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A NOTE ON ASYMPTOTIC DISTRIBUTION-FREE TESTS FOR SUBHYPOTHESES IN MULTIPLE LINEAR REGRESSION[1]

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The purpose of this note is to show that asymptotically distribution-free (ADF) rank tests proposed by Koul [*Ann. Math. Statist.* **41** (1970), 1273-1281] for testing subhypotheses in multiple linear regression when based on symmetric scores are asymptotically totally inefficient and can even be inconsistent for symmetric parent distributions. Also, the condition given in his Theorem 2.4 for a test to be ADF is not necessary.

1. Introduction. For every $n(\geqq 1)$, let $\{Y_{n1}, \cdots, Y_{nn}\}$ be a sequence of independent random variables, such that

$$F_{ni}(x) = P\{Y_{ni} \leqq x\} = F(x - \beta_1 x_{ni}(1) - \beta_2 x_{ni}(2)), \qquad i = 1, \cdots, n; \tag{1.1}$$

$x_{ni}(j)$, $i = 1, \cdots, n$, $j = 1, 2$ are known constants, and β_1, β_2 are unknown parameters. Our problem is to test

$$H_0: \beta_1 = 0 \quad \text{vs.} \quad H_1: \beta_1 \neq 0 \quad (\text{or} < \text{or} > 0), \tag{1.2}$$

where β_2 is treated as a nuisance parameter.

Koul (1970) considered an arbitrary estimator $\hat{\beta}_n$ of β_2, such that $|n^{\frac{1}{2}}(\hat{\beta}_n - \beta_2)| = O_p(1)$, and considered the following procedure. Let

$$Y_{ni}(b) = Y_{ni} - bx_{ni}(2), \qquad \hat{Y}_{ni} = Y_{ni}(\hat{\beta}_n), \qquad i = 1, \cdots, n, \tag{1.3}$$

and let $R_{ni}(b)$ (or $\hat{R}_{ni}$) be the rank of $Y_{ni}(b)$ (or $\hat{Y}_{ni}$) among the n values in (1.3). Let $\bar{x}_n(j) = n^{-1} \sum_{i=1}^n x_{ni}(j)$, $j = 1, 2$, and let

$$\begin{aligned} S_n^{(1)}(b) &= n^{-1} \textstyle\sum_{i=1}^n (x_{ni}(1) - \bar{x}_n(1))\phi(R_{ni}(b)/(n+1)), \\ \hat{S}_n^{(1)} &= S_n^{(1)}(\hat{\beta}_n), \end{aligned} \tag{1.4}$$

where $\phi(u)$ is absolutely continuous and bounded inside $[0, 1]$. He showed that under certain conditions, $\hat{S}_n^{(1)}$ is ADF under H_0 in (1.2), and hence, a test for H_0 can be based on $\hat{S}_n^{(1)}$. He showed that if F is double exponential and $\phi(u)$ is Freund-Ansary or Mood type (which are both symmetric), then the corresponding $\hat{S}_n^{(1)}$ is ADF. To this result, we add the following.

THEOREM 1. *For the entire class of bounded, absolutely continuous and symmetric scores (i.e., for $\phi(u) = \phi(1-u)$, $u \in [0, 1]$), Koul's test based on $S_n^{(1)}$ is* ADF, *but is totally asymptotically inefficient (in the sense that for local alternatives, the power is equal to the level of significance) when the underlying F is symmetric.*

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THEOREM 2. *Whenever* $\phi(u) = \phi(1-u)$, $u \in [0, 1]$, *Koul's* ADF *tests can also be inconsistent for a fixed* $\beta_1 \neq 0$, *no matter whether F is symmetric or not.*

Let us now define

$$S_n^{(2)}(b) = n^{-1} \sum_{i=1}^{n} (x_{ni}(2) - \bar{x}_n(2))\phi(R_{ni}(b)/(n+1)) , \tag{1.5}$$

so that when $\phi(u)$ is non-decreasing, $S_n^{(2)}(b)$ is non-increasing in $b: -\infty < b < \infty$. Let then

$$\hat{\beta}_{n,1} = \sup\{b : S_n^{(2)}(b) > 0\} , \qquad \hat{\beta}_{n,2} = \inf\{b : S_n^{(2)}(b) < 0\} ; \tag{1.6}$$

$$\hat{\beta}_n = [\hat{\beta}_{n,1} + \hat{\beta}_{n,2}]/2 . \tag{1.7}$$

Finally, let

$$\phi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u)) , \qquad u \in [0, 1] , \tag{1.8}$$

$$b_n(\phi, f) = [n^{-1} \sum_{i=1}^{n} (x_{ni}(1) - \bar{x}_n(1))(x_{ni}(2) - \bar{x}_n(2))] b(\phi, f) ; \tag{1.9}$$

$$b(\phi, f) = \int_0^1 \phi(u, f)\phi(u)\, du = -\int_{-\infty}^{\infty} f'(x)\phi(F(x))\, dx . \tag{1.10}$$

In his Theorem 2.4, Koul (1970) has shown that a sufficient condition for $\hat{S}_n^{(1)}$ to be ADF is that $b_n(\phi, f) = 0$. In this context, we have the following:

THEOREM 3. *Whenever* $\phi(u)$ *is monotonic in* $u \in [0, 1]$, *and* $\hat{\beta}_n$ *is defined by* (1.6)—(1.7), $\hat{S}_n^{(1)}$ *is* ADF, *no matter whether* $b_n(\phi, f)$ *is equal to* 0 *or not.*

The proofs of the theorems are sketched in Section 2. It clearly follows from the first two theorems that symmetric scores (such as Mood's where $\phi(u) = (u - \frac{1}{2})^2$) are unsuitable for this problem, while from the last theorem, it follows that for monotonic scores (such as Wilcoxon's where $\phi(u) = u - \frac{1}{2}$), Koul's condition that $b_n(\phi, f) = 0$ can easily be avoided when the estimator $\hat{\beta}_n$ is based on the same scores.

2. Outline of the proofs.

(i) PROOF OF THEOREM 1. ϕ symmetric implies that $\phi(u) = \phi(1-u)$, $u \in [0, 1]$, while F symmetric implies that $\phi(u, f) = -\phi(1-u, f)$, $u \in [0, 1]$. Hence, by (1.10),

$$b(\phi, f) = 0 \Rightarrow b_n(\phi, f) = 0 . \tag{2.1}$$

Consider then the sequence of local alternatives $\{H_n\}$, where

$$H_n : \beta_1 = \beta_1^{(n)} = n^{-\frac{1}{2}}\theta , \qquad \theta \neq 0 . \tag{2.2}$$

Now, by Theorem 2.3 of Koul (1970), under (2.1),

$$n^{\frac{1}{2}}\hat{S}_n^{(1)} = n^{\frac{1}{2}} S_n^{(1)}(\beta_2) + o_p(1) , \qquad \text{as } n \to \infty , \tag{2.3}$$

where $S_n^{(1)}(\beta_2)$ is defined by (1.4) for $b = \beta_2$. Since by (1.1) and (1.3), $Y_{ni}(\beta_2)$ has the distribution $F(x - \beta_1 x_{ni}(1))$, for $i = 1, \cdots, n$, by the well-known results of Hájek (1962), under (2.2),

$$\mathscr{L}(n^{\frac{1}{2}} S_n^{(1)}(\beta_2) - \theta b_n(\phi, f) \mid H_n) \to \mathscr{N}(0, A^2) , \tag{2.4}$$

where

$$A^2 = \int_0^1 \phi^2(u)\,du - \left(\int_0^1 \phi(u)\,du\right)^2 . \tag{2.5}$$

As shown in (2.1), under the hypothesis of Theorem 1, $b_n(\phi, f) = 0$, and hence, by (2.3) and (2.4), we conclude that for all finite θ, the asymptotic power of the test based on $\hat{S}_n^{(1)}$ remains equal to α, the level of significance. That is, the test is ADF but totally inefficient. □

(ii) PROOF OF THEOREM 2. Let τ_α be the upper $100\alpha\%$ point of the standard normal distribution. Then, we are to show that

$$P\{|n^{\frac{1}{2}}\hat{S}_n^{(1)}| > A\tau_{\alpha/2} \mid \beta_1 \neq 0\} \tag{2.6}$$

does not necessarily converge to 1 as $n \to \infty$. This we show by means of the following example.

Let $n = 2m$, $x_{n1}(1) = \cdots = x_{nm}(1) = -1$, $x_{nm+1}(1) = \cdots = x_{nn}(1) = 1$, $\beta_1 = 2 + \varepsilon$, $\varepsilon > 0$, and let $F(x)$ be such that $F(-1) = 0$, $F(1) = 1$, (the latter can always be made by choosing first an absolutely continuous df G with finite Fisher information, and letting then $F(x) = [G(x) - G(-1)]/[G(1) - G(-1)]$ for $x \in [-1, 1]$, and $F(x) = 0$ for $x \leqq -1$, $F(x) = 1$, for $x \geqq 1$). Then, proceeding as in Koul (1970), we have

$$\max_{1\leq i\leq n} |[Y_{ni} - \hat{\beta}_n x_{ni}(2)] - [Y_{ni} - \beta_2 x_{ni}(2)]| = o_p(1) , \qquad \text{as } n \to \infty . \tag{2.7}$$

and hence, for every $\varepsilon > 0$, there exists an n_ε, such that for $n \geqq n_\varepsilon$, the right-hand side of (2.7) is less than ε, with probability approaching to unity, as $n \to \infty$. Since the variation of F is contained entirely within the range $(-1, 1)$ and $\beta_1 = 2 + \varepsilon$, $\varepsilon > 0$, it follows that for $n \geqq n_\varepsilon$,

$$Y_{ni} - \hat{\beta}_n x_{ni}(2) < Y_{ni} - \hat{\beta}_n x_{nj}(2) , \qquad i = 1, \cdots, m,\ j = m+1, \cdots, n , \tag{2.8}$$

with a probability arbitrarily close to unity as $n \to \infty$. Hence,

$$P\{|\hat{S}_n^{(1)} - n^{-1} \textstyle\sum_{i=1}^{m} [\phi((i+m)/2m) - \phi(i/2m)]| = 0\} \to 1 \qquad \text{as } n \to \infty . \tag{2.9}$$

Since $\phi(u) = \phi(1-u)$ for all $u \in [0, 1]$, by (2.9),

$$n^{\frac{1}{2}}\hat{S}_n^{(1)} = 0 \quad \text{with probability tending to 1} \qquad \text{as } n \to \infty , \tag{2.10}$$

which shows that (2.6) does not converge to 1 as $n \to \infty$, so the test is inconsistent. The same proof holds for $\beta_1 < -2$ or for $|\beta_1| > 2$. □

(iii) PROOF OF THEOREM 3. Let $C_{jj'}^{(n)} = n^{-1} \sum_{i=1}^{n} (x_{ni}(j) - \bar{x}_n(j))(x_{ni}(j') - \bar{x}_n(j'))$, for $j, j' = 1, 2$. Then by Theorem 2.3 of Koul (1970), as $n \to \infty$,

$$n^{\frac{1}{2}}[\hat{S}_n^{(1)} - S_n^{(1)}(\beta_2)] = -n^{\frac{1}{2}}(\hat{\beta}_n - \beta_2) b_n(\phi, f) + o_p(1) . \tag{2.11}$$

On the other hand, under (1.2), $\beta_1 = 0$, so that when $\hat{\beta}_n$ is defined by (1.6) and (1.7), by the fundamental results of Jurečková (1969), as $n \to \infty$,

$$n^{\frac{1}{2}}|S_n^{(2)}(\beta_2) - (\hat{\beta}_n - \beta_2) b_n(\phi, f)| = 0_p(1) . \tag{2.12}$$

Consequently, by (2.11) and (2.12), under (1.2), as $n \to \infty$,

$$n^{\frac{1}{2}}\hat{S}_n^{(1)} = n^{\frac{1}{2}}[S_n^{(1)}(\beta_2) - S_n^{(2)}(\beta_2)] + o_p(1) . \tag{2.13}$$

Finally, under (1.2), $n^{\frac{1}{2}}[S_n^{(1)}(\beta_2), S_n^{(2)}(\beta_2)]$ jointly has asymptotically a bivariate normal distribution with mean (0, 0) and covariance matrix $A^2\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, where $C_{jj'} = \lim_{n\to\infty} C_{jj'}^{(n)}$, $j, j' = 1.2$. Hence, by (2.13), under H_0 in (1.2),

$$\mathscr{L}(n^{\frac{1}{2}}\hat{S}_n^{(1)}) \to \mathscr{N}(0, A^2(C_{11} + C_{22} - 2C_{12})) , \qquad \text{as } n \to \infty , \tag{2.14}$$

and as A^2 as well as the $C_{jj'}$ are known quantities, the result follows. □

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