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Emergent Spacetime from Modular Motives

ROLF SCHIMMRIGK [†]

Indiana University South Bend
1700 Mishawaka Ave., South Bend, IN 46634

ABSTRACT:

The program of constructing spacetime geometry from string theoretic modular forms is extended to Calabi-Yau varieties of dimensions two, three, and four, as well as higher rank motives. Modular forms on the worldsheet can be constructed from the geometry of spacetime by computing the L-functions associated to omega motives of Calabi-Yau varieties, generated by their holomorphic n -forms via Galois representations. The modular forms that emerge from the Ω -motive and other motives of the intermediate cohomology are related to characters of the underlying rational conformal field theory. The converse problem of constructing space from string theory proceeds in the class of diagonal theories by determining the motives associated to modular forms in the category of pure motives with complex multiplication. The emerging picture indicates that the L-function can be interpreted as a map from the geometric category of motives to the category of conformal field theories on the worldsheet.

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Math: 11G25 Varieties over finite fields; 11G40 L-functions; 14G10 Zeta functions; 14G40 Arithmetic Varieties

Phys: 11.25.-w Fundamental strings; 11.25.Hf Conformal Field Theory; 11.25.Mj Compactification

[†]email: netahu@yahoo.com

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1 Introduction

The present paper continues the program of applying methods from arithmetic geometry to the problem of understanding the question how spacetime emerges in string theory. The goal is to construct a direct relation between the physics on the worldsheet and the geometry of the extra dimensions. One way to formulate this question is by asking whether it is possible to explicitly determine the geometry of the compact dimensions from the building blocks of the two-dimensional string structure. In this general, but vague, form the problem of constructing an emergent geometry in string theory could have been formulated more than thirty years ago. The reason that it was not can probably be traced to both the lack of a concrete framework, and the lack of useful tools. The framework of the heterotic string of the 1980s, in combination with the web of dualities between different string models discovered the 1990s, motivates a more concrete version of this problem, which aims at the relation between Calabi-Yau varieties and worldsheet physics given by exactly solvable conformal field theories. Both, Calabi-Yau varieties and exactly solvable field theories define rich structures, raising a number of problems which have not been addressed in the past.

The key ingredient of the program pursued here is the modular invariance of the theory. From a spacetime physics perspective it is initially somewhat surprising that this feature of string theory should turn out to provide a useful tool for the understanding of its geometric consequences, because it is the modular invariance of the two-dimensional theory that a priori appears most difficult to explain from a geometric perspective. By now there exists a fair amount of evidence that shows that methods from arithmetic geometry provide promising tools for this problem, at least in lower dimensions. The main purpose of the present paper is to generalize previous results by constructing a class of motives for all Calabi-Yau manifolds (and Fano varieties of special type), independent of any specific construction, and to analyze their modularity properties in the context of weighted Fermat varieties (manifolds of Brieskorn-Pham type). As a consequence, string theoretic modularity emerges for varieties of dimensions three and four, relevant for string, M- and F-theory, including motives of higher rank).

The basic problem of extending modularity results for L-functions in dimensions larger than one is made difficult by the fact that no generalization of the elliptic modularity theorem [1, 2] is known, even conjecturally. This makes even the first step, of constructing modular forms from algebraic varieties, nontrivial. There exists, however, a program, associated most closely with the name Langlands, that suggests that even in higher dimensions the Hasse-Weil L-functions of geometric structures have modular properties in a generalized sense. It is expected

in particular that associated to each cohomology group is an automorphic representation, leading to an automorphic form. The class of automorphic L-functions contains a special type of objects, the standard L-functions, which generalize the Hecke L-functions [3]. Modularity of Hecke L-functions is known in virtue of their analytic continuation and their functional equations [4, 5]. Langlands' vision thus is based on results obtained by Artin and Hecke. While Artin considered representations ρ of the Galois group $\text{Gal}(K/\mathbb{Q})$ of a number field K to define L-functions $L(\rho, s)$, Hecke had previously introduced L-functions based on certain characters χ associated to number fields (called Größencharaktere by Hecke, also called algebraic Hecke characters), whose structure was motivated by an attempt to establish modularity of the L-function. It turned out that these a priori different concepts lead to the same object in the sense that Artin's L-functions and Hecke's L-functions agree [6, 7]. Langlands' conjecture involves a generalization of the Artin-Hecke framework to $\text{GL}(n)$. More precisely, the connection between geometry and arithmetic can be made because representations of the Galois group can be constructed by considering the ℓ -adic cohomology as a representation space. This strategy has proven difficult to implement for higher n in general, and in particular in the context of obtaining a string theoretic interpretation of geometric modular forms beyond the case of elliptic curves and rigid Calabi-Yau varieties.

For varieties of higher dimensions the results obtained so far indicate that it is more important to identify irreducible pieces of low rank in the cohomology groups, and to consider the L-functions of these subspaces. The difficulty here is that at present there exists no general framework that provides guidance for the necessary decomposition of the full cohomology groups. Nevertheless, the Langlands program suggests that modularity, and more generally automorphy, are phenomena that transcend the framework of elliptic curves, and one can ask the question whether the methods described in [8, 9, 10] to establish modularity relations between elliptic curves and conformal field theories can be generalized to higher dimensional varieties. Results in this direction have been obtained for extremal K3 surfaces of Brieskorn-Pham type in ref. [11].

In the present paper the string modularity results obtained previously are extended to all higher dimensions that are of physical relevance. The idea is to consider particular subgroups of the intermediate cohomology group of a variety, defined by the representation of the Galois group associated to the manifold. For manifolds of Calabi-Yau and special Fano type there exists at least one nontrivial orbit, defined by the holomorphic n -form in the case of Calabi-Yau spaces, and the corresponding cohomology group in the case of special Fano manifold. This orbit will be called the omega motive. The strategy developed here is completely general

and can be applied to any Calabi-Yau variety, as well as Fano varieties of special type. In later sections the framework developed will be applied to Calabi-Yau varieties of Brieskorn-Pham type, the class of varieties for which Gepner [12] originally discovered a relation between the spectra of a certain type of conformal field theory and the cohomology of the manifolds. For this class the Ω -motive, as well as the other submotives, can be determined explicitly.

A simplifying characteristic of the class of extremal K3 surfaces of Brieskorn-Pham type considered in [11] is that their Ω -motive is of rank two. This is not the case in general, and it is therefore of interest to see whether it is possible to extend the string analysis of [11] to K3 motives of higher rank.

Define the K3 surface

$$X_2^{12} = \left\{ (z_0 : z_1 : z_2 : z_3) \in \mathbb{P}_{(2,3,3,4)} \mid z_0^6 + z_1^4 + z_2^4 + z_3^3 = 0 \right\} \quad (1)$$

and denote by $E^4 \subset \mathbb{P}_{(1,1,2)}$ and $E^6 \subset \mathbb{P}_{(1,2,3)}$ the two weighted Fermat curves of degree four and six, respectively [10]. The following result is shown.

Theorem 1. *The L-series of X_2^{12} is given by the Mellin transform of product of the modular forms f_{E^4} and f_{E^6} of the elliptic curves E^4 and E^6*

$$L_\Omega(X_2^{12}, s) = L(f_{E^4} \otimes f_{E^6}, s). \quad (2)$$

The precise meaning of the tensor product of modular forms will become clear below.

In dimension three consider the manifolds

$$\begin{aligned} X_3^6 &= \{z_0^6 + z_1^6 + z_2^6 + z_3^6 + z_4^3 = 0\} \subset \mathbb{P}_{(1,1,1,1,2)} \\ X_3^{12} &= \{z_0^6 + z_1^6 + z_2^6 + z_3^4 + z_4^4 = 0\} \subset \mathbb{P}_{(2,2,2,3,3)} \end{aligned} \quad (3)$$

and denote by $f_{X_2^{6A}}$ the modular form of the Ω -motive of the weighted Fermat K3 surface $X_2^{6A} \subset \mathbb{P}_{(1,1,1,3)}$, determined in [11] and described in Section 6. The following results hold.

Theorem 2.

1) *The inverse Mellin transform of the Ω -motivic $L_\Omega(X_3^6, s)$ of the threefold X_3^6 is a cusp form $f_\Omega(q)$ of weight four and level 108 for Hecke's congruence subgroup. The L-function of the intermediate cohomology group $H^3(X_3^6)$ decomposes into modular factors as*

$$L(H^3(X_3^6), s) = L(f_\Omega, s) \cdot \prod_i L(f_i \otimes \chi_i, s), \quad (4)$$

where $f_i \in S_2(\Gamma_0(N_i))$ with $N_i = 27, 144, 432$, and χ_i are twist characters (that can be trivial).

2) The Ω -motivic L-series of X_3^{12} is the Mellin transform of the modular forms associated to $E^4 \subset \mathbb{P}_{(1,1,2)}$ and the extremal weighted Fermat surface $X_2^{6A} \subset \mathbb{P}_{(1,1,1,3)}$

$$L_\Omega(X_3^{12}, s) = L(f_{E^4} \otimes f_{X_2^{6A}}, s). \quad (5)$$

The L-series of the remaining part of the intermediate cohomology $H^3(X_3^{12})$ decomposes into a product of factors that include L-series of modular forms of weight two and levels $N = 27, 64, 144, 432$, possibly including a twist.

As a last example the modularity of the Ω -motive of the degree six fourfold

$$X_4^6 = \left\{ (z_0 : \cdots : z_5) \in \mathbb{P}_5 \mid \sum_{i=0}^5 z_i^6 = 0 \right\} \quad (6)$$

is shown to be modular.

Theorem 3. *The inverse Mellin transform of the L-function of the Ω -motive of X_4^6 is of the form*

$$f_\Omega(X_4^6, q) = f_{27}(q) \otimes \chi_3 \quad (7)$$

where $f_{27}(q)$ is a cusp Hecke eigenform of weight $w = 5$ and level $N = 27$, and χ_3 is the Legendre character. There exists an algebraic Hecke character ψ_{27} with congruence ideal $\mathfrak{m} = (3)$ such that the motivic L-series is its $L_\Omega(X_4^6, s) = L(\psi_{27}^4, s) \otimes \chi_3$.

The basic question raised by these results is whether the Ω -motive is string automorphic in general. In a larger context, one may view the L-function as a link between the geometry of spacetime and the physics of the worldsheet. One way to make this idea more explicit is by viewing L as a functor from the category of Fano varieties of special type (or rather their motives) to the category of superconformal field theories. The evidence obtained so far supports this perspective for a physical interpretation of L-functions. The notion e.g. of composing motives then translates into a corresponding composition of conformal field theories. A concrete example is the motivic tensor structure which maps into a tensor structure for conformal field theories. The basic tensor structure of motives is described in L-function terms by the Rankin-Selberg convolution $L(f_1 \otimes f_2, s)$ of the modular forms f_i of the modular motives M_i , and also leads to the symmetric square of modular forms. Denote by $\Theta_\chi = \prod_i \Theta_i \otimes \chi$ twisted products constructed from modular forms Θ_i on the string worldsheet. The motivic L-functions $L(M(X), s)$ that emerge from Calabi-Yau varieties and special Fano varieties can be expressed in terms of string modular L-functions of the type

$$L(\Theta_\chi, s), \quad L(\text{Sym}^r \Theta_\chi, s), \quad L(\Theta_{\chi_1}^1 \otimes \Theta_{\chi_2}^2, s), \quad (8)$$

where $L(\text{Sym}^r f, s)$ describes the L-function associated to a symmetric tensor product of a modular form f .

The paper is organized as follows. Section 2 briefly introduces the necessary modular theoretic background. Section 3 describes the notion of a Grothendieck motive and defines the concept of Ω -motives for an arbitrary Calabi-Yau manifold, as well as for the class of Fano varieties of special type. This provides the framework for the relation between the geometry of spacetime and physics on the string worldsheet. Sections 4 and 5 describe the basic structure of the L-functions for Calabi-Yau surfaces and threefolds, derived from Artin's zeta function. Sections 6 briefly reviews the results for modular motives of rank two that appear as building blocks for the examples of higher dimension and higher rank described in Sections 7 through 11. Section 12 shows how the converse problem can be approached in the context of modular motives. Section 13 ends the paper with some final remarks.

2 Modularity

In order to make the paper more self-contained the paragraphs briefly summarize the types of modular forms that eventually are reflected in the geometry of weighted hypersurfaces of Calabi-Yau and special Fano type. The affine Lie algebraic forms introduced by Kac and Peterson provide the structures on the worldsheet, while certain types of modular Hecke L-series will arise from the arithmetic of the geometry.

2.1 Modular forms from affine Lie algebras

The simplest class of N=2 supersymmetric exactly solvable theories is built in terms of the affine SU(2) theory as a coset model

$$\frac{\text{SU}(2)_k \otimes \text{U}(1)_2}{\text{U}(1)_{k+2, \text{diag}}}. \quad (9)$$

Coset theories G/H lead to central charges of the form $c_G - c_H$, hence the supersymmetric affine theory at level k still has central charge $c_k = 3k/(k+2)$. The spectrum of anomalous dimensions $\Delta_{\ell, q, s}^k$ and U(1)-charges $Q_{\ell, q, s}^k$ of the primary fields $\Phi_{\ell, q, s}^k$ at level k is given by

$$\begin{aligned} \Delta_{\ell, q, s}^k &= \frac{\ell(\ell+2) - q^2}{4(k+2)} + \frac{s^2}{8} \\ Q_{\ell, q, s}^k &= -\frac{q}{k+2} + \frac{s}{2}, \end{aligned} \quad (10)$$

where $\ell \in \{0, 1, \dots, k\}$, $\ell + q + s \in 2\mathbb{Z}$, and $|q - s| \leq \ell$. Associated to the primary fields are characters defined as

$$\begin{aligned} \chi_{\ell,q,s}^k(\tau, z, u) &= e^{-2\pi i u} \text{tr}_{\mathcal{H}_{q,s}^\ell} e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{2\pi i J_0} \\ &= \sum c_{\ell,q+4j-s}^k(\tau) \theta_{2q+(4j-s)(k+2), 2k(k+2)}(\tau, z, u), \end{aligned} \quad (11)$$

where the trace is to be taken over a projection $\mathcal{H}_{q,s}^\ell$ to a definite fermion number (mod 2) of a highest weight representation of the (right-moving) $N = 2$ algebra with highest weight vector determined by the primary field. The expression of the rhs in terms of the string functions [13]

$$c_{\ell,m}^k(\tau) = \frac{\Theta_{\ell,m}^k(\tau)}{\eta^3(\tau)} \quad (12)$$

where $\eta(\tau)$ is the Dedekind eta function, and $\Theta_{\ell,m}^k(\tau)$ are the Hecke indefinite modular forms

$$\Theta_{\ell,m}^k(\tau) = \sum_{\substack{-|x| < y \leq |x| \\ (x,y) \text{ or } (\frac{1}{2}-x, \frac{1}{2}+y) \\ \in \mathbf{Z}^2 + (\frac{\ell+1}{2(k+2)}, \frac{m}{2k})}} \text{sign}(x) e^{2\pi i \tau ((k+2)x^2 - ky^2)} \quad (13)$$

and theta functions

$$\theta_{n,m}(\tau, z, u) = e^{-2\pi i m u} \sum_{\ell \in \mathbf{Z} + \frac{n}{2m}} e^{2\pi i m \ell^2 \tau + 2\pi i \ell z}. \quad (14)$$

is useful because it follows from this representation that the modular behavior of the $N = 2$ characters decomposes into a product of the affine $SU(2)$ structure in the ℓ index and into Θ -function behavior in the charge and sector index. It follows from the coset construction that the essential ingredient in the conformal field theory is the $SU(2)$ affine theory.

The issue of understanding emergent spacetime in string theory can now be reformulated in a more concrete way as the problem of relating string theoretic modular forms to geometric ones. It turns out that more important than the string functions are the associated $SU(2)$ theta functions $\Theta_{\ell,m}^k(\tau)$. These indefinite Hecke forms are associated to quadratic number fields determined by the level of the affine theory. They are modular forms of weight 1 and cannot, therefore, be identified with geometric modular forms. It turns out, however, that appropriate products lead to interesting motivic forms [8, 9, 10, 11].

2.2 Modular forms from algebraic Hecke characters

The modularity of the L-series determined in this paper follows from the fact that they can be interpreted in terms of Hecke L-series associated to Größencharaktere, defined by Jacobi

sums. A Größencharakter, or algebraic Hecke character) can be associated to any number field. Hecke's modularity discussion [14] has been extended by Shimura [15] and Ribet [16].

Let K be a number field and $\sigma : K \rightarrow \mathbb{C}$ denote an embedding.

Definition. A Größencharakter is a homomorphism $\psi : I_{\mathfrak{m}}(K) \rightarrow \mathbb{C}^\times$ from the fractional ideals of K prime to the congruence integral ideal \mathfrak{m} such that $\psi((z)) = \sigma(z)^{w-1}$, for all $z \in K^\times$ such that $z \equiv 1 \pmod{\times \mathfrak{m}}$. The type of behavior of ψ on the principal ideals is called the infinity type.

In the present case the cyclotomic Jacobi sums determined by the finite field Jacobi sums computed above arise from imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$, where where $-D$ is the discriminant of the field. In this case the structure of these characters simplifies. Denote by ψ an algebraic Hecke character of K and by $N\mathfrak{p}$ the norm of prime ideal \mathfrak{p} in the ring of integers \mathcal{O}_K . The Hecke L-series of ψ is defined by

$$L(\psi, s) = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \frac{1}{1 - \frac{\psi(\mathfrak{p})}{N\mathfrak{p}^s}}. \quad (15)$$

The modularity of the corresponding q -series $f(\psi, q) = \sum_n a_n q^n$ associated to the L-series via the Mellin transform is characterized by a Nebentypus character ϵ defined in terms of the Dirichlet character φ associated to K and a second Dirichlet character λ defined mod $N\mathfrak{m}$ by

$$\lambda(a) = \frac{\psi((a))}{\sigma(a)^{w-1}}, \quad a \in \mathbb{Z}. \quad (16)$$

The Nebentypus character ϵ is given by the product $\epsilon = \chi\lambda$ of these two characters. Modularity of the L-series follows from the following result of Hecke.

Theorem 4. Let ψ be a Größencharakter of the imaginary quadratic field K with infinity type σ^{w-1} . Define the coefficients c_n as

$$\sum_{\substack{(\mathfrak{a}, \mathfrak{m})=1 \\ \mathfrak{a} \text{ integral}}} \psi(\mathfrak{a}) q^{N\mathfrak{a}} =: \sum_{n=1}^{\infty} c_n q^n. \quad (17)$$

Then there exists a unique newform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight w and character $\epsilon = \lambda\varphi$ such that

$$a_p = c_p \quad \forall p \nmid DN\mathfrak{m}. \quad (18)$$

Of particular importance in this paper are algebraic Hecke characters associated to the Gauss field $\mathbb{Q}(\sqrt{-1})$ and the Eisenstein field $\mathbb{Q}(\sqrt{-3})$. For $\mathbb{Q}(\sqrt{-1})$ consider prime ideals $\mathfrak{p} = (z_{\mathfrak{p}})$

and define the character ψ_{32} by setting

$$\psi_{32}(\mathfrak{p}) = z_{\mathfrak{p}} \tag{19}$$

where the generator $z_{\mathfrak{p}}$ is determined by the congruence relation

$$z_{\mathfrak{p}} \equiv 1 \pmod{(2 + 2i)} \tag{20}$$

for the congruence ideal $\mathfrak{m} = (2 + 2i)$.

For the field $\mathbb{Q}(\sqrt{-3})$ two characters ψ_N , $N = 27, 36$ and their twists will appear. The congruence ideals here are given by

$$\begin{aligned} \mathfrak{m}_{27} &= (3) \\ \mathfrak{m}_{36} &= 1 + 2\xi_3 \end{aligned} \tag{21}$$

leading to the characters

$$\psi_N(\mathfrak{p}) = z_{\mathfrak{p}} \tag{22}$$

where the generator is determined uniquely by the congruence relations

$$z_{\mathfrak{p}} \equiv 1 \pmod{N}. \tag{23}$$

2.3 Rankin-Selberg products of modular forms

An important ingredient in the analysis of higher dimensional varieties is the fact that the L-series of motives carrying higher dimensional representations of the Galois group can sometimes be expressed in terms of L-functions of lower rank motives. This leads to Rankin-Selberg products of L-functions. This construction is quite general, but will be applied here only to products of L-series that are associated to Hecke eigenforms. If the L-series of two modular forms f, g of arbitrary weight and arbitrary levels are given by

$$\begin{aligned} L(f, s) &= \sum_n a_n n^{-s} \\ L(g, s) &= \sum_n b_n n^{-s}, \end{aligned} \tag{24}$$

it is natural to consider the naive Rankin-Selberg L-series associated to f, g as

$$L(f \times g, s) = \sum_n a_n b_n n^{-s}.$$

It turns out that a slight modification of this product defined has better properties, and is more appropriate for geometric constructions. If $f \in S_{w_1}(\Gamma_0(N), \epsilon)$ and $g \in S_{w_2}(\Gamma_0(N), \lambda)$ are cusp forms with characters ϵ and λ , respectively, the modified Rankin-Selberg product is defined as

$$L(f \otimes g, s) = L_N(\epsilon\lambda, 2s + 2 - (w + v))L(f \times g, s), \quad (25)$$

where $L_N(\chi, s)$ is the truncated Dirichlet L-series defined by the condition that $\chi(n) = 0$ if $(n, N) > 1$ [18].

Hecke showed that such forms f, g have Euler products given by

$$\begin{aligned} L(f, s) &= \prod_p [(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})]^{-1} \\ L(g, s) &= \prod_p [(1 - \gamma_p p^{-s})(1 - \delta_p p^{-s})]^{-1}, \end{aligned} \quad (26)$$

where $\alpha_p + \beta_p = a_p$, $\gamma_p + \delta_p = b_p$ and $\alpha_p \beta_p = p^{w-1}$, $\gamma_p \delta_p = p^{v-1}$. It can be shown that the modified Rankin-Selberg product has the Euler product

$$L(f \otimes g, s) = \prod_p [(1 - \alpha_p \gamma_p p^{-s})(1 - \alpha_p \delta_p p^{-s})(1 - \beta_p \gamma_p p^{-s})(1 - \beta_p \delta_p p^{-s})]^{-1}. \quad (27)$$

The tensor notation is at this point formal, but it will become clear that this product is indicative of a representation theoretic tensor product. Furthermore, it also describes the L-series of the tensor product $M_f \otimes M_g$ of motives M_f, M_g associated to the modular forms f, g via the constructions of Deligne [19], Jannsen [20] and Scholl [21]

$$L(f \otimes g, s) = L(M_f \otimes M_g, s).$$

The motivic tensor product will be described below.

The Rankin-Selberg products which will appear below involve modular forms of weight two and three, leading to rank four motives on Calabi-Yau varieties of dimension two and three.

2.4 Complex multiplication modular forms

A special class of modular forms that is relevant in this paper are forms which are sparse in the sense that a particular subset of the coefficients a_p of their Fourier expansion $f(q) = \sum_n a_n q^n$ vanish. A conceptual way to formulate was introduced by Ribet [16]. A complex multiplication (CM) modular form $f(q)$ is characterized by the existence of an imaginary quadratic number

field $K = \mathbb{Q}(\sqrt{-d})$ such that the coefficients a_p vanish for those rational primes p which are inert in K . It follows from this definition that any such form can be described by the inverse Mellin transform of an L-series associated to a Hecke Größencharakter, which is the view originally adopted by Hecke, and also Shimura.

Modular forms with complex multiplication are more transparent than general forms, in particular in the context of their associated geometry. This will become important further below in the construction of the Calabi-Yau motives from the conformal field theory on the worldsheet.

3 Ω -Motives

The results in the present and previous papers show that it is useful for modularity to consider the Galois orbit in the cohomology defined by the holomorphic n -form in a Calabi-Yau variety. This Galois orbit defines a geometric substructure of the manifold, called the Ω -motive, which is an example of a Grothendieck motive on these manifolds. Similar orbits can be considered in the context of so-called special Fano varieties considered in [22, 23, 24], whose modularity properties are analyzed in the context of mirror pairs of rigid Calabi-Yau varieties in [25]. The aim of the present section is to provide the background for the reconstruction of the motivic structure from the conformal field theory. The circle of ideas that is concerned with the relations between characters, modular forms, and motives extends beyond the class of weighted Fermat varieties, and it is useful to formulate the constructions in a general framework.

The outline of this section is to first describe the concept of Grothendieck motives, also called pure motives, then to define the notion of Ω -motives in complete generality, and finally to consider the Ω -motive in detail for weighted Fermat hypersurfaces, i.e. of Brieskorn-Pham type.

3.1 Grothendieck motives

The idea to construct varieties directly from the conformal field theory on the worldsheet without the intermediary of Landau-Ginzburg theories or sigma models becomes more complex as the number of dimensions increases. Mirror symmetry and other dualities show that it should not be expected that any particular model on the worldsheet should lead to a unique

variety. Rather, one should view manifolds as objects which can be build from irreducible geometric structures. This physical expectation is compatible with Grothendieck's notion of motives. The original idea for the existence of motives arose from a plethora of cohomology theories Grothendieck was led to during his pursuit of the Weil conjectures [26, 27, 28]. There are several ways to think of motives as structures that support these various cohomology theories, such as Betti, de Rham, étale, crystalline cohomology groups etc., and to view these cohomology groups as realization of motives.

Grothendieck's vision of motives as basic building blocks that support universal structures is based on the notion of correspondences. This is an old concept that goes back to Klein and Hurwitz in the late 19th century. The idea is to define a relation between two varieties by considering an algebraic cycle class on their product. In order to do so an algebraic cycle is defined as a finite linear combination of irreducible subvarieties $V_\alpha \subset X$ of codimension r of a variety X . The set of all these algebraic cycles defines a group

$$Z^r = \left\{ \sum_{\alpha} n_{\alpha} V_{\alpha} \mid V_{\alpha} \subset X. \right\} \quad (28)$$

This group is too large to be useful, hence one considers equivalence relations between its elements. There are a variety of such equivalence relations, resulting in quite different structures. The most common of these are rational, homological, and numerical equivalence. A description of these can be found in [29], but for the following it will not be important which of these is chosen. Given any of these equivalences one considers the group of equivalence classes

$$A^r(X) = Z^r(X) / \sim \quad (29)$$

of algebraic cycles to define the group of algebraic correspondences of degree r between manifolds X, Y of equal dimension d as

$$\text{Corr}^r(X, Y) = A^{d+r}(X \times Y). \quad (30)$$

Correspondences can be composed $f \cdot g$, leading to the notion of a projector p such that $p \cdot p = p$. The first step in the construction of Grothendieck motives is the definition of an effective motive, obtained by considering a pair defined by a variety and a projector $M = (X, p)$, where p is a projector in the ring of algebraic correspondences of degree zero, $p \in \text{Corr}^0(X, X)$. Maps between such objects are of the form [30]

$$\text{Hom}((X, p), (Y, q)) = q \circ \text{Corr}^0(X, Y)_{\mathbb{Q}} \circ p. \quad (31)$$

This formulation of morphisms between effective motives is equivalent to the original view of Grothendieck described in [28].

It is important for physical applications of motives to enlarge the class of effective motives by introducing twists of effective motives by powers of the inverse Lefschetz motive L . This is an effective motive defined as $L = (\mathbb{P}_1, 1 - Z)$, where Z is the cycle class $Z \in A^1(\mathbb{P}_1 \times \mathbb{P}_1)$ defined by the cycle $\mathbb{P}_1 \times \text{pt}$. It is possible to tensor effective motives M by L and its inverse. Combining the notions of effective motives and the Lefschetz motive leads to the concept of a Grothendieck motive.

Definition. A Grothendieck motive is a triple $M(m) = (X, p, m)$, where $M = (X, p)$ is an effective motive, and $m \in \mathbb{Z}$. $M(m) = (M, m)$ is the m -fold Tate twist of M . If $N(n) = (Y, q, n)$ is another motive morphisms are defined as

$$\text{Hom}(M(m), N(n)) := p \circ \text{Corr}^{n-m}(X, Y) \circ q. \quad (32)$$

The tensor product of two Grothendieck motives $M_i = (X_i, p_i, m_i), i = 1, 2$ is defined as

$$M_1 \otimes M_2 = (X_1 \times X_2, p_1 \oplus p_2, m_1 + m_2). \quad (33)$$

A discussion of the virtues and disadvantages of the various realizations in terms of specific equivalence relations can be found in [30, 31], building on earlier references, such as [26, 27, 28]. A more detailed discussion of motives can be found in [32].

3.2 Ω -motives for manifolds of Calabi-Yau and special Fano type

When considering the emergent geometry problem via string theoretic modular forms it is of interest to consider L-functions associated to motives of low rank, not of the full cohomology groups of a variety. For higher genus curves and higher dimensional varieties the experimental evidence [8, 9, 10, 11] suggests that the relevant physical information is encoded in subspaces of the cohomology. A possible strategy therefore is to consider the factorization of L-functions and to ask whether modular forms arise from the emerging pieces, and if so, whether these modular forms admit a string theoretic Kac-Moody interpretation.

This section describes a general strategy, valid for any Calabi-Yau variety X of dimension $\dim_{\mathbb{C}} X = d$, for decomposing the L-function of its intermediate cohomology $H^d(X)$, into pieces which lead to L-functions with integral coefficients. These L-functions then have the

potential, when modular, to admit a factorization into Kac-Moody theoretic modular forms along the lines discussed in [8, 9, 10] for elliptic curves and higher genus curves. The basic strategy outlined below is a generalization of the method described in [11], which was based on Jacobi sums associated to hypersurfaces embedded in weighted projective spaces.

The idea is to consider an orbit in the cohomology which is generated by the holomorphic d -form $\Omega \in H^{d,0}(X)$ via the action of the Galois group $\text{Gal}(K/\mathbb{Q})$ of the number field determined by the arithmetic properties of the variety as dictated by the Weil conjectures [33] proven by Grothendieck [34] and Deligne [35]. The resulting orbit of this action turns out to define a motive in the sense of Grothendieck, as will be shown further below. To see how the group structure appears in full generality it is necessary to briefly review the arithmetic structure of arbitrary Calabi-Yau varieties.

For a general smooth algebraic variety X reduced mod p the congruence zeta function of X/\mathbb{F}_q is defined by

$$Z(X/\mathbb{F}_q, t) \equiv \exp \left(\sum_{r \in \mathbb{N}} \#(X/\mathbb{F}_{q^r}) \frac{t^r}{r} \right). \quad (34)$$

Here the sum is over all finite extensions \mathbb{F}_{q^r} of \mathbb{F}_q of degree $r = [\mathbb{F}_{q^r} : \mathbb{F}_q]$. Per definition $Z(X/\mathbb{F}_q, t) \in 1 + \mathbb{Q}[[t]]$, but the expansion can be shown to be integer valued by writing it as an Euler product. The main virtue of $Z(X/\mathbb{F}_q, t)$ is that the numbers $N_{p^r} = \#(X/\mathbb{F}_{p^r})$ show a simple behavior, as a result of which the zeta function can be shown to be a rational function.

1) The first step is to consider the rational form of Artin's congruent zeta function. This leads to a link between the purely arithmetic geometric objects $N_{r,p} = \#(X/\mathbb{F}_{p^r})$ for all primes p and the cohomology of the variety. This was first shown for curves by F.K. Schmidt in the thirties in letters to Hasse [36][37][38]. Further experience by Hasse, Weil, and others led to the conjecture that this phenomenon is more general, culminating in the cohomological part of Weil conjecture. According to Weil [33] and Grothendieck [34] $Z(X/\mathbb{F}_p, t)$ is a rational function which can be written as

$$Z(X/\mathbb{F}_p, t) = \frac{\prod_{j=1}^d \mathcal{P}_p^{2j-1}(t)}{\prod_{j=0}^d \mathcal{P}_p^{2j}(t)}, \quad (35)$$

where

$$\mathcal{P}_p^0(t) = 1 - t, \quad \mathcal{P}_p^{2d}(t) = 1 - p^d t \quad (36)$$

and for $1 \leq j \leq 2d - 1$

$$\deg \mathcal{P}_p^j(t) = b^j(X), \quad (37)$$

where $b^j(X)$ denotes the j^{th} Betti number of the variety, $b^j(X) = \dim H_{\text{dR}}^j(X)$. The rationality of the zeta function was first shown by Dwork [39] by adelic methods.

The resulting building blocks given by the polynomials $\mathcal{P}_p^j(t)$ associated to the full cohomology group $H^j(X)$ are not useful in the present context, leading to L-series whose Mellin transforms in general cannot directly be identified with string theoretic modular forms of the type considered in [8, 9, 10, 11, 25] and in the present paper. The idea instead is to decompose these objects further, which leads to the factorization of the polynomials $\mathcal{P}_p^i(t)$.

2) The most difficult part of the Weil conjectures is concerned with the nature of the factorization of the polynomials

$$\mathcal{P}_p^j(t) = \prod_{i=1}^{b^j} (1 - \gamma_i^j(p)t). \quad (38)$$

Experience with Jacobi and Gauss sums in the context of diagonal weighted projective varieties indicates that the inverse eigenvalues $\gamma_i^j(p)$ are algebraic integers that satisfy the Riemann hypothesis

$$|\gamma_i^j(p)| = p^{j/2}, \quad \forall i. \quad (39)$$

It is this part of the Weil conjectures which resisted the longest, and was finally proved by Deligne [35].

3) Given the algebraic nature of the inverse roots of the polynomials $\mathcal{P}_p^j(t)$ one can consider the field $K = \mathbb{Q}(\{\gamma_i^j \mid j = 0, \dots, 2d-1, i = 1, \dots, b^j\})$. The field K is separable and therefore one can consider orbits within the cohomology with respect to the embedding monomorphisms. For n -dimensional Calabi-Yau varieties one can, in particular, consider the orbits \mathcal{O}_Ω associated to the holomorphic n -forms $\Omega \in H^{n,0}(X)$, while for special Fano manifolds of charge Q one can consider $\Omega \in H^{n-(Q-1), (Q-1)}(X)$. The orbits of these forms generated by the embeddings of the field K lead to a projection p_Ω on the intermediate cohomology, leading to the Grothendieck motive $M_\Omega = (X, p_\Omega, Q)$. This will be called the Ω -motive of the variety X in both, the Calabi-Yau case and the more general case of special Fano varieties.

4) Given the Ω -motive M_Ω one can combine the local factors of the zeta functions, leading to the motivic L-function of the variety

$$L_\Omega(X, s) := L(M_\Omega, s) = \prod_p \frac{1}{\mathcal{P}_p^\Omega(p^{-s})}, \quad (40)$$

where $\mathcal{P}_p^\Omega(t)$ are the polynomials described by the orbit of $\Omega \in H^{d-(Q-1), (Q-1)}(X)$. Denoting

the Weil number corresponding to Ω in (38) by γ_Ω the polynomial can be expressed as

$$\mathcal{P}_p^\Omega(t) = \prod_{\sigma \in \text{Hom}(K, \mathbb{C})} (1 - \sigma(\gamma_\Omega)t). \quad (41)$$

The σ -orbits defined via the embedding monomorphisms define traces of the number field K , which implies that the corresponding Ω -motives have L-functions $L_\Omega(X, s)$ with integral coefficients.

3.3 String theoretic modularity and automorphy

With the above structures in place we can ask in full generality for any Calabi-Yau variety the following

Questions.

When is the L-function $L_\Omega(X, s)$ of the Ω -motive M_Ω of a Calabi-Yau variety modular? Further, if it is modular, can $L_\Omega(X, s)$ be expressed in terms of string theoretic forms associated to Kac-Moody algebras?

Modularity of $L_\Omega(X, s)$ here is understood to include the usual linear operations on modular forms.

More generally this question can be raised for the class of motives associated to Fano varieties of special type. It has been shown in refs. [8, 9, 10, 11] that the answer to this question is affirmative at least sometimes in lower dimensions, and generalizations to higher dimensions will be established below. In such cases the L-function can be viewed as a map that takes motives and turns them into conformal field theoretic objects. This framework therefore leads to the following picture:

Conjecture:

The L-function provides a map from the category of Calabi-Yau motives (more general special Fano type motives) to the category of $N = 2$ supersymmetric conformal field theories.

The question and the conjecture can be raised in the more general context of automorphic representations. In this case the Langlands program leads to the expectation that every motive is automorphic, and at this level of speculation the question becomes whether the resulting automorphic forms have a string theoretic interpretation.

3.4 Grothendieck motives for Brieskorn-Pham varieties

In ref. [11] the notion of an Ω -motive was introduced because it leads to geometric theta series with coefficients that are rational integers. This allows in principle to compare these series with those obtained from (modified) characters in the conformal field theory. In a first approximation the Ω -motive can roughly be viewed via its realization in cohomology, given by the Galois orbit of the holomorphic forms on Calabi-Yau manifolds as well as Fano varieties of the special type considered in [22, 23, 24]. A more intrinsically geometric perspective is provided by Grothendieck's notion of a motive. The basic strategy to construct Grothendieck motives from Ω -motives is to construct algebraic correspondences via projectors that are associated to Galois orbits. These projectors define algebraic cycles which then can be used to define the correspondences. In the context of Fermat varieties such a transition has been constructed explicitly by Shioda [40] (see also [41, 17, 42]).

Consider a diagonal hypersurface X_n^d of degree d and dimension n in a weighted projective space with weights $(k_0, \dots, k_{n+1}) \in \mathbb{N}^{n+2}$. With $d_i = d/k_i$ one can consider the group $G_n^d = \prod_i(\mu_{d_i})$, where μ_{d_i} is the cyclic group with generator $\xi_{d_i} = e^{2\pi i/d_i}$. This group acts on the projective space as

$$gz = (\xi_{d_0}^{a_0} z_0, \dots, \xi_{d_{n+1}}^{a_{n+1}} z_{n+1}) \quad (42)$$

for a vector $a = (a_0, \dots, a_{n+1}) \in \mathbb{Z}^{n+2}$. Motives of Fermat type can be defined via projectors that are associated to the characters of the symmetry group. Denote the dual group of G_n^d by \widehat{G}_n^d and associate to $a \in \widehat{G}_n^d$ a projector p_a as

$$p_a = \frac{1}{|G_n^d|} \sum_{g \in G_n^d} a(g)^{-1} g, \quad (43)$$

where the character defined by a is given by

$$a(g) = \prod_{i=0}^{n+1} \xi_{d_i}^{a_i}. \quad (44)$$

Combining the projectors p_a within a $\text{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ -orbit then leads to

$$p_{\mathcal{O}} := \sum_{a \in \mathcal{O}} p_a. \quad (45)$$

The projectors $p_{\mathcal{O}}$ can now be regarded as algebraic cycles on $X \times X$ with rational coefficients by considering their associated graphs. By abuse of notation effective motives of Fermat type

can therefore be defined as objects that are determined by the Galois orbits \mathcal{O} as $(X_s^n, p_{\mathcal{O}})$. Including Tate twists then leads to Grothendieck motives

$$M_{\mathcal{O}} := (X, p_{\mathcal{O}}, m). \quad (46)$$

3.5 Ω -motives for CY and SF BP hypersurfaces

For varieties of Calabi-Yau and special Fano type there exists a particularly important cohomology group which can be parametrized explicitly. For Calabi-Yau manifolds of complex dimension n these forms take values in $H^{n,0}(X)$, while for special Fano varieties of complex dimension n and charge Q one has more generally $\Omega \in H^{n-(Q-1), (Q-1)}(X)$, where $Q \in \mathbb{N}$. The Ω -motive associated to the Galois orbit of Ω of a variety of charge Q will be denoted by

$$M_{\Omega}^Q = (X, p_{\Omega}, Q - 1). \quad (47)$$

For $Q = 1$ one recovers the Calabi-Yau case, which will be denoted by $M_{\Omega} = M_{\Omega}^1$.

3.6 Lower weight motives from higher dimensional varieties

The Ω -motive is not the only motive of Calabi-Yau and special Fano varieties that can lead to interesting modular forms. Instead, one can construct Grothendieck motives $M_{\mathcal{O}}(X)$ associated to Galois orbits \mathcal{O} that do not arise from the Ω -form, but instead come from motives represented by subgroups of the remainder of the intermediate cohomology group. The associated L-series $L(M_{\mathcal{O}}(X), s)$ may, or may not lead to interesting modular forms. In the case they do in the examples below, the resulting modular forms can be described as determined by Tate twists of forms

$$L(M_{\mathcal{O}}(X), s) = L(f_{\mathcal{O}}, s - 1),$$

where $f_{\mathcal{O}}(q)$ is a modular associated to the motive $M_{\mathcal{O}}(X)$. In general one would expect that these lower weight motives lead to Tate twists of automorphic forms of lower dimensional varieties that are embedded in the higher-dimensional manifolds.

4 L-functions for Calabi-Yau varieties of dimension 2,3, and 4

In this paper the focus is on Calabi-Yau manifolds in complex dimensions two, three and four. String theoretic examples of rank two and dimension one and two are considered in [8, 9, 10, 11]. Combining these examples with the varieties discussed in the present paper covers all physically interesting dimensions in string theory, M-theory, and F-theory.

A general, non-toroidal, Calabi-Yau 2-fold X_2 is a K3 surface, i.e with Betti numbers $b^1 = 0 = b^3$, leading to the zeta function

$$Z(X_2/\mathbb{F}_p, t) = \frac{1}{(1-t)\mathcal{P}_p^2(t)(1-p^2t)} \quad (48)$$

with $\deg(\mathcal{P}_p^2(t)) = b^2 = 22$. Expanding this rational form via

$$\mathcal{P}_p^2(t) = \sum_{i=0}^{22} \beta_i^2(p)t^i \quad (49)$$

leads to

$$\beta_1^2(p) = 1 + p^2 - N_{1,p}. \quad (50)$$

The only interesting L-function associated to a K3 surface therefore is associated to its second cohomology group, in particular its Ω -motivic piece, leading to $L_\Omega(X_2, s)$.

Calabi-Yau threefolds with finite fundamental group lead to zeta functions of the form

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}_p^3(t)}{(1-t)\mathcal{P}_p^2(t)\mathcal{P}_p^4(t)(1-p^3t)} \quad (51)$$

with

$$\begin{aligned} \deg(\mathcal{P}_p^3(t)) &= 2 + 2h^{(2,1)} \\ \deg(\mathcal{P}_p^2(t)) &= h^{(1,1)} \end{aligned}$$

This follows from the fact that for non-toroidal Calabi-Yau threefolds we have $b^1 = 0$. For Calabi-Yau threefolds with $h^{1,1} = 1$ the zeta function reduces to

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}_p^3(t)}{(1-t)(1-pt)(1-p^2t)(1-p^3t)} \quad (52)$$

and therefore becomes particularly simple. The coefficients $\beta_i^3(p)$ of the polynomial

$$\mathcal{P}_p^3(t) = \sum_{j=0}^{b^3(X)} \beta_j^3(p) t^j \quad (53)$$

are related to the cardinalities of the variety via the expansion

$$Z(X/\mathbb{F}_p, t) = 1 + N_{1,p}t + \frac{1}{2}(N_{1,p}^2 + N_{2,p})t^2 + \left(\frac{1}{3}N_{3,p} + \frac{1}{2}N_{1,p}N_{2,p} + \frac{1}{6}N_{1,p}^3 \right) t^3 + \mathcal{O}(t^4) \quad (54)$$

as

$$\begin{aligned} \beta_1^3(p) &= N_{1,p} - (1 + p + p^2 + p^3) \\ \beta_2^3(p) &= \frac{1}{2}(N_{1,p}^2 + N_{2,p}) - N_{1,p}(1 + p + p^2 + p^3) \\ &\quad + (1 + p + p^2 + p^3)^2 + (1 + p + p^2) + p^2(1 + p) + p^4(1 + p + p^2) \end{aligned} \quad (55)$$

etc. This procedure is useful because the knowledge of a finite number of terms in the L-function determines it uniquely.

In dimension four the cohomology of Calabi-Yau varieties is more complicated, leading to zeta functions

$$Z(X_4/\mathbb{F}_p, t) = \frac{\mathcal{P}_p^3(t)\mathcal{P}_p^5(t)}{(1-t)\mathcal{P}_p^2(t)\mathcal{P}_p^4(t)\mathcal{P}_p^6(t)(1-p^4t)} \quad (56)$$

for non-toroidal spaces, but the procedure is the same as above. For smooth hypersurfaces the cohomology groups except of degree given by the dimension are either trivial or inherited from the ambient space, leading to the intermediate L-function

$$L(X, s) = \prod_p \frac{1}{\mathcal{P}_p^4(p^{-s})} \quad (57)$$

as the only nontrivial factor.

5 L-functions via Jacobi sums

For the class of hypersurfaces of Brieskorn-Pham type it is possible to gain insight into the precise structure of the L-function by using a result of Weil [33] which expresses the cardinalities of the variety in terms of Jacobi sums of finite fields. In this context there the L-function of the Ω -motive of such weighted Fermat hypersurfaces can be made explicit.

For any degree vector $\underline{n} = (n_0, \dots, n_{s+1})$ and for any prime p define the numbers $d_i = (n_i, p-1)$ and the set

$$\mathcal{A}_s^{p,\underline{n}} = \left\{ (\alpha_0, \dots, \alpha_{s+1}) \in \mathbb{Q}^{s+2} \mid 0 < \alpha_i < 1, d_i \alpha_i = 0 \pmod{1}, \sum_i \alpha_i = 0 \pmod{1} \right\}. \quad (58)$$

Theorem 5. *The number of solutions of the smooth projective variety*

$$X_s = \left\{ (z_0 : z_1 : \dots : z_{s+1}) \in \mathbb{P}_{s+1} \mid \sum_{i=0}^{s+1} b_i z_i^{n_i} = 0 \right\} \subset \mathbb{P}_{s+1} \quad (59)$$

over the finite field \mathbb{F}_p is given by

$$N_p(X_s) = 1 + p + p^2 + \dots + p^s + \sum_{\alpha \in \mathcal{A}_s^{p,\underline{n}}} j_p(\alpha) \prod \bar{\chi}_{\alpha_i}(a_i), \quad (60)$$

where

$$j_p(\alpha) = \frac{1}{p-1} \sum_{\substack{u_i \in \mathbb{F}_q \\ u_0 + \dots + u_s = 0}} \chi_{\alpha_0}(u_0) \cdots \chi_{\alpha_s}(u_s). \quad (61)$$

With these Jacobi sums $j_q(\alpha)$ one defines the polynomials

$$\mathcal{P}_p^s(t) = (1 - p^{s/2}t)^{|s|} \prod_{\alpha \in \mathcal{A}_s^n} \left(1 - (-1)^s j_{p^f}(\alpha) \prod_i \bar{\chi}_{\alpha_i}(b^i) t^f \right)^{1/f} \quad (62)$$

and the associated L-function

$$L^{(j)}(X, s) = \prod_p \frac{1}{\mathcal{P}_p^j(p^{-s})}. \quad (63)$$

Here $|s| = 1$ if s is even and $|s| = 0$ if s is odd.

A slight modification of this result is useful even in the case of smooth weighted projective varieties because it can be used to compute the factor of the zeta function coming from the invariant part of the cohomology, when viewing these spaces as quotient varieties of projective spaces.

The Jacobi-sum formulation allows to write the L-function of the Ω -motive M_Ω of weighted Fermat hypersurfaces in a more explicit way. Define the vector $\alpha_\Omega = \left(\frac{k_0}{d}, \dots, \frac{k_{s+1}}{d} \right)$ corresponding to the holomorphic s -form, and denote its Galois orbit by $\mathcal{O}_\Omega \subset \mathcal{A}_s^n$. Then

$$L_\Omega(X, s) = \prod_p \prod_{\alpha \in \mathcal{O}_\Omega} (1 - (-1)^s j_{p^f}(\alpha) p^{-fs})^{-1/f}. \quad (64)$$

6 Modularity results for rank 2 motives in dimension one and two

The simplest possible framework in which this question can be raised is for toroidal compactifications, in particular Brieskorn-Pham curves E^d of degree d embedded in the weighted projective plane. The elliptic curves E^d are defined over the rational number \mathbb{Q} , and therefore modular in lieu of the Shimura-Taniyama conjecture, proven in complete generality in ref. [2], based on Wiles' breakthrough results [1]. This theorem says that any elliptic curve over the rational numbers is modular in the sense that the inverse Mellin transform of the Hasse-Weil L-function is a modular form of weight two for some congruent subgroup $\Gamma_0(N)$. This raises the question whether the modular forms derived from these Brieskorn-Pham curves are related in some way to the characters of the conjectured underlying conformal field theory models.

The conformal field theory on the string worldsheet is fairly involved, and a priori there are a number of different modular forms that could play a role in the geometric construction of the varieties. The first string theoretic modularity result showed that the modular form associated to the cubic Fermat curve $E^3 \subset \mathbb{P}_2$ factors into a product of $SU(2)$ -modular forms that arise from the characters of the underlying world sheet [8]. More precisely, the worldsheet forms that encode the structure of the compact spacetime geometry are the Hecke indefinite theta series associated to Kac-Moody theoretic string functions introduced by Kac and Peterson. For the remaining two elliptic weighted Fermat curves this relation requires a modification involving a twist character that is physically motivated by the number field generated by the quantum dimensions of the string model [9, 10]. The explicit structure of these one-dimensional results is described in Section 6. The modular forms that emerge from the weighted Fermat curves provide a string theoretic interpretation of the Hasse-Weil L-function of the exactly solvable Gepner models at central charge $c = 3$. Ref. [10] also identifies the criteria that lead to the derivation of these elliptic curves from the conformal field theory itself, with no a priori input from the geometry.

This Section briefly summarizes the results obtained in [8, 9, 10, 11], several of which appear as building blocks for the new examples considered in the remainder of this paper.

Explicitly, the class of elliptic Brieskorn-Pham curves is given by

$$\begin{aligned} E^3 &= \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0\} \\ E^4 &= \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_1^4 + z_2^2 = 0\} \end{aligned}$$

$$E^6 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,2,3)} \mid z_0^6 + z_1^3 + z_2^2 = 0\}. \quad (65)$$

The modular forms associated to these curves are cusp forms of weight two with respect to congruence groups of level N $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}. \quad (66)$$

Recalling the string theoretic Hecke indefinite theta series $\Theta_{\ell,m}^k$ defined in Section 2 in terms of the Kac-Peterson string functions $c_{\ell,m}^k(\tau)$, the geometric modular forms roughly decompose as

$$f(E^d, q) = \prod_i \Theta_{\ell_i, m_i}^{k_i}(q^{a_i}) \otimes \chi_d, \quad (67)$$

where $\chi_d(\cdot) = \left(\frac{\cdot}{d}\right)$ is the Legendre symbol. More precisely, given the worldsheet theta series $\Theta_{1,1}^1(\tau) = \eta^2(\tau)$ and $\Theta_{1,1}^2(\tau) = \eta(\tau)\eta(2\tau)$ the factorization takes the following form [10].

Theorem 6. *The inverse Mellin transforms $f(E^d, q)$ of the Hasse-Weil L -functions $L_{\mathrm{HW}}(E^d, s)$ of the curves $E^d, i = 3, 4, 6$ are modular forms $f(E^d, q) \in S_2(\Gamma_0(N))$, with $N = 27, 64, 144$ respectively. These cusp forms factor as*

$$\begin{aligned} f(E^3, q) &= \Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9) \\ f(E^4, q) &= \Theta_{1,1}^2(q^4)^2 \otimes \chi_2 \\ f(E^6, q) &= \Theta_{1,1}^1(q^6)^2 \otimes \chi_3. \end{aligned} \quad (68)$$

Consider the class of extremal K3 surfaces that can be constructed as weighted Fermat varieties

$$\begin{aligned} X_2^4 &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}, \\ X_2^{6A} &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_{(1,1,1,3)} \mid z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0\}, \\ X_2^{6B} &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_{(1,1,2,2)} \mid z_0^6 + z_1^6 + z_2^3 + z_3^3 = 0\}. \end{aligned} \quad (69)$$

These surfaces have motives that are modular and admit a string theoretic interpretation. Denote by $M(X) \subset H^2(X_2^d)$ the cohomological realization of a motive M , with $M_\Omega(X)$ the motive associated to the holomorphic 2-form, and let $L_\Omega(X, s) = L(M_\Omega(X), s)$ be the associated L -series, with $f_\Omega(X, q)$ denoting the inverse Mellin transform of $L_\Omega(X, s)$. The following result shows that the modular forms determined by the Ω -motives of these extremal K3 surfaces are determined by the string theoretic modular forms determined in Theorem 6.

Theorem 7. *Let $M_\Omega \subset H^2(X_2^d)$ be the irreducible representation of $\text{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ associated to the holomorphic 2-form $\Omega \in H^{2,0}(X_2^d)$ of the K3 surface X_2^d , where $d = 4, 6A, 6B$. Then the q -series $f_\Omega(X_2^d, q)$ of the L -functions $L_\Omega(X_2^d, s)$ are modular forms given by*

$$\begin{aligned} f_\Omega(X_2^4, q) &= \eta^6(q^4) \\ f_\Omega(X_2^{6A}, q) &= \vartheta(q^3)\eta^2(q^3)\eta^2(q^9) \\ f_\Omega(X_2^{6B}, q) &= \eta^3(q^2)\eta^3(q^6) \otimes \chi_3. \end{aligned} \tag{70}$$

These functions are cusp forms of weight three with respect to $\Gamma_0(N)$ with levels 16, 27 and 48, respectively. For X_2^4 and X_2^{6A} the L -functions can be written as

$$L_\Omega(X_2^d, s) = L(\psi_d^2, s), \tag{71}$$

where ψ_d are algebraic Hecke characters associated to cusp forms $f_d(q)$ of weight two and levels 64 and 27, respectively, given by the elliptic forms

$$\begin{aligned} f_4(\tau) &= f(E^4, q) = \eta^2(q^4)\eta^2(q^8) \otimes \chi_2 \\ f_{6A}(\tau) &= f(E^3, q) = \eta^2(q^3)\eta^2(q^9). \end{aligned} \tag{72}$$

For X_2^{6B} the L -series is given by $L_\Omega(X_2^{6B}, s) = L(\psi_{144}^2 \otimes \chi_3, s)$, leading to the cusp form of level 144

$$f_{6B}(\tau) = f(E^6, q) = \eta^4(q^6) \otimes \chi_3. \tag{73}$$

These results will enter in the discussion below of higher dimensional varieties.

7 A nonextremal K3 surface $X_2^{12} \subset \mathbb{P}_{(2,3,3,4)}$

String theoretic modularity of the class of extremal K3 surfaces of Brieskorn-Pham type has been established in [11]. Extremal K3 surfaces are characterized by the fact that their Picard number is maximal, i.e. $\rho = 20$.

A nonextremal example of a K3 surface is defined as the Brieskorn-Pham hypersurface (1) of degree twelve in the weighted projective space $\mathbb{P}_{(2,3,3,4)}$. The Galois group of the cyclotomic field $\mathbb{Q}(\mu_{12})$ has order four, hence the Ω -motive has rank four. The four Jacobi sums which parametrize this motive are given by

$$j_p(\sigma\alpha_\Omega), \quad \sigma \in \text{Gal}(\mathbb{Q}(\mu_{12})/\mathbb{Q}) \tag{74}$$

i.e. $j_p(\alpha)$ with

$$\alpha \in \left\{ \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3} \right), \left(\frac{5}{6}, \frac{1}{4}, \frac{1}{4}, \frac{2}{3} \right) \right\} \quad (75)$$

and their conjugates $\bar{\alpha} = \mathbf{1} - \alpha$, where $\mathbf{1}$ denotes the unit vector. The values of the independent Jacobi sums are collected for low p^f in Table 1.

p^f	$j_{p^f} \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3} \right)$	$j_{p^f} \left(\frac{5}{6}, \frac{1}{4}, \frac{1}{4}, \frac{2}{3} \right)$
13	$-3 + 4\sqrt{3} + (2 + 6\sqrt{3})i$	$-3 - 4\sqrt{3} + (2 - 6\sqrt{3})i$
25	$15 - 20i$	$15 - 20i$
37	$-5 - 12\sqrt{3} + (30 - 2\sqrt{3})i$	$-5 + 12\sqrt{3} + (30 + 2\sqrt{3})i$
49	$-7 - 28\sqrt{3}i$	$-7 + 28\sqrt{3}i$
61	$35 - 12\sqrt{3} - (42 + 10\sqrt{3})i$	$35 + 12\sqrt{3} - (42 - 10\sqrt{3})i$
73	$15 - 32\sqrt{3} + (40 + 12\sqrt{3})i$	$15 + 32\sqrt{3} + (40 - 12\sqrt{3})i$

Table 1. *Jacobi sums for the K3 surface $X_2^{12} \subset \mathbb{P}_{(2,3,3,4)}$.*

Using these results leads to the L-function of the Ω -motive

$$L_{\Omega}(X_2^{12}, s) \doteq 1 - \frac{12}{13^s} + \frac{30}{25^s} - \frac{20}{37^s} - \frac{14}{49^s} + \frac{140}{61^s} + \frac{60}{73^s} + \dots \quad (76)$$

Insight into the structure of this L-function can be obtained by noting that the surface X_2^{12} can be constructed via the twist map [43, 44] (see also [45, 46]) by considering

$$\Phi : \mathbb{P}_{(2,1,1)} \times \mathbb{P}_{(3,1,2)} \longrightarrow \mathbb{P}_{(3,3,2,4)} \quad (77)$$

defined by

$$((x_0, x_1, x_2), (y_0, y_1, y_2)) \mapsto \left(y_0^{1/3} x_1, y_0^{2/3} x_2, x_0^{1/2} y_1, x_0^{1/2} y_2 \right), \quad (78)$$

which on the product $E_-^4 \times E^6$ leads to the K3 surface of degree twelve X_2^{12} .

The coefficients of $L(E_-^4, s) = L(E^4, s) = \sum_n a_n(E^4)n^{-s}$ and $L(E^6, s) = \sum_n b_n(E^6)n^{-s}$ of the Hasse-Weil L-functions of the elliptic curves E^4 and E^6 , respectively, can be obtained by expanding the results of Theorem 6. Multiplying the coefficients $a_p(E^4)$ and $b_p(E^6)$ leads

$$a_p(E^4)b_p(E^6) = c_p(X_2^{12}). \quad (79)$$

For low primes the results are collected in Table 2.

p^f	13	25	37	49	61	73
$a_{p^f}(E^4)$	-6	-1	2	-7	10	-6
$b_{p^f}(E^6)$	2	-5	-10	9	14	-10
$c_{p^f}(X_2^{12})$	-12	30	-20	-14	140	60

Table 2. Coefficient comparison of the surface X_2^{12} and the curves E^4 and E^6 .

The Mellin transform of the L-function of both building blocks E^4 and E^6 of the surface X_2^{12} are given in terms of string theoretic theta functions as described in Theorem 6. The modular forms of E^4 and E^6 are both of complex multiplication type, leading to Hecke interpretation of the L-series in terms of Größencharaktere. They are both twists by Legendre symbols of characters ψ_{32} and ψ_{36} , as described in [10], leading to

$$\begin{aligned} L(E^4, s) &= L(\psi_{32} \otimes \chi_2, s) \\ L(E^6, s) &= L(\psi_{36} \otimes \chi_2, s), \end{aligned} \tag{80}$$

where the characters ψ_{32} and ψ_{36} are associated to the Gauss field $\mathbb{Q}(\sqrt{-1})$ and the Eisenstein field $\mathbb{Q}(\sqrt{-3})$ respectively, as described in §2.

The string interpretation of the Hasse-Weil L-series of E^4 and E^6 described in Theorem 6 therefore induces a string interpretation of the modular blocks of the K3 surface X_2^{12} . The CM property of these forms will allow a systematic discussion in Section 12 of the reverse construction of emergent space from the modular forms of the worldsheet field theory.

The factorization of the coefficients $c_p(X_2^{12})$ suggests that the rank four Ω -motive $M_\Omega(X_2^{12})$ is the tensor product of the elliptic motives of E^4 and E^6 , but a priori leaves open the precise nature of the L-function product. It turns out that the correct version is the modified Rankin-Selberg product considered in Section 2, applied to the case of two modular forms of weight two

$$L_\Omega(X_2^{12}, s) = L(M_{f_{64}} \otimes M_{f_{144}}, s),$$

where $M_{f_{N_i}}$ are the elliptic motives of f_{N_i} .

8 Modular motives of the Calabi-Yau threefold $X_3^6 \subset \mathbb{P}_{(1,1,1,1,2)}$

In this and the following section modularity is established for the Ω -motives of two Calabi-Yau threefolds. The two varieties considered lead to motives of ranks two and four.

8.1 The modular Ω -motive of X_3^6

Consider the Calabi-Yau variety X_3^6 defined as the double cover branched over the degree six Fermat surface in projective threespace \mathbb{P}_3 . This manifold can be viewed as a smooth degree six hypersurface of Brieskorn-Pham type in the weighted projective fourspace $\mathbb{P}_{(1,1,1,1,2)}$ as in (3). The Galois group of $\mathbb{Q}(\mu_6)$ has order two, hence the motive M_Ω has rank two. The values of the relevant Jacobi sum $j_{p^f}(\alpha_\Omega)$ with $\alpha_\Omega = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3})$ are collected in Table 3.

p^f	7	13	19	25	31	37
$j_{p^f}(\alpha_\Omega)$	$-\frac{17}{2} - \frac{19}{2}\sqrt{3}i$	$-\frac{89}{2} - \frac{17}{2}\sqrt{3}i$	$-\frac{107}{2} + \frac{73}{2}\sqrt{3}i$	125	$-154 - 45\sqrt{3}i$	$\frac{433}{2} - \frac{71}{2}\sqrt{3}i$

Table 3. Values for the Jacobi sum of the Ω -motive of $X_3^6 \subset \mathbb{P}_{(1,1,1,1,2)}$.

These Jacobi sums and their conjugates lead to the L-series of the motive of X_3^6

$$L_\Omega(X_3^6, s) \doteq 1 + \frac{17}{7^s} + \frac{89}{13^s} + \frac{107}{19^s} - \frac{125}{25^s} + \frac{308}{31^s} - \frac{433}{37^s} + \dots \quad (81)$$

The inverse Mellin transform of this L-series turns out to describe a modular form of weight $w = 4$ and level $N = 108$. This form $f_\Omega \in S_4(\Gamma_0(108))$ cannot be written as a product or a quotient of Dedekind eta-functions [47, 48], but it admits complex multiplication and $L_\Omega(X_3^6, s)$ can be written as the Hecke L-series of a twisted Größencharakter associated to the complex multiplication field $K = \mathbb{Q}(\sqrt{-3})$. Characters associated to K were considered in [10, 11] in the context of the elliptic Brieskorn-Pham curve $E^3 \subset \mathbb{P}_2$ and $E^6 \subset \mathbb{P}_{(1,2,3)}$ as well as modular K3 surfaces. The Hasse-Weil L-series $L(E^3, s)$ is a Hecke series for the character ψ_{27} considered in §2 which can be written as [8]

$$L(E^3, s) = L(\psi_{27}, s) = L(\Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9), s) \quad (82)$$

The character ψ_{27} turns out to be the fundamental building block of the L-series of the Ω -motive of X_3^6 .

For higher dimensional varieties it is possible to generalize such relations by considering powers of the Hecke character in order to obtain higher weight modular forms, as described above in Hecke's theorem. In the present case ψ_{27}^3 leads to a modular form which can be written in terms of the Dedekind eta function $\eta(q)$ as $\eta^8(q^3) \in S_4(\Gamma_0(9))$, which also appears as a motivic form in a number of geometries different from X_3^6 . In order to obtain the motivic L-series

$L_\Omega(X_3^6, s)$ computed above it is necessary to introduce a twist character. This can be chosen to be the cubic residue power symbol, denoted here by

$$\chi_2^{(3)}(p) := \left(\frac{2}{\mathfrak{p}}\right)_3, \quad (83)$$

where $\mathfrak{p}|p$ and the congruence ideal is chosen to be $\mathfrak{m} = (3)$. With this character the Hecke interpretation of the motivic L-function of X_3^6 takes the form

$$L_\Omega(X_3^6, s) = L(\psi_{27}^3 \otimes (\chi_2^{(3)})^2, s). \quad (84)$$

The inverse Mellin transform $f_\Omega(X_3^6, q)$ of this L-series is a modular form of weight 4 and level 108

$$f_\Omega(X_3^6, q) \doteq q + 17q^7 + 89q^{13} + 107q^{19} - 125q^{25} + 308q^{31} - 433q^{37} + \dots \quad (85)$$

which is a cusp form, i.e. $f_\Omega(X_3^6, q) \in S_4(\Gamma_0(108))$. This modular form is therefore of complex multiplication type in the sense of Ribet [16]. An explicit proof for the modular form $f(\psi_{27}, q) \in S_2(\Gamma_0(27))$ can be found in [10].

8.2 Lower weight modular motives of X_3^6

The degree six Calabi-Yau hypersurface X_3^6 provides an example of the phenomenon noted in Section 3 that the intermediate cohomology can lead to modular motives beyond the Ω -motive. For X_3^6 the motives are of rank two, given by the Galois group $\text{Gal}(\mathbb{Q}(\mu_6)/\mathbb{Q})$ with certain multiplicities and twists. Modulo these twists and multiplicities the group $H^{2,1}(X) \oplus H^{1,2}(X)$ leads to three different types of modular motives of weight two and rank two, denoted in the following by $M_A \in \{M_I, M_{II}, M_{III}\}$. The L-series $L(M_A, s)$ that result from these motives have coefficients \bar{a}_p^A that are all divisible by the prime p . By introducing the twisted coefficients $a_p^A = \bar{a}_p^A/p$, these L-series lead to modular forms of weight two $f_A \in S_2(\Gamma_0(N_A))$, where the level N_A is determined by the motive M_A

$$L(M_A(X_3^6), s) = L(f_A, s - 1).$$

The modular forms are of levels $N_A = 27, 144, 432$, the first two given by the curves E^3, E^6 described in Theorem 6

$$\begin{aligned} f_I(q) &= f(E^3, q) \\ f_{II}(q) &= f(E^6, q), \end{aligned} \quad (86)$$

while the level $N_{\text{III}} = 432$ form is given by

$$f_{\text{III}}(q) \doteq q - 5q^7 - 7q^{13} + q^{19} + 4q^{31} + \dots$$

It follows that the L-series $L(H^3(X_3^6), s)$ of the intermediate cohomology group decomposes into modular pieces in the sense that each factor arises from a modular form

$$L(H^3(X_3^6), s) = L(f_\Omega, s) \prod_i L(f_i \otimes \chi_i, s)^{a_i},$$

where $a_i \in \mathbb{N}$, $f_\Omega \in S_4(\Gamma_0(108))$ is as determined above, the $f_i(q)$ are modular forms of weight two and levels $N_i = 27, 144, 432$, and χ_i is a Legendre character (which can be trivial).

The Jacobi sums corresponding to the motives $M_{\text{I}}, M_{\text{II}}, M_{\text{III}}$ are listed in Table 4, together with the level N_A of the corresponding modular form $f_A \in S_2(\Gamma_0(N_A))$.

Type	Jacobi sum	Level N
I	$j_p\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$	27
II	$j_p\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{3}\right)$	144
III	$j_p\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$	432

Table 4. *The Jacobi sums of X_3^6 that lead to non-isogenic modular motives.*

The weight two modular forms f_A that emerge from X_3^6 have a natural geometric interpretation. The threefold X_3^6 contains divisors given by degree six Fermat curves $C^6 \subset \mathbb{P}_2$ and $\overline{C}^6 \subset \mathbb{P}_{(1,1,2)}$, obtained from the original hypersurface via intersections with coordinate hyperplanes. These curves are of genus ten and can be shown to decompose into ten elliptic factors of three different types $E_{\text{I}} = E^3$, $E_{\text{II}} = E^6$, and E_{III} an elliptic curve of conductor 432. Hence its L-function factors as

$$L(C^6, s) = L(E_{\text{I}}, s)L(E_{\text{II}}, s)^6L(E_{\text{III}}, s)^3,$$

taking into account their multiplicities. The curve \overline{C}^6 is of genus four and leads to the same modular forms, with different multiplicities.

9 The K3 fibration hypersurface $X_3^{12} \subset \mathbb{P}_{(2,2,2,3,3)}$

9.1 The Ω -motive of X_2^{12}

Consider the weighted Fermat hypersurface of degree twelve in $\mathbb{P}_{(2,2,2,3,3)}$ given in eq. (3). The Galois group of this variety is of order four, leading to an Ω -motive of rank four. The Jacobi sums that parametrize this motive are given by $j_p\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$, $j_p\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{4}, \frac{3}{4}\right)$ and their complex conjugates. The computation of these sums for low p^f are collected in Table 5.

p^f	$j_{p^f}\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$	$j_{p^f}\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{4}, \frac{3}{4}\right)$
13	$-\frac{3}{2} + 15\sqrt{3} + \left(1 + \frac{45}{2}\sqrt{3}\right) i$	$-\frac{3}{2} - 15\sqrt{3} - \left(1 - \frac{45}{2}\sqrt{3}\right) i$
25	$75 - 100i$	$75 + 100i$
37	$-\frac{47}{2} + 99\sqrt{3} + \left(141 + \frac{33}{2}\sqrt{3}\right) i$	$-\frac{47}{2} - 99\sqrt{3} - \left(141 - \frac{33}{2}\sqrt{3}\right) i$
49	$\frac{7}{2}(71 + 39\sqrt{3})i$	$\frac{7}{2}(71 + 39\sqrt{3})i$
61	$\frac{605}{2} + 27\sqrt{3} - \left(363 - \frac{45}{2}\sqrt{3}\right) i$	$\frac{605}{2} - 27\sqrt{3} + \left(363 + \frac{45}{2}\sqrt{3}\right) i$
73	$-\frac{291}{2} + 252\sqrt{3} - \left(388 + \frac{189}{2}\sqrt{3}\right) i$	$-\frac{291}{2} - 252\sqrt{3} + \left(388 - \frac{189}{2}\sqrt{3}\right) i$

Table 5. *Jacobi sums of the Ω -motive of X_3^{12} .*

Using these results leads to the expansion of the L-series

$$L_{\Omega}(X_3^{12}, s) \doteq 1 + \frac{6}{13^s} - \frac{150}{25^s} + \frac{94}{37^s} - \frac{497}{49^s} - \frac{1210}{61^s} + \frac{582}{73^s} + \dots \quad (87)$$

The structure of the Ω -motivic L-series of X_3^{12} can be understood by noting that the threefold is a K3 fibration with typical fiber X_2^{6A} given in (69). The interpretation of $L_{\Omega}(X_3^{12}, s)$ in terms of the fibration is also useful because it makes the complex multiplication structure of the associated modular form transparent. The threefold can be constructed explicitly as the quotient of a product of a torus E and a K3 surface

$$X = E \times \text{K3}/\iota, \quad (88)$$

where ι is an involution acting on the product. More precisely, the elliptic curve is given by the weighted

$$E_-^4 = \{x_0^2 - (x_1^4 + x_2^4 = 0)\} \subset \mathbb{P}_{(2,1,1)} \quad (89)$$

and the K3 surface is the generic fiber X_2^{6A} .

Applying the twist construction of [43, 44] gives first the map

$$\Phi : \mathbb{P}_{(2,1,1)} \times \mathbb{P}_{(3,1,1,1)} \longrightarrow \mathbb{P}_{(3,3,2,2,2)} \quad (90)$$

defined by

$$((x_0, x_1, x_2), (y_0, y_1, y_2, y_3)) \mapsto \left(y_0^{1/3} x_1, y_0^{1/3} x_2, x_0^{1/2} y_1, x_0^{1/2} y_2, x_0^{1/2} y_3 \right). \quad (91)$$

This map restricts on the product $E^4 \times X_2^{6A}$ to the threefold X_3^{12} .

The fibration structure suggests that the L–function of the threefold X_3^{12} can be understood in terms of those of its building blocks. The L-function of the K3 fiber of this threefold was determined in Theorem 7 to be given by the Mellin transform of the cusp form $f_\Omega(X_2^{6A}, q)$ of weight $w = 3$ and level $N = 27$, and the L-function of the quartic curve is given by $f(E^4, q) \in S_2(\Gamma_0(64))$ according to Theorem 6 [10].

A comparison of the coefficients of the L-function of the Ω –motive of X_3^{12} with those of its building blocks E^4 and X_2^{6A} should lead to a composite structure. Table 6 illustrates that this is indeed the case. The coefficients $a_p(E^4)$ arise from the L-series of the quartic E^4 , while the surface expansion $b_p(X_2^{6A})$ is that of (70) in Theorem 7, which leads to the expansion

$$f(X_2^{6A}, q) \doteq q - 13q^7 - q^{13} + 11q^{19} + 25q^{25} - 46q^{31} + 47q^{37} - 22q^{43} + 120q^{49} - 121q^{61} - 109q^{67} - 97q^{73} + \dots \quad (92)$$

It follows that the products $a_p(E^4)b_p(X_2^{6A})$ agree with the expansion coefficients $c_p(X_3^{12})$ of the threefold X_3^{12} .

p^f	13	25	37	49	61	73
$a_{p^f}(E^4)$	−6	−1	2	−7	10	−6
$b_{p^f}(X_2^{6A})$	−1	25	47	120	−121	−97
$c_{p^f}(X_3^{12})$	6	−150	94	−994	−1210	582

Table 6. Coefficient comparison for the threefold X_3^{12} .

The factorization of the Ω –motivic L-series $L_\Omega(X_3^{12}, s)$ also shows that its building block have complex multiplication. For the curve E^4 this was already discussed above in the context of the K3 surface X_2^{12} [10]. For the K3 surface X_2^{6A} it was shown in [11] that the modular form of Theorem 7 is the Mellin transform of the Hecke L-series of a Größencharakter ψ_{27}^2 of the Eisenstein field $\mathbb{Q}(\sqrt{-3})$

$$L(X_2^{6A}, s) = L(\psi_{27}^2, s) \quad (93)$$

where the character ψ_{27} has been defined in Section 2. The motivic L-series of both building blocks of X_3^{12} therefore are of complex multiplication type.

9.2 Lower weight modular forms of X_3^{12}

Similar to the degree six threefold X_3^6 the cohomology $H^{2,1} \oplus H^{1,2}$ of the degree twelve hypersurface X_3^{12} leads to modular forms of weight two. There are again Jacobi sums that lead to precisely the same modular forms $f_A(q)$, $A = \text{I, II, III}$ of weight two and levels $N_A = 27, 144, 432$, possibly including a twist, as for X_3^6 . This is expected because X_3^{12} contains the plane Fermat curve $C^6 \subset \mathbb{P}_2$ already encountered in X_3^6 . There is a further L-series of a rank two motive that is determined by the quartic elliptic weighted Fermat curve $E^4 \subset \mathbb{P}_{(1,1,2)}$ considered in Theorem 6

$$L_{\text{IV}}(X_3^{12}, s) = L(E^4, s - 1).$$

10 String modular rigid Calabi-Yau threefolds

The purpose of this section is to show that the modularity results of [8, 9] lead to a string modular interpretation of two rigid Calabi-Yau manifolds. It has been known for a long time in the context of lattice constructions of orbifolds that rigid Calabi-Yau manifolds are obtained for certain quotients of six-dimensional tori by discrete groups. In the context of zeta functions it is more useful to consider triple products of Brieskorn-Pham curves. Consider the elliptic Brieskorn-Pham curve of degree three $E^3 \subset \mathbb{P}_2$ and the quartic $E^4 \subset \mathbb{P}_{(1,1,2)}$. On the triple products $(E^d)^3$, $d = 3, 4$ there are group actions $\mathbb{Z}_d \times \mathbb{Z}_d$, where $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. The quotients of the triple products by these groups are singular and their resolutions $X_3^d = \text{res}((E^d)/\mathbb{Z}_d \times \mathbb{Z}_d)$ are rigid with

$$\begin{aligned} h^{1,1}(X_3^3) &= 84 \\ h^{1,1}(X_3^4) &= 90. \end{aligned} \tag{94}$$

The string modular forms of E^d are determined in terms of Hecke's indefinite modular forms by Theorem 6. Since the varieties X_3^d for $d = 3, 4$ are rigid it is to be expected that they are modular and that their modular forms can be expressed in terms of the Hecke indefinite modular forms $\Theta_{\ell, m}^k(\tau)$ as well. The fact that this is indeed the case follows from a result shown by Cynk and Hulek, who prove in [49] the following result.

Theorem 8. *The L-series of the rigid Calabi-Yau threefolds X_3^d , $d = 3, 4$ are given by cusp forms f_d of weight 4 with complex multiplication in $\mathbb{Q}(\sqrt{-d})$.*

The modular forms f_d are precisely the modular forms determined by the elliptic curves considered in Theorem 6. This can be seen as follows. Consider the character ψ_{27} defined in Section 2 and the twisted Größencharakter $\psi_{64} = \psi_{32} \otimes \chi_2$. It was shown in [10] that the modular forms of the elliptic curves E^d have complex multiplication in $\mathbb{Q}(\sqrt{-d})$ and that

$$L(E^d, s) = L(\psi_{d^3}, s). \quad (95)$$

For $d = 3$ the form $f_3 \in S_4(\Gamma_0(9))$ is the eta product $\eta^8(q^3)$, which can be written in terms of Hecke's indefinite forms $\Theta_{\ell, m}^k$ as $f_3(q) = \Theta_{1,1}^1(q^3)^4$, with the associated L-series given as

$$L(X_3^3, s) = L(f_3, s) = L(\psi_{27}^3, s). \quad (96)$$

For $d = 4$ the form f_4 is given by the cube of the character of conductor 64

$$L(X_3^4, s) = L(\psi_{64}^3, s). \quad (97)$$

It follows from these considerations that both rigid manifolds X_3^d , $d = 3, 4$ lead to Ω -motivic L-series that admit a string theoretic interpretation in terms of Hecke's indefinite modular forms.

This result is relevant in the context of mirror symmetry. From the point of view of the conformal field theory on the string worldsheet mirror theories are isomorphic. A string theoretic interpretation of a geometric modular form therefore leads to the expectation that the motivic L-function of a variety should be identical to that of its mirror. This problem is considered in [25] in the context of the rigid mirror manifolds considered above. It is shown there that one can associate mirror motives to the Ω -motives of the rigid varieties X_3^d , and that these motives are modular and lead to the same modular forms.

11 A modular Calabi-Yau fourfold $X_4^6 \subset \mathbb{P}_5$

Calabi-Yau varieties of complex dimension four are useful in the context of F-theory in four dimensions and M-theory in three dimensions. In this section it is shown that the Ω -motive of the fourfold of degree six in projective fivespace \mathbb{P}_5 defined by the Brieskorn-Pham hypersurface (6) is modular.

The Galois orbit is of length two, and the motivic L-function is described by computing the Jacobi sums $j_p(\alpha_\Omega)$ with $\alpha_\Omega = (\frac{1}{6}, \dots, \frac{1}{6})$. The results are collected in Table 7.

p^f	7	13	19	25	31	37
$j_{p^f}(\alpha_\Omega)$	$-\frac{71}{2} - \frac{39}{2}\sqrt{-3}$	$-\frac{337}{2} - \frac{15}{2}\sqrt{-3}$	$\frac{601}{2} + \frac{231}{2}\sqrt{-3}$	625	$-97 + 552\sqrt{-3}$	$-\frac{529}{2} - \frac{1551}{2}\sqrt{-3}$
β_{p^f}	-71	-337	601	625	-194	-529

Table 7. *Jacobi sums for X_4^6 .*

The resulting L-function is given by

$$L_\Omega(X_4^6, s) \doteq 1 - \frac{71}{7^s} - \frac{337}{13^s} + \frac{601}{19^s} + \frac{625}{25^s} - \frac{194}{31^s} - \frac{529}{37^s} + \dots \quad (98)$$

The associated q -expansion $f_\Omega(X_4^6, q)$ differs from the that of a newform of weight five and level 27

$$f_{27}(q) = q + 71q^7 - 337q^{13} - 601q^{19} + 194q^{31} - 529q^{37} + \dots \quad (99)$$

only in signs. These signs can be adjusted the quadratic character χ_3 defined by the Legendre symbol

$$\chi_3(p) = \left(\frac{3}{p}\right), \quad (100)$$

leading to

$$f_\Omega(X_4^6, q) = f_{27}(q) \otimes \chi_3. \quad (101)$$

This modular form can also be described as a Hecke L-series associated to the character ψ_{27} associated to the Eisenstein field $K = \mathbb{Q}(\sqrt{-3})$ and defined in Section 2 in (21). More precisely, the twisted Hecke L-series agrees with that of the Ω -motive of X_4^6

$$L_\Omega(X_4^6, s) = L(\psi_{27}^4, s) \otimes \chi_3. \quad (102)$$

Hence the motivic L-function is again a purely algebraic object and its fundamental structure is determined by the L-series of a Hecke indefinite theta series as noted in (82).

12 Emergent space from characters and modular forms

In the discussion so far the goal was to formulate a general framework of Ω -motives of varieties of Calabi-Yau type, and more generally, of special Fano type, in the context Grothendieck's

framework of motives, and to test the conjecture that these motives are string modular in the sense that it is possible to identify modular forms on the worldsheet whose Mellin transform agrees with the L-function of the resulting motives. The problem of constructing spacetime geometry from fundamental string input involves the inverse problem of this strategy.

The aim of the present section is to address this "space construction" problem. As already mentioned earlier, the idea here must be to obtain a construction of the motivic pieces of the compact varieties from the basic conformal field theoretic modular forms. There are several ways to think about these objects and the following remarks describe how these various constructions, which a priori are independent, fit together in the context of the Ω -motive of Brieskorn-Pham type varieties.

The main simplifying observation in the present context of weighted Fermat hypersurfaces is that all the motivic L-functions $L_\Omega(X, s)$ that have been encountered so far in the program initiated in [50, 8] and continued in [9, 10, 11, 51] lead to modular forms which are of complex multiplication type [16] (see [17] for geometric aspects of CM). The structure of such forms has been described in detail in [52, 10, 11]. The important point in the present context is that modular Ω -motives of CM type are algebraic objects whose L-functions are given by the Hecke L-series of a Größencharakter (possibly modulo a twist). This construction of algebraic Hecke characters from geometry can be inverted, and it is known how to construct motives directly from the characters. It is in particular possible to construct a Grothendieck motive of the form $M_\chi = (A_\chi, p_\chi)$, where A_χ is an abelian variety associated to the character χ by the theorem of Casselman [53], and p_χ is a projector associated to χ . This leads to an apparent problem of riches, because given any modular form $S_w(\Gamma_0(N))$ it is possible to construct a Grothendieck motive $M_f = (X_f, p_f)$ by considering the cohomology of an associated Kuga-Sato variety X_f , as shown by Deligne [19], Jannsen [20], and Scholl [21]. Combining the abelian motives and the Kuga-Sato motives with those of the Ω -motives thus leads to three a priori different motivic constructions associated to the algebraic Hecke characters encountered here and in the earlier papers.

It turns out that the motives M_χ , M_f and M_Ω all are isomorphic because they arise from the same CM modular form. This follows because motives associated to CM modular forms have CM, and one can generalize Faltings result that L-series characterize abelian varieties up to isogeny to CM motives, as shown e.g. by Anderson [54].

In the case of varieties of Brieskorn-Pham type it is possible to make the relation between abelian varieties and Ω -motives more concrete by noting that the cohomology of Fermat

varieties has a well-known inductive structure which was first noted by Shioda-Katsura [55] (see also Deligne [56]) in the context of Fermat varieties. This inductive structure allows to reduce the cohomology of higherdimensional varieties in terms of the cohomology of algebraic curves (modulo Tate twists). Hence the basic building blocks are abelian varieties derived from the Jacobians of these curves. It follows from results of Gross and Rohrlich [57] that these Jacobians factor into simple abelian varieties and that these abelian factors have complex multiplication. The final step in the construction is provided by the fact that the L-function of the abelian variety attached to χ by Casselman's result is given by the conjugates of the L-function of the Hecke character.

13 Further considerations

The goal of the program continued in this paper is to investigate the relation between the geometry of spacetime and the physics of the worldsheet by analyzing in some depth the connection between the modular symmetry encoded in exact models on the worldsheet and the modular symmetries that emerge from the nontrivial arithmetic structure of spacetime. The techniques introduced for this purpose provide a stronger, and more precise, alternative to the framework of Landau-Ginzburg theories and σ -models. The latter in particular presupposes the concept of an ambient space in which the string propagates, a notion that should emerge as a derived concept in a fundamental theory.

The focus of the results obtained in previous work and the present paper has been on the class of diagonal models, given by Gepner's construction. It would be interesting to extend these considerations to the more general class of Kazama-Suzuki models [58]. Of particular interest in that class are certain 'irreducible' models which are not tensor products, hence a single conformal field theoretic quotient suffices to saturate the necessary central charge. Such models exist for both K3 surfaces and Calabi-Yau threefolds, and establishing modularity in the sense described here would be a starting point for the exploration of modular points in the moduli space of nondiagonal varieties. Results in this direction would illuminate relations between different conformal field theories.

A second open problem is the analysis of families of varieties with respect to their modular properties. First steps in this direction have been taken in refs. [59, 60, 61] where the zeta functions for particular one-parameter families of Calabi-Yau threefolds are computed. It would be of interest to understand how the modular behavior of these families is related

to deformations along marginal directions of the associated conformal field theory. Such an analysis might be useful for the understanding of the conformal field theoretic behavior of conifold phase transitions.

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References

- [1] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. Math. **141** (1995) 443 – 551;
R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. **141** (1995) 553 – 572
- [2] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over \mathbb{Q} or Wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001) 843 – 939
- [3] R. Langlands, *L-Functions and automorphic representations*, Proc. Int. Cong. Math. 1978, Acad. Sci. Fennica, Helsinki, 1980
- [4] E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, Math. Ann. **112** (1936) 664 – 699
- [5] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967) 149 – 156
- [6] R.P. Langlands, *Problems in the theory of automorphic forms*, in LECTURES IN MODERN ANALYSIS AND APPLICATIONS III, LNM 170, Springer Verlag, 1970
- [7] R.P. Langlands, *Euler Products*, Yale University Press, 1971
- [8] R. Schimmrigk and S. Underwood, *The Shimura-Taniyama conjecture and conformal field theories*, J. Geom. Phys. **48** (2003) 169 – 189, [arXiv: hep-th/0211284]
- [9] M. Lynker and R. Schimmrigk, *Geometric Kac–Moody modularity*, J. Geom. Phys. **56** (2006) 843 – 863, [arXiv: hep-th/0410189]

- [10] R. Schimmrigk, *Arithmetic spacetime geometry from string theory*, Int. J. Mod. Phys. **A21** (2006) 6323 – 6350, [arXiv: hep-th/0510091]
- [11] R. Schimmrigk, *The Langlands program and string modular K3 surfaces*, Nucl. Phys. **B771** (2007) 143 – 166, [arXiv: hep-th/0603234]
- [12] D. Gepner, *Spacetime supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. **B296** (1988) 757 – 778
- [13] V.G. Kac and D.H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53** (1984) 125 – 264
- [14] E. Hecke, *Über die Dirichlet-Reihen mit Funktionalgleichungen und ihre Nullstellen auf der Mittelgraden*, Sitzungsberichte d. Bayerischen Akad. d. Wissenschaftern, 1937, 73 – 95
- [15] G. Shimura, *On Elliptic Curves with Complex Multiplication as Factors of the Jacobians of Modular Function Fields*, Nagoya Math. J. **43** (1971) 199 – 208
- [16] K.A. Ribet, *Galois representations attached to eigenforms with nebentypus*, in MODULAR FUNCTIONS OF ONE VARIABLE V, eds. J.-P. Serre and D. Zagier, LNM 601, Springer, 1977
- [17] N. Yui, *The L-series of Calabi-Yau orifolds of CM type*, with an appendix by Y. Goto, in MIRROR SYMMETRY V, AMS/IP Stud. Adv. Math. **38** (2006) pp. 185 – 252
- [18] Goro Shimura, *The special values of the zeta functions associated with cusp forms*, Com. Pure. Appl. Math. **29** (1976) 783 – 804
- [19] P. Deligne, *Formes modulaires et représentations ℓ -adique*, Séminaire Bourbaki **355** (février 1969), LNM **179**, Springer 1971, pp. 139 – 172
- [20] U. Jannsen, *Mixed motives and algebraic K-theory*, LNM 1400, Springer 1990
- [21] A.J. Scholl, *Motives from modular forms*, Inv. Math. **100** (1990) 419 – 430
- [22] R. Schimmrigk, *Critical string vacua from noncritical manifolds*, Phys. Rev. Lett. **70** (1993) 3688 – 3691, [arXiv: hep-th/9211001]
- [23] P. Candelas, E. Derrick and L. Parkes, *Generalized Calabi-Yau manifolds and the mirror of a rigid manifold*, Nucl. Phys. **B407** (1993) 115 – 154, [arXiv: hep-th/9304045]
- [24] R. Schimmrigk, *Mirror symmetry and string vacua from a special class of Fano varieties*, Int. J. Mod. Phys. **A11** (1996) 3049 – 3096, [arXiv: hep-th/9405086]
- [25] S. Kharel, M. Lynker and R. Schimmrigk, *String modular motives for mirrors of rigid Calabi-Yau varieties*, in MODULAR FORMS AND STRING DUALITY, Fields Inst. Commun. **54** (2008) pp. 47 – 63
- [26] Yu.I. Manin, *Correspondences, motifs and monoidal transformations*, Math. USSR Sbornik **6** (1968) 439 – 470
- [27] M. Demazure, *Motifs des variétés algebrique*, Séminaires Bourbaki, exp. 365, 1969

- [28] S. Kleiman, *Motives*, in ALGEBRAIC GEOMETRY, Oslo 1970, F. Oort (ed.), pp. 53 – 82, Groningen 1972
- [29] J.P. Murre, *Introduction to the theory of motives*, Bolletino U.M.I. **10-A** (1996) 477 – 489
- [30] U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992) 447 – 452
- [31] A.J. Scholl, *Classical Motives*, in: MOTIVES, Proc. Symp. Pure Math **55**, eds. U. Jannsen, S. Kleiman, and J.-P. Serre, Amer. Math. Soc., 1994
- [32] U. Jannsen, S. Kleiman and J.-P. Serre, *Motives*, Proc. Symp. Pure Math **55**, Amer. Math. Soc., 1994
- [33] A. Weil, *Number of solutions of equations in finite fields*, Bull. Am. Math. Soc. **55** (1949) 497 – 508
- [34] A. Grothendieck, *Formulé de Lefschetz et rationalité de fonction de L*, Séminaire Bourbaki **279**, 1964/1965, 1 – 15
- [35] P. Deligne, *La conjecture de Weil I*, Publ. Math. IHES **43** (1974) 273 – 307
- [36] F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p* , Math. Z. **33** (1931) 1 – 32
- [37] H. Hasse, *Beweis des Analogons der Riemannsches Vermutung für die Artinschen und F.K. Schmidtschen Kongruenzzetafunktionen in gewissen elliptischen Fällen. Vorläufige Mitteilung*, Nachrichten v.d. Gesellschaft d. Wiss. zu Göttingen I **42** (1933) 253 – 262
- [38] H. Hasse, *Über die Kongruenzzetafunktionen. Unter Benutzung von Mitteilungen von Prof. Dr. F.K. Schmidt und Prof. Dr. E. Artin*, S. Ber. Preuß. Akad. Wiss. H. **17** (1934) 250 – 263
- [39] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960) 631 – 648
- [40] T. Shioda, *Some observations on Jacobi sums*, in GALOIS REPRESENTATIONS AND ARITHMETIC GEOMETRY, Advanced Studies in Pure Mathematics **12**, 1987, ed. Y. Ihara
- [41] F. Gouvea and N. Yui, *Arithmetic of diagonal hypersurfaces over finite fields*, London Math. Soc., 1995
- [42] S. Kadir and N. Yui, *Motives and mirror symmetry for Calabi-Yau orbifolds*, in MODULAR FORMS AND STRING DUALITY, Fields Inst. Commun. **54** (2008) pp. 3 – 46
- [43] B. Hunt and R. Schimmrigk, *Heterotic gauge structure of type II $K3$ fibrations*, Phys. Lett. **B381** (1996) 427 – 436, [arXiv: hep-th/9512138]
- [44] B. Hunt and R. Schimmrigk, *$K3$ Fibered Calabi-Yau threefolds I: The twist map*, Int. J. Math. **10** (1999) 833 – 866
- [45] C. Voisin, *Miroirs et involutions sur les surfaces $K3$* , Journées de Géométrie Algébrique d’Orsay, Astérisque **218** (1993), 273 – 323

- [46] C. Borcea, *K3 Surfaces with involutions and mirror pairs of Calabi-Yau manifolds*, in MIRROR SYMMETRY, II, 717 – 743, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997
- [47] D. Dummit, H. Kisilvesky and J. McKay, *Multiplicative properties of η -functions* Contemp. Math. **45** (1985) 89 – 98
- [48] Y. Martin, *Multiplicative η -Quotients*, Trans. Amer. Math. Soc. **348** (1996) 4825 – 4856
- [49] S. Cynk and K. Hulek, *Construction and examples of higher-dimensional modular Calabi-Yau manifolds*, [arXiv: math.AG/0509424]
- [50] R. Schimmrigk, *Calabi-Yau arithmetic and rational conformal field theories*, J. Geom. Phys. **44** (2003) 555 – 569, [arXiv: hep-th/0111226]
- [51] R. Schimmrigk, *A modularity test of elliptic mirror symmetry*, Phys. Lett. **B655** (2007) 84 – 89, [arXiv: 0705.2427 (hep-th)]
- [52] M. Lynker, R. Schimmrigk and S. Stewart, *Complex multiplication of exactly solvable Calabi-Yau varieties*, Nucl. Phys. **B700** (2004) 463 – 489, [arXiv: hep-th/0312319]
- [53] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton UP 1971
- [54] G.W. Anderson, *Cyclotomy and an extension of the Taniyama group*, Compositio Math. **57** (1986) 153 – 217
- [55] T. Shioda and T. Katsura, *On Fermat varieties*, Tohoku Math. J. **31** (1979) 97 – 115
- [56] P. Deligne, *Hodge cycles on abelian varieties*, in HODGE CYCLES, MOTIVES, AND SHIMURA VARIETIES, eds. P. Deligne, J.S. Milne, A. Ogus and K.-y. Shih, LNM 900, Springer Verlag, 1982
- [57] B. Gross, *On the periods of abelian integrals and a formula of Chowla and Selberg*, Invent. Math. **45** (1978) 193 – 211, (with an appendix by D.E. Rohrlich)
- [58] Y. Kazama and H. Suzuki, *New $N=2$ superconformal field theories and superstring compactification*, Nucl. Phys. **B321** (1989) 232
- [59] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields I*, [arXiv: hep-th/0012233]
- [60] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields II*, in CALABI-YAU VARIETIES AND MIRROR SYMMETRY, 121 – 157, Fields Inst. Commun., 38, Amer. Math. Soc., 2003, [arXiv: hep-th/0402133]
- [61] S.N. Kadir, *The arithmetic of Calabi-Yau manifolds and mirror symmetry*, [arXiv: hep-th/0409202]; *Arithmetic mirror symmetry for a two-parameter family of Calabi-Yau manifolds*, in MIRROR SYMMETRY V, AMS/IP Stud. Adv. Math. **38** (2006) 35 – 86