where $C^{\mu\nu\alpha\beta}_{\sigma\tau}$ is the Lorentz group structure constant (2.24). Equation (3.20) means that $b_i^{\mu\nu}$ is a realization of the adjoint representation of the Lorentz group Lie algebra (Racah, 1965).

Now the Lagrangian depends on $\dot{\varphi}_i$ only through $\dot{\Lambda}\Lambda^{\mathrm{T}} = \Lambda\sigma\Lambda^{\mathrm{T}}$. Thus if we define

$$S^{\mu\nu} = \frac{\partial L}{\partial \sigma_{\mu\nu}} \tag{3.21}$$

we may write

$$T^i = \frac{\partial L}{\partial \dot{\varphi}_i} = \frac{1}{2}\frac{\partial \sigma^{\mu\nu}}{\partial \dot{\varphi}_i}\frac{\partial L}{\partial \sigma^{\mu\nu}} = \frac{1}{2}\, a_i^{\mu\nu}(\varphi)\, S_{\mu\nu}\,. \tag{3.22}$$

From Eq. (3.19), we have also

$$S^{\mu\nu} = b_i^{\mu\nu}(\varphi)\, T^i. \tag{3.23}$$

$S^{\mu\nu}$ is thus a combination of canonical coordinates and canonical momenta somewhat similar to the spinless Lorentz group generator $(x^\mu P^\nu - x^\nu P^\mu)$.

Using Eqs. (3.14) to (3.23), we can now express the Poisson brackets (3.9) in terms of $S^{\mu\nu}$ and $\Lambda^\mu_\nu$. The result is

$$\begin{aligned}\{A, B\} = {} & \frac{\partial A}{\partial r_\mu}\frac{\partial B}{\partial P^\mu} - \frac{\partial A}{\partial P^\mu}\frac{\partial B}{\partial r_\mu} + S^{\mu\nu}\frac{\partial A}{\partial S^\mu{}_\lambda}\frac{\partial B}{\partial S^{\nu\lambda}} \\ & + \Lambda^\mu{}_\alpha \frac{\partial A}{\partial \Lambda^{\mu\beta}}\frac{\partial B}{\partial S_{\alpha\beta}} - \Lambda^\mu{}_\alpha \frac{\partial B}{\partial \Lambda^{\mu\beta}}\frac{\partial A}{\partial S_{\alpha\beta}}\,,\end{aligned} \tag{3.24}$$

where $\Lambda$ and $S$ are to be considered as independent variables. We then find the canonical Poisson brackets

$$\begin{aligned} &\{P^\mu, x^\nu\} = -g^{\mu\nu} \\ &\{S^{\mu\nu}, S^{\alpha\beta}\} = + C^{\mu\nu\alpha\beta}_{\sigma\tau} S^{\sigma\tau} \\ &\{S^{\mu\nu}, \Lambda^{\alpha\beta}\} = \Lambda^{\alpha\nu} g^{\mu\beta} - \Lambda^{\alpha\mu} g^{\nu\beta} \\ &\{P^\mu, P^\nu\} = \{x^\mu, x^\nu\} = \{\Lambda^{\mu\alpha}, \Lambda^{\nu\beta}\} = 0\,. \end{aligned} \tag{3.25}$$

Therefore $S^{\mu\nu}$ obeys the Lorentz algebra by itself and generates the right (space-axis) Lorentz transformation on $\Lambda^{\alpha\beta}$. Note that

$$\{S^{\mu\nu}, \Lambda^{\alpha\beta}\Lambda^\gamma_\beta\} = 0\,.$$

Thus the brackets (3.24) are compatible with the strong validity of the constraints (3.1) even when all of the $S^{\mu\nu}$ and $\Lambda^{\alpha\beta}$ are treated as independent variables when taking partial derivatives. *In effect, we have sidestepped one stage of the Dirac procedure* by starting with brackets which are consistent with Eq. (3.1).

Next, we show how to rewrite the Euler equations (3.8) in terms of $S^{\mu\nu}$ and $\sigma^{\mu\nu}$. Using Eqs. (3.17) and (3.22), we find

$$0 = \frac{1}{2}\,\partial_j\, a_i^{\mu\nu}\,\dot{\varphi}_j\, S_{\mu\nu} + \frac{1}{2}\,\dot{S}_{\mu\nu}\, a_i^{\mu\nu} - \Lambda^\alpha_\beta\, a_i^{\beta\gamma}\frac{\partial L}{\partial \Lambda_{\alpha\gamma}} - \frac{1}{2}\, S_{\alpha\beta}\,\partial_i\, a_j^{\alpha\beta}\,\dot{\varphi}_j\,.$$

Now Eqs. (3.18) and (3.19) let us reexpress this as

$$\dot{S}^{\mu\nu} - S^{\mu\lambda}\,\sigma_\lambda{}^\nu + \sigma^{\mu\lambda}\,S_\lambda{}^\nu - \Lambda^{\lambda\mu}\,\frac{\partial L}{\partial \Lambda^\lambda_\nu} + \Lambda^{\lambda\nu}\,\frac{\partial L}{\partial \Lambda^\lambda_\mu} = 0\,. \tag{3.26}$$

One can verify explicitly that the choice (3.6) for the form of the Lagrangian implies

$$\begin{aligned} &- S^{\mu\lambda}\,\sigma_\lambda{}^\nu + \sigma^{\mu\lambda}\,S_\lambda{}^\nu - \Lambda^{\lambda\mu}\,\frac{\partial L}{\partial \Lambda^\lambda_\nu} + \Lambda^{\lambda\nu}\,\frac{\partial L}{\partial \Lambda^\lambda_\mu} \\ &= u^\mu\,P^\nu - u^\nu\,P^\mu. \end{aligned} \tag{3.27}$$

Thus the spin Euler equation can also be written

$$\dot{S}^{\mu\nu} + u^\mu\,P^\nu - u^\nu\,P^\mu = 0\,. \tag{3.28}$$

In order to forge a connection between Eqs. (3.26-28) and the familiar equations for the rotator (Goldstein, 1950).

$$\dot{\mathbf{L}}_{\text{space}} = \dot{\mathbf{L}}_{\text{body}} + \boldsymbol{\omega}_{\text{space}} \times \mathbf{L}_{\text{space}} = (\text{external torque})\,, \tag{3.29}$$

we define the body-fixed spin

$$\bar{S}^{\mu\nu} = \Lambda^{\mu\alpha}\,S_{\alpha\beta}\,\Lambda^{\nu\beta}. \tag{3.30}$$

Then when we refer $\bar{S}^{\mu\nu}$ to the space coordinate system, we find that

$$\dot{S}^{\mu\nu} = \Lambda^{\alpha\mu}\,\dot{\bar{S}}_{\alpha\beta}\,\Lambda^{\beta\nu} - \sigma^{\mu\lambda}\,S_\lambda{}^\nu + S^{\mu\lambda}\,\sigma_\lambda{}^\nu. \tag{3.31}$$

Comparing to Eq. (3.26), we find that the motion of the body-fixed spin in its own frame is

$$\dot{\bar{S}}^{\mu\nu} = -\Lambda^{\mu\lambda}\,\frac{\partial L}{\partial \Lambda^\lambda_\nu} + \Lambda^{\nu\lambda}\,\frac{\partial L}{\partial \Lambda^\lambda_\mu}\,. \tag{3.32}$$

Finally, we observe that if we define

$$M^{\mu\nu} = x^\mu\,P^\nu - x^\nu\,P^\mu + S^{\mu\nu}, \tag{3.33}$$

then the Euler equations (3.8) and (3.28) imply

$$\frac{dM^{\mu\nu}}{d\tau} = 0\,. \tag{3.34}$$

We now show that $M^{\mu\nu}$ is precisely the conserved generator of right (space-axis) Lorentz transformations. Under a right Lorentz transformation, the Lagrangian is invariant, so

$$\begin{aligned} 0 = \delta_\omega L &= \frac{\partial L}{\partial x^\mu}\,\delta_\omega x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\,\delta_\omega \dot{x}^\mu + \frac{\partial L}{\partial \varphi_i}\,\delta_\omega \varphi_i + \frac{\partial L}{\partial \dot{\varphi}_i}\,\delta_\omega \dot{\varphi}_i \\ &= \frac{d}{d\tau}\,(P_\mu\,\delta_\omega x^\mu + T^i\,\delta_\omega \varphi_i)\,, \end{aligned} \tag{3.35}$$

where we used the Euler equations (3.8). Now the infinitesimal right Lorentz transformation $\delta_\omega$ has the effect

$$\begin{aligned} \delta_\omega x^\mu &= \omega_{\alpha\beta}(x_\alpha g^{\beta\mu} - x^\beta g^{\alpha\mu}) \\ \delta_\omega \Lambda^{\mu\nu} &= \omega_{\alpha\beta}(\Lambda^{\mu\alpha} g^{\nu\beta} - \Lambda^{\mu\beta} g^{\nu\alpha}) . \end{aligned} \tag{3.36}$$

From Eq. (3.17), we find

$$\delta_\omega \varphi_i = \tfrac{1}{2} b_i^{\mu\nu} \Lambda_{\lambda\mu} \delta_\omega \Lambda^\lambda{}_\nu = \omega_{\alpha\beta} b_i^{\alpha\beta}. \tag{3.37}$$

For arbitrary $\omega_{\alpha\beta}$, Eq. (3.35) implies Eq. (3.34) and $M^{\mu\nu}$ is the conserved Noether current of right Lorentz transformations. By explicit computation using Eqs. (3.25), we see that $M^{\mu\nu}$ and $P^\mu$ obey the Poincaré algebra (2.25).

## B. Constraints on Top Lagrangian

We now examine the constraints which must appear in the top Lagrangian if it is to describe a reasonable physical system. First we define the set of ten right Poincaré invariants

$$\begin{aligned} a_i = (&\Lambda^0{}_\nu \dot{x}^\nu , \Lambda^1{}_\nu \dot{x}^\nu , \Lambda^2{}_\nu \dot{x}^\nu , \Lambda^3{}_\nu \dot{x}^\nu , \\ &\dot{\Lambda}^{0\nu} \Lambda^1{}_\nu , \dot{\Lambda}^{0\nu} \Lambda^2{}_\nu , \dot{\Lambda}^{0\nu} \Lambda^3{}_\nu , \dot{\Lambda}^{2\nu} \Lambda^3{}_\nu , \dot{\Lambda}^{3\nu} \Lambda^1{}_\nu , \dot{\Lambda}^{1\nu} \Lambda^2{}_\nu) \end{aligned} \tag{3.38}$$

composed of the variables (3.4) which appear in the Lagrangian. The derivatives of L with respect to these variables will be written

$$L_i(a) = \frac{\partial L(a)}{\partial a_i} . \tag{3.39}$$

$\tau$-reparametrization invariance of the action is assured by the requirement that L be *homogeneous of degree one* in the velocities; $L(a)$ is thus taken to satisfy Euler's differential equation for homogeneous functions,

$$L(a) = \sum_{i=1}^{10} a_i L_i \equiv a_i L_i . \tag{3.40}$$

Since the ten momenta $P^\mu$, $S^{\mu\nu}$ are homogeneous of degree *zero* in the velocities $a_i$, at most nine momenta can be independent; there must exist a *trajectory constraint* relating the momenta,

$$f(\Lambda^\mu{}_\nu P^\nu, \Lambda^{\mu\alpha} S_{\alpha\beta} \Lambda^{\nu\beta}) \approx 0 . \tag{3.41}$$

We will restrict ourselves in what follows to constraints of the form

$$P^2 + M^2(\bar{S}^{\mu\nu}) \approx 0 \tag{3.42a}$$

where $M^2$ is *not* a constant but is a function of the body-fixed angular momentum $\bar{S} = \Lambda S \Lambda^T$ satisfying

$$M^2(\bar{S}^{ij}, \bar{S}^{0i}) = + M^2(\bar{S}^{ij}, -\bar{S}^{0i}) . \tag{3.42 b}$$

The motivation for this choice will become apparent shortly.

Next, we must force our top to have only the usual three spin degrees of freedom in the nonrelativistic limit. In order to eliminate three of the six components of $S^{\mu\nu}$, *we hereafter restrict ourselves to Lagrangians for which the relation*

$$V^\mu \equiv S^{\mu\nu} P_\nu \approx 0 \tag{3.43}$$

*and Eq.* (3.42) *follow from the form of the Lagrangian alone.* (Since $P_\mu V^\mu \equiv 0$, Eq. (3.43) contains only three independent constraints). By choosing the constraint (3.43), we effectively pick the physical meaning of the coordinate $x^\mu$ of our spinning particle. According to Pryce (1948), Eq. (3.43) identifies $x^\mu$ as a body's *rest frame center of momentum*, which transforms as a four-vector.

Equation (3.43) must for consistency have a weakly vanishing derivative:

$$\dot{V}^\mu = \dot{S}^{\mu\nu} P_\nu + S^{\mu\nu} \dot{P}_\nu \approx 0 . \tag{3.44}$$

Combining this information with the Euler equations (3.8) and (3.28), we find

$$u^\mu \approx P^\mu \left( \frac{u \cdot P}{P^2} \right) . \tag{3.45}$$

Putting this back into Eq. (3.28), we have

$$\dot{S}^{\mu\nu} \approx 0 = \text{(external torque)}, \tag{3.46}$$

thus completing the connection between Eqs. (3.29) and (3.31).

Now we note the general identity

$$S^{*\mu\lambda} S_\lambda^{\ \nu} = S^{\mu\lambda} S_\lambda^{*\nu} = -\frac{1}{4} g^{\mu\nu} (S^{\alpha\beta} S^*_{\alpha\beta})$$

where

$$S^{*\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} S_{\lambda\sigma} ,$$

and $\varepsilon^{\mu\nu\lambda\sigma}$ is the totally antisymmetric pseudo-tensor, with $\varepsilon^{0123} = +1$. Multiplying by $P_\mu$, we find that the constraint (3.43) directly implies

$$S^{\mu\nu} S^*_{\mu\nu} \approx 0 . \tag{3.47}$$

Thus, for example, from Eqs. (3.45) and (3.26),

$$\begin{aligned} S^*_{\mu\nu} \Lambda^{\lambda\mu} \frac{\partial L}{\partial \Lambda^\lambda_{\ \nu}} &\approx 0 , \\ \bar{S}^*_{\mu\nu} \bar{S}^{\mu\nu} &\approx 0 . \end{aligned} \tag{3.48}$$

(Note that

$$(\bar{S}^{\mu\nu})^* = \tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,\Lambda_{\alpha\gamma}\,S^{\gamma\delta}\,\Lambda_{\beta\delta} = \Lambda^{\mu\alpha}\left(\tfrac{1}{2}\,\varepsilon_{\alpha\beta\gamma\delta}\,S^{\gamma\delta}\right)\Lambda^{\nu\beta}$$
$$= \overline{(S^{*\mu\nu})}$$

due to $\Lambda^{\mu\alpha}\,\Lambda^{\nu\beta}\,\Lambda^{\lambda\gamma}\,\Lambda^{\sigma\delta}\,\varepsilon_{\alpha\beta\gamma\delta} = \mathrm{Det}\,\Lambda\,\varepsilon^{\mu\nu\lambda\sigma} = \varepsilon^{\mu\nu\lambda\sigma}$).

We will not deal here with the problem of finding Lagrangians satisfying Eq. (3.43); procedures for doing so are given in Hanson and Regge (1974) for the case of the spherical top. A simple example of a suitable spherical top Lagrangian is

$$(3.49\,a)\qquad L = -\,2^{-\frac{1}{2}}\{Au^2 - B\sigma\cdot\sigma + [(A\,u^2 - B\sigma\cdot\sigma)^2 - 8\,B\,(Au\sigma\sigma u - 2\,B\,\mathrm{Det}\,\sigma)]^{\frac{1}{2}}\}^{\frac{1}{2}}$$

where $u^2 = u_\mu\,u^\mu$, $\sigma\cdot\sigma = \sigma^{\mu\nu}\,\sigma_{\mu\nu}$, $u\sigma\sigma u = u_\mu\,\sigma^{\mu\nu}\,\sigma_{\nu\lambda}\,u^\lambda$, $\mathrm{Det}\,\sigma = (\sigma^{\mu\nu}\,\sigma^*_{\mu\nu})^2/16$, and A and B are constants. Direct computation confirms that the constraints

$$S^{\mu\nu}\,P_\nu \approx 0$$
$$(3.49\,b)\qquad P^\mu\,P_\mu - \frac{A}{4B}\,S_{\alpha\beta}\,S^{\alpha\beta} - A \approx 0$$

follow from the form of the Lagrangian alone. Lagrangians also exist for which the mass and spin are *separate* unrelated constants:

$$(3.50)\qquad P^\mu\,P_\mu \approx A$$
$$\tfrac{1}{2}\,S_{\mu\nu}\,S^{\mu\nu} \approx B\,.$$

## C. Dirac Treatment of Top Constraints

We now consider the dynamics of a relativistic top whose Lagrangian gives rise to the following weak constraints among the canonical momenta:

$$(3.51)\qquad P^2 + M^2(\bar{S}) \approx 0$$
$$V^\mu = S^{\mu\nu}\,P_\nu \approx 0\,.$$

The canonical Hamiltonian is

$$(3.52)\qquad H_c = P^\mu\,u_\mu + T^i\,\dot{\varphi}_i - L = P^\mu\,u_\mu + \tfrac{1}{2}\,\sigma_{\mu\nu}\,S^{\mu\nu} - L \equiv 0$$

and so may be taken as the initial first class Hamiltonian. However, the constraints (3.51) are not all first class, since

$$(3.53)\qquad \{P^2 + M^2, V^\mu\} \approx 0$$
$$\{V^\mu, V^\nu\} \approx S^{\mu\nu}\,P^2\,.$$

There are nevertheless two first class combinations of the constraints (3.51), which we take to be

$$\begin{aligned} \psi_1 &= P^2 + M^2(\bar{S}) \approx 0 \\ \psi_2 &= S^*_{\alpha\beta} S^{\alpha\beta} \approx 0 \end{aligned} \tag{3.54}$$

We observed in Eq. (3.47) that $\psi_2 \approx 0$ follows directly from $V^\mu \approx 0$. Our first class Hamiltonian may now be taken to be

$$H = v_1(\tau)\,\psi_1 + v_2(\tau)\,\psi_2\,. \tag{3.55}$$

Since consistency requires

$$\dot{V}^\mu = \{ V^\mu, H \} \approx 0\,,$$

any multiples of $V^\mu$ which might be added to H must have vanishing coefficients and there are no secondary constraints.

Defining the tensor

$$I_{\alpha\beta} = \frac{\partial M^2(S)}{\partial S^{\alpha\beta}} \tag{3.56}$$

we compute the velocities to be

$$\begin{aligned} u^\mu &= \{ x^\mu\,, H \} = 2v_1\, P^\mu \\ \sigma^{\mu\nu} &= \Lambda^{\lambda\mu}\{\Lambda_\lambda{}^\nu, H\} = v_1\, I^{\mu\nu} + 4\, v_2\, S^{*\mu\nu}\,. \end{aligned} \tag{3.57}$$

As in the spinless case, we may set $v_1 = (-u^2)^{\frac{1}{2}}/(2M)$, and similarly restrict the form of $v_2$. Nevertheless, there remains an arbitrariness in $v_1$ and $v_2$ which is eliminated only when we choose gauge constraints.

Equations (3.51) represent four independent constraints. The two combinations (3.54) are first class while the two remaining independent combinations, say

$$\begin{aligned} V^1 &\approx 0 \\ V^2 &\approx 0 \end{aligned} \tag{3.58}$$

are *second class*:

$$\begin{aligned} C^{ij} &= \{ V^i\,, V^j \} = P_\mu\, P^\mu \begin{vmatrix} 0 & S^{12} \\ S^{21} & 0 \end{vmatrix} \\ (C^{-1})^{ij} &= \frac{1}{P^\mu\, P_\mu\, S^{12}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{3.59}$$

Strictly speaking, we must replace all Poisson brackets by the corresponding Dirac brackets (1.21) even in our preliminary first-class Hamiltonian system.

Now let us proceed directly to impose a gauge condition which is effectively the " phase space conjugate " of the constraint $V^\mu \approx 0$ on the momenta. Physically, we want $\Lambda^\mu{}_\nu$ to be a pure rotation in the rest frame of $P^\mu$, just as $S^{\mu\nu}$ becomes a pure three-angular-momentum in the rest frame. We therefore conjecture that

$$\xi^\mu = \Lambda_0{}^\mu - \frac{P^\mu}{P} \approx 0 \tag{3.60}$$

is a consistent constraint choice, where

$$P = (-P^\mu P_\mu)^{\frac{1}{2}} \tag{3.61}$$

and $\Lambda^0{}_0 = \Lambda_0{}^0 = -\Lambda^{00} = 1$ for a pure space rotation. Since

$$P\dot{\xi}^\mu \xi_\mu + 2 P^\mu \xi_\mu = 0 , \tag{3.62}$$

only three of the four constraints (3.60) are independent.

Now we use Eqs. (3.31), (3.46) and (3.57) to show

$$\dot{\bar{S}}^{0i} \approx v_1 (I^{0\lambda} \bar{S}_\lambda{}^i - \bar{S}^{0\lambda} \bar{I}_\lambda{}^i) . \tag{3.63}$$

where $\bar{I}^{\mu\nu} = \Lambda^{\mu\alpha} I_{\alpha\beta} \Lambda^{\nu\beta}$. Then if Eq. (3.42 *b*) holds and $M^2(\bar{S})$ is *even* under $\bar{S}^{0i} \to -\bar{S}^{0i}$, $\bar{I}^{0i}(\bar{S})$ must be *odd*:

$$\frac{\partial M^2}{\partial \bar{S}_{0i}} = \bar{I}^{0i}(\bar{S}^{ij}, \bar{S}^{0i}) = -\bar{I}^{0i}(\bar{S}^{ij}, -\bar{S}^{0i}) . \tag{3.64}$$

We conclude that if we set

$$\bar{S}^{0i} \approx 0 \tag{3.65}$$

at some point $\tau_0$, we have also

$$\bar{I}^{0i} = 0 . \tag{3.66}$$

Then from Eq. (3.63) we know that $\dot{\bar{S}}^{0i} \approx 0$ and so $\bar{S}^{0i}$ vanishes *for all* $\tau$. Equation (3.65) is therefore an *invariant relation* provided we restrict ourselves to trajectories obeying Eq. (3.42).

We now define

$$\rho^\mu = \Lambda^{\mu\nu} P_\nu .$$

Since $V^i \approx 0$ implies

$$\bar{S}^{ij} \rho_j \approx 0 ,$$

we must have

$$\rho^i = \alpha \bar{S}^{*0i} \equiv \alpha \bar{J}^i . \tag{3.67}$$

From Eq. (3.57), the equation of motion for $\rho^\mu$ is

$$\dot{\rho}^\mu = + v_1 \bar{I}^{\mu\nu} \rho_\nu + 4 v_2 \bar{S}^{*\mu\nu} \rho_\nu .$$

We now choose the *gauge*

$$\rho^0 \approx P \equiv (- P_\mu P^\mu)^{\frac{1}{2}},$$

so that

$$\rho^\mu \rho_\mu \equiv P^2 \approx P^2 + \alpha^2 \bar{J}^2$$

or

$$\alpha = 0 .$$

Thus from Eq. (3.67), we find

$$\rho^i \approx 0 .$$

The equation of motion of $\rho^i$ is now

$$0 \approx \dot{\rho}^i \approx v_1 \bar{I}^{ij} \rho_j - 4 v_2 \bar{S}^{*i0} P \approx + 4 v_2 P \bar{S}^{*0i} .$$

Consistency then forces

$$v_2 = 0 ,$$

so that the gauge choice $\rho^0 \approx P$ has fixed one of the arbitrary functions in the Hamiltonian as expected. Furthermore

$$\rho^\mu = \Lambda^{\mu\nu} P_\nu \approx \delta_0^\mu P = - g^{\mu 0} P$$

which is exactly equivalent to Eq. (3.50). We conclude that the constraints

$$\xi^\mu = \Lambda_0{}^\mu - P^\mu / P \approx 0$$

consist of *one gauge condition* and *two invariant relations.*

We may now compute preliminary Dirac brackets consistent with the six independent constraints

$$\begin{aligned} V^i &= S^{i\nu} P_\nu \approx 0 \\ \xi^i &= \Lambda_0{}^i - P^i / P \approx 0 \end{aligned} \tag{3.68}$$

which we write as

$$\varphi^i = (V^1, V^2, V^3, \xi^1, \xi^2, \xi^3), \qquad i = 1, \cdots, 6 .$$

The matrices needed to compute the preliminary brackets are

$$C^{ij} = \{\varphi^i, \varphi^j\} = \begin{vmatrix} S^{ij} P^\mu P_\mu & - (P\delta^{ij} + P^i P^j / P) \\ (P\delta^{ij} + P^i P^j / P) & 0 \end{vmatrix} \tag{3.69}$$

and its inverse

$$(3.70)\qquad (C^{-1})^{ij} = \begin{vmatrix} 0 & \frac{1}{P}\left(\delta^{ij} - \frac{P^i P^j}{(P^0)^2}\right) \\ -\frac{1}{P}\left(\delta^{ij} - \frac{P^i P^j}{(P^0)^2}\right) & -\left(S^{ij} + \frac{P^i S^{jk} P^k - P^j S^{ik} P^k}{(P^0)^2}\right) \end{vmatrix}$$

where $P = (-P^\mu P_\mu)^{\frac{1}{2}}$.

We may put our brackets into a manifestly covariant form by rewriting $(C^{-1})_{ij}$ as

$$(3.71)\qquad (C^{-1})_{ij} = T_{i\alpha}\,(C^{-1})^{\alpha\beta}\,T_{j\beta}, \qquad \begin{cases} i, j = 1, \cdots, 6 \\ \alpha, \beta = 1, \cdots, 8 \end{cases}$$

where

$$(3.72\,a)\qquad T_{i\alpha} = \begin{vmatrix} T_{k\mu} & 0 \\ 0 & T_{k\mu} \end{vmatrix}, \qquad \begin{cases} k = 1, 2, 3 \\ \mu = 0, 2, 2, 3 \end{cases}$$

and

$$(3.72\,b)\qquad T_{k\mu} = \begin{vmatrix} -P^1/P^0 & +1 & 0 & 0 \\ -P^2/P^0 & 0 & +1 & 0 \\ -P^3/P^0 & 0 & 0 & +1 \end{vmatrix}.$$

The matrix $C^{\alpha\beta}$ and its inverse are simply

$$(3.73)\qquad C^{\alpha\beta} = \begin{vmatrix} P_\alpha P^\alpha S^{\mu\nu} & -Pg^{\mu\nu} \\ Pg^{\mu\nu} & 0 \end{vmatrix}$$

$$(3.74)\qquad (C^{-1})^{\alpha\beta} = \begin{vmatrix} 0 & \frac{g^{\mu\nu}}{P} \\ -\frac{g^{\mu\nu}}{P} & -S^{\mu\nu} \end{vmatrix}$$

$T_{k\mu}$ is now used to convert $V_k$ and $\xi_k$ into $V_\mu$ and $\xi_\mu$ by using the identities $P^\mu V_\mu = 0$ and $P^\mu \xi_\mu = -P\xi^\mu \xi_\mu$ as follows:

$$(3.75)\qquad \begin{aligned} \{A, V^k\}\,T_{k\mu} &\approx \{A, V^k\,T_{k\mu}\} = \{A, V_\mu\}, \\ \{A, \xi^k\}\,T_{k\mu} &\approx \{A, \xi^k\,T_{k\mu}\} \approx \{A, \xi_\mu\}. \end{aligned}$$

Defining the eight component object

$$\varphi^\alpha = (V^0, V^1, V^2, V^3, \xi^0, \xi^1, \xi^2, \xi^3) \qquad \alpha = 1, \cdots, 8,$$

we may use Eq. (1.21) to write the preliminary Dirac brackets as

$$(3.76)\qquad \{A, B\}' \approx \{A, B\} - \frac{1}{P}\{A, V^\mu\}\{\xi_\mu, B\} + \\ + \frac{1}{P}\{A, \xi^\mu\}\{V_\mu, B\} + \{A, \xi_\mu\}\,S^{\mu\nu}\{\xi_\nu, B\},$$

where we use a prime to distinguish the preliminary brackets from the final brackets.

The explicit formulas for the primed brackets of the canonical variables are

(3.77)

1. $\{ P^\mu, x^\nu \}' = -g^{\mu\nu}$
2. $\{ x^\mu, x^\nu \}' = -S^{\mu\nu}/(P_\alpha P^\alpha)$
3. $\{ P^\mu, P^\nu \}' = 0$
4. $\{ P^\mu, S^{\nu\lambda} \}' = 0$
5. $\{ x^\mu, S^{\nu\lambda} \}' = (S^{\mu\nu} P^\lambda - S^{\mu\lambda} P^\nu)/(P_\alpha P^\alpha)$
6. $\{ S^{\mu\nu}, S^{\alpha\beta} \}' = S^{\mu\alpha}(g^{\nu\beta} - P^\nu P^\beta/P_\lambda P^\lambda) - S^{\mu\beta}(g^{\nu\alpha} - P^\nu P^\alpha/P_\lambda P^\lambda)$
   $+ S^{\alpha\nu}(g^{\mu\beta} - P^\mu P^\beta/P_\lambda P^\lambda) - S^{\beta\nu}(g^{\mu\alpha} - P^\mu P^\alpha/P_\lambda P^\lambda)$
7. $\{ \Lambda^{\mu\nu}, x^\alpha \}' = \Lambda^\mu{}_\beta (P^\nu g^{\alpha\beta} - P^\beta g^{\alpha\nu})/(P_\lambda P^\lambda)$
8. $\{ \Lambda^{\mu\nu}, P^\alpha \}' = 0$
9. $\{ \Lambda^{\mu\nu}, \Lambda^{\alpha\beta} \}' = 0$
10. $\{ \Lambda^{\mu\nu}, S^{\alpha\beta} \}' = \Lambda^{\mu\alpha}(g^{\nu\beta} - P^\nu P^\beta/P_\lambda P^\lambda) - \Lambda^{\mu\beta}(g^{\nu\alpha} - P^\nu P^\alpha/P_\lambda P^\lambda)$
    $+ \Lambda^\mu{}_\sigma P^\sigma (g^{\alpha\nu} P^\beta - g^{\beta\nu} P^\alpha)/(P_\lambda P^\lambda).$

These brackets are now compatible with setting the constraints $V^\mu$ and $\xi^\mu$ identically zero. Note that the position variables $x^\mu$ have *nonzero* brackets (Pryce, 1948). Nevertheless, the Poincaré algebra (2.25) is still satisfied in the primed bracket system and $x^\mu$ transforms like a four-vector.

Since $V^\mu$ and $\xi^\mu$ vanish strongly, $\bar{S}^{0i}$ vanishes also:

$$\bar{S}^{0i} = \Lambda^0{}_\alpha S^{\alpha\beta} \Lambda^i{}_\beta = -P_\alpha S^{\alpha\beta} \Lambda^i{}_\beta (-P^2)^{-\frac{1}{2}} = 0 . \tag{3.78}$$

Furthermore, from Eq. (3.64) we find

$$0 = \bar{I}^{0\mu} = \Lambda^0{}_\alpha I^{\alpha\beta} \Lambda^\mu{}_\beta = -P_\alpha I^{\alpha\beta} \Lambda^\mu{}_\beta (-P^2)^{-\frac{1}{2}} . \tag{3.79}$$

As a consequence, $I^{\alpha\beta}$ now obeys

$$I^{\alpha\beta} P_\beta = 0 . \tag{3.80}$$

We can form a Hamiltonian from the remaining first class constraint,

$$H = v(\tau)(P^\mu P_\mu + M^2(\bar{S}^{ij})) . \tag{3.81}$$

$M^2$ no longer depends on $\bar{S}^{0i}$ because of Eq. (3.78).
The velocities are simply

$$\begin{aligned} u^\mu &= \{ x^\mu, H \}' = 2 v P^\mu \\ \sigma^{\mu\nu} &= \Lambda^{\lambda\mu} \{ \Lambda_\lambda{}^\nu, H \}' = v I^{\mu\nu} . \end{aligned} \tag{3.82}$$

We now eliminate all arbitrary functions by selecting a gauge constraint conjugate to the trajectory constraint. Making use of the iterative property of the Dirac brackets, we choose the final constraints

$$\begin{aligned} \varphi_1 &= P^\mu P_\mu + M^2(\bar{S}^{ij}) \approx 0 \\ \varphi_2 &= x^0 - \tau \approx 0 \end{aligned} \tag{3.83}$$

and impose them on the *primed* brackets (3.77).

The required matrices are

$$\begin{aligned} C^{ij} &= \{\varphi_i, \varphi_j\}' = \begin{bmatrix} 0 & -2P^0 \\ 2P^0 & 0 \end{bmatrix} \\ C^{-1}_{ij} &= \frac{1}{2P^0}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned} \tag{3.84}$$

Equation (1.21) with all Poisson brackets replaced by the primed brackets (3.77) gives the form of the final Dirac brackets, which we now denote with stars. The results are

(3.85)

1. $\{P^\mu, x^\nu\}^* = -g^{\mu\nu} + g^{\mu 0} P^\nu/P^0$
2. $\{x^\mu, x^\nu\}^* = +S^{\mu\nu}/M^2 - (P^\mu S^{0\nu} - P^\nu S^{0\mu})/(M^2 P^0) =$
$= S^{\mu\nu}/M^2 - (P^\nu S^{\mu i} P^i - P^\mu S^{\nu i} P^i)/(M^2 (P^0)^2)$
3. $\{P^\mu, P^\nu\}^* = 0$
4. $\{P^\mu, S^{\nu\lambda}\}^* = 0$
5. $\{x^\mu, S^{\nu\lambda}\}^* = -\frac{1}{M^2}(S^{\mu\nu} P^\lambda - S^{\mu\lambda} P^\nu) + \frac{P^\mu}{P^0 M^2}(S^{0\nu} P^\lambda - S^{0\lambda} P^\nu)$
6. $\{S^{\mu\nu}, S^{\alpha\beta}\}^* = S^{\mu\alpha}(g^{\nu\beta} + P^\nu P^\beta/M^2) - S^{\mu\beta}(g^{\nu\alpha} + P^\nu P^\alpha/M^2)$
$+ S^{\alpha\nu}(g^{\mu\beta} + P^\mu P^\beta/M^2) - S^{\beta\nu}(g^{\mu\alpha} + P^\mu P^\alpha/M^2)$
7. $\{\Lambda^{\mu\nu}, x^\alpha\}^* = -\frac{1}{M^2}\Lambda^\mu_\lambda (P^\nu g^{\alpha\lambda} - P^\lambda g^{\alpha\nu}) +$
$+ \frac{1}{P^0 M^2} P^\alpha \Lambda^\mu_\lambda (P^\nu g^{0\lambda} - P^\lambda g^{0\nu}) - \frac{1}{2P^0 M^2} S^{0\alpha} \Lambda^\mu_\lambda I^{\lambda\nu}$
8. $\{\Lambda^{\mu\nu}, P^\alpha\}^* = -\frac{1}{2P^0} g^{0\alpha} \Lambda^\mu_\lambda I^{\lambda\nu}$
9. $\{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\}^* = -\frac{1}{2P^0 M^2}\Lambda^\mu_\lambda (P^\lambda g^{0\nu} - P^\nu g^{0\lambda}) \Lambda^\alpha_\gamma I^{\gamma\beta} +$
$+ \frac{1}{2P^0 M^2}\Lambda^\alpha_\gamma (P^\gamma g^{0\beta} - P^\beta g^{0\gamma}) \Lambda^\mu_\lambda I^{\lambda\nu}$
10. $\{\Lambda^{\mu\nu}, S^{\alpha\beta}\}^* = \Lambda^{\mu\alpha} g^{\nu\beta} - \Lambda^{\mu\beta} g^{\nu\alpha} - \frac{1}{2P^0 M^2}\Lambda^\mu_\lambda I^{\lambda\nu}(P^\alpha S^{0\beta} - P^\beta S^{0\alpha}) -$
$- \frac{1}{M^2}\Lambda^\mu_\lambda (P^\nu P^\alpha g^{\lambda\beta} - P^\nu P^\beta g^{\lambda\alpha} + P^\lambda P^\beta g^{\nu\alpha} - P^\lambda P^\alpha g^{\nu\beta})$

where $M = (-P^\mu P_\mu)^{\frac{1}{2}} = [M^2(\bar{S}^{ij})]^{\frac{1}{2}}$, $I_{\alpha\beta} = \partial M^2/\partial S^{\alpha\beta}$ and $I^{\alpha\beta} P_\beta = 0$.

The Hamiltonian is

$$H = P^0 = (\mathbf{P}^2 + M^2(\bar{S}^{ij}))^{\frac{1}{2}} \tag{3.86}$$

while Hamilton's equations of motion are

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + \{A, (\mathbf{P}^2 + M^2(\bar{S}^{ij}))^{\frac{1}{2}}\}' \\ &= \frac{\partial A}{\partial t} + \{A, P^0\}^*. \end{aligned} \tag{3.87}$$

Here we have used the traditional notation $x^0 \equiv t$ to emphasize that $x^0$ is now a parameter and not a canonical variable.

The Poincaré group generators are now written

$$\begin{aligned} P^0 &= H \\ P^i &= P^i \\ M^{0i} &= tP^i - x^i H + S^{0i} = tP^i - x^i H - S^{ij} P^j/H \\ M^{ij} &= x^i P^j - x^j P^i + S^{ij} . \end{aligned} \tag{3.88}$$

The star brackets of these quantities obey the Poincaré algebra Eq. (2.23) and the theory is again Poincaré-covariant. Applying Eq. (3.87), we confirm that the generators (3.88) are constants of the motion with respect to $x^0 = t$.

The transformation of $x^\mu$ under the Poincaré group is altered in the star brackets. $x^0 = t$ is now just a parameter, while $\boldsymbol{x}$ transforms as follows:

$$\begin{aligned} \{P^0, x^i\}^* &= \{H, x^i\}^* = -P^i/H \\ \{P^i, x^j\}^* &= -\delta^{ij} \\ \{M^{ij}, x^k\}^* &= \delta^{ik} x^j - \delta^{jk} x^i \\ \{M^{0i}, x^k\}^* &= -t\delta^{ik} + x^i P^k/H . \end{aligned} \tag{3.89}$$

The extra term in $\{M^{0i}, x^k\}^*$ is just the correction discussed in Chapter 2 which maintains the validity of the constraints (3.83) in the new Lorentz frame.

## D. Quantum Mechanics

In principle, the system of single-particle positive-energy quantum-mechanical operators describing the relativistic top can be deduced directly from the final Dirac brackets (3.85). In practice, however, this can be quite difficult unless we are able to construct from the original set of variables a reduced set of independent variables (1.85) obeying *canonical* brackets. For the top, the appropriate variables correspond to the Pryce-Newton-Wigner variables (Pryce, 1935: Newton and Wigner, 1949) supplemented by the Euler angles. We define

$$S^i = \tfrac{1}{2} \varepsilon^{ijk} S^{jk}$$

and take our new variables to be

$$
\begin{aligned}
(3.90)\qquad & q^0 = t \\
& \boldsymbol{q} = \boldsymbol{x} - \frac{\mathbf{S}\times\mathbf{P}}{H(H+M)} = x^i - \frac{S^{0i}}{H+M} \\
& \mathbf{J} = \frac{M}{H}\,\mathbf{S} + \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S})}{H(H+M)} \\
& R^{ij} = \Lambda^{ij} - \Lambda^{i0}\,P^j/(H+M) \\
& \qquad = \Lambda^{ij} - \frac{\Lambda^{ik}\,P^k\,P^j}{H(H+M)}\,. \\
& \mathbf{P} = \mathbf{P} \\
& P^0 = H = (\mathbf{P}^2 + M^2(\bar{\mathbf{J}}))^{\frac{1}{2}}.
\end{aligned}
$$

Here $M^2(\bar{S}^{ij})$ can be taken to be a function only of

$$
(3.91)\qquad \bar{J}^i = R^{ij}\,J^j
$$

because

$$
(3.92)\qquad \bar{S}^{ij} = R^{ik}(\varepsilon^{klm}\,J^m)\,R^{jl} = \varepsilon^{ijk}\,\bar{J}^k\,.
$$

Here we have observed that $R^{ij}$ is an orthonormal matrix,

$$
(3.93)\qquad R^{ik}\,R^{jk} = R^{ki}\,R^{kj} = \delta^{ij}
$$

and used the identity $\varepsilon^{ijk}\,R^{ai}\,R^{bj}\,R^{ck} = \varepsilon^{abc}\,\mathrm{Det}\,R = \varepsilon^{abc}$.

Defining

$$
(3.94)\qquad I^i = \frac{\partial M^2(\bar{\mathbf{J}})}{\partial J^i} = \frac{\partial M^2(\bar{\mathbf{J}})}{\partial \bar{J}^l}\,R^{li} = \bar{I}^l\,R^{li},
$$

we find by direct computation from Eqs. (3.90) and (3.85) the following brackets:

$$
\begin{aligned}
(3.95)\quad 1.\quad & \{P^i, q^j\}^* = -\delta^{ij} \\
& \{H, q^j\}^* = -P^j/H \\
2.\quad & \{q^i, q^j\}^* = 0 \\
3.\quad & \{P^\mu, P^\nu\}^* = 0 \\
4.\quad & \{P^\mu, J^j\}^* = 0 \\
5.\quad & \{q^i, J^j\}^* = 0 \\
6.\quad & \{J^i, J^j\}^* = \varepsilon^{ijk}\,J^k \\
7.\quad & \{R^{ij}, q^k\}^* = 0 \\
8.\quad & \{R^{ij}, P^k\}^* = 0 \\
& \{R^{ij}, H\}^* = -\frac{1}{2H}\,R^{ik}\,\varepsilon^{jkl}\,I^l \\
9.\quad & \{R^{ij}, R^{lm}\}^* = 0 \\
10.\quad & \{J^i, R^{lj}\}^* = \varepsilon^{ijk}\,R^{lk}.
\end{aligned}
$$

Equations (3.95) now form the basis for our quantum mechanical system. We begin by choosing the Hilbert space norm

$$(\Phi\ ,\ \Phi) = \int \mathrm{d}^3\, x \Phi^*(\boldsymbol{x}\ ,\ t)\ \Phi(\boldsymbol{x}\ ,\ t)\ . \tag{3.96}$$

We then take

$$\begin{aligned} & t = q^0 = c\text{-number} \\ & \mathrm{M}^2 \equiv -\ \mathrm{P}^\mu\ \mathrm{P}_\mu = \mathrm{M}^2(\bar{\mathbf{J}}) = q\text{-number}, \end{aligned} \tag{3.97}$$

and the Hamiltonian

$$\mathrm{H} = (\mathbf{P}^2 + \mathrm{M}^2(\bar{\mathbf{J}}))^{\frac{1}{2}}, \tag{3.98}$$

where the twelve independent operators are

$$q^i,\ \mathrm{P}^i,\ \mathrm{J}^i \quad \text{and three of the } \mathrm{R}^{ij}.$$

Starting from the Dirac brackets (3.95), we postulate the commutation relations

$$\begin{aligned} & i[\mathrm{P}^i, q^j] = \delta^{ij} \\ & i[\mathrm{P}^i, \mathrm{J}^j] = i[\mathrm{P}^i, \mathrm{P}^j] = i[\mathrm{P}^i, \mathrm{R}^{ik}] = 0 \\ & i[q^i, q^j] = i[q^i, \mathrm{J}^j] = i[q^i, \mathrm{R}^{jk}] = 0 \\ & i[\mathrm{J}^i, \mathrm{J}^j] = -\varepsilon^{ijk}\ \mathrm{J}^k \\ & i[\mathrm{J}^i, \mathrm{R}^{lj}] = -\varepsilon^{ijk}\ \mathrm{R}^{lk} \\ & i[\mathrm{R}^{ij}, \mathrm{R}^{lm}] = 0\ . \end{aligned} \tag{3.99}$$

We also observe that

$$\begin{aligned} & i[\mathrm{J}^i, \bar{\mathrm{J}}^j] = 0 \\ & i[\bar{\mathrm{J}}^i, \bar{\mathrm{J}}^j] = +\varepsilon^{ijk}\ \bar{\mathrm{J}}^k \\ & i[\bar{\mathrm{J}}^i, \mathrm{R}^{jl}] = +\varepsilon^{ijk}\ \mathrm{R}^{kl}\ , \end{aligned} \tag{3.100}$$

so $\bar{\mathbf{J}}$ commutes with right vectors and transforms left vectors. The Hamiltonian (3.98) has the commutation relations

$$\begin{aligned} & i[\mathrm{H}\ ,\ q^i] = \mathrm{P}^i/\mathrm{H} \\ & i[\mathrm{H}\ ,\ \mathrm{P}^i] = 0 \\ & i[\mathrm{H}\ ,\ \mathrm{J}^i] = 0 \end{aligned} \tag{3.101}$$

We omit the explicit computation of $i[\mathrm{H}\ ,\ \mathrm{R}^{ij}]$ due to nontrivial ordering problems.

The Poincaré group algebra is generated by the operators H, $\mathbf{P}$, and

$$\begin{aligned} & \mathrm{M}^{ij} = q^i\ \mathrm{P}^j - q^j\ \mathrm{P}^i + \varepsilon^{ijk}\ \mathrm{J}^k \\ & \mathrm{M}^{0i} = t\mathrm{P}^i - \tfrac{1}{2}(q^i\ \mathrm{H} + \mathrm{H}q^i) + \frac{\varepsilon^{ijk}\ \mathrm{J}^j\ \mathrm{P}^k}{\mathrm{H} + \mathrm{M}}\ . \end{aligned} \tag{3.102}$$

The Hamilton equations of motion for an operator A are

$$\frac{dA}{dt} = i[H, A] + \frac{\partial A}{\partial t} . \tag{3.103}$$

An explicit realization of the algebras (3.99) and (3.100) in terms of differential operators is the following:

$$\begin{aligned} P^l &= \frac{1}{i}\frac{\partial}{\partial q^l} \\ J^l &= \frac{1}{i}\varepsilon^{lmn} R^{km}\frac{\partial}{\partial R^{kn}} \\ \bar{J}^l &= i\varepsilon^{lmn} R^{mk}\frac{\partial}{\partial R^{nk}} . \end{aligned} \tag{3.104}$$

To get explicit expressions for **J** and **J** in terms of the three independent Euler angles, we first define the $3\times 3$ matrices

$$[L_l]_{jk} = \frac{1}{i}\varepsilon_{ljk} \tag{3.105}$$

realizing the algebra of the $J^l$'s, and then write $R^{ij}$ as

$$R^{ij}(\alpha, \beta, \gamma) = e^{-i\alpha L_3} e^{-i\beta L_2} e^{-i\gamma L_3} \tag{3.106}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix} .$$

Then we see that **J** may be written

$$\begin{aligned} J^1 &= i\left(-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \sin\gamma\frac{\partial}{\partial\beta} + \cos\gamma\cot\beta\frac{\partial}{\partial\gamma}\right) \\ J^2 &= i\left(\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cos\gamma\frac{\partial}{\partial\beta} - \sin\gamma\cot\beta\frac{\partial}{\partial\gamma}\right) \\ J^3 &= i\frac{\partial}{\partial\gamma} \end{aligned} \tag{3.107}$$

while **J** is

$$\begin{aligned} \bar{J}^1 &= i\left(-\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} - \sin\alpha\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right) \\ \bar{J}^2 &= i\left(-\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right) \\ \bar{J}^3 &= i\frac{\partial}{\partial\alpha} . \end{aligned} \tag{3.108}$$

Thus the Schrödinger picture wave equation which is the counterpart of the Heisenberg picture equation (3.103) is

$$H\psi(t, \boldsymbol{x}; R^{ij}) = (-\nabla^2 + M^2(\mathbf{J}))^{\frac{1}{2}}\psi = i\frac{\partial\psi}{\partial t}, \tag{3.109}$$

where $\mathbf{J}$ is taken to be the operator (3.104) or (3.108). (We make no attempt here to resolve ordering problems which may arise when $M^2(\mathbf{J})$ is expressed as an operator). Equation (3.109) is nonlocal; a local equation, which could be the starting point for a quantum field theory of relativistic tops, might take the form

$$\left(-\frac{\partial^2}{\partial x_\mu\,\partial x^\mu} + M^2(\mathbf{J})\right)\psi(x^\mu; R^{ij}) = 0. \tag{3.110}$$

We conclude with the remark that Dirac's procedures can also be carried out for the electrically charged relativistic top. However, since this system becomes exceedingly complicated and is still imperfectly understood, we will not attempt to treat it here (see Hanson and Regge, 1974).

## 4. STRING MODEL

Our examples thus far have dealt only with point-particle mechanics, where all canonical variables depend only on a single parameter $\tau$. As our first example of a field theory with continuous degrees of freedom, we examine the relativistic "string model" (Goddard *et al.*, 1973; for a review, see Rebbi, 1974). We take as the action functional the reparametrization-invariant expression for a *surface* embedded in a D-dimensional Minkowski space, analogous to the reparametrization-invariant *line element* (2.4) chosen as the point-particle action. The Nambu action is then

$$S[x^\mu] = -N\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left[\left(\frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\sigma}\right)^2 - \left(\frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}\right)\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\sigma}\right)\right]^{1/2}. \tag{4.1}$$

N is a normalization with dimensions of Planck's constant divided by length-squared, and one conventionally chooses

$$N = \frac{1}{2\pi\alpha'}, \qquad \sigma_1 = 0, \qquad \sigma_2 = \pi. \tag{4.2}$$

The D canonical coordinates $x^\mu(\tau, \sigma)$ now are two-dimensional field densities with continuous degrees of freedom labeled by $\sigma$. Any transformation of the form

$$\tau \to \tilde{\tau}(\tau, \sigma), \qquad \sigma \to \tilde{\sigma}(\tau, \sigma) \tag{4.3}$$

leaves the integrand of (4.1) unchanged.

A metric is induced in the $(\tau, \sigma)$ space by its embedding in $x^\mu$-space. However, we can avoid explicit use of this metric by working exclusively with $\tau, \sigma$, and the quantities

$$\dot{x}^\mu = v^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \tau}$$
$$x'^\mu = u^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \sigma} \tag{4.4 a}$$

and

$$E = v^2, \qquad G = u^2, \qquad F = u \cdot v. \tag{4.4 b}$$

Then we may write the Lagrangian density of the string as

$$\begin{aligned} \mathscr{L} &= -N\{(u\cdot v)^2 - v^2 u^2\}^{1/2} \\ &= -N\{F^2 - EG\}^{1/2}. \end{aligned} \tag{4.5}$$

The canonical momentum $\mathscr{P}^\mu$ is then written

$$\begin{aligned} \mathscr{P}^\mu = \frac{\partial \mathscr{L}}{\partial \dot{x}_\mu} = \frac{\partial \mathscr{L}}{\partial v_\mu} &= \left(-\frac{N^2}{\mathscr{L}}\right)(v^\mu u^2 - u^\mu (v\cdot u)) \\ &= -N^2 \mathscr{L}^{-1} G v^\mu + N^2 \mathscr{L}^{-1} G u^\mu. \end{aligned} \tag{4.6}$$

It is convenient to define also

$$\begin{aligned} \Pi^\mu = \frac{\partial \mathscr{L}}{\partial x'_\mu} = \frac{\partial \mathscr{L}}{\partial u_\mu} &= \left(-\frac{N^2}{\mathscr{L}}\right)(u^\mu v^2 - v^\mu (v\cdot u)) \\ &= -N^2 \mathscr{L}^{-1} E u^\mu + N^2 \mathscr{L}^{-1} F v^\mu. \end{aligned} \tag{4.7}$$

Now we require the variations of the action functional (4.1) to vanish. For the moment, we continue to allow $\sigma_1$ and $\sigma_2$ to depend on $\tau$, thus obtaining

$$\begin{aligned} 0 = \delta S = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma\, \delta x_\mu \left(-\frac{\partial \mathscr{P}^\mu}{\partial \tau} - \frac{\partial \Pi^\mu}{\partial \sigma}\right) \\ + \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma\, (\delta x_\mu \mathscr{P}^\mu)\Big|_{\tau=\tau_1}^{\tau=\tau_2} + \int_{\tau_1}^{\tau_2} d\tau\, (\delta\sigma(\tau)\mathscr{L})\Big|_{\sigma=\sigma_1(\tau)}^{\sigma=\sigma_2(\tau)} \\ + \int_{\tau_1}^{\tau_2} d\tau \left\{\delta x_\mu(\sigma_2)\left[\Pi^\mu(\sigma_2) - \frac{d\sigma_2}{d\tau}\mathscr{P}^\mu(\sigma_2)\right] - \delta x_\mu(\sigma_1)\left[\Pi^\mu(\sigma_1) - \frac{d\sigma_1}{d\tau}\mathscr{P}^\mu(\sigma_1)\right]\right\}. \end{aligned} \tag{4.8}$$

The Euler equations are therefore

$$\frac{\partial \mathscr{P}^\mu}{\partial \tau} + \frac{\partial \Pi^\mu}{\partial \sigma} = 0, \tag{4.9}$$

while the boundary conditions are

$$\begin{aligned} &\delta x^\mu(\tau_1\,,\,\sigma) = \delta x^\mu(\tau_2\,,\,\sigma) = 0 \\ &\Pi^\mu(\tau\,,\,\sigma_i) - \frac{d\sigma_i}{d\tau}\mathscr{P}^\mu(\tau\,,\,\sigma_i) = 0\,, \qquad i = 1\,,\,2\,, \\ &\mathscr{L}(\tau\,,\,\sigma_1) = \mathscr{L}(\tau\,,\,\sigma_2) = 0\,. \end{aligned} \tag{4.10}$$

Hereafter, we will treat mainly the conventional case with $\tau$-independent boundaries in the $d\sigma$ integral and will choose

$$\begin{aligned} \sigma_1 &= 0 \\ \sigma_2 &= \pi\,. \end{aligned} \tag{4.11}$$

The boundary conditions that we need to use explicitly then become

$$\begin{aligned} &\Pi^\mu(\tau\,,\,\sigma = 0) = 0 \\ &\Pi^\mu(\tau\,,\,\sigma = \pi) = 0\,. \end{aligned} \tag{4.12}$$

## A. System without Gauge Constraints

From Eq. (4.6), we find the following two primary constraints,

$$\psi_1 = \mathscr{P}\cdot\mathscr{P} + N^2 u\cdot u \approx 0 \tag{4.13 a}$$

$$\psi_2 = \mathscr{P}\cdot u \approx 0\,. \tag{4.13 b}$$

A dynamical relation,

$$\Pi\cdot\Pi + N^2 v\cdot v = 0 \tag{4.14}$$

follows from Eq. (4.7). Equations (4.12) and (4.14) imply that $v^2 = 0$ at $\sigma = 0\,,\,\pi$. If we define our canonical Poisson brackets to be

$$\begin{aligned} &\{\mathscr{P}^\mu(\tau\,,\,\sigma)\,,\,x^\nu(\tau\,,\,\sigma')\} = -g^{\mu\nu}\,\delta(\sigma - \sigma') \\ &\{\mathscr{P}^\mu(\tau\,,\,\sigma)\,,\,\mathscr{P}^\nu(\tau\,,\,\sigma')\} = 0 \\ &\{x^\mu(\tau\,,\,\sigma)\,,\,x^\nu(\tau\,,\,\sigma')\} = 0\,, \end{aligned} \tag{4.15}$$

then we find that $\psi_1$ and $\psi_2$ are *first class*:

$$\begin{aligned} &\{\psi_1(\tau\,,\,\sigma)\,,\,\psi_1(\tau\,,\,\sigma')\} = 4N^2(\psi_2(\tau\,,\,\sigma) + \psi_2(\tau\,,\,\sigma'))\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma') \approx 0 \\ &\{\psi_2(\tau\,,\,\sigma)\,,\,\psi_1(\tau\,,\,\sigma')\} = (\psi_1(\tau\,,\,\sigma) + \psi_1(\tau\,,\,\sigma'))\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma') \approx 0 \\ &\{\psi_2(\tau\,,\,\sigma)\,,\,\psi_2(\tau\,,\,\sigma')\} = (\psi_2(\tau\,,\,\sigma) + \psi_2(\tau\,,\,\sigma'))\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma') \approx 0\,. \end{aligned} \tag{4.16}$$

(See Appendix B for a discussion of the relation between Eqs. (4.16) and Eqs. (1.83)). Hereafter, the explicit dependence of the canonical variables on $\tau$ will be dropped if no confusion arises.

Now we examine the canonical "energy-momentum tensor" in $(\tau, \sigma)$ space. If we let $u^\mu_1 = v^\mu = \dot{x}^\mu$, $u^\mu_2 = u^\mu = x'^\mu$, with $u^\mu_a = (u^\mu_1, u^\mu_2)$, the tensor may be taken as

$$[\theta_c]^a_b = \frac{\partial \mathscr{L}}{\partial u^\mu_a} u^\mu_b - \delta^a_b \mathscr{L} \tag{4.17}$$

$$= \begin{vmatrix} \mathscr{P}^\mu v_\mu - \mathscr{L} & \mathscr{P}^\mu u_\mu \\ \Pi^\mu v_\mu & \Pi^\mu u_\mu - \mathscr{L} \end{vmatrix} \approx 0 .$$

Formally, the canonical tensor is conserved,

$$\frac{\partial [\theta_c]^1{}_a}{\partial \tau} + \frac{\partial [\theta_c]^2{}_a}{\partial \sigma} = 0 .$$

Since $[\theta_c]^a_b \approx 0$ is obviously first class, we may add linear combinations of the first class constraints (4.13) to form the density which is integrated to give the first class generators of gauge-like transformations in $\tau$ and $\sigma$. Thus we define the total "energy-momentum tensor" in $(\tau, \sigma)$ space to be

$$\theta^a_b = U^a_b \psi_1 + V^a_b \psi_2 \approx 0 , \tag{4.18}$$

so the generators of $\tau$ and $\sigma$ transformations are

$$H_\tau = \int_0^\pi d\sigma \, \theta^1_1$$

$$H_\sigma = \int_0^\pi d\sigma \, \theta^1_2 . \tag{4.19}$$

$H_\tau$ and $H_\sigma$ have weakly vanishing brackets with one another because the $\psi_i$ are first class. The brackets with $x^\mu$, however, are nontrivial and we may place restrictions on $U^a_b$ and $V^a_b$ by requiring

$$\dot{x}^\mu \equiv v^\mu = \{ x^\mu , H_\tau \} = 2 U^1_1 \mathscr{P}^\mu + V^1_1 u^\mu$$

$$x'^\mu \equiv u^\mu = \{ x^\mu , H_\sigma \} = 2 U^1_2 \mathscr{P}^\mu + V^1_2 u^\mu . \tag{4.20}$$

Thus the definition (4.6) of $\mathscr{P}^\mu$ tells us that the choice

$$U^1_1 = -\frac{\mathscr{L}}{2 N^2 G} \qquad V^1_1 = \frac{F}{G}$$

$$U^1_2 = 0 \qquad V^1_2 = 1 . \tag{4.21}$$

reproduces (4.20).

In an ordinary scale-invariant two-dimensional field theory with the canonical tensor $[\theta_c]^a_b \neq 0$, there is an infinite class of symmetry generators found by integrating combinations of $[\theta_c]^a_b$ over an arbitrary function. These (nonvanishing) generators produce all of the transformations mapping the two-

dimensional space pseudo-conformally into itself (see, e.g., Fubini, Hanson and Jackiw, 1973). We may construct a similar class of generators for the string using the *weakly zero* tensor (4.18).

We begin by defining $u^\mu(\sigma)$ and $\mathscr{P}^\mu(\sigma)$ for $-\pi \leq \sigma \leq 0$ by

$$\begin{aligned} u^\mu(-\sigma) &= -u^\mu(\sigma) \\ \mathscr{P}^\mu(-\sigma) &= +\mathscr{P}^\mu(\sigma) \end{aligned} \tag{4.22}$$

(Rebbi, 1974). Then we may write the generator of all $\tau$ and $\sigma$ reparametrizations as the functional

$$\begin{aligned} (4.23\,a)\quad L[f] &= \frac{1}{4N}\int_0^\pi d\sigma\{[f(\sigma)+f(-\sigma)](\mathscr{P}^2+N^2u^2)+2N[f(\sigma)-f(-\sigma)]\mathscr{P}\cdot u\} \\ &= \frac{1}{4N}\int_{-\pi}^{\pi} d\sigma f(\sigma)(\mathscr{P}+Nu)^2, \end{aligned}$$

where

$$L[f] \approx 0. \tag{4.23\,b}$$

The functionals $L[f]$ form a closed algebra in the Poisson brackets,

$$\{L[f], L[g]\} = L[fg' - f'g]. \tag{4.24}$$

The familiar Virasoro functionals are essentially the Fourier components of (4.23) and are defined as

$$L_n \equiv L[e^{in\sigma}] = \frac{1}{4N}\int_{-\pi}^{\pi} d\sigma\, e^{in\sigma}(\mathscr{P}+Nu)^2. \tag{4.25}$$

Classically, the $L_n$ obey the algebra

$$\{L_m, L_n\} = -i(m-n)L_{n+m}. \tag{4.26}$$

We emphasize that the $L_n$ are *weakly zero* because they are formed from the first class primary constraints (4.13).

We now turn to the generators of the Poincaré group transformations of the fields $x^\mu(\tau, \sigma)$. Since the $x^\mu(\tau, \sigma)$, $\mu = 0, \cdots, D-1$, are D separate scalar fields in the two-dimensional $(\tau, \sigma)$ space, symmetries acting on the index $\mu$ are more like internal isospin symmetries than spacetime symmetries of a conventional field theory.

We compute the Noether currents generating Poincaré transformations of the $x^\mu$ by examining the corresponding variation of the Lagrangian density, with boundary conditions (4.10):

$$\begin{aligned} 0 = \delta\mathscr{L} &= \mathscr{P}^\mu\,\delta v_\mu + \Pi^\mu\,\delta u_\mu \\ &= \frac{\partial}{\partial\tau}(\mathscr{P}^\mu\,\delta x_\mu) + \frac{\partial}{\partial\sigma}(\Pi^\mu\,\delta x_\mu). \end{aligned} \tag{4.27}$$

If $\delta x^\mu$ is an infinitesimal translation,

$$\delta x^\mu = \varepsilon^\mu \tag{4.28 a}$$

we find the usual Euler equations

$$\frac{\partial \mathscr{P}^\mu}{\partial \tau} + \frac{\partial \Pi^\mu}{\partial \sigma} = 0 . \tag{4.28 b}$$

The $\tau$-independent " charge " generating infinitesimal translations of $x^\mu$ is therefore

$$P^\mu(\tau) = \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \, \mathscr{P}^\mu(\tau , \sigma) = \int_0^\pi d\sigma \, \mathscr{P}^\mu(\tau , \sigma) . \tag{4.29}$$

We revert for a moment to $\tau$-dependent boundaries for $\sigma$ and note that both the Euler equations and the boundary conditions (4.10) enter into the derivation of the $\tau$-independence (conservation) of $P^\mu$ :

$$\begin{aligned} \frac{dP^\mu}{d\tau} &= \frac{d\sigma_2}{d\tau} \mathscr{P}^\mu(\sigma_2) - \frac{d\sigma_1}{d\tau} \mathscr{P}^\mu(\sigma_1) - \int_{\sigma_1}^{\sigma_2} d\sigma \, \frac{\partial \Pi^\mu}{\partial \sigma} \\ &= \frac{d\sigma_2}{d\tau} \mathscr{P}^\mu(\sigma_2) - \Pi^\mu(\sigma_2) - \frac{d\sigma_1}{d\tau} \mathscr{P}^\mu(\sigma_1) + \Pi^\mu(\sigma_1) = 0 . \end{aligned} \tag{4.30}$$

If $\delta x^\mu$ is an infinitesimal Lorentz transfomation,

$$\delta x^\mu = \omega_{\alpha\beta} (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \tag{4.31 a}$$

we find

$$\frac{\partial}{\partial \tau} (x^\mu \mathscr{P}^\nu - x^\nu \mathscr{P}^\mu) + \frac{\partial}{\partial \sigma} (x^\mu \Pi^\nu - x^\nu \Pi^\mu) = 0 . \tag{4.31 b}$$

This equation also follows from Eqs. (4.6–4.7) and the Euler equations. The generator of infinitesimal Lorentz transformations is thus

$$M^{\mu\nu} = \int_0^\pi d\sigma \, (x^\mu \mathscr{P}^\nu - x^\nu \mathscr{P}^\mu) . \tag{4.32}$$

Using Eq. (4.31 *b*) and the boundary conditions (4.12), we find that $M^{\mu\nu}$ is conserved:

$$\frac{dM^{\mu\nu}}{d\tau} = 0 : \tag{4.33}$$

The canonical Poisson brackets (4.15) may now be used to show that the full Poincaré algebra is satisfied,

$$\begin{aligned} &\{ P^\alpha , P^\beta \} = 0 \\ &\{ M^{\mu\nu} , P^\alpha \} = g^{\mu\alpha} P^\nu - g^{\nu\alpha} P^\mu \\ &\{ M^{\mu\nu} , M^{\alpha\beta} \} = C^{\mu\nu\alpha\beta}_{\sigma\tau} M^{\sigma\tau} \end{aligned} \tag{4.34}$$

where $C^{\mu\nu\alpha\beta}_{\sigma\tau}$ is the structure constant (2.24). It is also clear the $x^\mu$ transforms as

$$\begin{aligned} &\{ P^\mu , x^\nu (\tau , \sigma)\} = -g^{\mu\nu} \\ &\{ M^{\mu\nu}, x^\alpha (\tau , \sigma)\} = g^{\mu\alpha} x^\nu (\tau , \sigma) - g^{\nu\alpha} x^\mu (\tau , \sigma) , \end{aligned} \tag{4.35}$$

so that $P^\mu$ and $M^{\mu\nu}$ indeed generate the infinitesimal Poincaré transformations of $x^\mu (\tau , \sigma)$; no integrals of the first class constraints (4.13) need to be added to the canonical expressons for $P^\mu$ and $M^{\mu\nu}$.

## B. Orthonormal Gauge Constraints

We will now choose a convenient set of gauge constraints to fix the scale of $\tau$ and $\sigma$ relative to $x^\mu$. We begin by taking an arbitrary constant vector $\lambda_\mu$ and examining the consequences of the gauge choice

$$\lambda_\mu \left( x^\mu (\tau , \sigma) - \frac{P^\mu \tau}{N\pi} \right) \approx 0 \tag{4.36 a}$$

$$\lambda_\mu \left( \mathscr{P}^\mu (\tau , \sigma) - \frac{P^\mu}{\pi} \right) \approx 0 . \tag{4.36 b}$$

N is defined as usual as the normalization appearing in the action (4.1) and $P^\mu$ is the translation generator (4.29). Note that the conditions (4.36) are nonlocal in $\sigma$.

If we differentiate (4.36 *a*) with respect to $\tau$ and $\sigma$, we find

$$\begin{aligned} &\lambda \cdot v \approx \frac{\lambda \cdot P}{N\pi} \approx \frac{\lambda \cdot \mathscr{P}}{N} \\ &\lambda \cdot u \approx 0 . \end{aligned} \tag{4.37}$$

Equation (4.36 *b*) and the Euler equations imply

$$\frac{\partial (\lambda \cdot \mathscr{P})}{\partial \tau} = - \frac{\partial (\lambda \cdot \Pi)}{\partial \sigma} \approx 0 , \tag{4.38}$$

so the boundary condition (4.12) requires

$$\lambda \cdot \Pi (\tau , \sigma) \approx 0 \tag{4.39}$$

for *all* $\sigma$, not just $\sigma = 0 , \pi$.

Next we observe that the definitions (4.6) and (4.7) of $\mathscr{P}^\mu$ and $\Pi^\mu$, together with Eq. (4.37), imply that

$$\begin{aligned} \lambda \cdot \mathscr{P} &\approx - (\lambda \cdot v) N^2 G / \mathscr{L} \\ &\approx - (\lambda \cdot \mathscr{P}) NG / \mathscr{L} \not\approx 0 . \end{aligned} \tag{4.40}$$

In addition, Eq. (4.39) requires

$$\lambda \cdot \Pi \approx (\lambda \cdot v) N^2 F / \mathscr{L} \approx 0 . \tag{4.41}$$

We first conclude that

$$F \equiv v \cdot u \approx 0 . \tag{4.42 a}$$

Secondly, Eq. (4.40) tells us that $NG/\mathscr{L} \approx -1$, so we discover that

$$E + G = v^2 + u^2 \approx 0 . \tag{4.42 b}$$

By definition, Eqs. (4.42) show that the gauge choice (4.36) has resulted in an *orthornormal parametrization* of the surface swept out by the string.

The value of the Lagrangian can now be written

$$\begin{aligned} \mathscr{L} &= -N(F^2 - EG)^{1/2} \approx -NG \approx +NE \\ &\approx \frac{N}{2}(E - G) \equiv \frac{N}{2}(v^2 - u^2) \end{aligned} \tag{4.43}$$

so the following sign convention results:

$$\begin{aligned} \mathscr{P}^\mu &\approx N v^\mu = N\dot{x}^\mu \\ \Pi^\mu &\approx -N u^\mu = -N x'^\mu . \end{aligned} \tag{4.44}$$

The Euler equations and boundary conditions now take the form

$$\frac{\partial^2 x^\mu}{\partial \tau^2} - \frac{\partial^2 x^\mu}{\partial \sigma^2} \approx 0 \tag{4.45 a}$$

$$\frac{\partial x^\mu}{\partial \sigma} \approx 0 , \qquad \sigma = 0 , \pi , \tag{4.45 b}$$

where we must continue to bear in mind the orthonormality conditions (4.42).

*The solutions of Eq.* (4.45) *have no physical meaning unless they are consistent with the constraints.*

It is worthwhile to remark that imposing Eqs. (4.42) alone does not fix the gauge completely in the way that (4.36) does. By making a reparametrization

$$\tau , \sigma \to \tilde{\tau}(\tau , \sigma) , \tilde{\sigma}(\tau , \sigma)$$

such that

$$\begin{aligned} \frac{\partial \tilde{\sigma}}{\partial \sigma} &= \frac{\partial \tilde{\tau}}{\partial \tau} \\ \frac{\partial \tilde{\sigma}}{\partial \tau} &= \frac{\partial \tilde{\tau}}{\partial \sigma} \end{aligned} \tag{4.46 a}$$

where

$$\frac{\partial^2 \tilde{\tau}}{\partial \tau^2} - \frac{\partial^2 \tilde{\tau}}{\partial \sigma^2} = 0 , \tag{4.46 b}$$

we may preserve Eqs. (4.42) ,

$$\frac{\partial x^\mu}{\partial \tilde\tau}\,\frac{\partial x_\mu}{\partial \tilde\sigma} \approx 0$$

$$\frac{\partial x^\mu}{\partial \tilde\tau}\,\frac{\partial x_\mu}{\partial \tilde\tau} + \frac{\partial x_\mu}{\partial \tilde\sigma}\,\frac{\partial x^\mu}{\partial \tilde\sigma} \approx 0 ,$$

where $x^\mu = x^\mu(\tau(\tilde\tau , \tilde\sigma) , \sigma(\tilde\tau , \tilde\sigma))$. Thus we may remain in an orthonormal gauge while making a different linear combination of the $x^\mu(\tau , \sigma)$ proportional to $\tau$ in (4.36 *a*).

## C. Dirac Brackets

We now proceed to develop a set of Dirac brackets compatible with the primary constraints (4.13), the gauge choice (4.36), and the boundary conditions (4.12). Only then can we take Eqs. (4.45) to be strongly valid. We first extend the canonical variables to all values of their arguments by defining a new periodic Poisson bracket

$$\{\mathscr{P}^\mu(\tau , \sigma) , x^\nu(\tau , \sigma')\} = -g^{\mu\nu}\Delta(\sigma , \sigma') \tag{4.47}$$

where

$$\Delta(\sigma , \sigma') \equiv \Delta_+(\sigma , \sigma')$$

and

$$\Delta_\pm(\sigma , \sigma') = \sum_{n=-\infty}^{\infty}[\delta(\sigma - \sigma' + 2n\pi) \pm \delta(\sigma + \sigma' + 2n\pi)] \tag{4.48 a}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(e^{in(\sigma-\sigma')} \pm e^{in(\sigma+\sigma')}) ,$$

$$\int_0^\pi d\sigma\, \Delta_\pm(\sigma , \sigma') f(\sigma) = f(\sigma') , \qquad 0 \le \sigma' \le \pi .$$

We will need the following properties of $\Delta_\pm$ ,

$$\frac{\partial\Delta(\sigma , \sigma')}{\partial\sigma} = -\frac{\partial\Delta_-(\sigma , \sigma')}{\partial\sigma'} \tag{4.48 b}$$

$$\frac{\partial\Delta(\sigma , \sigma')}{\partial\sigma'} = -\frac{\partial\Delta_-(\sigma , \sigma')}{\partial\sigma}$$

and

$$\partial\Delta/\partial\sigma = 0 \qquad \text{at} \quad \sigma = 0 , \pi$$

$$\partial\Delta/\partial\sigma' = 0 \qquad \text{at} \quad \sigma' = 0 , \pi \tag{4.48 c}$$

$$\partial^2\Delta/\partial\sigma\,\partial\sigma' = 0 \qquad \text{at} \quad \sigma = 0 , \pi$$

$$\partial^2\Delta/\partial\sigma^2 \neq 0 \qquad \text{at} \quad \sigma = 0 , \pi .$$

Thus we see that Eq. (4.47) is compatible with the boundary condition (4.45 *b*).

Our full set of constraints is now

$$
\begin{aligned}
&\varphi_1(\sigma) = \mathscr{P}\cdot u \approx 0\\
&\varphi_1(\sigma) = \lambda\cdot x - \tau\lambda\cdot \mathrm{P}/\mathrm{N}\pi \approx 0\\
&\varphi_3(\sigma) = \mathscr{P}^2 + \mathrm{N}^2 u^2 \approx 0\\
&\varphi_4(\sigma) = \lambda\cdot\mathscr{P} - \lambda\cdot \mathrm{P}/\pi \approx 0.
\end{aligned}
\tag{4.49}
$$

The matrix of Poisson brackets is

$$
\mathrm{C}_{ij}(\sigma,\sigma') = \{\varphi_i(\sigma),\varphi_j(\sigma')\} \approx
$$

$$
\approx \begin{vmatrix}
0 & 0 & 0 & \frac{\lambda\cdot\mathrm{P}}{\pi}\frac{\partial}{\partial\sigma}\Delta(\sigma,\sigma')\\
0 & 0 & \frac{2\lambda\cdot\mathrm{P}}{\pi}\Delta(\sigma,\sigma') & \lambda^2\left(\Delta(\sigma,\sigma')-\frac{1}{\pi}\right)\\
0 & -\frac{2\lambda\cdot\mathrm{P}}{\pi}\Delta(\sigma,\sigma') & 0 & 0\\
-\frac{\lambda\cdot\mathrm{P}}{\pi}\frac{\partial}{\partial\sigma'}\Delta(\sigma,\sigma') & -\lambda^2\left(\Delta(\sigma,\sigma')-\frac{1}{\pi}\right) & 0 & 0
\end{vmatrix}
\tag{4.50}
$$

Here we have made use of the constraints (4.49) after computing the brackets, as we are always entitled to do.

To compute the Dirac brackets, we need the inverse matrix $\mathrm{C}^{-1}_{ij}(\sigma,\sigma')$ defined by

$$
\int_0^\pi \mathrm{d}\sigma''\,\mathrm{C}_{ik}(\sigma,\sigma'')\,\mathrm{C}^{-1}_{kj}(\sigma'',\sigma') = \mathrm{I}_{ij}(\sigma,\sigma') =
\tag{4.51}
$$

$$
= \begin{vmatrix}
\Delta_-(\sigma,\sigma') & 0 & 0 & 0\\
0 & \Delta_+(\sigma,\sigma') & 0 & 0\\
0 & 0 & \Delta_+(\sigma,\sigma') & 0\\
0 & 0 & 0 & \Delta_+(\sigma,\sigma')
\end{vmatrix}.
$$

The appearance of $\Delta_-$ in the first diagonal entry of the generalized unit matrix $\mathrm{I}_{ij}(\sigma,\sigma')$ and $\Delta_+$ in the remaining positions is dictated by the boundary conditions. If we differentiate (4.51) with respect to $\sigma$ and evaluate the result at $\sigma = 0$ or $\sigma = \pi$, Eqs. (4.50) and (4.48) imply

$$
\frac{\partial}{\partial\sigma}\int \mathrm{d}\sigma''\,\mathrm{C}_{14}(\sigma,\sigma'')\,\mathrm{C}^{-1}_{41}(\sigma'',\sigma')\Big|_{\sigma=0,\pi} = \frac{\partial}{\partial\sigma}\,\mathrm{I}_{11}(\sigma,\sigma')\Big|_{\sigma=0,\pi} \neq 0\,.
\tag{4.52}
$$

The corresponding expressions for the other diagonal elements of $\mathrm{I}_{ij}$ vanish and they accordingly may be set consistently equal to $\Delta_+(\sigma,\sigma')$.

To determine $\mathrm{I}_{11}$, we first observe that the solution of

$$
\int \mathrm{d}\sigma''\,\mathrm{C}_{41}(\sigma,\sigma')\,\mathrm{C}^{-1}_{41}(\sigma'',\sigma') = \mathrm{I}_{44}(\sigma,\sigma') = \Delta_+(\sigma,\sigma')
$$