Flatness of the commutator map on Special Linear Groups

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Submitted to the faculty of the University Graduate School

in partial fulfillment of the requirements

for the degree

Doctor of Philosophy

in the Department of Mathematics,

Indiana University

July, 2018

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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May 10, 2018

ACKNOWLEDGMENTS

First of all, I owe my thoroughly heartfelt gratitude to my advisor, Professor Michael Larsen, for his invaluable guidance, encouragement and continued help throughout the years of promoting my academic qualification and the completion of this thesis. I sincerely thank him for introducing me to this dynamic area of representation theory which leads to far reaching mathematics. It not only motivates my current academic endeavor but also a more grand working scheme in the following years. I also thank Professor Larry Moss, Professor Noah Snyder and Professor Matthias Strauch for generously sparing their time and energy on serving as my thesis defense committee.

In addition, I am greatly thankful for IU math department which makes it available both financially and spiritually for me to continue my postgraduate study and appreciate a much more vigorous academic environment. My gratitude extends to everyone in the department.

Moreover personally, I thank my parents Rong Lu and Yuqiong Yang who have been supporting my studying for over twenty years by all means despite their limited financial sources; my wife Keping Du whose wholehearted life coordinating and accommodating makes the years we spent together a crystal cherish; and lastly my angelic pretty baby daughter Rubia Lu, who presents to me as the most wonderful gift in the happiest days of my life.

Zhipeng Lu

FLATNESS OF THE COMMUTATOR MAP ON SPECIAL LINEAR GROUPS

This thesis contributes to the study of the fibers of the commutator map on special linear groups in characteristic zero. Specifically, we show that the fibers over non-central elements all have the same dimension. Also we explain that the fibers over central elements can be of larger dimension and compute how large. We use the character tables of finite general linear groups constructed by J.A. Green to count solutions to the commutator equation $[x, y] = g$ over finite fields and use algebraic geometry to go from characteristic p to characteristic 0. To deal with fibers over central elements, we compute the orbits of the conjugation action of GL_n on these fibers.

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Introduction

This thesis is devoted to studying fibers of commutator maps over algebraic groups, specifically,

- for $m \geq 1$, let $[,]^m : \mathcal{G}^m \times \mathcal{G}^m \to \mathcal{G}$ via $[a,b]^m = [a_1, b_1] \cdots [a_m, b_m]$ with $a = (a_1, \cdots, a_m), b = (b_1, \cdots, b_m) \in \mathcal{G}^m$ and $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$, what can we tell about size of the fibers $([,]^m)^{-1}(\mathfrak{g})$ for any $\mathfrak{g} \in \mathcal{G}$, say for $\mathcal{G}$ an algebraic group over $\mathbb{C}$?

There are many interesting algebraic questions on commutators, for instance, the Ore conjecture, which says in a finite non-abelian simple group, every element is a commutator. The conjecture was proved by Liebeck, O'Brien, Shalev and Tiep in [23].

Besides the simple algebraic definition above, the question is also rooted in a geometric background. The fundamental group of a compact Riemann surface of genus g S^g has the presentation

$$\pi_1(S^g) = \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

For any group G, every $\rho \in Hom(\pi_1(S^g), G)$ is determined by $\rho(a_1), \rho(b_i)$ which could be arbitrary and the relation

$$[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = 1.$$

Hence

$$Hom(\pi_1(S^g), G) \leftrightarrow \phi_g^{-1}(1),$$

i.e. fiber of the commutator map over the identity can be identified with the representation space of a surface group into G. What about any other fiber $\phi_g^{-1}(\mathfrak{g})$ over arbitrary $\mathfrak{g} \in G$?

This leads to considering Riemann surface deleting a point. Using the uniformization above, $S^g \setminus \{*\}$ has deformation retract to the wedge of $2g$ circles. Then the fundamental group of $S^g \setminus \{*\}$ is just

$F_{2g} = \mathbb{Z} * \cdots * \mathbb{Z}$, the free group generated by $2g$ letters, which can also be presented as

$$\pi_1(S^g \setminus \{*\}) = \langle a_1, b_1, \cdots, a_g, b_g, c \mid [a_1, b_1] \cdots [a_g, b_g]c = 1 \rangle.$$

Now for any group G, any $\forall \rho \in Hom(\pi_1(S^g \setminus \{*\}), G)$ is determined by $\rho(a_1), \rho(b_i), \rho(c)$ which could be arbitrary and the relation

$$[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = \rho(c)^{-1}.$$

Hence

$$Hom(\pi_1(S^g \setminus \{*\}), G) \leftrightarrow \phi_g^{-1}(G),$$

i.e. any fiber $\phi_g^{-1}(\mathfrak{g}), \forall \mathfrak{g} \in G$ determines the representation space of $\pi_1(S^g \setminus \{*\})$ into G of type with the free quantifier c mapped to x. In particular, $\phi_g^{-1}(1) \leftrightarrow Hom(\pi_1(S^g), G)$.

When G is an algebraic group, then any representation space $\phi_g^{-1}(\mathfrak{g})$, for any $\mathfrak{g} \in G$, is an algebraic set. Moreover if G is (connected) over $\mathbb{C}$ (Lie group), then $\phi_g^{-1}(\mathfrak{g})$ inherits both the Zariski and (locally) Euclidean topology of G^{2g}. The key example is as follows:

- Let $G = GL_n(\mathbb{C})$, then

  $$\phi_g : GL_n(\mathbb{C})^{2g} \to GL_n(\mathbb{C})$$

  has image $SL_n(\mathbb{C})$. The thesis focuses on studying dimension of the fibers $\phi_g^{-1}(\mathfrak{g}), \forall \mathfrak{g} \in SL_n(\mathbb{C})$, which relates to the moduli space of complex vector bundles over $S^g \setminus \{*\}$ for $g = 1$, the flat torus case.

For any $\rho \in Hom(\pi_1(S^g \setminus \{*\}), GL_n(\mathbb{C})), n \geq 1$, a complex representation of the surface group, we can define a complex vector bundle over $S^g \setminus \{*\}$ as follows:

let X be the universal cover of $S^g \setminus \{*\}$ with deck transformation group $\pi = \pi_1(S^g \setminus \{*\})$, then

$X \times \mathbb{C}^n$ is equipped with a π-action

$$\alpha \cdot (x, v) = (\alpha \cdot x, \rho(\alpha)v), \forall \alpha \in \pi, x \in X, v \in \mathbb{C}^n.$$

Since π acts properly and freely on X hence also $X \times \mathbb{C}^n$, the resulting quotient

$$(X \times \mathbb{C}^n)/\pi \to X/\pi = S^g \setminus \{*\},$$

is a rank n bundle, called the *flat bundle with holonomy* ρ, denoted by E_ρ. Note that E_ρ inherits a *flat connection* (zero curvature) from the trivial bundle.

Especially when $g = 1$, $S^g \setminus \{*\}$ deformation retracts to a wedge of two circles, which has a universal cover as the Cayley graph T of $F_2 = \mathbb{Z} * \mathbb{Z}$. Hence any flat bundle with holonomy ρ can be seen as the quotient of $T \times \mathbb{C}^n$. Actually, T looks like a fractal tree which has four branches at each node, see page 77 of Hatcher [13]. In particular, F_2 is linearly realizable, say by the Sanov representation

$$\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle.$$

Hence the flat structure of E_ρ's over $S^g \setminus \{*\}$ ($g = 1$) naturally comes from the trivial bundle $T \times \mathbb{C}^n$ quotient by linear representation of F_2. Similar for any $g \geq 2$.

Inversely, we can construct a linear representation of π from a flat vector bundle by holonomy. Let E be a rank n vector bundle over $S^g \setminus \{*\}$ and ∇ be connection on E. Given any smooth loop $\gamma : [0, 1] \to S^g \setminus \{*\}$, based at x, the connection defines a linear and invertible *parallel transport* $P_\gamma : E_x \to E_x$ along the loop, hence a linear transformation in $GL(E_x) = GL_n(\mathbb{C})$. Define

$$Hol_x(\nabla) = \{P_\gamma \in GL(E_x) \mid \gamma \text{ a loop based at } x\}.$$

Clearly any other base point gives a conjugation of the above group, hence up to isomorphism, we denote it by $Hol(\nabla)$ and call it the *holonomy group* of the connection. If ∇ is flat, contractible loops gives trivial transport, resulting a surjective group homomorphism $\pi \to Hol(\nabla) \subset \mathrm{GL}_n(\mathbb{C})$ which sends $[\gamma]$ to P_γ. This gives a representation of π into $GL_n(\mathbb{C})$ with image $Hol(\nabla)$. By construction, the flat bundle of holonomy E_ρ, $\forall \rho \in Hom(\pi, GL_n(\mathbb{C}))$ with the natural flat connection, gives back the representation ρ through holonomy as above.

By this far, we see the following identifications

$$\phi_g^{-1}(SL_n(\mathbb{C})) \leftrightarrow Hom(\pi, GL_n(\mathbb{C}))$$

$$\leftrightarrow \{\text{flat complex vector bundles over } S^g \setminus \{*\}\},$$

via

$$\phi_g^{-1}(\mathfrak{g}) \leftrightarrow \{\rho : \pi \to GL_n(\mathbb{C}), \rho(c) = \mathfrak{g}^{-1}\}$$

$$\leftrightarrow \{\text{complex vector bundle with flat connection } \nabla \text{ of type } \nabla_c = \mathfrak{g}^{-1}\}.$$

More generally for any Lie group G, we can consider principal G-bundles over Riemann surfaces and establish the correspondence in a similar way. Then flatness of the commutator map in the geometric setting, basically means that the flat complex vector bundles (or generally principal G-bundles) of types in consideration have equal dimension.

For $g \geq 2$, it was proved that $\phi_g = [,]^g : G^{2g} \to G$ is flat for any simple algebraic group G. Thus, spaces of principal G-bundles of any type over hyperbolic surfaces all have the same dimension $(2g-1)\dim G$. Jun Li proved in [19] the flatness of $[,]^g$ for $g > 1$ and any connected complex semi-simple Lie group $\mathcal{G}$, hence shows that those moduli spaces of E_ρ basically have the same size for hyperbolic surfaces. Moreover, Liebeck and Shalev in [21] gave a new proof of Li's result by studying zeta function $\zeta_{\mathcal{G}(\mathbb{Z}/p)}(s)$. We will introduce the latter approach following Aizenbud-Avni [1] and try to illustrate the thesis work for $g = 1$ and $G = SL_n(\mathbb{C})$, i.e. complex vector bundles over the flat torus in the geometric setting, in scheme of the approach.

First, we have a formula by Frobenius [8]: Let G be a finite group, and let $k \geq 1$ be an integer. Then $\forall \mathfrak{g} \in G$, the number of solutions in G^{2k} to the equation $[x_1, y_1] \cdots [x_k, y_k] = \mathfrak{g}$ is equal to

$$|G|^{2k-1} \sum_{\chi \in Irr(G)} \frac{\chi(\mathfrak{g})}{\chi(1)^{2k-1}},$$

where $|G|$ denotes the order of the group and $Irr(G)$ the set of all irreducible characters of G.

Second, in Liebeck and Shalev's paper [22], they studied for any finite group G the associated "zeta function", $\forall t > 0$

$$\zeta^G(t) = \sum_{\chi \in Irr(G)} \chi(1)^{-t},$$

and showed that for any simple algebraic group G defined over a finite field $\mathbb{F}_q$, the zeta function $\zeta^{G(\mathbb{F}_{q^m})}(t) < C$ is uniformly bounded by some constant C for any integer $m \geq 1$ and real $t > 1$.

Rewrite the definition $\phi_{G,k} = [,]^k : G^{2k} \to G$ for any group G, we will outline a proof of the following result: for G group scheme over $\mathbb{Z}$ such that the generic fiber is simple and $k \geq 2$, $\phi_{G,k}$ is flat.

We first prove it for $G(\mathbb{F}_{q^m})$, for $m \geq 1$. By Frobenius' formula and Liebeck-Shalev's result on zeta function ($k \geq 2$)

$$\begin{aligned}
|\phi^{-1}_{G(\mathbb{F}_{q^m}),k}(\mathfrak{g})| &= |G(\mathbb{F}_{q^m})|^{2k-1} \left| \sum_{\chi \in Irr(G(\mathbb{F}_{q^m}))} \frac{\chi(\mathfrak{g})}{\chi(1)^{2k-1}} \right| \\
&\leq |G(\mathbb{F}_{q^m})|^{2k-1} \sum_{\chi \in Irr(G(\mathbb{F}_{q^m}))} \frac{1}{\chi(1)^{2k-2}} \\
&= |G(\mathbb{F}_{q^m})|^{2k-1} \zeta^{G(\mathbb{F}_{q^m})}(2k-2) \leq C|G(\mathbb{F}_{q^m})|^{2k-1}.
\end{aligned}$$

Now we use Lang-Weil bound: suppose V be a variety over $\bar{\mathbb{F}}_p$, $\dim V = f$, and V has e components of top dimension f, then there is a power q_0 of p such that

$$|V(q)| = (e + o(1))q^f$$

for all powers q of q_0.

Here the dimension is the Krull dimension over $\bar{\mathbb{F}}_p$, which is same with the Krull dimension over $\mathbb{F}_q$ for almost all power q of p. Thus for any $\mathfrak{g} \in G$,

$$\dim \phi_{G,k}^{-1}(\mathfrak{g}) \leq (2k-1)\dim G,$$

which actually implies $= (2k-1)\dim G$.

To extend the result to characteristic 0, we resort to the following theorem by Grothendieck (9.2.6.1 of EGA IV [6]) that if $f : X \to S$ is a scheme morphism of finite presentation, then the function $s \mapsto \dim(f^{-1}(s))$ is locally constructible. Hence first $\dim G$ is generically the same. Now for any $\mathfrak{g} \in G(\bar{\mathbb{Q}})$, let $X = \phi_{G,k}^{-1}(\mathfrak{g})$

$$= \{(x_1, y_1, \cdots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = \mathfrak{g}\}.$$

There are large enough number fields $K/\mathbb{Q}$ such that $\mathfrak{g} \in G(K)$, hence for any good prime p, $X_p = X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\ \mathbb{F}_p = X(\mathbb{F}_p)$ contains $\phi_{G,k}^{-1}(\mathfrak{g}_q)$ as its closed points for some $\mathfrak{g}_q \in G(\mathbb{F}_q)$. Thus $\dim X_p = (2k-1)\dim G$ for almost all fibers (over good finite characteristics p). Again by Grothendieck's theorem, for the generic fiber,

$$\dim X = \dim \phi_{G,k}^{-1}(\mathfrak{g}) = (2k-1)\dim G.$$

To show

$$\dim \phi_{G,k}^{-1}(\mathfrak{g}) = (2k-1)\dim G, \forall \mathfrak{g} \in G(\mathbb{C}),$$

we notice that the Zariski closure $X_{\mathfrak{g}} = \{\sigma(\mathfrak{g}) \mid \sigma \bar{\in} Aut(\mathbb{C})\}$ is defined over $\mathbb{Q}$, and $X_{\mathfrak{g}}(\bar{\mathbb{Q}})$ is Zariski dense in $X_{\mathfrak{g}}$. We showed for any $\mathfrak{h} \in X_{\mathfrak{g}}(\bar{\mathbb{Q}}), \dim \phi_{G,k}^{-1}(\mathfrak{h}) = (2k-1)\dim G$, hence again by Grothendieck's theorem applied to $\phi_{G,k} : G^{2k} \to G$, we get the result.

Finally we turn the equi-dimension property into flatness by Ex. 3.10.9 of Hartshorne [12] that for $f : X \to Y$ a morphism of varieties, Y regular, X Cohen-Macaulay, and that every fiber of f has dimension equal to $\dim X - \dim Y$, f is flat.

However, when $k = 1$, the above arguments all work except that $\zeta^{G(\mathbb{F}_{q^m})}(2k-2) = \zeta^{G(\mathbb{F}_{q^m})}(0)$. Hence we can't use zeta functions to control the character sum in Frobenius' formula

$$|\phi_{G(\mathbb{F}_{q^m}),1}^{-1}(\mathfrak{g})| = |G(\mathbb{F}_{q^m})| \sum_{\chi \in Irr(G(\mathbb{F}_{q^m}))} \frac{\chi(\mathfrak{g})}{\chi(1)},$$

unless we can estimate the character ratio $\dfrac{\chi(\mathfrak{g})}{\chi(1)}$ more precisely. To do this, we need the generic character table of $G(\mathbb{F}_q)$. Fortunately for $GL_n(\mathbb{F}_q)$, it was calculated by J.A. Green. Moreover, unlike $k \geq 2$, dimension of the fibers over central elements may be higher than those over non-central elements. The thesis deals with these issues for $G = SL_n(\mathbb{C})$ and shows that fibers of the commutator map on $\mathrm{SL}_n(\mathbb{C})$ almost all have the same dimension except over the finite number of non-primitive central elements, hence sizes of those moduli spaces of holonomy bundles over a torus with one hole are mostly about the same. We will first study the commutator map on the finite quasi-simple groups $\mathrm{GL}_n(\mathbb{F}_q)$ and show that most fibers have the same sizes using character estimates of which the starting point is Frobenius theorem on character sums as in [8]. This can be seen as a strengthening of Ore's conjecture in this special case. The basic tool for estimating the character sum of $\mathrm{GL}_n(\mathbb{F}_q)$ is Green's character formula in [10]. Then to transform the result to $\mathrm{GL}_n(\mathbb{C})$ by Lang-Weil bound which turns counting of q-rational points to information of dimension over $\bar{\mathbb{F}}_p$, and exploiting the structural morphism $f : [,]^{-1}(\mathfrak{g}) \to \mathrm{Spec}\,\mathbb{Z}$ which builds a flat families of varieties over each characteristic and suffices us to just prove for finite characteristics. Hence the thesis will be presented as follows

- Chapter 1 introduces basics of representation theory of finite groups, especially Frobenius character formula and irreducible characters of symmetric groups which will be used throughout the thesis.

- Chapter 2 devotes to explanation of Green's character formula for $\mathrm{GL}_n(\mathbb{F}_q)$ which is the main tool for estimating character sums. Green's notations and conventions are basically maintained to ease reader's reference.

- Chapter 3 includes character tables and the detailed computation of sizes of fibers of the commutator map on $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_3(\mathbb{F}_q)$, which motivates the general treatment of $\mathrm{GL}_n(\mathbb{C})$.

- Chapter 4 discusses character estimates in details and consists of the major part of the thesis. Especially, it resorts to a crucial estimate from [20] and dramatically reduce the amount of computation.

- Chapter 5 finally translates the result from $\mathrm{GL}_n(\mathbb{F}_q)$ first by Lang-Weil bound (see Lemma 7.1 of [21]) to $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$, then through the flat family given by the natural morphism to Spec $\mathbb{Z}$ and Lefschetz's principle to $\mathrm{GL}_n(\mathbb{C})$.

At the end all relevant references are listed under the Bibliography.

CHAPTER 1

Representation of finite groups

This chapter briefly reviews the representation theory of finite groups, developed mainly by Frobenius, Burnside, Schur and Brauer from the end of the nineteenth century. As general references, please check Serre [31], Fulton-Harris [9], Etingof et al. [7] and D.E. Littlewood [24].

1.1 Preliminaries of representation theory

In a general sense, representations of a group are ways of manifesting actions by the group on various objects. One example is Cayley's theorem stating that any group can be embedded into permutation groups, say via the action of the group on itself, which can also be viewed as embedding of the group into a permutation group. In this thesis we consider only the *matrix representation*, i.e. linear action of groups on finite dimensional vector spaces.

Definition 1.1.1. A *representation* of group $\mathcal{G}$ (not necessarily finite) on an n-dimensional complex vector space V is a homomorphism $\rho : \mathcal{G} \to Aut(V) = \mathrm{GL}_n(\mathbb{C})$ (with a basis specified), via the action of $\mathcal{G}$ on V by left multiplication of $\rho(\mathfrak{g}), \forall \mathfrak{g} \in \mathcal{G}$. A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under the action of $\rho(\mathcal{G})$. A representation V is called *irreducible* if it has no proper nonzero subrepresentation.

Example 1.1.2. For any group $\mathcal{G}$, let $\mathbb{C}[\mathcal{G}] = \oplus_{\mathfrak{g} \in \mathcal{G}} \mathbb{C} \cdot \mathfrak{g}$ be the group algebra, which is a vector space endowed with a multiplication

$$(\sum_{\mathfrak{g}\in\mathcal{G}} a_{\mathfrak{g}}\mathfrak{g})(\sum_{\mathfrak{g}\in\mathcal{G}} b_{\mathfrak{g}}\mathfrak{g}) = \sum_{\mathfrak{g}\in\mathcal{G}}(\sum_{\mathfrak{h}_1\mathfrak{h}_2=\mathfrak{g}} a_{\mathfrak{h}_1}b_{\mathfrak{h}_2})g$$

for any finite sums, which induces an action of $\mathcal{G}$ by left multiplication on the group algebra, which makes $\mathbb{C}[\mathcal{G}]$ a representation of $\mathcal{G}$, finite dimensional if $\mathcal{G}$ is finite. This is the so called *regular representation.*

From 1.1.1 there directly arises various computational aspects of linear algebra, one significant of which is the following

Definition 1.1.3. Let $\rho : \mathcal{G} \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Then the naturally assigned complex valued function $\chi_\rho : \mathcal{G} \to \mathbb{C}$ given by

$$\chi_\rho(\mathfrak{g}) = trace(\rho(\mathfrak{g})), \forall \mathfrak{g} \in \mathcal{G},$$

is called the *character* of the representation ρ. The number $n = \chi_\rho(1)$ is called the *degree* of the character. If $n = 1$, the character is said to be *linear*. By an *irreducible character* we mean the character of an irreducible representation. The set of all irreducible characters of $\mathcal{G}$ is denoted by $Irr(\mathcal{G})$.

Remark 1.1.1. If every element of $\mathcal{G}$ has finite order, then for any $\mathfrak{g} \in \mathcal{G}$, every eigenvalue of $\rho(\mathfrak{g})$ must be a root of unity. Hence by definition, it is immediate that $\chi(\mathfrak{g}^{-1}) = \bar{\chi(}\mathfrak{g})$. Moreover, for any finite group, its representations can all be realized as unitary representations, see 3.6 of Etingof et al. [7].

Example 1.1.4. (1) For any group $\mathcal{G}$, the representation $\rho : \mathcal{G} \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ given by $\rho(\mathfrak{g}) = 1, \forall \mathfrak{g} \in \mathcal{G}$, is an irreducible representation with character $\chi = 1$, called the *trivial character*, denoted by 1, and the representation is called the trivial representation.

(2) If $\mathcal{G}$ is a finite group, then the character of its regular representation has

$$\chi(\mathfrak{g}) = \begin{cases} |\mathcal{G}| & \text{for } g = 1 \\ 0 & \text{for } g \neq 1, \end{cases}$$

since elements of $\mathcal{G}$ act on the basis of $\mathbb{C}[\mathcal{G}]$ as permutations.

(3) Actually (2) is a special case of the so called permutation characters. Let $\mathcal{G}$ act on a finite set $\Omega = \{\omega_1, \cdots, \omega_k\}$ and denote $\mathbb{C}[\Omega]$ the free vector space with basis $\omega_1, \cdots, \omega_k$, then the action of $\mathcal{G}$ on Ω extends linearly to $\mathbb{C}[\Omega]$, i.e.

$$\mathfrak{g} \cdot \sum_{i=1}^{k} a_i \omega_i = \sum_{i=1}^{k} a_i (\mathfrak{g} \cdot \omega_i), \forall \mathfrak{g} \in \mathcal{G}, a_1, \cdots, a_k \in \mathbb{C},$$

which gives a representation $\mathcal{G} \to \mathrm{GL}_k(\mathbb{C})$, called the natural representation. Clearly its character is

$$\chi(\mathfrak{g}) = \text{the number of fixed points of } \mathfrak{g} \text{ on } \Omega.$$

It is called the permutation character given by the action of $\mathcal{G}$ on Ω.

Definition 1.1.5. A complex valued function $f : \mathcal{G} \to \mathbb{C}$ on any group $\mathcal{G}$ is a *class function*, if $f(\mathfrak{h}^{-1}\mathfrak{g}\mathfrak{h}) = f(\mathfrak{g}), \forall \mathfrak{g}, \mathfrak{h} \in \mathcal{G}$, i.e. the value of a class function depends only on conjugacy classes. For any two class functions χ_1, χ_2 on a finite group $\mathcal{G}$, we can define an inner product as follows

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|\mathcal{G}|} \sum_{\mathfrak{g} \in \mathcal{G}} \chi_1(\mathfrak{g}) \bar{\chi_2}(\mathfrak{g}),$$

where $\bar{\chi}$ denotes the complex conjugate of χ.

Immediately from the definition, a character is a *class function.*

With those basic notions defined, two cornerstones support the representation theory of finite groups, Schur's lemma and Maschke's theorem.

Theorem 1.1.6 (Schur's lemma)**.** Let $(V_1, \rho_1), (V_2, \rho_2)$ be irreducible representations of a group $\mathcal{G}$ (not necessarily finite) and $\varphi : V_1 \to V_2$ a homomorphism of representations, i.e. $\varphi(\rho_1(\mathfrak{g})v) = \rho_2(\mathfrak{g})\varphi(v), \forall \mathfrak{g} \in \mathcal{G}, v \in V_1$. Then

(1) Either φ is an isomorphism, or $\varphi = 0$.

(2) If $V_1 = V_2$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$ and I the identity.

Schur's lemma implies representation vector spaces of abelian groups have less content in the following sense

Corollary 1.1.6.1. Every irreducible finite dimensional representation of an abelian group $\mathcal{G}$ is 1-dimensional. In particularly any irreducible character is a group homomorphism from $\mathcal{G}$ to $\mathbb{C}^{\times}$.

Proof. Let (V, ρ) be irreducible, then any $\mathfrak{g} \in \mathcal{G}$ induces a endomorphism of V as follows

$$\rho(\mathfrak{g})(\rho(\mathfrak{h})v) = \rho(\mathfrak{g}\mathfrak{h})v = \rho(\mathfrak{h}\mathfrak{g})v = \rho(\mathfrak{h})(\rho(\mathfrak{g})v), \forall \mathfrak{h} \in \mathcal{G}, v \in V.$$

By Schur's lemma, $\rho(\mathfrak{g}) = \lambda \cdot I$ is a scalar operator on V. Hence every subspace of V is a subrepresentation. Since V is irreducible, it follows that $\dim V = 1$. Then $\chi_\rho(\mathfrak{g}) = \lambda$, which defines a multiplicative homomorphism $\mathcal{G} \to \mathbb{C}^{\times}$. □

For abelian groups, $Irr(\mathcal{G})$ forms a group, called the *dual group* of $\mathcal{G}$ and $Irr(\mathcal{G}) \simeq \mathcal{G}$. For details, see chapter 6 of Apostol [2].

Schur's lemma also implies that for a finite group the set of class function on it forms an inner product space having the irreducible characters as an orthonormal basis defined in 1.1.5. Specifically

Corollary 1.1.6.2 (Schur's orthogonality)**.** For any $\chi, \chi' \in Irr(\mathcal{G})$ for any finite group $\mathcal{G}$,

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

For a proof see Theorem 3.9 of Etingof et al. [7]. It can also be generalized to locally compact groups (for integrable class functions), by Pontryagin duality [30] and by the Peter-Weyl theorem [18].

Theorem 1.1.7 (Maschke's theorem)**.** Let $\mathcal{G}$ be a finite group. Then there is an isomorphism of representations $\mathbb{C}[\mathcal{G}] \simeq \oplus_i V_i^{\dim(V_i)}$, summing over all the irreducible representations V_i of $\mathcal{G}$. In particular,

$$|\mathcal{G}| = \sum_i \dim(V_i)^2 = \sum_{\chi \in Irr(\mathcal{G})} \chi(1)^2.$$

Also, there are as many irreducible representations as conjugacy classes of $\mathcal{G}$. Further, every representation of $\mathcal{G}$ is a direct sum of irreducible representations.

For a proof, see Theorem 3.1 of Etingof et al. [7].

1.2 Induced character and Mackey's formula

Given a representation (V, ρ) of a group $\mathcal{G}$ and a subgroup $\mathcal{H} \subset \mathcal{G}$, the restriction of the group action on V to $\mathcal{H}$ gives a representation of the subgroup. This is a representation of $\mathcal{H}$ with the same vector space V and the restricted homomorphism $\rho_{|\mathcal{H}}$. There is an adjoint construction producing a representation of a group from a representation of its subgroup.

Definition 1.2.1. If $\mathcal{G}$ is a finite group, $\mathcal{H} \subset \mathcal{G}$ a subgroup, and (V, ρ) a representation of $\mathcal{H}$. Suppose $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$ is a set of representatives in $\mathcal{G}$ of the left cosets in $\mathcal{G}/\mathcal{H}$, and for any $\mathfrak{g} \in \mathcal{G}, \mathfrak{g}\mathfrak{g}_i = \mathfrak{g}_{j_i}\mathfrak{h}_i$ for some $\mathfrak{h}_i \in \mathcal{H}, 1 \leq j_i \leq n, \forall i$. Then the *induced representation* $Ind_{\mathcal{H}}^{\mathcal{G}} V$ is the representation of $\mathcal{G}$ with the vector space

$$Ind_{\mathcal{H}}^{\mathcal{G}} V = \bigoplus_{i=1}^{n} \mathfrak{g}_i V,$$

and the action $\rho_{Ind_{\mathcal{H}}^{\mathcal{G}}V}$ of $\mathcal{G}$ via

$$\mathfrak{g}\cdot\sum_{i=1}^{n}\mathfrak{g}_i v_i=\sum_{i=1}^{n}\mathfrak{g}_{j_i}\rho(\mathfrak{h}_i)v_i$$

where $v_i\in V,\forall i$.

Proposition 1.2.2. With the above notations,

$$\dim(Ind_{\mathcal{H}}^{\mathcal{G}}V)=\dim(V)\frac{|\mathcal{G}|}{|\mathcal{H}|}.$$

Also if $\mathcal{K}\subset\mathcal{H}\subset\mathcal{G}$ are groups and V a representation of $\mathcal{K}$, then

$$Ind_{\mathcal{H}}^{\mathcal{G}}(Ind_{\mathcal{K}}^{\mathcal{H}}V)=Ind_{\mathcal{K}}^{\mathcal{G}}V.$$

Example 1.2.3. Let $\mathcal{H}$ be a subgroup of a finite group $\mathcal{G}$, and $\rho:\mathcal{H}\to\mathbb{C}^{\times}$ a 1-dimensional representation. Then

$$e_\rho=\frac{1}{|\mathcal{H}|}\sum_{\mathfrak{g}\in\mathcal{H}}\rho(\mathfrak{g})^{-1}\mathfrak{g}\in\mathbb{C}[\mathcal{H}]$$

is an idempotent and $Ind_{\mathcal{H}}^{\mathcal{G}}V\simeq\mathbb{C}[\mathcal{G}]e_\rho$.

Notably the character χ of $Ind_{\mathcal{H}}^{\mathcal{G}}V$ has the following expression

Theorem 1.2.4 (Mackey's formula)**.** Suppose $x_1,\cdots,x_n$ is a set of representatives of the right cosets $\mathcal{H}\backslash\mathcal{G}$ and χ_V the character of the representation V, then

$$\chi(\mathfrak{g})=\sum_{x_i\mathfrak{g}x_i^{-1}\in\mathcal{H}}\chi_V(x_i\mathfrak{g}x_i^{-1}),\forall\mathfrak{g}\in\mathcal{G}.$$

By Mackey's formula, it is easy to check the degree $\chi(1)=|\mathcal{H}\backslash\mathcal{G}|\chi_V(1)=\dim(V)\frac{|\mathcal{G}|}{|\mathcal{H}|}$, which coincides with the proposition above.

1.3 The Frobenius theorem

In this section we recall a theorem of Frobenius [8].

Proposition 1.3.1. For any finite group $\mathcal{G}$, let $\mathfrak{g}, \mathfrak{h} \in \mathcal{G}$ and $Z_{\mathfrak{g}}$ denote the centralizer of $\mathfrak{g}$ in $\mathcal{G}$. Then

$$\sum_{\chi \in Irr(\mathcal{G})} \chi(\mathfrak{g})\bar{\chi}(\mathfrak{h}) = \begin{cases} |Z_{\mathfrak{g}}| & \text{if } \mathfrak{g} \text{ is conjugate to } \mathfrak{h} \\ 0, & \text{otherwise.} \end{cases}$$

For a proof, see Theorem 3.9 of [7].

Proposition 1.3.2. Let $\mathcal{G}$ be a finite group and χ the character of the irreducible representation V_χ of $\mathcal{G}$, let

$$\psi_\chi = \frac{\chi(1)}{|\mathcal{G}|} \sum_{\mathfrak{g} \in \mathcal{G}} \chi(\mathfrak{g}) \cdot \mathfrak{g}^{-1} \in \mathbb{C}[\mathcal{G}].$$

Then ψ_χ acts on V_χ as identity while as null map on any other irreducible representation $V_{\chi'}$ of $\mathcal{G}$ if $\chi \neq \chi' \in Irr(\mathcal{G})$. Also ψ_χ is idempotent, i.e. $\psi_\chi^2 = \psi_\chi$, and $\psi_\chi \psi_{\chi'} = 0$ for $\chi \neq \chi' \in Irr(\mathcal{G})$. In particular,

$$\chi'(\psi_\chi) = \frac{\chi(1)}{|\mathcal{G}|} \sum_{\mathfrak{g} \in \mathcal{G}} \chi(\mathfrak{g})\bar{\chi'}(\mathfrak{g}) = \chi(1)\langle \chi, \chi' \rangle = \begin{cases} \chi(1) & \text{if } \chi' = \chi \\ 0 & \text{if } \chi' \neq \chi. \end{cases}$$

Proof. Directly compute $\chi'(\psi_\chi)$ using Schur's orthogonality. □

Corollary 1.3.2.1. With the same notations, $\forall \mathfrak{g} \in \mathcal{G}$, $\mathfrak{g} \cdot \psi_\chi = \frac{\chi(1)}{|\mathcal{G}|} \sum_{\mathfrak{h} \in \mathcal{G}} \chi(\mathfrak{h}) \cdot \mathfrak{g}\mathfrak{h}^{-1} \in \mathbb{C}[\mathcal{G}]$ acts on V_χ as a permutation while as null map on any other $V_{\chi'}$ for $\chi \neq \chi' \in Irr(\mathcal{G})$. In particular,

$$\chi'(\mathfrak{g} \cdot \psi_\chi) = \begin{cases} \chi(\mathfrak{g}) & \text{if } \chi' = \chi \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to present Frobenius' theorem.

Theorem 1.3.3 (Frobenius [8])**.** For any finite group $\mathcal{G}$, $\forall \mathfrak{g} \in \mathcal{G}$ and any integer $k \geq 1$, the number of solutions $(x_1, \cdots, x_k, y_1, \ldots, y_k) \in \mathcal{G}^k \times \mathcal{G}^k$ to the equation $[x_1, y_1] \cdots [x_k, y_k] = x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_k y_k x_k^{-1} y_k^{-1} = \mathfrak{g}$, denoted by $N_{k,g}$, is

$$N_{k,\mathfrak{g}} = |\mathcal{G}|^{2k-1} \sum_{\chi \in Irr(\mathcal{G})} \frac{\chi(\mathfrak{g})}{\chi(1)^{2k-1}}$$

Proof. We prove it for $k = 1$ then use induction in general.

First, note that if $[x, y] = [x_1, y] = xyx^{-1}y^{-1} = x_1 y x_1^{-1} y^{-1}$, then $xyx^{-1} = x_1 y x_1^{-1}$, i.e. $y = (x^{-1}x_1) y (x^{-1}x_1)^{-1}$, thus $x^{-1}x_1 \in Z_y$, where Z_y denotes the centralizer of y in $\mathcal{G}$. Hence for any fixed $y \in \mathcal{G}$, the number of solutions to the equation $[x, y] = \mathfrak{g}$ is either $|Z_y|$ or 0.

Second, we employ the fact from Proposition 1.3.1 that

$$\sum_{\chi \in Irr(\mathcal{G})} \chi(y) \bar{\chi(z)} = \begin{cases} |Z_y| & \text{if } z \text{ is conjugate to } y \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with the analysis above and using a variation that $[x, y] = \mathfrak{g} \Leftrightarrow xyx^{-1} = \mathfrak{g}y$, i.e. $\mathfrak{g}y$ is conjugate to y, we get another expression of $N_{1,\mathfrak{g}}$

$$N_{1,\mathfrak{g}} = \sum_{y \in \mathcal{G}} \sum_{\chi \in Irr(\mathcal{G})} \chi(\mathfrak{g}y) \bar{\chi(y)} = \sum_{\chi \in Irr(\mathcal{G})} \sum_{y \in \mathcal{G}} \chi(\mathfrak{g}y) \bar{\chi(y)}.$$

Third, we focus on the inner sum $\sum_{y \in \mathcal{G}} \chi(\mathfrak{g}y) \bar{\chi(y)}$, which is just the value of χ on $\phi_{\mathfrak{g},\chi} := \sum_{y \in \mathcal{G}} \chi(\mathfrak{g}y) \cdot y^{-1} \in \mathbb{C}[\mathcal{G}]$. We can also transform it into

$$\phi_{\mathfrak{g},\chi} = \sum_{y \in \mathcal{G}} \chi(\mathfrak{g}y) \cdot y^{-1} = \sum_{y \in \mathcal{G}} \chi(\mathfrak{g}y) \cdot (\mathfrak{g}y)^{-1} \cdot \mathfrak{g} = \sum_{y \in \mathcal{G}} \chi(y) \cdot y^{-1} \cdot \mathfrak{g},$$

i.e. $\phi_{\mathfrak{g},\chi} = \frac{|\mathcal{G}|}{\chi(1)}\mathfrak{g}\cdot\psi_\chi$. Hence by Proposition 1.3.2, $\chi(\phi_{\mathfrak{g},\chi}) = |\mathcal{G}|\frac{\chi(\mathfrak{g})}{\chi(1)}$. In conclusion,

$$N_{1,\mathfrak{g}} = \sum_{\chi\in Irr(\mathcal{G})} \chi(\phi_{\mathfrak{g},\chi}) = |\mathcal{G}| \sum_{\chi\in Irr(\mathcal{G})} \frac{\chi(\mathfrak{g})}{\chi(1)}.$$

Now suppose the formula is true for $\le k$, then by

$$[x_1,y_1]\cdots[x_k,y_k][x_{k+1},y_{k+1}] = \mathfrak{g} \Leftrightarrow [x_1,y_1]\cdots[x_k,y_k] = \mathfrak{g}[y_{k+1},x_{k+1}]$$

we get

$$\begin{aligned}
N_{k+1,\mathfrak{g}} &= \sum_{\mathfrak{h}\in\mathcal{G}} N_{k,\mathfrak{g}\mathfrak{h}} N_{1,\mathfrak{h}} = \sum_{\mathfrak{h}\in\mathcal{G}} \left(|\mathcal{G}|^{2k-1} \sum_{\chi\in Irr(\mathcal{G})} \frac{\chi(\mathfrak{g}\mathfrak{h})}{\chi(1)^{2k-1}} \right) \left(|\mathcal{G}| \sum_{\chi\in Irr(\mathcal{G})} \frac{\chi(\mathfrak{h})}{\chi(1)} \right) \\
&= |\mathcal{G}|^{2k} \sum_{\chi,\chi'\in Irr(\mathcal{G})} \frac{1}{\chi(1)^{2k-1}\chi'(1)} \sum_{\mathfrak{h}\in\mathcal{G}} \chi(\mathfrak{g}\mathfrak{h})\chi'(\mathfrak{h}) \\
&= |\mathcal{G}|^{2k} \sum_{\chi,\chi'\in Irr(\mathcal{G})} \frac{1}{\chi(1)^{2k-1}\chi'(1)} \chi'(\phi_{\mathfrak{g},\chi}) \\
&= |\mathcal{G}|^{2k+1} \sum_{\chi\in Irr(\mathcal{G})} \frac{\chi(\mathfrak{g})}{\chi(1)^{2k+1}},
\end{aligned}$$

in which the last equality is deduced again from Proposition 1.3.2. Then by induction, the formula is true for all $k \ge 1$. □

Example 1.3.4. Let $\mathcal{G}$ be a finite abelian group. Since $\chi(1) = 1, \forall\chi \in Irr(\mathcal{G})$, we get

$$N_{1,\mathfrak{g}} = |\mathcal{G}| \sum_{\chi\in Irr(\mathcal{G})} \chi(\mathfrak{g}) = \begin{cases} |\mathcal{G}|^2 & \text{if } \mathfrak{g} = 1 \\ 0 & \text{if } \mathfrak{g} \neq 1, \end{cases}$$

by the orthogonality in Proposition 1.3.1. This coincides with the obvious facts for abelian groups that $[\mathfrak{g},\mathfrak{h}] = 1, \forall \mathfrak{g},\mathfrak{h} \in \mathcal{G}$.

1.4 Characters of symmetric groups

The last section of the first chapter is devoted to introducing the Frobenius character formula for symmetric groups, which will be used in the computation of character estimates later. Some standard notations for symmetric groups and partitions need to be fixed first.

Definition 1.4.1. A partition λ of any integer $n \geq 1$ is a row of positive numbers $l_1 \geq l_2 \geq \cdots \geq l_k \in \mathbb{Z}_+$ with sum $l_1 + \cdots + l_k = n$, denoted by $\lambda = \{l_1, \cdots, l_k\}$ and each l_i is called a part of the partition. $|\lambda| = n$ always means λ is a partition of n. Another notation for a partition is $\lambda = \{1^{r_1} 2^{r_2} \cdots\}$, meaning that λ consists of r_1 parts equal to 1, r_2 parts equal to 2, etc. The following notations will also be used:

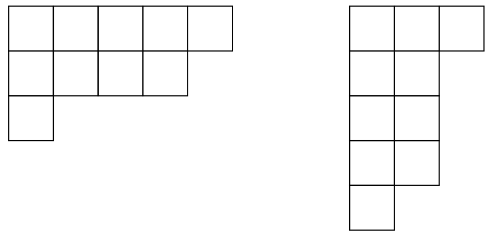
$$m \cdot \{1^{r_1} 2^{r_2} \cdots\} = \{(m)^{r_1} (2m)^{r_2} \cdots\},$$

$$\{1^{r_1} 2^{r_2} \cdots\} + \{1^{s_1} 2^{s_2} \cdots\} = \{1^{r_1+s_1} 2^{r_2+s_2} \cdots\}.$$

Definition 1.4.2 (Young diagram)**.** To each partition $\lambda = \{l_1, \cdots, l_k\}$ with $l_1 \geq l_2 \geq \cdots \geq l_k$, we associate a union of n unit squares in k rows, called its Young diagram, denoted by Y_λ, in which the first row has l_1 squares, second row has l_2 squares, $\ldots$, and k-th row has l_k squares. The conjugate of Y_λ is its transpose, i.e. the first column has L_1 squares, second column has l_2 squares, etc, and the corresponding partition is called the conjugate of λ.

Definition 1.4.3 (Young tableau)**.** A Young tableau corresponding to a Young diagram is obtained by filling the numbers $1, \cdots, n$ into the n unit squares one for each in any way. By convention, a standard Young tableau often denoted by T_λ for a given partition λ, is obtained by filling the numbers in increasing order, left to right, top to bottom.

Example 1.4.4. Let $\lambda = \{5, 4, 1\}$, then its Young diagram and the conjugate look like



and the conjugate of λ is hence $\{3, 2, 2, 2, 1\} = \{1^1 2^3 3^1\}$. The Young tableau of λ

1	2	4	7	8
3	5	6	9	
10				

is standard.

Definition 1.4.5 (Young projectors)**.** Given a Young tableau T_λ, define two subgroups of S_n as follows:

(1) The row subgroup $\mathcal{P}_\lambda$: the subgroup which maps every element of $\{1, \cdots, n\}$ into an element lying in the same row in T_λ.

(2) The column subgroup $\mathcal{Q}_\lambda$: the subgroup which maps every element of $\{1, \cdots, n\}$ into an element lying in the same column in T_λ.

($\mathcal{P}_\lambda \cap \mathcal{Q}_\lambda = \{1\}$ since their intersection fixes every element of $\{1, \cdots, n\}$.)

Define the Young projectors (row projector and column projector) as:

$$a_\lambda := \frac{1}{|\mathcal{P}_\lambda|} \sum_{\mathfrak{g} \in \mathcal{P}_\lambda} \mathfrak{g} \in \mathbb{C}[S_n],$$

$$b_\lambda := \frac{1}{|\mathcal{Q}_\lambda|} \sum_{\mathfrak{g} \in \mathcal{Q}_\lambda} (-1)^{\mathfrak{g}} \mathfrak{g} \in \mathbb{C}[S_n],$$

where $(-1)^{\mathfrak{g}}$ denotes the sign of the permutation $\mathfrak{g}$.

Definition 1.4.6 (Specht Module)**.** Set $c_\lambda = a_\lambda b_\lambda$, then the subspace $\mathcal{V}_\lambda := \mathbb{C}[S_n] c_\lambda$ of $\mathbb{C}[S_n]$ is called the Specht module associated to the partition λ of n.

Remark 1.4.1. Clearly for any two Young tableaux of a given partition λ of n the row subgroups and column subgroups can be conjugated by a common permutation of S_n. The same is true for the Young projectors. This shows that Specht modules of Young tableaux of a given partition are all isomorphic.

Example 1.4.7. (1) For $\lambda = \{n\}$, $\mathcal{P}_{\{n\}} = S_n$, $\mathcal{Q}_{\{n\}} = \{1\}$, and so $b_{\{n\}} = 1, c_{\{n\}} = a_{\{n\}} = \frac{1}{n!}\sum_{\mathfrak{g}\in S_n}\mathfrak{g}$, which is invariant under the action of $\mathbb{C}[S_n]$, i.e. $\forall \mathfrak{g} \in S_n, \mathfrak{g}\cdot c_{\{n\}} = c_{\{n\}}$. Hence $\mathcal{V}_{\{n\}} = \mathbb{C}[S_n]c_{\{n\}} = \mathbb{C}\cdot c_{\{n\}}$, a 1-dimensional subspace, on which S_n acts trivially as identity, i.e. $\mathcal{V}_{\{n\}}$ is the trivial representation.
(2) For $\lambda = \{1^n\}$, $\mathcal{P}_{\{1^n\}} = \{1\}$, $\mathcal{Q}_{\{n\}} = S_n$, and so $a_{\{1^n\}} = 1, c_{\{1^n\}} = b_{\{1^n\}} = \frac{1}{n!}\sum_{\mathfrak{g}\in S_n}(-1)^{\mathfrak{g}}\mathfrak{g}$. Clearly, for any $\mathfrak{h}\in S_n$,

$$\mathfrak{h}\cdot c_{\{1^n\}} = \frac{1}{n!}\sum_{\mathfrak{g}\in S_n}(-1)^{\mathfrak{g}}\mathfrak{h}\mathfrak{g} = \frac{1}{n!}\sum_{\mathfrak{g}\in S_n}(-1)^{\mathfrak{h}^{-1}\mathfrak{g}}\mathfrak{g} = \frac{1}{n!}(-1)^{\mathfrak{h}}\sum_{\mathfrak{g}\in S_n}(-1)^{\mathfrak{g}}\mathfrak{g} = (-1)^{\mathfrak{h}}\cdot c_{\{1^n\}}.$$

Hence $\dim \mathcal{V}_{\{1^n\}} = 1$ and the action of S_n on it gives the sign character, denoted by χ_{sgn}.

In general, the irreducible representations of S_n are classified by Specht modules:

Theorem 1.4.8. Any Specht module $\mathcal{V}_\lambda$ for a partition λ of n is an irreducible representation of S_n under left multiplication. Every irreducible representation of S_n is isomorphic to $\mathcal{V}_\lambda$ for a unique λ.

For a proof, see Theorem 4.36 of Etingof et al. [7].

Remark 1.4.2. By Maschke's Theorem 1.1.7, there are as many irreducible representations as conjugacy classes of a finite group. For the symmetric group S_n, the above theorem shows that an irreducible representation is uniquely determined by a partition of n. On the other hand, any conjugacy class of S_n can be determined by a partition as follows:

Any permutation of $1, \cdots, n$ can be decomposed into a composition of disjoint cycles, say with r_1 many 1-cycles (fixed numbers), r_2 many 2-cycles, etc, and any two permutations decomposable into same numbers of disjoint k-cycles for any k are conjugate. Hence such a conjugacy class of S_n is determined by the partition $\{1^{r_1}2^{r_2}\cdots\}$ of n.

Now we are ready to present the Frobenius character formula for symmetric groups. First, a conventional notation for character values will be fixed as follows

Definition 1.4.9. For any partitions λ, ρ of n, χ^λ denotes the character of $\mathcal{V}_\lambda$ and by χ^λ_ρ the value of χ^λ at the conjugacy class of S_n determined by ρ. Also given a row of variables $x = (x_1, \cdots, x_N)$, $\forall m \geq 0$, let

$$H_m(x) = \sum_{i=1}^{N} x_i^m,$$

and

$$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

Theorem 1.4.10 (Frobenius character formula)**.** Suppose $\lambda = \{l_1, \cdots, l_k\}, \rho = \{1^{r_1}2^{r_2}\cdots\}$ are partitions of n, and $N \geq k$, then χ^λ_ρ is the coefficient of $\prod_{j=1}^{N} x_j^{l_j+N-j}$ in the polynomial

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{r_m}.$$

Equivalently, χ^λ_ρ is the coefficient of $\prod_{j=1}^{N} x_j^{l_j}$ in the Laurent polynomial

$$\prod_{1 \leq i < j \leq N} \left(1 - \frac{x_i}{x_j}\right) \prod_{m \geq 1} H_m(x)^{r_m}.$$

For a proof, see Theorem 4.47 of Etingof et al. [7]. There is also a detailed discussion in Chapter V of D.E. Littlewood [24].

Example 1.4.11. (1) For $\lambda = \{n\}$, N can be chosen as $N = 1$, then there's only one variable, denoted by x. In this case, $\Delta(x) = 1, \prod_{j=1}^{N} x_j^{l_j+N-j} = x^n$, and for any partition $\rho = \{1^{r_1}2^{r_2}\cdots\}$ of n, $\prod_{m\geq 1} H_m(x)^{r_m} = x^n$, hence $\Delta(x)\prod_{m\geq 1} H_m(x)^{r_m} = x^n$ and by Frobenius character formula

$$\chi_\rho^{\{n\}} = \text{the coefficient of } x^n \text{ in } x^n = 1.$$

This coincides with Example 1.4.8 where $\mathcal{V}_{\{n\}}$ is shown to be the trivial representation.

(2) For $\lambda = \{n-1, 1\}$ $(n \geq 2)$, choose the Young tableau as

1	2	$\cdots$	n-1
n			

The row subgroup $\mathcal{P}_{\{n-1,1\}} = S_{n-1}$ is the stabilizer of n, hence

$$a_\lambda = \frac{1}{(n-1)!}\sum_{\mathfrak{g}\in S_{n-1}} \mathfrak{g},$$

which is fixed by all $\mathfrak{g} \in S_{n-1}$. Then $U_\lambda = \mathbb{C}[S_n]\cdot a_\lambda \simeq \mathbb{C}[S_n/S_{n-1}]\cdot a_\lambda$, where $\mathbb{C}[S_n/S_{n-1}]$ denotes the n-dimensional free vector space. Clearly this is the natural representation (see Example 1.1.5 (3)) of S_n, which gives the permutation character of the action of S_n on $\{1, \cdots, n\}$.

Actually (2) is a special case of the more general result on induction of the trivial representation

Lemma 1.4.12. Let $\mathcal{G}$ act transitively on a finite set Ω and $\omega \in \Omega$, then $Ind_{\mathcal{G}_\omega}^{\mathcal{G}} 1$ gives the permutation character of the action, where $\mathcal{G}_\omega$ denotes the stabilizer of ω in $\mathcal{G}$ and 1 the trivial representation.

Proof. Since $\mathcal{G}$ acts transitively on Ω, the left cosets $\mathcal{G}/\mathcal{G}_\omega$, say with representatives $\mathfrak{g}_1, \cdots, \mathfrak{g}_k$, are in one to one correspondence with the elements of Ω via $\mathfrak{g}_i\mathcal{G}_\omega \mapsto \mathfrak{g}_i\mathcal{G}_\omega\omega = \mathfrak{g}_i\omega$. Hence for any $\mathfrak{g} \in \mathcal{G}$, left multiplication by $\mathfrak{g}$ on the cosets $\mathcal{G}/\mathcal{G}_\omega$ gives an action isomorphic to the one on Ω. Then by 1.2.1, $Ind_{\mathcal{G}_\omega}^{\mathcal{G}} 1$ must be the representation given by the action of $\mathcal{G}$ on Ω and its character must be the permutation character. This can also be verified by investigating the induced character using

Mackey's formula 1.2.4. □

By the Frobenius character formula, the first non-trivial irreducible character of S_n can be computed as follows

Proposition 1.4.13. The character of $\mathcal{V}_{\{n-1,1\}}$ is $\chi^{\{n-1,1\}} = \chi_{perm} - 1$, where χ_{perm} denotes the permutation character given by the action of S_n on $\{1, \cdots, n\}$ and 1 the trivial character. Actually, $U_{\{n-1,1\}} = \mathbb{C}[S_n] \cdot a_\lambda = \mathcal{V}_{\{n-1,1\}} \oplus 1$, where 1 is the trivial representation.

Proof. For $\lambda = \{n-1, 1\}$, N can be chosen as 2 and variables x_1, x_2. In this case, $\Delta(x) = x_1 - x_2, \prod_{j=1}^{N} x_j^{l_j+N-j} = x_1^n x_2$, and for any partition $\rho = \{1^{r_1} 2^{r_2} \cdots\}$ of n, $\prod_{m\geq 1} H_m(x)^{r_m} = \prod_{m\geq 1}(x_1^m + x_2^m)^{r_m}$, hence

$$\Delta(x) \prod_{m\geq 1} H_m(x)^{r_m} = (x_1 - x_2) \prod_{m\geq 1} (x_1^m + x_2^m)^{r_m} = x_1 \prod_{m\geq 1} (x_1^m + x_2^m)^{r_m} - x_2 \prod_{m\geq 1} (x_1^m + x_2^m)^{r_m}.$$

The second term on the right gives $-x_2 x_1^n$ for any partition ρ. For the first term, to get $x_1^n x_2$, there must be parts equal to 1 in ρ and the coefficient of $x_1^n x_2$ equals the number of 1's in ρ, which is the number of fixed points of the permutation corresponding to ρ (as in 1.4.10) on $1, \cdots, n$. Thus the coefficient of $x_1^n x_2$ in the first term is the value of the permutation character $\chi_{perm}(\rho)$ (see Example 1.1.5 (3)) and

$$\chi^{\{n-1,1\}} = \chi_{perm} - 1,$$

where 1 denotes the trivial character.

To verify the latter, notice that $\dim \mathcal{V}_{\{n-1,1\}} = \chi^{\{n-1,1\}}_{\{1^n\}} = n-1$, and also any permutation character for any finite group $\mathcal{G}$ acting transitively on a finite set Ω can be computed as

$$\begin{aligned}\langle \chi_{perm}, 1 \rangle &= \frac{1}{|\mathcal{G}|} \sum_{\mathfrak{g}\in\mathcal{G}} \chi_{perm}(\mathfrak{g}) \\ &= \frac{1}{|\mathcal{G}|} \sum_{\mathfrak{g}\in\mathcal{G}} \#\{\omega \in \Omega \mid \mathfrak{g} \cdot \omega = \omega\}\end{aligned}$$

$$= \frac{1}{|\mathcal{G}|} \sum_{\omega \in \Omega} \#\{\mathfrak{g} \in \mathcal{G} \mid \mathfrak{g} \cdot \omega = \omega\}$$
$$= \frac{1}{|\mathcal{G}|} \sum_{\omega \in \Omega} |\mathcal{G}_\omega| = \frac{1}{|\mathcal{G}|} \sum_{\omega \in \Omega} \frac{|\mathcal{G}|}{|\Omega|}$$
$$= 1,$$

hence the trivial representation 1 is a constituent of the natural representation $U_{\{n-1,1\}}$. Counting dimensions, by Maschke's theorem 1.1.7

$$U_{\{n-1,1\}} = \mathcal{V}_{\{n-1,1\}} \oplus 1.$$

□

Remark 1.4.3. $\mathcal{V}_{\{n-1,1\}}$ is actually the standard representation of S_n. In general, for any finite group $\mathcal{G}$ acting transitively on a finite set Ω and its natural representation $\mathbb{C}[\Omega]$, let $V = \{\sum_{\omega \in \Omega} a_\omega \cdot \omega \in \mathbb{C}[\Omega] \mid \sum_{\omega \in \Omega} a_\omega = 0\}$, called the standard representation of the action $\mathcal{G}$ on Ω. Then by the computation in the proof above,

$$\mathbb{C}[\Omega] = V \oplus 1.$$

Further, it can be shown that V is always irreducible as follows:

First, for general action (not necessarily transitive) of any group $\mathcal{H}$ on a finite set X, a similar computation as above gives Burnside's lemma (see Theorem 3.22 of [29]) which says

$$\frac{1}{|\mathcal{H}|} \sum_{\mathfrak{h} \in \mathcal{H}} \#\{x \in X \mid \mathfrak{h} \cdot x = x\} = |X/\mathcal{H}|,$$

where $|X/\mathcal{H}|$ denotes the number of orbits.

Second, since the diagonal action of $\mathcal{G}$ on $\Omega^2 = \Omega \times \Omega$ has two orbits, $\{(\omega, \omega) \mid \omega \in \Omega\}$ and

$\{(\omega, \omega') \mid \omega \neq \omega' \in \Omega\}$, then by Burnside's lemma

$$\langle \chi_{perm}, \chi_{perm} \rangle = \frac{1}{|\mathcal{G}|} \sum_{\mathfrak{g} \in \mathcal{G}} \chi_{perm}^2(\mathfrak{g}) = |\mathbb{C}[\Omega]^2/\mathcal{G}| = 2.$$

Hence

$$\begin{aligned}
\langle \chi_V, \chi_V \rangle &= \langle \chi_{perm} - 1, \chi_{perm} - 1 \rangle \\
&= \langle \chi_{perm}, \chi_{perm} \rangle - \langle \chi_{perm}, 1 \rangle - \langle 1, \chi_{perm} \rangle + \langle 1, 1 \rangle \\
&= 2 - 1 - 1 + 1 = 1.
\end{aligned}$$

In general, the calculation of character values of symmetric groups using the Frobenius character formula 1.4.10 is tedious. There are other explicit character formulas for symmetric groups, for instance the Murnaghan–Nakayama rule (see Stanley [32]), which will not be of use in the thesis. Rather, an exploitation of the conjugation of partitions gives an interesting result which we will put to good use later.

Theorem 1.4.14. If μ is the partition conjugate to λ, then

$$\chi_\rho^\mu = \chi_\rho^\lambda \chi_\rho^{\{1^n\}}.$$

For a proof, see 5.3.V of D.E. Littlewood [24]. In particular, this gives

Corollary 1.4.14.1.

$$\chi^{\{1^{n-2}2^1\}} = \chi^{\{n-1,1\}} \chi_{sgn} = (\chi_{perm} - 1)\chi_{sgn},$$

where χ_{perm} denotes the permutation character, 1 the trivial character and χ_{sgn} the sign character.

Proof. Directly by Theorem 1.4.17 together with Proposition 1.4.15 and Example 1.4.8 (2). □

For a detailed treatment of representation theory of symmetric groups and Specht modules, check James [15] and James, Kerber [16].

CHAPTER 2

The characters of finite general linear groups

This chapter introduces the classification of irreducible characters of finite general linear groups, following the conventions of Green's original work [10]. In that paper, a number of special characters are first constructed with the help of Brauer's theorem [5], then by analogy of Schur functions (see VII of D.E. Littlewood [24]), general irreducible characters can be constructed based on those special characters. Especially, some fundamental cancellation properties of Green's polynomials and counting methods of p-modules are used to establish irreducibility of the resulting characters, which will also be utilized in this thesis.

As to study of finite dimensional representations of complex general linear groups, please check Schur-Weyl duality in Etingof et al. [7] or Weyl [34], and polynomial representations in Green [11].

2.1 Basic definitions and notations

This section introduces all necessary notations following Green [10]. There are some variants of Green's original notations to suit a modern treatment, which will be listed as follows:

Definition 2.1.1. For a fixed prime number p and any power $q = p^r$, $\mathbb{F}_q$ denotes the finite field with q elements. For any $n \in \mathbb{Z}_+, \mathrm{GL}_n(\mathbb{F}_q)$ denotes the general linear group $Aut_{\mathbb{F}_q}(\mathbb{F}_q^n)$. ($\mathfrak{F} := \mathbb{F}_q, \mathfrak{F}_d := \mathbb{F}_{q^d}$, $GL(n, q)$ or $\mathfrak{G}_n := \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{S}_n := S_n$ were used in [10].)

Recall previous notations on partitions in 1.4.1, there are the following definitions:

Definition 2.1.2 (Jordan canonical form over $\mathbb{F}_q$)**.** For any polynomial $f(x)=x^d-a_{d-1}x^{d-1}-\cdots-a_0$ over $\mathbb{F}_q$ and any positive integer m, define the matrix

$$U(f)=U_1(f)=\begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1 \\ a_0 & a_1 & a_2 & ... & a_{d-1} \end{pmatrix}$$

and

$$U_m(f)=\begin{pmatrix} U(f) & I_d & 0 & \\ 0_d & U(f) & I_d & \\ ... & ... & ... & 0 \\ 0_d & ... & U(f) & I_d \\ 0_d & ... & ... & U(f) \end{pmatrix}.$$

Also, if $\lambda=\{l_1,l_2,...,l_m\}$ is a partition of n, we denote

$$U_\lambda(f)=diag(U_{l_1}(f),U_{l_2}(f),...,U_{l_m}(f)).$$

Then for any matrix $A\in\mathrm{GL}_n(\mathbb{F}_q)$ with characteristic polynomial decomposed over $\mathbb{F}_q$ as $f_A=f_1^{k_1}f_2^{k_2}...f_N^{k_N}$, where $f_1,f_2,...,f_N$ are distinct irreducible polynomials over $\mathbb{F}_q$, we have

$$A{\sim}diag(U_{\nu_1}(f_1),U_{\nu_2}(f_2),...,U_{\nu_N}(f_N))$$

in which $\nu_1,\nu_2,...,\nu_N$ are partitions of $k_1,k_2,...,k_N$ respectively determined by A. Hence each conjugacy class c of $\mathrm{GL}_n(\mathbb{F}_q)$ is determined by a set of irreducible polynomials and the corresponding partitions, and can be represented by

$$c\sim(f_1^{\nu_1}\cdots f_N^{\nu_N}),$$

in which $\sum_i^N \deg(f_i)|\nu_i| = n$ with $|\nu|$ the sum of all the parts in the partition ν. Sometimes for short, we will just write $c = (\cdots f^{\nu(f)} \cdots)$, which assigns a partition $\nu(f)$ to each irreducible f (if it does not occur in the characteristic polynomial of c then $\nu(f)$ is null).

Definition 2.1.3 ($p-$module)**.** For any matrix $A \in M_n(\mathbb{F}_q)$, $V_A(q)$ denotes the module over the subring of $M_n(\mathbb{F}_q)$ generated by A, with abelian group structure of $\mathbb{F}_q^n$, i.e. $V_A(q) := (\mathbb{F}_q^n, A)$ with A acting on $\mathbb{F}_q^n$. Then a submodule of $V_A(q)$ can be just characterized as an invariant subspace of $\mathbb{F}_q^n$ under the action of A. For any partition λ, denote by $V_\lambda := V_{U_\lambda(f)}(q)$ for $f(x) = x$. (Actually for any $f_\alpha(x) = x - \alpha$, $\forall \alpha \in \mathbb{F}_q$, $V_{U_\lambda(f)}(q) \simeq V_{U_\lambda(f_\alpha)}(q)$, so we don't distinguish them and V_λ is well-defined.) Also by definition 2.1.2, if f is an irreducible polynomial of degree d over $\mathbb{F}_q$, then $U(f)$ generates the field $\mathbb{F}_{q^d}$ in $M_d(\mathbb{F}_q)$. Hence $V_{U_\lambda(f)}(q) \simeq V_\lambda(q^d)$ as modules over $M_{d|\lambda|}(\mathbb{F}_q)$.

Definition 2.1.4 (Substitution into conjugacy class)**.** Let $\rho = \{1^{r_1} 2^{r_2} \cdots\}$ be the partition with r_1 parts equal to 1, r_2 parts equal to 2 etc, and let $c \sim (f_1^{\nu_1} \cdots f_N^{\nu_N})$ be a conjugacy class in $\mathrm{GL}_n(\mathbb{F}_q)$, as in 2.1.2. Let α be a map sending partitions to rows of irreducible polynomials over $\mathbb{F}_q$, such that (Green used the notation $\rho\alpha$)

$$\alpha(\rho) = (f_{1,1}, \ldots f_{1,r_1}; f_{2,1}, \cdots, f_{2,r_2}; \cdots),$$

in which the polynomials may not be all distinct, and

$$\deg(f_{i,j}) \mid i, \forall i \geq 1, j \leq r_i.$$

In other words, it is a *substitution* of every part of the partition by an irreducible polynomial of degree dividing the part. For a part τ of ρ (denoted as $\tau \in \rho$), $\alpha(\tau)$ (or $\tau\alpha$ in Green's notation) denotes the polynomial the part is sent to by α. For each irreducible polynomial f over $\mathbb{F}_q$, $\forall k \geq 1$, let $\mathfrak{r}_k(\alpha(\rho), f) = \#\{(i,j) \mid k \cdot \deg(f) = i, j \leq r_i, f_{i,j} = f\}$, i.e. the number of parts of ρ sent to f

by α, and $\rho(\alpha, f)$ be the partition $\{1^{\mathfrak{r}_1(\alpha,f)}2^{\mathfrak{r}_2(\alpha,f)}\cdots\}$, i.e.

$$\deg(f)\cdot\rho(\alpha, f) := \{(\deg(f))^{\mathfrak{r}_1(\alpha,f)}(2\deg(f))^{\mathfrak{r}_2(\alpha,f)}\cdots\}$$

is the sub-partition of ρ consisting of parts sent to f by α. Then we say α is a *substitution* of ρ into c if $|\rho(\alpha, f_l)| = |\nu_l|, \forall l = 1, \cdots, N$.

Example 2.1.5. Clearly, for any $q, n \geq 1$ and partition ρ of n, there is a substitution of ρ into $I_n \sim ((t-1)^{\{1^n\}})$ which sends all parts to $t-1$.

Suppose $q = 2, n = 10$, $\rho = \{1^1 2^3 3^1\}$, and $c \sim ((t+1)^{\nu_1}(t^2+t+1)^{\nu_2}(t^3+t+1)^{\nu_3})$ a conjugacy class in $\mathrm{GL}_{10}(\mathbb{F}_2)$ with $\nu_1 = \{1^3 2^1\}, \nu_2 = \{1\}, \nu_3 = \{1\}$, then

$$\alpha(\rho) = (t+1; t+1, t+1, t^2+t+1; t^3+t+1)$$

is a substitution of ρ into c, since $\rho(\alpha, t+1) = \{1^1 2^2\}$ is a partition of $|\nu_1| = 5$, and $\rho(\alpha, t^2+t+1) = \rho(\alpha, t^3+t+1) = \{1\} = \nu_2 = \nu_3$.

Definition 2.1.6 (Simplex and dual class)**.** For an integer $s \geq 1$, if $g = \{k, kq, \cdots, kq^{s-1}\}$ is a set of distinct residues modulo $(q^s - 1)$, we call it a *simplex* (over $\mathbb{F}_q$) of degree s (denoted by $\deg(g) = s$, or simply $d(g) = s$), or an $s-$*simplex*, and we call any residue in the set a *root* of g. A *dual class* of degree $n \geq 1$ is represented by $e = (g_1^{\nu_1}\cdots g_N^{\nu_N})$ with g_i simplices, ν_i partitions, and $\sum_{i=1}^N \deg(g_i)|\nu_i| = n$.

Remark 2.1.1. There are exactly as many $s-$simplexes as irreducible polynomials of degree s over $\mathbb{F}_q$ since $s-$simplexes consist of exponents of roots with respect to a fixed primitive element of $\mathbb{F}_{q^s}$ for irreducible polynomials of degree s over $\mathbb{F}_q$. Moreover, the notion of *substitution* applies to dual classes accordingly if we replace irreducible polynomials by simplices. (Green calls a substitution into dual classes a *dual substitution.*)

Definition 2.1.7 $(z_\rho, \mathfrak{n}_\lambda, \phi_n, N_\lambda, g^\lambda_{\lambda_1,\lambda_2,\cdots,\lambda_k}, Q^\lambda_\rho, T_{s,e}(k:\gamma))$**.** For any partitions $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_k$, $\rho = \{1^{r_1}2^{r_2}\cdots\}$, $|\rho| = |\lambda| = |\lambda_1|+|\lambda_2|+\cdots+|\lambda_k|$, and $\theta : \mathbb{F}^\times_{q^{n!}} \to \mathbb{C}^\times$ a fixed faithful homomorphism for all our purpose later on, we define the numbers

$$z_\rho = 1^{r_1} \cdot r_1! \cdot 2^{r_2} \cdot r_2! \cdot \cdots,$$

$$\mathfrak{n}_\lambda = \sum (i-1) l_i = \sum \binom{k_i}{2}, \text{ if } l_1 \geq l_2 \geq \cdots \text{ are all the parts in } \lambda \text{ and}$$

k_i's are parts in the dual partition of λ,

$$\phi_n(t) = (1-t)(1-t^2)\cdots(1-t^n),$$

N_λ denotes the number of parts in λ.

(*Hall's polynomial*)

$$g^\lambda_{\lambda_1,\lambda_2,\cdots,\lambda_k}(q) = \text{number of chains } V_\lambda = V_0 \supset V_1 \supset \cdots \supset V_{k-1} \supset V_k = 0 \text{ of}$$

$$\text{submodules of } V_\lambda \text{ such that } V_{i-1}/V_i \simeq V_{\lambda_i} \text{ (as in 2.1.3)}, \ \forall i = 1, \cdots, k;$$

(*Green's polynomial*)

$$Q^\lambda_\rho(q) = \sum g^\lambda_{\lambda_1,\lambda_2,\cdots}(q)\phi_{N_{\lambda_1}-1}(q)\phi_{N_{\lambda_2}-1}(q)\cdots, \text{ with the sum being over all}$$

rows of partitions $(\lambda_1, \lambda_2, \cdots)$ s.t. $\lambda_1, \cdots, \lambda_{r_1}$ are partitions of $1, \lambda_{r_1+1}, \cdots,$ $\lambda_{r_1+r_2}$ are partitions of 2 etc.

In addition, for any positive numbers s, e, integer k and any $\gamma \in \mathbb{F}^\times_{q^{n!}}$ with $\deg(\gamma)|se$ where $\deg(\gamma)$ denotes the degree of its minimal polynomial over $\mathbb{F}_q$ (in other words, $\gamma \in \mathbb{F}_{q^{se}}$), we define

$$T_{s,e}(k:\gamma) = \theta^k(\gamma^{1+q^s+\cdots+q^{(e-1)s}}) + \theta^{qk}(\gamma^{1+q^s+\cdots+q^{(e-1)s}})$$

$$+\cdots+\theta^{q^{s-1}k}(\gamma^{1+q^s+\cdots+q^{(e-1)s}}).$$

For any *conjugate* γ' of γ over $\mathbb{F}_q$, i.e. any other root of the minimal polynomial of γ, $T_{s,e}(k:\gamma') = T_{s,e}(k:\gamma)$.

Definition 2.1.8 (Parabolic induction for $\mathrm{GL}_n(\mathbb{F}_q)$)**.** Let $n = s_1 + s_2 + \cdots + s_k$ with $s_i \geq 1$ and $\mathbb{F}_q^n = V_0 > V_1 > V_2 > \cdots > V_k = 0$ be a flag of vector spaces such that $\dim(V_{i-1}/V_i) = s_{i-1}, \forall i = 1, \cdots, k$. Then matrices of the subgroup $G_{s_1,\cdots,s_k} \leq \mathrm{GL}_n(\mathbb{F}_q)$ which leave the flag invariant are all of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2,k} \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}$$

with $A_{ii} \in \mathrm{GL}_{s_i}(\mathbb{F}_q), \forall i = 1, \cdots, k$. If χ_i is a character of $\mathrm{GL}_{s_i}(\mathbb{F}_q)$, then χ defined via

$$\chi(A) = \chi_1(A_{11})\chi_2(A_{22})\ldots\chi_k(A_{kk})$$

is a character of $G_{s_1,\cdots,s_k}$. Then we define $\chi_1 \circ \chi_2 \circ \cdots \circ \chi_k$ to be the character $Ind_{G_{s_1,\cdots,s_k}}^{\mathrm{GL}_n(\mathbb{F}_q)}\chi$.

Green showed basic properties and gave an explicit formula of the parabolic induction as follows

Proposition 2.1.9 (Theorem 2, Lemma 2.5 and 2.6 in [10])**.** With notations as in the above definitions, the parabolic induction operation $\circ$ is multi-linear, associative and symmetric. Moreover for any conjugacy class $c = (\cdots f^{\nu(f)}\cdots)$ of $\mathrm{GL}_n(\mathbb{F}_q)$,

$$\chi_1 \circ \chi_2 \circ \cdots \circ \chi_k(c) = \sum g^c_{c_1,\cdots,c_k}\chi_1(c_1)\chi_2(c_2)\cdots\chi_k(c_k),$$

summing over all rows of conjugacy classes $c_i = (\cdots f^{\nu_i(f)}\cdots)$, respectively of $\mathrm{GL}_{s_i}(\mathbb{F}_q), i = 1, \cdots, k$, in which

$$g^c_{c_1,\cdots,c_k} = \prod_f g^{\nu(f)}_{\nu_1(f),\cdots,\nu_k(f)}(q^{\deg(f)}).$$

2.2 Irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$

Now Green's classification of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ can be stated as follows

Theorem 2.2.1 (Theorem 14 in [10])**.** Irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ are in 1-1 correspondence with dual classes of degree n. For any dual class $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$, denote the corresponding character by I_e. Then for any conjugacy class $c \sim (f_1^{\nu_1} \cdots f_N^{\nu_N})$ of $\mathrm{GL}_n(\mathbb{F}_q)$ we have

$$I_e(c) = \left(\prod_{i=1}^{N} I_{(g_i^{\lambda_i})}\right)(c) = (-1)^{n-\sum|\lambda_i|} \sum_{\rho,m,M} \chi(m,e)Q(M,c)B_\rho(h^\rho m : \xi^\rho M).$$

In the above formula, $\prod$ denotes the parabolic induction on irreducible characters $I_{(g_i^{\lambda_i})}$ of $\mathrm{GL}_{\deg(g_i)|\lambda_i|}$, $i = 1, \cdots, k$, $\sum$ is over all partitions ρ with at least one (dual) substitution m into e and at least one substitution M into c,

$$\chi(m,e) = \prod_{i=1}^{k} \frac{1}{z_{\rho(m,g_i)}} \chi^{\lambda_i}_{\rho(m,g_i)},$$

$$Q(M,c) = \prod_{i=1}^{N} \frac{1}{z_{\rho(M,f_i)}} Q^{\nu_i}_{\rho(M,f_i)}(q^{\deg(f_i)}),$$

for any partition $\rho = \{1^{r_1}2^{r_2}\cdots\}$, row of integers $h^\rho = (h_{11}, \cdots, h_{1r_1}; h_{21}, \cdots, h_{2,r_2}; \cdots)$ and row of variables $\xi^\rho(\xi_{11}, \cdots, \xi_{1r_1}; \xi_{21}, \cdots, \xi_{2,r_2}; \cdots)$,

$$B_\rho(h^\rho : \xi^\rho) = \prod_d \sum_{\sigma \in S_{r_d}} S_d(h_{d1} : \xi_{d\sigma(1)}) \cdots S_d(h_{dr_d} : \xi_{d\sigma(r_d)}),$$

in which $S_d(h:\xi) := \theta^h(\xi) + \theta^{qh}(\xi) + \theta^{q^2h}(\xi) + \cdots + \theta^{q^{d-1}h}(\xi)$ (with θ fixed since 2.1.7). Moreover, the degree of I_e is (1 will be short notation for I_n throughout)

$$\begin{aligned} I_e(1) &= \frac{\phi_n(q)}{\prod_{i=1}^{k} \phi_{\deg(g_i)|\lambda_i|}(q)} \prod_{i=1}^{k} \phi_{\deg(g_i)|\lambda_i|}(q)\{\lambda_i : q^{\deg(g_i)}\} \\ &= \phi_n(q) \prod_{i=1}^{k} \{\lambda_i : q^{\deg(g_i)}\}, \end{aligned}$$

where

$$\{\lambda : q\} = q^{\mathfrak{n}_\lambda} \frac{\prod_{1\le r<s\le t}(q^{l_r - l_s - r + s} - 1)}{\prod_{r=1}^{t} \phi_{l_r+p-r}(q)},$$

for any partition $\lambda = (l_1, l_2, \cdots, l_t)$ with $l_1 \ge l_2 \ge \cdots \ge l_t \ge 1$.

Definition 2.2.2. For a dual class $e = (g^\lambda)$ with a single simplex g, a character of the form I_e is called a *primary character*.

Remark 2.2.1. By the theorem, we can deduce that linear characters of $\mathrm{GL}_n(\mathbb{F}_q)$ ($q > 2$ or $n > 2$) are all of the form $\theta^k(\det(A)),$ for $k = 1, \cdots, q-1$, where $\det(A)$ is the determinant of A. Actually, to get $I_e(1) = 1$ we can only have $e = (g^\lambda)$ with a single simplex g, i.e. a primary character. Then $|\phi_{\deg(g)|\lambda|}(q)/\phi_\lambda(q^{\deg(g)})| = 1$ implies $\deg(g) = 1$, hence $\lambda = \{n\}$ and the character must be linear. With the above theorem, irreducible characters can be classified according to their corresponding dual classes following Green's classification of conjugacy classes

Definition 2.2.3. Two dual classes $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$ and $e' = ((g_1')^{\lambda_1'} \cdots (g_l')^{\lambda_l'})$ of degree n are of *same type*, if $k = l$ and there is a permutation $\sigma \in S_k$ such that $\deg(g_i) = \deg(g'_{\sigma(i)})$ and $\lambda_i = \lambda'_{\sigma(i)}, \forall i = 1, \cdots, k$. We denote it by $e \simeq e'$ and also $I_e \simeq I_{e'}$ for the corresponding characters. Especially two *primary* characters $e = (g^\lambda)$ and $e' = ((g')^{\lambda'})$ are of the same type if and only if $\deg(g) = \deg(g')$ (which we also write $g' \simeq g$) and $\lambda = \lambda'$.

Remark 2.2.2. There is a 1-1 correspondence between degree s simplexes and degree s irreducible polynomials over $\mathbb{F}_q$ (Lemma 7.7 in [10]), hence

$$\#\{s - simplexes\} = \#\{\text{degree } s \text{ irreducible polynomials in } \mathbb{F}_q[x]\}$$

$$= \frac{1}{s} \sum_{t|s} \mu(t) q^{s/t} = O(q^s).$$

Clearly for general $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$, we have

$$\#\{e' \mid e' \simeq e\} = O(q^{\sum_i^k \deg(g_i)}).$$

With this typification, by Theorem 2.2.1, it is direct that

Proposition 2.2.4. Characters of the same type have the same degree. Also suitable substitutions in their character formulas are in 1-1 correspondence.

2.3 Frobenius sum and character estimate

By Theorem 1.3.3, for any finite group $\mathcal{G}$,

$$\left|[,]^{-1}(\mathfrak{g})\right| = |\mathcal{G}| \sum_{\chi \in Irr(\mathcal{G})} \frac{\chi(\mathfrak{g})}{\chi(1)}, \forall \mathfrak{g} \in \mathcal{G}.$$

It is convenient to define by Theorem 2.2.1

Definition 2.3.1 (Frobenius sum)**.** For any $n \in \mathbb{Z}_+$ and $\mathfrak{g} \in \mathrm{GL}_n(\mathbb{F}_q)$, let

$$\mathcal{S}(\mathfrak{g}) = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{\chi(\mathfrak{g})}{\chi(1)} = \sum_{\deg(e)=n} \frac{I_e(\mathfrak{g})}{I_e(1)},$$

where the sum is taken over all the dual classes of degree n. This is called the Frobenius sum of $\mathfrak{g}$ for $\mathrm{GL}_n(\mathbb{F}_q)$. Since characters are class functions, we also use the notation $\mathcal{S}(c)$ for any conjugacy class c of $\mathrm{GL}_n(\mathbb{F}_q)$. The Frobenius sum $\mathcal{S}(c)$ can be grouped into partial sums over characters of the same type, i.e. for a given dual class e of degree n, let

$$\mathcal{S}_e(c) = \sum_{e' \simeq e} \frac{I_{e'}(c)}{I_{e'}(1)} = \frac{1}{I_e(1)} \sum_{e' \simeq e} I_{e'}(c),$$

called the partial Frobenius sum for c over the type e.

It can be defined for any finite group, but in the thesis $\mathcal{S}(c)$ is only treated for finite general linear groups. The notion is clearly related to the general notion of *representation zeta function* for topological groups, see Liebeck, Shalev [22] and Aizenbud, Avni [1].

Example 2.3.2. If $e = (g^{\{1^n\}})$ for some simplex g with $deg(g) = 1$, the partial Frobenius sum $\mathcal{S}_e$ is taken over all linear characters and by Remark 2.2.3

$$\mathcal{S}_e(c) = \frac{1}{I_e(1)} \sum_{e' \simeq e} I_{e'}(c) = \sum_{k=1}^{q-1} \theta^k(\det(c)) = \begin{cases} q-1 & \text{if } \det(c) = 1 \\ 0 & \text{if } \det(c) \neq 1. \end{cases}$$

Clearly if $\det(c) \neq 1$, then $\mathcal{S}_e(c) = 0$ by Theorem 1.3.3.

Now we define the following concept for Frobenius sums in general

Definition 2.3.3 (Numerical flatness)**.** Let $\varphi : X \to Y$ be a morphism of schemes which can be defined over $\mathbb{F}_q$. If for any $y \in Y(\mathbb{F}_q)$, $\varphi^{-1}(y)$ as a subset of $X(\mathbb{F}_q)$ has cardinality $|\varphi^{-1}(y)| \sim q^m$ for a fixed m not depending on y or q, then we say φ is numerically flat.

In our case, we consider $[,] : \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q) \to \mathrm{SL}_n(\mathbb{F}_q)$. Actually, we consider $Y = \mathrm{SL}_n(\mathbb{F}_q) \smallsetminus C(\mathrm{SL}_n(\mathbb{F}_q))$ and $X = [,]^{-1}(\mathrm{SL}_n(\mathbb{F}_q) \smallsetminus C(\mathrm{SL}_n(\mathbb{F}_q)))$, where $C(\mathrm{SL}_n(\mathbb{F}_q))$ denotes the center. Clearly to prove numerical flatness for the commutator map, it suffices to prove, for any partial Frobenius sum $\mathcal{S}_e(c)$ of any given dual class type e and any non-central c

$$\sum_{e' \simeq e} \frac{I_{e'}(c)}{I_{e'}(1)} = \frac{1}{I_e(1)} \sum_{e' \simeq e} I_{e'}(c) = O(q)$$

since the number of types is independent of q (just depending on n). Especially for primary characters of the type $I_{(g^\lambda)}$, we need to show the partial sum

$$\sum_{g' \simeq g} \frac{I_{((g')^\lambda)}(c)}{I_{((g')^\lambda)}(1)} = \frac{1}{I_{(g^\lambda)}(1)} \sum_{g' \simeq g} I_{((g')^\lambda)}(c) = O(q).$$

Because $\#\{g' \mid g' \simeq g\} = O(q^{\deg(g)})$, it suffices to show for any non-central c

Theorem 2.3.4 (Primary numerical flatness)**.**

$$\left|\frac{I_{(g^\lambda)}(c)}{I_{(g^\lambda)}(1)}\right| \lesssim q^{1-\deg(g)}.$$

For $\deg(g) = 1$, we have

$$\left|\frac{I_{g^\lambda}(c)}{I_{g^\lambda}(1)}\right| \leq q^{1-\deg(g)} = q^{1-1} = 1,$$

which is trivially true for any conjugacy class c by the definition of degree of a character. Hence we always consider $\deg(g) \geq 2$ for the primary case. Generally because $\#\{e' \mid e' \simeq e\} = O(q^{\sum_i \deg(g_i)})$, for any dual class $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$ it suffices to prove

Theorem 2.3.5 (General numerical flatness)**.**

$$\left|\frac{I_e(c)}{I_e(1)}\right| \lesssim q^{1-\sum_{i=1}^k \deg(g_i)}.$$

In Chapter 3, the above two theorems will be established for $n = 2, 3$ by explicit calculations, which inspires a treatment for general n later in Chapter 4.

CHAPTER 3

Numerical flatness on $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_3(\mathbb{F}_q)$

For $n = 1$, the general linear group is just the multiplicative group $\mathbb{F}_q^\times$, a cyclic (abelian) group. As shown in Example 1.3.4, the Frobenius sum is zero for non-identity elements, so there's no numerical flatness for $n = 1$. For $n = 2, 3$, the character tables of $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_3(\mathbb{F}_q)$ can be directly computed by Theorem 2.2.1. It turns out, by detailed computation of the Frobenius sums, numerical flatness holds in these two cases, which will be generalized to general $n \geq 2$ in Chapter 4.

We need the following notations throughout this chapter. Let θ be a fixed faithful homomorphism of $\mathbb{F}_{q^{n!}}$ into $\mathbb{C}$. Also

Definition 3.0.6.

$\mathfrak{K}_1 = \{0, 1, \ldots, q-2\}$;

$\mathfrak{K}_2$ is the subset of $\{1, \cdots, q^2-1\}$ such that for any $l \in \mathfrak{K}_2, q+1 \nmid l$ or equivalently l and lq are different modulo q^2-1;

$\mathfrak{K}_3$ is the subset of $\{1, \cdots, q^3-1\}$ such that $\forall k \in \mathfrak{K}_3, k, kq, kq^2$ are distinct modulo q^3-1.

Also $l \neq k \neq m$ means they are all distinct.

3.1 Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

This section recalls the character table for $\mathrm{GL}_2(\mathbb{F}_q)$. We need a detailed classification of conjugacy classes and irreducible characters or equivalently dual classes. For detailed information about representations of $\mathrm{GL}_2(\mathbb{F}_q)$, please check Etingof et al. [7] or Fulton and Harris [9].

3.1.1 Conjugacy class of $\mathrm{GL}_2(\mathbb{F}_q)$

For $n = 2$, there are three possible forms the characteristic polynomial of a matrix $A \in \mathrm{GL}_2(\mathbb{F}_q)$ can have, i.e.

1. $f(t) = (t - \alpha)^2$ for some $\alpha \in \mathbb{F}_q{}^{\times}$;

2. $f(t) = (t - \alpha)(t - \beta)$ for distinct $\alpha, \beta \in \mathbb{F}_q{}^{\times}$;

3. $f(t) = (t^2 + b_1 t + b_0)$ for some irreducible $t^2 + b_1 t + b_0$ over $\mathbb{F}_q$.

Then the conjugacy classes can be represented as follows

1. In this case, there are two subcases as follows

(i) $c = ((t - \alpha)^{\{1^2\}})$, i.e. the matrices

$$A \sim U_{\{1^2\}}(t - \alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

(ii) $c = ((t - \alpha)^{\{2\}})$, i.e. the matrices

$$A \sim U_2(t - \alpha) = U_{\{1^2\}}(t - \alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

2. $c = ((t - \alpha)^{\{1\}}(t - \beta)^{\{1\}})$, i.e. the matrices

$$A \sim U_2(t - \alpha) = U_{\{1^2\}}(t - \alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

3. In this case, clearly $A \sim U(f)$ for $f = t^2 + b_1 t + b_0$ irreducible over $\mathbb{F}_q$. Consider the conjugation over $\mathbb{F}_{q^2}$, in which $f(t)$ decomposes as $f(t) = (t - r)(t - r^q)$ for some $r \in \mathbb{F}_{q^2}$ (but not in $\mathbb{F}_q$), then

$b_0 = r^{1+q}, b_1 = -r - r^q$. Therefore $c = (f^{\{1\}})$ consists of matrices

$$A \sim \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r + r^q \end{pmatrix}.$$

3.1.2 Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

According to the typification as in Definition 2.2.4, all types of irreducible characters of $\mathrm{GL}_2(\mathbb{F}_q)$ can be grouped corresponding to the following 4 types of dual classes

1. $e = (g^\lambda)$ with $\deg(g) = 1, |\lambda| = 2$. Hence $g = \{k\}$ for some $k \in \mathfrak{K}_1$ and there are 2 types of those primary characters corresponding to the two partitions of 2

$$(\{k\}^{\{2\}}), (\{k\}^{\{1^2\}}).$$

2. $e = (g_1^{\{1\}} g_2^{\{1\}})$ with $\deg(g_1) = \deg(g_2) = 1$. Hence $g_1 = \{k\}, g_2 = \{l\}$ for $k \neq l \in \mathfrak{K}_1$ ($k < l$ to avoid repetition) and the only type of such characters corresponds to

$$(\{k\}^{\{1\}}\{l\}^{\{1\}}).$$

3. $e = (g^{\{1\}})$ with $\deg(g) = 2$. Hence $g = \{k, kq\}$ for some $k \in \mathfrak{K}_2$ and the only type of such characters corresponds to

$$(\{k, kq\}^{\{1\}}).$$

With the above classification of conjugacy classes and irreducible characters, by Theorem 2.2.1, the character table of $\mathrm{GL}_2(\mathbb{F}_q)$ is computed as follows (see 28.5 Theorem of James and Liebeck [17])

Type	1.(i)	1.(ii)	2.	3.	Range
Reps of $\mathfrak{g}$	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -r^{1+q} & r+r^q\end{pmatrix}$	$\alpha \neq \beta \in \mathbb{F}_q^\times, r \in \mathbb{F}_{q^2}$
$(\{k\}^{\{2\}})$	$\theta^{2k}(\alpha)$	$\theta^{2k}(\alpha)$	$\theta^k(\alpha\beta)$	$\theta^k(r^{1+q})$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1^2\}})$	$q\theta^{2k}(\alpha)$	0	$\theta^k(\alpha\beta)$	$-\theta^k(r^{1+q})$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1\}})$	$(q+1)\theta^{-k+l}(\alpha)$	$\theta^{-k+l}(\alpha)$	$\theta(\alpha^{-k}\beta^{-l}) + \theta(\alpha^{-l}\beta^{-k})$	0	$k, l \in \mathfrak{K}_1, k < l$
$(\{k, kq\}^{\{1\}})$	$(q-1)\theta^k(\alpha)$	$-\theta^k(\alpha)$	0	$-\theta^k(r) - \theta^{kq}(r)$	$k \in \mathfrak{K}_2, kq \notin \mathfrak{K}_2$

Table 3.1: The character table of $\mathrm{GL}_2(\mathbb{F}_q)$

3.2 Numerical flatness on $\mathrm{SL}_2(\mathbb{F}_q)$

Now we are ready to use the Frobenius character formula (Theorem 1.3.3) and the above character table to compute the number of solutions to the commutator equation $[x, y] = \mathfrak{g}$ in $\mathrm{GL}_2(\mathbb{F}_q)$.

3.2.1 Calculation of partial Frobenius sums on $\mathrm{GL}_2(\mathbb{F}_q)$

It is necessary that matrices on the right hand side of the commutator equation must have $\det(\mathfrak{g}) = 1$. Denote by $\mathcal{S}_j, j = 1, \cdots, 4$, the partial Frobenius sums on each of the 4 types of characters in order from top to bottom listed in the character table.

For the conjugacy classes of type 1.(i) (the central elements, especially $\alpha = 1$ which gives the identity). Since $\det(\mathfrak{g}) = 1$, $\alpha^2 = 1$, i.e. $\alpha = -1$. Here are the details of calculation on $\mathcal{S}_j$'s

$$\mathcal{S}_1 = \sum_{k \in \mathfrak{K}_1} \frac{\theta^{2k}(\alpha)}{\theta^{2k}(1)} = \sum_{k=0}^{q-2} 1 = q - 1,$$

$$\mathcal{S}_2 = \sum_{k \in \mathfrak{K}_1} \frac{q\theta^{2k}(\alpha)}{q\theta^{2k}(1)} = \sum_{k=0}^{q-2} 1 = q - 1,$$

$$\begin{aligned}
\mathcal{S}_3 &= \sum_{k<l \in \mathfrak{K}_1} \frac{(q+1)\theta^{-k+1}(\alpha)}{(q+1)\theta^{-k+l}(1)} = \sum_{0 \le k < l \le q-2} \theta^{-k+1}(\alpha) \\
&= \frac{1}{2}\left(\sum_{0 \le k \le q-2} \theta^{-k}(\alpha) \sum_{0 \le l \le q-2} \theta^{l}(\alpha) - \sum_{0 \le k = l \le q-2} \theta^{-k+l}(\alpha) \right) \\
&= \frac{1}{2}\left(\frac{\theta^{-(q-1)}(\alpha) - 1}{\theta^{-1}(\alpha) - 1} \cdot \frac{\theta^{(q-1)}(\alpha) - 1}{\theta(\alpha) - 1} - (q-1) \right)
\end{aligned}$$

$$= \frac{1}{2}\left(0 \cdot 0 - (q-1)\right) = -\frac{q-1}{2},$$

$$\begin{aligned}
\mathcal{S}_4 &= \sum_{k \in \mathfrak{K}_2, kq \notin \mathfrak{K}_2} \frac{(q-1)\theta^k(\alpha)}{(q-1)\theta^k(1)} \\
&= \frac{1}{2}\left(\sum_{1 \le k \le q^2-1} - \sum_{k(q+1), 1 \le k \le q-1} \right) \theta^k(\alpha) \\
&= \frac{1}{2}\left(\theta(\alpha) \frac{\theta^{(q^2-1)}(\alpha) - 1}{\theta(\alpha) - 1} - (q-1) \right) \\
&= -\frac{q-1}{2}.
\end{aligned}$$

Hence

$$\mathcal{S}(1.(i)) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 = q-1.$$

For the conjugacy classes of type 1.(ii). Again $\det(\mathfrak{g}) = 1$, then $\alpha = \pm 1$. Here are the details of calculation on $\mathcal{S}_j$'s

$$\mathcal{S}_1 = \sum_{k \in \mathfrak{K}_1} \frac{\theta^{2k}(\alpha)}{\theta^{2k}(1)} = q-1,$$

$$\mathcal{S}_2 = 0,$$

$$\begin{aligned}
\mathcal{S}_3 &= \sum_{k < l \in \mathfrak{K}_1} \frac{\theta^{-k+l}(\alpha)}{(q+1)\theta^{-k+l}(1)} \\
&= \frac{1}{2(q+1)} \left(\sum_{0 \le k \le q-2} \theta^{-k}(\alpha) \sum_{0 \le l \le q-2} \theta^{l}(\alpha) - \sum_{0 \le k = l \le q-2} \theta^{-k+l}(\alpha) \right) \\
&= \frac{1}{2(q+1)} \left(\frac{\theta^{-(q-1)}(\alpha) - 1}{\theta^{-1}(\alpha) - 1} \cdot \frac{\theta^{(q-1)}(\alpha) - 1}{\theta(\alpha) - 1} - (q-1) \right) \\
&= -\frac{q-1}{2(q+1)},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}_4 &= \sum_{k\in\mathfrak{K}_2, kq\notin\mathfrak{K}_2} \frac{-\theta^k(\alpha)}{(q-1)\theta^k(1)} \\
&= -\frac{1}{q-1}\frac{1}{2}\left(\sum_{1\le k\le q^2-1} - \sum_{k(q+1),1\le k\le q-1}\right)\theta^k(\alpha) \\
&= -\frac{1}{2(q-1)}\left(\theta(\alpha)\frac{\theta^{(q^2-1)}(\alpha)-1}{\theta(\alpha)-1} - (q-1)\right) \\
&= -\frac{-(q-1)}{2(q-1)} = \frac{1}{2}.
\end{aligned}
$$

Hence

$$
\mathcal{S}(1.(ii)) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 = q-1+0+\frac{q-1}{2(q+1)} + \frac{1}{2} = q - \frac{1}{q+1}.
$$

For the conjugacy classes of type 2.. Again $\det(\mathfrak{g}) = \alpha\beta = 1$, i.e. $\beta = \alpha^{-1}$. Here are the details of calculation on $\mathcal{S}_j$'s

$$
\mathcal{S}_1 = \sum_{k\in\mathfrak{K}_1} \frac{\theta^k(\alpha\beta)}{\theta^{2k}(1)} = q-1,
$$

$$
\mathcal{S}_2 = \sum_{k\in\mathfrak{K}_1} \frac{\theta^k(\alpha\beta)}{q\theta^{2k}(1)} = \frac{q-1}{q},
$$

$$
\begin{aligned}
\mathcal{S}_3 &= \sum_{k<l\in\mathfrak{K}_1} \frac{\theta(\alpha^{-k}\beta^{-l}) + \theta(\alpha^{-l}\beta^{-k})}{(q+1)\theta^{-k+l}(1)} \\
&= \frac{1}{q+1}\left(\sum_{0\le k\le q-2}\theta^{-k}(\alpha)\sum_{0\le l\le q-2}\theta^l(\beta) - \sum_{0\le k=l\le q-2}\theta^{-k+l}(\alpha)\right) \\
&= \frac{1}{q+1}\left(\frac{\theta^{-(q-1)}(\alpha)-1}{\theta^{-1}(\alpha)-1}\cdot\frac{\theta^{(q-1)}(\beta)-1}{\theta^{-1}(\beta)-1} - (q-1)\right) \\
&= -\frac{q-1}{q+1},
\end{aligned}
$$

$$
\mathcal{S}_4 = 0.
$$

Hence

$$
\mathcal{S}(2.) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 = q-1+\frac{q-1}{q} - \frac{q-1}{q+1} + 0 = q-1+\frac{q-1}{q(q+1)}.
$$

For the conjugacy classes of type 3.. Again $\det(\mathfrak{g}) = r^{1+q} = 1$, for $r \in \mathbb{F}_{q^2}$. Here are the details of calculation on $\mathcal{S}_j$'s

$$\mathcal{S}_1 = \sum_{k \in \mathfrak{K}_1} \frac{\theta^k(r^{1+q})}{\theta^{2k}(1)} = q - 1,$$

$$\mathcal{S}_2 = \sum_{k \in \mathfrak{K}_1} \frac{-\theta^k(r^{1+q}}{q\theta^{2k}(1)} = -\frac{q-1}{q},$$

$$\mathcal{S}_3 = 0,$$

$$\begin{aligned}
\mathcal{S}_4 &= \sum_{k \in \mathfrak{K}_2, kq \notin \mathfrak{K}_2} \frac{-\theta^k(r) - \theta^{kq}(r)}{(q-1)\theta^k(1)} \\
&= -\frac{1}{q-1}\frac{1}{2}\left(\sum_{1 \le k \le q^2-1} - \sum_{k(q+1), 1 \le k \le q-1} \right)(\theta^k(r) + \theta^{kq}(r)) \\
&= -\frac{1}{2(q-1)}\theta(r)\frac{\theta^{(q^2-1)}(r) - 1}{\theta(\alpha) - 1} - \frac{1}{2(q-1)}\theta^q(r)\frac{\theta^{q(q^2-1)}(r) - 1}{\theta^q(\alpha) - 1} + \frac{2(q-1)}{2(q-1)} \\
&= -0 - 0 + 1 = 1.
\end{aligned}$$

Hence

$$\mathcal{S}(3.) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 = q - 1 - \frac{q-1}{q} + 0 + 1 = q - 1 + \frac{1}{q}.$$

3.2.2 Numerical flatness on $\mathrm{SL}_2(\mathbb{F}_q)$

The last subsection shows by computation the following more explicit result

Theorem 3.2.1 (Numerical flatness on $\mathrm{SL}_3(\mathbb{F}_q)$)**.** $\mathcal{S}(\mathfrak{g}) = q + O(1), \forall \mathfrak{g} \in \mathrm{SL}_3(\mathbb{F}_q)$ non-central, hence the commutator map on $\mathrm{SL}_3(\mathbb{F}_q)$ is numerically flat.

This is better than what is needed for numerical flatness, i.e. not only $\mathcal{S}(\mathfrak{g}) = O(q)$ but $\mathcal{S}(\mathfrak{g}) = q + O(1)$. Note also that for non-identity central element as in 1.(i), i.e. $\mathfrak{g} = -I$, the Frobenius sum is $q - 1$, also about the same size with the non-central elements. In general, the central elements behave differently from the non-central ones which will be discussed in more detail in Chapter 5.

3.3 Character table of $\mathrm{GL}_3(\mathbb{F}_q)$

This section recalls the character table for $\mathrm{GL}_3(\mathbb{F}_q)$. We start with a classification of conjugacy classes and irreducible characters or equivalently dual classes.

3.3.1 The conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$

For $n = 3$, there are five possible forms the characteristic polynomial of a matrix $A \in \mathrm{GL}_3(\mathbb{F}_q)$ can have, i.e.

1. $f(t) = (t-\alpha)^3$ for some $\alpha \in {\mathbb{F}_q}^{\times}$;

2. $f(t) = (t-\alpha)^2(t-\beta)$ for distinct $\alpha, \beta \in {\mathbb{F}_q}^{\times}$;

3. $f(t) = (t-\alpha)(t-\beta)(t-\gamma)$ for distinct $\alpha, \beta, \gamma \in {\mathbb{F}_q}^{\times}$;

4. $f(t) = (t^2 + b_1 t + b_0)(t-\alpha)$ for some irreducible $t^2 + b_1 t + b_0$ over $\mathbb{F}_q$;

5. $f(t) = t^3 + b_2 t^2 + b_1 t + b_0$ irreducible over $\mathbb{F}_q$.

Before entering into the details, the following notation will be handy to use

Definition 3.3.1 (Exclusive degree)**.** For any $d \geq 1$, $r \propto \mathbb{F}_{q^d}$ means $r \in \mathbb{F}_{q^d}$ but $r \notin \mathbb{F}_{q^{d'}}, \forall d' \mid d, d' \neq d$.

Now, the conjugacy classes can be classified as follows

1. In this case, there are three subcases

(i) $c = ((t-\alpha)^{\{1^3\}})$, i.e. the matrices

$$A \sim U_{\{1^3\}}(t-\alpha) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

(ii) $c = ((t-\alpha)^{\{12\}})$, i.e. the matrices

$$A \sim U_{\{12\}}(t-\alpha) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

(iii) $c = ((t-\alpha)^{\{3\}})$, i.e. the matrices

$$A \sim U_3(t-\alpha) = U_{\{3\}}(t-\alpha) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}.$$

2. In this case, there are two subcases as follows

(i) $c = ((t-\alpha)^{\{1^2\}}(t-\beta)^{\{1\}})$, i.e. the matrices in the standard Jordan form (Definition 2.1.2)

$$A \sim diag(U_{\{1^2\}}(t-\alpha), U_{\{1\}}(t-\beta)) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

(ii) $c = ((t-\alpha)^{\{2\}}(t-\beta)^{\{1\}})$, i.e. the matrices

$$A \sim diag(U_{\{2\}}(t-\alpha), U_{\{1\}}(t-\beta)) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

3. $c = ((t-\alpha)^{\{1\}}(t-\beta)^{\{1\}}(t-\gamma)^{\{1\}})$, i.e. the matrices

$$A \sim diag(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

4. In this case, let $f_1(t) = t^2 + b_1 t + b_0$. The matrices are $A \sim diag(U_1(f_1), \alpha)$, in which $f_1(t)$ decomposes as $f_1(t) = (t-r)(t-r^q)$ for some $r \propto \mathbb{F}_{q^2}$, then $b_0 = r^{1+q}$ and $b_1 = -r - r^q$, therefore $c = (f_1^{\{1\}}(t-\alpha)^{\{1\}})$ consists of matrices

$$A \sim \begin{pmatrix} 0 & 1 & 0 \\ -b_0 & -b_1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -r^{1+q} & r + r^q & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

5. Lastly, let $f(t) = t^3 + b_2 t^2 + b_1 t + b_0 = (t-s)(t-s^q)(t-s^{q^2})$ for some $s \propto \mathbb{F}_{q^3}$. Then $b_0 = -s^{1+q+q^2}, b_1 = s^{1+q} + s^{1+q^2} + s^{q+q^2}$ and $b_2 = -s - s^q - s^{q^2}$, so

$$A \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^{1+q+q^2} & -s^{1+q} - s^{1+q^2} - s^{q+q^2} & s + s^q + s^{q^2} \end{pmatrix}.$$

3.3.2 Character table of $\mathrm{GL}_3(\mathbb{F}_q)$

Now we are ready to give a full classification of all the irreducible characters of $\mathrm{GL}_3(\mathbb{F}_q)$ by Theorem 2.2.1 (Green's Theorem 14 [10]). According to the typification as in 2.2.3, all types of irreducible characters of $\mathrm{GL}_3(\mathbb{F}_q)$ can be grouped corresponding to the following 8 types of dual classes

1. $e = (g^\lambda)$ with $\deg(g) = 1, |\lambda| = 3$. Hence $g = \{k\}$ for some $k \in \mathfrak{K}_1$ and there are 3 types of those primary characters corresponding the 3 partitions of 3

$$(\{k\}^{\{3\}}), (\{k\}^{\{12\}}), (\{k\}^{\{1^3\}}).$$

2. $e = (g_1^{\lambda_1} g_2^{\lambda_2})$ with $\deg(g_1) = \deg(g_2) = 1, g_1 \neq g_2$ and $|\lambda_1| + |\lambda_2| = 3$. Hence $g_1 = \{k\}, g_2 = \{l\}$ for $k \neq l \in \mathfrak{K}_1$, and there are 2 types of such characters

$$(\{k\}^{\{1\}}\{l\}^{\{2\}}), (\{k\}^{\{1\}}\{l\}^{\{1^2\}}).$$

3. $e = (g_1^{\{1\}} g_2^{\{1\}} g_3^{\{1\}})$ with $\deg(g_i) = 1, i = 1, 2, 3$. Hence $g_1 = \{k\}, g_2 = \{l\}, g_3 = \{m\}$ for $k \neq l \neq m \in \mathfrak{K}_1$ and the only type of such characters is

$$(\{k\}^{\{1\}}\{l\}^{\{1\}}\{m\}^{\{1\}}).$$

4. $e = (g_1^{\{1\}} g_2^{\{1\}})$ with $\deg(g_1) = 1, \deg(g_2) = 2$. Hence $g_1 = \{k\}, k \in \mathfrak{K}_1$, $g_2 = \{l, lq\}, l \in \mathfrak{K}_2$ and the only type of such characters is

$$(\{k\}^{\{1\}}\{l, lq\}^{\{1\}}).$$

5. $e = (g^{\{1\}})$ with $\deg(g) = 3$. Hence $g = \{k, kq, kq^2\}$ for $k \in \mathfrak{K}_3$ and the only type of such characters is

$$(\{k, kq, kq^2\}^{\{1\}}).$$

With the above classification of conjugacy classes and irreducible characters, by Theorem 2.2.1, the character table of $\mathrm{GL}_3(\mathbb{F}_q)$ can be presented as follows (see Table 5.12 of [3]).

Type	1.(i)	1.(ii)	1.(iii)	Range
Reps of $\mathfrak{g}$	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^\times$
$(\{k\}^{\{3\}})$	$\theta^{3k}(\alpha)$	$\theta^{3k}(\alpha)$	$\theta^{3k}(\alpha)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{12\}})$	$(q^2+q)\theta^{3k}(\alpha)$	$q\theta^{3k}(\alpha)$	0	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1^3\}})$	$q^3\theta^{3k}(\alpha)$	0	0	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{2\}})$	$(q^2+q+1)\theta^{k+2l}(\alpha)$	$(q+1)\theta^{k+2l}(\alpha)$	$\theta^{k+2l}(\alpha)$	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1^2\}})$	$q(q^2+q+1)\theta^{k+2l}(\alpha)$	$q\theta^{k+2l}(\alpha)$	0	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1\}}\{m\}^{\{1\}})$	$(q+1)(q^2+q+1)\theta^{k+l+m}(\alpha)$	$(2+1)q\theta^{k+l+m}(\alpha)$	$\theta^{k+l+m}(\alpha)$	$k \neq l \neq m \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l, lq\}^{\{1\}})$	$(q^3-1)\theta^{k+l}(\alpha)$	$-\theta^{k+l}(\alpha)$	$-\theta^{k+l}(\alpha)$	$k \in \mathfrak{K}_1, l \in \mathfrak{K}_2, ql \notin \mathfrak{K}_2$
$(\{k, kq, kq^2\}^{\{1\}})$	$(q-1)^2(q+1)\theta^k(\alpha)$	$-(q-1)\theta^k(\alpha)$	$\theta^k(\alpha)$	$k \in \mathfrak{K}_3, kq \notin \mathfrak{K}_3, kq^2 \notin \mathfrak{K}_3$

Table 3.2: The character table of $\mathrm{GL}_3(\mathbb{F}_q)$

Type	2.(i)	2.(ii)	3.	Range
Reps of $\mathfrak{g}$	$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \beta\end{pmatrix}$	$\begin{pmatrix}\alpha & 1 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \beta\end{pmatrix}$	$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{pmatrix}$	$\alpha \neq \beta \neq \gamma \in \mathbb{F}_q^\times$
$(\{k\}^{\{3\}})$	$\theta^k(\alpha^2\beta)$	$\theta^k(\alpha^2\beta)$	$\theta^k(\alpha\beta\gamma)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{12\}})$	$(q+1)\theta^k(\alpha^2\beta)$	$\theta^k(\alpha^2\beta)$	$2\theta^k(\alpha\beta\gamma)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1^3\}})$	$q\theta^k(\alpha^2\beta)$	0	$\theta^k(\alpha\beta\gamma)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{2\}})$	$(q+1)\theta(\alpha^{k+l}\beta^l)$ $+\theta(\alpha^{2l}\beta^k)$	$\theta(\alpha^{k+l}\beta^l)$ $+\theta(\alpha^{2l}\beta^k)$	$\theta(\alpha^k\beta^l\gamma^l)+\theta(\alpha^l\beta^k\gamma^l)$ $\theta(\alpha^l\beta^l\gamma^k)$	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1^2\}})$	$(q+1)\theta(\alpha^{k+l}\beta^l)$ $+q\theta(\alpha^{2l}\beta^k)$	$\theta^{k+l}(\alpha)\theta^l(\beta)$	$\theta(\alpha^k\beta^l\gamma^l)+\theta(\alpha^l\beta^k\gamma^l)$ $\theta(\alpha^l\beta^l\gamma^k)$	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1\}}\{m\}^{\{1\}})$	$(q+1)(\theta(\alpha^{k+l}\beta^m)$ $+\theta(\alpha^{k+m}\beta^l)$ $+\theta(\alpha^{l+m}\beta^k))$	$\theta(\alpha^{k+l}\beta^m)$ $+\theta(\alpha^{l+m}\beta^k)$ $+\theta(\alpha^{k+m}\beta^l)$	$\theta(\alpha^k\beta^l\gamma^m)+\theta(\alpha^k\beta^m\gamma^l)$ $+\theta(\alpha^m\beta^l\gamma^k)+\theta(\alpha^m\beta^k\gamma^l)$ $+\theta(\alpha^l\beta^k\gamma^m)+\theta(\alpha^l\beta^m\gamma^k)$	$k \neq l \neq m \in \mathfrak{K}_1$
$(\{k\}^{\{1,lq\}}\{l\}^{\{1\}})$	$(q-1)\theta(\alpha^l\beta^k)$	$-\theta(\alpha^l\beta^k)$	0	$k \in \mathfrak{K}_1, l \in \mathfrak{K}_2, ql \notin \mathfrak{K}_2$
$(\{k, kq, kq^2\}^{\{1\}})$	0	0	0	$k \in \mathfrak{K}_3, kq \notin \mathfrak{K}_3, kq^2 \notin \mathfrak{K}_3$

Type	4.	5.	Range
Reps of $\mathfrak{g}$	$\begin{pmatrix}0 & 1 & 0\\ -r^{q+1} & r+r^{1+q} & 0\\ 0 & 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ s^{1+q+q^2} & -\sum_{i=0}^2 s^{q^i(1+q)} & \sum_{i=0}^2 s^{q^i}\end{pmatrix}$	$\alpha \in \mathbb{F}_q^\times, r \in \mathbb{F}_{q^2}, s \in \mathbb{F}_{q^3}$
$(\{k\}^{\{3\}})$	$\theta^k(\alpha r^{q+1})$	$\theta^{k(q^2+q+1)}(s)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{12\}})$	0	$-\theta^{k(q^2+q+1)}(s)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1^3\}})$	$-\theta^k(\alpha r^{q+1})$	$\theta^{k(q^2+q+1)}(s)$	$k \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{2\}})$	$\theta(\alpha^k r^{l(q+1)})$	0	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1^2\}})$	$-\theta(\alpha^k r^{l(q+1)})$	0	$k \neq l \in \mathfrak{K}_1$
$(\{k\}^{\{1\}}\{l\}^{\{1\}}\{m\}^{\{1\}})$	0	0	$k \neq l \neq m \in \mathfrak{K}_1$
$(\{k\}^{\{1,lq\}}\{l\}^{\{1\}})$	$-\theta^k(\alpha)(\theta^l(r)+\theta^{ql}(r))$	0	$k \in \mathfrak{K}_1, l \in \mathfrak{K}_2, ql \notin \mathfrak{K}_2$
$(\{k, kq, kq^2\}^{\{1\}})$	0	$\theta^k(s)+\theta^{kq}(s)+\theta^{kq^2}(s)$	$k \in \mathfrak{K}_3, kq \notin \mathfrak{K}_3, kq^2 \notin \mathfrak{K}_3$

Table 3.3: The character table of $\mathrm{GL}_3(\mathbb{F}_q)$ (continued)

3.4 Numerical flatness on $\mathrm{SL}_3(\mathbb{F}_q)$

Now it is ready to use the Frobenius character formula (Theorem 1.3.3) and the above character table to compute the numbers of solutions to the commutator equation $[x, y] = \mathfrak{g}$ in $\mathrm{GL}_3(\mathbb{F}_q)$.

3.4.1 Calculation of partial Frobenius sums on $\mathrm{GL}_3(\mathbb{F}_q)$

It is necessary that matrices on the right hand side of the equation must have $\det(\mathfrak{g}) = 1$. Denote by $\mathcal{S}_j, j = 1, ..., 8$, the partial Frobenius sums of $\dfrac{\chi(g)}{\chi(1)}$ for each of the 8 types of characters in order from top to bottom. To write it more compactly, $\hat{\alpha}$ is used instead of $\theta(\alpha)$.

For the conjugacy classes of type 1.(i) (except for $\alpha = 1$). Since $\det(\mathfrak{g}) = 1$, it follows that $\alpha^3 = 1$. Also note that $\alpha^{q-1} = 1$ always. Here are the details of calculation on $\mathcal{S}_j$'s ($\sum_l^*$ means l

is taken over the corresponding range in the character table)

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \sum_{k=0}^{q-2} \hat{\alpha}^{3k} = \sum_{k=0}^{q-2} 1 = q-1,$$

$$\mathcal{S}_4 = \mathcal{S}_5 = \sum_{k,l=0,l\neq k}^{q-2} \hat{\alpha}^{k+2l} = \sum_{k=0}^{q-2} \hat{\alpha}^k \left(\frac{1-\hat{\alpha}^{2q-2}}{1-\hat{\alpha}^2} - \hat{\alpha}^{2k} \right) = 1-q,$$

$$\begin{aligned}
\mathcal{S}_6 &= \sum_{0\leq k<l<m\leq q-2} \hat{\alpha}^{k+l+m} = \frac{1}{3!} \left(\sum_{k,l,m=0}^{q-2} \hat{\alpha}^{k+l+m} - 3\mathcal{S}_4 - \mathcal{S}_1 \right) \\
&= \frac{1}{6} \left(0 - 3(1-q) - (q-1)\right) \\
&= \frac{q-1}{3},
\end{aligned}$$

$$\mathcal{S}_7 = \sum_{k=0}^{q-2} \sum_{l}^{*} \hat{\alpha}^{k+l} = \left(\frac{1-\hat{\alpha}^{q-1}}{1-\hat{\alpha}} \right) \sum_{l}^{*} \hat{\alpha}^l = 0,$$

$$\begin{aligned}
\mathcal{S}_8 = \sum_{k}^{*} \hat{\alpha}^k &= \frac{1-\hat{\alpha}^{q^3}}{1-\hat{\alpha}} - \hat{\alpha}^{q^2+q+1} \frac{1-\hat{\alpha}^{q^3-1}}{1-\hat{\alpha}^{q^2+q+1}} - \sum_{i=0}^{q-1} \hat{\alpha}^i \sum_{j=0}^{q^2-1} \hat{\alpha}^{jq} \\
&\quad - \sum_{i=0}^{q^2-1} \hat{\alpha}^i \sum_{j=0}^{q-1} \hat{\alpha}^{jq^2} + \sum_{k=0}^{q-1} \hat{\alpha}^{k(q^2+q+1)} \\
&= \frac{1}{3}\mu(3)(q-1).
\end{aligned}$$

Hence

$$\mathcal{S}(1.(i)) = \sum_{j=1}^{8} \mathcal{S}_j = 3(q-1) + 2(1-q) + \frac{1}{3}(q-1) + 0 - \frac{1}{3}(q-1) = q-1.$$

For the conjugacy classes of type 1.(ii). Assume the same for α as in 1.(i) except that there could be $\alpha = 1$.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{\alpha}^{3k} = q-1,$$

$$\mathcal{S}_2 = \frac{1}{q^2+q} \sum_{k=0}^{q-2} q\hat{\alpha}^{3k} = \frac{q-1}{q+1},$$

$$\mathcal{S}_3 = 0,$$

$$\mathcal{S}_4 = \frac{1}{q^2+q+1} \sum_{k,l=0,l\neq k}^{q-2} (q+1)\hat{\alpha}^{k+2l} = \frac{1-q^2}{q^2+q+1},$$

$$\mathcal{S}_5 = \frac{1}{q(q^2+q+1)} \sum_{k,l=0,l\neq k}^{q-2} q\hat{\alpha}^{k+2l} = \frac{1-q}{q^2+q+1},$$

$$\mathcal{S}_6 = \frac{1}{(q+1)(q^2+q+1)} \sum_{0\le k<l<m\le q-2} (2q+1)\hat{\alpha}^{k+l+m} = \frac{(q-1)(2q+1)}{3(q+1)(q^2+q+1)},$$

$$\mathcal{S}_7 = \frac{1}{q^3-1} \sum_{k=0}^{q-2} \sum_{l}^{*} (-\hat{\alpha}^{k+l}) = 0,$$

$$\mathcal{S}_8 = \frac{1}{(q-1)^2(q+1)} \sum_{k}^{*} (1-q)\hat{\alpha}^{k} = \frac{1}{q^2-1}.$$

Hence

$$\mathcal{S}(1.(ii)) = \sum_{j=1}^{8} \mathcal{S}_j = q + o\left(\frac{1}{q}\right).$$

For the conjugacy classes of type 1.(iii). Assume the same for α as in 1.(i) except that there could be $\alpha = 1$.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{\alpha}^{3k} = q-1,$$

$$\mathcal{S}_2 = \mathcal{S}_3 = 0,$$

$$\mathcal{S}_4 = \frac{1}{q^2+q+1} \sum_{k,l=0,l\neq k}^{q-2} \hat{\alpha}^{k+2l} = \frac{1-q}{q^2+q+1},$$

$$\mathcal{S}_5 = 0,$$

$$\mathcal{S}_6 = \frac{1}{(q+1)(q^2+q+1)} \sum_{0\le k<l<m\le q-2} \hat{\alpha}^{k+l+m} = \frac{q-1}{3(q+1)(q^2+q+1)},$$

$$\mathcal{S}_7 = \frac{1}{q^3-1} \sum_{k=0}^{q-2} \sum_{l}^{*} (-\hat{\alpha}^{k+l}) = 0,$$

$$\mathcal{S}_8 = \frac{1}{(q-1)^2(q+1)} \sum_{k}^{*} \hat{\alpha}^{k} = \frac{-1}{(q-1)^2(q+1)}.$$

Hence

$$\mathcal{S}(1.(iii)) = \sum_{j=1}^{8} \mathcal{S}_j = q - 1 + o\left(\frac{1}{q}\right).$$

For the conjugacy classes of type 2.(i). In this case it is necessary that $\alpha^2\beta = 1$, or $\beta = \alpha^{-2}$, but $\alpha \neq 1$, otherwise the matrix will be the identity.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{\alpha}^{2k}\hat{\beta}^k = \sum_{k=0}^{q-2} 1 = q - 1,$$

$$\mathcal{S}_2 = \frac{1}{q^2+q} \sum_{k=0}^{q-2} (q+1)\hat{\alpha}^{2k}\hat{\beta}^k = \frac{q-1}{q},$$

$$\mathcal{S}_3 = \frac{1}{q^3} \sum_{k=0}^{q-2} q\hat{\alpha}^{2k}\hat{\beta}^k = \frac{q-1}{q^2},$$

$$\begin{aligned}
\mathcal{S}_4 &= \frac{1}{q^2+q+1} \sum_{k,l=0,l\neq k}^{q-2} \Big((q+1)\hat{\alpha}^{k+l}\hat{\beta}^l + \hat{\alpha}^{2l}\hat{\beta}^k\Big) \\
&= \frac{1}{q^2+q+1} \sum_{k,l=0,l\neq k}^{q-2} \Big((q+1)\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}\Big) \\
&= \frac{1}{q^2+q+1} \left(\sum_{k,l=0}^{q-2} ((q+1)\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}) - (q+1)(q-1) - (q-1)\right) \\
&= \begin{cases} -\dfrac{3(q-1)}{q^2+q+1} & \text{if } \alpha^2 = 1 \\[2ex] -\dfrac{(q-1)(q+2)}{q^2+q+1} & \text{if } \alpha^2 \neq 1, \end{cases}
\end{aligned}$$

$$\mathcal{S}_5 = \frac{1}{q(q^2+q+1)} \sum_{k,l=0,l\neq k}^{q-2} \Big((q+1)\hat{\alpha}^{k+l}\hat{\beta}^l + q\hat{\alpha}^{2l}\hat{\beta}^k\Big)$$

$$= \frac{1}{q(q^2+q+1)} \left(\sum_{k,l=0}^{q-2} \left((q+1)\hat{\alpha}^{k-l} + q\hat{\alpha}^{2l-2k} \right) - (q+1)(q-1) - q(q-1) \right)$$

$$= \begin{cases} \dfrac{q(q-1)^2 - (q-1)(2q+1)}{q(q^2+q+1)} = \dfrac{(q-1)(q^2-3q-1)}{q(q^2+q+1)} & \text{if } \alpha^2 = 1 \\ \\ -\dfrac{(q-1)(2q+1)}{q(q^2+q+1)} & \text{if } \alpha^2 \neq 1, \end{cases}$$

$$\mathcal{S}_6 = \frac{1}{(q+1)(q^2+q+1)} \sum_{0 \le k < l < m \le q-2} (q+1) \left(\hat{\alpha}^{k+l}\hat{\beta}^m + \hat{\alpha}^{k+m}\hat{\beta}^l + \hat{\alpha}^{l+m}\hat{\beta}^k \right)$$

$$= \frac{1}{(q+1)(q^2+q+1)} \sum_{0 \le k < l < m \le q-2} (q+1) \left(\hat{\alpha}^{k+l-2m} + \hat{\alpha}^{k+m-2l} + \hat{\alpha}^{l+m-2k} \right)$$

$$= \frac{1}{3!(q^2+q+1)} (\sum_{k,l,m} - \sum_{k \neq l = m} \left(2\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k} \right) - \sum_{l \neq k = m} \left(2\hat{\alpha}^{l-k} + \hat{\alpha}^{2k-2l} \right)$$

$$- \sum_{m \neq k = l} \left(2\hat{\alpha}^{m-l} + \hat{\alpha}^{2l-2m} \right) - \sum_{k=l=m} 3)$$

$$= \frac{1}{3!(q^2+q+1)} \left(3 \sum_{k,l,m} \hat{\alpha}^{k+l-2m} - 3 \sum_{k \neq l} (2\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}) - 3(q-1) \right)$$

$$= \begin{cases} -\dfrac{(q-1)(q-3)}{2(q^2+q+1)} & \text{if } \alpha^2 = 1 \\ \\ \dfrac{q-1}{q^2+q+1} & \text{if } \alpha^2 \neq 1, \end{cases}$$

$$\mathcal{S}_7 = \frac{1}{q^3-1} \sum_{k=0}^{q-2} \sum_{l}^{*} (q-1)\hat{\alpha}^l \hat{\beta}^k$$

$$= \frac{1}{q^2+q+1} \sum_{k=0}^{q-2} \sum_{l}^{*} \hat{\alpha}^{l-2k}$$

$$
= \begin{cases} -\dfrac{(q-1)}{q^2+q+1} & \text{if } \alpha^2 = 1 \\ \\ 0 & \text{if } \alpha^2 \neq 1, \end{cases}
$$

$$
\mathcal{S}_8 = 0.
$$

Hence

$$
\mathcal{S}(2.(i)) = \sum_{j=1}^{8} \mathcal{S}_j = \begin{cases} q-1+1+1-\frac{1}{2}+o\left(\dfrac{1}{q}\right) = q+\dfrac{1}{2}+o\left(\dfrac{1}{q}\right) & \text{if } \alpha^2 = 1 \\ \\ q-1+1-1+o\left(\dfrac{1}{q}\right) = q-1+o\left(\dfrac{1}{q}\right) & \text{if } \alpha^2 \neq 1. \end{cases}
$$

For the conjugacy classes of type 2.(ii). In this case it is necessary that $\alpha^2\beta = 1,$ or $\beta = \alpha^{-2},$ but $\alpha \neq 1$, otherwise the matrix will be the identity.

$$
\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{\alpha}^{2k}\hat{\beta}^k = q-1,
$$

$$
\mathcal{S}_2 = \frac{1}{q^2+q} \sum_{k=0}^{q-2} \hat{\alpha}^{2k}\hat{\beta}^k = \frac{q-1}{q^2+q},
$$

$$
\mathcal{S}_3 = 0,
$$

$$
\begin{aligned}
\mathcal{S}_4 &= \frac{1}{q^2+q+1} \sum_{k,l=0, l\neq k}^{q-2} \left(\hat{\alpha}^{k+l}\hat{\beta}^l + \hat{\alpha}^{2l}\hat{\beta}^k\right) \\
&= \frac{1}{q^2+q+1} \sum_{k,l=0, l\neq k}^{q-2} \left(\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}\right) \\
&= \frac{1}{q^2+q+1} \left(\sum_{k,l=0}^{q-2} (\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}) - (q-1) - (q-1) \right)
\end{aligned}
$$

$$= \begin{cases} \dfrac{(q-1)(q-3)}{q^2+q+1} & \text{if } \alpha^2 = 1 \\ \\ -\dfrac{2(q-1)}{q^2+q+1} & \text{if } \alpha^2 \neq 1, \end{cases}$$

$$\begin{aligned} \mathcal{S}_5 &= \frac{1}{q(q^2+q+1)} \sum_{k,l=0,l\neq k}^{q-2} \left(\hat{\alpha}^{k+l}\hat{\beta}^{l}\right) \\ &= \frac{1}{q(q^2+q+1)} \left(\sum_{k,l=0}^{q-2} (\hat{\alpha}^{k-l}) - (q-1) \right) \\ &= -\frac{q-1}{q(q^2+q+1)}, \end{aligned}$$

$$\begin{aligned} \mathcal{S}_6 &= \frac{1}{(q+1)(q^2+q+1)} \sum_{0\leq k<l<m\leq q-2} \left(\hat{\alpha}^{k+l}\hat{\beta}^{m} + \hat{\alpha}^{k+m}\hat{\beta}^{l} + \hat{\alpha}^{l+m}\hat{\beta}^{k}\right) \\ &= \frac{1}{(q+1)(q^2+q+1)} \sum_{0\leq k<l<m\leq q-2} \left(\hat{\alpha}^{k+l-2m} + \hat{\alpha}^{k+m-2l} + \hat{\alpha}^{l+m-2k}\right) \\ &= \frac{1}{3!(q+1)(q^2+q+1)} (\sum_{k,l,m} - \sum_{k\neq l=m} (2\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}) - \sum_{l\neq k=m} (2\hat{\alpha}^{l-k} + \hat{\alpha}^{2k-2l}) \\ &\quad - \sum_{m\neq k=l} (2\hat{\alpha}^{m-l} + \hat{\alpha}^{2l-2m}) - \sum_{k=l=m} 3) \\ &= \frac{1}{3!(q+1)(q^2+q+1)} \left(3\sum_{k,l,m} \hat{\alpha}^{k+l-2m} - 3\sum_{k\neq l} (2\hat{\alpha}^{k-l} + \hat{\alpha}^{2l-2k}) - 3(q-1) \right) \\ &= \begin{cases} -\dfrac{(q-1)(q-3)}{2(q+1)(q^2+q+1)} & \text{if } \alpha^2 = 1 \\ \\ \dfrac{q-1}{(q+1)(q^2+q+1)} & \text{if } \alpha^2 \neq 1, \end{cases} \end{aligned}$$

$$\mathcal{S}_7 = \frac{1}{q^3-1}\sum_{k=0}^{q-2}\sum_{l}^{*}\left(-\hat{\alpha}^l\hat{\beta}^k\right)$$
$$= \frac{1}{q^3-1}\sum_{k=0}^{q-2}\sum_{l}^{*}\left(-\hat{\alpha}^{l-2k}\right)$$
$$= \begin{cases} \frac{q-1}{q^3-1} & \text{if } \alpha^2 = 1 \\ \\ 0 & \text{if } \alpha^2 \neq 1, \end{cases}$$

$$\mathcal{S}_8 = 0.$$

Hence

$$\mathcal{S}(2.(ii)) = \sum_{j=1}^{8}\mathcal{S}_j = \begin{cases} q-1+1+o(\frac{1}{q}) = q+o\left(\frac{1}{q}\right) & \text{if } \alpha^2 = 1 \\ \\ q-1+o(\frac{1}{q}) = q-1+o\left(\frac{1}{q}\right) & \text{if } \alpha^2 \neq 1. \end{cases}$$

For the conjugacy classes of type 3. In this case it is necessary that $\alpha\beta\gamma = 1$. Note that there can only be one of α, β, γ with value 1, since otherwise the matrix will be identity or of type $2.(i)$.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2}\hat{\alpha}^k\hat{\beta}^k\hat{\gamma}^k = \sum_{k=0}^{q-2} 1 = q-1,$$

$$\mathcal{S}_2 = \frac{1}{q^2+q}\sum_{k=0}^{q-2} 2\hat{\alpha}^k\hat{\beta}^k\hat{\gamma}^k = \frac{2(q-1)}{q^2+q},$$

$$\mathcal{S}_3 = \frac{1}{q^3}\sum_{k=0}^{q-2}\hat{\alpha}^k\hat{\beta}^k\hat{\gamma}^k = \frac{q-1}{q^3},$$

$$\mathcal{S}_4 = \frac{1}{q^2+q+1}\sum_{k,l=0,l\neq k}^{q-2}\left(\hat{\alpha}^k\hat{\beta}^l\hat{\gamma}^l + \hat{\alpha}^l\hat{\beta}^k\hat{\gamma}^l + \hat{\alpha}^l\hat{\beta}^l\hat{\gamma}^k\right)$$

$$= \frac{1}{q^2+q+1} \sum_{k,l=0,l\neq k}^{q-2} (\hat{\alpha}^{k-l} + \hat{\beta}^{k-l} + \hat{\gamma}^{k-l})$$

$$= \frac{1}{q^2+q+1} \left(\sum_{k,l=0}^{q-2} (\hat{\alpha}^{k-l} + \hat{\beta}^{k-l} + \hat{\gamma}^{k-l}) - (q-1) - (q-1) - (q-1) \right)$$

$$= \begin{cases} \frac{(q-1)(q-4)}{q^2+q+1} & \text{if } 1 \in \{\alpha, \beta, \gamma\} \\ \\ -\frac{3(q-1)}{q^2+q+1} & \text{if } 1 \notin \{\alpha, \beta, \gamma\}, \end{cases}$$

$$\mathcal{S}_5 = \frac{1}{q(q^2+q+1)} \sum_{k,l=0,l\neq k}^{q-2} \left(\hat{\alpha}^k \hat{\beta}^l \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^k \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^l \hat{\gamma}^k \right)$$

$$= \begin{cases} \frac{(q-1)(q-4)}{q(q^2+q+1)} & \text{if } 1 \in \{\alpha, \beta, \gamma\} \\ \\ -\frac{3(q-1)}{q(q^2+q+1)} & \text{if } 1 \notin \{\alpha, \beta, \gamma\}, \end{cases}$$

$$\mathcal{S}_6 = \frac{1}{(q+1)(q^2+q+1)} \sum_{0\leq k<l<m\leq q-2} \left(\hat{\alpha}^k \hat{\beta}^l \hat{\gamma}^m + \hat{\alpha}^k \hat{\beta}^m \hat{\gamma}^l + \hat{\alpha}^m \hat{\beta}^l \hat{\gamma}^k + \hat{\alpha}^m \hat{\beta}^k \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^k \hat{\gamma}^m + \hat{\alpha}^l \hat{\beta}^m \hat{\gamma}^k \right)$$

$$= \frac{1}{(q+1)(q^2+q+1)} \sum_{0\leq k<l<m\leq q-2} \sum_{\tau\in\mathfrak{S}_3} \hat{\alpha}^{\tau(k)} \hat{\beta}^{\tau(l)} \hat{\gamma}^{\tau(m)}$$

$$= \frac{1}{(q+1)(q^2+q+1)} \sum_{0\leq k<l<m\leq q-2} \sum_{\tau\in\mathfrak{S}_3} \hat{\alpha}^{\tau(k)-\tau(m)} \hat{\beta}^{\tau(l)-\tau(m)}$$

$$= \frac{1}{3!(q+1)(q^2+q+1)} \sum_{\tau\in\mathfrak{S}_3} (\sum_{k,l,m} \hat{\alpha}^{\tau(k)-\tau(m)} \hat{\beta}^{\tau(l)-\tau(m)} - \sum_{k\neq l=m} \hat{\alpha}^{\tau(k)-\tau(m)} - \sum_{l\neq k=m} \hat{\beta}^{\tau(l)-\tau(m)}$$

$$- \sum_{m\neq k=l} \hat{\alpha}^{\tau(k)-\tau(m)} \hat{\beta}^{\tau(l)-\tau(m)} - \sum_{k=l=m} 3)$$

$$= \frac{1}{3!(q+1)(q^2+q+1)} \sum_{\tau\in\mathfrak{S}_3} \left(- \sum_{k\neq l=m} \hat{\alpha}^{\tau(k)-\tau(m)} - \sum_{l\neq k=m} \hat{\beta}^{\tau(l)-\tau(m)} - \sum_{m\neq k=l} \hat{\gamma}^{\tau(m)-\tau(k)} - 3(q-1) \right)$$

$$= \frac{1}{3!(q+1)(q^2+q+1)} \sum_{\tau \in \mathfrak{S}_3} \left(-\sum_{k,m=0}^{q-2} \hat{\alpha}^{\tau(k)-\tau(m)} - \sum_{l,m=0}^{q-2} \hat{\beta}^{\tau(l)-\tau(m)} - \sum_{m,k=0}^{q-2} \hat{\gamma}^{\tau(m)-\tau(k)} \right)$$

$$= \begin{cases} -\dfrac{(q-1)^2}{(q+1)(q^2+q+1)} & 1 \in \{\alpha, \beta, \gamma\} \\ \\ 0 & 1 \notin \{\alpha, \beta, \gamma\}, \end{cases}$$

$$\mathcal{S}_7 = \mathcal{S}_8 = 0.$$

Hence

$$\mathcal{S}(3.) = \sum_{j=1}^{8} \mathcal{S}_j = \begin{cases} q - 1 + 1 + o\left(\dfrac{1}{q}\right) = q + o\left(\dfrac{1}{q}\right) & \text{if } 1 \in \{\alpha, \beta, \gamma\} \\ \\ q - 1 + o\left(\dfrac{1}{q}\right) = q - 1 + o\left(\dfrac{1}{q}\right) & \text{if } 1 \notin \{\alpha, \beta, \gamma\}. \end{cases}$$

For the conjugacy classes of type 4. In this case it is necessary that $\alpha r^{q+1} = 1$. Note that $r \propto \mathbb{F}_{q^2}$, $r^{q+1} \in \mathbb{F}_q$, and $r + r^q \in \mathbb{F}_q$.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{\alpha}^k \hat{r}^{k(q+1)} = \sum_{k=0}^{q-2} 1 = q - 1,$$

$$\mathcal{S}_2 = 0,$$

$$\mathcal{S}_3 = \frac{1}{q^3} \sum_{k=0}^{q-2} \left(-\hat{\alpha}^k \hat{r}^{k(q+1)} \right) = -\frac{q-1}{q^3},$$

$$\begin{aligned} \mathcal{S}_4 &= \frac{1}{q^2+q+1} \sum_{k,l=0, l \neq k}^{q-2} \hat{\alpha}^k \hat{r}^{l(q+1)} \\ &= \frac{1}{q^2+q+1} \sum_{k,l=0, l \neq k}^{q-2} \hat{\alpha}^{k-l} \\ &= \frac{1}{q^2+q+1} \sum_{k,l=0}^{q-2} \left(\hat{\alpha}^{k-l} - (q-1) \right) \end{aligned}$$

$$= \begin{cases} \dfrac{(q-1)(q-2)}{q^2+q+1} & \text{if } \alpha = 1 \\ \\ -\dfrac{q-1}{q^2+q+1} & \text{if } \alpha \neq 1, \end{cases}$$

$$\mathcal{S}_5 = \frac{1}{q(q^2+q+1)} \sum_{k,l=0,l\neq k}^{q-2} \left(-\hat{\alpha}^k \hat{r}^{l(q+1)}\right)$$

$$= \begin{cases} -\dfrac{(q-1)(q-2)}{q(q^2+q+1)} & \text{if if } \alpha = 1 \\ \\ \dfrac{q-1}{q(q^2+q+1)} & \text{if } \alpha \neq 1, \end{cases}$$

$$\mathcal{S}_6 = 0,$$

$$\mathcal{S}_7 = \frac{1}{q^3-1} \sum_{k=0}^{q-2} \sum_{l}^{*} \left(-\hat{\alpha}^k \left(\hat{r}^l + \hat{r}^{ql}\right)\right) = \frac{1}{q^3-1} \left(\frac{1-\hat{\alpha}^{q-1}}{1-\hat{\alpha}}\right) \sum_{l}^{*} \left(-\hat{r}^l - \hat{r}^{ql}\right),$$

$$= \begin{cases} -\dfrac{2}{q^2+q+1} & \text{if } \alpha = 1 \\ \\ 0 & \text{if } \alpha \neq 1, \end{cases}$$

$$\mathcal{S}_8 = 0.$$

Hence

$$\mathcal{S}(4.) = \sum_{j=1}^{8} \mathcal{S}_j = \begin{cases} q - 1 + 1 + o\left(\dfrac{1}{q}\right) = q + o\left(\dfrac{1}{q}\right) & \text{if } \alpha = 1 \\ \\ q - 1 + o\left(\dfrac{1}{q}\right) & \text{if } \alpha \neq 1. \end{cases}$$

For the conjugacy classes of type 5. In this case it is necessary that $s^{1+q+q^2} = 1$. Note that

$s \propto \mathbb{F}_{q^3}$.

$$\mathcal{S}_1 = \sum_{k=0}^{q-2} \hat{s}^{k(q^2+q+1)} = \sum_{k=0}^{q-2} 1 = q-1,$$

$$\mathcal{S}_2 = \frac{1}{q^2+q} \sum_{k=0}^{q-2} \left(-\hat{s}^{k(q^2+q+1)}\right) = -\frac{q-1}{q^2+q+1},$$

$$\mathcal{S}_3 = \frac{1}{q^3} \sum_{k=0}^{q-2} \hat{s}^{k(q^2+q+1)} = \frac{q-1}{q^3},$$

$$\mathcal{S}_4 = \mathcal{S}_5 = \mathcal{S}_6 = \mathcal{S}_7 = 0,$$

$$\mathcal{S}_8 = \frac{1}{(q-1)^2(q+1)} \sum_{k}^{*} \left(\hat{s}^k + \hat{s}^{kq} + \hat{s}^{kq^2}\right) = 0.$$

Hence

$$\mathcal{S}(5.) = \sum_{j=1}^{8} \mathcal{S}_j = q - 1 + o\left(\frac{1}{q}\right).$$

Thus, we conclude that the commutator map on $\mathrm{SL}_3(\mathbb{F}_q) \smallsetminus \{1\}$ is numerically flat.

3.4.2 Numerical flatness on $\mathrm{SL}_3(\mathbb{F}_q)$

The last subsection shows by computation the following more explicit result

Theorem 3.4.1 (Numerical flatness on $\mathrm{SL}_3(\mathbb{F}_q)$)**.** $\mathcal{S}(\mathfrak{g}) = q + O(1)$, for any non-identity $\mathfrak{g} \in \mathrm{SL}_3(\mathbb{F}_q)$, hence the commutator map on $\mathrm{SL}_3(\mathbb{F}_q)$ is numerically flat.

This is better than what is needed for numerical flatness, i.e. not only $\mathcal{S}(\mathfrak{g}) = O(q)$ but $\mathcal{S}(\mathfrak{g}) = q + O(1)$. Unfortunately, this is not true for general n as shown in Chapter 5.

CHAPTER 4

Character estimates for $\mathrm{GL}_n(\mathbb{F}_q)$

4.1 Specification of $B_\rho(h^\rho m : \xi^\rho M)$

Before proving two estimates of character ratios in Theorem 2.3.4 and Theorem 2.3.5, and the numerical flatness on $\mathrm{SL}_n(\mathbb{F}_q) \setminus C(\mathrm{SL}_n(\mathbb{F}_q))$, $B_\rho(h^\rho m : \xi^\rho M)$ appearing in Theorem 2.2.1 needs to be evaluated more explicitly. For details, please check section 6 of Green [10]. Specifically, we need

Proposition 4.1.1. (Green's theorem 12 in [10])

For any primary character $I_{(g^\lambda)}$ with $g = \{k, kq, \cdots, kq^{s-1}\}$ an s-simplex and any conjugacy class $c \sim (f_1^{\nu_1} \cdots f_l^{\nu_l})$ in $\mathrm{GL}_n(\mathbb{F}_q)$ with $\deg(f_i) = F_i, i = 1, \cdots, l$, we have for any dual substitution m and substitution M

$$B_\rho(h^\rho m : \xi^\rho M) = \prod_{i=1}^{l} \prod_{\tau \in \rho(M, f_i)} T_{s,\tau F_i/s}(k : \xi_{f_i})$$

$$= \prod_{i=1}^{l} \prod_{\tau \in \rho(M, f_i)} \sum_{j=0}^{s-1} \theta^{kq^j} \left(\xi_{f_i}^{(q^{F_i \tau} - 1)/(q^s - 1)} \right)$$

in which $\tau \in \rho(M, f)$ means τ is a part of the partition $\rho(M, f)$ (as in Definition 6), ξ_f any root of the irreducible polynomial f. Any suitable ρ must be of the form $s \cdot \pi$ for some partition π of n/s, and for each such ρ we have the unique substitution m which sends all parts of ρ to g. In addition, $F_i \cdot \rho(M, f_i) = s \cdot \pi_{f_i}$ for some sub-partition π_{f_i} of π, i.e. $\pi_{f_i} = (F_i/s) \cdot \rho(M, f_i)$.

Then by Theorem 1.3.3, the partial Frobenius sum $\mathcal{S}_e(c)$ for any primary type $e = (g^\lambda)$ can be rewritten as

$$\begin{aligned}\mathcal{S}_e(c) &= \frac{1}{I_e(1)}(-1)^{n-|\lambda|}\sum_{g'\simeq g} I_{((g')^\lambda)}(c)\\ &= \frac{1}{I_e(1)}\sum_{|\pi|=n/s,M}\frac{1}{z_{s\cdot\pi}}\chi^\lambda_{s\cdot\pi}Q(M,c)\sum_{g'\simeq g}\prod_{i=1}^{l}\prod_{\tau\in\rho(M,f_i)}\sum_{j=0}^{s-1}\theta^{kq^j}\left(\xi_{f_i}^{(q^{F_i\tau}-1)/(q^s-1)}\right)\\ &= \frac{1}{I_e(1)}\sum_{|\pi|=n/s,M}\frac{1}{z_{s\cdot\pi}}\chi^\lambda_{s\cdot\pi}Q(M,c)B_\rho(g^\lambda,m,M), \qquad (4.1.1)\end{aligned}$$

in which any dual substitution m into $e = ((g')^\lambda)$ is determined by the partition π, $Q(M,c)$ as in Theorem 2.2.1 while $B_\rho(g^\lambda, m, M)$ in (4.1.1) is defined as

Definition 4.1.2. With all notations in the proposition above,

$$B_\rho(g^\lambda,m,M) := \sum_{g'\simeq g}\prod_{i=1}^{l}\prod_{\tau\in\rho(M,f_i)}\sum_{j=0}^{s-1}\theta^{kq^j}\left(\xi_{f_i}^{(q^{F_i\tau}-1)/(q^s-1)}\right).$$

Since we are summing over all s-simplices (for the fixed partition λ) in $B_\rho(g^\lambda, m, M)$, the root k (together with its *conjugates* $kq, \cdots, kq^{s-1}$) run through all numbers $t \in \{1, 2, \cdots, q^s - 1\}$ which make $t, tq, \cdots, tq^{s-1}$ all distinct modulo $q^s - 1$, by the definition of simplex. We denote the set of such numbers in $\{1, \cdots, q^s - 1\}$ by $\mathfrak{K}_s$, i.e.

Definition 4.1.3 ($\mathfrak{K}_s$)**.**

$$\mathfrak{K}_s := \{1 \le t \le q^s - 1 \mid t \in g \text{ for some simplex } g \text{ of } \deg(g) = s\}.$$

Actually, for a fixed primitive element ξ of $\mathbb{F}_{q^s}^\times$,

$$\mathfrak{K}_s = \{1 \le t \le q^s - 1 \mid \mathbb{F}_q[\xi^t] = \mathbb{F}_{q^s}\},$$

by the correspondence between $s-$simplexes and irreducible polynomials of degree s over $\mathbb{F}_q$.

Example 4.1.4. As defined in Chapter 3, simply $\mathfrak{K}_1 = \{1, 2, \cdots, q-1\}$. For $s = 2$, we need $t \not\equiv tq \ (mod\ q^2-1)$, i.e. $q^2 - 1 \nmid t(q-1)$ or $q+1 \nmid t$, hence

$$\mathfrak{K}_2 = \{1, 2, \cdots, q^2-1\} \setminus \{q+1, 2(q+1), \cdots, (q-1)(q+1) = q^2-1\}.$$

Now $B_\rho(g^\lambda, m, M)$ defined above can be rewritten as (maintaining notations from above)

$$\begin{aligned} B_\rho(g^\lambda, m, M) &= \frac{1}{s} \sum_{t\in\mathfrak{K}_s} \prod_{i=1}^{l} \prod_{\tau\in\rho(M,f_i)} \sum_{j=0}^{s-1} \theta^{tq^j} \left(\xi_{f_i}^{(q^{F_i\tau}-1)/(q^s-1)} \right) \\ &= \frac{1}{s} \sum_{u\in\{0,\cdots,s-1\}^\pi} \sum_{t\in\mathfrak{K}_s} \theta^t \left(\xi_*^{\sum_{\tau\in\pi} c_{\tau M} q^{u(\tau)} (q^{s\tau}-1)/(q^s-1)} \right) \\ &= \frac{1}{s} \sum_{u\in\{0,\cdots,s-1\}^\pi} \sum_{t\in\mathfrak{K}_s} \theta^t(\xi^u) \end{aligned} \tag{4.1.2}$$

Here ξ_* is a fixed primitive element of $\mathbb{F}_{q^{n!}}$, $\xi_*^{c_{\tau M}} := \xi_{\tau M}$ is any root of the irreducible polynomial τM (the part τ of ρ is sent to by the substitution M as in Definition 2.1.4), and for any function $u : \pi \to \{0, \cdots, s-1\}$ which sends each part of the partition π to a number, we denote

Definition 4.1.5.

$$\xi^u := \xi_*^{\sum_{\tau\in\pi} c_{\tau M} q^{u(\tau)}(q^{s\tau}-1)/(q^s-1)} = \prod_{\tau\in\pi} \xi_{\tau M}^{q^{u(\tau)}(q^{s\tau}-1)/(q^s-1)}.$$

Note that we need the functions u and $\xi_{\tau M}^{q^{u(\tau)}}$ to distinguish different choices of roots of the polynomials τM.

Thus more explicitly we have for $B_\rho(g^\lambda, m, M)$

Lemma 4.1.6. With all the notations above,

$$B_\rho(g^\lambda, m, M) \lesssim |\mathfrak{K}_s| = O(q^s).$$

Also for any class $c = (f^\nu)$ with $f(x) = x - \xi$ for some $\xi \in \mathbb{F}_q$, we have

$$B_\rho(g^\lambda, m, M) = s^{N_\pi - 1} \sum_{t \in \mathfrak{K}_s} \theta^t(\xi^{n/s}).$$

Proof. Since c is primary and $\xi \in \mathbb{F}_q$, we have $\forall \tau \in \pi, \tau M = f$ and $\xi_{\tau M}^{q^{u(\tau)}} = \xi^{q^{u(\tau)}} = \xi$, hence for any function u,

$$\xi^u = \prod_{\tau \in \pi} \xi^{(q^{s\tau}-1)/(q^s-1)} = \prod_{\tau \in \pi} \xi^{1+q^s+\cdots+q^{(\tau-1)s}} = \prod_{\tau \in \pi} \xi^\tau = \xi^{|\pi|} = \xi^{n/s}.$$

Then by (4.2)

$$B_\rho(g^\lambda, m, M) = \frac{1}{s} \sum_{u \in \{0,\cdots,s-1\}^\pi} \sum_{t \in \mathfrak{K}_s} \theta^t(\xi^u)$$

$$= \frac{1}{s} s^{N_\pi} \sum_{t \in \mathfrak{K}_s} \theta^t(\xi^{n/s}) = s^{N_\pi - 1} \sum_{t \in \mathfrak{K}_s} \theta^t(\xi^{n/s}).$$

□

4.2 Exponential sum over $\mathfrak{K}_s$

Now look at $\mathfrak{K}_s$ closer. If $tq^i \equiv tq^j \mod (q^s - 1)$ for some $1 \le i < j \le s-1$, then $(q^s - 1) \mid sq^i(q^{j-i} - 1)$, or $(q^s - 1) | i(q^{j-i} - 1)$. Furthermore we have

Lemma 4.2.1. For any positive integers a, b, c and q a power of any prime p we have

$$(q^a - 1, q^b - 1) = q^{(a,b)} - 1$$

and if $c|b$

$$\left(q^a - 1, \frac{q^b - 1}{q^c - 1}\right) = (q^{(a,c)} - 1, \frac{b}{[(a,b),c]})\frac{q^{(a,b)} - 1}{q^{(a,c)} - 1},$$

where (a, b) denotes their greatest common divisor and $[a, b]$ their least common multiplier.

Proof. Suppose $a < b$ and $b = d_1 a + b_1$ with $0 \le b_1 < a$. Multiply $q^a - 1$ by $q^{(d_1-1)a}$ and subtract by it from $q^b - 1$ we get

$$(q^a - 1, q^b - 1) = (q^a - 1, q^{d_1 a + b_1} - 1 - q^{d_1 a} + q^{(d_1-1)a})$$

$$= (q^a - 1, q^{d_1 a}(q^{b_1} - 1) + q^{(d_1-1)a} - 1) = (q^a - 1, q^{b_1} - 1),$$

then continue the Euclidean algorithm to get the first result. As to the second equality, we first prove the case where $a \mid c$, for which we have

$$\left(q^a - 1, \frac{q^b - 1}{q^c - 1}\right) = (q^a - 1, q^{c(b/c-1)} + q^{c(b/c-2)} + \cdots + q^c + 1)$$

$$= (q^a - 1, 1 + 1 + \cdots + 1 + 1) = \left(q^a - 1, \frac{b}{c}\right),$$

since $q^{at} \equiv 1 \mod (q^a - 1)$. In general we reduce it to

$$\left(q^a - 1, \frac{q^b - 1}{q^c - 1}\right) = \frac{1}{q^c - 1}\left((q^a - 1)(q^c - 1), q^b - 1\right)$$

$$= \frac{q^{(a,b)} - 1}{q^c - 1}\left(\frac{q^a - 1}{q^{(a,b)} - 1}(q^c - 1), \frac{q^b - 1}{q^{(a,b)} - 1}\right)$$

$$= \frac{q^{(a,b)} - 1}{q^c - 1}\left(q^c - 1, \frac{q^b - 1}{q^{(a,b)} - 1}\right) \cdots\cdots (*)$$

by the former result. But then by putting the outer multiplier in we get

$$\left(q^a - 1, \frac{q^b - 1}{q^c - 1}\right) = \left(q^{(a,b)} - 1, \frac{q^b - 1}{q^c - 1}\right).$$

Hence we just need to work out the case where $a \mid b$. Also we can just consider $a < c$ and $c = d_1 a + c_1$ for some $0 \le c_1 < a$, otherwise we just flip them using the equality $(*)$. Now we can just prove it by induction on a, but it is better to show the idea behind it as follows. Still by $q^{at} \equiv 1 (mod q^a - 1)$

we can rewrite the *gcd* as

$$\left(q^a-1, \frac{q^b-1}{q^{d_1a+c_1}-1}\right) = \left(q^a-1, \sum_{i=0}^{b/c-1} q^{(d_1a+c_1)i}\right)$$

$$= \left(q^a-1, \sum_{i=0}^{b/c-1} q^{c_1 i}\right) = \left(q^a-1, \frac{q^{bc_1/c}-1}{q^{c_1}-1}\right)$$

$$= \frac{q^a-1}{q^{c_1}-1}\left(q^{c_1}-1, \frac{q^{bc_1/c}-1}{q^a-1}\right).$$

Now if $c_1 \mid a$, then $(a,c)=c_1$ and by our previous result

$$\left(q^a-1, \frac{q^b-1}{q^c-1}\right) = \frac{q^a-1}{q^{c_1}-1}\left(q^{c_1}-1, \frac{bc_1}{ca}\right)$$

$$= \frac{q^a-1}{q^{(a,c)}-1}\left(q^{(a,c)}-1, \frac{b}{[a,c]}\right),$$

which is what we want. If $c_1 \nmid a$, we can repeat the same Euclidean procedure again and it is easy to check that we will again get the promised result. This keeps on until we get to the *gcd*. Tracing back our reduction, we get the second equality of the lemma. □

Clearly

$$\{1 \le t \le q^s-1\} \setminus \mathfrak{K}_s = \{1 \le t \le q^s-1, t = l\frac{q^s-1}{q^r-1}, \exists r|s, 1 \le r < s\}$$
$$= \bigcup_{1\le r<s, r|s} \frac{q^s-1}{q^r-1}\{1 \le t \le q^r-1\},$$

while for any $r|s, r'|s$,

$$\frac{q^s-1}{q^r-1}\{1 \le t \le q^r-1\} \cap \frac{q^s-1}{q^{r'}-1}\{1 \le t \le q^{r'}-1\}$$

$$= \frac{q^s-1}{q^{(r,r')}-1}\{1 \le t \le q^{(r,r')}-1\}.$$

Also, by the inclusion-exclusion principle (for any $\xi \in \mathbb{F}_{q^s}$)

Lemma 4.2.2 (Sum over $\mathfrak{K}_s$ and definition of id number)**.**

$$\sum_{t\in\mathfrak{K}_s} \theta^t(\xi) = \sum_{r|s} \mu(s/r) \sum_{t=1}^{q^r-1} \theta^{t(q^s-1)/(q^r-1)}(\xi) = \sum_{r|\mathrm{id}_s(\xi)} \mu(s/r)(q^r-1),$$

in which $\mu()$ is the Möbius function. Here $\mathrm{id}_s(\xi)$ denotes the largest divisor $r \mid s$ such that $\xi^{(q^s-1)/(q^r-1)} = 1$, called the *identity* number with respect to s. Clearly the *id* of 1 is just s. Also, if there is no divisor $r \mid s$ such that $\xi^{(q^s-1)/(q^r-1)} = 1$, we let $\mathrm{id}_s(\xi) = 0$ and then $\sum_{t\in\mathfrak{K}_s} \theta^t(\xi) = 0$.

4.3 Character ratio for central element $\neq 1$

Keep notations from the last section that $e = (g^\lambda)$ is a primary dual class with $\deg(g) = s$, etc. Now for any $1 \neq \xi \in \mathbb{F}_q$, $\xi^{(q^s-1)/(q^r-1)} = \xi^{s/r}, \forall r \mid s$. So to decide $\mathrm{id}_s(\xi)$ (as in Lemma 4.2.2), it is necessary to know if the order of ξ (in the multiplicative group $\mathbb{F}_q^\times$, denoted as $ord(\xi)$) divides s. If $ord(\xi) \nmid s$ then $\mathrm{id}_s(\xi) = 0$; if $ord(\xi) \mid s$, we can compute for any $w \in \mathbb{Z}$ that

$$\mathrm{id}_s(\xi^w) = \frac{s}{ord(\xi^w)} = \frac{s \cdot \gcd(ord(\xi), w)}{ord(\xi)} = \frac{sw}{lcm(ord(\xi), w)} \tag{4.3.1}$$

Then for any central element of the class $c \sim ((x-\xi)^{\{1^n\}})$ which will be denoted as ξ throughout, using (4.3.1) and the lemma above (note that $\rho = s \cdot \pi$)

$$B_\rho(g^\lambda, m, M) = s^{N_\pi - 1} \sum_{t\in\mathfrak{K}_s} \theta^t(\xi^{n/s}) = s^{N_\pi-1} \sum_{r|\frac{n}{lcm(ord(\xi),n/s)}} \mu(s/r)(q^r-1)$$

and the partial sum (in the central element case the substitution M into ξ is also determined by π)

$$\sum_{g'\simeq g} I_{(g')^\lambda}(\xi) = (-1)^{n-|\lambda|} \frac{1}{s} \sum_{|\pi|=n/s} \frac{1}{z_\pi} \chi^\lambda_{s\cdot\pi} Q(M, c) \sum_{r|\frac{n}{lcm(ord(\xi),n/s)}} \mu(s/r)(q^r-1),$$

in which z_π as in 2.1.7 and $Q(M,c)$ as in Theorem 2.2.1.

If $\xi = 1$, by the character formula in Theorem 2.2.1 and the above lemma, the degree of the primary characters can be expressed as

$$I_{(g)^\lambda}(1) = I_{(g')^\lambda}(1) = (-1)^{n-|\lambda|} \sum_{|\pi|=n/s} \frac{1}{z_\pi} \chi^\lambda_{s\cdot\pi} Q(M,c), \tag{4.3.2}$$

for any $g' \simeq g$. Note that directly from the formula in Theorem 2.2.1 we have for Green's polynomials

Lemma 4.3.1. $Q(M,c) = Q(M,1)$ for any central $c = \xi I_n$ of $\mathrm{GL}_n(\mathbb{F}_q)$.

Hence

$$\sum_{g'\simeq g} \frac{I_{(g')^\lambda}(\xi)}{I_{(g')^\lambda}(1)} = \frac{1}{s} \sum_{t\in\mathfrak{K}_s} \theta^t(\xi^{n/s}) = \frac{1}{s} \sum_{r|\frac{n}{lcm(ord(\xi),n/s)}} \mu(s/r)(q^r - 1) \tag{4.3.3}$$

Also for any dual class $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$ with $s_i = \deg(g_i)$ all distinct,

$$\{e' \simeq e\} = \coprod_{i=1}^{k} \{s_i - \text{simplexes}\},$$

in which $\coprod$ denotes concatenation of words. Hence it deduces from (4.4) that

$$\begin{aligned} \sum_{e'\simeq e} \frac{I_{e'}(\xi)}{I_{e'}(1)} &= \frac{1}{s_1 \cdots s_k} \prod_{i=1}^{k} \sum_{t\in\mathfrak{K}_{s_i}} \theta^t(\xi^{|\lambda_i|}) \\ &= \frac{1}{s_1 \cdots s_k} \prod_{i=1}^{k} \sum_{r|\frac{s_i|\lambda_i|}{lcm(ord(\xi),|\lambda_i|)}} \mu(s_i/r)(q^r - 1) \end{aligned} \tag{4.3.4}$$

If s_i are not all distinct, the set of dual classes of the same type does not have such a splitting decomposition, but it can be expressed by an inclusion-exclusion-principle-formula of the sets

$\coprod_{i=1}^{k}\{s_i - \text{simplexes}\}$. In general using the induced character formula as in Proposition 2.1.10, the partial Frobenius sum can be written as

Proposition 4.3.2. For any dual class $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$,

$$\sum_{e' \simeq e} \frac{I_{e'}(\xi)}{I_{e'}(1)} = \sum_{e' \simeq e} \prod_{i=1}^{k} \theta^{s(g_i')}(\xi^{|\lambda_i|}),$$

in which $s(g_i')$ denotes a root of the simplex g_i' occurring in the dual class e'.

For general central elements, the sum in the proposition above is not easy to calculate. However for primitive central elements

Proposition 4.3.3. The Frobenius sum $\mathcal{S}(\xi) = O(q)$ for $ord(\xi) = n$.

Proof. We just need to prove it for each type of characters (dual classes), since the number of types is independent of q. For any primary dual class $e = (g^\lambda)$ with $\deg(g) = s, n = s|\lambda|$, equation (6) shows

$$\sum_{g' \simeq g} \frac{I_{(g')^\lambda}(\xi)}{I_{(g')^\lambda}(1)} = \frac{1}{s}(q-1)$$

for $\text{id}_s(\xi) = 1 = \dfrac{n}{[ord(\xi), n/s]} = 1$ by $ord(\xi) = n$. Also (7) shows for any dual class $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$ with $k \geq 2$ and $s_i = \deg(g_i)$ all distinct,

$$\sum_{e' \simeq e} \frac{I_{e'}(\xi)}{I_{e'}(1)} = 0,$$

for $s_i|\lambda_i| < n = ord(\xi)$ and $\text{id}_{s_i}(\xi) = 0$. This implies by the inclusion-exclusion principle that if there are some $s_i \neq s_j$, the partial Frobenius sum is still zero. Then together with the inclusion-exclusion principle, we can show that the partial Frobenius sum over all dual classes of the same type with $k \geq 2$ and at least some $s_i \neq s_j$, is also zero. Hence we just need to consider the types with $k \geq 2$ and $s_1 = \cdots = s_k = s$. In this case, again by the inclusion-exclusion principle, we have

$$\sum_{e' \simeq e} \frac{I_{e'}(\xi)}{I_{e'}(1)} = \frac{1}{s^k} \prod_{i=1}^{k} \sum_{r | \frac{s|\lambda_i|}{lcm(ord(\xi), |\lambda_i|)}} \mu(s/r)(q^r - 1) + \sum_{e'' < e} (\pm 1) \sum_{e' \simeq e''} \frac{I_{e'}(\xi)}{I_{e'}(1)},$$

in which $e'' < e$ means any dual class of the form $(g_1^{\lambda'_1} \cdots g_l^{\lambda'_l})$ with $l < k$, $\deg(g_i) = s$ and λ'_j a sum of some partitions among $\lambda_1, \cdots, \lambda_k$, $\forall j = 1, \cdots, l$. In this order, a smallest class is of the type $((g')^{\lambda_1 + \cdots + \lambda_k})$ for any $\deg(g') = s$. Again similar to (7) the first term on the right hand side is zero, so that

$$\sum_{e' \simeq e} \frac{I_{e'}(\xi)}{I_{e'}(1)} = \sum_{e'' < e} (\pm 1) \sum_{e' \simeq e''} \frac{I_{e'}(\xi)}{I_{e'}(1)}.$$

Then for any $e'' < e$ we can do the same thing using inclusion-exclusion principle until we get down to the smallest type which gives $O(q)$ by (6) as we did in the beginning of the proof. There are apparently $O(1)$ many partial sums of the smallest type at the end, hence $\mathcal{S}(\xi) = O(1) \cdot O(q) = O(q)$. □

If n is a prime, this implies

Corollary 4.3.3.1. If n is prime, then the commutator map is numerically flat over $\mathrm{SL}_n(\mathbb{F}_q) \setminus \{I_n\}$, hence flat over $\mathrm{SL}_n(\mathbb{C}) \setminus \{I_n\}$.

4.4 Character values of unipotent classes

Besides central central elements, the next simplest is for unipotent classes (matrices with diagonal entries all equal). First it is easy to evaluate them on some special types of characters, i.e. $I_{g^{\{n/s\}}}$ with $s = \deg(g)$, which serves as a special case of the next section.

Consider any conjugacy class $c \sim (f^\lambda)$ with $\deg(f) = 1$, i.e. $c \sim ((x - \xi)^\lambda)$ for some $\xi \in \mathbb{F}_q$ and partition λ of n. Then similar to the central case, by Theorem 1.3.3, (4.1) and (4.3), it is direct to compute that $\forall J \in \{0, \cdots, s-1\}^{N_\pi}, \xi_J = \xi^{|\lambda|}, \mathrm{id}_s(\xi_J) = n/[ord(\xi), |\lambda|]$ and so by $\chi_\rho^{\{n\}} = 1$

(Example 1.4.13 (1))

$$\sum_{g'\sim g} I_{g\{n/s\}}(c) = \left(\frac{1}{s}\sum_{r|(n/lcm(ord(\xi),|\lambda|))} \mu\left(\frac{s}{r}\right)(q^r-1)\right)\sum_{|\pi|=|\lambda|}\frac{1}{z_\pi}Q^\lambda_{s\cdot\pi}(q),$$

where Q^λ_ρ for any partitions λ, ρ and z_π are defined in Definition 2.1.8. Define the latter sum as

Definition 4.4.1 (Green's sum)**.** The sum

$$Q^\lambda_s := \sum_{|\pi|=|\lambda|/s}\frac{1}{z_\pi}Q^\lambda_{s\cdot\pi}(q)$$

is called *Green's sum* with respect to a partition λ and a divisor s of $|\lambda|$.

If $\lambda = \{1^n\}$, Q^λ_s is just the degree of $I_{g\{n/s\}}$ (say by (4.3.2)). For non-central classes, we prove a *cancellation proposition* for the special case where $\lambda = \{1^{n-2}2\}$ that

Proposition 4.4.2 (First Cancellation Proposition)**.** For $n/s = 2$ and $\lambda = \{1^{n-2}2\}$, the sum

$$Q^\lambda_s = \sum_{|\pi|=2}\frac{1}{z_\pi}Q^\lambda_{s\cdot\pi}(q) = \frac{1}{z_{\{1^2\}}}Q^\lambda_{\{n/2,n/2\}}(q) + \frac{1}{z_{\{2\}}}Q^\lambda_{\{n\}}(q) \lesssim q^{\mathfrak{n}_\lambda-1},$$

where $z_\pi, \mathfrak{n}_\lambda$ are defined as in Definition 2.1.8.

To prove the proposition, there needs some results about Green's polynomials

Lemma 4.4.3 (Appendix tables of $Q^\lambda_\rho(q)$ in [10])**.** For any partitions $\lambda, \rho = \{1^{r_1}2^{r_2}\cdots k^{r_k}\}$ of n, we have

$$Q^{\{n\}}_\rho(q) = 1,$$

$$Q^{\{1^n\}}_\rho(q) = \frac{\phi_n(q)}{(1-q)^{r_1}(1-q^2)^{r_2}\cdots(1-q^k)^{r_k}},$$

$Q^\lambda_{\{n\}}(q) = \phi_{N_\lambda-1}(q)$ (N_λ is the number of parts in λ as in Definition 8),

$Q^\lambda_\rho(q)$ has leading term $\chi^\lambda_\rho q^{\mathfrak{n}_\lambda}$ ($\mathfrak{n}_\lambda$ as defined in Definition 8).

Proof of Proposition 4.4.2. As the leading term of $Q^\lambda_\rho(q)$ is $\chi^\lambda_\rho q^{\mathfrak{n}_\lambda}$, the leading coefficient c_λ of $Q^\lambda_{(s=n/2)}$ is

$$c_\lambda = \frac{1}{2}(\chi^\lambda_{\{(n/2)^2\}} + \chi^\lambda_{\{n\}}),$$

since by 2.1.7 $z_{\{1^2\}} = 1 \cdot 2! = 2$ and $z_{\{2\}} = 2 \cdot 1! = 2$.

Now the Frobenius character formula 1.4.10 for symmetric groups can be used to find the explicit values. Simply, χ^λ_ρ is the coefficient of x^λ in the Lorentz series

$$\prod_{i<j}(1 - \frac{x_i}{x_j}) \prod_{m \geq 1} H_m(x)^{r_m}),$$

where $x = (x_1, \cdots, x_N), H_m(x) = x_1^m + \cdots + x_N^m, \rho = (1^{r_1} 2^{r_2} \cdots)$, for any $N \geq N_\lambda$. In our case we can set $N = n-1$ and then look for coefficients of $x^\lambda = x_1^2 x_2 \cdots x_{n-1}$.

For $\rho = \{n\}$, the target term in $\prod_{i<j}(1 - \frac{x_i}{x_j}) \prod_{m\geq 1} H_m(x)^{r_m} = \prod_{i<j}(1 - \frac{x_i}{x_j})(x_1^n + \cdots + x_{n-1}^n)$ is

$$(-\frac{x_2}{x_1})(-\frac{x_3}{x_1}) \cdots (-\frac{x_{n-1}}{x_1}) x_1^n = (-1)^{n-2} x_1^2 x_2 \cdots x_{n-1}$$

and it is easy to see there is no other like term. Since n is even $(n/G = 2)$ we see that $\chi^\lambda_{\{n\}} = (-1)^{n-2} = 1$.

For $\rho = \{n/2, n/2\}$, $\prod_{i<j}(1 - \frac{x_i}{x_j}) \prod_{m\geq 1} H_m(x)^{r_m} = \prod_{i<j}(1 - \frac{x_i}{x_j})(x_1^{n/2} + \cdots + x_{n-1}^{n/2})^2$. There are two target terms

$$(-\frac{x_2}{x_1})(-\frac{x_3}{x_1}) \cdots (-\frac{x_{n-1}}{x_1}) x_1^n$$

and

$$(-\frac{x_2}{x_1}) \cdots (-\frac{x_{j-1}}{x_1})(-\frac{x_{j+1}}{x_j}) \cdots (-\frac{x_{n-1}}{x_j}) \cdot 2 x_1^{n/2} x_j^{n/2},$$

in which there must be $j = n/2$ to get $x_1^2 x_2 \cdots x_{n-1}$. Combining the two coefficients we get $\chi^\lambda_{\{n/2,n/2\}} = (-1)^{n-2} + (-1)^{n-3} \cdot 2 = 1 - 2 = -1$.

Altogether we proved

$$c_\lambda = \frac{1}{2}(\chi^\lambda_{\{n/2,n/2\}} + \chi^\lambda_{\{n\}}) = \frac{1}{2}(-1+1) = 0,$$

hence $Q_s^\lambda \lesssim q^{\mathfrak{n}_\lambda - 1}$ for $n/s = 2$. $\square$

Another perspective comes from Corollary 1.4.17.1. Actually, for $\lambda = \{1^{n-2}2\}$, $\chi^\lambda = (\chi_{perm} - 1)\chi_{sgn}$ where χ_{perm} is the permutation character and χ_{sgn} the sign character. For $n \geq 2$, $\chi_{perm}(\{(n/2)^2\}) = 0$, and for $n \geq 1, \chi_{sgn}(\{n\}) = 0$. However, $\chi_{sgn}(\{(n/2)^2\}) = 1$ while $\chi_{sgn}(\{n\}) = -1$ (n is even). Hence

$$c_\lambda = \frac{1}{2}(\chi^\lambda_{\{n/2,n/2\}} + \chi^\lambda_{\{n\}}) = \frac{1}{2}(-1+1) = 0,$$

and $Q_s^\lambda \lesssim q^{\mathfrak{n}_\lambda - 1}$ for $n/s = 2$.

Although the above result suits our purpose on estimating the character ratio, by applying the Littlewood-Richardson rule (see 6.3 Theorem V in [24] or 1.9 in [25]) it is possible to find exact formulas of Q_s^λ when $s = n/2$ and $s = n/3$ for $\lambda = \{1^{n-2}2\}$, even though the full power of it is not needed. The original rule is about expressing product of Schur functions, but by Green's homomorphism between the algebra of Schur functions and that of class functions (see section 7 of [10]), a weaker version of the rule for Hall's polynomials can be stated as follows

Lemma 4.4.4. Let λ, μ, ν be partitions. Then $g^\nu_{\lambda\mu} = 0$ if either of the Young diagrams of λ, μ is not included in that of ν.

Proposition 4.4.5 (A precise version of Proposition 4.4.2)**.**

$$Q_s^{\{1^{n-2}2\}} = \frac{\phi_{n-2}(q)}{1-q^s}, \text{ for } s = n/2.$$

To prove the proposition, there needs a decomposition rule on Green's polynomials as follows

Lemma 4.4.6 (Green's lemma 4.4 in [10])**.** For any given partitions ρ, σ and ν with $|\rho| + |\sigma| = |\nu|$

$$Q^{\nu}_{\rho+\sigma} = \sum_{|\lambda|=|\rho|,|\mu|=|\sigma|} g^{\nu}_{\lambda\mu} Q^{\lambda}_{\rho} Q^{\mu}_{\sigma},$$

where $\rho + \sigma$ is the partition consisting of all parts of ρ and σ together.

Proof of Proposition 4.4.5. By the above lemma, with $\nu = \{1^{n-2}2\}$, by Littlewood-Richardson rule, to make $g^{\nu}_{\lambda\mu} \neq 0$ there must be $\lambda \subset \nu$ and $\mu \subset \nu$, i.e. their Young diagrams are included in that of ν. For $n/s = 2$, we have then

$$Q^{\{1^{n-2}2\}}_{\{s^2\}} = g^{\{1^{n-2}2\}}_{\{1^s\}\{1^s\}} Q^{\{1^s\}}_{\{s\}} Q^{\{1^s\}}_{\{s\}} + 2g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^s\}} Q^{\{1^s\}}_{\{s\}} Q^{\{1^{s-2}2\}}_{\{s\}}$$

$$+ g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^{s-2}2\}} Q^{\{1^{s-2}2\}}_{\{s\}} Q^{\{1^{s-2}2\}}_{\{s\}},$$

in which we used commutativity $g^{\nu}_{\lambda\mu} = g^{\nu}_{\mu\lambda}$. Fortunately, it is not hard to compute those Hall polynomials with those simple indexes. By definition, $g^{\nu}_{\lambda\mu}$ is the number of flags $V_\nu \supset V_1 \supset V_2 = 0$ such that $V_\nu/V_1 \simeq V_\lambda$ and $V_1/V_2 = V_1 \simeq V_\mu$.

We first compute $g^{\{1^{n-2}2\}}_{\{1^s\}\{1^s\}}$. Let $\nu = \{1^{n-2}2\}, \lambda = \mu = \{1^s\}$ with $s = n/2$, and $e_i = (\delta_{i1}, \cdots, \delta_{in})$ (with the i-th coordinate 1 and others 0) be the standard basis so that $Ae_i = e_i$ for $i \leq n-1$ and $Ae_n = e_{n-1} + e_n$ if $A = ((t-1)^{\{1^{n-2}2\}})$. The two conditions on flags mean A acts trivially on the s-dimensional subspace V_1 and the quotient V_ν/V_1. Suppose $V_1 = \langle f_1, \cdots, f_s\rangle$ with $f_k = \sum_{j=1}^{n} a_{kj} e_j, k = 1, \cdots, s$, then $0 = Af_k - f_k = a_{kn} e_{n-1}$, i.e. $a_{kn} = 0, \forall k$, which means $V_1 \subset \langle e_1, \cdots, e_{n-1}\rangle$. Then A acts trivially on the quotient means for $\forall v, Av - v \in V_1$, especially $Ae_n - e_n = e_{n-1} \in V_1$. Hence we can show the two conditions are equivalent to $V_1 = \langle e_1\rangle \oplus W$ for any $s-1$ dimensional subspace $W \subset \langle e_1, \cdots, e_{n-2}\rangle$. Thus $g^{\{1^{n-2}2\}}_{(1^s)(1^s)}$ is the number of $s-1$ dimensional subspace in $\mathbb{F}_q^{n-2}$, i.e. the size of the Grassmannian $Gr(s-1,n-2)(\mathbb{F}_q)$. Hence for $s=n/2$,

$$g^{\{1^{n-2}2\}}_{\{1^s\}\{1^s\}} = |Gr(s-1,n-2)(\mathbb{F}_q)| = \frac{\phi_{n-2}(q)}{\phi^2_{n/2-1}(q)}.$$

Second for $g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^s\}}$, we keep the previous notations except $\mu=\{1^{s-2}2\}$. Suppose $f_k, k=1,\cdots,s$ is a basis of V_1 such that $Af_k=f_k$, for $k\leq s-1$, while $Af_s=f_{s-1}+f_s$, then $\{f_1,\cdots,f_{s-1}\}\subset\langle e_1,\cdots,e_{n-1}\rangle$ and $Af_s-f_s=a_{sn}e_{n-1}=f_{s-1}$ which forces $a_{sn}\neq 0$. This means $e_{n-1}\in V_1$ and $V_1\nsubseteq\langle e_1,\cdots,e_{n-1}\rangle$. Again A acting trivially on V_ν/V_1 just means $e_{n-1}\in V_1$. Hence we can show the two conditions in this case are equivalent to $V_1=\langle e_{n-1}\rangle\oplus W$ for any $s-1$ dimensional subspace $W\subset\langle e_1,\cdots,e_{n-2},e_n\rangle$ such that $W\nsubseteq\langle e_1,\cdots,e_{n-2}\rangle$. Then the number of such subspaces is $|Gr(s-1,n-1)(\mathbb{F}_q)|-|Gr(s-1,n-2)(\mathbb{F}_q)|$, hence

$$g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^s\}} = \frac{\phi_{n-1}(q)}{\phi_{n/2-1}(q)\phi_{n/2}(q)} - \frac{\phi_{n-2}(q)}{\phi^2_{n/2-1}(q)}.$$

Thirdly we can show $g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^{s-2}2\}}=0$. We keep the notations except $\lambda=\mu=\{1^{s-2}2\}$ and again similarly we get $e_{n-1}\in V_1$. We consider any extended basis $f_1,\cdots,f_s,g_1,\cdots,g_s$ of V_ν such that by the action of A on V_ν/V_1 we can have $Ag_k-g_k=b_{kn}e_{n-1}\in V_1, k\leq s-1$ and $Ag_s-(g_{s-1}+g_s)=b_{sn}e_{n-1}-g_{s-1}\in V_1$, in which $g_k=\sum_{j=1}^n b_{kj}e_j$. But since $e_{n-1}\in V_1$, from the last equation we get $g_{s-1}\in V_1$ $(s\geq 2)$, a contradiction. Hence there are no compatible flags.

Finally by $Q^\lambda_{\{n\}}(q)=\phi_{N_\lambda-1}(q)$ we can compute that

$$Q^{\{1^{n-2}2\}}_{(s=n/2)} = \frac{1}{2}\phi_{n-2}(q) + \frac{1}{2}g^{\{1^{n-2}2\}}_{\{1^s\}\{1^s\}}\phi^2_{n/2-1}(q) + g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^s\}}\phi_{n/2-1}(q)\phi_{n/2-2}(q)$$

$$+\frac{1}{2}g^{\{1^{n-2}2\}}_{\{1^{s-2}2\}\{1^{s-2}2\}}\phi^2_{n/2-2}(q)$$

$$=\frac{1}{2}\phi_{n-2}(q)+\frac{1}{2}\frac{\phi_{n-2}(q)}{\phi^2_{n/2-1}(q)}\phi^2_{n/2-1}(q)+$$

$$+\left(\frac{\phi_{n-1}(q)}{\phi_{n/2-1}(q)\phi_{n/2}(q)} - \frac{\phi_{n-2}(q)}{\phi^2_{n/2-1}(q)}\right)\phi_{n/2-1}(q)\phi_{n/2-2}(q)$$

$$= \frac{1}{2}\phi_{n-2}(q) + \frac{1}{2}\phi_{n-2}(q)\frac{1+q^s}{1-q^s}$$

$$= \frac{\phi_{n-2}(q)}{1-q^s} \sim q^{(n^2-4n+2)/2}.$$

□

Moreover, following the above proof verbatim

Corollary 4.4.6.1. $\forall r \geq 2, s \geq 2,$

$$g^{\{1^{r+s-2}2\}}_{\{1^r\}\{1^s\}}(q) = |Gr(r-1, r+s-2)(\mathbb{F}_q)| = \frac{\phi_{r+s-2}(q)}{\phi_{r-1}(q)\phi_{s-1}(q)},$$

$$g^{\{1^{r+s-2}2\}}_{\{1^{r-2}2\}\{1^s\}} = |Gr(r-1, r+s-1)(\mathbb{F}_q)| - |Gr(r-1, r+s-2)(\mathbb{F}_q)|$$

$$= \frac{\phi_{r+s-2}(q)q^s(1-q^{r-1})}{\phi_{r-1}(q)\phi_{s-1}(q)(1-q^s)},$$

$$g^{\{1^{r+s-2}2\}}_{\{1^{r-2}2\}\{1^{s-2}2\}} = 0.$$

Moreover,

$$Q^{\{1^{r+s-2}2\}}_{\{rs\}}(q) = \phi_{r+s-2}(q)\frac{1-q^{r+s}}{(1-q^r)(1-q^s)},$$

and especially,

$$Q^{\{1^{2r-2}2\}}_{\{r^2\}}(q) = \phi_{2r-2}(q)\frac{1+q^r}{1-q^r}.$$

Now using the induction rule on Hall polynomials and the above counting method, it can be computed that

Proposition 4.4.7 (Second Cancellation Proposition)**.** For $s = n/3 \geq 2$,

$$Q^{\{1^{n-2}2\}}_{s\cdot\{1^3\}} = \phi_{n-2}(q)\frac{1-q^n}{(1-q^{n/3})^3}$$

and

$$Q_s^{\{1^{n-2}2\}} = \frac{1}{3!}Q_{s\cdot\{1^3\}}^{\{1^{n-2}2\}} + \frac{1}{1\cdot 2}Q_{s\cdot\{12\}}^{\{1^{n-2}2\}} + \frac{1}{3}Q_{s\cdot\{3\}}^{\{1^{n-2}2\}}$$

$$= \frac{\phi_{n-2}(q)}{(1-q^{n/3})^2(1+q^{n/3})}.$$

Proof. By the corollary we have

$$Q_{s\cdot\{1^3\}}^{\{1^{n-2}2\}} = g_{\{1^s\}\{1^{2s}\}}^{\{1^{n-2}2\}} Q_{\{s\}}^{\{1^s\}} Q_{\{s^2\}}^{\{1^{2s}\}} + g_{\{1^s\}\{1^{2s-2}2\}}^{\{1^{n-2}2\}} Q_{\{s\}}^{\{1^s\}} Q_{\{s^2\}}^{\{1^{2s-2}2\}}$$

$$+g_{\{1^{s-2}2\}\{1^{2s}\}}^{\{1^{n-2}2\}} Q_{\{s\}}^{\{1^{s-2}2\}} Q_{\{s^2\}}^{\{1^{2s}\}} + g_{\{1^{s-2}2\}\{1^{2s-2}2\}}^{\{1^{n-2}2\}} Q_{\{s\}}^{\{1^{s-2}2\}} Q_{\{s^2\}}^{\{1^{2s-2}2\}},$$

$$= \frac{\phi_{n-2}(q)}{\phi_{s-1}(q)\phi_{2s-1}(q)}\phi_{s-1}(q)\phi_{2s}(q)/(1-q^s)^2$$

$$+\frac{\phi_{n-2}(q)q^s(1-q^{2s-1})}{\phi_{2s-1}(q)\phi_{s-1}(q)(1-q^s)}\phi_{s-1}(q)\phi_{2s-2}(q)\frac{1+q^s}{1-q^s}$$

$$+\frac{\phi_{n-2}(q)q^{2s}(1-q^{s-1})}{\phi_{s-1}(q)\phi_{2s-1}(q)(1-q^{2s})}\phi_{s-2}(q)\phi_{2s}(q)/(1-q^s)^2 + 0$$

$$= \phi_{n-2}(q)\frac{1-q^{2s}}{(1-q^s)^2} + \phi_{n-2}(q)\frac{q^s(1+q^s)}{(1-q^s)^2} + \phi_{n-2}(q)\frac{q^{2s}}{(1-q^s)^2}$$

$$= \phi_{n-2}(q)\frac{1+q^s+q^{2s}}{(1-q^s)^2} = \phi_{n-2}(q)\frac{1-q^n}{(1-q^{n/3})^3}.$$

Finally we can compute

$$Q_{(s=n/3)}^{\{1^{n-2}2\}} = \frac{1}{s}\frac{\phi_{n-2}(q)(1-q^n)}{(1-q^{n/3})^3} + \frac{1}{2}\frac{\phi_{n-2}(q)(1-q^n)}{(1-q^{n/3})(1-q^{2n/3})} + \frac{1}{3}\phi_{n-2}(q)$$

$$= \frac{\phi_{n-2}(q)}{(1-q^{n/3})^2(1+q^{n/3})} \sim q^{(n^2-5n+2)/2}.$$

□

In general as a corollary

Corollary 4.4.7.1. $\forall r, s, t \geq 2$,

$$Q_{\{rst\}}^{\{1^{r+s+t-2}2\}} = \phi_{r+s+t-2}(q) \frac{1-q^{r+s+t}}{(1-q^r)(1-q^s)(1-q^t)}.$$

Moreover, for general $r_i \geq 2, i = 1, \cdots, m$ and $r = \sum_i r_i$, it can be proved inductively that

$$Q_{\{r_1 \cdots r_m\}}^{\{1^{r-2}2\}} = \phi_{r-2}(q) \frac{1-q^r}{\prod_i (1-q^{r_i})}.$$

4.5 Partial Frobenius sum over type $(g^{\{v\}})$

Now look at the simplest primary characters which correspond to $(g^{\{v\}})$ with $\deg(g) = s$ and $sv = n$ over *non-central* conjugacy classes of $\mathrm{SL}_n(\mathbb{F}_q)$. If $s = 1$ then it is a linear character and evaluated at 1 on any matrix in $\mathrm{SL}_n(\mathbb{F}_q)$. Hence

$$\sum_{\deg(g)=1} \frac{I_{(g^{\{n\}})}(c)}{I_{(g^{\{n\}})}(1)} = |\mathfrak{K}_1| = q-1, \forall c \in \mathrm{SL}_n(\mathbb{F}_q).$$

Consider $s \geq 2$. By Theorem 1.3.3 and evaluating $B_\rho(g^\lambda, m, M)(c)$ in section 4.1 (note that $\chi_\rho^{\{v\}} = 1$ for any partition ρ of v)

$$\begin{aligned}
I_{(g^{\{v\}})}(1) \cdot \mathcal{S}_{(g^{\{v\}})}(c) &= (-1)^{n-v} \sum_{g' \simeq g} I_{((g')^{\{v\}})}(c) \\
&= \sum_{|\pi|=v, M} \frac{1}{z_{s\cdot\pi}} Q(M,c) \sum_{g' \simeq g} \prod_{i=1}^{l} \prod_{r \in \rho(M,f_i)} \sum_{j=0}^{s-1} \theta^{s_{g'} q^j} \left(\xi_{f_i}^{(q^{F_i r}-1)/(q^s-1)} \right) \\
&= \sum_{|\pi|=v, M} \frac{1}{z_{s\cdot\pi}} Q(M,c) B_\rho(g^\lambda, m, M)(c) \qquad (4.5.1)
\end{aligned}$$

whose magnitude is determined by $Q(M,c)$'s. Also its degree can be computed as

$$I_{(g^{\{v\}})}(1) = \phi_n(q)/\phi_v(q^s) \sim q^{n(n+1)/2 - s\cdot v(v+1)/2} = q^{(n/2)(n-n/s)} \qquad (4.5.2)$$

Now there comes the major estimate by Liebeck et al. as follows

Proposition 4.5.1. (Theorem 3.1 in [20])

For any $n \geq 5$ (we will treat $n = 2, 3, 4$ as separate cases), any non-central element c in $\mathrm{GL}_n(\mathbb{F}_q)$ and any of its irreducible character χ,

$$|\chi(c)/\chi(1)| \lesssim |\chi(1)|^{-\frac{1}{2n}}.$$

By the proposition, for $\chi = I_{(g^{\{v\}})}$, we have

$$|I_{(g^{\{v\}})}(1)|^{-\frac{1}{2n}} \sim q^{(n/2)(n-n/s)\cdot(-\frac{1}{2n})} = q^{\frac{n}{4s}(1-s)}.$$

So if $n/s \geq 4$, we have $|\chi(c)/\chi(1)| \lesssim q^{1-s}$, which is needed for Theorem 2.3.4. Hence the three discrete cases where $n/s = 1, 2, 3$ are all left over. We need some results about $\mathfrak{n}_\lambda$ in Definition 2.1.8 first

Lemma 4.5.2. For any $m \in \mathbb{Z}_+$ and any partition λ,

$$\begin{aligned}
&\mathfrak{n}_{\{m\}} = 0;\\
&\mathfrak{n}_\lambda \leq \mathfrak{n}_{\{1^m\}} = \binom{m}{2}, \text{ for } |\lambda| = m;\\
&\mathfrak{n}_\lambda \leq \mathfrak{n}_{\{1^{m-2}2\}} = \binom{m-1}{2}, \text{ for } |\lambda| = m \text{ with } N_\lambda \leq m-1.
\end{aligned}$$

Now consider it case by case for general non-central conjugacy classes in $\mathrm{SL}_n(\mathbb{F}_q)$.

(i) $n/s = 1$ or $s = n$. There is only one suitable partition, $s \cdot \{1\} = \{n\}$, and a substitutable conjugacy class c must only have one irreducible polynomial such that $c = (f^\nu)$ with $\deg(f) = F \mid n$.

Then by Lemma 4.4.3

$$Q(M,c) = Q^{\nu}_{\{n/F\}}(q^F) = \phi_{N_\nu - 1}(q^F) \sim q^{FN_\nu(N_\nu-1)/2},$$

compared with $I_{(g^{\{v\}})}(1)$ in (4.8), to achieve Theorem 2.3.4 there needs

$$Q(M,c)/I_{g^{\{n\}}}(1) \sim q^{FN_\nu(N_\nu-1)/2-n(n-1)/2} \leq q^{1-n},$$

i.e. we want

$$FN_\nu(N_\nu - 1) - n(n-1) - 2(1-n) = FN_\nu(N_\nu - 1) - (n-1)(n-2) \leq 0.$$

If $\deg(f) = F = 1$, then $\nu \neq \{1^n\}$ to avoid being central, so that $N_\nu \leq n-1$ and

$$FN_\nu(N_\nu - 1) - (n-1)(n-2) \leq (n-1)(n-2) - (n-1)(n-2) = 0;$$

if $F \geq 2$, then

$$FN_\nu(N_\nu - 1) - (n-1)(n-2) \leq n(n/2-1) - (n-1)(n-2)$$

$$= -n^2/2 + 2n - 2 = -(n-2)^2/2 \leq 0.$$

Thus by those two really niche estimations we proved it for $n/s = 1$ over non-central classes.

(ii) $n/s = 2$, the suitable partitions are $s \cdot \{1^2\} = \{(n/2)^2\}, s \cdot \{2\} = \{n\}$. Then the substitutable classes are $c = (f^\nu)$ or $c = (f_1^{\nu_1} f_2^{\nu_2})$. Here consider $c = (f^\nu)$ and leave the other to case **(iv)**. For non-unipotent classes (equivalent to $F = \deg(f) \geq 2$), by Lemma 4.4.3 that $Q^\nu_\rho(q) \lesssim q^{\mathfrak{n}_\nu}$, to achieve Theorem 2.3.4 it is necessary to show

$$Q(M,c)/I_{(g^{\{n\}})}(1) \sim q^{F\mathfrak{n}_\nu - n(n-2)/2} \leq q^{1-s} = q^{1-n/2}.$$

But again similar to the second estimation in **(i)** there is

$$2F\mathfrak{n}_\nu - n(n-2) - 2(1-n/2) \le F|\nu|(|\nu|-1) - (n-1)(n-2)$$

$$\le n(n/2-1) - (n-1)(n-2) = -\frac{1}{2}(n-2)^2 \le 0.$$

For $\deg(f) = F = 1$, the unipotent classes, there needs

$$Q(M,c)/I_{(g^{\{n\}})}(1) \lesssim q^{\mathfrak{n}_\nu - n(n-2)/2} \le q^{1-n/2}.$$

By $\mathfrak{n}_\nu \le (n-1)(n-2)/2$ for $\nu \ne \{1^n\}$ (again to avoid being central)

$$\mathfrak{n}_\nu - n(n-2)/2 \le (n-1)(n-2)/2 - n(n-2)/2 = (-n+2)/2 = 1-s,$$

in which the equality can only be achieved by the extreme case $\nu = \{1^{n-2}2\}$. However in that case by the first cancellation proposition (Proposition 4.4.2)

$$\frac{I_{(g^{\{v\}})}(c)}{I_{g^{\{v\}}}(1)} \lesssim q^{\mathfrak{n}_\nu - 1 - n(n-2)/2} = q^{-n/2},$$

and even better by Proposition 4.4.5 we have

$$\frac{I_{(g^{\{v\}})}(c)}{I_{(g^{\{v\}})}(1)} \sim q^{(n-1)(n-2)/2 - s - n(n-2)/2} = q^{1-2s} = q^{1-n}.$$

Hence in summary for $n/s = 2, c = (f^\nu)$ non-central we have

$$\frac{I_{(g^{\{v\}})}(c)}{I_{(g^{\{v\}})}(1)} \lesssim q^{-s},$$

which is better than what we want for Theorem 2.3.4 ($\lesssim q^{1-s}$).

(iii) $n/s = 3$. Then the suitable partitions are $s \cdot \{1^3\} = \{(n/3)^3\}, s \cdot \{12\} = \{n/3, 2n/3\}, s \cdot \{3\} = \{n\}$, and the substitutable classes are $c = (f^\nu), c = (f_1^{\nu_1} f_2^{\nu_2})$, or $c = (f_1^{\nu_1} f_2^{\nu_2} f_3^{\nu_3})$. Again consider the primary conjugacy classes and leave the other two cases to **(iv)**. Then similar for $F = \deg(f) \geq 2$ it needs to show

$$Q(M, c)/I_{(g^{\{n\}})}(1) \sim q^{F\mathfrak{n}_\nu - n(n-3)/2} \leq q^{1-n/3}.$$

i.e. there needs

$$\begin{aligned} 6F\mathfrak{n}_\nu - 3n(n-3) + 2n - 6 &\leq 3n(n/F - 1) - (3n-2)(n-3) \\ &= (3/F - 3)n^2 + 8n - 6 \leq 0 \end{aligned} \tag{4.5.3}$$

If $F = 2$, then $n \geq 6$ ($F|n$ and $n/s = 3$) so that (4.9) becomes

$$(3/F - 3)n^2 + 8n - 6 \leq -3/2(n^2 - 16n/3 + 4) = -3/2((n - 8/3)^2 - 28/9)$$

$$\leq -3/2(100/9 - 28/9) = -12 < 0,$$

since the parabola decreases as n gets bigger than $8/3$, so it suits our purpose.

If $F = 3$, (4.9) becomes

$$(3/F - 3)n^2 + 8n - 6 = -2n^2 + 8n - 6 = -2(n-2)^2 + 2,$$

which is exactly 0 for $n = 3$ (then $s = 1$) and negative for $n = 3m$ with $m \geq 2$. Hence again it suits our purpose.

If $F \geq 4$, then $n \geq 6$ (so that $3 \mid n$ and $n \geq 4$), (4.9) becomes

$$(3/F - 3)n^2 + 8n - 6 < -2n^2 + 8n - 6 = -2(n-2)^2 + 2 < -30.$$

If $F = 1$ and $\nu \neq \{1^n\}$ (to avoid being central) or $\{1^{n-2}2\}$, then $\mathfrak{n}_\nu \leq (n-2)(n-3)/2 + 1$ so that

$$6F\mathfrak{n}_\nu - 3n(n-3) + 2n - 6 \leq 3(n-2)(n-3) + 6 - (3n-2)(n-3) = 6 - 4(n-3) < 0$$

for $n > 3$ (then $n \geq 6$ for $n/s = 3$). If $n = 3$, then there can only be $\nu = \{3\}$, so that $\mathfrak{n}_\nu = 0$ and

$$6F\mathfrak{n}_\nu - 3n(n-3) + 2n - 6 = 0.$$

If $\nu = \{1^{n-2}2\}$, by the second cancellation proposition (Proposition 4.4.7) (on $Q_s^{\{1^{n-2}2\}}$)

$$\frac{I_{(g^{\{v\}})}(c)}{I_{(g^{\{v\}})}(1)} \sim q^{(n^2-5n+2)/2-n(n-3)/2} = q^{1-n}.$$

In summary there is for $n/s = 3$ and $c = (f^\nu)$,

$$\frac{I_{(g^{\{v\}})}(c)}{I_{(g^{\{v\}})}(1)} \lesssim q^{1-s}.$$

(iv) Now let's consider the general case $c \sim (f_1^{\lambda_1} \cdots f_k^{\lambda_k})$ for $k \geq 2$. Denote $F_i = \deg(f_i), |\lambda_i| = m_i, \mathfrak{n}_i = F_i m_i$ so that $\sum_i F_i m_i = \sum_i \mathfrak{n}_i = n$. Then $Q(M, c) = \prod_i Q^{\lambda_i}_{\rho(M,f_i)}(q^{F_i})$. Again by the estimate by Liebeck et al. as in Proposition 4.5.1, it only needs to consider $n/s = 1, 2, 3$. Then $k \leq 3$ because the suitable partitions can have at most 3 parts ($s \cdot \{1^3\} = \{(n/3)^3\}$) and substitutable conjugacy classes can not have more irreducible polynomial factors in its characteristic polynomial.

① For $n/s = 1$, $I_{(g^{\{v\}})}(c) = 0$ because there is no substitution of $s \cdot \{1\} = \{n\}$ into c (number of parts in potential partitions must not be less than the number of irreducible polynomials in c).

② For $n/s = 2$, to have non-zero character values, there must be $k = 2$ and $\mathfrak{n}_1 = \mathfrak{n}_2 = s = n/2$. Then

$$Q^{\lambda_i}_{\rho(M,f_i)}(q^{F_i}) \lesssim q^{F_i \mathfrak{n}_{\lambda_i}} \leq q^{\mathfrak{n}_i(m_i-1)/2} = q^{(n/2)(n/(2F_i)-1)/2}$$

and

$$Q(M,c) \lesssim q^{n(n/2-1)/2}.$$

Compared with the degree we get

$$\left|\frac{Q(M,c)}{I_{(g^{\{v\}})}(1)}\right| \lesssim q^{n(n/2-1)/2-n(n/s+1)/2-n(n+1)/2} \leq q^{1-s-(n^2/2-3n/2+1)},$$

which satisfies Theorem 2.3.4 for $n \geq 2$ when $n^2/2-3n/2+1=(n-1)(n-2)/2 \geq 0$.

③ For $n/s=3$, $k \leq 3$. To have substitutions from $s \cdot \{1^3\}$ or $s \cdot \{12\}$ (again no substitutions from $s \cdot \{3\} = \{n\}$), there must be either $k=2, \mathfrak{n}_1 = s = n/3, \mathfrak{n}_2 = 2s = 2n/3$, or $k=3, \mathfrak{n}_1 = \mathfrak{n}_2 = \mathfrak{n}_3 = s = n/3$.

ⓐ $k=2$, then

$$Q(M,c) \lesssim q^{s(s-1)/2+2s(2s-1)/2} = q^{(5n^2/9-n)/2}$$

and

$$\frac{|Q(M,c)|}{|I_{(g^{\{v\}})}(1)|} \lesssim q^{(5n^2/9-n)/2-n(n/s+1)/2-n(n+1)/2} = q^{1-s-(2n^2/9-4n/3+1)},$$

which suits Theorem 2.3.4 for $n \geq 6$. (This is fine, because to have $n/s=3$ and $s \geq 2$, there must be $n \geq 6$.)

ⓑ $k=3$, then

$$Q(M,c) \lesssim q^{3s(s-1)/2} = q^{(n^2/3-n)/2}$$

and

$$\frac{|Q(M,c)|}{|I_{(g^{\{v\}})}(1)|} \lesssim q^{(n^2/3-n)/2-n(n/s+1)/2-n(n+1)/2} = q^{1-s-(n^2/3-4n/3+1)},$$

which satisfies Theorem 2.3.4 for $n \geq 3$ when $n^2/3-4n/3+1=(n-1)(n-3)/3 \geq 0$.

Hence using the above crude estimation it is clear for any non-central class c

$$\frac{|I_{(g^{\{v\}})}(c)|}{|I_{(g^{\{v\}})}(1)|} \lesssim q^{1-s} \tag{4.5.4}$$

for any simplex g with $s = \deg(g) \geq 2$ and $sv = n$.

4.6 For general primary characters

To deal with general primary characters $I_{(g^{\lambda})}$, it is necessary to prove

Lemma 4.6.1 (Degree comparison lemma). $|I_{(g^{\lambda})}(1)| \gtrsim |I_{(g^{\{v\}})}(1)|$, for any $|\lambda| = v = n/s$ with $\deg(g) = s$.

Proof. By Green's lemma 7.4 in [10] or Theorem 1.3.3 we have

$$I_{(g^{\lambda})}(1) = \phi_n(q)\{\lambda : q^s\},$$

in which

$$\{\lambda : q\} = \frac{q^{l_2+2l_3+\cdots} \prod_{1\leq r<t\leq u}(1-q^{l_r-l_t-r+t})}{\prod_{r=1}^{u} \phi_{l_r+u-r}(q)},$$

if $\lambda = \{l_1, l_2, \cdots, l_u\}$ with $l_1 \geq l_2 \geq \cdots \geq l_u > 0$. To prove the desired result, we just need to show $\{\lambda : q\} \gtrsim \phi_{|\lambda|}^{-1}(q)$. Rather than directly compare them, we comprise to show

$$\{\lambda : q\}\phi_{|\lambda|}(q) \gtrsim \{\lambda : q\} \prod_{r=1}^{u} \phi_{l_r}(q) \gtrsim 1.$$

By the above formula we get

$$\{\lambda : q\} \prod_{r=1}^{u} \phi_{l_r}(q) \sim q^{l_2+2l_3+\cdots+\sum_{1\leq r<t\leq u}(l_r-l_t-r+t)-\sum_{r=1}^{u}\sum_{t=1}^{u-r}(l_r+t)}.$$

We compute the middle sum that

$$\sum_{1\le r<t\le u}(l_r - l_t - r + t) = \sum_{r=1}^{u}(u-r)l_r - \sum_{r=1}^{u}(r-1)l_r + \sum_{r=1}^{u}(u-2r+1)(u-r),$$

then the starting sum that

$$l_2 + 2l_3 + \cdots = \sum_{r=1}^{u}(r-1)l_r,$$

and the last sum that

$$\sum_{r=1}^{u}\sum_{t=1}^{u-r}(l_r + t) = \sum_{r=1}^{u}[(u-r)l_r + (u-r+1)(u-r)/2].$$

Then combining all those we get

$$\sum_{r=1}^{u}(u-r)(u-3r+1)/2.$$

Then this sum is supposed to be zero and interestingly it is always zero for $u \ge 1$. Thus we proved the desired result. □

On the other hand, the estimate in Proposition 4.4.7 covers for $n/s \ge 4$ and general non-central conjugacy classes. Hence one again just needs to consider the cases $n/s = 1, 2, 3$. By the degree comparison lemma above we find all the arguments on the simplest primary case involving only $Q(M,c)/I_{(g^{\{n/s\}})}(1)$ in section 4.5 can apply. Hence it works for all the cases of **(i)-(iv)** which needs only the estimation for single $Q(M,c)$ but not $Q_s^{\{1^{n-2}2\}}$, i.e. in **(iii)** for $\lambda \ne \{n/s = 3\}$ and $c = ((x-\xi)^{\{1^{n-2}2\}})$. In this case the best estimation by Proposition 4.4.5 gives

$$Q(M,c) \sim \phi_{n-2}(q) \sim q^{(n-1)(n-2)/2},$$

while the direct comparison with $I_{(g^{\{n/s=3\}})}(1)$ gives

$$Q(M,c)/I_{(g^{\{n/s=3\}})}(1) \sim q^{(n-1)(n-2)/2-n(n-3)/2} = q.$$

But for $\lambda = \{l_1, \cdots, l_u\} \neq \{n/s = 3\}$ it can be seen from proof of the degree comparison lemma that

$$I_{g^\lambda}(1)/I_{g^{\{3\}}}(1) \gtrsim \phi_3(q^s)/\prod_{r=1}^{u} \phi_{l_r}(q^s)$$

$$= \begin{cases} \phi_3(q^s)/\phi_1^3(q^s) \sim q^{3s} & \text{if } \lambda = \{1^3\} \\ \phi_3(q^s)/\phi_1(q^s)\phi_2(q^s) \sim q^{2s} & \text{if } \lambda = \{12\}. \end{cases}$$

Hence we get $Q(M,c)/I_{(g^\lambda)}(1) \sim q^{1-3s}$ or q^{1-2s} respectively.

4.7 General irreducible characters Corresponding to any complexes

Now look at the general case for $I_{(g_1^{\lambda_1}\cdots g_k^{\lambda_k})}$ with $s_i = \deg(g_i), b_i = |\lambda_i|, m_i = s_i b_i$ and $n = \sum_{i=1}^{k} m_i$. We need to prove the following

Proposition 4.7.1. For any non-central conjugacy class c of $n \times n$ matrices, we have

$$\left| \frac{I_e(c)}{I_e(1)} \right| \leq q^{1-\sum_{i=1}^{k} s_i}.$$

(When $k = 1$ we can see it is compatible with the result about primary characters.)

Proof. For any complex $e = (g_1^{\lambda_1} \cdots g_k^{\lambda_k})$, we have $I_e = I_{(g_1^{\lambda_1})} \circ \cdots \circ I_{(g_k^{\lambda_k})}$, which denotes the parabolic induction. If χ is induced from the character $I_{(g_1^{\lambda_1})}$ of $\mathrm{GL}_{m_1}(\mathbb{F}_q)$, by the induction formula in Proposition 2.1.10 (which is a variant of Mackey's formula 1.2.4), for any matrix C in the class we have

$$\chi(C) = \sum I_{(g_1^{\lambda_1})}(DCD^{-1}),$$

in which the sum is over all cosets $D \in \mathrm{GL}_n(\mathbb{F}_q)/\mathrm{GL}_{m_1}(\mathbb{F}_q)$ such that $DCD^{-1} \in \mathrm{GL}_{m_1}(\mathbb{F}_q) \subset \mathrm{GL}_n(\mathbb{F}_q)$, i.e. $\mathrm{GL}_{m_1}(\mathbb{F}_q)DC = \mathrm{GL}_{m_1}(\mathbb{F}_q)D$. Note that the cosets of $\mathrm{GL}_n(\mathbb{F}_q)/\mathrm{GL}_{m_1}(\mathbb{F}_q)$ are in one-to-one correspondence with flags $\mathbb{F}_q^n = V_0 \supset V_1 \supset 0$ with $\dim V_0/V_1 = m_1$, we can see that the suitable ones in the sum correspond to the flags with V_1 fixed by C (i.e. a submodule of the $p-$module V_C, with $\mathbb{F}_q^n$ acted on by C).

Hence inductively we can see that

$$I_e(C) = \sum_{\mathcal{F}} I_{(g_1^{\lambda_1})}(C_1) \cdots I_{(g_k^{\lambda_k})}(C_k),$$

in which the sum is over flags $\mathcal{F} = (V_0 \supset V_1 \supset \cdots \supset V_k = 0)$ with $\dim V_{i-1}/V_i = m_i, CV_i = V_i, i = 1, \cdots, k$, while $C_i = C|_{V_{i-1}/V_i}$ and $V_{i-1}/V_i \simeq V_{C_i}$. Denote the set of such flags by $\mathcal{F}_{m,C}$ where $m = (m_1, \cdots, m_k)$, and for each subset $S \subseteq \{1, \cdots, k\}$, let $\mathcal{F}_{m,C,S} = \{(V_i) \in \mathcal{F}_{m,C} \mid C|_{V_{j-1}/V_j} \text{ is scalar}, \forall j \in S\}$. Also, if C is the identity, denote $\mathcal{F}_m := \mathcal{F}_{m,1}$. Then in this context $I_e(1) = |\mathcal{F}_m| I_{(g_1^{\lambda_1})}(1) \cdots I_{(g_k^{\lambda_k})}(1)$ and

$$\left|\frac{I_e(C)}{I_e(1)}\right| = \frac{\sum_{\mathcal{F}\in\mathcal{F}_{m,C}} I_{(g_1^{\lambda_1})}(C_1) \cdots I_{(g_k^{\lambda_k})}(C_k)}{|\mathcal{F}_m|}$$

$$= \frac{1}{|\mathcal{F}_m|} \cdot \left| \sum_{\mathcal{F}\in\mathcal{F}_{m,C}} \frac{I_{(g_1^{\lambda_1})}(C_1) \cdots I_{(g_k^{\lambda_k})}(C_k)}{I_{(g_1^{\lambda_1})}(1) \cdots I_{(g_k^{\lambda_k})}(1)} \right|$$

$$\leq \frac{1}{|\mathcal{F}_m|} \cdot \sum_{S\subseteq\{1,\cdots,k\}} \sum_{\mathcal{F}\in\mathcal{F}_{m,C,S}} \left| \frac{I_{(g_1^{\lambda_1})}(C_1) \cdots I_{(g_k^{\lambda_k})}(C_k)}{I_{(g_1^{\lambda_1})}(1) \cdots I_{(g_k^{\lambda_k})}(1)} \right|.$$

Since we proved $\dfrac{I_{(g_i^{\lambda_i})}(C_i)}{I_{(g_i^{\lambda_i})}(1)} \lesssim q^{1-s_i}$, for all non-central (non-scalar) C_i, $\forall S$ we have

$$\sum_{\mathcal{F}\in\mathcal{F}_{m,C,S}} \left| \frac{I_{(g_1^{\lambda_1})}(C_1) \cdots I_{(g_k^{\lambda_k})}(C_k)}{I_{(g_1^{\lambda_1})}(1) \cdots I_{(g_k^{\lambda_k})}(1)} \right| \lesssim |\mathcal{F}_{m,C,S}| q^{\sum_{i\notin S}(1-s_i)}.$$

Now we **claim** that

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} \lesssim q^{1-n+\sum_{i\notin S}(m_i-1)}.$$

Actually, if we use Green's notations, $|\mathcal{F}_{m,C}| = g^{C}_{C_1,\cdots,C_k}$, the Hall's polynomial. If $C \sim (f_1^{\nu(f_1)} \cdots f_l^{\nu(f_l)})$, then simply by direct sum decomposition of V_C or Proposition 2.1.10

$$g^{C}_{C_1,\cdots,C_k} = \prod_{j=1}^{l} g^{v(f_j)}_{\nu_1(f_j)\cdots\nu_k(f_j)}(q^{\deg(f_j)}),$$

if $C_i = (f_1^{\nu_i(f_1)} \cdots f_l^{\nu_i(f_l)})$.

First we consider the case where $S = \{1, \cdots, k\}$. Then $C_i = ((t-\xi_i)^{\{1^{m_i}\}}), i = 1, \cdots, k$, in which $\xi_i \in \mathbb{F}_q$. Without loss of generality we can assume $\xi_1 = \cdots = \xi_{i_1} = \eta_1$, $\xi_{i_1+1} = \cdots = \xi_{i_2} = \eta_2, \cdots, \xi_{i_{l-1}+1} = \cdots = \xi_k = \eta_l$, and $C \sim ((t-\eta_1)^{\nu_1} \cdots (t-\eta_l)^{\nu_l})$ with $|\nu_j| = m_{i_{j-1}+1} + \cdots + m_{i_j}$, in which for convenience we denote $0 = i_0 < i_1 < i_2 < \cdots < i_{l-1} < i_l = k$. Then

$$g^{C}_{C_1,\cdots,C_k} = \prod_{j=1}^{l} g^{v_j}_{\{1^{m_{i_{j-1}+1}}\}\cdots\{1^{m_{i_j}}\}}(q)$$

$$\lesssim q^{\sum_{j=1}^{l}(\mathfrak{n}_{\nu_j} - m_{i_{j-1}+1}(m_{i_{j-1}+1}-1)/2 - \cdots - m_{i_j}(m_{i_j}-1)/2)}$$

$$= q^{\sum_{j=1}^{l}\mathfrak{n}_{\nu_j} - \sum_{i=1}^{k} m_i(m_i-1)/2}.$$

by Green's theorem 4 [10] that $g^{\lambda}_{\lambda_1,\cdots,\lambda_k}(q)$ has leading term $c^{\lambda}_{\lambda_1,\cdots,\lambda_k} q^{\mathfrak{n}_\lambda - \mathfrak{n}_{\lambda_1} - \cdots - \mathfrak{n}_{\lambda_k}}$. Also we can compute

$$|\mathcal{F}_m| = |Gr(m_k, n)||Gr(m_{k-1}, n-m_k)| \cdots |Gr(m_1, n-m_k-\cdots-m_2)|$$

$$= \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{\prod_{i=1}^{k} |\mathrm{GL}_{m_i}(\mathbb{F}_q)| q^{\sum_{j=1}^{k-1} m_j(n-m_1-\cdots-m_j)}} = \frac{\phi_n(q)}{\prod_{i=1}^{k}\phi_{m_i}(q)}$$

$$\sim q^{n(n-1)/2 - \sum_{i=1}^{k} m_i(m_i-1)/2}.$$

Then we get

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} \lesssim q^{\sum_{j=1}^{l} \mathfrak{n}_{\nu_j} - n(n-1)/2}.$$

Now if $l = 1$ and $C \sim ((t-\eta)^{\nu})$, since C is not central we must have $\nu \neq \{1^n\}$, and so $\mathfrak{n}_{\nu} \leq (n-1)(n-2)/2$. Then we get

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} \lesssim q^{\mathfrak{n}_{\nu} - n(n-1)/2} \leq q^{(n-1)(n-2)/2 - n(n-1)/2} = q^{1-n},$$

which is exactly what we want. It is clear that for $l \geq 2$, $\sum_{j=1}^{l} \mathfrak{n}_{\nu_j}$ is even smaller, so the result is valid for $S = \{1, \cdots, k\}$ with any possible non-central matrix.

If $S \neq \{1, \cdots, n\}$, we prove it *by induction* on k. First for $k = 1$, then $n = m_1$ and S is either $\{1\}$ or $\emptyset$. But if $S = \{1\}$ then C must be central. So we can only have $S = \emptyset$, and the result is trivially true because $1 - n + \sum_{i \notin S}(m_i - 1) = 1 - n + (m_1 - 1) = -n + m_1 = 0$. For all $k \geq 2$, assume the result for $1, \cdots, k-1$ and any n. We separate it into two cases as follows

① $k \in S$, then for any flag $(\mathbb{F}_q^n = V_0 \supset V_1 \supset \cdots \supset V_k = 0) \in \mathcal{F}_{m,C,S}$, the flag $(\mathbb{F}_q^{n-m_k} = V_0/V_{k-1} \supset V_1/V_{k-1} \supset \cdots \supset V_{k-1}/V_{k-1} = 0)$ belongs to $\mathcal{F}_{(m_1,\cdots,m_{k-1}),C|_{V_0/V_{k-1}},S \smallsetminus \{k\}}$, denoted by $\mathcal{F}'_{m,C,S}$. If $S \neq \{1, \cdots, k\}$ then $S \smallsetminus \{k\} \neq \{1, \cdots, k-1\}$ and we can use induction to get

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} = \frac{\left|\mathcal{F}'_{m,C,S}\right|}{|\mathcal{F}_m|} = \frac{\left|\mathcal{F}'_{m,C,S}\right|}{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|} \cdot \frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|}$$

$$\lesssim q^{1-(n-m_k)+\sum_{i \in \{1,\cdots,k-1\} \smallsetminus S}(m_i-1)} \cdot \frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|}$$

$$= q^{1-n+m_k+\sum_{i \notin S}(m_i-1)} \frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|}.$$

With the above formula we can compute that

$$\frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|} = \frac{\phi_{n-m_k}(q)\phi_{m_k}(q)}{\phi_n(q)}$$

$$\sim q^{(n-m_k)(n-m_k+1)/2+m_k(m_k+1)/2-n(n+1)/2} = q^{(1-n)m_k}.$$

Therefore,

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} \lesssim q^{1-n+\sum_{i\notin S}(m_i-1)+m_k+(1-n)m_k} \leq q^{1-n+\sum_{i\notin S}(m_i-1)},$$

since $m_k + (1-n)m_k = (2-n)m_k \leq 0$ for $n \geq k \geq 2$.

② $k \notin S$. If $S = \{1, \cdots, k-1\}$ and we look at the quotient flags as in ①, we get into the case we consider before so then

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} = \frac{\left|\mathcal{F}'_{m,C,S}\right|}{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|} \cdot \frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|}$$

$$\lesssim q^{1-(n-m_k)+(1-n)m_k} = q^{1-n+(m_k-1)+1-(n-1)m_k} \leq q^{1-n+(m_k-1)},$$

since $n \geq k \geq 2$ and $m_k \geq 1$. For $S \neq 1, \cdots, k-1$ we use induction to get

$$\frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} = \frac{\left|\mathcal{F}'_{m,C,S}\right|}{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|} \cdot \frac{\left|\mathcal{F}_{(m_1,\cdots,m_{k-1})}\right|}{|\mathcal{F}_m|}$$

$$\lesssim q^{1-(n-m_k)+\sum_{i\notin\{1,\cdots,k-1\}\setminus S}(m_i-1)+(1-n)m_k}$$

$$= q^{1-n+\sum_{i\notin S}(m_i-1)+1-(n-1)m_k} \leq q^{1-n+\sum_{i\notin S}(m_i-1)},$$

which is again what we want. Hence altogether we can prove the **claim**.

Finally, we can conclude that

$$\left|\frac{I_e(C)}{I_e(1)}\right| \lesssim \frac{|\mathcal{F}_{m,C,S}|}{|\mathcal{F}_m|} q^{\sum_{i\notin S}(1-s_i)} \lesssim q^{1-n+\sum_{i\notin S}(m_i-1)+\sum_{i\notin S}(1-s_i)}$$

$$= q^{1-n+\sum_{i\notin S}(m_i-s_i)} \leq q^{1-\sum_{i=1}^{k} s_i}.$$

□

4.8 Proof of Numerical flatness on $\mathrm{SL}_n(\mathbb{F}_q)$

To prove numerical flatness on $\mathrm{SL}_n(\mathbb{F}_q)$, we further need the Lang-Weil bound to turn the above arithmetic result into a dimension result.

Proposition 4.8.1 ([21], lemma 7.1)**.** Let V be a variety over $\bar{\mathbb{F}}_p$ which can be defined over $\mathbb{F}_q$. Suppose $\dim V = f$, and that V has e components of top dimension f. For any power q of p, let $V(q)$ denote the set of $\mathbb{F}_q - rational$ points in V. Then there is a power q_0 of p such that

$$|V(q)| = (e + o(1))q^f$$

for all powers q of q_0.

Through 3.1-3.7 we proved

$$\sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{\chi(\mathfrak{g})}{\chi(1)} \lesssim q,$$

which by Frobenius counting formula (Proposition 3) gives

$$|[,]^{-1}(\mathfrak{g})| \lesssim q|\mathrm{GL}_n(\mathbb{F}_q)| \sim q^{n^2+1},$$

for any $\mathfrak{g}$ of $\mathrm{SL}_n(\mathbb{F}_q)$ which is non-central, or $\mathfrak{g} \sim ((x-\xi)^{\{1^n\}})$ with ξ a primitive n-th root of unity. Hence for any such $\mathfrak{g}$ in $\mathrm{SL}_n(\bar{\mathbb{F}}_p)$, $[,]^{-1}(\mathfrak{g}) \subset \mathrm{GL}_n(\bar{\mathbb{F}}_p) \times \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ is a subvariety, and $\left|[,]^{-1}(\mathfrak{g})(q)\right| \lesssim q^{n^2+1}$. By Lang-Weil, this is to say

$$\dim[,]^{-1}(\mathfrak{g}) \leq n^2 + 1.$$

On the other hand, $[,] : \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathrm{SL}_n(\mathbb{F}_q)$ is a surjective morphism by Theorem 1 in [28], then $[,] : \mathrm{GL}_n(\bar{\mathbb{F}}_p) \times \mathrm{GL}_n(\bar{\mathbb{F}}_p) \to \mathrm{SL}_n(\bar{\mathbb{F}}_p)$ is also surjective. Hence by standard counting of dimension of fibers (see Theorem 4.1 of [14])

$$\dim[,]^{-1}(c) \geq \dim \mathrm{GL}_n \times \mathrm{GL}_n - \dim \mathrm{SL}_n = \dim \mathrm{GL}_n + 1 = n^2 + 1.$$

Altogether we must have numerical flatness for those $\mathfrak{g}$, i.e.

$$\dim[,]^{-1}(\mathfrak{g}) = n^2 + 1, \text{ over } \bar{\mathbb{F}}_p.$$

This implies for $\forall \mathfrak{g}_p \in \mathrm{SL}_n(\mathbb{F}_q) \smallsetminus \{\xi I_n, \xi \in \mathbb{F}_q$ not primitive n-th root of $1\}$,

$$|[,]^{-1}(\mathfrak{g})| \sim q^{n^2+1}.$$

CHAPTER 5

Flatness of the commutator map on $\mathrm{GL}_n(\mathbb{C})$ and related topics

5.1 Proof of Geometric flatness

We have dealt with numerical flatness in the previous sections, now we turn to prove geometric flatness of the commutator map. Now let $V_{\mathfrak{g},p}$ be the fiber in $\mathrm{GL}_n(\bar{\mathbb{F}}_p)\times\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ of the commutator map over $\mathfrak{g}\in\mathrm{SL}_n(\bar{\mathbb{F}}_p)\setminus\{\xi\cdot I_n,\xi \text{ is not primitive } n\text{-th root}\}$. In last section of the previous chapter, we proved $\dim V_{\mathfrak{g},p}=n^2+1$.

To prove flatness of the commutator map on $\mathrm{SL}_n(\mathbb{C})$, we introduce Grothendieck's theorem on constructible dimension of fibers:

Proposition 5.1.1 (Grothendieck's theorem, see [6], 9.2.6.1)**.** If $f:X\to S$ is a morphism of finite presentation, then the function $s\mapsto dim(f^{-1}(s))$ is locally constructible.

Now for any $\mathfrak{g}\in\mathrm{SL}_n(\mathbb{Q})\setminus\{\xi I_n,\xi\in\mathbb{Q}\}$, let $X=V_{\mathfrak{g}}$, where $V_{\mathfrak{g}}$ is the fiber $[,]^{-1}(\mathfrak{g})$ in $\mathrm{GL}_n\times\mathrm{GL}_n$, i.e. defined by a set of quadratic equations given by

$$xy=\mathfrak{g}yx.$$

By multiplying an integer scalar N to make all entries of $\mathfrak{g}$ integers which have no common factors, then clearly $V_{\mathfrak{g}}$ can be defined over $\mathbb{Z}$ (of finite presentation) as an algebraic variety, given by

$$Nxy=(N\mathfrak{g})yx \text{ (with } N\mathfrak{g}\in M_n(\mathbb{Z})).$$

Denote by $\mathfrak{g}_p$ the matrix in $\mathrm{SL}_n(\mathbb{F}_p)$ with all entries of $\mathfrak{g}$ modulo p. Let $f : V_{\mathfrak{g}} \to \operatorname{Spec} \mathbb{Z}$ be the structure morphism, and we denote the function $s \mapsto \dim(f^{-1}(s))$ on $S = \operatorname{Spec} \mathbb{Z}$ by h. Since for any point (p) ((0) if generic) of $\operatorname{Spec} \mathbb{Z}$, the fiber $f^{-1}((p)) = V_{\mathfrak{g}}(\mathbb{F}_p)$ is just the variety defined over $\mathbb{F}_p$ ($\mathbb{Q}$ if generic), which contains the fiber of $V_{\mathfrak{g}_p} = [,]^{-1}(\mathfrak{g}_p) \subset \mathrm{GL}_n(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$ as a dense set of closed points. (This is because $V_{\mathfrak{g}_p} \to V_{\mathfrak{g}}(\mathbb{F}_p)$ is a morphism of finite type between Jacobson schemes, which then sends closed points to closed points, see 10.4.6 and 10.4.7 of EGA IV [6].) Since the set of closed points are Zariski dense (see exercise 3.17 of Hartshorne [12]), hence we can apply our result of numerical flatness that for those co-finitely many characteristics p such that $\mathfrak{g}_p \in \mathrm{SL}_n(\mathbb{F}_p) \setminus \{\xi I_n, \xi \in \mathbb{F}_p$ not primitive n-th root of unity$\}$, $\dim(f^{-1}((p))) = n^2 + 1$. Then by Grothendieck's theorem, $h^{-1}(n^2 + 1)$ is a (locally) constructible subset of $\operatorname{Spec} \mathbb{Z}$, which therefore must contain the generic point (0). Thus we know

$$\dim V_{\mathfrak{g}}(\mathbb{Q}) = n^2 + 1.$$

Now using the similar argument as above, we go on to show $\forall \mathfrak{g} \in \mathrm{SL}_n(\bar{\mathbb{Q}}) \setminus \{\xi \cdot I_n \mid \xi \in \bar{\mathbb{Q}}\}$ and $V_{\mathfrak{g}} = [,]^{-1}(\mathfrak{g})$ fiber of the commutator map over $\bar{\mathbb{Q}}$, there's also $\dim V_{\mathfrak{g}} = n^2 + 1$. Since the entries of $\mathfrak{g}$ falls in some finite extension of $\mathbb{Q}$, $V_{\mathfrak{g}}$ is again definable over $\mathbb{Z}$, and we have the structure morphism $f : V_{\mathfrak{g}} \to \operatorname{Spec} \mathbb{Z}$. Then similarly for any point (p) of $\operatorname{Spec} \mathbb{Z}$, $f^{-1}((p)) = V_{\mathfrak{g}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p = V_{\mathfrak{g}}(\mathbb{F}_p)$ contains $[,]^{-1}(\mathfrak{g}_q) \subset \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q)$ for some power q of p and some matrix $\mathfrak{g}_q \in \mathrm{SL}_n(\mathbb{F}_q)$ as its closed points. Again by density of the closed points, applying numerical flatness for those co-finitely many characteristics p such that $\mathfrak{g}_q$ is not central or non-primitive n-th root of unity, we have $\dim f^{-1}((p)) = n^2 + 1$. Then by Grothendieck's theorem, $\dim V_{\mathfrak{g}} = n^2 + 1$.

By Lefschetz's principle (see [4]) we can cheaply extend the result to $\mathbb{C}$, i.e. $\forall \mathfrak{g} \in \mathrm{SL}_n(\mathbb{C}) \setminus \{\xi I_n, \xi \in \mathbb{C}$ not primitive n-th root of unity$\}$ we have

$$\dim [,]^{-1}(\mathfrak{g}) = n^2 + 1.$$

To make it more precise, we notice that the Zariski closure $X_{\mathfrak{g}} = \overline{\{\sigma(\mathfrak{g}) \mid \sigma \in Aut(\mathbb{C})\}}$ is defined over $\mathbb{Q}$, and $X_{\mathfrak{g}}(\bar{\mathbb{Q}})$ is Zariski dense in $X_{\mathfrak{g}}$. We showed for any $\mathfrak{h} \in X_{\mathfrak{g}}(\bar{\mathbb{Q}}), \dim[,]^{-1}(\mathfrak{h}) = \dim \mathrm{GL}_n(\mathbb{C})$, hence again by Grothendieck's theorem applied to $[,] : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$, we get the result.

Now we use the Cohen-Macaulay machinery to turn the equi-dimension argument into a flatness argument:

Proposition 5.1.2 (Hartshorne [12], Ex. 3.10.9)**.** Let $f : X \to Y$ be a morphism of varieties (over some field k). Assume that Y is regular, X is Cohen-Macaulay, and that every fiber of f has dimension equal to $\dim X - \dim Y$. Then f is flat.

We let

$$X = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}),$$

and

$$Y = \mathrm{SL}_n(\mathbb{C}) \setminus \{\xi \cdot I_n \mid \xi \text{ is not a primitive } n\text{-th root of } 1\},$$

and f be the commutator map, then they clearly satisfy all the prerequisites of the theorem above, hence we get the flatness of f.

5.2 Fibers of commutator maps over central elements in $\mathrm{SL}_n(\mathbb{C})$

5.2.1 For primitive roots of unity

Following the process in the last section, Proposition 4.3.3 suffices it to prove Theorem 2 for a central element $\xi := \xi \cdot I_n \in \mathrm{GL}_n(\mathbb{C})$ with ξ any primitive $n-$th root of unity. Actually directly counting dimension in $\mathbb{C}$ of fibers of the commutator map over such central elements is also manageable and we can show

Theorem 5.2.1. For $[,] : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ and any primitive $n-$th root $\xi \in \mathbb{C}$ of 1, the fiber over ξ is a disjoint union of orbits

$$[,]^{-1}(\xi) = \bigsqcup_{a,b \in \mathbb{C}^\times} \mathrm{GL}_n(\mathbb{C}) \cdot (a\sigma, b\tau_n(\xi)),$$

in which $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by conjugation on both entries, $\sigma = (12\cdots n)$ the $n-$cycle in the permutation group S_n and $\tau_n(\xi) = diag(1, \xi, \cdots, \xi^{n-1})$. Consequently,

$$\dim_{\mathbb{C}}[,]^{-1}(\xi) = \dim_{\mathbb{C}} \mathrm{GL}_n(\mathbb{C}) + 1.$$

Proof. First, by computation $[\sigma, \tau_n(\xi)] = \sigma\tau_n(\xi)\sigma^{-1}\tau_n(\xi)^{-1} = \xi$ using σ represented by the permutation matrix $(a_{i,j} = \delta_{i+1,j})_{n\times n}$. For simplicity we set $\delta_{n+1,1} = 1$ and consider indexes all modulo n.

Second, suppose there are $x, y \in \mathrm{GL}_n(\mathbb{C})$ such that $xyx^{-1}y^{-1} = \xi$, then $xyx^{-1} = \xi y$. WLOG, we assume y is in its Jordan canonical form and has spectrum $\{\lambda_1, \cdots, \lambda_n\}$. Now that y is similar to ξy, we must have $\xi\lambda_1 = \lambda_{i_1}$ with $i_1 \neq 1$, otherwise $\lambda_1 = 0$. Then $\xi^2\lambda_1 = \xi\lambda_{i_1} = \lambda_{i_2}$, with $i_2 \neq 1, i_1$, and subsequently, $\forall k = 1, \cdots, n-1, \xi^k\lambda_1 = \lambda_{i_k}$ with $i_k \neq 1, i_1, \cdots, i_{k-1}$, until $\xi^n\lambda_1 = \lambda_1 = \lambda_{i_n}$. Thus $(1 i_1 i_2 \cdots i_{n-1})$ forms a $n-cycle$ and the spectrum of y is actually $\lambda_1, \lambda_1\xi, \cdots, \lambda_1\xi^{n-1}$. Hence y is diagonalizable and by conjugation we can make $y = c\tau_n(\xi)$ for some $c \in \mathbb{C}^\times$. Then the equation becomes $x\tau_n(\xi)x^{-1} = \xi\tau_n(\xi)$, and compared with $\sigma\tau_n(\xi)\sigma^{-1} = \xi\tau_n(\xi)$ we get $(\sigma^{-1}x)\tau_n(\xi) = \tau_n(\xi)(\sigma^{-1}x)$. Clearly the matrices commuting with $\tau_n(\xi)$ must be diagonal, so $x = \sigma b$ for some $b = diag(b_1, \cdots, b_n)$.

Finally, we claim that if $\det b = 1$ then there is a diagonal matrix $d = diag(d_1, \cdots, d_n)$ such that $\sigma b = d\sigma d^{-1}$. Simply by computation we have

$$d\sigma d^{-1} = (a_{i,j} = \delta_{i+1,j} d_i d_j^{-1})_{n\times n}, \;\; \sigma b = (a_{i,j} = \delta_{i+1,j} b_{i+1})_{n\times n},$$

to set them equal we must have

$$d_1 = b_2 d_2, d_2 = b_3 d_3, \cdots, d_{n-1} = b_n d_n, d_n = b_1 d_1.$$

So given any $a \in \mathbb{C}^\times$, there is a solution for d as

$$d = diag(a, ab_2^{-1}, ab_2^{-1}b_3^{-1}, \cdots, ab_2^{-1} \cdots b_{n-1}^{-1}, ab_2^{-1} \cdots b_{n-1}^{-1} b_n^{-1} = ab_1),$$

in which $detb = b_1 b_2 \cdots b_n = 1$ makes the last entry valid. Hence in general we have $\sigma b = (\det b)^{1/n} d\sigma d^{-1}$. Clearly for each $a, b \in \mathbb{C}^\times, \mathrm{GL}_n(\mathbb{C}) \cdot (a\sigma, b\tau_n(\xi))$ is a distinct orbit and each solution to the commutator equation falls into some of those orbits.

Consequently, since the stabilizer of any $(a\sigma, b\tau_n(\xi))$ is just the center $\mathbb{C}^\times \cdot I_n$, we have

$$\dim_{\mathbb{C}}[,]^{-1}(\xi) = 2 + \dim_{\mathbb{C}} \mathrm{GL}_n(\mathbb{C}) - 1 = \dim_{\mathbb{C}} \mathrm{GL}_n(\mathbb{C}) + 1.$$

□

5.2.2 For non-primitive root of unity in $\mathrm{SL}_n(\mathbb{C})$

Over $\mathbb{C}$ with the $\mathrm{GL}_n(\mathbb{C})$ action, we investigate the following example and check that

$$[,]^{-1}(\xi^2) \supset \bigsqcup_{a, b_1 \neq b_2, c \in \mathbb{C}^\times} \mathrm{GL}_4(\mathbb{C}) \cdot (a\sigma^2, diag(b_1, b_2\xi, b_1, b_2\xi) + cE_{1,2} + c\xi E_{3,4}),$$

which has dimension $\geq \dim_{\mathbb{C}} \mathrm{GL}_4(\mathbb{C}) + 2$, note that $E_{i,j}$ denotes the matrix with entry $a_{i,j} = 1$ and all other entries 0. This suggests that for non-primitive root central elements, their fiber of the commutator map could be of bigger size than non-central elements.

Keep the notations as in section 5.1, for non-primitive roots we first prove the following

Theorem 5.2.2. For $n = ml, l \geq 1$, D the set of all semi-simple elements in $\mathrm{GL}_n(\mathbb{C})$, we have

$$[,]^{-1}(\xi^m) \cap (\mathrm{GL}_n(\mathbb{C}) \times D) = \bigsqcup_{b_1,\cdots,b_m \in \mathbb{C}^\times, A \in \mathrm{GL}_n(\mathbb{C})/C_{b_1,\cdots,b_m}} \mathrm{GL}_n(\mathbb{C}) \cdot (\sigma^m A, \tau_n^{b_1,\cdots,b_m}(\xi)),$$

in which

$$\tau_n^{b_1,\cdots,b_m}(\xi) = diag(b_1, \cdots, b_m, \xi^m b_1, \cdots, \xi^m b_m, \cdots, \xi^{(l-1)m} b_1, \cdots, \xi^{(l-1)m} b_m),$$

and $C_{b_1,\cdots,b_m} = C(\tau_n^{b_1,\cdots,b_m}(\xi)) \cap \mathrm{SL}_n^{m_1,\cdots,m_k}(\mathbb{C})$ with $C(\tau_n^{b_1,\cdots,b_m}(\xi))$ the centralizer of $\tau_n^{b_1,\cdots,b_m}(\xi)$ and $\mathrm{SL}_n^{m_1,\cdots,m_k}(\mathbb{C})$ defined by (5.2) below.

Consequently we have

$$dim_{\mathbb{C}}[,]^{-1}(\xi^m) \cap (\mathrm{GL}_n(\mathbb{C}) \times D) = \dim \mathrm{GL}_n(\mathbb{C}) + m.$$

Proof. First simply by computation we have $[\sigma^m, \tau_n^{b_1,\cdots,b_m}(\xi)] = \xi^m$.

Second, using the similar argument as in the previous proof to deal with $xyx^{-1} = \xi^m y$ with y semi-simple, we can see the action of ξ^m on the spectrum of y is partitioned into m many $l-$cycles, hence we can make $y = \tau_n^{b_1,\cdots,b_m}(\xi)$ for some $b_1, \cdots, b_m \in \mathbb{C}^\times$ by conjugation. Then compared with $\sigma^m \tau_n^{b_1,\cdots,b_m}(\xi)\sigma^{-m} = \xi^m \tau_n^{b_1,\cdots,b_m}(\xi)$ we get

$$(\sigma^{-m} x)\tau_n^{b_1,\cdots,b_m}(\xi) = \tau_n^{b_1,\cdots,b_m}(\xi)(\sigma^{-m} x),$$

i.e. $\sigma^{-m} x = A \in C = C(\tau_n^{b_1,\cdots,b_m}(\xi))$, the centralizer of $\tau_n^{b_1,\cdots,b_m}(\xi)$. We need to know whether $[x = \sigma^m A, \tau_n^{b_1,\cdots,b_m}(\xi)]$ belongs to a different orbit from $[\sigma^m, \tau_n^{b_1,\cdots,b_m}(\xi)]$ under the conjugation action of $\mathrm{GL}_n(\mathbb{C})$. Knowing that y can always be conjugated to $\tau_n^{b_1,\cdots,b_m}(\xi)$, we actually just need to know the orbits under conjugation action of C.

First, we have $[C\sigma^{-m}, C] \subset C \cap \mathrm{SL}_n(\mathbb{C})$, i.e. C acts on $\sigma^m C$ via conjugation: $\forall A, B \in C$,

$$\begin{aligned}
&(\sigma^m A)^{-1} B (\sigma^m A) B^{-1} \tau_n^{b_1,\cdots,b_m}(\xi) \\
=&(\sigma^m A)^{-1} B \sigma^m \tau_n^{b_1,\cdots,b_m}(\xi) A B^{-1} \\
=&\xi^m (\sigma^m A)^{-1} B \tau_n^{b_1,\cdots,b_m}(\xi) \sigma^m A B^{-1} \\
=&\xi^m A^{-1} \sigma^{-m} \tau_n^{b_1,\cdots,b_m}(\xi) B \sigma^m A B^{-1} \\
=&\xi^m \xi^{-m} A^{-1} \tau_n^{b_1,\cdots,b_m}(\xi) \sigma^{-m} B (\sigma^m A) B^{-1} \\
=&\tau_n^{b_1,\cdots,b_m}(\xi) (\sigma^m A)^{-1} B (\sigma^m A) B^{-1} \\
\Rightarrow&(\sigma^m A)^{-1} B (\sigma^m A) B^{-1} \in C \\
\Rightarrow&B(\sigma^m A) B^{-1} \in \sigma^m A C = \sigma^m C.
\end{aligned} \tag{5.2.1}$$

(Actually it is easy to see that any word consisting of σ^m, σ^{-m} and elements of C belongs to C if σ^m appears as many times as σ^{-m}.) Now we want to count the orbits of this action. More specifically, by

$$B(\sigma^m A)B^{-1} = \sigma^m A' \Leftrightarrow [(\sigma^m A)^{-1}, B] = A^{-1}A',$$

$\sigma^m A$ and $\sigma^m A'$ belong to the same orbit if and only if their difference $A^{-1}A' \in [(\sigma^m A)^{-1}, C]$. We have shown in the proof of Theorem 5.2.1 that for $m = 1$, the action is transitive on $C \cap \mathrm{SL}_n(\mathbb{C})$, and $[\sigma, C] = C \cap \mathrm{SL}_n(\mathbb{C})$. In general we can show that $[\sigma^m, C] = C \cap \mathrm{SL}_{m_1,\cdots,m_k}(\mathbb{C})$ for any $m \mid n$, $C = C(\tau_n^{b_1,\cdots,b_m}(\xi))$ and $\mathrm{SL}_n^{m_1,\cdots,m_k}(\mathbb{C})$ will be defined by (5.2) in the following.

We notice that $C = C(\tau_n^{b_1,\cdots,b_m}(\xi)) \subset \mathrm{GL}_m(\mathbb{C})^l$. More specifically, looking b_i as representatives of the l-cycles, if for $m_1 + \cdots + m_k = m$ with $m_i \geq 1$ we have $b_1 = b_2 = \cdots = b_{m_1} = \beta_1$; $b_{m_1+1} = \cdots = b_{m_1+m_2} = \beta_2$; $\cdots$; $b_{m_1+\cdots+m_{k-1}+1} = \cdots = b_{m_1+\cdots+m_{k-1}+m_k} = b_m = \beta_k$ with $\beta_i\beta_j^{-1} \neq \xi^{rm}$ for any integer r, then

$$C \sim (\mathrm{GL}_{m_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{m_k}(\mathbb{C}))^l.$$

Thus for any $A, B \in C$, they are of the form

$$A \sim diag(A_{11}, \cdots, A_{1k}; \cdots; A_{l1}, \cdots, A_{lk}),\ A_{ij} \in \mathrm{GL}_{m_j}(\mathbb{C}), \forall 1 \le i \le l, 1 \le j \le k,$$

$$B \sim diag(B_{11}, \cdots, B_{1k}; \cdots; B_{l1}, \cdots, B_{lk}),\ B_{ij} \in \mathrm{GL}_{m_j}(\mathbb{C}), \forall 1 \le i \le l, 1 \le j \le k$$

and

$$\begin{aligned} A &= \sigma^{-m} B \sigma^m B^{-1} \\ &= diag(B_{l1}B_{11}^{-1}, \cdots, B_{lk}B_{1k}^{-1}; B_{11}B_{21}^{-1}, \cdots, B_{2,k}^{-1}; \cdots; B_{(l-1)1}B_{l1}^{-1}, \cdots, B_{(l-1)k}B_{lk}^{-1}) \\ &\Rightarrow A_{1j}A_{2j}\cdots A_{lj} = B_{lj}B_{1j}^{-1}B_{1j}B_{2j}^{-1}\cdots B_{(l-1)j}B_{lj}^{-1} = I_n,\ \forall 1 \le j \le k. \end{aligned} \tag{5.2.2}$$

Conversely, if A satisfies the condition (5.2), then $\forall B_j \in \mathrm{GL}_{m_j}(\mathbb{C}), 1 \le j \le k$, setting $B_{ij} = A_{ij}^{-1} \cdots A_{1j}^{-1} B_j, \forall 1 \le i \le k$, we have $A = \sigma^{-m} B \sigma^m B^{-1}$. Denote by $\mathrm{SL}_n^{m_1,\cdots,m_k}(\mathbb{C})$ the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of matrices with diagonal blocks satisfying (5.2). This shows that orbits of the conjugation action of C on $\sigma^m C$ are classified by $\mathrm{GL}_{m_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{m_k}(\mathbb{C})$.

Together with the conjugation by $\mathrm{GL}_n(\mathbb{C})$ it enables us to see that if $(x, y) \in [,]^{-1}(\xi^m)$ with y semi-simple then it falls in some orbit of the form $\mathrm{GL}_n(\mathbb{C}) \cdot (\sigma^m A, \tau_n^{b_1,\cdots,b_m}(\xi))$, for some $A \in \mathrm{GL}_n(\mathbb{C})/(C \cap \mathrm{SL}_n^{m_1,\cdots,m_k}(\mathbb{C}))$ which is one-to-one corresponding to $\mathrm{GL}_{m_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{m_k}(\mathbb{C})$.

Now we count the dimension of the fiber over ξ^m. By (5.2) we see that any matrix in the centralizer of $(\sigma^m, \tau_n^{b_1,\cdots,b_m}(\xi))$ must have the form

$$A \sim diag(A_{11}, \cdots, A_{1k}; \cdots; A_{11}, \cdots, A_{1k}),$$

i.e. A_{ij} are all the same for $i = 1, \cdots, l$ and a fixed j. Hence the dimension of any such centralizer is $\sum_{i=1}^k \dim \mathrm{GL}_{m_i}(\mathbb{C})$, and

$$\dim \mathrm{GL}_n(\mathbb{C}) \cdot (\sigma^m, \tau_n^{b_1,\cdots,b_m}(\xi)) = \dim \mathrm{GL}_n(\mathbb{C}) - \sum_{i=1}^k \dim \mathrm{GL}_{m_i}(\mathbb{C}),$$

For centralizers of $(\sigma^m A, \tau_n^{b_1,\cdots,b_m}(\xi))$, by solving the linear equations given by $B\sigma^m AB^{-1} = \sigma^m A$ in $(M_{m_1}(\mathbb{C}) \times M_{m_k}(\mathbb{C}))^l$, we can see they all have the same dimension as above. Consequently the dimension of the union of all such type of orbits is then

$$\sum_{i=1}^k \dim \mathrm{GL}_{m_i}(\mathbb{C}) + k + \dim \mathrm{GL}_n(\mathbb{C}) - \sum_{i=1}^k \dim \mathrm{GL}_{m_i}(\mathbb{C}) = k + \dim \mathrm{GL}_n(\mathbb{C}),$$

which achieves maximum when $k = m$.

□

For $[x, y] = xyx^{-1}y^{-1} = \xi^m$ in $\mathrm{GL}_n(\mathbb{C})$ with y not semi-simple, we can use the multiplicative Jordan decomposition (see Chapter VI of Humphreys [14]) to reformulate the commutator equation as follows

$$\begin{aligned} & xyx^{-1} = \xi^m y = xy_s x^{-1} xy_u x^{-1} = \xi^m y_s y_u \\ \Leftrightarrow & xy_s x^{-1} = \xi^m y_s \text{ and } xy_u x^{-1} = y_u \\ \Leftrightarrow & [x, y_s] = \xi^m \text{ and } [x, y_u] = 1. \end{aligned}$$

By Theorem 5.2.2 which deals with the first equation, further confined by the second equation, we have

Theorem 5.2.3. With the notations from above (including the proof),

$$[,]^{-1}(\xi^m) = \bigcup_{|\lambda_1|+\cdots+|\lambda_k|=m} \; \bigcup_{\beta_1,\cdots,\beta_k \in \mathbb{C}^\times, A \in \mathrm{GL}_n(\mathbb{C})/C_{b_1,\cdots,b_m}} \mathrm{GL}_n(\mathbb{C}) \cdot (\sigma^m A, \tau_{\lambda_1,\cdots,\lambda_k}^{\beta_1,\cdots,\beta_k}(\xi)),$$

in which

$$\tau^{\beta_1,\cdots,\beta_k}_{\lambda_1,\cdots,\lambda_k}(\xi) = diag(U_{\lambda_1}(t-\beta_1),\cdots,U_{\lambda_k}(t-\beta_k); U_{\lambda_1}(t-\beta_1\xi^m),\cdots,U_{\lambda_k}(t-\beta_k\xi^m);$$

$$\cdots; U_{\lambda_1}(t-\beta_1\xi^{(l-1)m}),\cdots,U_{\lambda_k}(t-\beta_k\xi^{(l-1)m})),$$

for any partitions $|\lambda_i| = m_i$, $m = m_1 + m_2 + \cdots + m_k$ with $m_i \geq 1$ and $\beta_1, \beta_2, \cdots, \beta_k$ have distinct l-cycles, i.e. $\beta_i\beta_j^{-1} \neq \xi^{rm}, \forall i \neq j, 0 \leq r \leq l-1$. (Note that if $k = m$, then $\lambda_i = \{1\}$ and $\tau^{\beta_1,\cdots,\beta_k}_{\lambda_1,\cdots,\lambda_k}(\xi) = \tau^{\beta_1,\cdots,\beta_k}_n(\xi)$.)

Consequently we have

$$dim_{\mathbb{C}}[,]^{-1}(\xi^m) = \dim \mathrm{GL}_n(\mathbb{C}) + m.$$

5.2.3 For non-primitive root of unity in $\mathrm{SL}_n(\mathbb{F}_q)$

Besides the results obtained in section 4.3, in general, for $ord(\xi) \mid n$ with $1 < ord(\xi) < n$, i.e. non-primitive $n-$th roots of unity, the fiber $[,]^{-1}(\xi)$ could be of higher dimension. Say for $n = 4, ord(\xi) = 2$, by mere computation on basically 11 cases using Proposition 20, we get $|[,]^{-1}(\xi)| = 2q^2 + O(q)$. Using the Cohn-Macaulay machinery and Lefschetz principle in section 5.1, we can verify Theorem 5.2.3 when $n = 4, m = 2$.

It is computationally complicated to verify it for general n and $m \mid n$. However, for $m = n$, i.e. the fiber over the identity, the verification of Theorem 5.2.3 directly comes from the fact that the number of conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$ is a degree n polynomial in q, hence by Frobenius character formula (Theorem 1.3.3), $\dim_q[,]^{-1}(1) = \dim_q \mathrm{GL}_n(\mathbb{F}_q) + n$. Then again by the machinery in section 5.1 we get $\dim[,]^{-1}(1) = \dim \mathrm{GL}_n(\mathbb{C}) + n$.

Remark 5.2.1. At the end of section 1 in [10], Green gave the generating function for the number of conjugacy classes (denoted by $c(n,q)$) of $\mathrm{GL}_n(\mathbb{F}_q)$, but did not give the explicit formula which shows that $c(n,q)$ is degree n polynomials with constant rational coefficients. For smaller $n = 2, 3, 4,$

we can count from the known character tables from [7], [3] and [33] that

$$c(2,q) = q^2 - 1,$$
$$c(3,q) = q^3 - q,$$
$$c(4,q) = q^4 - \frac{2}{3}q^3 + \frac{5}{2}q^2 - \frac{23}{6}q + 1.$$

For $p \geq 3$, $\mathrm{GL}_4(\mathbb{F}_q)$ has 11 types of conjugacy classes. When $q = 2$ it has 9 types of conjugacy classes since there are only 3 linear polynomials and 1 degree 2 irreducible polynomials over $\mathbb{F}_2$, i.e. $q^2 + q + 1$, which is consistent with $c(4,2) = 14$.

5.3 Line bundles over Riemann Surfaces

In the introduction, we related spaces of n-dimensional representations of the fundamental group π of a compact Riemann surface with one point deleted to the moduli spaces of rank n vector bundles over the surface. For torus with one point deleted, $\pi = \langle A, B, C \mid [A,B]C = 1 \rangle$. By all the previous work which proves the geometric flatness of the commutator map, we showed that the moduli spaces of E_ρ, the flat bundle of holonomy ρ, for representation $\rho : \pi \to \mathrm{GL}_n(\mathbb{C})$ of type $\rho(C) = \mathfrak{g} \in \mathrm{SL}_n(\mathbb{C})$ non-central, are all about the same size. More specifically, by Narasimhan and Seshadri's work [26], those moduli spaces are parametrized by complex manifolds and those manifolds have the same dimension by the work of this thesis. By the previous two sections, we see that the parametrization manifolds for representation types $\rho(C)$ central in $\mathrm{SL}_n(\mathbb{C})$ may be of higher dimension, and especially for the trivial type with $\rho(C) = 1$, i.e. the moduli space of those vector bundles over a torus (without marked points), its parametrization manifold has much higher dimension.

We explain a little bit more explicitly. For $n = 1$, we look at $\rho : \pi \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ and the only type is $\rho(C) = 1$, i.e. the line bundles over the torus with one point deleted are same with line bundles over the torus. In other words, there are flat line bundle of holonomy arising from representations of the torus group acting on the Riemann sphere with a branch point (branch type). For $n \geq 2$, there are flat bundles of holonomy arising as branch type and by section 5.2, the space of such flat bundles E_ρ over the torus is of dimension $\dim \mathrm{GL}_n(\mathbb{C}) + n$.

5.4 Lefschetz Principle

In this section, we briefly introduce a baby version of Lefschetz Principle that we mentioned in section 5.1. For details, there is a thorough "metamathematical analysis" in Barwise and Eklof [4].

Proposition 5.4.1. Let $\mathtt{k}$ be an algebraically closed field, and $\mathbb{K}$ an algebraically closed extension of $\mathtt{k}$. Let P be any first-order statement in the language of fields. Then P is true for $\mathtt{k}$ if and only if it is true for $\mathbb{K}$.

First, a first-order theory means a theory modeled by a set, finite arity functions on variables evaluated as true or false. A first-order logic statement is a true or false sentence quantified on individuals. First-order theories basically captures all mathematics of finite provability. In our case of section 5.1, the set consists of the general linear groups and the fibers of commutator map which are all algebraic varieties over $\bar{\mathbb{Q}}$ and $\mathbb{C}$. Our statement is about the validity of the equality of dimensions for individuals of fibers of commutator map, hence a first-order statement. Since $\mathbb{C}$ is an algebraically closed extension of $\bar{\mathbb{Q}}$, Lefschetz principle certainly works in our case. All our work in Chapter 4 was trying to translate the first-order statement over finite characteristic to first-order statement over characteristic zero, to establish a local-global principle for the theory of this specific problem.

Bibliography

Bibliography

[1] Avraham Aizenbud, Nir Avni, *Representation growth and rational singularities of the moduli space of local systems* Inventiones Mathematicae. 24 (2016), p.245-316.

[2] Tom M. Apostol, *Introduction to Analytic Number Theory* (1976), Springer-Verlag, New York.

[3] ABM Basheer, *Character Table s of the General Linear Group and Some of its Subgroups* (2008), Thesis.

[4] J. Barwise, P. Eklof, *Lefschetz's principle*, Journal of Algebra, Volume 13, Issue 4, December 1969, Pages 554-570.

[5] R. Brauer, *A characterization of the characters of groups of finite order*, Ann. of Math. vol. 57 (1953) pp. 357-377.

[6] Alexandre Grothendieck, Jean Dieudonné (1966). *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie.* Publications Mathématiques de l'IHÉS. 28: 5–255.

[7] Pavel I Etingof et al. *Introduction to representation theory*, Amer. Math. Soc. 59 (2011).

[8] F.G. Frobenius, *Über Gruppencharaktere*, Sitzber. Preuss. Akad. Wiss (1896) 985-1021; reprinted in Gesammelte Abhandlungen, Vol. 3 (Springer, Heidelberg, 1968) 1-37.

[9] William Fulton, Joe Harris, *Representation Theory: a fist course* (1991), Springer-Verlag.

[10] J. A. Green, *The characters of the finite general linear group*, Trans. Amer. Math. Soc. 80 (1955), 402-447.

[11] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics, 830, Springer, Berlin, 1980.

[12] Robin Hartshorne, *Algebraic Geometry* (1977), Springer-Verlag. ISBN 0-387-90244-9.

[13] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge 2002.

[14] J. E. Humphreys, *Linear Algebraic Groups*, GTM, Vol. 21, Springer-Verlag, New York/Berlin, 1975.

[15] Gordon James, *The representation theory of the symmetric groups* (1978), Lecture Notes in Mathematics, 682, Berlin, New York, Springer-Verlag.

[16] Gordon James, Adalbert Kerber, *The representation theory of the symmetric group* (1981), Encyclopedia of Mathematics and its Applications, 16, Addison-Wesley Publishing Co.

[17] Gordan James, Martin Liebeck, *Representations and Characters of Groups*, second edition (2012), Cambridge University Press.

[18] A. W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton Landmarks in Mathematics, Princeton University Press, 2001.

[19] Jun Li, *The space of surface group representations*, Manuscripta Math. 78 (1993), no. 3, 223-243.

[20] M.W. Liebeck, *Character ratios for finite groups of Lie type, and applications*, American Mathematical Society.

[21] Martin W. Liebeck, Aner Shalev, *Fuchsian groups, finite simple groups and representation varieties*, Invent. Math. 159 (2005), no. 2, 317-367.

[22] Martin W. Liebeck, Aner Shalev, *Character degrees and random walks in finite groups of Lie type*, Proc. London Math. Soc. (3) 90 (2005), no.1, 61-86.

[23] Martin W. Liebeck, E.A. O'Brien, Aner Shalev, Pham Huu Tiep, *The Ore conjecture*, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 4, 939-1008.

[24] Dudley E. Littlewood, *The theory of group characters and matrix representations of groups*, Oxford, 1940.

[25] I. G. Macdonald, *Symmetric functions and Hall polynomials*. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[26] M.S. Narasimhan, C.S. Seshadri, *Stable and Unitary Vector Bundles on a Compact Riemann Surface*, Annals of Mathematics, Second Series, Vol. 82, No. 3 (Nov. 1965), 540-567.

[27] O. Ore, *Some remarks on commutators*, Proc. Amer. Math. Soc. 2 (1951), 307-314.

[28] Martin W. Liebeck, E.A. O'Brien, Aner Shalev, and Pham Huu Tiep, *Commutators in finite quasisimple groups* (2011), BULLETIN OF LONDON MATHEMATICAL SOCIETY, Vol: 43, Pages: 1079-1092, ISSN: 0024-6093

[29] Joseph Rotman, *An introduction to the theory of groups* (1995), Springer-Verlag

[30] Walter Rudin, *Fourier Analysis on Groups*, Wiley: New York, 1962.

[31] J.P. Serre, *Représentations linéaires des groupes finis* (1971), Hermann, as *Linear Representations of Finite Groups* (1977), Springer-Verlag.

[32] Richard P. Stanley, *Enumerative Combinatorics* (Volume 2), Cambridge University Press 1999.

[33] Robert Steinberg, *The representations of* $GL(3,q), GL(4,q), PGL(3,q)$ *and* $PGL(4,q)$, Canad. J. Math. 3 (1951), 225-235.

[34] Hermann Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N. J. 1939.

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