\, dx$,
- $\epsilon \int_0^1 x u_{xxxx}\, dx = -\epsilon \int_0^1 u_{xxx}\, dx + \epsilon u_{xxx} x\big|_{x=0}^{x=1} = -\epsilon u_{xx}\big|_{x=1} + \epsilon u_{xxx}\big|_{x=1}$.

Hence

$$u_x^\epsilon\big|_{x=0} + u_{xx}^\epsilon\big|_{x=1} - \frac{1}{2}\int_0^1 (u^\epsilon)^2\, dx - \epsilon u_{xx}^\epsilon\big|_{x=1} + \epsilon u_{xxx}^\epsilon\big|_{x=1} = \int_0^1 g^\epsilon x\, dx. \tag{4.64}$$

Integrating (4.62) in x from $\tilde{x}$ to 1, we obtain

- $\int_{\tilde{x}}^1 u_{xxx}\, dx = u_{xx}\big|_{x=1} - u_{xx}$,
- $\int_{\tilde{x}}^1 u u_x\, dx = -\frac{1}{2}u^2$,
- $\epsilon \int_{\tilde{x}}^1 u_{xxxx}\, dx = \epsilon u_{xxx}\big|_{x=1} - \epsilon u_{xxx}$.

Hence

$$u_{xx}^\epsilon\big|_{x=1} - u_{xx}^\epsilon - \frac{1}{2}(u^\epsilon)^2 + \epsilon u_{xxx}^\epsilon\big|_{x=1} - \epsilon u_{xxx}^\epsilon = \int_{\tilde{x}}^1 g^\epsilon\, dx. \tag{4.65}$$

Then (4.64) and (4.65) imply

$$u_x^\epsilon\big|_{x=0} - \frac{1}{2}\int_0^1 (u^\epsilon)^2\, dx - \epsilon u_{xx}^\epsilon\big|_{x=1} + u_{xx}^\epsilon + \frac{1}{2}(u^\epsilon)^2 + \epsilon u_{xxx}^\epsilon = \int_0^1 g^\epsilon x\, dx - \int_{\tilde{x}}^1 g^\epsilon\, dx,$$

which we rewrite as

$$u_{xx}^\epsilon + \epsilon u_{xxx}^\epsilon = \epsilon u_{xx}^\epsilon\big|_{x=1} + h^\epsilon, \tag{4.66}$$

where

$$h^\epsilon = -u_x^\epsilon\big|_{x=0} + \frac{1}{2}\int_0^1 (u^\epsilon)^2\, dx - \frac{1}{2}(u^\epsilon)^2 + \int_0^1 g^\epsilon x\, dx - \int_{\tilde{x}}^1 g^\epsilon\, dx. \tag{4.67}$$

Now we estimate the term $(u^\epsilon)^2$ in (4.67). Since

$$|(u^\epsilon)^2|^2 \le |u^\epsilon|_{L^4(\mathcal{M})}^4 \lesssim (\text{by } H^{3/4}(\mathcal{M}) \subset L^4(\mathcal{M}) \text{ in 3D}) \lesssim |\nabla u^\epsilon|^3 |u^\epsilon|,$$

we have

$$\int_0^{T_*} |(u^\epsilon)^2|^2\,ds \lesssim \int_0^{T_*} |\nabla u^\epsilon|^3 |u^\epsilon|\,ds \le (\text{ by } (4.21)) \lesssim C(\mu)^4 T_*.$$

Thus

$$(u^\epsilon)^2 \text{ is bounded independently of } \epsilon \text{ in } L^2(0,T_*;L^2(\mathcal{M})). \tag{4.68}$$

Applying $(4.7)_4$, (4.63) and (4.68) to (4.67) we find

$$h^\epsilon \text{ is bounded independently of } \epsilon \text{ in } L^2(0,T_*,\,L^2(\mathcal{M})). \tag{4.69}$$

Multiplying (4.66) by u^ϵ_{xx}, integrating in x from 0 to 1, we obtain

$$\bullet\ \epsilon\int_0^1 u_{xxx}u_{xx}\,dx = \frac{\epsilon}{2}u^2_{xx}\big|_{x=1},$$
$$\bullet\ \epsilon\int_0^1 u_{xx}\big|_{x=1} u_{xx}\,dx = \epsilon u_{xx}\big|_{x=1}\int_0^1 u_{xx}\,dx = -\epsilon u_{xx}\big|_{x=1} u_x\big|_{x=0};$$

hence we arrive at

$$\begin{aligned}\int_0^1 (u^\epsilon_{xx})^2\,dx + \frac{\epsilon}{2}(u^\epsilon_{xx})^2\big|_{x=1} &= -\epsilon u^\epsilon_{xx}\big|_{x=1} u^\epsilon_x\big|_{x=0} + \int_0^1 u^\epsilon_{xx}\,h^\epsilon\,dx,\\ &\le \frac{\epsilon}{4}(u^\epsilon_{xx}\big|_{x=1})^2 + c'\epsilon(u^\epsilon_x\big|_{x=0})^2 + \frac{1}{2}|u^\epsilon_{xx}|^2_{L^2(I_x)} + \frac{1}{2}|h^\epsilon|^2_{L^2(I_x)}.\end{aligned}$$

Thus

$$\frac{1}{2}\int_0^1 (u^\epsilon_{xx})^2\,dx + \frac{\epsilon}{4}(u^\epsilon_{xx}\big|_{x=1})^2 \le c'\epsilon(u^\epsilon_x\big|_{x=0})^2 + \frac{1}{2}|h^\epsilon|^2_{L^2(I_x)}. \tag{4.70}$$

We integrate both sides of (4.70) in $I_{x^\perp}$ and then in time from 0 to T_*; by (4.69) and $(4.7)_4$ we obtain (5.35). As in Remark 4.1.1, we thus have (4.60).

Now since

$$\begin{aligned}|u^\epsilon u^\epsilon_x|^2 &\le |u^\epsilon|^2_{L^4(\mathcal{M})}|u^\epsilon_x|^2_{L^4(\mathcal{M})}\\ &\lesssim (\text{by } H^{3/4}(\mathcal{M})\subset L^4(\mathcal{M}) \text{ in 3D})\\ &\lesssim |u^\epsilon|^{1/2}|\nabla u^\epsilon|^{3/2}|u^\epsilon_x|^{1/2}[u^\epsilon]_2^{3/2}\\ &\lesssim |u^\epsilon|^2|\nabla u^\epsilon|^4 + [u^\epsilon]_2^2,\end{aligned}$$

hence we obtain

$$\int_0^{T_*} |u^\epsilon u_x^\epsilon|^2 \, ds \lesssim \int_0^{T_*} |u^\epsilon|^2 |\nabla u^\epsilon|^4 \, ds + \int_0^{T_*} [u]_2^2 \, ds.$$

This together with (4.21) and (4.60) implies (4.59). □

4.2 Passage to the Limit

Using a compactness argument, we can pass to the limit in (4.4). Hence we obtain (1.1), with a function $u \in \mathcal{C}^1([0,T_*];L^2(\mathcal{M})) \cap L^2(0,T;H_0^1(\mathcal{M}))$. Then we rewrite (1.1) as

$$u_{xxx} = -u_t - \Delta^\perp u_x - cu_x - uu_x - f. \tag{4.71}$$

From (4.19), (4.24) and (4.59), we infer that each term in the right-hand-side of (4.71) belongs to $L^2(0,T_*;L^2(\mathcal{M}))$, and hence

$$u_{xxx} \in L^2(0,T_*;L^2(\mathcal{M})). \tag{4.72}$$

Now we are ready to state the main result: the local existence of strong solutions.

Theorem 4.2.1. *The assumptions are the same as in Proposition 4.1.1, that is (4.5), (4.6), (4.8) and (4.13)-(4.17). We suppose also that the following compatibility conditions hold:*

$$u_0 = 0 \text{ on } \partial\mathcal{M}, \; u_{0x}\big|_{x=1} = 0, \; u_{0yy}\big|_{y=\pm\frac{\pi}{2}} = u_{0zz}\big|_{z=\pm\frac{\pi}{2}} = 0, \tag{4.73}$$

$$u_{t0} = 0 \text{ on } \partial\mathcal{M}, \; \frac{\partial u_{t0}}{\partial x}\Big|_{x=1} = 0, \; \frac{\partial^2 u_{t0}}{\partial y^2}\Big|_{y=\pm\frac{\pi}{2}} = \frac{\partial^2 u_{t0}}{\partial z^2}\Big|_{z=\pm\frac{\pi}{2}} = 0, \tag{4.74}$$

where $u_{t0} = -\Delta u_{0x} - u_0 u_{0x} - cu_{0x} + f(0)$. Then there exists a local strong solution to (1.1)-(2.4) on some time interval $[0,T_*)$, $T_* > 0$ depending only on the data as in Proposition 4.1.1, such that

$$\nabla u, u_{yy}, u_{zz}, u_t \in L^\infty(0,T_*;L^2(\mathcal{M})), \tag{4.75}$$

$$u \in L^2(0,T_*;D(A) \cap \Xi \cap H^3(I_x;L^2(I_{x^\perp})) \cap H^3(I_{x^\perp};L^2(I_x))), \tag{4.76}$$

$$u_t \in L^2(0, T_*; H^1(\mathcal{M})). \tag{4.77}$$

Moreover, we have for every $t \in (0, T_*)$,

$$u_{yy}(t)\big|_{y=\pm\frac{\pi}{2}} = u_{zz}(t)\big|_{z=\pm\frac{\pi}{2}} = 0. \tag{4.78}$$

Remark 4.2.1. *We have proven that all the spatial derivatives of the third order of u are in $L^2(0, T_*; L^2(\mathcal{M}))$, except for u_{xxy} and u_{xxz}.*

Proof. We rewrite (1.1) as

$$Au = -u_t - uu_x - f; \tag{4.79}$$

from (4.19) and (4.59) we know that each term on the right-hand side of (4.79) belongs to $L^2(0, T_*; L^2(\mathcal{M}))$. Hence Au belongs to the same space. We also know that $u_x(1,\, x^{\perp},\, t) = 0$, $t \in [0, T]$ as prove in Proposition 2.3.1. Hence we obtain that $u \in L^2(0, T_*; D(A))$. This together with (4.3), we deduce that $u \in L^2(0, T_*; \Xi)$.

By (4.22), we know that u_{yyy}, u_{zzz} both belong to $L^2(0, T_*; L^2(\mathcal{M}))$. Hence we can apply the trace theorem and pass to the limit on the boundary conditions in (3.4) to obtain (4.78).

The other results can be deduced directly from (4.19)-(4.24) and (4.72). □

Remark 4.2.2. *As for the periodic case, that is, (1.1) and the boundary and initial conditions (2.2), (2.3) and (2.5), the results are exactly the same as in the Dirichlet case discussed above. The reasoning is totally the same and therefore we skip it.*

Chapter 5

Global Existence of Strong Solutions in $3D$

We now establish the global existence of strong solutions in $3D$. First we need to assume different boundary conditions at x boundaries, that is, we suppose

$$u(0,\, x^{\perp},\, t) = u(1,\, x^{\perp},\, t), \tag{5.1}$$

$$u_x(0,\, x^{\perp},\, t) = u_x(1,\, x^{\perp},\, t), \quad u_{xx}(0,\, x^{\perp},\, t) = u_{xx}(1,\, x^{\perp},\, t). \tag{5.2}$$

For the boundary conditions in the y and z directions, we will choose the same boundary conditions as before, that is, either the Dirichlet boundary conditions in (2.4) or the periodic boundary conditions in (2.5).

Similarly, we will focus on the initial and boundary value problem (1.1), (5.1), (5.2) and (2.3) supplemented with the boundary condition (2.4), that is, the Dirichlet case on the $x^{\perp}$ boundaries, and we will make some remarks on the extension to the periodic boundary condition case.

We recall the notations $|\cdot|$ and $(\cdot,\cdot)$ for the norm and the inner product of $L^2(\mathcal{M})$, and by $[\cdot]_2$ the following seminorm which will be useful in the sequel:

$$\left(\int_{\mathcal{M}} u_{xx}^2 + u_{yy}^2 + u_{yy}^2 \, d\,\mathcal{M}\right)^{1/2} =: [u]_2, \quad u \in H^2(\mathcal{M}). \tag{5.3}$$

To establish the existence of strong solutions in dimensions 2 and 3, we use the parabolic regularization as in Chapter 3, but with different boundary conditions. For the sake of simplicity we only treat the more complicated case when $d = 2$.

5.1 Parabolic Regularization

To begin with, we recall the parabolic regularization (4.4) in Section 4.1 supplemented with the boundary conditions (5.1), (5.2) and (2.4) and the additional boundary conditions

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \Delta \dfrac{\partial u^\epsilon}{\partial x} + c\dfrac{\partial u^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon L u^\epsilon = f, \\ \\ u^\epsilon(0) = u_0, \end{cases} \tag{5.4}$$

where

$$L u^\epsilon := \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4},$$

and

$$u^\epsilon_{xxx}(0,\, x^\perp,\, t) = u^\epsilon_{xxx}(1,\, x^\perp,\, t), \tag{5.5}$$

$$u^\epsilon_{yy} = 0 \text{ at } y = \pm\frac{\pi}{2},\;\; u^\epsilon_{zz} = 0 \text{ at } z = \pm\frac{\pi}{2}. \tag{5.6}$$

Note that from (5.2) and (5.5) we infer

$$u^\epsilon_{x^j}(0,\, x^\perp,\, t) = u^\epsilon_{x^j}(1,\, x^\perp,\, t), \quad j = 1,\, 2,\; 3. \tag{5.7}$$

We also note that since $u^\epsilon_{yy}\big|_{x=0}^{x=1} = u^\epsilon_{zz}\big|_{x=0}^{x=1} = 0$, (5.7) is equivalent to

$$\Delta u^\epsilon\big|_{x=0}^{x=1} = 0. \tag{5.8}$$

5.2 Estimates Independent of ϵ

We establish the estimates independent of ϵ for various norms of the solutions.

5.2.1 L^2 Estimate Independent of ϵ

We first show a bound independent of ϵ for u^ϵ in $L^\infty(0,T;L^2(\mathcal{M}))$.

Lemma 5.2.1. *We assume that*

$$u_0 \in L^2(\mathcal{M}), \tag{5.9}$$

$$f \in L^2(0,T;L^2(\mathcal{M})). \tag{5.10}$$

Then for every $T > 0$ the following estimates independent of ϵ hold:

$$u^\epsilon \ \textit{is bounded in}\ L^\infty(0,T;\ L^2(\mathcal{M})), \tag{5.11}$$

$$\sqrt{\epsilon}\, u^\epsilon \ \textit{is bounded in}\ L^2(0,T;\ H^2(\mathcal{M})). \tag{5.12}$$

Proof. As in Section 3.1, we multiply (5.4) with u, integrate over $\mathcal{M}$ and integrate by parts, dropping the superscript ϵ for the moment we find:

$$\begin{aligned}
\bullet \int_{\mathcal{M}} \Delta u_x\, u\, d\mathcal{M} + \int_{\mathcal{M}} c u_x\, u\, d\mathcal{M} &= (\text{thanks to (5.1)}) \\
&= -\int_{\mathcal{M}} \nabla u_x\, \nabla u\, d\mathcal{M} \\
&= -\frac{1}{2}\int_{I_{x^\perp}} (\nabla u)^2\Big|_{x=0}^{x=1} dx^\perp \\
&= (\text{thanks to (5.1) and (5.7)}) \\
&= 0,
\end{aligned} \tag{5.13}$$

$$\bullet \int_{\mathcal{M}} u u_x\, u\, d\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) d\mathcal{M} = (\text{thanks to (5.1)}) = 0,$$

$$\begin{aligned}
\bullet\ \epsilon \int_{\mathcal{M}} u_{xxxx}\, u\, d\mathcal{M} &= (\text{thanks to (5.1) and (5.7)}) \\
&= -\epsilon \int_{\mathcal{M}} u_{xxx}\, u_x\, d\mathcal{M} \\
&= (\text{thanks to (5.7)}) \\
&= \epsilon \int_{\mathcal{M}} u_{xx}^2\, d\mathcal{M},
\end{aligned}$$

$$\begin{aligned}
\bullet\ \epsilon \int_{\mathcal{M}} (u_{xxxx} + u_{yyyy} + u_{zzzz})\ u\, d\mathcal{M} &= \epsilon \int_{\mathcal{M}} u_{xx}^2 + u_{yy}^2 + u_{yy}^2\, d\mathcal{M} \\
&= (\text{thanks to (5.3)}) \\
&= \epsilon [u]_2^2,
\end{aligned}$$

$$\bullet \int_{\mathcal{M}} f u\, d\mathcal{M} \le \frac{1}{2}|f|^2 + \frac{1}{2}|u|^2.$$

Hence we find

$$\frac{d}{dt}|u^\epsilon(t)|^2 + 2\epsilon[u^\epsilon]_2^2 \le |f|^2 + |u^\epsilon|^2.$$

Using the Gronwall lemma we classically infer

$$\sup_{t\in(0,T)} |u^\epsilon(t)|^2 + \epsilon \int_0^T [u^\epsilon]_2^2 \, dt \le const := \mu_1, \tag{5.14}$$

where μ_i indicates a constant depending only on the data u_0, f, etc, whereas C' below is an absolute constant. These constants may be different at each occurrence. Let us admit for the moment the following:

Lemma 5.2.2.

$$|u^\epsilon|^2_{H^2(\mathcal{M})} \le C' \left([u^\epsilon]_2^2 + |u^\epsilon|^2\right). \tag{5.15}$$

By the previous lemma, we have

$$\begin{aligned}
\epsilon \int_0^T |u^\epsilon|^2_{H^2(\mathcal{M})} \, dt &\le C' \left(\epsilon \int_0^T [u^\epsilon]_2^2 \, dt + \epsilon \int_0^T |u^\epsilon|^2 \, dt\right) \\
&\le C' \left(\epsilon \int_0^T [u^\epsilon]_2^2 \, dt + \epsilon \, T \sup_{t\in(0,T)} |u^\epsilon(t)|^2\right) \\
&\le \text{(thanks to (5.14))} \\
&\le const := \mu_2,
\end{aligned}$$

which implies (5.12). Thus Lemma 5.2.1 is proven once we have proven Lemma (5.2.2).

Proof of Lemma 5.2.2. We first observe that using the generalized Poincaré inequality (see [44]) we have

$$|u^\epsilon_x - \int_0^1 u^\epsilon_x \, dx|_{L^2(I_x)} \le C' |u^\epsilon_{xx}|_{L^2(I_x)}. \tag{5.16}$$

Thanks to (5.1), we have $\int_0^1 u^\epsilon_x \, dx = u^\epsilon|_{x=0}^{x=1} = 0$, and hence (5.16) implies

$$|u^\epsilon_x|_{L^2(I_x)} \le C' |u^\epsilon_{xx}|_{L^2(I_x)}.$$

Squaring both sides and integrating both sides on $I_{x^\perp}$, we find

$$|u^\epsilon_x| \le C' |u^\epsilon_{xx}|. \tag{5.17}$$

Similarly we can show that $|u_y^\epsilon| \le C'|u_{yy}^\epsilon|$ and $|u_z^\epsilon| \le C'|u_{zz}^\epsilon|$, which implies

$$|\nabla u^\epsilon| \le C'[u^\epsilon]_2. \tag{5.18}$$

Next we see that, for smooth functions

$$\begin{aligned} |u_{xy}^\epsilon|^2 &= \text{(thanks to (5.1) and (5.7))} \\ &= -\int_{\mathcal{M}} u_y^\epsilon u_{xxy}^\epsilon d\mathcal{M} \\ &= \text{(thanks to (2.4))} \\ &= \int_{\mathcal{M}} u_{yy}^\epsilon u_{xx}^\epsilon d\mathcal{M} \\ &\le |u_{xx}^\epsilon|^2 + |u_{yy}^\epsilon|^2 \le [u^\epsilon]_2^2. \end{aligned} \tag{5.19}$$

Similarly we can prove that $|u_{xz}^\epsilon| \le [u^\epsilon]_2$ and $|u_{zy}^\epsilon| \le [u^\epsilon]_2$, and hence

$$|u_{xy}^\epsilon|^2 + |u_{xz}^\epsilon|^2 + |u_{yz}^\epsilon|^2 \le C'[u^\epsilon]_2^2. \tag{5.20}$$

Then inequality (5.19) and (5.20) extend by continuity to all H^2 function periodic in x and satisfying (2.4) and (5.6). Finally from (5.20) and (5.18) we deduce (5.15). □

5.2.2 H^1 Estimate Independent of ϵ

Now we establish the key observation, a bound independent of ϵ for ∇u^ϵ in $L^\infty(0,T;L^2(\mathcal{M}))$.

Proposition 5.2.1. *Under the same assumptions as in Lemma 5.2.1, we further suppose that*

$$u_0 \in H^1(\mathcal{M}) \cap L^3(\mathcal{M}), \tag{5.21}$$

$$f \in L^2(0,T;H^2(I_x;\ H^2 \cap H_0^1(I_{x^\perp}))) \cap L^2(0,T;L^\infty(\mathcal{M})), \tag{5.22}$$

and f and f_x assume the periodic boundary conditions on $x = 0,\ 1$. Then for every $T > 0$, the following estimates independent of ϵ hold:

$$u^\epsilon \text{ is bounded in } L^\infty(0,T;\ H^1(\mathcal{M})), \tag{5.23}$$

$$\sqrt{\epsilon}\,\nabla u_{xx}^\epsilon,\ \sqrt{\epsilon}\,\nabla u_{yy}^\epsilon,\ \sqrt{\epsilon}\,\nabla u_{zz}^\epsilon \text{ are bounded in } L^2(0,T;L^2(\mathcal{M})). \tag{5.24}$$

Proof. We multiply (5.4) with $-\Delta u^\epsilon - \frac{1}{2}(u^\epsilon)^2$, integrate over $\mathcal{M}$ and integrate by parts. Firstly we show the calculation details of the multiplication by Δu^ϵ, integration over $\mathcal{M}$ and integration by parts (dropping the super index of ϵ for the moment):

- $$\int_{\mathcal{M}} u_t\,\Delta u\,d\mathcal{M} = -\int_{\mathcal{M}} \nabla u_t\,\nabla u\,d\mathcal{M} + \int_{\partial\mathcal{M}} u_t\,\frac{\partial u}{\partial n}\,d\,\partial\mathcal{M}$$
  $$= \text{(thanks to (5.1) and (5.7))} = -\int_{\mathcal{M}} \nabla u_t\,\nabla u\,d\mathcal{M}$$
  $$= -\frac{1}{2}\frac{d}{dt}\,|\nabla u|^2\,,$$
- $$\int_{\mathcal{M}} \Delta u_x\,\Delta u\,d\mathcal{M} = \frac{1}{2}\int_{I_{x^\perp}} (\Delta u)^2\Big|_{x=0}^{x=1}\,d\,I_{x^\perp} = \text{(thanks to (5.8))} = 0,$$
- $$\int_{\mathcal{M}} u_x\,\Delta u\,d\mathcal{M} = \int_{\mathcal{M}} u_x\,u_{xx} + u_x\,\Delta^\perp u\,d\mathcal{M} = \text{(thanks to (5.1))}$$
  $$= \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{(u_x)^2}{2}\right)\,d\mathcal{M} - \int_{\mathcal{M}} \nabla^\perp u_x\,\nabla^\perp u\,d\mathcal{M}$$
  $$= \text{(thanks to (5.7) and (5.1))} = 0,$$
- $$\int_{\mathcal{M}} u_{xxxx}\,u_{xx}\,d\,\mathcal{M} = \text{(thanks to (5.7))} = -\int_{\mathcal{M}} u_{xxx}^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{xxxx}\,u_{yy}\,d\,\mathcal{M} = \text{(thanks to (5.1)-(2.4) and (5.7))} = -\int_{\mathcal{M}} u_{xxy}^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{yyyy}\,\Delta u\,d\,\mathcal{M} = \text{(thanks to (2.4) and (5.6))} = -\int_{\mathcal{M}} u_{yyy}\,\Delta u_y\,d\,\mathcal{M}$$
  $$= \text{(thanks to (5.6))} = \int_{\mathcal{M}} u_{yy}\,\Delta u_{yy}\,d\,\mathcal{M}$$
  $$= -\int_{\mathcal{M}} (\nabla u_{yy})^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} f\,\Delta u\,d\mathcal{M} = \text{(thanks to (5.22))} = \int_{\mathcal{M}} \Delta f\,u\,d\mathcal{M},$$

Hence we find after changing the sign,

$$\frac{1}{2}\frac{d}{dt}\,|\nabla u^\epsilon|^2 - \int_{\mathcal{M}} u^\epsilon u_x^\epsilon\,\Delta u^\epsilon\,d\mathcal{M} + \epsilon[\nabla u^\epsilon]_2^2 = -\int_{\mathcal{M}} \Delta f\,u^\epsilon\,d\mathcal{M}. \tag{5.25}$$

Next we show the calculation details of the multiplication by $(u^\epsilon)^2$, integrating over $\mathcal{M}$

and integrating by parts:

- $\displaystyle\int_{\mathcal{M}} u_t u^2 \, d\,\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial t}\left(\frac{u^3}{3}\right) d\,\mathcal{M} = \frac{1}{3}\frac{d}{dt}\left(\int_{\mathcal{M}} u^3 \, d\mathcal{M}\right),$
- $\displaystyle\int_{\mathcal{M}} \Delta u_x \, u^2 \, d\mathcal{M} = -2\int_{\mathcal{M}} \Delta u \, u u_x \, d\mathcal{M} + \int_{I_{x^\perp}} \Delta u \, u^2\big|_{x=0}^{x=1} d\,I_{x^\perp}$

  $\displaystyle = \text{(thanks to (5.8) and (5.1))}$

  $\displaystyle = -2\int_{\mathcal{M}} \Delta u \, u u_x \, d\mathcal{M},$

- $\displaystyle\int_{\mathcal{M}} u_x \, u^2 \, d\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) d\mathcal{M} = \frac{1}{3}\int_{I_{x^\perp}} u^3\big|_{x=0}^{x=1} d\,I_{x^\perp} = \text{(by (5.1))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u u_x \, u^2 \, \mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{u^4}{4}\right) d\mathcal{M} = \frac{1}{4}\int_{I_{x^\perp}} u^4\big|_{x=0}^{x=1} d\,I_{x^\perp} = \text{(by (5.1))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u_{xxxx} \, u^2 \, d\,\mathcal{M} = \text{(thanks to (5.1) and (5.7))} = -2\int_{\mathcal{M}} u_{xxx} \, u_x \, u \, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{yyyy} \, u^2 \, d\,\mathcal{M} = \text{(thanks to (2.4))} = -2\int_{\mathcal{M}} u_{yyy} \, u_y \, u \, d\,\mathcal{M},$

Hence we find

$$\begin{aligned} &\frac{1}{3}\frac{d}{dt}\left(\int_{\mathcal{M}} (u^\epsilon)^3 \, d\mathcal{M}\right) - 2\int_{\mathcal{M}} \Delta u^\epsilon \, u^\epsilon u_x^\epsilon \, d\mathcal{M} \\ =&2\epsilon\int_{\mathcal{M}} u_{xxx}^\epsilon \, u_x^\epsilon \, u^\epsilon + u_{yyy}^\epsilon \, u_y^\epsilon \, u^\epsilon + u_{zzz}^\epsilon \, u_z^\epsilon \, u^\epsilon \, d\,\mathcal{M} + \int_{\mathcal{M}} f\,(u^\epsilon)^2 \, d\mathcal{M}. \end{aligned} \tag{5.26}$$

Adding (5.25) to (5.26) multiplied by $-1/2$, we see that the term $\int_{\mathcal{M}} \Delta u^\epsilon \, u^\epsilon u_x^\epsilon \, d\mathcal{M}$ get canceled, which yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\,|\nabla u^\epsilon|^2 + \epsilon[\nabla u^\epsilon]_2^2 =&\frac{1}{6}\,\frac{d}{dt}\left(\int_{\mathcal{M}} (u^\epsilon)^3 \, d\mathcal{M}\right) \\ &- \epsilon\int_{\mathcal{M}} u_{xxx}^\epsilon \, u_x^\epsilon \, u^\epsilon + u_{yyy}^\epsilon \, u_y^\epsilon \, u^\epsilon + u_{zzz}^\epsilon \, u_z^\epsilon \, u^\epsilon \, d\,\mathcal{M} \\ &- \int_{\mathcal{M}} \Delta f u^\epsilon \, d\mathcal{M} - \frac{1}{2}\int_{\mathcal{M}} f\,(u^\epsilon)^2 \, d\mathcal{M}. \end{aligned}$$

Integrating both sides in time from 0 to t, we obtain for every $t \in (0,T)$,

$$
\begin{aligned}
&\frac{1}{2}\left|\nabla u^{\epsilon}(t)\right|^{2}+\epsilon \int_{0}^{t}[\nabla u^{\epsilon}]_{2}^{2}\, ds\\
=&\frac{1}{6}\int_{\mathcal{M}}\left(u^{\epsilon}(t)\right)^{3}\, d\mathcal{M}+\kappa_{0}\\
&-\epsilon \int_{0}^{t}\int_{\mathcal{M}} u^{\epsilon}_{xxx}\, u^{\epsilon}_{x}\, u^{\epsilon}+u^{\epsilon}_{yyy}\, u^{\epsilon}_{y}\, u^{\epsilon}+u^{\epsilon}_{zzz}\, u^{\epsilon}_{z}\, u^{\epsilon}\, d\mathcal{M}\, ds\\
&-\int_{0}^{t}\int_{\mathcal{M}}\Delta f u^{\epsilon}\, d\mathcal{M}\, ds-\frac{1}{2}\int_{0}^{t}\int_{\mathcal{M}} f\left(u^{\epsilon}\right)^{2}\, d\mathcal{M}\, ds,
\end{aligned}
\tag{5.27}
$$

where

$$
\kappa_{0}:=\frac{1}{2}\left|\nabla u_{0}\right|^{2}-\frac{1}{6}\int_{\mathcal{M}} u_{0}^{3}\, d\mathcal{M}.
$$

We estimate each term on the right-hand-side of (5.27); we will use here the interpolation space $H^{1/2}(\mathcal{M})$ as defined in [31] where it is shown that $H^{1/2}(\mathcal{M}) \subset L^{3}(\mathcal{M})$ in dimension 3 with a continuous embedding. Dropping the superscript ϵ for the moment we then find:

$$
\begin{aligned}
\left|\frac{1}{6}\int_{\mathcal{M}} u^{3}(t)\, d\mathcal{M}\right| &\leq \frac{1}{6}|u(t)|^{3}_{L^{3}(\mathcal{M})}\\
&\leq C'|u(t)|^{3}_{H^{1/2}(\mathcal{M})}\\
&\leq C'|u(t)|^{3/2}\left|\nabla u(t)\right|^{3/2}\\
&\leq C'|u(t)|^{6}+\frac{1}{4}\left|\nabla u(t)\right|^{2},
\end{aligned}
$$

$$
\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_{xxx}\, u_{x}\, u\, d\,\mathcal{M}\right| &\leq \epsilon\left|u_{xxx}\right|\left|u_{x}\, u\right|\\
&\leq C'\epsilon\left|u_{x}\, u\right|^{2}+\frac{\epsilon}{10}|u_{xxx}|^{2}\\
&\leq C'\epsilon|u|^{2}_{L^{4}(\mathcal{M})}|u_{x}|^{2}_{L^{4}(\mathcal{M})}+\frac{\epsilon}{10}|u_{xxx}|^{2}\\
&\leq (\text{by } H^{3/4}(\mathcal{M}) \subset L^{4}(\mathcal{M}) \text{ in 3D})\\
&\leq C'\epsilon|u|^{1/2}|\nabla u|^{3/2}|u_{x}|^{1/2}|u_{x}|^{3/2}_{H^{1}(\mathcal{M})}+\frac{\epsilon}{10}|u_{xxx}|^{2}\\
&\leq C'\epsilon|u|^{1/2}|\nabla u|^{2}|u|^{3/2}_{H^{2}(\mathcal{M})}+\frac{\epsilon}{10}|u_{xxx}|^{2},
\end{aligned}
\tag{5.28}
$$

$$
\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_{yyy}\, x\, u_{y}\, u\, d\,\mathcal{M}\right| &\leq (\text{by similar estimates as above})\\
&\leq C'\epsilon|u|^{1/2}|\nabla u|^{2}|u|^{3/2}_{H^{2}(\mathcal{M})}+\frac{\epsilon}{10}|u_{yyy}|^{2},
\end{aligned}
$$

$$\epsilon\left|\int_{\mathcal{M}} u_{zzz}\, x\, u_z\, u\, d\,\mathcal{M}\right| \le \text{(by similar estimates as above)}$$

$$\le C'\epsilon|u|^{1/2}|\nabla u|^2|u|^{3/2}_{H^2(\mathcal{M})} + \frac{\epsilon}{10}|u_{zzz}|^2,$$

$$\left|\int_{\mathcal{M}} \Delta f\, u\, d\mathcal{M}\right| \le |\Delta f|^2 + |u|^2\,,$$

$$\left|\int_{\mathcal{M}} f u^2\, d\mathcal{M}\right| \le |f|_{L^\infty(\mathcal{M})}|u|^2 \le |f|^2_{L^\infty(\mathcal{M})} + |u|^4.$$

Collecting the above estimates, along with (5.27) we observe that the terms with third-order derivatives in the RHS of (5.28) and the following two inequalities can be canceled by a term on the LHS of (5.27). Thus (5.27) now yields

$$\begin{aligned}
\frac{1}{4}|\nabla u^\epsilon(t)|^2 + \frac{\epsilon}{10}\int_0^t [\nabla u^\epsilon]_2^2\, ds \le{}& \int_0^t \left(1 + C'\epsilon|u^\epsilon|^{1/2}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}\right)|\nabla u^\epsilon(s)|^2\, ds \\
&+ C'|u^\epsilon(t)|^6 + \kappa_0 + \int_0^t |\Delta f|^2\, ds \\
&+ \int_0^t |u^\epsilon|^2 + |u^\epsilon|^4\, ds + \int_0^t |f|^2_{L^\infty(\mathcal{M})}\, ds \\
\le{}&\text{(thanks to (5.14))} \\
\le{}& \int_0^t \left(1 + C'\epsilon\mu_1^{1/4}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}\right)|\nabla u^\epsilon(s)|^2\, ds \\
&+ C'\mu_1^3 + \kappa_0 + |f|^2_{L^2(0,T;H_0^2(\mathcal{M}))} \\
&+ (\mu_1 + \mu_1^2)T + |f|^2_{L^2(0,T;L^\infty(\mathcal{M}))}.
\end{aligned} \tag{5.29}$$

In particular, setting $\sigma^\epsilon(t) := 1 + C'\epsilon\mu_1^{1/4}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}$, from (5.29) we deduce

$$\begin{aligned}
\frac{1}{4}|\nabla u^\epsilon(t)|^2 + \frac{\epsilon}{10}\int_0^t [\nabla u^\epsilon]_2^2\, ds \le{}& \int_0^t \sigma^\epsilon(s)|\nabla u^\epsilon(s)|^2\, ds \\
&+ C'\mu_1^3 + \kappa_0 + |f|^2_{L^2(0,T;H_0^2(\mathcal{M}))} \\
&+ (\mu_1 + \mu_1^2)T + |f|^2_{L^2(0,T;L^\infty(\mathcal{M}))}.
\end{aligned} \tag{5.30}$$

Since $|u^\epsilon|^{3/2}_{H^2(\mathcal{M})} \le |u^\epsilon|^2_{H^2(\mathcal{M})} + C'$, we find

$$\int_0^T \sigma^\epsilon(s)\,ds \le T + C'\epsilon\mu_1^{1/4} \int_0^T \left(|u^\epsilon|^2_{H^2(\mathcal{M})} + C'\right) ds$$

$$\le \text{(thanks to (5.12))}$$

$$\le \text{const} := \mu_3.$$

We can then apply the Gronwall inequality to (5.30) to obtain

$$\sup_{t\in(0,T)} |\nabla u^\epsilon(t)|^2 + \frac{\epsilon}{10}\int_0^T [\nabla u^\epsilon]_2^2\,ds \le const := \mu_4. \tag{5.31}$$

This together with (5.11) implies (5.23) and (5.24). □

5.2.3 Estimates Independent of ϵ for u^ϵ_{xxx} and $u^\epsilon u^\epsilon_x$

For the sake of the passage to the limit on the boundary conditions and the compactness argument, we now derive bounds independent of ϵ for u^ϵ_{xxx} and $u^\epsilon\, u^\epsilon_x$. In particular, to obtain the estimates for u^ϵ_{xxx}, we first deduce a bound independent of ϵ for $\epsilon\, u^\epsilon_{xxxx}$ in $L^2(0,T;L^2(\mathcal{M}))$.

Proposition 5.2.2. *Under the same assumptions as in Proposition 5.2.1, we further suppose that*

$$u_{0xx} \in L^2(\mathcal{M}), \tag{5.32}$$

$$f_{xxx} \in L^2(0,T;L^2(\mathcal{M})), \tag{5.33}$$

and f_{xx} assume the periodic boundary condition on $x = 0,\ 1$. Then we have the following bounds independent of ϵ,

$$\epsilon\,[u_{xx}]_2 \text{ is bounded in } L^2(0,T;L^2(\mathcal{M})), \tag{5.34}$$

$$u^\epsilon u^\epsilon_x \text{ is bounded in } L^\infty(0,T;L^{3/2}(\mathcal{M})). \tag{5.35}$$

$$u^\epsilon_{xxx} \text{ is bounded in } L^{3/2}(I_x; H_t^{-1}(0,T;\ H^{-4}(I_{x^\perp}))), \tag{5.36}$$

Proof. For notational simplicity, we will drop the super index ϵ in the calculations. Multiplying (5.4) by u^ϵ_{xxxx}, integrating over $\mathcal{M}$ and integrating by parts we find:

- $\displaystyle\int_{\mathcal{M}} u_t\, u_{xxxx}\, d\mathcal{M} = \text{(thanks to (5.1) and (5.7))} = \frac{1}{2}\frac{d}{dt}|u_{xx}|^2,$
- $\displaystyle\int_{\mathcal{M}} \Delta u_x\, u_{xxxx}\, d\mathcal{M} = \text{(thanks to (5.1), (5.7) and (2.4))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u_x u_{xxxx}\, d\mathcal{M} = \text{(thanks to (5.1), (5.7) and (2.4))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u u_x u_{xxxx} d\mathcal{M} = -\int_{\mathcal{M}} u_x^2 u_{xxx} d\mathcal{M} - \int_{\mathcal{M}} u u_{xx} u_{xxx} d\mathcal{M} = \frac{5}{2}\int_{\mathcal{M}} u_x u_{xx}^2 d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{yyyy} u_{xxxx}\, d\,\mathcal{M} = \text{(thanks to (5.1), (5.7) and (2.4))} = \int_{\mathcal{M}} u_{xxyy}^2\, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{zzzz} u_{xxxx}\, d\,\mathcal{M} = \int_{\mathcal{M}} u_{xxzz}^2\, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} f u_{xxxx}\, d\mathcal{M} = -\int_{\mathcal{M}} f_{xxx}\, u_x\, d\mathcal{M} \le |f_{xxx}|^2 + |u_x|^2.$

Hence we find

$$\frac{1}{2}\frac{d}{dt}|u^\epsilon_{xx}|^2 + \epsilon[u^\epsilon_{xx}]_2^2 \le \frac{5}{2}\int_{\mathcal{M}} u^\epsilon_x\ (u^\epsilon_{xx})^2\ d\mathcal{M} + |f_{xxx}|^2 + |u^\epsilon_x|^2.$$

Multiplying both sides by ϵ we obtain

$$\frac{\epsilon}{2}\frac{d}{dt}|u^\epsilon_{xx}|^2 + \epsilon^2[u^\epsilon_{xx}]_2^2 \le \frac{5\epsilon}{2}\int_{\mathcal{M}} u^\epsilon_x\ (u^\epsilon_{xx})^2\ d\mathcal{M} + \epsilon|f_{xxx}|^2 + \epsilon|u^\epsilon_x|^2. \tag{5.37}$$

We estimate the first term on the right-hand side of (5.37) and find

$$\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_x u_{xx}^2 d\mathcal{M}\right| &\le \epsilon\,|u_x||u_{xx}|^2_{L^4(\mathcal{M})}\\
&\le C'\epsilon\,|u_x||u_{xx}|^{1/2}|\nabla u_{xx}|^{3/2}\\
&\le \text{(the intermediate derivative theorem } |u_{xx}|^2 \le |u_x||u_{xxx}|)\\
&\le C'\epsilon|u_x|^{5/4}|u_{xxx}|^{1/4}|\nabla u_{xx}|^{3/2}\\
&\le C'\epsilon|u_x|^{5/4}|\nabla u_{xx}|^{7/4}\\
&\le \text{(thanks to (5.31))}\\
&\le C'\epsilon\mu_4^{5/8}|\nabla u_{xx}|^{7/4}.
\end{aligned}$$

This along with (5.37) implies

$$\frac{\epsilon}{2}\frac{d}{dt}|u^\epsilon_{xx}|^2 + \epsilon^2[u^\epsilon_{xx}]_2^2 \le C'\epsilon\,\mu_4^{5/8}|\nabla u^\epsilon_{xx}|^{7/4} + \epsilon|f_{xxx}|^2 + \epsilon\mu_4.$$

Integrating both sides in t from 0 to T, we find

$$\epsilon^2\int_0^T [u^\epsilon_{xx}]_2^2\,dt \le \frac{\epsilon}{2}|u_{0xx}|^2 + C'\mu_4^{5/8}\int_0^T \epsilon|\nabla u^\epsilon_{xx}|^{7/4}\,dt + \epsilon|f_{xxx}|^2_{L^2(0,T;L^2(\mathcal{M}))} + \epsilon\mu_4 T. \quad (5.38)$$

By (5.24) we see that $\int_0^T \epsilon\,|\nabla u^\epsilon_{xx}|^{7/4}\,dt \le C'\int_0^T \epsilon\left(|\nabla u^\epsilon_{xx}|^2 + 1\right)dt \le const := \mu_6$. This along with (5.38) implies (5.34).

Now since

$$\int_{\mathcal{M}} (uu_x)^{3/2}\,d\mathcal{M} \le C'|u|^{3/2}_{L^6(\mathcal{M})}|u_x|^{3/2} \le (H^1(\mathcal{M}) \subset L^6(\mathcal{M}) \text{ in } 3D) \le C'|u|^3_{H^1},$$

this along with (5.23) implies (5.35), and hence

$$u^\epsilon u^\epsilon_x \text{ is bounded in } L^{3/2}(I_x; L^{3/2}((0,T)\times I_{x^\perp})). \quad (5.39)$$

Finally rewriting (5.4) we find

$$u^\epsilon_{xxx} = -u^\epsilon_t - \Delta^\perp u^\epsilon_x - cu^\epsilon_x - u^\epsilon\,u^\epsilon_x - \epsilon\,u^\epsilon_{xxxx} - \epsilon\,u^\epsilon_{yyyy} - \epsilon\,u^\epsilon_{zzzz}. \quad (5.40)$$

Thanks to (5.34), we see that $\epsilon u^\epsilon_{xxxx}$ remains bounded in $L^2(0,T;\,L^2(\mathcal{M}))$. Moreover since u^ϵ remains bounded in $L^\infty(0,T;H^1(\mathcal{M}))$, we find that each term on the right-hand side of (5.40) except for $u^\epsilon\,u^\epsilon_x$ remains bounded at least in the space $L^2(I_x; H_t^{-1}(0,T;\,H^{-4}(I_{x^\perp})))$. This together with (5.39) implies that each term on the right-hand side of (5.40) remains bounded at least in $L^{3/2}(I_x; H_t^{-1}(0,T;\,H^{-4}(I_{x^\perp})))$. Thus we obtain (5.36) from (5.40). □

5.3 The Main Result

Using a compactness argument, we can pass to the limit in (5.4) and obtain (1.1), with a function $u \in \mathcal{C}([0,T];H^1(\mathcal{M}))$. Moreover, from (5.36) we infer that u^ϵ_{xxx} converges weakly in the space $L^{3/2}(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp})))$, hence by the trace theorem and Mazur's theorem, we deduce that $u^\epsilon_{x^j}(0, x^\perp, t)$ and $u^\epsilon_{x^j}(1, x^\perp, t)$ converge weakly in $H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp}))$, $j = 1,\ 2$. Thus from (5.7) we obtain (5.2).

Now we are ready to state the main result of the article by collecting all the previous estimates.

Theorem 5.3.1. *The assumptions are the same as in Proposition 5.2.2, that is, (5.9), (5.10), (5.21), (5.22), (5.32), (5.33), and f and f_{x^j} assume the periodic boundary conditions on $x = 0,\ 1,\ j = 1, 2$. Then the initial and boundary value problem for the ZK equation, that is, (1.1),* (5.1)*, (5.2), (2.3) and (2.4), possesses at least a solution u:*

$$u \in \mathcal{C}([0,T]; H^1(\mathcal{M})) \cap W^{3,3/2}(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp}))). \tag{5.41}$$

Remark 5.3.1. *We can obtain stronger regularity for $\bar{u}(x^\perp, t) := \int_0^1 u(x, x^\perp, t)\, dx$. Integrating (1.1) in x from* 0 *to* 1*, we find by (5.1) and (5.2)*

$$\frac{\partial \bar{u}}{\partial t} = \bar{f}. \tag{5.42}$$

Thus $u = \bar{u} + v$, where $\bar{u}$ satisfies (5.42), and v satisfies $\bar{v} = 0$ and (5.41).

5.4 Discussions about the Uniqueness of Solutions

Let u and v be two solutions of (1.1), (5.1), (5.2), (2.3), and (2.4) and let $w = u - v$. Letting $\bar{w}(x^\perp, t) := \int_0^1 u(x, x^\perp, t)\, dx$, we see that $\dfrac{\partial \bar{w}}{\partial t} = 0$ and hence

$$\bar{w}(t) = 0, \quad \forall\ t \in [0, T]. \tag{5.43}$$

However, it is not clear if we can further prove that $w(t) = 0,\ \ \forall\ t \in [0,T]$. Firstly, the ideas in the proof of existence can not be extended to prove the uniqueness because the structure of the nonlinear term is changed. Secondly, the methods in Chapter 3 are not applicable due to the lack of assumptions on the boundary condition u_x at $x = 1$. For the same reason, the proof of the local existence in Chapter 4 fails as well, which prevents us from using the methods in [7].

To conclude, the uniqueness of solutions in both dimensions 2 and 3 are still open due to the partially hyperbolic feature of this model.

Remark 5.4.1. *As for the periodic case, that is, (1.1) and the boundary and initial conditions (5.1), (5.2), (2.5) and (2.3), the results are the same as in the Dirichlet case. We skip the very similar reasoning.*

Chapter 6

Stochastic ZK Equation

We consider the stochastic ZK equation with multiplicative noise

$$du + (\Delta u_x + cu_x + uu_x)\,dt = f\,dt + \sigma(u)\,dW(t), \tag{6.1}$$

evolving in a rectangular or parallelepiped domain. We assume that f is a deterministic function, and the white noise driven stochastic term $\sigma(u)\,dW(t)$ is in general state dependent.

The boundary conditions are the same as in Chapter 3, that is, we assume (2.2)-(2.5).

Again for the simplicity of the presentation, we will mostly study the Dirichlet case (6.1), (2.2)-(2.4). We will just make some remarks concerning the closely related space periodic case when (2.4) is replaced by (2.5).

6.1 Stochastic Framework

In order to define the term $\sigma(u)\,dW(t)$ in (6.1), we recall some basic notions and notations of stochastic analysis from [11]. For further details and background, see e.g. [37], [18], [17], [2] and [9].

To begin with we fix a stochastic basis

$$\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W_k\}_{k\geq 1}), \tag{6.2}$$

that is a filtered probability space with $\{W_k\}_{k\geq 1}$ a sequence of independent standard one-dimensional Brownian motions relative to $\{\mathcal{F}_t\}_{t\geq 0}$. In order to avoid unnecessary complications below we may assume that $\mathcal{F}_t$ is complete and right continuous (see [9]).

We fix a separable Hilbert space $\mathfrak{U}$ with an associated orthonormal basis $\{e_k\}_{k\geq 1}$. We may formally define W by taking $W = \sum_{k=1}^{\infty} W_k e_k$. As such W is said to be a 'cylindrical Brownian motion' evolving over $\mathfrak{U}$.

We next recall some basic definitions and properties of spaces of Hilbert-Schmidt operators. For this purpose we suppose that X is any separable Hilbert space with the associated norm and inner product written as $|\cdot|_X$, $(\cdot,\cdot)_X$. We denote by

$$L_2(\mathfrak{U},\, X) = \left\{ R \in L(\mathfrak{U},\, X) : \; \sum_k |Re_k|_X^2 < \infty \right\},$$

the space of Hilbert-Schmidt operators from $\mathfrak{U}$ to X. We know that the definition of $L_2(\mathfrak{U},\, X)$ is independent of the choice of the orthonormal basis $\{e_k\}_{k\geq 1}$ in X. By endowing this space with the inner product $\langle R, T\rangle_{L_2(\mathfrak{U},X)} = \sum_k \langle Re_k, Te_k\rangle_X$, we may consider $L_2(\mathfrak{U}, X)$ as itself being a Hilbert space. Again this scalar product can be shown to be independent of the orthonormal basis $\{e_k\}_{k\geq 1}$.

We also define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 := \left\{ v = \sum_{k\geq 0} a_k e_k : \; \sum_k \frac{a_k^2}{k^2} < \infty \right\},$$

endowed with the norm $|v|_{\mathfrak{U}_0}^2 := \sum_k a_k^2/k^2$, $v = \sum_k a_k e_k$. Note that the embedding of $\mathfrak{U} \subset \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, using standard martingale arguments combined with the fact that each W_k is almost surely continuous (see [9]) we obtain that, for almost every $\omega \in \Omega$, $W(\omega) \in \mathcal{C}([0,T], \mathfrak{U}_0)$.

Given an X-valued predictable process $\Psi \in L^2(\Omega;\, L^2((0,T),\, L_2(\mathfrak{U},\, X))$, one may define the Itō stochastic integral

$$M_t := \int_0^t \Psi\, dW = \sum_k \int_0^t \Psi_k\, dW_k,$$

as an element in $\mathcal{M}_X^2$, that is the space of all X-valued square integrable martingales. In the sequel we will use the Burkholder-Davis-Gundy inequality which takes the form

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\int_0^s \Psi dW(t)\right|_X^r\right) \leq c_1\,\mathbb{E}\left[\left(\int_0^T ||\Psi||^2_{L_2(\mathfrak{U},\,X)}\,dt\right)^{r/2}\right], \tag{6.3}$$

valid for any $r \geq 1$. Here c_1 is an absolute constant depending only on r.

Conditions Imposed on σ, f and u_0. Given any pair of Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, we denote by $Bnd_u(\mathcal{X}_1,\,\mathcal{X}_2)$, the collections of all continuous mappings

$$\Psi : \mathcal{X}_1 \to \mathcal{X}_2, \tag{6.4}$$

such that

$$||\Psi(u)||_{\mathcal{X}_2} \leq c_B(1 + ||u||_{\mathcal{X}_1}), \quad u \in \mathcal{X}_1, \tag{6.5}$$

for some constant c_B. In addition, if

$$||\Psi(u) - \Psi(v)||_{\mathcal{X}_2} \leq c_U||u - v||_{\mathcal{X}_1}, \quad \forall u,\, v \in \mathcal{X}_1, \tag{6.6}$$

for some constant c_U, we say that $\Psi \in Lip_u(\mathcal{X}_1,\,\mathcal{X}_2)$. In the sequel we will consider time dependent families of such mappings $\Psi = \Psi(t)$ and require that (6.5) and (6.6) hold for *a.e.* t with the same constants c_B, c_U for all t's under consideration.

We shall assume throughout the work that

$$\sigma : [0,\,\infty) \times L^2(\mathcal{M}) \to L_2(\mathfrak{U},\, L^2(\mathcal{M})). \tag{6.7}$$

Here $\mathfrak{U}$ and $L_2(\mathfrak{U}, L^2(\mathcal{M}))$ are as introduced above. Moreover we assume that for *a.e.* t,

$$\sigma(t) \in Bnd_u(L^2(\mathcal{M}),\, L_2(\mathfrak{U}, L^2(\mathcal{M}))) \cap Bnd_u(\Xi_1,\, L_2(\mathfrak{U},\, \Xi_1)), \tag{6.8}$$

and

$$\sigma(t) \in Lip_u(L^2(\mathcal{M}),\, L_2(\mathfrak{U}, L^2(\mathcal{M}))), \tag{6.9}$$

where

$$\Xi_1 := \left\{ u \in H^2(\mathcal{M}) \cap H_0^1(\mathcal{M}),\ u_x\big|_{x=1} = 0 \right\}. \tag{6.10}$$

When proving pathwise uniqueness of martingale solutions and the existence of pathwise solutions in Section 6.3.2, we will additionally suppose that for a.e. t,

$$\sigma(t) \in Lip_u(L^2(\mathcal{M}),\, L_2(\mathfrak{U},\, \Xi_1)). \tag{6.11}$$

Furthermore in the sequel σ is a measurable function of t and all the corresponding norms of $\sigma(t)$ are essentially (a.e.) bounded in time.

Finally we state the assumptions for the initial condition u_0 and for f. On the one hand, in Section 3.1, where we consider only the case of martingale solutions, since the stochastic basis is an unknown of the problem, we will only be able to specify u_0 as an initial probability measure μ_{u_0} on the space $L^2(\mathcal{M})$ such that

$$\int_{L^2(\mathcal{M})} |u|_{L^2(\mathcal{M})}^6 \, d\,\mu_{u_0}(u) < \infty, \tag{6.12}$$

and we assume that f is deterministic,

$$f = f(x, x^\perp, t) \in L^6(0, T;\, L^2(\mathcal{M})). \tag{6.13}$$

On the other hand, for pathwise uniqueness and the existence of pathwise solutions in Section 6.3.2, where the stochastic basis $\mathcal{S}$ is fixed in advance we assume that, relative to this basis, u_0 is an $L^2(\mathcal{M})$-valued random variable such that

$$u_0 \in L^7(\Omega;\, L^2(\mathcal{M})) \text{ and } u_0 \text{ is } \mathcal{F}_0 \text{ measurable}, \tag{6.14}$$

and f is deterministic,

$$f = f(x, x^\perp, t) \in\ L^7(0, T;\, L^2(\mathcal{M})). \tag{6.15}$$

6.2 Regularized Stochastic ZK Equation

As indicated above we consider the Dirichlet case, i.e. (6.1), (2.2)-(2.4). The domain is $\mathcal{M} = I_x \times (-\pi/2, \pi/2)^d$, in $\mathbb{R}^{d+1}$ with $d = 1$ or 2. In order to study this system, we will use a parabolic regularization of equation (6.1), as in the previous chapters. That is, for $\epsilon > 0$ "small", we consider the stochastic parabolic equation of the 4-th order in space:

$$\begin{cases} du^\epsilon + \left[\Delta u_x^\epsilon + cu_x^\epsilon + u^\epsilon u_x^\epsilon + \epsilon\left(\frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4}\right)\right] dt = f^\epsilon\, dt + \sigma(u^\epsilon)\, dW(t), \\ u^\epsilon(0) = u_0^\epsilon, \end{cases} \tag{6.16}$$

supplemented with the boundary conditions (2.2), (2.4) and the additional boundary conditions

$$u_{yy}^\epsilon\big|_{y=\pm\frac{\pi}{2}} = u_{zz}^\epsilon\big|_{z=\pm\frac{\pi}{2}} = 0, \tag{6.17}$$

$$u_{xx}^\epsilon\big|_{x=0} = 0. \tag{6.18}$$

In the case of martingale solutions in Section 3.1, observing that the space $L^2(\Omega;\, \Xi_1) \cap L^{22/3}(\Omega;\, L^2(\mathcal{M}))$ is dense in $L^6(\Omega;\, L^2(\mathcal{M}))$, we can use e.g. the Fourier series to construct an approximate family $\{u_0^\epsilon\}_{\epsilon>0}$ which is $\mathcal{F}_0$ measurable, such that, as $\epsilon \to 0$:

$$u_0^\epsilon \in L^2(\Omega;\, \Xi_1) \cap L^{22/3}(\Omega;\, L^2(\mathcal{M})), \tag{6.19}$$

$$u_0^\epsilon \to u_0 \text{ in } L^6(\Omega;\, L^2(\mathcal{M})). \tag{6.20}$$

Similarly there exists a family of deterministic functions $\{f^\epsilon\}_{\epsilon>0}$ such that as $\epsilon \to 0$:

$$f^\epsilon \in L^{22/3}(0,T;\, L^2(\mathcal{M})), \tag{6.21}$$

$$f^\epsilon \to f \text{ in } L^6(0,T;\, L^2(\mathcal{M})). \tag{6.22}$$

In the case of pathwise solutions in Section 6.3.2, in the same way we can deduce the existence of the approximate families $\{u_0^\epsilon\}_{\epsilon>0}$ and $\{f^\epsilon\}_{\epsilon>0}$ satisfying (6.19) and (6.21) respectively, and such that as $\epsilon \to 0$:

$$u_0^\epsilon \to u_0 \text{ in } L^7(\Omega;\, L^2(\mathcal{M})), \tag{6.23}$$

$$f^\epsilon \to f \text{ in } L^7(0,T;\, L^2(\mathcal{M})). \tag{6.24}$$

For notational convenience, as in Chapter 3, we recast (6.16) in the form

$$\begin{cases} du^\epsilon = (-Au^\epsilon - B(u^\epsilon) - \epsilon\, Lu^\epsilon + f^\epsilon)\, dt + \sigma(u^\epsilon)\, dW(t), \\ u^\epsilon(0) = u_0^\epsilon, \end{cases} \tag{6.25}$$

where

$$\begin{aligned} &Au = \Delta u_x + cu_x, && \forall\, u \in D(A), \\ &B(u,v) = uv_x \in H^{-1}(\mathcal{M}), && \forall\, u \in L^2(\mathcal{M}),\ v \in H^1(\mathcal{M}), \\ &Lu = u_{xxxx} + u_{yyyy} + u_{zzzz}, && \forall\, u \in H^4(\mathcal{M}). \end{aligned} \tag{6.26}$$

with $D(A) = \big\{u \in L^2(\mathcal{M}):\ Au \in L^2(\mathcal{M}),\ u = 0 \text{ on } \partial\mathcal{M}, u_x = 0\ at\ x = 1\big\}$. We recall Theorem 2.2.1, which shows that if $u \in L^2(\mathcal{M})$ and $Au \in L^2(\mathcal{M})$ then the traces of u on $\partial\mathcal{M}$ and of u_x at $x = 1$ make sense.

Remark 6.2.1. *As mentioned in the Introduction, although we can rewrite (6.1) as*

$$du + (Au + B(u))\, dt = f\, dt + \sigma(u)\, dW(t), \tag{6.27}$$

which is similar to the equation studied in [11], the models are actually different. Indeed, the operator A does not satisfy the assumptions in [11]; for example, A is not symmetric. In fact, for the adjoint A^* and its domain $D(A^*)$, we have

$$\begin{aligned} D(A^*) &= \{\bar{u} \in L^2(\mathcal{M}):\ A\bar{u} \in L^2(\mathcal{M}),\, \bar{u} = 0\ on\ \partial\mathcal{M},\, \bar{u}_x = 0\ at\ x = 0\}, \\ A^*\bar{u} &= -(\Delta\bar{u}_x + c\bar{u}_x),\ \ \bar{u} \in D(A^*). \end{aligned} \tag{6.28}$$

For more details see Section 2.3.2.

6.2.1 Definition of Solutions

We first introduce the necessary operators and functional spaces. We will denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm of $L^2(\mathcal{M})$. The space Ξ_1 defined in (6.10) is endowed with the scalar product and norm $[\cdot,\cdot]_2$, $[\cdot]_2$:

$$[u,v]_2 = (u_{xx}, v_{xx}) + (u_{yy}, v_{yy}) + (u_{zz}, v_{zz}),$$

$$[u]_2^2 = |u_{xx}|^2 + |u_{yy}|^2 + |u_{zz}|^2, \tag{6.29}$$

which make it a Hilbert space. Note that since $|\Delta u| + |u|$ is a norm on $H_0^1 \cap H^2$ equivalent to the H^2-norm, $[\cdot]_2$ is a norm on Ξ_1 equivalent to the H^2-norm. Thanks to the Riesz theorem, we can associate to the scalar product $[\cdot,\cdot]_2$ the isomorphism $\mathcal{L}$ from Ξ_1 onto $\Xi_1{}'$, where $\mathcal{L}$ denotes the abstract operator corresponding to the differential operator L. Then considering the Gelfand triple $\Xi_1 \subset H := L^2(\mathcal{M}) \subset \Xi_1{}'$, we introduce $\mathcal{L}^{-1}(H)$ the domain of $\mathcal{L}$ in H, which is the space

$$\Xi_2 = \left\{ u \in \Xi_1 \cap H^4(\mathcal{M}),\ u_{yy}\big|_{y=\pm\frac{\pi}{2}} = u_{zz}\big|_{z=\pm\frac{\pi}{2}} = u_{xx}\big|_{x=0} = 0 \right\}. \tag{6.30}$$

The operator $\mathcal{L}^{-1}$ is self adjoint and compact in H. It possesses an orthonormal set of eigenvectors which is complete in H, and which we denote by $\{\phi_i\}_{i\geq 1}$. Note that all the ϕ_i belong to Ξ_2 which is the domain of $\mathcal{L}$ in H. Hence we have

$$(\mathcal{L}u, v) = [u,v]_2,\ u \in \Xi_2,\ v \in \Xi_1.$$

We now introduce the following definitions.

Definition 6.2.1. *(Global martingale solutions for the regularized ZK equation) Fix an* $\epsilon > 0$. *For the case of martingale solutions, we only specify the measure* $\mu_{u_0^\epsilon}$ *to be the probability measure of* u_0^ϵ *on* Ξ_1 *which satisfies*

$$\int_{L^2(\mathcal{M})} |u|^{22/3}\, d\,\mu_{u_0^\epsilon}(u) < \infty, \tag{6.31}$$

$$\int_{\Xi_1} |u|^2\, d\,\mu_{u_0^\epsilon}(u) < \infty, \tag{6.32}$$

and f^ϵ *and* σ *satisfy (6.21), (6.8) and (6.9) respectively.*

A pair $(\tilde{\mathcal{S}}, \tilde{u}^\epsilon)$ *is a global martingale solution to the regularized stochastic ZK equation (6.16)-(6.18), (2.2) and (2.4) (in the Dirichlet case), if* $\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}, \{\tilde{W}^k\}_{k\geq 1})$ *is a stochastic basis, and* $\tilde{u}^\epsilon(\cdot) : \tilde{\Omega} \times [0,\infty) \to \Xi_1$ *is an* $\{\tilde{\mathcal{F}}_t\}$ *adapted process such that:*

$$\tilde{u}^\epsilon \in L^{22/3}(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^\infty([0,T];\, \Xi_1) \cap L^2(0,T;\, \Xi_2)), \tag{6.33}$$

and

$$\tilde{u}^\epsilon(\cdot,\omega) \in \mathcal{C}([0,T];\, L^2_w(\mathcal{M}))\ \tilde{\mathbb{P}} - a.s., \tag{6.34}$$

where $L^2_w(\mathcal{M})$ *is* $L^2(\mathcal{M})$ *equipped with the weak topology, and the law of* $\tilde{u}^\epsilon(0)$ *is* $\mu_{u_0^\epsilon}$*, defined as above, i.e.* $\mu_{u_0^\epsilon}(E) = \tilde{\mathbb{P}}(\tilde{u}^\epsilon(0) \in E)$*, for all Borel subsets* E *of* Ξ_1*, and finally* $\tilde{u}^\epsilon$ *almost surely satisfies*

$$\tilde{u}^\epsilon(t) + \int_0^t (A\tilde{u}^\epsilon + B(\tilde{u}^\epsilon) + \epsilon L\tilde{u}^\epsilon - f^\epsilon)\, ds = \tilde{u}^\epsilon(0) + \int_0^t \sigma(\tilde{u}^\epsilon)\, d\tilde{W}, \tag{6.35}$$

as an equation in $L^2(\mathcal{M})$ *for every* $0 \leq t \leq T$*.*

Definition 6.2.2. *(Global pathwise solutions for the regularized ZK equation; Uniqueness)*

Let $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W^k\}_{k\geq 1})$ *be a fixed stochastic basis and assume that* u_0^ϵ*,* σ *and* f^ϵ *satisfy (6.19), (6.8), (6.9) and (6.21).*

(i) For any fixed $\epsilon > 0$*, a random process* u^ϵ *is a global pathwise solution to (6.16)-(6.18), (2.2) and (2.4) if* u^ϵ *is an* $\mathcal{F}_t$ *adapted process in* $L^2(\mathcal{M})$ *so that (relative to the fixed-given-basis* $\mathcal{S}$*) (6.33)-(6.35) hold.*

(ii) Global pathwise solutions of (6.16)-(6.18), (2.2) and (2.4) are said to be global (pathwise) unique if given any pair of pathwise solutions u^ϵ*,* v^ϵ *which coincide at* $t = 0$ *on a subset* Ω_0 *of* Ω*,* $\Omega_0 = \{u^\epsilon(0) = v^\epsilon(0)\}$*, then*

$$\mathbb{P}\{\mathbf{1}_{\Omega_0}(u^\epsilon(t) = v^\epsilon(t))\} = 1,\ 0 \leq t \leq T. \tag{6.36}$$

In the sequel, we will prove that there exists a unique global pathwise solution u^ϵ to (6.16)-(6.18), (2.2) and (2.4), which is sufficiently regular for the calculations in Section 3.1

to be fully legitimate without any need of further regularization. The existence of such a solution is basically classical (see e.g. [9], [17], [18] and [11]) for a parabolic problem like this, but we will make partly explicit the construction of u^ϵ because we need to see how the estimates depend or not on ϵ.

6.2.2 Pathwise Solutions in Dimensions 2 and 3

With the above definitions, we can state the main result of section 6.2:

Theorem 6.2.1. *When $d = 1, 2$, suppose that, relative to a fixed given stochastic basis $\mathcal{S}$, u_0^ϵ satisfies (6.19), and that f^ϵ and σ satisfy (6.21), (6.8) and (6.9), with $\epsilon > 0$ fixed arbitrary. Then there exists a unique global pathwise solution u^ϵ which satisfies (6.16) and the boundary conditions (2.2), (2.4), (6.17) and (6.18).*

To prove this theorem, we first use a Galerkin scheme to derive the estimates indicating a compactness argument based on fractional Sobolev spaces and tightness properties of the truncated sequence. Then by the Skorokhod embedding theorem (see Theorem 2.4 in [9], also [3] and [26]) we can pass to the limit in the Galerkin truncation and hence obtain the global existence of martingale solutions. Finally we deduce the existence of global pathwise solutions using pathwise uniqueness of martingale solutions and the Gyöngy-Krylov Theorem (Theorem 7.5.1 of the Appendix). Here we will only present in details the derivation of the estimates, which will be utilized in the subsequent investigations of the stochastic ZK equation in Section 6.3.1.

We start the proof of Theorem 6.2.1 by introducing the Galerkin system. We define P^n as the orthogonal projector from $L^2(\mathcal{M})$ onto H^n, the space spanned by the first n eigenfunctions of $\mathcal{L}$, ϕ_1, ..., ϕ_n. We consider the Galerkin system as follows

$$\begin{cases} du^n + (A^n u^n + B^n(u^n))\, dt + \epsilon L u^n \, dt = f^n \, dt + \sigma^n(u^n)\, dW(t), \\ u^n(0) = P_n u_0, \end{cases} \tag{6.37}$$

where u^n maps $\Omega \times [0, T]$ into H^n, $A^n u^n := P^n A u^n$, $B^n(u^n) := P^n B(u^n)$, and $\sigma^n(u^n) := P^n(\sigma(u^n))$. In (6.37), ϵ being fixed, we write for simplicity u^n for $u^{\epsilon,n}$ and f^n for $f^{\epsilon,n}$.

Equation (6.37) is equivalent to a system of n stochastic differential equations for the components of u^n and it is a classical result that there exists a unique regular pathwise solution $u^n = u^{\epsilon,n}$ such that

$$u^n \in L^2(\Omega;\, \mathcal{C}(0,T;\, H^n)). \tag{6.38}$$

Estimates Independent of ϵ and n

We first derive the following estimates on $u^{\epsilon,n}$ independent of ϵ and n.

Lemma 6.2.1. *With the same assumptions as in Theorem 6.2.1, if u_0^ϵ and f^ϵ satisfy (6.20) and (6.22) respectively, then the following estimates hold for $u^n = u^{\epsilon,n}$ independently of ϵ and n:*

$$u_x^{\epsilon,n}\big|_{x=0} \text{ remains bounded in } L^2(\Omega;\, L^2(0,T;\, L^2(I_{x^\perp}))), \tag{6.39}$$

$$\sqrt{\epsilon}u^{\epsilon,n} \text{ remains bounded in } L^2(\Omega;\, L^2(0,T;\, \Xi_1)), \tag{6.40}$$

$$u^{\epsilon,n} \text{ remains bounded in } L^6(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))). \tag{6.41}$$

If we further assume that (6.23) and (6.24) hold, then

$$u^{\epsilon,n} \text{ remains bounded in } L^7(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))), \tag{6.42}$$

with the bounds in (6.39)-(6.42) independent of both ϵ and n.

Proof. We start by applying the Itō formula to (6.37). This yields

$$d|u^n|^2 = 2\,(u^n,\, \mathcal{N}^n(u^n))\; dt + 2\,(u^n,\, \sigma^n(u^n)\, dW(t)) + ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\, dt, \tag{6.43}$$

where $\mathcal{N}^n(u^n) := -A^n u^n - B^n(u^n) - \epsilon\, L u^n + f^n$, and

$$\begin{aligned}(u^n,\, \mathcal{N}^n(u^n)) &= -(u^n, A^n u^n) - (u^n, B^n(u^n)) - \epsilon(u^n,\, Lu^n) + (u^n, f^n)\\ &= -(u^n, Au^n) - (u^n, B(u^n)) - \epsilon(u^n,\, Lu^n) + (u^n, f^n).\end{aligned} \tag{6.44}$$

To compute the right-hand side of (6.44) we remember that by (6.38), for every t, $u^n(t) \in H^n = span\ (\phi_1, ..., \phi_n) \in \Xi_2$ a.s., since all the ϕ_i belong to Ξ_2. Hence in particular $u^n(t)$ satisfies the boundary conditions in (6.30). We drop the super index n for the moment and perform the following calculations a.s. exactly as in Chapter 3, then we find

$$\begin{aligned} d\,|u^n|^2 &+ \left(\left|u^n_x\right|_{x=0}\Big|^2_{L^2(I_{x^\perp})} + 2\epsilon[u^n]_2^2 \right) dt \\ &= 2(f^n,\, u^n)\, dt + ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt + 2\,(u^n,\, \sigma^n(u^n)\, dW(t))\,. \end{aligned} \tag{6.45}$$

Integrating both sides from 0 to s with $0 \le s \le r \le T$, taking the supremum over $[0, r]$, we have

$$\begin{aligned} \sup_{0\le s\le r} |u^n(s)|^2 &+ \int_0^r \left(\left|u^n_x\right|_{x=0}\Big|^2_{L^2(I_{x^\perp})} + 2\epsilon[u^n]_2^2 \right) dt \\ &\le |u^n_0|^2 + 2\int_0^r |(f^n,\, u^n)|\, dt + \int_0^r ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt \\ &\quad + 2\sup_{0\le s\le r}\left| \int_0^s (u^n,\, \sigma^n(u^n)\, dW(t)) \right|. \end{aligned} \tag{6.46}$$

Raising both sides to the power $p/2$ for $p \ge 2$, then taking expectations, we obtain with the Minkowski inequality and Fubini's Theorem

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le r} |u^n(s)|^p &\lesssim \mathbb{E}|u^n_0|^p + 2\,\mathbb{E}\int_0^r |(f^n,\, u^n)|^{p/2}\, dt \\ &\quad + \mathbb{E}\int_0^r ||\sigma^n(u^n)||^p_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt \\ &\quad + 2\mathbb{E}\left(\sup_{0\le s\le r}\left| \int_0^s (u^n,\, \sigma^n(u^n)\, dW(t)) \right| \right)^{p/2}, \end{aligned} \tag{6.47}$$

where $\lesssim$ means $\le$ up to an *absolute* multiplicative constant. Here and below c' indicates an absolute constant, whereas η, κ, and the κ_i indicate constants depending on the data u_0, f, etc. These constants may be different at each occurrence. We estimate the terms on the right-hand side of (6.47) a.s. and for a.e. t:

$$|(f^n,\, u^n)|^{p/2} \le |f^n|^{p/2}|u^n|^{p/2} \le |u^n|^p + |f^n|^p,$$

$$||\sigma^n(u^n)||^p_{L_2(\mathfrak{U}, L^2(\mathcal{M}))} \le \text{ by (6.8) } \le c_B^p(|u^n| + c')^p \lesssim |u^n|^p + c';$$

for the stochastic term, we use the Burkholder-Davis-Gundy inequality (see (6.3)) and (6.8)

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le r}\left|\int_0^s (u^n,\, \sigma^n(u^n)\, dW(t))\right|\right)^{p/2} \\
&\le \mathbb{E}\sup_{0\le s\le r}\left|\int_0^s (u^n,\, \sigma^n(u^n)\, dW(t))\right|^{p/2} \\
&\le c_1\,\mathbb{E}\left[\left(\int_0^r |u^n|^2\, \|\sigma^n(u^n)\|^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\, dt\right)^{p/4}\right] \\
&\lesssim \mathbb{E}\left[\left(\sup_{0\le s\le r}|u^n|^2\int_0^r 1+|u^n|^2\, dt\right)^{p/4}\right] \\
&\le \frac{1}{2}\,\mathbb{E}\sup_{0\le s\le r}|u^n|^p + c'\,\mathbb{E}\int_0^r |u^n|^p\, dt + c'.
\end{aligned}$$

Applying the above estimates to (6.47), we obtain

$$\frac{1}{2}\,\mathbb{E}\sup_{0\le s\le r}|u^n(s)|^p \le \mathbb{E}\,|u_0^n|^p + c'\,\mathbb{E}\int_0^r |u^n(t)|^p\, dt + \mathbb{E}\int_0^r |f^n(t)|^p\, dt + c'.^{1} \tag{6.48}$$

Since $\mathbb{E}\int_0^r |u^n(t)|^p\, dt \le \int_0^r \mathbb{E}\sup_{0\le l\le t}|u^n(l)|^p\, dt$, setting $\mathbb{E}\sup_{0\le s\le r}|u^n(s)|^p =: U(r)$, with (6.48) we deduce

$$U(r) \le U(0) + c'\int_0^r U(t)dt + \int_0^r \mathbb{E}\,|f^n(t)|^p\, dt + c',$$

for every $0 \le r \le T$. Hence applying the (deterministic) Gronwall lemma, we obtain for $p \ge 2$,

$$\mathbb{E}\sup_{0\le r\le T}|u^n(r)|^p \lesssim \mathbb{E}|u_0^n|^p + \mathbb{E}\int_0^T |f^n(t)|^p\, dt + c'. \tag{6.49}$$

Letting $p = 6$, thanks to (6.20) and (6.22), we deduce that

$$\mathbb{E}\sup_{0\le r\le T}|u^n(r)|^6 \le \kappa_1, \tag{6.50}$$

for a constant κ_1 depending only on u_0, f, T and σ, and independent of ϵ and n; this implies (6.41). Similarly, setting $p = 7$ in (6.49), we infer (6.42) from (6.23) and (6.24). Finally, setting $p = 2$ in (6.48), along with (6.46) we obtain (6.39) and (6.40). □

[1] Note that here $f^{\epsilon,n}$ is actually independent of $\omega \in \Omega$ and the symbol $\mathbb{E}$ in front of the corresponding term is not needed. However in Section 6.3.2 we will use another version of this calculation in which $f^{\epsilon,n}$ is replaced by g^{ϵ} which depends on ω; hence we leave $\mathbb{E}$ in front of the term involving $f^{\epsilon,n}$ in view of the calculations in Section 6.3.2.

Estimates Dependent on ϵ

We now derive estimates independent of n only, that is, valid for fixed ϵ.

Lemma 6.2.2. *With the same assumptions as in Theorem 6.2.1, the following estimates hold for $u^n = u^{\epsilon,n}$, as $n \to \infty$ and $\epsilon > 0$ remains fixed:*

$$u^{\epsilon,n} \text{ remains bounded in } L^{22/3}(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))), \tag{6.51}$$

$$u^{\epsilon,n} \text{ remains bounded in } L^2(\Omega;\, L^\infty(0,T;\, \Xi_1)), \tag{6.52}$$

$$u^{\epsilon,n} \text{ remains bounded in } L^2(\Omega;\, L^2(0,T;\, \Xi_2)). \tag{6.53}$$

Proof. Setting $p = 22/3$ in (6.49), we infer (6.51) from (6.19) and (6.21).

Returning to (6.37), we apply the Itō formula to (6.37) and obtain an evolution equation for the Ξ_1 norm:

$$d[u^n]_2^2 = 2\,(Lu^n,\, \mathcal{N}(u^n))\; dt + 2\,(Lu^n,\, \sigma^n(u^n)\, dW(t)) + ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U},\, \Xi_1)}\, dt, \tag{6.54}$$

where $\mathcal{N}^n(u^n)$ has been defined before. Similar to (6.44) we have a.s. and for a.e. t:

$$(Lu^n,\, \mathcal{N}^n(u^n)) = -(Lu^n, Au^n) - (Lu^n, B(u^n)) - \epsilon|Lu^n|^2 + (Lu^n, f^n). \tag{6.55}$$

By (6.54) and (6.55) we deduce

$$\begin{aligned} d[u^n]_2^2 + \epsilon|Lu^n|^2\, dt = &- 2(Lu^n, Au^n)\, dt - 2(Lu^n, B(u^n))\, dt + 2(Lu^n, f^n)\, dt \\ &+ 2\,(Lu^n,\, \sigma^n(u^n)\, dW(t)) + ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U},\, \Xi_1)}\, dt. \end{aligned} \tag{6.56}$$

Integrating both sides from 0 to s with $0 \le s \le T$, taking the supremum over $[0,T]$, then taking expectations, we arrive at

$$\begin{aligned} &\mathbb{E} \sup_{0\le s\le T} [u^n]_2^2 + \epsilon\, \mathbb{E} \int_0^T |Lu^n|^2\, dt \\ \le\, &\mathbb{E}\, |u_0^n|^2 + 2\,\mathbb{E} \int_0^T |(Lu^n, Au^n)|\; dt + 2\,\mathbb{E} \int_0^T |(Lu^n, B(u^n))|\; dt + 2\,\mathbb{E} \int_0^T |(Lu^n, f^n)|\; dt \\ &+ 2\,\mathbb{E} \sup_{0\le s\le T} \int_0^s (Lu^n,\, \sigma^n(u^n)\, dW(t)) + \mathbb{E} \int_0^T ||\sigma^n(u^n)||^2_{L_2(\mathfrak{U},\, \Xi_1)}\, dt. \end{aligned} \tag{6.57}$$

We estimate a.s. each term on the right-hand side of (6.57); we emphasize that the estimates depend on ϵ but not on n or on $\omega \in \Omega$:

$$
\begin{aligned}
|(Lu^n, Au^n)| &\leq |Lu^n||u^n|_{H^3(\mathcal{M})} \leq |Lu^n||u^n|_{H^2}^{1/4}|u^n|_{H^4}^{3/4} \\
&\leq |Lu^n|^{7/4}[u^n]_2^{1/4} \leq \frac{\epsilon}{4}|Lu^n|^2 + \eta(\epsilon)[u^n]_2^2,
\end{aligned}
$$

where $\eta(\epsilon)$ depends on ϵ. For the term $|(Lu^n, B(u^n))|$, we first estimate a.s. $|B(u^n)|$ in dimension three:

$$
\begin{aligned}
|u^n u_x^n| &\leq |u^n|_{H_0^1(\mathcal{M})}^{3/2}|u^n|_{H^2(\mathcal{M})}^{1/2} \\
&\leq (\text{by interpolation in dimension three, } |u^n|_{H_0^1(\mathcal{M})} \lesssim |u^n|^{3/4}|u^n|_{H^4(\mathcal{M})}^{1/4}, \\
&\qquad \text{and } |u^n|_{H^2(\mathcal{M})} \lesssim |u^n|^{1/2}|u^n|_{H^4(\mathcal{M})}^{1/2}) \qquad (6.58) \\
&\lesssim |u^n|^{9/8}|u^n|_{H^4(\mathcal{M})}^{3/8}|u^n|^{1/4}|u^n|_{H^4(\mathcal{M})}^{1/4} \\
&\lesssim |u^n|^{11/8}|Lu^n|^{5/8}.
\end{aligned}
$$

Hence

$$
|(Lu^n, B(u^n))| \leq |B(u^n)|\,|Lu^n| \leq (\text{by (6.58)}) \lesssim |u^n|^{11/8}|Lu^n|^{13/8} \lesssim \frac{\epsilon}{4}|Lu^n|^2 + \eta(\epsilon)|u^n|^{22/3},
$$

where $\eta(\epsilon)$ depends on ϵ.

For the stochastic term, we have

$$
\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq T}\left|\int_0^s (Lu^n,\, \sigma^n(u^n)\, dW(t))\right| &\leq (\text{by the Burkholder-Davis-Gundy inequality (6.3)}) \\
&\leq c_1\, \mathbb{E}\left[\left(\int_0^T |Lu^n|^2\, ||\sigma^n(u^n)||_{L_2(\mathfrak{U},L^2(\mathcal{M}))}^2\, dt\right)^{1/2}\right] \\
&\leq c_1 c_B^2\, \mathbb{E}\left[\left(\int_0^T |Lu^n|^2\,(1+|u^n|^2)\, dt\right)^{1/2}\right] \\
&\leq \eta(\epsilon)\, \mathbb{E} \sup_{0\leq s\leq T} |u^n|^2 + \frac{\epsilon}{4}\, \mathbb{E}\int_0^T |Lu^n|^2\, dt + \eta(\epsilon),
\end{aligned}
$$

where $\eta(\epsilon)$ depends on ϵ.

For the term $\mathbb{E}\int_0^T ||\sigma^n(u^n)||_{L_2(\mathfrak{U},\,\Xi_1)}^2\, dt$, we infer from (6.8) that

$$
\mathbb{E}\int_0^T ||\sigma^n(u^n)||_{L_2(\mathfrak{U},\,\Xi_1)}^2\, dt \lesssim \mathbb{E}\int_0^T \left(1+[u^n]_2^2\right)\, dt.
$$

Collecting all the above estimates, along with (6.57) we deduce

$$
\begin{aligned}
\frac{1}{2}\,\mathbb{E}\sup_{0\le s\le T}[u^n(s)]_2^2 &+ \frac{\epsilon}{4}\,\mathbb{E}\int_0^T |Lu^n|^2\,dt \\
&\le \mathbb{E}\,[u_0^n]_2^2 + \eta(\epsilon)\,\mathbb{E}\int_0^T [u^n(t)]_2^2\,dt + \eta(\epsilon)\,\mathbb{E}\int_0^T |u^n(t)|^{22/3}\,dt \\
&\quad + \eta(\epsilon)\,\mathbb{E}\sup_{0\le s\le T}|u^n|^2 + \eta(\epsilon)\,\mathbb{E}\int_0^T |f^n(t)|^2\,dt + \eta(\epsilon).
\end{aligned}
\tag{6.59}
$$

Hence we can apply (6.51) and (6.40) to (6.59), and we obtain (6.52) and (6.53). Thus we have completed the proof of Lemma 6.2.2. □

Estimates in Fractional Sobolev Spaces.

We will apply the compactness result based on fractional Sobolev spaces in Lemma 7.3.1 (of the Appendix) with

$$
\mathcal{Y} := L^2(0,T;\,H_0^1(\mathcal{M})) \cap W^{\alpha,2}(0,T;\,\Xi_2'), \quad 0<\alpha<\frac{1}{2}, \tag{6.60}
$$

where Ξ_2' is the dual of Ξ_2 relative to $L^2(\mathcal{M})$. For that purpose we will need the following estimates on fractional derivatives of $u^{\epsilon,n}$.

Lemma 6.2.3. *With the same assumptions as in Theorem 6.2.1, we have*

$$
\mathbb{E}|u^{\epsilon,n}|_{\mathcal{Y}} \le \kappa_2(\epsilon), \tag{6.61}
$$

$$
\mathbb{E}\left|u^{\epsilon,n}(t) - \int_0^t \sigma^n(u^{\epsilon,n})\,dW(s)\right|^2_{H^1(0,T;\,\Xi_2')} \le \kappa_3, \tag{6.62}
$$

$$
\mathbb{E}\left|\int_0^t \sigma(u^{\epsilon,n})\,dW(s)\right|^2_{W^{\alpha,6}(0,T;\,L^2(\mathcal{M}))} \le \kappa_4, \quad \forall\,\alpha<\frac{1}{2}, \tag{6.63}
$$

where $\kappa_2(\epsilon)$ is independent of n (but may depend on ϵ and other data), while κ_3 and κ_4 depend only on u_0, f, T and σ, and are independent of ϵ and n.

Proof. We can rewrite (6.37) as

$$
\begin{aligned}
u^n(t) =& u_0^n - \int_0^t A^n u^n ds - \int_0^t B^n(u^n) ds \\
& - \epsilon\int_0^t Lu^n ds + \int_0^t f^n ds + \int_0^t \sigma^n(u^n) dW(s) \\
:=& J_1^n + J_2^n + J_3^n + J_4^n + J_5^n + J_6^n.
\end{aligned}
\tag{6.64}
$$

For J_2^n, fixing $u^\sharp \in D(A^*)$ we have a.s. and for a.e. t

$$\left|(A^n u^n,\, u^\sharp)\right| = \left|(u^n,\, A^* P^n u^\sharp)\right| \le |u^n| \left|P^n u^\sharp\right|_{D(A^*)} \le (\text{since } \Xi_2 \subseteq D(A^*)) \le |u^n||u^\sharp|_{\Xi_2}.$$

Hence

$$|A^n u^n|_{\Xi_2'} \lesssim |u^n|. \tag{6.65}$$

With (6.65) and (6.41) we obtain

$$\mathbb{E}\,|J_2^n|^6_{W^{1,6}(0,T;\,\Xi_2')} \text{ is bounded independently of } n \text{ and } \epsilon. \tag{6.66}$$

For J_3^n, firstly we observe that $\forall\ u^\sharp \in \Xi_2$ (dropping the super index n for the moment),

$$\begin{aligned}
\left|(B(u),\, u^\sharp)\right| &= \left|\int_{\mathcal{M}} \frac{\partial}{\partial x}(\frac{u^2}{2})\, u^\sharp \, d\,\mathcal{M}\right| = \frac{1}{2}\left|\int_{\mathcal{M}} u^2 u^\sharp_x \, d\,\mathcal{M}\right| \\
&\le \frac{1}{2}|u|^2 |u^\sharp_x|_{L^\infty(\mathcal{M})} \\
&\le (\text{with } H^3(\mathcal{M}) \subset L^\infty(\mathcal{M}) \text{ in dimension 3}) \\
&\lesssim |u|^2 |u^\sharp_x|_{H^3(\mathcal{M})} \\
&\lesssim |u|^2 |u^\sharp|_{\Xi_2};
\end{aligned} \tag{6.67}$$

hence

$$|(B^n(u^n),\, u^\sharp)| = |(B(u^n),\, P^n u^\sharp)| \lesssim |u^n|^2 |P^n u^\sharp|_{\Xi_2} \le |u^n|^2 |u^\sharp|_{\Xi_2}, \tag{6.68}$$

which implies that $|B^n(u^n)|_{\Xi_2'} \lesssim |u^n|^2$. This along with (6.41) implies that

$$\mathbb{E}|B^n(u^n)|^2_{L^2(0,T;\,\Xi_2')} \text{ is bounded independently of } n \text{ and } \epsilon, \tag{6.69}$$

and hence

$$\mathbb{E}|J_3^n|^2_{H^1(0,T;\,\Xi_2')} \text{ is bounded independently of } n \text{ and } \epsilon. \tag{6.70}$$

For J_4^n, we have, $\forall\ u^\sharp \in \Xi_2$, $\left|\left(Lu^n,\, u^\sharp\right)\right| = \left|\left(u^n,\, Lu^\sharp\right)\right| \le |u^n||Lu^\sharp|$. Hence $|Lu^n|_{\Xi_2'} \lesssim |u^n|$. Thus

$$\mathbb{E}\int_0^T |Lu^n|^2_{\Xi_2'}\, dt \le 2\,\mathbb{E}\int_0^T |u^n|_2^2\, dt.$$

Multiplying both sides by ϵ^2, we obtain with (6.41)

$$\mathbb{E}|J_4^n|^2_{H^1(0,T;\,\Xi_2')} \text{ is bounded independently of } n \text{ and } \epsilon. \tag{6.71}$$

For J_6^n, Lemma 7.4.2 implies that, $\forall\ \alpha < \dfrac{1}{2}$,

$$\begin{aligned}\mathbb{E}\left|\int_0^t \sigma^n(u^n(s))\,dW(s)\right|^6_{W^{\alpha,6}(0,T;\,L^2(\mathcal{M}))} &\lesssim \mathbb{E}\int_0^t |\sigma^n(u^n(s))|^6_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,ds\\ &\lesssim \mathbb{E}\int_0^t |\sigma(u^n(s))|^6_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,ds\\ &\leq (\text{by } (6.8))\\ &\leq c'\,c_B^6\,\mathbb{E}\int_0^t (1+|u^n|)^6 ds.\end{aligned}$$

This together with (6.41) implies that

$$\mathbb{E}\,|J_6^n|^2_{W^{\alpha,6}(0,T;\,L^2(\mathcal{M}))} \text{ is bounded independently of } n \text{ and } \epsilon,\ \ \forall\ \alpha < \frac{1}{2}. \tag{6.72}$$

Hence we obtain (6.63). Collecting the estimates (6.66) and (6.70)-(6.72), we obtain

$$\mathbb{E}|u^n|_{W^{\alpha,2}(0,T;\,\Xi_2')} \text{ is bounded independently of } n \text{ and } \epsilon, \quad \alpha < \frac{1}{2}. \tag{6.73}$$

By (6.52) we deduce

$$\mathbb{E}|u^n|_{L^2(0,T;\,H_0^1(\mathcal{M}))} \text{ is bounded independently of } n, \tag{6.74}$$

but the bounds may depend on ϵ. From (6.73) and (6.74) we obtain (6.61).

Observing from (6.64) that $u^n(t) - \int_0^t \sigma^n(u^n)\,dW(s) = J_1^n + J_2^n + J_3^n + J_4^n + J_5^n$, and applying (6.66), (6.70) and (6.71), we obtain (6.62) as desired. □

Remark 6.2.2. *See Lemma 6.3.3 below for a variant of the proof of Lemma 6.2.3 leading to the analogue of bounds in (6.61)-(6.63) but independent of ϵ. Note however that the proof in Lemma 6.3.3 for u^ϵ can not be applied here to $u^{\epsilon,n}$, because multiplication by $\sqrt{1+x}$ does not commute with P^n, which prevents us from deducing for now the estimates derived from (6.84) below.*

Proof of Theorem 6.2.1. The rest of the proof of Theorem 6.2.1 is classical (see e.g. [18] and [11]). Applying Lemma 7.3.1 (of the Appendix) and Chebychev's inequality to the estimates (6.61)-(6.63), we can use the same technic as that for the proof of Lemma 4.1 in [11] to derive the compactness and tightness properties of the sequences $(u^{\epsilon,n}(t), W(t))$ in n for fixed ϵ. Then we apply the Skorokhod embedding theorem to construct some subsequence $\{(u^{\epsilon,n_k}(t), W(t))\}$ that converges strongly as $n_k \to \infty$, upon shifting the underlying probability basis. Then we pass to the limit on the Galerkin truncation (6.37) as $n_k \to \infty$ (ϵ fixed). Note that we do not need to worry about passing to the limit on the boundary conditions, because they are all well-defined (and conserved) thanks to (6.53). Thus, we have established the existence of martingale solutions to the regularized stochastic ZK equation (6.16)-(6.18), (2.2) and (2.4) in the sense of Definition 6.2.1.

As for the pathwise solutions, we first prove the pathwise uniqueness of martingale solutions, and then by the Gyöngy-Krylov Theorem we obtain the global existence of pathwise solutions in the sense of Definition 6.2.2.

To conclude, we have completed the proof of Theorem 6.2.1. □

We will develop these steps below in more details in the more complicated case when $\epsilon \to 0$.

6.3 Passage to the Limit as $\epsilon \to 0$

We now aim to study the stochastic solutions to the ZK equation basically by passing to the limit as $\epsilon \to 0$ in (6.16) and the boundary conditions (2.2), (2.4), (6.17) and (6.18).

Definition of solutions of the stochastic ZK equation. The definition of the martingale and pathwise solutions for the stochastic ZK equation are essentially the same as that for the regularized equation, with the necessary changes in the assumptions, equations and the function spaces.

Definition 6.3.1. *(Global Martingale Solutions) Let μ_{u_0} be the probability measure of u_0 given as in (6.12) on $L^2(\mathcal{M})$ and assume that (6.8), (6.9) and (6.13) hold.*

A global martingale solution to the stochastic ZK equation (6.1), (2.2)-*(2.4) (in the Dirichlet case) is defined as in Definition 6.2.1 as a pair $(\tilde{\mathcal{S}}, \tilde{u})$, such that*

$$\tilde{u} \in L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^2(0,T;\, H_0^1(\mathcal{M}))), \tag{6.75}$$

$$\tilde{u}(\cdot,\omega) \in \mathcal{C}([0,T];\, L_w^2(\mathcal{M}))\ \tilde{\mathbb{P}} - a.s., \tag{6.76}$$

and $\tilde{u}$ satisfying almost surely

$$\tilde{u}(t) + \int_0^t (\Delta\tilde{u}_x + c\tilde{u}_x + \tilde{u}\tilde{u}_x)ds = \tilde{u}(0) + \int_0^t f\, ds + \int_0^t \sigma(\tilde{u})\, d\tilde{W}(s); \tag{6.77}$$

the equality in (6.77) is understood in the sense of distributions on $\mathcal{D}(\mathcal{M})$ for every $t \in [0,T]$.

Moreover $\tilde{u}$ vanishes on $\partial\mathcal{M}$ (since $\tilde{u} \in L^2(\tilde{\Omega};\, L^2(0,T;\, H_0^1(\mathcal{M})))$) and $\tilde{u}_x\big|_{x=1} = 0$. For the latter, we observe that according to Lemma 6.3.5 below, $\tilde{u}_x\big|_{x=1} = 0$ makes sense in a suitable space for any $\tilde{u}$ satisfying (6.75) and (6.77).

Definition 6.3.2. *(Global Pathwise Solutions; Uniqueness)*

Let $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W^k\}_{k\geq 1})$ be a fixed stochastic basis and suppose that u_0 is an $L^2(\mathcal{M})$-valued random variable (relative to $\mathcal{S}$) satisfying (6.14). We suppose that σ and f satisfy (6.8), (6.9), (6.11) and (6.15).

(i) A global pathwise solution u of (6.1) and (2.2)-*(2.4) is defined as in Definition 6.2.2 with (6.33)-(6.35) replaced by (6.75)-(6.77). Also note that u vanishes on $\partial\mathcal{M}$ (because $u \in L^2(\Omega;\, L^2((0,T);\, H_0^1(\mathcal{M})))$ and $u_x\big|_{x=1} = 0$ which makes sense for the same reasons as for the martingale solution.*

(ii) Global pathwise uniqueness is defined in the same way as in Definition 6.2.2.

The strategy is the same as that in the case of the regularized stochastic ZK equation in Section 6.2.2: we first derive the global existence of martingale solutions, then prove the pathwise uniqueness of martingale solutions and hence deduce the existence of global pathwise solutions.

6.3.1 Martingale Solutions in Dimensions 2 and 3

All the subsequent proofs are valid for $d = 1$ or 2, except for (6.81) and (6.83), and for the uniqueness in Section 6.3.2, which are only valid for $d = 1$ (space dimension two).

Theorem 6.3.1. *When $d = 1$ or 2, suppose that μ_0 satisfies (6.12), that σ and f maintain (6.8), (6.9) and (6.13). Then there exists a global martingale solution $(\tilde{\mathcal{S}}, \tilde{u})$ of (6.1) and (2.2)-(2.4) in the sense of Definition 6.3.1.*

Furthermore, when $d = 1$, and if additionally f and σ satisfy (6.15) and (6.11), then the martingale solution is pathwise unique (see Proposition 6.3.3 below).

To prove Theorem 6.3.1, similar to the case of the regularized stochastic ZK equation, we first derive the estimates leading to weak convergence, then using the Skorokhod embedding theorem we upgrade the weak convergence into the strong convergence, with the probability basis shifted. Special measures will be taken to pass to the limit in the boundary conditions.

Estimates and Developments Independent of ϵ.

We begin the proof of Theorem 6.3.1 by deriving the estimates on u^ϵ valid as $\epsilon \to 0$. We observe that we can prove the estimates in (6.39)-(6.42) under the new assumptions in Theorem 6.3.1.

Lemma 6.3.1. *With the assumptions of Theorem 6.3.1, when $d = 1, 2$, we have the following estimates valid as $\epsilon \to 0$:*

$$u_x^\epsilon\big|_{x=0} \text{ remains bounded in } L^2(\Omega;\, L^2(0, T;\, L^2(I_{x^\perp}))), \tag{6.78}$$

$$\sqrt{\epsilon} u^\epsilon \text{ remains bounded in } L^2(\Omega;\, L^2(0, T;\, \Xi_1)), \tag{6.79}$$

$$u^\epsilon \textit{ remains bounded in } L^6(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))), \tag{6.80}$$

If we additionally assume that u_0 and f satisfy (6.14) and (6.15), then we have

$$u^\epsilon \textit{ remains bounded in } L^7(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))). \tag{6.81}$$

Proof. The estimates follow from (6.39)-(6.41) (or (6.42)) by passing to the lower limit first in n and then in ϵ using the lower semicontinuity of the norms; indeed e.g. to show (6.80), with (6.41) we obtain $|u^\epsilon|_{L^6(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M})))} \le \liminf_n |u^{\epsilon,n}|_{L^6(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M})))} \le \kappa_1'$, for a constant κ_1' independent of ϵ. □

Lemma 6.3.2. *The assumptions are those of Theorem 6.3.1 with $d = 1$ or 2. We have the following estimates valid as $\epsilon \to 0$:*

$$u^\epsilon \textit{ remains bounded in } L^2(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M}))). \tag{6.82}$$

If furthermore we suppose that u_0 and f satisfy (6.14) and (6.15), and $d = 1$, then we have

$$u^\epsilon \textit{ remains bounded in } L^{7/2}(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M}))). \tag{6.83}$$

Remark 6.3.1. *We will use (6.81) and (6.83) only when dealing with the pathwise uniqueness (see the calculations leading to (6.156) below).*

Proof of Lemma 6.3.2. The proof does not follow promptly from the estimates on $u^{\epsilon,n}$ as that of (6.78)-(6.81), but they are derived directly from the solutions u^ϵ of the regularized equations; this is in fact the reason for which we introduced this regularization. Note that the solutions u^ϵ are sufficiently regular for the following calculations to be valid.

We start by multiplying (6.25) with $\sqrt{1+x}$, to find

$$d(\sqrt{1+x}\, u^\epsilon) = \sqrt{1+x}\, \mathcal{N}(u^\epsilon)\, dt + \sqrt{1+x}\, \sigma(u^\epsilon)\, dW(t), \tag{6.84}$$

where again $\mathcal{N}(u^\epsilon) := -Au^\epsilon - B(u^\epsilon) - \epsilon\, Lu^\epsilon + f^\epsilon$. Applying the Itō formula to (6.84), we obtain

$$
\begin{aligned}
d|\sqrt{1+x}\,u^\epsilon|^2 = & \,2\left(\sqrt{1+x}\,u^\epsilon,\ \sqrt{1+x}\,\mathcal{N}(u^\epsilon)\right)\,dt \\
& + 2\left(\sqrt{1+x}\,u^\epsilon,\ \sqrt{1+x}\,\sigma(u^\epsilon)\,dW(t)\right) \\
& + ||\sqrt{1+x}\,\sigma(u^\epsilon)||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt.
\end{aligned} \tag{6.85}
$$

With exactly the same calculations as in the deterministic case (see Section 3.1), performed a.s. and for a.e. t, we have:

$$
\begin{aligned}
2&\left(\sqrt{1+x}\,u^\epsilon,\ \sqrt{1+x}\,\mathcal{N}(u^\epsilon)\right) \\
&= -|\nabla u^\epsilon|^2 - 2|u^\epsilon_x|^2 - (1-2\epsilon)\left|u^\epsilon_x\big|_{x=0}\right|^2_{L^2(I_{x^\perp})} \\
&\quad - 2\epsilon\left(|\sqrt{1+x}\,u^\epsilon_{xx}|^2 + |\sqrt{1+x}\,u^\epsilon_{yy}|^2 + |\sqrt{1+x}\,u^\epsilon_{zz}|^2\right)\\
&\quad + 2(f,\,(1+x)\,u^\epsilon) + \frac{2}{3}\int_{\mathcal{M}} (u^\epsilon)^3\,d\mathcal{M} + c|u^\epsilon|^2.
\end{aligned} \tag{6.86}
$$

Integrating both sides of (6.85) in t from 0 to s, $0 \le s \le T$, we find with (6.86) that when say $\epsilon \le 1/4$,

$$
\begin{aligned}
\int_0^s |\nabla u^\epsilon|^2\,dt \le |\sqrt{1+x}\,u^\epsilon_0|^2 &+ 2\int_0^s (f^\epsilon,\,(1+x)u^\epsilon)\,dt + \frac{2}{3}\int_0^s |u^\epsilon|^3_{L^3(\mathcal{M})}\,dt \\
&+ c\int_0^s |u^\epsilon|^2\,dt + \int_0^s ||\sqrt{1+x}\,\sigma(u^\epsilon)||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt \\
&+ 2\int_0^s \left((1+x)u^\epsilon,\,\sigma(u^\epsilon)\,dW(t)\right).
\end{aligned} \tag{6.87}
$$

For the first term on the right-hand side, using $H^{1/2}(\mathcal{M}) \subset L^3(\mathcal{M})$ in dimension three, we have $|u^\epsilon|^3_{L^3(\mathcal{M})} \le c'|u^\epsilon|^{3/2}|\nabla u^\epsilon|^{3/2} \le \frac{1}{4}|\nabla u^\epsilon|^2 + c'|u^\epsilon|^6$; hence taking expectations on both sides of (6.87) and using Hölder's inequality, we obtain

$$
\begin{aligned}
\frac{1}{2}\,\mathbb{E}\int_0^s |\nabla u^\epsilon|^2\,dt \lesssim 2\,\mathbb{E}|\,u^\epsilon_0|^2 &+ \mathbb{E}\int_0^s |f^\epsilon|^2\,dt + c'\,\mathbb{E}\int_0^s |u^\epsilon|^6\,dt + c' \\
&+ \mathbb{E}\int_0^s ||\sqrt{1+x}\,\sigma(u^\epsilon)||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt.
\end{aligned} \tag{6.88}
$$

Here the stochastic term vanishes. We find with (6.80) and (6.88)

$$
\mathbb{E}\int_0^T |\nabla u^\epsilon|^2\,dt \le \kappa_5, \tag{6.89}
$$

for a constant κ_5 depending only on u_0, f, T and σ, and independent of ϵ; this implies (6.82).

Returning to (6.87), when $d = 1$, we have

$$\begin{aligned}\int_0^s |u^\epsilon|^3_{L^3(\mathcal{M})} dt &\leq (H^{1/3}(\mathcal{M}) \subset L^3(\mathcal{M}) \text{ in dimension 2}) \\ &\leq \int_0^s |u^\epsilon|^2 |\nabla u^\epsilon| dt\, d\,\mathcal{M} \\ &\leq c' \sup_{0\leq t\leq s} |u^\epsilon(t)|^4 + \frac{1}{3}\left(\int_0^s |\nabla u^\epsilon| dt\right)^2 .\end{aligned} \tag{6.90}$$

Hence (6.87) implies

$$\begin{aligned}\frac{1}{2}\int_0^s |\nabla u^\epsilon|^2\, dt \leq 2|u_0^\epsilon|^2 + \int_0^s |f^\epsilon|^2\, dt + c' \int_0^s |u^\epsilon|^2\, dt + c' \sup_{0\leq t\leq s} |u^\epsilon(t)|^4 + c' \\ + 2\int_0^s ((1+x)u^\epsilon,\, \sigma(u^\epsilon)\, dW(t)) .\end{aligned} \tag{6.91}$$

Taking the supremum over $[0, T]$, raising both sides to the power $7/4$, then taking expectations, we obtain with Minkowski's inequality and Fubini's Theorem:

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left(\int_0^T |\nabla u^\epsilon|^2\, dt\right)^{7/4} &\lesssim \mathbb{E}|u_0^\epsilon|^{7/2} + \mathbb{E}\int_0^T |f^\epsilon|^{7/2}\, dt \\ &+ \mathbb{E}\int_0^T |u^\epsilon|^{7/2}\, dt + \mathbb{E} \sup_{0\leq s\leq T} |u^\epsilon(s)|^7 + c' \\ &+ 2\,\mathbb{E}\left[\sup_{0\leq s\leq T} \int_0^s |((1+x)u^\epsilon,\, \sigma(u^\epsilon)\, dW(t))|\right]^{7/4} .\end{aligned} \tag{6.92}$$

For the stochastic term, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq s\leq T} \int_0^s |((1+x)u^\epsilon,\, \sigma(u^\epsilon)\, dW(t))|\right]^{7/4} \\ &\qquad \leq \mathbb{E} \sup_{0\leq s\leq T} \left|\int_0^s ((1+x)u^\epsilon,\, \sigma(u^\epsilon)\, dW(t))\right|^{7/4} \\ &\qquad \leq (\text{by the Burkholder-Davis-Gundy inequality (6.3)}) \\ &\qquad \leq c_1\, \mathbb{E}\left[\left(\int_0^T |u^\epsilon|^2\, ||\sigma(u^\epsilon)||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt\right)^{7/8}\right] \\ &\qquad \lesssim \mathbb{E}\left[\left(\int_0^T |u^\epsilon|^4\, dt\right)^{7/8}\right] + c'.\end{aligned}$$

This together with (6.92) implies

$$\frac{1}{2}\mathbb{E}\left(\int_0^T |\nabla u^\epsilon|^2\,dt\right)^{7/4} \lesssim \mathbb{E}|u_0^\epsilon|^{7/2} + \mathbb{E}\int_0^T |f^\epsilon|^{7/2}\,dt + \mathbb{E}\int_0^T |u^\epsilon|^{7/2}\,dt + \mathbb{E}\sup_{0\le s\le T}|u^\epsilon(s)|^7 + \mathbb{E}\int_0^r |u^\epsilon|^4\,dt + c'. \tag{6.93}$$

Hence (6.93) and (6.81) imply

$$\mathbb{E}\left(\int_0^T |\nabla u^\epsilon|^2\,dt\right)^{7/4} \le \kappa_6, \tag{6.94}$$

for a constant κ_6 depending only on u_0, f, T and σ, and independent of ϵ; this implies (6.83). The proof of Lemma 6.3.2 is complete. □

Estimates in fractional Sobolev spaces.

Lemma 6.3.3. *With the same assumptions as in Theorem 6.3.1 and $d = 1, 2$, we have*

$$\mathbb{E}|u^\epsilon|^2_{\mathcal{Y}} \le \kappa_7, \tag{6.95}$$

$$\mathbb{E}\left|u^\epsilon(t) - \int_0^t \sigma(u^\epsilon)\,dW(s)\right|^2_{H^1(0,T;\,\Xi_2')} \le \kappa_8, \tag{6.96}$$

$$\mathbb{E}\left|\int_0^t \sigma(u^\epsilon)\,dW(s)\right|^2_{W^{\alpha,6}(0,T;\,L^2(\mathcal{M}))} \le \kappa_9, \quad \forall\ \alpha < \frac{1}{2}, \tag{6.97}$$

where $\mathcal{Y}$ is defined as in (6.60), and κ_7, κ_8 and κ_9 are independent of ϵ.

Proof. By repeating the proof of Lemma 6.2.3 (see Remark 6.2.2), we see that we can obtain for u^ϵ the estimates analog to (6.61)-(6.63) independent of ϵ. The only point is to derive the estimate of $\mathbb{E}|u^\epsilon|_{L^2(0,T;\,H_0^1(\mathcal{M}))}$ being bounded independently of ϵ (see (6.74) correspondingly). For that we just need the estimate (6.82). Hence Lemma 6.3.3 is proven.

□

Compactness Arguments for $\{(u^\epsilon, W)\}_{\epsilon>0}$

With these estimates independent of ϵ in hand, we can establish the compactness of the family $(u^\epsilon(t), W(t))$. For this purpose we consider the following phase spaces:

$$\mathcal{X}_u = L^2(0,T;\,L^2(\mathcal{M})) \cap \mathcal{C}(0,T;\,H^{-5}(\mathcal{M})), \quad \mathcal{X}_W = \mathcal{C}(0,T;\,\mathfrak{U}_0), \quad \mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W. \tag{6.98}$$

We then define the probability laws of $u^\epsilon(t)$ and $W(t)$ respectively in the corresponding phase spaces:

$$\mu_u^\epsilon(\cdot) = \mathbb{P}(u^\epsilon \in \cdot), \tag{6.99}$$

and

$$\mu_W(\cdot) = \mu_W^\epsilon(\cdot) = \mathbb{P}(W \in \cdot). \tag{6.100}$$

This defines a family of probability measures $\mu^\epsilon := \mu_u^\epsilon \times \mu_W^\epsilon$ on the phase space $\mathcal{X}$. We now show that this family is tight in ϵ. More precisely:

Lemma 6.3.4. *We suppose that $d = 1, 2$, and the hypotheses of Theorem 6.3.1 hold. Consider the measures μ^ϵ on $\mathcal{X}$ defined according to (6.99) and (6.100). Then the family $\{\mu^\epsilon\}_{\epsilon>0}$ is tight and therefore weakly compact over the phase space $\mathcal{X}$.*

Proof. We can use the same technic as in the proof of Lemma 4.1 in [11]. The main idea is to apply Lemma 7.3.1 (of the Appendix) and Chebychev's inequality to (6.95)-(6.97). □

Strong convergence as $\epsilon \to 0$. Since the family of measures $\{\mu^\epsilon\}$ associated with the family $(u^\epsilon(t), W(t))$ is weakly compact on $\mathcal{X}$, we deduce that μ^ϵ converges weakly to a probability measure μ on $\mathcal{X}$ up to a subsequence. We can apply the Skorokhod embedding theorem (see Theorem 2.4 in [9], also [3] and [26][2]) to deduce the strong convergence of a further subsequence, that is :

Proposition 6.3.1. *Suppose that μ_0 is a probability measure on $L^2(\mathcal{M})$ that satisfies (6.12). Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and a subsequence ϵ_k of random vectors $(\tilde{u}^{\epsilon_k}, \tilde{W}^{\epsilon_k})$ with values in $\mathcal{X}$ ($\mathcal{X}$ defined in (6.98)) such that*

(i) $(\tilde{u}^{\epsilon_k}, \tilde{W}^{\epsilon_k})$ have the same probability distributions as $(u^{\epsilon_k}, W^{\epsilon_k})$.

[2]particularly in [26], the theorem applies to $\mathcal{X}$ as a Polish space, that is, a separable completely metrizable topological space.

(ii) $(\tilde{u}^{\epsilon_k}, \tilde{W}^{\epsilon_k})$ *converges almost surely as* $\epsilon_k \to 0$*, in the topology of* $\mathcal{X}$*, to an element* $(\tilde{u}, \tilde{W}) \in \mathcal{X}$*, i.e.*

$$\tilde{u}^{\epsilon_k} \to \tilde{u} \text{ strongly in } L^2(0,T;\, L^2(\mathcal{M}))) \cap \mathcal{C}([0,T];\, H^{-5}(\mathcal{M})) \text{ a.s.}, \tag{6.101}$$

$$\tilde{W}^{\epsilon_k} \to \tilde{W} \text{ strongly in } \mathcal{C}([0,T];\, \mathfrak{U}_0) \text{ a.s.}, \tag{6.102}$$

where $(\tilde{u}, \tilde{W})$ *has the probability distribution* μ*.*

(iii) $\tilde{W}^{\epsilon_k}$ *is a cylindrical Wiener process, relative to the filtration* $\tilde{\mathcal{F}}_t^{\epsilon_k}$*, given by the completion of the* σ*-algebra generated by* $\{(\tilde{u}^{\epsilon_k}(s), \tilde{W}^{\epsilon_k}(s));\ s \le t\}$*.*

(iv) For each fixed ϵ_k*,* $\tilde{u}^{\epsilon_k} \in L^2(\tilde{\Omega};\, L^2(0,T;\, \Xi_2))$*. Moreover, all the statistical estimates on* u^{ϵ_k} *are valid for* $\tilde{u}^{\epsilon_k}$*, in particular, (6.80) and (6.82) hold.*

(v) Each pair $(\tilde{u}^{\epsilon_k}, \tilde{W}^{\epsilon_k})$ *satisfies (6.16) as an equation in* $L^2(\mathcal{M})$ *a.s., and satisfies the boundary conditions (2.2), (2.4), (6.17) and (6.18) thanks to (iv), that is,* $\tilde{u}^{\epsilon_k}(t)$ *is adapted to* $\tilde{\mathcal{F}}_t^{\epsilon_k}$*, and*

$$\begin{cases} d\tilde{u}^{\epsilon_k} = (-A\tilde{u}^{\epsilon_k} - B(\tilde{u}^{\epsilon_k}) - \epsilon_k\, L\tilde{u}^{\epsilon_k} + f^{\epsilon_k})\, dt + \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k}(t), \\ \tilde{u}^{\epsilon_k} = 0 \text{ on } \partial\mathcal{M}, \quad \tilde{u}_x^{\epsilon_k}\big|_{x=1} = 0,\ \tilde{u}_{xx}^{\epsilon_k}\big|_{x=0} = \tilde{u}_{yy}^{\epsilon_k}\big|_{y=\pm\frac{\pi}{2}} = \tilde{u}_{zz}^{\epsilon_k}\big|_{z=\pm\frac{\pi}{2}} = 0, \\ \tilde{u}^{\epsilon_k}(0) = \tilde{u}_0^{\epsilon_k}. \end{cases} \tag{6.103}$$

Proof. (i) and (ii) follow directly from the Skorokhod embedding theorem.

To prove (iv), we first observe that thanks to Lemma 7.6.2 (of the Appendix), the space $L^2(0,T;\, \Xi_2)$ is a Borel set in the space $\mathcal{X}_u$, and hence the integration $\int_{L^2(0,T;\,\Xi_2)} |u|^2\, d\,\mu_u^{\epsilon_k}(u)$ makes sense, and by (i) we have for each ϵ_k,

$$\mathbb{E}|u^{\epsilon_k}|^2_{L^2(0,T;\,\Xi_2)} = \int_{L^2(0,T;\,\Xi_2)} |u|^2\, d\,\mu_u^{\epsilon_k}(u) = \tilde{\mathbb{E}}|\tilde{u}^{\epsilon_k}|^2_{L^2(0,T;\,\Xi_2)} < (\text{by } (6.33)) < \infty.$$

In the same way we would prove that all estimates on u^ϵ are valid for $\tilde{u}^{\epsilon_k}$, particularly (6.80) and (6.82).

To prove (v), we define

$$\tilde{M}^{\epsilon_k}$$
$$:= \int_0^T \left| \tilde{u}^{\epsilon_k}(t) + \int_0^t A\tilde{u}^{\epsilon_k} + B(\tilde{u}^{\epsilon_k}) + \epsilon_k L\tilde{u}^{\epsilon_k} - f^{\epsilon_k} ds - \tilde{u}^{\epsilon_k}(0) - \int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k}(s) \right|^2 dt;$$

then we can use the exact same technique in [2] to prove $\tilde{\mathbb{E}}\, \frac{\tilde{M}^{\epsilon_k}}{1+\tilde{M}^{\epsilon_k}} = 0$. Hence we obtain (6.103). □

Passage to the Limit

Now equipped with the strong convergences in (6.101), we can consider passing to the limit on the regularized equation $(6.103)_1$ as $\epsilon_k \to 0$. Note that $(6.103)_1$ is the version of (6.16) provided by the Skorokhod embedding theorem.

Thanks to (6.80) and (6.82), we deduce the existence of an element

$$\tilde{u} \in L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^2(0,T;\, H_0^1(\mathcal{M}))), \tag{6.104}$$

and a subsequence still denoted as ϵ_k such that

$$\tilde{u}^{\epsilon_k} \rightharpoonup \tilde{u} \text{ weak-star in } L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))), \tag{6.105}$$

and

$$\tilde{u}^{\epsilon_k} \rightharpoonup \tilde{u} \text{ weakly in } L^2(\tilde{\Omega};\, L^2(0,T;\, H_0^1(\mathcal{M}))). \tag{6.106}$$

Fixing $u^\sharp \in \Xi_2$, by (6.106) and (6.105) we can pass to the limit in the linear terms.

For the nonlinear term, for every $u^\sharp \in \Xi_2$, we write a.s. and for a.e. t:

$$\begin{aligned}
&\Big| \int_0^t \Big(B(\tilde{u}^{\epsilon_k}) - B(\tilde{u}), u^\sharp \Big) \, ds \Big| \\
&\quad = \frac{1}{2} \Big| \int_0^t \Big((\tilde{u}^{\epsilon_k} - \tilde{u})(\tilde{u}^{\epsilon_k} + \tilde{u}), u^\sharp_x \Big) \, ds \Big| \\
&\quad \le \frac{1}{2} \int_0^t |\tilde{u}^{\epsilon_k} - \tilde{u}| \, |\tilde{u}^{\epsilon_k} + \tilde{u}| \, |u^\sharp_x|_{L^\infty(\mathcal{M})} \, ds \\
&\quad \le \text{(by the same calculations as in (6.67))} \\
&\quad \le \frac{1}{2} \int_0^t |\tilde{u}^{\epsilon_k} - \tilde{u}| \, |\tilde{u}^{\epsilon_k} + \tilde{u}| \, |u^\sharp|_{\Xi_2} \, ds \\
&\quad \le \frac{1}{2} |u^\sharp|_{\Xi_2} \left(\int_0^T |\tilde{u}^{\epsilon_k} - \tilde{u}|^2 \, ds \right)^{1/2} \left(\int_0^T |\tilde{u}^{\epsilon_k} + \tilde{u}|^2 \, ds \right)^{1/2}.
\end{aligned} \tag{6.107}$$

Thus with (6.101) and (6.80), we deduce that

$$\int_0^t \Big(B(\tilde{u}^{\epsilon_k}), u^\sharp \Big) \, ds \to \int_0^t \Big(B(\tilde{u}), u^\sharp \Big) \, ds \quad \text{for } a.e. \ (\tilde{\omega}, t) \in \tilde{\Omega} \times (0,T). \tag{6.108}$$

We next establish the convergence for the nonlinear term in the space $L^1(\tilde{\Omega} \times (0,T))$. We calculate as in (6.68),

$$\mathbb{E} \int_0^T \Big| \int_0^t \Big(B(\tilde{u}^{\epsilon_k}), u^\sharp \Big) \, ds \Big|^2 dt \lesssim \mathbb{E} \int_0^T |\tilde{u}^{\epsilon_k}|^4 |u^\sharp|^2_{\Xi_2} \, ds \lesssim |u^\sharp|^2_{\Xi_2} \mathbb{E} \int_0^T |\tilde{u}^{\epsilon_k}|^4 \, ds.$$

Thus by (6.80), we have

$$\left\{ \int_0^t \Big(B(\tilde{u}^{\epsilon_k}), u^\sharp \Big) \, ds \right\}_{\epsilon_k > 0} \text{ is uniformly integrable for all } \epsilon_k \text{ in } L^1(\tilde{\Omega} \times (0,T)).$$

Hence thanks to the Vitali convergence theorem, we conclude that

$$\int_0^t \Big\langle B(\tilde{u}^{\epsilon_k}), u^\sharp \Big\rangle \, ds \to \int_0^t \Big\langle B(\tilde{u}), u^\sharp \Big\rangle \, ds \quad \text{in } L^1(\tilde{\Omega}) \times (0,T). \tag{6.109}$$

For the stochastic term, by (6.101) we obtain

$$|\tilde{u}^{\epsilon_k} - \tilde{u}|^2 \to 0, \quad \text{for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times (0,T). \tag{6.110}$$

Thus, along with (6.9) we deduce

$$|\sigma(\tilde{u}^{\epsilon_k}) - \sigma(\tilde{u})|_{L_2(\mathfrak{U},H)} \to 0, \quad \text{for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times (0,T).$$

On the other hand, we observe that

$$\sup_{\epsilon_k} \mathbb{E}\left(\int_0^T |\sigma(\tilde{u}^{\epsilon_k})|^6_{L_2(\mathfrak{U},H)}\, ds \right) \lesssim \sup_{\epsilon_k} \mathbb{E}\left(\int_0^T (1+|\tilde{u}^{\epsilon_k}|^6)\, ds \right),$$

where we made use of (6.8). We therefore infer from (6.41) that $|\sigma(\tilde{u}^{\epsilon_k})|_{L_2(\mathfrak{U},H)}$ is uniformly integrable for ϵ_k in $L^q(\tilde{\Omega} \times (0,T))$ for any $q \in [1,6)$. With the Vitali convergence theorem we deduce that, for all such $q \in [1,6)$,

$$\sigma(\tilde{u}^{\epsilon_k}) \to \sigma(\tilde{u}) \text{ in } L^q(\tilde{\Omega};\, L^q((0,T), L_2(\mathfrak{U},H))). \tag{6.111}$$

Particularly (6.111) implies the convergence in probability of $\sigma(\tilde{u}^{\epsilon_k})$ in $L^2((0,T), L_2(\mathfrak{U},H))$. Thus, along with the assumption (6.102), we apply Lemma 7.4.1 (of the Appendix) and deduce that

$$\int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k} \to \int_0^t \sigma(\tilde{u})\, d\tilde{W}, \text{ in probability in } L^2((0,T);\, L^2(\mathcal{M})). \tag{6.112}$$

By the Vitali convergence theorem using the estimates involving (6.3) and (6.111), from (6.112) we infer a stronger convergence result:

$$\int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k} \to \int_0^t \sigma(\tilde{u})\, d\tilde{W}, \text{ in } L^2(\tilde{\Omega};\, L^2((0,T); L^2(\mathcal{M})). \tag{6.113}$$

Hence we can pass to the limit in (6.16), and obtain (6.77) as an equation in Ξ_2'

For the initial condition, since (6.101) and (6.104) imply that $\tilde{u}^\epsilon \in L^\infty(0,T;\, L^2(\mathcal{M}))$ $\cap\, \mathcal{C}([0,T];\, H^{-5}(\mathcal{M}))$ a.s., hence $\tilde{u}^\epsilon$ is weakly continuous with values in $L^2(\mathcal{M})$ a.s.; then (6.76) follows.

Having shown that the limit $\tilde{u}$ almost surely satisfies (6.77) in the sense of distributions on $\mathcal{D}(\mathcal{M})$, we want now to address the question of the boundary conditions. We need to be more careful because of the lack of regularity (see Lemma 6.3.5 below).

Passage to the limit on the boundary conditions. Since $\tilde{u} \in L^2(0,T;\, H^1_0(\mathcal{M}))$ a.s. (see (6.106)), we deduce that $\tilde{u}$ satisfies the Dirichlet boundary conditions. Hence there remains to show that the boundary condition

$$\tilde{u}_x\big|_{x=1} = 0, \tag{6.114}$$

is satisfied almost surely. This boundary condition is the object of Lemma 6.3.5 below where we show that $\tilde{u}_x\big|_{x=1}$ is well defined when $\tilde{u} \in L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^2(0,T;\, H^1(\mathcal{M})))$, and satisfies an equation like (6.77).

Lemma 6.3.5. *We assume that* $\tilde{u} \in L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^2(0,T;\, H^1(\mathcal{M})))$ *satisfies (6.77) almost surely in the sense of distributions on* $\mathcal{D}(\mathcal{M})$*, for every* $0 \le t \le T$*. Then*

$$\tilde{u}_x, \tilde{u}_{xx} \in \mathcal{C}_x(I_x;\, \mathcal{B}), \quad \textit{where } \mathcal{B} = L^{5/4}(\tilde{\Omega};\; H^{-3}((0,T) \times I_{x^\perp}), \tag{6.115}$$

and, in particular,

$$\tilde{u}_x\big|_{x=0,1} \textit{ and } \tilde{u}_{xx}\big|_{x=0,1}, \tag{6.116}$$

are well defined in $\mathcal{B}$*.*

Proof. If $\tilde{u}$ almost surely satisfies (6.77), then $\tilde{U} := \int_0^t \tilde{u}\, ds$ satisfies

$$\frac{\partial \tilde{U}}{\partial t} + \Delta \frac{\partial \tilde{U}}{\partial x} + c\frac{\partial \tilde{U}}{\partial x} = F \;\; a.s., \tag{6.117}$$

where $F := \tilde{u}_0 - \int_0^t B(\tilde{u})\, ds + \int_0^t f\, ds + \int_0^t \sigma(\tilde{u})\, d\tilde{W}(s)$.

For the term $\int_0^t B(\tilde{u})\, ds$, we note that by (4.10) in [40],

$$|\tilde{u}\tilde{u}_x|_{L^{9/8}(\mathcal{M})} \le |\tilde{u}|^{2/3} |\nabla \tilde{u}|^{4/3}, \text{ for } a.e.\ t \text{ and } a.s..$$

Hence we have a.s.

$$\begin{aligned} \left|\int_0^t B(\tilde{u})\, ds\right|^{5/4}_{L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))} &= \int_0^T |\tilde{u}\tilde{u}_x|^{5/4}_{L^{9/8}(\mathcal{M})}\, dt \\ &\lesssim \int_0^T \left(|\tilde{u}|^{5/6}\right)^6 + \left(|\nabla \tilde{u}|^{5/3}\right)^{6/5} dt \\ &\lesssim \int_0^T |\tilde{u}|^5 + |\nabla \tilde{u}|^2\, dt. \end{aligned} \tag{6.118}$$

Since $\tilde{u} \in L^6(\tilde{\Omega};\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega};\, L^2(0,T;\, H^1(\mathcal{M})))$, taking expectations on both sides of (6.118) we have

$$\tilde{\mathbb{E}} \left| \int_0^t B(\tilde{u})\, ds \right|^{5/4}_{L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))} < \infty,$$

that is

$$\begin{aligned} &\int_0^t B(\tilde{u})\, ds \text{ belongs to } L^{5/4}(\tilde{\Omega};\, L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))), \\ &\text{and hence belongs to } L^{5/4}(I_x;\, L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})). \end{aligned} \tag{6.119}$$

For the term $\int_0^t \sigma(\tilde{u})\, d\tilde{W}(s)$, from (6.113) we deduce that $\int_0^t \sigma(\tilde{u})\, d\tilde{W}(s)$ belongs to $L^2(\tilde{\Omega};\, L^2(0,T;\, L^2(\mathcal{M}))$.

Applying the above estimates, we obtain that

$$\begin{aligned} &F \text{ belongs to } L^{5/4}(\tilde{\Omega};\, L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))), \\ &\quad \text{and hence belongs to } L^{5/4}(I_x;\, L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp}))). \end{aligned} \tag{6.120}$$

Hence Lemma 7.2.2 (of the Appendix) applies with $p = 5/4$ and $\mathcal{E} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})$, and from (7.13) we have

$$\tilde{U}_x \text{ and } \tilde{U}_{xx} \text{ belong to } \mathcal{C}_x(I_x;\, L^{5/4}(\tilde{\Omega};\, H^{-2}((0,T) \times (I_{x^\perp}))). \tag{6.121}$$

Since $\tilde{U}_x(t) = \int_0^t \tilde{u}_x\, ds$, we have $\frac{d\tilde{U}_x(t)}{dt} = \tilde{u}_x(t)$; differentiation in time maps continuously $H^{-2}(0,T)$ into $H^{-3}(0,T)$ and from (6.121) we thus infer (6.115) and (6.116). $\square$

We now need to show that the boundary condition $\tilde{u}_x^{\epsilon_k}\big|_{x=1} = 0$, "passes to the limit" to imply (6.114). The idea is to apply Lemma 7.2.3 (of the Appendix) to $\tilde{U}^{\epsilon_k}(t) := \int_0^t \tilde{u}^{\epsilon_k}\, ds$. Rewriting (6.103) in an integral form and rearranging, we obtain a.s.

$$\begin{aligned} &\tilde{u}^{\epsilon_k}(t) + \int_0^t \Delta \tilde{u}_x^{\epsilon_k}\, ds + c \int_0^t \tilde{u}_x^{\epsilon_k}\, ds + \epsilon_k \int_0^t L\tilde{u}^{\epsilon_k}\, ds \\ &\qquad = \tilde{u}_0^{\epsilon_k} - \int_0^t B(\tilde{u}^{\epsilon_k})\, ds + \int_0^t f^{\epsilon_k}\, ds + \int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k}(s). \end{aligned} \tag{6.122}$$

Hence for almost every $\tilde{\omega}$, $\tilde{U}^{\epsilon_k}$ satisfies the linearized parabolic regularized equation:

$$\begin{cases} \dfrac{\partial \tilde{U}^{\epsilon_k}}{\partial t} + \Delta \dfrac{\partial \tilde{U}^{\epsilon_k}}{\partial x} + c \dfrac{\partial \tilde{U}^{\epsilon_k}}{\partial x} + \epsilon_k L \tilde{U}^{\epsilon_k} = F^{\epsilon_k}, \\ \tilde{U}^{\epsilon_k}\big|_{x=0} = \tilde{U}^{\epsilon_k}\big|_{x=1} = \tilde{U}^{\epsilon_k}_x\big|_{x=1} = \tilde{U}^{\epsilon_k}_{xx}\big|_{x=0} = 0, \end{cases} \tag{6.123}$$

where $F^{\epsilon_k} := \tilde{u}_0^{\epsilon_k} - \int_0^t B(\tilde{u}^{\epsilon_k})\, ds + \int_0^t f^{\epsilon_k}\, ds + \int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k}(s)$.

For the term $\int_0^t B(\tilde{u}^{\epsilon_k})\, ds$, by the same calculations as those leading to (6.118), we infer from (6.80) and (6.82) that

$$\tilde{\mathbb{E}} \left| \int_0^t B(\tilde{u}^{\epsilon_k})\, ds \right|^{5/4}_{L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))} \text{ is bounded independently of } \epsilon_k. \tag{6.124}$$

By (6.113) we deduce that $\int_0^t \sigma(\tilde{u}^{\epsilon_k})\, d\tilde{W}^{\epsilon_k}(s)$ remains bounded in $L^2(\tilde{\Omega};\, L^2((0,T);\, L^2(\mathcal{M}))$. Collecting all the previous estimates we conclude that $\tilde{\mathbb{E}}|F^{\epsilon_k}|^{5/4}_{L^{5/4}(0,T;\, L^{9/8}(\mathcal{M}))}$ is bounded independently of ϵ_k, and hence

$$F^{\epsilon_k} \text{ is bounded independently of } \epsilon_k \text{ in } L^{5/4}(I_x;\, L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})).$$

Applying Lemma 7.2.3 (of the Appendix) with $p = 5/4$, $\tilde{\mathcal{E}} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})$ and $\tilde{\mathcal{B}} = L^2(\tilde{\Omega};\, H_t^{-1}(0,T;\, L^2(I_{x^\perp}))) + L^2(\tilde{\Omega};\, L_t^2(0,T;\, H^{-4}(I_{x^\perp}))) + L^{5/4}(\tilde{\Omega} \times (0,T) \times (I_{x^\perp}))$, we deduce that $\tilde{U}^{\epsilon_k}_x\big|_{x=1}$ converges to $\tilde{U}_x\big|_{x=1}$ weakly in $\tilde{\mathcal{B}}$. Hence

$$\tilde{U}_x\big|_{x=1}(t) = 0, \tag{6.125}$$

a.s. and for a.e. $t \in (0,T)$. Since $\tilde{U}_x\big|_{x=1}(t) = \int_0^t \tilde{u}_x\big|_{x=1}\, ds$, thanks to the Lebesgue differentiation theorem, we infer from (6.125) that $\tilde{u}_x\big|_{x=1}(t) = 0$ a.s. and for a.e. $t \in (0,T)$. Thus we have finished the proof of Theorem 6.3.1. □

6.3.2 Pathwise Solutions in Dimension 2

We aim to establish the existence of pathwise solutions when $d = 1$, that is:

Theorem 6.3.2. *When $d = 1$, assume that, relative to a fixed stochastic basis $\mathcal{S}$, u_0 satisfies (6.14), and that σ and f satisfy (6.8), (6.9), (6.11) and (6.15) respectively. Then* *there exists a unique global pathwise solution u which satisfies (6.1) and (2.2)-(2.4) in the sense of Definition 6.3.2.*

To prove this theorem, we first establish the pathwise uniqueness of martingale solutions and then apply the Gyöngy-Krylov Theorem (Theorem 7.5.1 of the Appendix). The difficulty lies in deducing the pathwise uniqueness due to a lack of regularity of the martingale solutions (see (6.75) and (6.76)). Adapting the idea from the deterministic case (see Chapter 3), we introduce a preliminary result concerning the existence and uniqueness of global pathwise solutions to the linearized stochastic ZK equation with additive noise. More importantly, we establish an energy inequality, which leads to a suitable estimate of the difference of the solutions for the application of the version of the stochastic Gronwall lemma given in Lemma 7.7.2 below.

Linearized Stochastic ZK Equation with Additive Noise ($d = 1$)

Proposition 6.3.2. *When $d = 1$, let $\mathcal{S}$ be a fixed stochastic basis, that is*

$$\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W^k\}_{k\geq 1}).$$

We consider the linearized stochastic ZK equation ($c = 0$),

$$\begin{cases} d\mathcal{R} + \Delta\mathcal{R}_x\, dt = g\, dt + h\, dW(t), \\ \mathcal{R}(0) = \mathcal{R}_0, \end{cases} \tag{6.126}$$

with the boundary conditions (2.2) and (2.4) for $\mathcal{R}$. We assume that

$$\mathcal{R}_0 \in L^2(\Omega;\, L^2(\mathcal{M})), \tag{6.127}$$

and h and g are given predictable processes relative to the stochastic basis $\mathcal{S}$, such that

$$g \in L^2(\Omega;\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M}))) \cap L^2(\Omega;\, L^2(0,T;\, \Xi_2')), \tag{6.128}$$

and

$$h \in L^2(\Omega;\, L^2(0,T;\, L_2(\mathfrak{U}, L^2(\mathcal{M})))) \cap L^2(\Omega;\, L^2(0,T;\, L_2(\mathfrak{U}, \Xi_1))). \tag{6.129}$$

Then there exists a unique global pathwise solution $\mathcal{R}$ to (6.126) which satisfies (2.2) and (2.4), and such that

$$\mathcal{R} \in L^2(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M}))), \tag{6.130}$$

and

$$\mathcal{R}(\cdot,\omega) \in \mathcal{C}([0,T];\, L^2_w(\mathcal{M}))\ a.s.. \tag{6.131}$$

Furthermore $\mathcal{R}$ satisfies the following energy inequality for any stopping time τ_b with $0 \leq \tau_b \leq T$,

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\sup_{0\leq s\leq \tau_b} |\mathcal{R}(s)|^2 + \mathbb{E}\int_0^{\tau_b} |\nabla \mathcal{R}|^2\, dt \\ &\quad \leq \mathbb{E}|\mathcal{R}(0)|^2 + 2\,\mathbb{E}\int_0^{\tau_b} |(g,\, (1+x)\mathcal{R})|\, dt + c'\,\mathbb{E}\int_0^{\tau_b} ||h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\, dt. \end{aligned} \tag{6.132}$$

Proof. We will first show the existence of the solutions, which is similar to that of the nonlinear case, but only easier because the use of a compactness argument and the derivation of strong convergence are not necessary for the linearized model. Then we will verify the uniqueness of the solutions, which is direct since the noise is additive. More precisely, the difference of two solutions satisfies a deterministic equation depending on the parameters $\omega \in \Omega$. Finally, we will deduce the energy inequality (6.132) utilizing the duality between the spaces to which g and $\mathcal{R}$ each belongs.

We start by proving the existence of pathwise solutions with application of the parabolic regularization:

$$\begin{cases} d\mathcal{R}^\epsilon + (\Delta \mathcal{R}^\epsilon_x + \epsilon L \mathcal{R}^\epsilon)\, dt = g^\epsilon\, dt + h\, dW(t), \\ \mathcal{R}^\epsilon(0) = \mathcal{R}^\epsilon_0, \end{cases} \tag{6.133}$$

supplemented with the boundary conditions (2.2), (2.4) and the additional boundary conditions (6.17), (6.18). As in Section 6.2, there exist $\{\mathcal{R}^\epsilon_0\}_{\epsilon>0}$, a family of elements in the space $L^2(\Omega;\, \Xi_1) \cap L^{22/3}(\Omega;\, L^2(\mathcal{M}))$ which are $\mathcal{F}_0$ measurable, and such that, as $\epsilon \to 0$,

$$\mathcal{R}_0^\epsilon \to \mathcal{R}_0 \text{ in } L^2(\Omega;\, L^2(\mathcal{M})) \text{ strongly;} \tag{6.134}$$

and there exist $\{g^\epsilon\}_{\epsilon>0}$, a family of predictable processes relative to the stochastic basis $\mathcal{S}$, so that

$$g^\epsilon \in L^\infty(\Omega;\, L^{22/3}(0,T;\, L^2(\mathcal{M}))), \tag{6.135}$$

$$g^\epsilon \to g \text{ in } L^2(\Omega;\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M}))) \text{ strongly as } \epsilon \to 0. \tag{6.136}$$

Since (6.129) corresponds to (6.8) and (6.9), we can use a proof similar to that of Theorem 6.2.1 to deduce the existence and uniqueness of the global pathwise solution $\mathcal{R}^\epsilon$ for each fixed ϵ. Note that for the proof of existence, although g^ϵ depends on ω, it will not be a problem for us; this is essentially because we can prove the existence of a *pathwise* solution without referring to any compactness argument.

In the sequel, we will derive the estimates independent of ϵ, then pass to the limit on the parabolic regularization, where again we need to pay special attention to the boundary conditions.

(i) Preliminary estimates independent of ϵ*.* We will prove the following bounds on $\mathcal{R}^\epsilon$ as $\epsilon \to 0$:

$$\mathcal{R}^\epsilon \text{ remains bounded in } L^2(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))), \tag{6.137}$$

$$\mathcal{R}^\epsilon \text{ remains bounded in } L^2(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M}))). \tag{6.138}$$

We start by multiplying both sides of (6.133) by $\sqrt{1+x}$ and applying the Itō formula, we find

$$\begin{aligned} d|\sqrt{1+x}\,\mathcal{R}^\epsilon|^2 &= 2\,(\sqrt{1+x}\,\mathcal{R}^\epsilon,\, \sqrt{1+x}\,Q(\mathcal{R}^\epsilon))\,dt \\ &\quad + 2(\sqrt{1+x}\,\mathcal{R}^\epsilon,\, \sqrt{1+x}\,h\,dW(t)) + ||\sqrt{1+x}\,h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt, \end{aligned} \tag{6.139}$$

where $Q(\mathcal{R}^\epsilon) := -\Delta\mathcal{R}_x^\epsilon - \epsilon\, L\mathcal{R}^\epsilon + g^\epsilon$.

Let some stopping times τ_a, τ_b be given so that $0 \le \tau_a \le \tau_b \le T$; we integrate (6.139) from τ_a to s and take the supremum over $[\tau_a, \tau_b]$. After taking expected values, and by the same calculations as those leading to (6.87), we obtain that when $\epsilon \le \dfrac{1}{4}$,

$$\begin{aligned}\mathbb{E} \sup_{\tau_a \le s \le \tau_b} |\mathcal{R}^\epsilon(s)|^2 + \mathbb{E} \int_{\tau_a}^{\tau_b} |\nabla \mathcal{R}^\epsilon|^2 dt \le 2\, \mathbb{E}\, |\mathcal{R}^\epsilon(\tau_a)|^2 + 2\, \mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon,\, (1+x)\mathcal{R}^\epsilon)|\; dt \\ + 2\, \mathbb{E} \sup_{\tau_a \le s \le \tau_b} \int_{\tau_a}^{s} ((1+x)\mathcal{R}^\epsilon, h\, dW(t)) \\ + 2\, \mathbb{E} \int_{\tau_a}^{\tau_b} ||h||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt.\end{aligned} \tag{6.140}$$

For the stochastic term, we have

$$\begin{aligned}\mathbb{E} \sup_{\tau_a \le s \le \tau_b} \left| \int_{\tau_a}^{s} ((1+x)\mathcal{R}^\epsilon,\, h\, dW(t)) \right| &\lesssim \text{(by the Burkholder-Davis-Gundy inequality)} \\ &\lesssim \mathbb{E} \left[\left(\int_{\tau_a}^{\tau_b} |\mathcal{R}^\epsilon|^2\, ||h||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt \right)^{1/2} \right] \\ &\le \frac{1}{4}\, \mathbb{E} \sup_{\tau_a \le s \le \tau_b} |\mathcal{R}^\epsilon|^2 + c'\, \mathbb{E} \int_{\tau_a}^{\tau_b} ||h||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt.\end{aligned}$$

Hence (6.140) implies

$$\begin{aligned}\frac{1}{2}\, \mathbb{E} \sup_{\tau_a \le s \le \tau_b} |\mathcal{R}^\epsilon(s)|^2 + \mathbb{E} \int_{\tau_a}^{\tau_b} |\nabla \mathcal{R}^\epsilon|^2\, dt \le 2\, \mathbb{E}\, |\mathcal{R}^\epsilon(\tau_a)|^2 \\ + 2\, \mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon,\, (1+x)\mathcal{R}^\epsilon)|\; dt \\ + c'\, \mathbb{E} \int_{\tau_a}^{\tau_b} ||h||^2_{L_2(\mathfrak{U}, L^2(\mathcal{M}))}\, dt.\end{aligned} \tag{6.141}$$

To estimate the term $\mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon,\, (1+x)\mathcal{R}^\epsilon)|\; dt$, we observe that a.s.

$$\begin{aligned}|(g^\epsilon,\, (1+x)\mathcal{R}^\epsilon)| &\le |g^\epsilon|_{L^{4/3}(\mathcal{M})} |(1+x)\mathcal{R}^\epsilon|_{L^4(\mathcal{M})} \\ &\le \text{(by Sobolev embedding in dimension 2)} \\ &\le |g^\epsilon|_{L^{4/3}(\mathcal{M})} |\nabla \mathcal{R}^\epsilon|^{1/2} |\mathcal{R}^\epsilon|^{1/2} \\ &\le c' |g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})} |\mathcal{R}^\epsilon|^{2/3} + \frac{1}{4} |\nabla \mathcal{R}^\epsilon|^2 \\ &\le c' |g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})} \big(|\mathcal{R}^\epsilon|^2 + 1 \big) + \frac{1}{4} |\nabla \mathcal{R}^\epsilon|^2.\end{aligned}$$

Applying the above estimates to (6.141) we obtain

$$\begin{aligned} &\frac{1}{2}\,\mathbb{E}\sup_{\tau_a\le s\le\tau_b}|\mathcal{R}^\epsilon(s)|^2+\frac{1}{2}\,\mathbb{E}\int_{\tau_a}^{\tau_b}|\nabla\mathcal{R}^\epsilon|^2\,dt\\ &\qquad\qquad\le 2\,\mathbb{E}\,|\mathcal{R}^\epsilon(\tau_a)|^2+2c'\,\mathbb{E}\int_{\tau_a}^{\tau_b}|g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})}|\mathcal{R}^\epsilon|^2\,dt\\ &\qquad\qquad\quad+\mathbb{E}\int_{\tau_a}^{\tau_b}2c'|g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})}+c'||h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt. \end{aligned}\tag{6.142}$$

Thanks to (6.135) and (6.129), we can apply the stochastic Gronwall lemma (Lemma 7.7.1 below) to (6.142) to find

$$\begin{aligned} &\frac{1}{2}\,\mathbb{E}\sup_{0\le s\le T}|\mathcal{R}^\epsilon(s)|^2+\frac{1}{2}\,\mathbb{E}\int_0^T|\nabla\mathcal{R}^\epsilon|^2\,dt\\ &\qquad\qquad\lesssim\mathbb{E}\,|\mathcal{R}^\epsilon_0|^2+\mathbb{E}\int_0^T|g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})}+||h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt. \end{aligned}\tag{6.143}$$

Thanks to (6.136), we have $|g^\epsilon|_{L^2(\Omega;\,L^{4/3}(0,T;\,L^{4/3}(\mathcal{M})))}\le|g|_{L^2(\Omega;\,L^{4/3}(0,T;\,L^{4/3}(\mathcal{M})))}+c'$. Hence $\mathbb{E}\int_0^T|g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})}\lesssim\mathbb{E}\left[\int_0^T|g^\epsilon|^{4/3}_{L^{4/3}(\mathcal{M})}\,dt\right]^{3/2}+c'\le|g|^2_{L^2(\Omega;\,L^{4/3}(0,T;\,L^{4/3}(\mathcal{M})))}+c'$; thus (6.143) implies that

$$\begin{aligned} &\frac{1}{2}\,\mathbb{E}\sup_{0\le s\le T}|\mathcal{R}^\epsilon(s)|^2+\frac{1}{2}\,\mathbb{E}\int_0^T|\nabla\mathcal{R}^\epsilon|^2\,dt\\ &\qquad\lesssim\mathbb{E}\,|\mathcal{R}_0|^2+|g|^2_{L^2(\Omega;\,L^{4/3}(0,T;\,L^{4/3}(\mathcal{M})))}+c'+||h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt. \end{aligned}$$

Hence along with (6.128) and (6.129), we obtain (6.137) and (6.138).

(ii) Estimates in fractional Sobolev spaces. By the same proof as for Lemma 6.3.3, with $\mathcal{Y}$ defined as in (6.60), we derive the following estimates independent of ϵ.

$$\mathbb{E}|\mathcal{R}^\epsilon|^2_{\mathcal{Y}}<\kappa_8,\tag{6.144}$$

with κ_8 independent of ϵ. This estimate will be useful to prove the continuity in time in (6.131).

(iii) Passage to the limit as $\epsilon_k\to 0$. With (6.137), (6.138) and (6.144), we deduce the following weak convergences, for a subsequence $\epsilon_k\to 0$:

$$\mathcal{R}^{\epsilon_k}\rightharpoonup\mathcal{R}\text{ weakly in }L^2(\Omega;\,L^2(0,T;\,H^1_0(\mathcal{M})))\cap L^2(\Omega;\,W^{\alpha,2}(0,T;\,\Xi_2')),\tag{6.145}$$

$$\mathcal{R}^{\epsilon_k} \rightharpoonup \mathcal{R} \text{ weak star in } L^2(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))). \tag{6.146}$$

We can thus pass to the weak limit in (6.133) and obtain

$$\left\langle \mathcal{R}(t), \mathcal{R}^\sharp \right\rangle + \int_0^t \left\langle \Delta\mathcal{R}_x - g, \mathcal{R}^\sharp \right\rangle ds = \left\langle \mathcal{R}_0, \mathcal{R}^\sharp \right\rangle + \int_0^t \left\langle h, \mathcal{R}^\sharp \right\rangle dW, \tag{6.147}$$

for almost every $(\omega, t) \in \Omega \times (0,T)$ and every $\mathcal{R}^\sharp \in \Xi_2$.

To pass to the limit on the boundary conditions (2.2) and (2.4), we use the same idea as in Section 6.3.1. Firstly, we can prove an analogue of Lemma 6.3.5; that is, $\mathcal{R}_x\big|_{x=1}$ is well defined if $\mathcal{R} \in L^2(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))) \cap L^2(\Omega;\, L^2(0,T;\, H^1(\mathcal{M})))$, and satisfies an equation like (6.147). To show this, we just need to observe that thanks to (6.128), Lemma 7.2.2 applies with $p = 4/3$ and $\mathcal{E} = L^{4/3}(\Omega \times (0,T) \times I_{x^\perp})$. Secondly, we can pass to the limit on the boundary conditions applying Lemma 7.2.3 (of the Appendix) with $p = 4/3$, $\tilde{\mathcal{E}} = L^{4/3}(\Omega \times (0,T) \times I_{x^\perp})$ and $\tilde{\mathcal{B}} = L^2(\Omega;\, H_t^{-1}(0,T;\, L^2(I_{x^\perp}))) + L^2(\Omega;\, L_t^2(0,T;\, H^{-4}(I_{x^\perp}))) + L^{4/3}(\Omega \times (0,T) \times (I_{x^\perp}))$.

To prove (6.131), we infer from (6.145) that $\mathcal{R} \in W^{\alpha,2}(0,T;\, \Xi_2') \cap L^\infty(0,T;\, L^2(\mathcal{M}))$ *a.s.*, and hence $\mathcal{R} \in \mathcal{C}(0,T;\, H^{-5}(\mathcal{M}))) \cap L^\infty(0,T;\, L^2(\mathcal{M}))$ a.s.. Thus $\mathcal{R}$ is weakly continuous with values in $L^2(\mathcal{M})$ almost surely, which implies (6.131).

To conclude, we have proven the existence of a global pathwise solution $\mathcal{R}$ which satisfies (6.126), (2.2) and (2.4).

(iv) Global pathwise uniqueness. We assume that $\mathcal{R}_1, \mathcal{R}_2$ are two solutions of (6.126); setting $\mathcal{R} = \mathcal{R}_1 - \mathcal{R}_2$, we subtract the equation (6.126) for $\mathcal{R}_1$ from that for $\mathcal{R}_2$; we obtain that almost surely

$$\begin{cases} \dfrac{\partial \mathcal{R}}{\partial t} + \Delta\mathcal{R}_x = 0, \\ \mathcal{R}_0 = 0. \end{cases} \tag{6.148}$$

With (6.130), we have $\mathcal{R} \in L^\infty(0,T;\, L^2(\mathcal{M})) \cap L^2(0,T;\, H_0^1(\mathcal{M}))$ *a.s.*. Hence we can apply Lemma 3.2.1 and deduce that $\frac{d}{dt}|\mathcal{R}|^2 \leq 0$ for a.e. $\omega \in \Omega$ and $t \geq 0$; thus $\mathcal{R}(\omega) = 0$ follows whenever $\mathcal{R}_0(\omega) = 0$.

(v) Passage to the limit to obtain energy inequality (6.132). From (6.141), we obtain when $\tau_a = 0$,

$$\frac{1}{2}\,\mathbb{E}\sup_{0\le s\le\tau_b}|\mathcal{R}^\epsilon(s)|^2 + \mathbb{E}\int_0^{\tau_b}|\nabla\mathcal{R}^\epsilon|^2\,dt \le 2\,\mathbb{E}\,|\mathcal{R}_0^\epsilon|^2 + 2\,\mathbb{E}\int_0^{\tau_b}|(g^\epsilon,\,(1+x)\mathcal{R}^\epsilon)|\;dt \\ + c'\,\mathbb{E}\int_0^{\tau_b}||h||^2_{L_2(\mathfrak{U},L^2(\mathcal{M}))}\,dt. \tag{6.149}$$

We infer from (6.145) and (6.146) that for any τ_b with $0 \le \tau_b \le T$,

$$\mathcal{R}^{\epsilon_k}\mathbf{1}_{t\le\tau_b} \rightharpoonup \mathcal{R}\mathbf{1}_{t\le\tau_b} \text{ weakly in } L^2(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M}))),$$

$$\mathcal{R}^{\epsilon_k}\mathbf{1}_{t\le\tau_b} \rightharpoonup \mathcal{R}\mathbf{1}_{t\le\tau_b} \text{ weak star in } L^2(\Omega;\, L^\infty(0,T;\, L^2(\mathcal{M}))),$$

and hence we can pass to the lower limit on the left-hand-side of (6.149). To pass to the limit on the term $\mathbb{E}\,|\mathcal{R}_0^\epsilon|^2$, we use (6.134).

For the term $\mathbb{E}\int_0^{\tau_b}|(g^\epsilon,\,(1+x)\mathcal{R}^\epsilon)|\;dt$, we first note that in dimension 2,

$$\begin{aligned}\left(\int_0^T|\mathcal{R}^\epsilon|^4_{L^4(\mathcal{M})}\,ds\right)^{1/4} &\le \left(\int_0^T|\mathcal{R}^\epsilon|^2|\nabla\mathcal{R}^\epsilon|^2\,ds\right)^{1/4}\\ &\le \sup_{0\le s\le T}|\mathcal{R}^\epsilon(s)|^{1/2}\left(\int_0^T|\nabla\mathcal{R}^\epsilon|^2\,ds\right)^{1/4}\\ &\le 2\sup_{0\le s\le T}|\mathcal{R}^\epsilon(s)| + 2\left(\int_0^T|\nabla\mathcal{R}^\epsilon|^2\,ds\right)^{1/2}.\end{aligned}$$

Squaring both sides and taking the expectations we can use (6.137) and (6.138) to obtain that, as $\epsilon\to 0$,

$$\mathcal{R}^\epsilon \text{ remains bounded in } L^2(\Omega;\, L^4(0,T;\, L^4(\mathcal{M}))), \tag{6.150}$$

and hence a subsequence of $\mathcal{R}^\epsilon$ converges weakly in the space $L^2(\Omega;\, L^4(0,T;\, L^4(\mathcal{M})))$, which is the dual of $L^2(\Omega;\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M})))$. Since

$$\mathbb{E}\int_0^{\tau_b}|(g^\epsilon,\,(1+x)\mathcal{R}^\epsilon)|\,dt = \mathbb{E}\int_0^T|\mathbf{1}_{t\le\tau_b}(g^\epsilon,\,(1+x)\mathcal{R}^\epsilon)|\,dt = \mathbb{E}\int_0^T|(g^\epsilon\mathbf{1}_{t\le\tau_b},\,(1+x)\mathcal{R}^\epsilon)|\,dt,$$

we see that with (6.136), $g^\epsilon\mathbf{1}_{t\le\tau_b} \to g\mathbf{1}_{t\le\tau_b}$ strongly in $L^2(\Omega;\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M})))$, and hence the convergence of $\mathbb{E}\int_0^{\tau_b}|(g^\epsilon,\,(1+x)\mathcal{R}^\epsilon)|\;dt$ follows.

Thus we can pass to the lower limit on the left-hand side of (6.149) and to the limit on the right-hand side of (6.149), and thus deduce (6.132). Hence we have completed the proof of Proposition 6.3.2. □

Global Pathwise Uniqueness for the Full Stochastic ZK Equation ($d = 1$)

The following result establishes the pathwise uniqueness of martingale solutions to (6.1) and (2.2)-(2.4).

Proposition 6.3.3. *When $d = 1$, suppose that $(\tilde{\mathcal{S}}, \tilde{u})$ and $(\tilde{\mathcal{S}}, \tilde{v})$ are two global martingale solutions of (6.1),* (2.2)-*(2.4), relative to the same stochastic basis. We assume that the conditions imposed in Definition 6.3.2 hold. We define*

$$\Omega_0 = \{\tilde{u}(0) = \tilde{v}(0)\}.$$

Then $\tilde{u}$, $\tilde{v}$ are indistinguishable on Ω_0 in the sense that

$$\tilde{\mathbb{P}}(\mathbf{1}_{\Omega_0}(\tilde{u}(t) = \tilde{v}(t))) = 1, \ \ \forall\ 0 \leq t \leq T. \tag{6.151}$$

Proof. We will mainly use (6.132) from Proposition 6.3.2 and the version of the stochastic Gronwall lemma given in Lemma 7.7.2 below. We define $\mathcal{R} = \tilde{u} - \tilde{v}$. Due to the bilinear term $B(\tilde{u})$, when attempting to estimate $\mathcal{R}$, the terms that involve only $\tilde{u}$ or $\tilde{v}$ will arise. To deal with this issue we define the stopping times

$$\begin{aligned} \tau^{(m)} &= \inf_{t\geq 0}\left\{\sup_{s\in[0,t]}|\tilde{u}|^2 + \int_0^t |\nabla \tilde{u}|^2\,ds + \sup_{s\in[0,t]}|\tilde{v}|^2 + \int_0^t |\nabla \tilde{v}|^2\,ds \geq m\right\} \\ &= \sup_{t\geq 0}\left\{\sup_{s\in[0,t]}|\tilde{u}|^2 + \int_0^t |\nabla \tilde{u}|^2\,ds + \sup_{s\in[0,t]}|\tilde{v}|^2 + \int_0^t |\nabla \tilde{v}|^2\,ds \leq m\right\}. \end{aligned} \tag{6.152}$$

We deduce from (6.104) that $\lim_{m\to\infty}\tau^{(m)} = \infty$. Define $\bar{\mathcal{R}} = \mathbf{1}_{\Omega_0}\mathcal{R}$, and the result will follow once we show that for any m,

$$\tilde{\mathbb{E}}\left(\sup_{[0,\tau^{(m)}\wedge T]}|\bar{\mathcal{R}}|^2\right) = 0. \tag{6.153}$$

Subtracting the equation (6.1) for $\tilde{v}$ from that for $\tilde{u}$, multiplying both sides by $\mathbf{1}_{\Omega_0}$, we arrive at the following equation for $\bar{\mathcal{R}}$,

$$\begin{cases} d\bar{\mathcal{R}} + \Delta\bar{\mathcal{R}}_x\,dt = \left(-c\bar{\mathcal{R}}_x + \mathbf{1}_{\Omega_0}(B(\tilde{v}) - B(\tilde{u}))\right)\,dt + \mathbf{1}_{\Omega_0}(\sigma(\tilde{u}) - \sigma(\tilde{v}))\,d\tilde{W}(t), \\ \bar{\mathcal{R}}(0) = 0. \end{cases} \tag{6.154}$$

Hence together with the stochastic basis $\tilde{\mathcal{S}}$, we can regard $\bar{\mathcal{R}}$ as a global pathwise solution to (6.154) written as (6.126) with the boundary conditions (2.2) and (2.4) , where $g = -c\bar{\mathcal{R}}_x + \mathbf{1}_{\Omega_0}(B(\tilde{v}) - B(\tilde{u}))$ and $h = \mathbf{1}_{\Omega_0}\,(\sigma(\tilde{u}) - \sigma(\tilde{v}))$. To apply Proposition 6.3.2, now we only need to show that (6.128) and (6.129) are satisfied. We infer from (6.69) that $g \in L^2(\tilde{\Omega};\, L^2(0,T;\, \Xi_2'))$. To show that g also belongs to the space $L^2(\tilde{\Omega};\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M})))$, we first note that $\mathcal{R}_x \in L^2(\tilde{\Omega};\, L^2((0,T);\, L^2(\mathcal{M})))$. Next we estimate $B(u)$. By the Sobolev embedding theorem in dimension 2, we deduce that $|B(\tilde{u})|_{L^{4/3}(\mathcal{M})} \leq c'|\tilde{u}|_{L^4(\mathcal{M})}|\tilde{u}_x| \leq c'|\tilde{u}|^{1/2}|\nabla\tilde{u}|^{1/2}|\tilde{u}_x|$, and hence almost surely

$$\begin{aligned} \left(\int_0^T |B(\tilde{u})|^{4/3}_{L^{4/3}(\mathcal{M})}\,dt\right)^{3/2} &\leq c_2\left(\int_0^T |\tilde{u}|^{2/3}|\nabla\tilde{u}|^2\,dt\right)^{3/2} \\ &\leq c'\left(\sup_{t\in[0,T]}|\tilde{u}|^{2/3}\int_0^T |\nabla\tilde{u}|^2\,dt\right)^{3/2} \\ &= c'\sup_{t\in[0,T]}|\tilde{u}|\left(\int_0^T |\nabla\tilde{u}|^2\,dt\right)^{3/2} \\ &\leq c'\sup_{t\in[0,T]}|\tilde{u}|^7 + \left(\int_0^T |\nabla\tilde{u}|^2\,dt\right)^{7/4}. \end{aligned} \tag{6.155}$$

Since (6.81) and (6.83) imply that $\tilde{u}$ and $\tilde{v}$ both belong to the space $L^7(\tilde{\Omega};\, L^\infty((0,T);\, H_0^1(\mathcal{M})))\cap L^{7/2}(\tilde{\Omega};\, L^2((0,T);\, H_0^1(\mathcal{M})))$, taking expectations on both sides of (6.155) we obtain

$$B(\tilde{u}) \text{ belong to } L^2(\tilde{\Omega};\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M}))). \tag{6.156}$$

To conclude we infer that $g \in L^2(\tilde{\Omega};\, L^{4/3}(0,T;\, L^{4/3}(\mathcal{M})))$. We infer from (6.11) that

$$||h||_{L_2(\mathfrak{U};\,\Xi_1)} = ||\sigma(\tilde{u}) - \sigma(\tilde{v})||_{L_2(\mathfrak{U};\,\Xi_1)} \leq c_U|\mathcal{R}|,$$

which implies that $h \in L^2(\tilde\Omega;\, L^2(0,T;\, L_2(\mathfrak{U}, \Xi_1)))$. Similarly, By (6.9) we can deduce that $h \in L^2(\tilde\Omega;\, L^2(0,T;\, L_2(\mathfrak{U}, L^2(\mathcal{M}))))$. Thus we have proven that h satisfies (6.129).

Now Proposition 6.3.2 applies, and we obtain (6.132) for any τ_b with $0 \le \tau_b \le \tau^{(m)} \wedge T$, $\tau^{(m)}$ defined as in (6.152) (for notation simplicity, we will write $\tau^{(m)} := \tau^{(m)} \wedge T$ from now on). We then estimate the right-hand side of (6.132). Thanks to (6.152), we see that

$$\mathcal{R}(\cdot \wedge \tau^{(m)}) \in L^\infty(\tilde\Omega;\, L^\infty((0,T);\, L^2(\mathcal{M}))) \cap L^\infty(\tilde\Omega;\, L^2((0,T);\, H_0^1(\mathcal{M}))), \tag{6.157}$$

and hence the following calculations are all legitimate for $t \in (0, \tau^{(m)})$. We observe that a.s. and for a.e. t:

$$\begin{aligned}
\big|(g,\,(1+x)\bar{\mathcal{R}})\big| &= \big|-c(\bar{\mathcal{R}}_x,\,(1+x)\bar{\mathcal{R}}) + (B(\tilde v) - B(\tilde u)),\,(1+x)\bar{\mathcal{R}})\big| \\
&= \left|\frac{c}{2}|\bar{\mathcal{R}}|^2 + \left(\bar{\mathcal{R}}^2,\,(1+x)\tilde u_x - \frac{1}{2}(\tilde v + (1+x)\tilde v_x)\right)\right| \\
&\le (\text{with } \gamma(t) = |\tilde u_x(t)| + |\tilde v(t)| + |\tilde v_x(t)|) \\
&\le \frac{c}{2}|\bar{\mathcal{R}}|^2 + c'\gamma(t)|\bar{\mathcal{R}}|^2_{L^4(\mathcal{M})} \\
&\le (\text{by interpolation } H^{1/2} \subset L^4 \text{ in dimension 2}) \\
&\le \frac{c}{2}|\bar{\mathcal{R}}|^2 + c'\gamma(t)|\bar{\mathcal{R}}||\nabla\bar{\mathcal{R}}| \\
&\le \frac{1}{2}|\nabla\bar{\mathcal{R}}|^2 + c'\gamma^2(t)|\bar{\mathcal{R}}|^2.
\end{aligned} \tag{6.158}$$

Applying (6.158) to (6.132), with $\bar{\mathcal{R}}(0) = 0$ we obtain

$$\frac{1}{2}\,\tilde{\mathbb{E}} \sup_{0\le s\le\tau_b} |\bar{\mathcal{R}}(s)|^2 + \frac{1}{2}\,\mathbb{E}\int_0^{\tau_b} |\nabla\bar{\mathcal{R}}|^2\,dt \le c'\,\tilde{\mathbb{E}}\int_0^{\tau_b} \gamma^2(t)|\bar{\mathcal{R}}|^2\,dt, \tag{6.159}$$

for any stopping time τ_b with $0 \le \tau_b \le \tau^{(m)}$. Along with (6.157), the version of the stochastic Gronwall Lemma given in Lemma 7.7.2 below applies. Hence we obtain (6.153). This completes the proof of Proposition 6.3.3. □

Thanks to the pathwise uniqueness of martingale solutions, we can apply the Gyöngy-Krylov Theorem to prove the existence of the pathwise solutions (for more details, see [11]).

Proof of Theorem 6.3.2. We consider the families $(u^\epsilon, u^{\epsilon'}, W)$, where u^ϵ and $u^{\epsilon'}$ are pathwise solutions to the parabolic regularization (6.16)-(6.18), (2.2) and (2.4). Then by

(6.99) and (6.100), we can define the joint distributions of $(u^\epsilon, u^{\epsilon'}, W)$ as $\nu^{\epsilon,\epsilon'} = \mu_u^\epsilon \times \mu_u^{\epsilon'} \times \mu_W$ on the phase space $\mathcal{X}_u \times \mathcal{X}_u \times \mathcal{X}_W$ ($\mathcal{X}_u$ and $\mathcal{X}_W$ defined in (6.98)). With the same argument as for Lemma 6.3.4, we can show that the family $\{\nu^{\epsilon,\epsilon'}\}$ is tight in ϵ and ϵ'. By the Skorokhod embedding theorem, we deduce the existence of a family $(\tilde{u}^\epsilon, \tilde{\tilde{u}}^{\epsilon'}, \tilde{W})$ defined on a different probability space which converges almost surely to an element $(\tilde{u}, \tilde{\tilde{u}}, \tilde{W})$. By the same proof as for Proposition of 6.3.1, we can show that $(\tilde{u}^\epsilon, \tilde{W})$ and $(\tilde{\tilde{u}}^{\epsilon'}, \tilde{W})$ both satisfy (i)-(v). In particular, $(\tilde{u}^\epsilon, \tilde{\tilde{u}}^{\epsilon'})$ have the same probability distributions, $\mu_u^\epsilon \times \mu_u^{\epsilon'}$, as $(u^\epsilon, u^{\epsilon'})$, and the family $\{\mu_u^\epsilon \times \mu_u^{\epsilon'}\}_{\epsilon,\epsilon'>0}$ is tight and hence converges weakly to a probability measure μ_1, defined by $\mu_1(\cdot) = \mathbb{P}(\tilde{u}, \tilde{\tilde{u}} \in \cdot)$. It is clear that $\tilde{u}$ and $\tilde{\tilde{u}}$ are both martingale solutions to (6.1) and (2.2)-(2.4), hence by the pathwise uniqueness (Proposition 6.3.3), $\tilde{u} = \tilde{\tilde{u}}$ a.s.. Thus

$$\mu_1(\{(u,v) \in \mathcal{X}_u \times \mathcal{X}_u : u = v\}) = \mathbb{P}(\tilde{u} = \tilde{\tilde{u}} \text{ in } \mathcal{X}_u) = 1.$$

We can apply the Gyöngy-Krylov Theorem (Theorem 7.5.1 of the Appendix) and deduce that the original family u^ϵ defined on the initial probability space converges in probability, and hence converges almost surely up to a subsequence, to an element u in the topology of $\mathcal{X}_u$. Thus we can pass to the limit on the regularized equation as explained in details in Section 6.3.1. To conclude we have established the existence of a pathwise solutions to (6.1), (2.2)-(2.4), and we have completed the proof of Theorem 6.3.2. □

Remark 6.3.2. *For the space periodic case, that is, (6.1) and the boundary and initial conditions (2.2), (2.3) and (2.5), the results will be the same with the Dirichlet case. The reasoning will be similar as in Chapter 3.*

Chapter 7

Appendix

7.1 Space $\Sigma(\mathcal{M})$ and its Dual $\Sigma'(\mathcal{M})$

To derive and justify equation (2.9) we need to introduce the concept of distributions on

$$\tilde{\mathcal{M}} = I_x \times \mathbb{R}_y \times \mathbb{R}_z, \tag{7.1}$$

which are periodic in y and z (and usual distributions in x). This concept combines the concept of distributions on an open set (see [41]) with that of periodic distributions (see [41] and [19]).

We consider the space $\Sigma(\mathcal{M})$ of functions which are $\mathcal{C}^\infty$ in $\tilde{\mathcal{M}}$, periodic in the y and z directions, periodic with all the derivatives, with period 2π, and which are compactly supported in I_x. This space is equipped with the (metrizable) topology of uniform convergence on any compact set of $\tilde{\mathcal{M}}$ of the functions and all its derivatives. The dual $\Sigma'(\mathcal{M})$ of $\Sigma(\mathcal{M})$ is the desired space of distributions. One can define derivatives on this space and perform the usual operations performed on distributions. Also, as in the case of $\mathcal{D}(\mathcal{M})$, we can show that the space spanned by $\varphi\,\omega_{k_1}(y)\,\omega_{k_2}(z)$ with $\varphi \in \mathcal{D}(I_x)$ is dense in $\Sigma(\mathcal{M})$, where $\{\omega_{k_1}(y)\}$ is an orthonormal basis of $L^2(I_y)$ and a smooth periodic functions with period 2π, and the same thing for $\{\omega_{k_2}(z)\}$ in $L^2(I_z)$. We will also assume below that the ω_{k_1} are the eigenfunctions of the operators d^2/dy^2 on I_y (sine and cosine functions) which form an orthonormal basis of $L^2(I_y)$, and the ω_{k_2} are the eigenfunctions of the operators d^2/dz^2 on

I_z which form an orthonormal basis of $L^2(I_z)$.

If a function u belongs to $L^2(\mathcal{M})$, let $\bar{u}$ be its extension to $\tilde{\mathcal{M}}$ by periodicity in y and z and let

$$u = \sum_k \widehat{u}_k(x)\, \omega_{k_1}(y)\, \omega_{k_2}(z) \tag{7.2}$$

be its expansion in $L^2(I_y \times I_z)$. This expansion being convergent in $L^2(\mathcal{M})$ as observed before, it also converges to u in $\Sigma'(\mathcal{M})$, which simply means that

$$\int_{I_x} \int_{I_y \times I_z} u_N(x,y,z)\, \psi(x,y,z)\, dxdydz$$

converges to

$$\int_{I_x} \int_{I_y \times I_z} u(x,y,z)\, \psi(x,y,z)\, dxdydz,\ \forall \psi \in \Sigma(\mathcal{M}),$$

where

$$u_N(x,y,z) = \sum_{|k| \le N} \widehat{u}_k(x)\, \omega_{k_1}(y)\, \omega_{k_1}(z),\ N \in \mathbb{N}. \tag{7.3}$$

With these in mind we can state the following:

Lemma 7.1.1. *For $u \in L^2(\mathcal{M})$ as in (7.2), we have*

$$\langle \Delta u_x + c u_x,\, \varphi\, \omega_{k_1}(y)\, \omega_{k_2}(z) \rangle = -C_d\, \langle \varphi''' - (\lambda_k - c)\varphi',\, \widehat{u}_k \rangle,\ \ \forall \varphi \in \mathcal{D}(I_x),$$

where $\lambda_k = \lambda_{k_1} + \lambda_{k_2}$, and C_d is a constant depending only on the dimension.

Hence

$$\langle \Delta u_x + c u_x,\ \omega_{k_1}(y)\, \omega_{k_2}(z) \rangle = C_d(\widehat{u}_k'''(x) - (\lambda_k - c)\widehat{u}_k'(x)), \tag{7.4}$$

in the sense of distributions on $\mathcal{D}(I_x)$.

Proof. We observe that, for $\varphi \in \mathcal{D}(I_x)$, $\varphi\, \omega_{k_1}(y)\, \omega_{k_2}(z) \in \Sigma(\mathcal{M})$ and then classically

$$\begin{aligned}
\langle \Delta u_x + c u_x,\ \varphi\, \omega_{k_1}\, \omega_{k_2} \rangle &= -\langle u,\ (\Delta + c)\partial_x(\varphi\, \omega_{k_1}\, \omega_{k_2}) \rangle \\
&= -\langle u,\ (\varphi''' - (\lambda_k - c)\varphi')\, \omega_{k_1}\, \omega_{k_2} \rangle \\
&= -C_d \langle \widehat{u}_k,\ \varphi''' - (\lambda_k - c)\varphi' \rangle \\
&= C_d \langle \widehat{u}_k''' - (\lambda_k - c)\widehat{u}_k',\ \varphi \rangle,
\end{aligned}$$

and the conclusion follows. □

Corollary 7.1.1. *Under the hypothesis of Lemma 7.1.1, if we further assume that* $\Delta u_x + cu_x \in L^2(\mathcal{M})$*, then*

$$\widehat{u}_k''' - (\lambda_k - c)\widehat{u}_k' = \widehat{g}_k, \tag{7.5}$$

holds in $L^2(\mathcal{M})$*, where* $g := \Delta u_x + cu_x$*, and* $\widehat{g}_k(x)$ *are the Fourier coefficients of* g *as in (7.2).*

Proof. Since $g \in L^2(\mathcal{M})$, $\langle g,\ \omega_{k_1}\,\omega_{k_2}\rangle = C_d\,\widehat{g}_k$. Hence by (7.4), we obtain that (7.5) holds in the sense of distributions on $\mathcal{D}(I_x)$.

Define

$$Y_k = \widehat{u}_k'' - (\lambda_k - c)\widehat{u}_k - \int_0^x \widehat{g}_k(\xi)d\xi. \tag{7.6}$$

Then (7.5) implies $Y_k' = 0$. Hence $Y_k = C(k)$ *a.e.*, with $C(k)$ a constant independent of x. Thus $\widehat{u}_k'' \in L^2(I_x)$. By the intermediate derivatives theorem, $\widehat{u}_k' \in L^2(I_x)$. Hence (7.5) implies

$$\widehat{u}_k''' \in L^2(I_x). \tag{7.7}$$

Thus (7.5) holds in $L^2(I_x)$.

Remark 7.1.1. *Note that we will also briefly use the space* $\Sigma'((0,T)\times\Omega)$ *corresponding to distributions that are regular distributions in* x *and* t*, and periodic in* y *or* y *and* z*.*

7.2 Trace Results

The following is a slight generalization used in the article of a trace result from [40] allowing the value $p = 1$.

Lemma 7.2.1. *Let* Y *be a reflexive Banach space and let* $p \geq 1$*. Assume that two sequences* *of functions* u^ϵ, $g^\epsilon \in L^p_x(I_x;Y)$ *satisfy*

$$\begin{cases} u^\epsilon_{xxx} + \epsilon u^\epsilon_{xxxx} = g^\epsilon, \\ u^\epsilon(0) = u^\epsilon(1) = u^\epsilon_x(1) = u^\epsilon_{xx}(0) = 0, \end{cases} \tag{7.8}$$

with g^ϵ *bounded in* $L^p_x(I_x;Y)$ *as* $\epsilon \to 0$. *Then* u^ϵ_{xx} *(and hence* u^ϵ_x*, and* u^ϵ*) is bounded in* $L^\infty_x(I_x;Y)$ *as* $\epsilon \to 0$. *Furthermore, for any subsequences* $u^\epsilon \to u$ *converging (strongly or weakly) in* $L^q_x(I_x;Y)$, $1 \le q < \infty$, $u^\epsilon_x(1)$ *converges to* $u_x(1)$ *in* Y *(weakly at least), and hence* $u_x(1) = 0$.

Proof. Firstly, by the same proof as in [40], we obtain

$$|u^\epsilon_{xx}(x)|_Y \le c'|g^\epsilon|_{L^1(Y)}, \tag{7.9}$$

which shows that u^ϵ_{xx} remains bounded in $L^\infty(0,1;Y)$ as $\epsilon \to 0$.

Secondly, we prove the weak convergence of $u^\epsilon_x(1)$. Since u^ϵ_{xx} is bounded in $L^\infty_x(I_x;Y)$ as $\epsilon \to 0$, applying the Banach-Alaugu theorem to the reflexive Banach space $L^r_x(I_x;Y)$, $1 < r < \infty$, we deduce that $u_{xx} \rightharpoonup v$ weakly in $L^r_x(I_x;Y)$ for some function $v \in L^r_x(I_x;Y)$. Since we know that $u^\epsilon \to u$ strongly or weakly in $L^q_x(I_x;Y)$, it is easy to see that $v = u_{xx} \in L^q_x(I_x;Y)$. Hence we have

$$u^\epsilon_{xx} \rightharpoonup u_{xx} \text{ weakly in } L^q_x(I_x;Y). \tag{7.10}$$

By the trace theorem and Mazur's theorem, $u_x(1)$ converges to $u_x(1)$ weakly in Y.

Remark 7.2.1. *Note that in (7.10),* q *can not be* ∞*, and hence we can not pass to the limit on the boundary term* $u^\epsilon_{xx}(0)$ *based on the argument above. This is fine because we do not care about passing to the limit on this boundary term as we do not necessarily have* $u_{xx}(0) = 0$ *in the Z-K equation.*

The following trace result is an extension of the linear case of Lemma 3.1.1.

Lemma 7.2.2. *Let u be a random process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. F is a given function such that*

$$F \in L^p(I_x; \mathcal{E}), \text{ where } \mathcal{E} = L^p(\Omega \times (0,T) \times I_{x^\perp}), \; p > 1. \tag{7.11}$$

We assume that $u \in L^2(\Omega; L^2(0,T; H^1(\mathcal{M})))$ satisfies almost surely the following linear equation

$$u_t + \Delta u_x + cu_x = F, \tag{7.12}$$

that is, almost surely we have

$$u(t) + \int_0^t (\Delta u_x + cu_x) ds = u(0) + \int_0^t F\, ds,$$

in the sense of distributions on $\mathcal{D}(\mathcal{M})$ for every $0 \leq t \leq T$. Then

$$u_x, u_{xx} \in \mathcal{C}_x(I_x; \mathcal{B}), \;\; \mathcal{B} = L^2(\Omega; H^{-2}((0,T) \times I_{x^\perp})) \cap \mathcal{E}. \tag{7.13}$$

and, in particular,

$$u_x\big|_{x=0,1} \text{ and } u_{xx}\big|_{x=0,1}, \tag{7.14}$$

are well defined in $\mathcal{B}$.

Proof. We write equation (7.12) in the form

$$u_{xxx} = F - cu_x - \Delta^\perp u_x - u_t.$$

Then clearly we have

$$u_{xxx} \in L_x^{p\wedge 2}(I_x; L^2(\Omega; H^{-2}((0,T) \times I_{x^\perp})) \cap \mathcal{E}), \;\; p \geq 1, \tag{7.15}$$

which implies that (7.13) holds. □

We use Lemma 7.2.2 in the proof of Lemma 6.3.5 with $p = 5/4$ and $\mathcal{E} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})$, and in the proof of Proposition 6.3.2 with $p = 4/3$ and $\mathcal{E} = L^{4/3}(\Omega \times (0,T) \times I_{x^\perp})$.

We are now ready to prove the following trace result generalized from the argument in Lemma 7.2.1.

Lemma 7.2.3. *Let $\{u^\epsilon\}_{\epsilon>0}$ be a family of random processes, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the following linearized regularized equation*

$$\begin{cases} u_t^\epsilon + \Delta u_x^\epsilon + c u_x^\epsilon + \epsilon L u^\epsilon = F^\epsilon, \\ u^\epsilon\big|_{x=0} = u^\epsilon\big|_{x=1} = u_x^\epsilon\big|_{x=1} = u_{xx}^\epsilon\big|_{x=0} = 0, \end{cases} \tag{7.16}$$

where F^ϵ remains bounded in the reflexive Banach space $L^p(I_x, \tilde{\mathcal{E}})$, with $\tilde{\mathcal{E}} := L^p(\Omega \times (0,T) \times I_{x^\perp})$, $p > 1$. We suppose that u^ϵ almost surely satisfies (7.16), that is, almost surely we have

$$u^\epsilon(t) + \int_0^t (\Delta u_x^\epsilon + c u_x^\epsilon + \epsilon L u^\epsilon - F^\epsilon)\, ds = u^\epsilon(0),$$

in the sense of distributions on $\mathcal{D}(\mathcal{M})$ for every $0 \le t \le T$. We assume that u^ϵ converges weakly to some u in $L^2(\Omega;\, L^2(0,T;\, H_0^1(\mathcal{M})))$ as $\epsilon \to 0$, then $u_x^\epsilon(1)$ converges to $u_x(1)$ in $\tilde{\mathcal{B}}$ specified below, and hence $u_x(1) = 0$.

Proof. By (7.16) we have

$$u_{xxx}^\epsilon + \epsilon u_{xxxx}^\epsilon = F^\epsilon - u_t^\epsilon - c u_x^\epsilon - \Delta^\perp u_x^\epsilon - \epsilon u_{yyyy}^\epsilon - \epsilon u_{zzzz}^\epsilon.$$

We call the right hand side g^ϵ. It is easy to observe that, since u^ϵ remains bounded in $L^2(I_x;\, L^2(\Omega \times (0,T) \times I_{x^\perp})$ as $\epsilon \to 0$, then g^ϵ remains bounded in the reflexive Banach space $L_x^{p\wedge 2}(I_x;\, \tilde{\mathcal{B}})$, $p > 1$, where

$$\tilde{\mathcal{B}} = L^2(\Omega;\, H_t^{-1}(0,T;\, L^2(I_{x^\perp}))) + L^2(\Omega;\, L_t^2(0,T;\, H^{-4}(I_{x^\perp}))) + \tilde{\mathcal{E}}. \tag{7.17}$$

Thus we can apply Lemma 7.2.1 with this space $\tilde{\mathcal{B}}$ and obtain the convergence of the boundary term $u_x^\epsilon(1)$. □

Lemma 7.2.3 is applied in Section 6.3.1 with $p = 5/4$ and $\tilde{\mathcal{E}} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x^\perp})$ and in the proof of Proposition 6.3.2, with $p = 4/3$, $\tilde{\mathcal{E}} = L^{4/3}(\Omega \times (0,T) \times I_{x^\perp})$ and $\tilde{\mathcal{B}} = L^2(\Omega;\, H_t^{-1}(0,T;\, L^2(I_{x^\perp}))) + L^2(\Omega;\, L_t^2(0,T;\, H^{-4}(I_{x^\perp}))) + L^{4/3}(\Omega \times (0,T) \times (I_{x^\perp}))$.

7.3 Compact Embedding Theorems

We recall the theorems from [18] and [17] (see also [43] for Lemma 7.3.1).

Definition 7.3.1. *(The Fractional Derivative Space) We assume that* H *is a separable Hilbert space. Given* $\tilde{p} \geq 2$, $\alpha \in (0,1)$, $W^{\alpha,\tilde{p}}(0,T;\,H)$ *denotes the Sobolev space of all* $h \in L^{\tilde{p}}(0,T;\,H)$ *such that*

$$\int_0^T \int_0^T \frac{|h(t)-h(s)|_H^{\tilde{p}}}{|t-s|^{1+\alpha\tilde{p}}}\,dt\,ds < \infty, \tag{7.18}$$

which is endowed with the Banach norm

$$|h|_{W^{\alpha,\tilde{p}}(0,T;\,H)} = \left(\int_0^T |h(t)|_H^{\tilde{p}}\,dt + \int_0^T \int_0^T \frac{|h(t)-h(s)|_H^{\tilde{p}}}{|t-s|^{1+\alpha\tilde{p}}}\,dt\,ds\right)^{1/\tilde{p}} < \infty. \tag{7.19}$$

Lemma 7.3.1. *(i) Let* $\mathcal{E}_0 \subset \mathcal{E} \subset \mathcal{E}_1$ *be Banach spaces,* $\mathcal{E}_0$ *and* $\mathcal{E}_1$ *reflexive, with continuous injections and a compact embedding of* $\mathcal{E}_0$ *in* $\mathcal{E}$. *Let* $1 < p < \infty$ *and* $\alpha \in (0,1)$ *be given. Let* $\mathcal{Y}$ *be the space*

$$\mathcal{Y} := L^p(0,T;\,\mathcal{E}_0) \cap W^{\alpha,p}(0,T;\,\mathcal{E}_1), \tag{7.20}$$

endowed with the natural norm. Then the embedding of $\mathcal{Y}$ *in* $L^p(0,T;\,\mathcal{E})$ *is compact.*

(ii) If $\mathcal{E} \subset \bar{\mathcal{E}}$ *are two Banach spaces with* $\mathcal{E}$ *compactly embedded in* $\bar{\mathcal{E}}$, $1 < p < \infty$ *and* $\alpha \in (0,1)$ *satisfy*

$$\alpha p > 1,$$

then the space $W^{\alpha,p}(0,T;\,\mathcal{E})$ *is compactly embedded into* $\mathcal{C}([0,T];\,\bar{\mathcal{E}})$.

7.4 Convergence Theorem for the Noise Term

The following convergence theorem for the stochastic integrals is used to facilitate the passage to the limit in the parabolic regularization approximation. The statements and proofs can be found in [2], [23] and [11].

Lemma 7.4.1. *Let* $\{\Omega, \mathcal{F}, \mathbb{P}\}$ *be a fixed probability space, and* $\mathcal{X}$ *a separable Hilbert space. Consider a sequence of stochastic bases* $\mathcal{S}_n := (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t\geq 0}, \mathbb{P}, W^n)$*, such that each* W^n *is a cylindrical Brownian motion (over* $\mathfrak{U}$*) with respect to* $\{\mathcal{F}_t^n\}_{t\geq 0}$*. We suppose that the* $\{G^n\}_{n\geq 1}$ *are a sequence of* $\mathcal{X}$*-valued* $\mathcal{F}_t^n$ *predictable processes so that* $G^n \in L^2((0,T);\, L_2(\mathfrak{U}, \mathcal{X}))$ *a.s.. Finally consider* $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ *and a function* $G \in L^2((0,T);\, L_2(\mathfrak{U},\, \mathcal{X}))$*, which is* $\mathcal{F}_t$ *predictable. If*

$$W^n \to W \text{ in probability in } \mathcal{C}([0,T];\, \mathfrak{U}_0),$$

$$G^n \to G \text{ in probability in } L^2((0,T);\, L_2(\mathfrak{U}, \mathcal{X})),$$

then

$$\int_0^t G^n\, dW^n \to \int_0^t G\, dW \text{ in probability in } L^2((0,T);\, \mathcal{X}).$$

Then we have the following lemma based on the Burkholder-Davis-Gundy inequality and the notion of fractional derivatives in Definition 7.3.1 (whose proof can be found in [18]).

Lemma 7.4.2. *Let* $q \geq 2$*,* $\alpha > \frac{1}{2}$ *be given so that* $q\alpha > 1$*. Then for any predictable process* $h \in L^q(\Omega \times (0,T);\, L_2(\mathfrak{U}, H))$*, we have*

$$\int_0^t h(s)\, dW(s) \in L^q(\Omega;\, W^{\alpha,q}(0,T;\, H)),$$

and there exists a constant $c' = c'(q,\alpha) \geq 0$ *independent of* h *such that*

$$\mathbb{E}\left|\int_0^t h(s)\, dW(s)\right|^q_{W^{\alpha,q}(0,T;\, H)} \leq c'(q,\alpha)\, \mathbb{E}\int_0^t |h(s)|^q_{L_2(\mathfrak{U},H)}\, ds. \tag{7.21}$$

7.5 Some Probability Tools

We recall the Gyöngy-Krylov Theorem from [23], which is used in proving the existence of pathwise solutions.

Theorem 7.5.1. *Let $\mathcal{X}$ be a Polish space. Suppose that $\{Y_m\}$ is a sequence of $\mathcal{X}$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mu_{k,m}\}_{k,m\geq 1}$ be the sequence of joint laws of $\{Y_m\}_{m\geq 1}$, that is*

$$\mu_{k,m}(E) := ((Y_k, Y_m) \in E), \quad E \in \mathcal{B}(\mathcal{X} \times \mathcal{X}).$$

Then $\{Y_m\}$ converges in probability if and only if for every subsequence of joint probability measures, $\{\mu_{k_l,m_l}\}_{l\geq 0}$, there exists a further subsequence which converges weakly to a probability measure μ such that

$$\mu(\{(u,v) \in \mathcal{X} \times \mathcal{X} : u = v\}) = 1. \tag{7.22}$$

7.6 The Jakubowski-Skorokhod Representation Theorem

We recall the following result from [35].

Lemma 7.6.1. *Let $\mathcal{A}_1$ be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : \mathcal{A}_1 \to \mathcal{R}$ that separate points of $\mathcal{A}_1$. Let $\mathcal{A}_2$ be a Polish space, that is, a separable completely metrizable topological space, and let $I : \mathcal{A}_2 \to \mathcal{A}_1$ be a continuous injection. Then $I(B)$ is a Borel set in $\mathcal{A}_1$ whenever B is Borel in $\mathcal{A}_2$.*

The following result is a special case of Lemma 7.6.1.

Lemma 7.6.2. *Let $\mathcal{A}_1$ be a separable Hilbert space. Assume that $\mathcal{A}_2$ is a separable Hilbert space continuously injected into $\mathcal{A}_1$. Then $\mathcal{A}_2$ is a Borel set of $\mathcal{A}_1$.*

Proof. Firstly, it is clear that any separable Hilbert space $\mathcal{A}_1$ satisfies the hypotheses of Lemma 7.6.1. Since $\mathcal{A}_2$ is a separable Hilbert space, hence it is a Polish space. Now in Lemma 7.6.1, let B be $\mathcal{A}_2$, which of course is a Borel set of $\mathcal{A}_2$. Then $I(B) = I(\mathcal{A}_2) = \mathcal{A}_2$ is a Borel set in $\mathcal{A}_1$ thanks to Lemma 7.6.1. □

We use Lemma 7.6.2 in the proof of Proposition 6.3.1.

7.7 An Adapted Stochastic Gronwall Lemma

We first recall the stochastic Gronwall lemma from [21] (see also [34]), then we present a variant result which is used in the proof of Proposition 6.3.3.

Lemma 7.7.1. *Fix $T > 0$. We assume that*

$$X, Y, Z, M : [0, T) \times \Omega \to \mathbb{R},$$

are real valued, non-negative stochastic processes. Let $0 \le \tau < T$ be a stopping time so that

$$\mathbb{E} \int_0^{\tau} (MX + Z)\, ds < \infty. \tag{7.23}$$

We suppose, moreover that for some fixed constant κ,

$$\int_0^{\tau} M\, ds < \kappa, \quad a.s.. \tag{7.24}$$

Suppose that for all stopping times τ_a, τ_b with $0 \le \tau_a \le \tau_b \le \tau$ we have

$$\mathbb{E} \left(\sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y\, ds \right) \le C_0\, \mathbb{E} \left(X(\tau_a) + \int_{\tau_a}^{\tau_b} (MX + Z)\, ds \right), \tag{7.25}$$

where C_0 is a constant independent of the choice of τ_a, τ_b. Then

$$\mathbb{E} \left(\sup_{t \in [\tau_0, \tau]} X + \int_{\tau_0}^{\tau} Y\, ds \right) \le C\, \mathbb{E} \left(X(0) + \int_{\tau_0}^{\tau} Z\, ds \right), \tag{7.26}$$

where $C = C(C_0, T, \kappa)$.

When $X(0) = 0$ and $Z = 0$ we can weaken the hypotheses by requiring that (7.25) only holds for $\tau_a = 0$ and all τ_b, $0 \le \tau_b \le \tau$. We then obtain

Lemma 7.7.2. *We assume that $X(0) = 0$ and $Z = 0$ in Lemma 7.7.1 and that (7.25) holds only for $\tau_a = 0$ and all τ_b, $0 \le \tau_b \le \tau$, that is:*

$$\mathbb{E} \left(\sup_{t \in [0, \tau_b]} X + \int_0^{\tau_b} Y\, ds \right) \le C_0\, \mathbb{E} \left(X(0) + \int_0^{\tau_b} MX\, ds \right), \tag{7.27}$$

where C_0 is a constant independent of the choice of τ_b. Then the calculation (7.26) holds true and reduces to

$$\mathbb{E} \sup_{t\in[0,\tau]} X = \mathbb{E} \int_0^\tau Y\, ds = 0. \tag{7.28}$$

Proof. Step 1. We first show how to construct a finite sequence of stopping times

$$0 < \tau_1 < ... < \tau_N < \tau_{N+1} = \tau \quad a.s.,$$

so that

$$\int_{\tau_j}^{\tau_{j+1}} M\, ds < \frac{1}{2C_0} \;\; a.s., \;\; \forall\, j = 1, ..., N. \tag{7.29}$$

We construct the sequence inductively. We start with time 0. We assume that τ_{j-1} is found. Then define

$$\tau_j := \inf_{t\geq 0} \left\{ \int_{\tau_{j-1}}^t M\, ds < \frac{1}{2C_0} \right\} \wedge \tau,$$

and $\tau_j > 0$ is well-defined since $M > 0$ and it satisfies (7.24). Hence we have

$$\int_{\tau_{j-1}}^{\tau_j} M\, ds \geq \frac{1}{2C_0}, \quad \forall\, j \geq 1 \text{ such that } \tau_j < \tau. \tag{7.30}$$

Now we claim that there exists a finite integer N such that $\tau_N = \tau$, and

$$N \leq 2C_0\kappa + 1, \;\; a.s.. \tag{7.31}$$

We show this by contradiction; suppose that (7.31) is not true, then $N - 1 > 2C_0\kappa$, and hence

$$\int_0^{\tau_{N+1}} M\, ds = \sum_{j=1}^{N-1} \int_{\tau_j}^{\tau_{j+1}} M\, ds + \int_{\tau_N}^{\tau} M\, ds \geq \text{ with (7.30)} \geq (N-1)\frac{1}{2C_0} > \kappa.$$

But this contradicts with (7.24). Hence (7.31) is proven, and we can choose the integer $N = \ulcorner 2C_0\kappa + 1 \urcorner$.

Step 2. Letting $\tau_b = \tau_1$ in (7.27), we have

$$\mathbb{E}\left(\sup_{t\in[0,\tau_1]} X + \int_0^{\tau_1} Y\, ds \right) \leq C_0\, \mathbb{E} \int_0^{\tau_1} MX\, ds; \tag{7.32}$$

from (7.32), (7.23) and (7.29) we infer

$$\mathbb{E}\left(\frac{1}{2}\sup_{t\in[0,\tau_1]} X + \int_0^{\tau_1} Y\,ds\right) = 0. \tag{7.33}$$

Thanks to (7.33), for every $\tau_b \geq \tau_1$ a.s., we find that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_b]} X = \mathbb{E}\sup_{t\in[\tau_1,\tau_b]} X,\\
&\mathbb{E}\int_0^{\tau_b} Y\,ds = \mathbb{E}\int_{\tau_1}^{\tau_b} Y\,ds, \qquad (7.34)\\
&\mathbb{E}\int_0^{\tau_b} MX\,ds = \mathbb{E}\int_{\tau_1}^{\tau_b} MX\,ds.
\end{aligned}$$

Thus from (7.34) and (7.27), we infer that for every $\tau_b \geq \tau_1$ a.s.,

$$\mathbb{E}\left(\sup_{t\in[\tau_1,\tau_b]} X + \int_{\tau_1}^{\tau_b} Y\,ds\right) \leq C_0\,\mathbb{E}\left(\int_{\tau_1}^{\tau_b} MX\,ds\right). \tag{7.35}$$

Setting $\tau_b = \tau_2$ in (7.35), we have

$$\mathbb{E}\left(\sup_{t\in[\tau_1,\tau_2]} X + \int_{\tau_1}^{\tau_2} Y\,ds\right) \leq C_0\,\mathbb{E}\left(\int_{\tau_1}^{\tau_2} MX\,ds\right); \tag{7.36}$$

with (7.36), (7.23) and (7.29) we deduce

$$\mathbb{E}\left(\frac{1}{2}\sup_{t\in[\tau_1,\tau_2]} X + \int_{\tau_1}^{\tau_2} Y\,ds\right) = 0. \tag{7.37}$$

Hence by finite induction up to N we obtain (7.28). $\square$

Bibliography

[1] E.S. Baykova and A. Faminskii. On initial-boundary value problem in a strip for generalized two-dimensional Zakharov-Kuznetsov equation. *Advances Diff. Equations*, (18(7–8)):663–686, 2013.

[2] A. Bensoussan. Stochastic Navier-Stokes equations. *Acta Appl. Math.*, 38(3):267–304, 1995.

[3] Patrick Billingsley. *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, second edition, 1986.

[4] J. L. Bona, W. G. Pritchard, and L. R. Scott. An evaluation of a model equation for water waves. *Philos. Trans. Roy. Soc. London Ser. A*, 302(1471):457–510, 1981.

[5] J. L. Bona, W. G. Pritchard, and L. R. Scott. A comparison of solutions of two model equations for long waves. In *Fluid dynamics in astrophysics and geophysics (Chicago, Ill., 1981)*, volume 20 of *Lectures in Appl. Math.*, pages 235–267. Amer. Math. Soc., Providence, R.I., 1983.

[6] J. L. Bona and R. Smith. The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287):555–601, 1975.

[7] C. Cao and E. Titi. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. *Ann. of Math. (2)*, 166(1):245–267, 2007.

[8] Thierry Colin and Marguerite Gisclon. An initial-boundary-value problem that approximate the quarter-plane problem for the Korteweg-de Vries equation. *Nonlinear Anal.*, 46(6, Ser. A: Theory Methods):869–892, 2001.

[9] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[10] Anne de Bouard and Arnaud Debussche. On a stochastic Korteweg-de Vries equation with homogeneous noise. In *Séminaire: Équations aux Dérivées Partielles. 2007–2008*, Sémin. Équ. Dériv. Partielles, pages Exp. No. V, 15. École Polytech., Palaiseau, 2009.

[11] Arnaud Debussche, Nathan Glatt-Holtz, and Roger Temam. Local martingale and pathwise solutions for an abstract fluids model. *Phys. D*, 240(14-15):1123–1144, 2011.

[12] Arnaud Debussche and Jacques Printems. Effect of a localized random forcing term on the Korteweg-de Vries equation. *J. Comput. Anal. Appl.*, 3(3):183–206, 2001.

[13] G. G. Doronin and N. A. Larkin. Exponential decay for the linear Zakharov-Kuznetsov equation without critical domain restrictions. *Appl. Math. Lett.*, 27:6–10, 2014.

[14] A. V. Faminskii. The Cauchy problem for the Zakharov-Kuznetsov equation. *Differentsial prime nye Uravneniya*, 31(6):1070–1081, 1103, 1995.

[15] A. V. Faminskii. On the nonlocal well-posedness of a mixed problem for the Zakharov-Kuznetsov equation. *Sovrem. Mat. Prilozh.*, (38):135–148, 2006.

[16] Andrei V. Faminskii. Well-posed initial-boundary value problems for the Zakharov-Kuznetsov equation. *Electron. J. Differential Equations*, pages No. 127, 23, 2008.

[17] F. Flandoli. An introduction to 3d stochastic fluid dynamics. In *SPDE in Hydrodynamic: Recent Progress and Prospects*, volume 1942 of *Lecture Notes in Mathematics*, pages 51–150. Springer Berlin/Heidelberg, 2008.

[18] F. Flandoli and D. Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields*, 102(3):367–391, 1995.

[19] G. B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[20] Wa Gao and Jun Dong Bao. Exact solutions for a $(2+1)$-dimensional stochastic KdV equation. *J. Jilin Univ. Sci.*, 44(1):46–49, 2006.

[21] N. Glatt-Holtz and M. Ziane. Strong pathwise solutions of the stochastic Navier-Stokes system. *Advances in Differential Equations*, 14(5-6):567–600, 2009.

[22] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[23] I. Gyöngy and N. Krylov. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields*, 105(2):143–158, 1996.

[24] Russell L. Herman and Andrew Rose. Numerical realizations of solutions of the stochastic KdV equation. *Math. Comput. Simulation*, 80(1):164–172, 2009.

[25] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[26] A. Jakubowski. The almost sure Skorokhod representation for subsequences in non-metric spaces. *Teor. Veroyatnost. i Primenen.*, 42(1):209–216, 1997.

[27] E.W. Laedke and K.H. Spatschek. Growth rates of bending solitons. *J. Plasma Phys.*, 26(3):469–484, 1982.

[28] Nikolai A. Larkin and Eduardo Tronco. Regular solutions of the 2D Zakharov-Kuznetsov equation on a half-strip. *J. Differential Equations*, 254(1):81–101, 2013.

[29] F. Linares, D. Lannes, and Jean-Claude Saut. The Cauchy problem for the Euler-Poisson system and derivation of the Zakharov-Kuznetsov equation. *Progress in Nonlinear Differential Equations and their Applications, M. Cicognani, FL. Colombini and D. Del Santo Eds.*, Vol 84:183–215, 2013.

[30] Felipe Linares and Jean-Claude Saut. The Cauchy problem for the 3D Zakharov-Kuznetsov equation. *Discrete Contin. Dyn. Syst.*, 24(2):547–565, 2009.

[31] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[32] Jacques L. Lions. *Problèmes aux limites dans les équations aux dérivées partielles*, volume 1962 of *Deuxième édition. Séminaire de Mathématiques Supérieures, No. 1 (Été.* Les Presses de l'Université de Montréal, Montreal, Que., 1965.

[33] Qing Liu. A modified Jacobi elliptic function expansion method and its application to Wick-type stochastic KdV equation. *Chaos Solitons Fractals*, 32(3):1215–1223, 2007.

[34] R. Mikulevicius and B. L. Rozovskii. Stochastic Navier-Stokes equations for turbulent flows. *SIAM J. Math. Anal.*, 35(5):1250–1310, 2004.

[35] Martin Ondreját. Stochastic nonlinear wave equations in local Sobolev spaces. *Electron. J. Probab.*, 15:no. 33, 1041–1091, 2010.

[36] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[37] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[38] Francis Ribaud and Stéphane Vento. A note on the Cauchy problem for the 2D generalized Zakharov-Kuznetsov equations. *C. R. Math. Acad. Sci. Paris*, 350(9-10):499–503, 2012.

[39] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[40] Jean-Claude Saut and Roger Temam. An initial boundary-value problem for the Zakharov-Kuznetsov equation. *Adv. Differential Equations*, 15(11-12):1001–1031, 2010.

[41] Laurent Schwartz. *Radon measures on arbitrary topological spaces and cylindrical measures*. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. Tata Institute of Fundamental Research Studies in Mathematics, No. 6.

[42] Walter A. Strauss. On continuity of functions with values in various Banach spaces. *Pacific J. Math.*, 19:543–551, 1966.

[43] R. Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1995.

[44] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

[45] V.E. Zakharov and E.A. Kuznetsov. On three-dimensional solitons. *Sov. Phys. JETP*, 30:285–286, 1974.

[46] Sheng Zhang and Hong-Qing Zhang. Fan sub-equation method for Wick-type stochastic partial differential equations. *Phys. Lett. A*, 374(41):4180–4187, 2010.

CURRICULUM VITAE OF CHUNTIAN WANG

Department of Mathematics and
The Institute for Scientific Computing & Applied Mathematics, Indiana University Bloomington

Tel: 812 855 3529
E-mail: wang211@umail.iu.edu
Web page: http://pages.iu.edu/~wang211/

Research Interests

Nonlinear Partial Differential Equations, Stochastic Partial Differential Equations, Fluid Dynamics, numerical analysis.

Education

Indiana University Bloomington
Ph.D, Applied Mathematics, Fall 2009 - May 2015
Thesis Advisors: Roger Temam & Nathan Glatt-Holtz
Thesis Title: "Initial and Boundary value problems for the Deterministic and Stochastic Zakharov--Kuznetsov Equation in a Bounded Domain"

Sichuan University, China
B.S., Applied Mathematics and Actuary, June 2008
M.S., Applied Mathematics, September 2008 - May 2009
Thesis Advisor: Tian Ma
Thesis Title: "Bifurcation of the Euler-Bernoulli pole"

Articles Published or Accepted in Refereed International Journals

[5] "*Time discrete approximation of weak solutions for stochastic equations of geophysical fluid dynamics and application*", with N. Glatt-Holtz & R. Temam, submitted.

[4] "*Martingale and pathwise solutions to the stochastic Zakharov-Kuznetsov equation with multiplicative noise*", with N. Glatt-Holtz & R. Temam, Discrete Contin. Dyn. Syst. Ser. B., 19 (2014), no. 4, 1047-1085.

[3] "*The existence of strong solutions to the 3D Zakharov-Kuznetsov equation in a bounded domain*", Discrete Contin. Dyn. Syst. A., 34 (2014), no. 11, 4897-4910.

[2] "*Local existence of strong solutions to the 3D Zakharov-Kuznetsov equation in a bounded domain*", Appl. Math. Optim., 69 (2014), no. 1, 1-19.

[1] "*An initial and boundary-value problem for the Zakharov-Kuznetsov equation in a bounded domain*", with J.- C. Saut & R. Temam, J. Math. Phys., Nov., 2012.

Work Submitted or in Preparation

[P1] "*Numerical analysis of the stochastic Navier-Stokes equations: stability and convergence of invariant measures*", preprint, with N. Glatt-Holtz & R. Temam.

Awards and Honors

Travel Award - Selected for participation internationally to the Heidelberg Laureate Forum (HLF). All fares covered by the HLF foundation and Oak Ridge Associated Universities (ORAU) (2014)

NSF Graduate Student Fellowship, Sponsor: Prof. Temam (Fall 2014)

Joseph and Frances Morgan Swain Fellowship - Indiana University (2014)

College of Arts and Sciences Travel Award - Indiana University (2014)

Schober Travel Award - Indiana University (2014)

NSF Graduate Student Fellowship, Sponsor: Prof. Temam (Fall 2013)

Schober Travel Award - Indiana University (2013)

Excellent Undergraduate Dissertation Award (top 5% of seniors) - Sichuan University (May 2008)

Invited Research Visits

Virginia Polytechnic Institute and State University, Aug., 2013. During the visit, I gave the talk "*A hyperbolic model from plasma physics: the Zakharov-Kuznetsov equation*".

Virginia Polytechnic Institute and State University, Nov. 2014.

Invited Presentations at Conferences

"*Global existence of strong solutions to 3D Zakharov-Kuznetsov equation in a bounded domain*", also invited to chair the Partial Differential Equations session at the Joint Mathematics Meetings, Baltimore, Jan. 2014

"*Time discrete approximation of weak solutions for stochastic equations of geophysical fluid dynamics and application*", at IPAM Long Program: Mathematics of Turbulence, Institute for Pure and Applied Mathematics, Oct. 2014

"*Martingale and pathwise solutions to the stochastic Zakharov-Kuznetsov equation with multiplicative noise*", at NSF-CBMS Regional Research Conference in the Mathematical Sciences, Oklahoma State University, July 2014,

"*Martingale and pathwise solutions to the stochastic Zakharov-Kuznetsov equation with multiplicative noise*", at the AMS sectional meeting, University of New Mexico, Apr. 2014

"*Some recent results of the 3D Zakharov-Kuznetsov equation in a bounded domain*", PDE Workshop, the University of Maryland, Jan. 2014,

"*Some recent results of the 3D Zakharov-Kuznetsov equation in a bounded domain*", at the AMS sectional meeting, University of California, Riverside, Nov. 2013

"*Some recent results of the 3D Zakharov-Kuznetsov equation in a bounded domain*", the AMS sectional meeting, University of Louisville, Oct. 2013

"*An initial and boundary-value problem for the Zakharov-Kuznetsov equation in a bounded domain*", the AMS sectional meeting, University of Arizona, Oct. 2012

Conferences and workshops (Attendance only)

"Heidelberg Laureate Forum (HLF)", Heidelberg, Germany, Sept. 2014, total expense supported by the HLF foundation and ORAU (19 people for all of Math and Computer Science for all the US and 200 internationally)

"IPAM Workshop: Mathematical Analysis of Turbulence", Institute for Pure and Applied Mathematics, Oct. 2014

"MSRI Summer Graduate School: Stochastic Partial Differential Equations", Mathematical Sciences Research Institute, July 2014 (selected for participation nationally)

"IMA Workshop: Theory and Applications of Stochastic PDEs", Institute for Mathematics and its Applications, Jan. 2013

"AIM Workshop: Stochastic in Geophysical Fluid Dynamics", The American Institute of Mathematics, Feb. 2013

"The 9th AIMS Conference on Dynamical Systems, Differential Equations and Applications", Orlando, Florida, July 2012

"Interdisciplinary Session on Deterministic and Stochastic Partial Differential Equations", the AMS sectional meeting, University of Notre Dame, Nov. 2010

Teaching Experiences

Indiana University (Fall 2009 - present):

Lecturing (Precalculus (M025), Spring 2014)

Main Lecturer (seminar classes "Stochastic Partial Differential Equations" (M741), Fall 2013)

Assisting (Finite Mathematics)

Grading (Probability, Stochastic Process, Vector Calculus, Linear Algebra)

References

Roger Meyer Temam
Indiana University Bloomington, Institute for Scientific Computing and Applied Mathematics
Rawles Hall, Bloomington, IN 47405-5701
Tel: 812 855 8521, Fax: 812 855 7850
temam@indiana.edu

Nathan Glatt-Holtz
Virginia Polytechnic Institute and State University, Blacksburg, VA 24061
Tel: 540 231 5296
negh@vt.edu

Shouhong Wang
Indiana University Bloomington, Rawles Hall, Bloomington, IN 47405
Tel: 812 855 8350
showang@indiana.edu

Mickaël Chekroun
Department of Atmospheric and Oceanic Sciences and Institute of Geophysics and Planetary Physics
University of California, Los Angeles, CA 90095-1565
mchekroun@atmos.ucla.edu

Igor Kukavica
University of Southern California, Los Angeles, CA 90089
Tel: 213 740 3781
kukavica@usc.edu

Richard Bradley
Indiana University Bloomington, Rawles Hall, Bloomington, IN 47405
Tel: 812 855 6371
bradleyr@indiana.edu

Linda McKinley
Indiana University Bloomington, Rawles Hall, Bloomington, IN 47405
Tel: 812 855 0144
holdenl@indiana.edu