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Classical kinematics for Lorentz violation

V. Alan Kostelecký [a,*], Neil Russell [b]

[a] Physics Department, Indiana University, Bloomington, IN 47405, USA
[b] Physics Department, Northern Michigan University, Marquette, MI 49855, USA

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ABSTRACT

Classical point-particle relativistic lagrangians are constructed that generate the momentum–velocity and dispersion relations for quantum wave packets in Lorentz-violating effective field theory.

1. Introduction

A promising prospect for experimental detection of new physics at the Planck scale is tiny Lorentz and CPT violation arising in an underlying unified theory such as strings [1]. At experimentally attainable energy scales, effective quantum field theory provides a useful tool for describing observable signals of Lorentz and CPT violation [2,3]. The comprehensive realistic effective field theory for Lorentz and CPT violation incorporating both the Standard Model and General Relativity is the Standard-Model Extension (SME) [4], which has been the basis for much theoretical work and for numerous sensitive experimental searches [5]. However, comparatively little is known about the corresponding classical Lorentz-violating kinematics, a topic central to subjects such as the behavior of quantum wave packets, the analysis of relativistic scattering and the motion of macroscopic bodies.

One useful approach to the classical relativistic kinematics of a quantum field theory is to introduce an analogue point-particle system with relativistic lagrangian L, which leads directly to various results such as the equations of motion for the classical trajectory, the momentum–velocity connection, and the dispersion relation. In the Lorentz-invariant case, the extensive literature on relativistic lagrangians dates from Planck's 1906 work on the free relativistic particle [6], Einstein's analysis of geodesics [7], and Frenkel's treatment of the effects of spin on trajectories [8]. Textbook applications of relativistic point-particle lagrangians include the compact description of the dispersion relations for a relativistic wave packet and for the center-of-mass motion of a relativistic macroscopic body, the treatment of systems involving particles propagating in various spacetimes and interacting via electromagnetic or other couplings, and the kinematical analysis of relativistic scattering problems [9].

For effective field theories with Lorentz violation, however, available kinematical results exist primarily at leading order and only for simple systems. It is known that the 3-velocity and 3-momentum are typically misaligned and that generic spin and momentum eigenstates may have ill-defined velocities [4,10,11]. The classical relativistic scattering problem with specified initial velocities and with interactions conserving 4-momentum involves the explicit momentum–velocity relationship, which is unknown for most systems with Lorentz violation. The kinematics of scattering in quantum field theory requires the exact propagator at all orders in Lorentz violation for external legs, essentially because the Lorentz-invariant states fail to span the asymptotic Hilbert space, and only a few processes have been analyzed (see, for example, Refs. [12–19]). The study of classical Lorentz-violating trajectories for bodies moving under electromagnetic fields and in post-newtonian gravity has been restricted to leading-order terms in Lorentz violation [20]. Perhaps the best understood kinematical feature of a Lorentz-violating quantum field theory is the dispersion relation for a quantum wave packet. The form of the dispersion relation generated by an effective field theory is constrained and depends on the intrinsic spin of the quantum fields [21,22]. In the single-fermion limit of the renormalizable sector of the SME in Minkowski spacetime, the exact dispersion relation has been obtained [10] and techniques to study it have been developed [23,24]. In quantum electrodynamics, the complete and exact dispersion relation for the photon, arising from all gauge-invariant

* Corresponding author.
E-mail address: kostelec@indiana.edu (V.A. Kostelecký).

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operators of arbitrary mass dimension, has been constructed for uniform Lorentz violation [22].

In this work, we present a method for constructing the classical point-particle lagrangians L corresponding to a given polynomial dispersion relation for a particle of mass m in the presence of uniform background fields generically denoted by k_x, where the subscript x denotes the relevant spacetime indices. The resulting lagrangians describe the classical kinematics associated with effective quantum field theories with background fields, and they permit a straightforward derivation of exact results such as the momentum–velocity relation, the dispersion relation, and the equations of motion for the particle trajectory. The method is directly applicable to effective field theories with Lorentz violation, including the SME and its various limits. Here, we use it to obtain some explicit lagrangians describing the classical kinematics of the single-fermion sector of the minimal SME in Minkowski spacetime, thereby resolving a number of the open issues described above.

The single-fermion sector of the minimal SME contains all Lorentz-violating quadratic operators for a Dirac fermion of mass dimensions three and four, which are controlled by dimension-one coefficients a_μ, b_μ, $H_{\mu\nu}$ and by dimensionless coefficients $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ, f_μ, and $g_{\lambda\mu\nu}$, respectively. The corresponding exact dispersion relation can be obtained from the generalized Dirac equation for plane waves of 4-momentum p_μ by imposing that the determinant of the Dirac operator vanishes [10]. It can be written in the compact form

$$\begin{aligned}0 = {}&\frac{1}{4}\big(V^2 - S^2 - A^2 - P^2\big)^2\\&+ 4\big[P(VTA) - S(V\widetilde{T}A) - VTTV + ATTA\big]\\&+ V^2A^2 - (V\cdot A)^2\\&- X\big(V^2 + S^2 - A^2 - P^2\big) - 2YSP + X^2 + Y^2,\end{aligned}\tag{1}$$

where the scalar quantity is $S = -m + e\cdot p$, the pseudoscalar is $P = f\cdot p$, the vector is $V_\mu = p_\mu + (cp)_\mu - a_\mu$, the axial-vector is $A_\mu = (dp)_\mu - b_\mu$, and the tensor is $T_{\mu\nu} = \frac{1}{2}(gp - H)_{\mu\nu}$. The two invariants of $T_{\mu\nu}$ are denoted $X \equiv T_{\mu\nu}T^{\mu\nu}$ and $Y \equiv T_{\mu\nu}\widetilde{T}^{\mu\nu}$, with the dual defined by $\widetilde{T}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}T^{\alpha\beta}$. Note that for vanishing coefficients for Lorentz violation the dispersion relation (1) reduces to the usual form $(p^2 - m^2)^2 = 0$, which is effectively quadratic. The quadratic nature is retained when the only nonzero coefficients are a_μ, $c_{\mu\nu}$, e_μ, and f_μ. However, the dispersion relation is generically quartic if b_μ, $d_{\mu\nu}$, $g_{\lambda\mu\nu}$, or $H_{\mu\nu}$ are nonzero.

2. Lagrangian construction

Consider a generic dispersion relation that takes the form of a polynomial equation $\mathcal{R}(p_\mu; m, k_x) = 0$ in the 4-momentum p_μ, with coefficients depending on the mass m and on the background fields k_x. For example, for the single-fermion limit of the Lagrange density for the minimal SME, $\mathcal{R}(p)$ is the polynomial of degree four in p_μ given explicitly in Eq. (1). If nonminimal quadratic Lorentz-violating operators of mass dimension d for a single fermion are also included, $(d-3)$ spacetime derivatives appear [22]. Each row of the determinant of the corresponding Dirac operator in momentum space is then of order $(d-3)$ in p_μ, so $\mathcal{R}(p)$ becomes a polynomial of degree $4(d-3)$ in p_μ.

The dispersion relation $\mathcal{R}(p) = 0$ expresses one on-shell condition on the four components of p_μ. It can be viewed as an equation constraining the energy p_0 as a function $p_0(p_j)$ of the 3-momentum p_j, $j = 1, 2, 3$, and it has multiple roots for p_0. For background fields k_x that are perturbative relative to the mass m in any given concordant frame [10], these roots include ones corresponding to the components of a general wave packet decomposed into positive- and negative-energy solutions with various spin projections. For example, the single-fermion limit of the minimal SME with perturbative Lorentz violation involves two spin projections and hence a total of four roots p_0, matching the quartic structure of the dispersion relation (1). The nonminimal version of this theory with operators up to mass dimension d can have $4(d-3)$ roots, but only four of these are perturbative. The remaining roots correspond to high-frequency solutions that are artifacts of effective field theory and can be disregarded for practical purposes. More generally, a massive quantum field of intrinsic spin quantum number j has $(2j+1)$ spin projections and so the dispersion relation $\mathcal{R}(p) = 0$ has a total of $2(2j+1)$ perturbative roots. Each root can be associated with a particle or antiparticle system and a corresponding lagrangian L. The challenge of interest here is therefore to construct lagrangians L for all the perturbative roots of a given dispersion relation $\mathcal{R}(p) = 0$.

Using the dispersion relation $\mathcal{R}(p) = 0$, we can find three more on-shell conditions by taking derivatives with respect to the 3-momentum and requiring that the group 3-velocity of the quantum wave packet matches the 3-velocity of the classical analogue particle, $-\partial p_0/\partial p_j = dx^j/dx^0$. If the path of the particle is parametrized by a path parameter λ, then $dx^j/dx^0 = (dx^j/d\lambda)/(dx^0/d\lambda)$ and we can impose $-\partial p_0/\partial p_j = u^j/u^0$, where $u^\mu \equiv dx^\mu/d\lambda$. The three derivative conditions $\partial\mathcal{R}/\partial p_j = 0$ obtained from the dispersion relation are typically nonlinear in the momenta. For example, the dispersion relation for the single-fermion limit of the minimal SME is quartic in p_μ, so the three conditions are cubic. In the nonminimal case with operators of mass dimension d, the maximum degree of the three conditions becomes $4(d-3)-1$.

To identify the point-particle theories leading to the dispersion relation $\mathcal{R}(p) = 0$ and the three conditions for u^j, we seek suitable lagrangian functionals $L(x^\mu(\lambda), u^\mu(\lambda), \lambda; m, k_x)$ of the spacetime 4-position $x^\mu(\lambda)$ and its derivative $u^\mu(\lambda)$ with respect to λ. We suppose that the physics of the particle system is independent of the path parametrization, so L has no explicit dependence on λ. The action $S = \int L\,d\lambda$ must also be reparametrization independent, so L must be homogeneous of degree one in u^μ. Invoking Euler's theorem for homogeneous functions yields $L = -p_\mu u^\mu$, where $p_\mu \equiv -\partial L/\partial u^\mu$ is the particle canonical 4-momentum expressed as a function of u^μ. By construction, this canonical 4-momentum is identified with the 4-momentum of the quantum wave packet. If the background fields k_x are uniform, as in the specific examples with Lorentz violation analyzed below, then the canonical 4-momentum p_μ is constant, the system conserves energy and momentum, and the lagrangians L have no explicit dependence on x^μ.

The choice of λ amounts to a choice of diffeomorphism gauge on the one-dimensional path manifold. On shell and for a timelike curve, the element of path length $d\lambda$ is related to the proper-time interval $d\tau$ obeying $d\tau^2 \equiv \eta_{\mu\nu}dx^\mu dx^\nu$ via $(d\tau/d\lambda)^2 = u_\mu u^\mu$. The choice of λ can therefore be understood as one constraint on the four components u^μ, leaving three independent degrees of freedom. Note that λ can be taken as the proper time τ, in which case u^μ becomes the usual on-shell particle 4-velocity, but that other choices can be more convenient in the presence of Lorentz violation.

Collecting the above results yields the five key equations

$$\mathcal{R}(p) = 0, \qquad u^j = -u^0\frac{\partial p_0}{\partial p_j}, \qquad L = -p_\mu u^\mu,\tag{2}$$

which suffice to determine $L = L(u^\mu)$. To obtain an explicit result, the five equations can be manipulated to eliminate the 4-momentum components, leaving a single equation that can be viewed as a polynomial for L,

$$\mathcal{P}\big(L; u^\mu, m, k_x\big) \equiv \sum_{n=0}^{N} c_n\big(u^\mu; m, k_x\big)L^n = 0. \tag{3}$$

The degree N of $\mathcal{P}(L)$ is bounded from above by the degrees of the conditions (2). For example, we find $N \leqslant 12$ in the single-fermion limit of the minimal SME. For the nonminimal case with operators of mass dimension d, the degree N of the polynomial $\mathcal{P}(L)$ satisfies $N \leqslant (4d-12)(4d-13)$.

The roots of the polynomial (3) are the candidate lagrangians $L(u^\mu; m, k_x)$ for the classical analogue systems with positive- and negative-energy particles. Only the roots representing perturbative deviations from the conventional relativistic point-particle lagrangians $L = \pm m\sqrt{u^2}$ are of physical interest. Other roots of $\mathcal{P}(L)$ correspond to candidate lagrangians for which the background fields are dominant and are therefore spurious in the present context. When the perturbative roots can be found analytically, they generate classical lagrangians that correctly reproduce the center-of-mass kinematics of the wave-packet components at all orders in the background fields.

Perturbativity of the background fields implies $p_\mu = p_\mu(u)$ can be inverted to yield the 4-velocity $u^\mu = u^\mu(p)$, while the uniformity of the background fields ensures conservation of the canonical 4-momentum p_μ as mentioned above. Since L has no explicit dependence on the path parameter λ, these results imply that

$$\dot{u}^\mu \equiv \frac{\partial u^\mu}{\partial \lambda} + \frac{\partial u^\mu}{\partial p_\nu}\frac{dp_\nu}{d\lambda} = 0. \tag{4}$$

It follows that each particle undergoes uniform motion in a straight line, so Newton's first law remains unchanged in the presence of constant coefficients for Lorentz violation. Effects from uniform Lorentz violation on the behavior of a single classical point particle that is otherwise free are therefore unobservable *per se* in Minkowski spacetime, confirming known results [22]. In general, physical effects can be detected experimentally only by comparison of two systems with differing properties. The systems may be particles of distinct flavor, different spin projections of a single particle, or identical particles with different momenta. Determining the physical implications of the explicit lagrangians obtained below therefore requires care in establishing which two or more quantities are being compared in a given experiment.

We remark in passing that the above technique for constructing lagrangians can in principle also be used if the dispersion relation involves nonuniform background fields, such as those arising naturally from curvature in Riemann spacetime or from nonzero electric and magnetic field strengths in quantum electrodynamics or in the nonabelian sector. However, exact dispersion relations are typically challenging to obtain for nonuniform fields. Perturbative constructions such as the Foldy–Wouthuysen method [25] can yield the dispersion relation to a specified order in the background fields, whereupon the method described here can generate the corresponding classical point-particle lagrangians.

3. Quadratic case

For a massive particle and uniform background fields, any dispersion relation quadratic in p_μ can be written in the suggestive form

$$(p+\kappa)\Omega(p+\kappa) = \mu^2, \tag{5}$$

where $\mu > 0$ is a mass-like scalar, κ_μ is a constant 4-vector shift of the momentum, and $\Omega^{\mu\nu}$ is a constant metric-like symmetric tensor. In the limit of vanishing background fields, $\Omega^{\mu\nu} \to \eta^{\mu\nu}$, $\kappa_\mu \to 0$, and $\mu \to m$. For perturbative background fields, Ω is invertible.

For this case, calculation with the five equations (2) yields a quadratic polynomial $\mathcal{P}(L)$ with the root for the particle being

$$L(u; \mu, \kappa, \Omega) = -\mu\sqrt{u\Omega^{-1}u} + \kappa \cdot u. \tag{6}$$

The second root has the form $L(u; -\mu, \kappa, \Omega)$ and corresponds to the antiparticle after reinterpretation. The canonical momentum for the particle is

$$p_\mu \equiv -\frac{\partial L}{\partial u^\mu} = \mu\frac{(\Omega^{-1}u)_\mu}{\sqrt{u\Omega^{-1}u}} - \kappa_\mu. \tag{7}$$

Notice that p_μ and u^μ generically fail to align and that the 4-momentum p_μ can be nonzero when the 3-velocity vanishes, features already noted for Lorentz violation [4,10,11]. The dispersion relation (5) can be recovered by manipulations of Eq. (7). The commonly used condition $u^2 = 1$ sets the path parameter to the particle proper time, but other choices are equally valid and leave the physics unaffected. One convenient choice is $d\lambda = \sqrt{(\Omega^{-1})_{\mu\nu}\,dx^\mu\,dx^\nu}$, which simplifies calculations and matches the proper time in the limit of vanishing background fields.

Next, we apply this formalism to the SME dispersion relation (1) restricted to nonzero coefficients a_μ, $c_{\mu\nu}$, e_μ, and f_μ. For this special case, we find $\Omega = (\delta + 2c + c^T c - ee - ff)$. The inverse $(\Omega^{-1})_{\mu\nu}$ can be constructed as an infinite series. For $c_{\mu\nu} = 0$, we find the comparatively simple lagrangian $L(u; m, a, e, f)$ given by

$$\begin{aligned} L = -\mu\bigg\{u^2 &+ \frac{1}{\Delta}\Big[\big(1-f^2\big)(e\cdot u)^2 + \big(1-e^2\big)(f\cdot u)^2 \\ &+ 2(e\cdot f)(e\cdot u)(f\cdot u)\Big]\bigg\}^{1/2} - a\cdot u \\ &+ \frac{1}{\Delta}\Big[\big(1-f^2\big)(m-e\cdot a) - (e\cdot f)(f\cdot a)\Big]e\cdot u \\ &+ \frac{1}{\Delta}\Big[(e\cdot f)(m-e\cdot a) - \big(1-e^2\big)(f\cdot a)\Big]f\cdot u, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu = \frac{1}{\sqrt{\Delta}}\Big[&\big(1-f^2\big)(m-e\cdot a)^2 \\ &- 2(e\cdot f)(f\cdot a)(m-e\cdot a) \\ &+ \big(1-e^2\big)(f\cdot a)^2\Big]^{1/2}, \end{aligned} \tag{9}$$

and where the determinant of $(\delta - ee - ff)$ is $\Delta = (1-e^2)(1-f^2) - (e\cdot f)^2$. If instead $c_{\mu\nu}$ is the only nonzero coefficient, then the dispersion relation reduces to $p(\delta + 2c + c^T c)p = m^2$, where $(c^T)_{\mu\nu} = c_{\nu\mu}$. Only the symmetric piece of the expression in parentheses can contribute, and we find $\Omega = (\delta + c_S)^2 - c_A^2 + [c_S, c_A]$, where $(c_S)_{\mu\nu} \equiv (c_{\mu\nu} + c_{\nu\mu})/2$ and $(c_A)_{\mu\nu} \equiv (c_{\mu\nu} - c_{\nu\mu})/2$. The corresponding lagrangian is

$$L(u; m, c) = -m\sqrt{u\big\{(\delta + c_S)^2 - c_A^2 + [c_S, c_A]\big\}^{-1}u}. \tag{10}$$

The leading correction to p_μ therefore appears at first order if c has a symmetric part but at second order if c is antisymmetric, matching the known result at the level of field theory [4].

In some limits of the SME, certain coefficients for Lorentz violation are unphysical and can be removed by field redefinitions [4,10,20,22,26–30]. For example, the Lagrange density for a single Dirac field with only a nonzero coefficient a_μ has no physical Lorentz violation because a_μ can be eliminated via a phase redefinition [4]. The classical particle then has lagrangian (8) with $e_\mu = f_\mu = 0$, in which a_μ appears only in the unphysical total-derivative term $-a\cdot u = -d(a\cdot x)/d\lambda$. When the field theory depends also on e_μ

and f_μ, the required phase shift is more complicated, but by inspection we can find it here exactly as the coefficient of u^μ in the result (8). As another example, consider the dispersion relation $p[(\delta+c)^2 - ff]p = m^2$ for symmetric $c_{\mu\nu}$ and f_μ. The f_μ field can be absorbed into a modified c-type coefficient $(c')_{\mu\nu}$ by matching the operator in brackets with $(\delta + c')^2$. We find

$$(c')^\mu{}_\nu = -\delta^\mu{}_\nu + \sqrt{\delta^\mu{}_\nu + 2c^\mu{}_\nu + (c^2)^\mu{}_\nu - f^\mu f_\nu}, \tag{11}$$

to be understood as an infinite matrix series, showing that $L(u; m, c_S, f) \equiv L(u; m, c')$. This expression reduces to the field-theoretic result given in Ref. [29] for the case $c_{\mu\nu} = 0$. In the special case of a lightlike f_μ and $c_{\mu\nu} = 0$, the match $(\delta + c')^2 = \delta - ff$ gives $(c')_{\mu\nu} \equiv -\frac{1}{2} f_\mu f_\nu$, again in agreement with Ref. [29].

4. Quartic case

When the coefficients b_μ, $d_{\mu\nu}$, $H_{\mu\nu}$, or $g_{\lambda\mu\nu}$ are nonzero, the classical properties of the single-fermion limit of the minimal SME become more intractable and very few results are known. Here, we apply the above general methods to explore some of these cases.

Consider first nonzero a_μ and b_μ coefficients, for which the dispersion relation $\mathcal{R}(p) \equiv [-(p-a)^2 + b^2 + m^2]^2 - 4[b\cdot(p-a)]^2 + 4b^2(p-a)^2 = 0$ is quartic. Some calculations with Eq. (2) yield an octic polynomial $\mathcal{P}(L)$ of the form (3), which factors into three pieces. The first piece gives the two particle lagrangians

$$L(u; m, a, b) = -m\sqrt{u^2} - a\cdot u \mp \sqrt{(b\cdot u)^2 - b^2 u^2}. \tag{12}$$

The second piece has the form $L(u; -m, a, b)$ and corresponds to the antiparticles after reinterpretation [23,31], while the third piece is spurious. The canonical 4-momenta for $L(u; m, a, b)$ are

$$p_\mu = \frac{m u_\mu}{\sqrt{u^2}} + a_\mu \pm \frac{(b\cdot u) b_\mu - b^2 u_\mu}{\sqrt{(b\cdot u)^2 - b^2 u^2}}. \tag{13}$$

The two particle lagrangians and two canonical momenta reflect the two particle spin projections in the quantum wave packet. However, the detailed match is subtle. Consider, for example, the case of timelike b_μ with particles at rest, for which the denominator of the last term in Eq. (13) vanishes. Choosing an observer frame in which $b_\mu = (b, 0, 0, 0)$ and adopting the proper-time parametrization, we find $\vec{p} = m\vec{u} \mp |b|\hat{u}$, revealing that the 3-momentum and 3-velocity are collinear but have noncoincident zeros. We can use spatial isotropy to choose $\vec{p}$ and $\vec{u}$ nonzero only along the 3 direction, giving $p^3 = mu^3 \mp |b|\,\mathrm{sign}(u^3)$. For p^3 to be a continuous function of u^3, it follows that the sign choice in Eq. (13) must change when u^3 changes sign.

The effect of b_μ on a Dirac fermion parallels that of minimally coupled torsion $T_{\alpha\beta\gamma}$ in a Riemann–Cartan spacetime [4,32,33]. The result (12) therefore can be adapted to yield the analogue classical lagrangian for a minimally coupled Dirac field in a uniform torsion background. For compatibility with the torsion literature, in this paragraph we adopt the notation and conventions of Ref. [33]. The correspondence between b_μ and the axial-vector projection $(T_A)^\mu \equiv \epsilon^{\alpha\beta\gamma\mu} T_{\alpha\beta\gamma}/6$ of the torsion tensor is $b_\mu = -3(T_A)_\mu/4$, which yields

$$L(u; T_A) = -m\sqrt{-u^2} \mp \frac{3}{4}\sqrt{\big[(T_A)\cdot u\big]^2 - (T_A)^2 u^2} \tag{14}$$

as the all-orders classical lagrangian determining the trajectory of the relativistic point particle in a uniform axial-torsion background.

As another example with a quartic dispersion relation, consider the case with only $H_{\mu\nu}$ nonzero, for which $\mathcal{R}(p) = (p^2 - m^2 + 2X)^2 - 8Xp^2 - 4pHHp + 4Y^2$. Calculations with $H_{\mu\nu}$ can be simplified by noting that all nontrivial observer scalars can be expressed in terms of u^2, the two invariants $X \equiv H_{\mu\nu}H^{\mu\nu}/4$ and $Y \equiv H_{\mu\nu}\widetilde{H}^{\mu\nu}/4$, and the quantity $\alpha \equiv uHHu$. For example, $uH\widetilde{H}u = -Yu^2$ and $uH^4u = Y^2u^2 - 2X\alpha$. Also, an observer basis can be chosen in which $(HH)^\mu{}_\nu$ is diagonal with first two entries $\sqrt{X^2+Y^2} - X$ and last two entries $-\sqrt{X^2+Y^2} - X$. This basis can be further refined via observer Lorentz transformations to impose $u^1 = u^2 = 0$, so calculations can be performed without loss of generality using only the two independent variables u^0, u^3. Nonetheless, the general case remains refractory, so we consider here three special instances.

First, when $Y = 0$ some calculation shows that the polynomial equation $\mathcal{P}(L)$ factorizes, with roots yielding the two particle lagrangians

$$L(u; m, H; X, Y=0) = -m\sqrt{u^2} \pm \sqrt{uHHu + 2Xu^2}, \tag{15}$$

along with $L(u; -m, H; X, Y=0)$ corresponding to the two antiparticle solutions. The particle canonical 4-momenta are

$$p_\mu = \frac{m u_\mu}{\sqrt{u^2}} \mp \frac{(uHH)_\mu + 2Xu_\mu}{\sqrt{uHHu + 2Xu^2}}. \tag{16}$$

The 3-momentum and 3-velocity are typically noncollinear and their zeros noncoincident. When $X < 0$, the dispersion relation can be solved for p_0 to give

$$p_0 = \sqrt{\left(\sqrt{(p_2)^2 + (p_3)^2} \pm \sqrt{-2X}\right)^2 + m^2 + (p_1)^2} \tag{17}$$

for the two positive sheets. The structure is similar to that reported in the case of timelike b_μ [10], except that the sheets touch when the canonical momentum vanishes in the 2–3 plane rather than in all three momentum directions. Since the derivatives $\partial p_0/\partial p_j$ are nonexistent at zero p_j, the energy–momentum space cannot be a manifold. An interesting open question is whether introducing an additional spin-analogue variable would resolve this singularity.

As the second special instance of the case with nonzero $H_{\mu\nu}$, consider $X = 0$. Some calculation reveals that $\mathcal{P}(L)$ becomes quartic in L^2,

$$\begin{aligned}
0 = {} & 4Y^2L^8 - 4Y^2\big(3m^2u^2 + 4\alpha\big)L^6 \\
& + \big[m^4\big(\alpha^2 + 12Y^2u^4\big) + 16m^2Y^2\alpha u^2 \\
& + 24Y^2\alpha^2 - 8Y^4u^4\big]L^4 \\
& - 2\big[m^6\big(\alpha^2 + 2Y^2u^4\big)u^2 + m^4\alpha\big(\alpha^2 + Y^2u^4\big) \\
& + 2m^2\big(5Y^4u^4 - Y^2\alpha^2\big)u^2 \\
& + 8Y^2\alpha^3 - 8Y^4\alpha u^4\big]L^2 \\
& + \big(m^4 + 4Y^2\big)\big(m^2\alpha u^2 - \alpha^2 + Y^2u^4\big)^2.
\end{aligned} \tag{18}$$

All eight solutions for $L(u; m, H; X = 0, Y)$ can be found using the standard solution for the roots of a quartic. Only the perturbative roots are of interest, corresponding to the two particle lagrangians and their antiparticle partners, but their explicit form is cumbersome. This example offers some intuition about the complexity of the classical lagrangians leading to the complete dispersion relation (1).

The third special case is a nonzero $H_{\mu\nu}$ with both observer invariants X and Y vanishing [34]. The dispersion relation for this case is the quartic $(p^2 - m^2)^2 = 4pHHp$. The corresponding polynomial $\mathcal{P}(L)$ can be obtained from Eq. (18) as the limit $Y \to 0$ while noting that $\alpha \neq 0$. The two particle lagrangians $L(u; m, H; X = 0, Y = 0)$ take the form of Eq. (15) with $X \to 0$, as expected.

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