Mathematical Caring Relations: A Challenging Case

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Abstract

Developed from Noddings's (2002) care theory, von Glasersfeld's (1995) constructivism, and Ryan and Frederick's (1997) notion of subjective vitality, a mathematical caring relation (MCR) is a quality of interaction between a student and a mathematics teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning. In this paper I examine the challenge of establishing an MCR with one mathematically talented eleven-year-old student, Deborah, during an 8-month constructivist teaching experiment with two pairs of eleven-year-old students, in which I (the author) was the teacher. Two characteristics of Deborah contributed to this challenge: her strong mathematical reasoning and her self-concept as a top mathematical knower. Two of my characteristics also contributed to the challenge: my request that Deborah engage in activity that was foreign to her, such as developing imagery for quantitative situations, and my assumption that Deborah's strong reasoning would allow her to operate in the situations I posed to her. The lack of trust she felt at times toward me and the lack of openness I felt at times toward her impeded our establishment of a MCR. Findings include a way to understand this dynamic and dissolve it to make way for more productive interaction.

Key Words: caring relations, mathematical caring relations, student-teacher relationships, mathematically talented students, mathematical self-concept

Mathematical Caring Relations: A Challenging Case

Noddings (2002) characterises caring relations as an evolving interaction based on receptivity and responsiveness: A teacher receives students' experiences and responds by helping students realise their goals and ideas. In turn, students receive the teacher's care and show some evidence of this reception—e.g., demonstrate renewed interest or increased energy, such as a "glow of well-being" (p. 28). This evidence from students is, in turn, what a teacher needs to receive in order to continue to care. So in Noddings's care theory, receptivity and responsiveness between teachers and students help them build the experience of relatedness, which psychologists view as a fundamental human need (Baumeister & Leary, 1995). Fulfilling this need can augment the level of positive energy a person experiences as available to the self, or a person's *subjective vitality* (Ryan & Frederick, 1997). Since Noddings describes the reception of care in terms of increased energy, feeling subjective vitality can be considered a marker of the establishment of caring relations.

Based on Noddings's (2002) care theory and the construct of subjective vitality, I define a *mathematical caring relation* (MCR) as a quality of interaction between a student and a mathematics teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning (Hackenberg, 2005b). Mathematics teachers may act as carers in general, but they start to act as *mathematical* carers when they hold their work of orchestrating mathematical learning for their students together with an orientation to monitor and respond to fluctuations in subjective vitality (Ryan & Frederick, 1997) that may accompany student-teacher interactions.

In this paper I examine the challenge of establishing an MCR with one mathematically talented eleven-year-old student, Deborah, during an 8-month constructivist teaching experiment with two pairs of eleven-year-old students, in which I (the author) was the teacher. Two characteristics of Deborah contributed to this challenge: her strong mathematical reasoning and her self-concept as a top mathematical knower. My responses to these characteristics—how I received them—led to a good deal of discomfort for both of us. The discomfort ranged from mild uneasiness to outright fatigue and tension, all of which could deplete our subjective vitality and weaken our MCR. Broadly, this paper highlights an understudied area in mathematics education: The influence of student-teacher relationships on interaction aimed toward mathematical learning (cf. Vithal, 2003). More specifically, this paper addresses this question: What are the challenges involved in establishing an MCR with a mathematically talented female student?

Student-Teacher Relationships

To respond to this question, I position my study in relation to prior research on student-teacher relationships. According to Davis (2003), student-teacher relationships have been studied from the perspectives of attachment theory, motivation theories, and socio-cultural theories. Since research based on attachment theory focuses on very young children, the latter two perspectives are more relevant for this study. Research based on motivation theories (e.g., Ryan, Stiller, & Lynch, 1994; Wentzel, 1997) has sought to uncover correlations between student-teacher relationships and various outcomes for students, usually at the general level of schooling rather than in particular academic domains. This research supports conclusions that supportive student-teacher relationships are positively correlated with outcomes such as achievement (Cornelius-White, 2007), appreciation of mathematics (Midgley, Feldlaufer, & Eccles, 1989), academic effort (Wentzel, 1997), and self-esteem (Ryan, Stiller, & Lynch, 1994). In contrast, research from a socio-cultural perspective (e.g., Averill & Clarke, 2006; Gay, 2000; Pianta, 1999) portrays relationship formation between students and teachers as embedded within larger contexts of school, community, and culture. For example, Averill and Clarke studied student and teacher perspectives on teacher caring to develop recommendations for cultivating effective student-teacher relationships in New Zealand mathematics classrooms.

Although my study has some commonalities with both perspectives, it also contrasts with their purposes. In contrast with research from the perspective of motivation theory, I aimed to understand how student-teacher relationships form. This purpose is similar to research from a socio-cultural perspective, but in contrast to that approach I focused intensively on student-teacher dyads rather than on larger units of analysis. In addition, my study examined the *teacher's* energetic fluctuations in the establishment of an MCR: Including the needs of both teacher and student in conceiving of care is a critical aspect of Noddings's (2002) theory. However, according to Davis (2003), little research on student-teacher relationships has investigated "…teachers' attempts to meet their own interpersonal needs within the classroom" (p. 226). Finally, in this study I aimed to study caring student-teacher relationships in the particular academic domain of mathematics, which is unusual in prior research (see Averill & Clarke, 2006, and Midgley, et al., 1989, for exceptions).

Mathematical Learning, Mathematical Caring, and Mathematical Self-Concepts

Based on these aims, I frame the paper using three conceptual components: my model of mathematical learning, my formulation of mathematical caring, and my understanding of students' mathematical self-concepts. I discuss learning first in order to set up terms that I use to characterise mathematical care.

Mathematical Learning

I conceive of mathematical learning in the context of accommodations—modifications or reorganizations—in schemes and operations (cf. Piaget, 1970; von Glasersfeld, 1995).

Operations are mental actions, such as partitioning a rectangle into some number of equal parts. Schemes are goal-directed "assemblages" of operations that consist of three parts: a perceived or assimilated situation, an activity, and a result that a person assimilates to her or his expectations. For example, one kind of multiplying scheme involves coordinating units within units (Steffe, 1992). Students who have constructed this kind of multiplying scheme can use it to find one-fifth of one-fourth of a rectangle by inserting fifths into each fourth of the rectangle, thus determining that the smallest parts are twentieths.

If the result of a scheme is as the person expects, the scheme is likely to close and no modification or reorganization need occur. However, if the result is unexpected in some way, the person may enter a state of perturbation: a disturbance that is not necessarily within conscious awareness, but that can be accompanied by surprise, interest, or even dismay (von Glasersfeld, 1995). So, experiencing a perturbation often entails an emotional response (Hackenberg, in review). To eliminate a perturbation, the person may make an accommodation in any of the three parts of her scheme and possibly construct a new scheme out of prior schemes.

The emotional response to a perturbation is also entwined with fluctuations subjective vitality, or an *energetic* response. Depletion of vital energy may occur if a person senses that he does not know what to do to eliminate the perturbation, or that such activity will be onerous. If depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity immediately or in the future. Perturbations can also provoke enhancement of subjective vitality in the form of a challenge, particularly if a person senses that she can meet that challenge, or that doing so will be enjoyable. If a feeling of subjective vitality is sufficient, the student's interest in a situation may prolong mathematical activity and open new opportunities for learning. If, over time, subjective vitality is enhanced (even though there may be times of depletion), the student may feel mathematically cared for.

Mathematical Caring

From a teacher's perspective, establishing MCRs with students involves a cyclical process of harmonising with students' schemes and energetic responses to mathematical activity; challenging students to expand their mathematical ways of operating;[i] assessing students' cognitive and energetic responses to these challenges; and recalibrating efforts to harmonise and challenge (cf. Noddings, 2002). Harmonising with students' schemes and energetic responses to mathematical activity is a dynamic process of making interpretations of students' current schemes and operations and basing interaction on those interpretations so that tasks posed to students are sensible to them (Hackenberg, in review). Thus harmonising with students' schemes and energetic responses to mathematical activity requires significant cognitive decentering. I view cognitive decentering as the practice of setting one's own concepts to the side in order to formulate a coherent description of the mental activity of a student that can be the basis for interaction with that student (cf. Steffe & Thompson, 2000).

From a student's perspective, participating in an MCR with a teacher involves being open to the teacher's interventions and pursuing questions of interest. Experiencing mathematical care can lead to feelings of positive energy available to the self (cf. Noddings, 2002), or subjective vitality. Subjective vitality can manifest as a positive aroused state (Ryan & Frederick, 1997), such as excitement or enjoyment, or as calm energy, a state of high energy combined with relaxation (Thayer, 2001). In contrast, feeling taxed, fatigued, or tense tiredness (a combination of tension and low energy, Thayer, 2001) are states where subjective vitality is depleted. When engaging in mathematical activity aimed toward learning, sustaining some depletion is often necessary for subsequent enhancement of energy. However, prolonged or severe depletion may contribute to students' withdrawal from mathematical activity and thus may curtail learning.

Mathematical Self-Concepts

Prolonged or severe depletion may also have an effect on the evaluations people make of their experiences, which is the main source of people's self-concepts. Researchers have advocated for distinguishing self-concepts in relation to academic domains because self-concept is not a monolithic construct (Marsh & Koller, 2003; Marsh & Shavelson, 1985). Although definitions differ, in general a *mathematical self-concept* is considered to be collective perceptions of one's ability to do and know mathematics, and such perceptions are formed in relation to others (Ingram, 2007; Marsh & Shavelson, 1985; Schunk, 1991).

Researchers have found that students' mathematical self-concepts are positively correlated with achievement (Craven, Marsh, & Burnett, 2003; Marsh & Koller, 2003). These findings hold true for talented/gifted students, although these students usually have higher mathematical self-concepts than the general student population (Hoge & Renzulli, 1993; Pajares & Graham, 1999). In particular, researchers have found that talented/gifted girls are biased toward a lack of confidence in their mathematical abilities (Junge & Dretzke, 1995; Pajares & Graham, 1999; Siegle & Reis, 1998), despite the fact that the self-perceptions of talented/gifted students tend to be more accurate than those of the general student population (Pajares, 1996).

Methodology and Methods of Inquiry

Prior to the study, I used my formulation of mathematical caring to guide my actions during the 8-month constructivist teaching experiment. In this section I give a brief overview of this methodology, how I adapted it to study MCRs, and specific methods I used.

Teaching Experiment Methodology

A central goal of constructivist teaching experiment methodology (Steffe & Thompson, 2000) is to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching. Since researchers' mathematical knowledge is likely insufficient to understand students' ways of operating, researchers aim to learn mathematical ways of operating from students. Researchers also make conjectures about how students might construct new schemes and test the conjectures through posing tasks in teaching episodes. Teaching practices include presenting students with problem situations, analyzing students' responses, and determining new situations that might allow students to make accommodations.

Because of the emphasis on learning mathematics from students while opening possibilities for them to construct new schemes, this type of teaching experiment methodology is well-suited for studying the establishment of MCRs: Built into the methodology are opportunities to harmonise with students' mathematical ways of operating and to challenge students to expand their ideas beyond what they may imagine possible. However, to study MCRs I adapted the methodology so that students' cognitive *and* affective responses influenced how I intervened in teaching episodes. The situations and tasks I posed were aimed at constructing a model of the students' mental activity—the usual goal of the methodology—as well as a model of students' energetic responses to our mathematical interactions. So, I attended to students' energy levels and curtailed our activity if depletion seemed too great, or prolonged our activity (e.g., by extending problems, or posing what I thought were greater challenges) if students seemed to be experiencing a balance between depletion and enhancement of subjective vitality.

In addition, I decided against asking students explicitly about their energetic responses to our interactions for two reasons. First, I was interested in studying energetic responses, and the management of those responses, in the moments of interaction. Energetic responses to interactions are not always within conscious awareness, so getting a “read” on students’ energetic states by asking them to articulate their responses may not be possible. Second, I was not interested in studying how students hold out their energetic responses for inspection, but in the energetic “flow” as the student-teacher interaction proceeded. I decided that explicit questioning could interfere with this flow.

My Teaching Experiment

Participants and structure. Four 11-year-old students from a rural middle school in the southeastern United States participated in my teaching experiment from October 2003 to May 2004 (Hackenberg, 2005a). They were selected to participate based on unrecorded selection interviews in which I assessed students’ multiplicative concepts. My aim was to select students who demonstrated a range of reasoning multiplicatively with whole numbers, so I invited and paired the students based on these initial assessments. The girls, Deborah and Bridget, were both Caucasian. Deborah was in the top-level mathematics class. Although the school had not “labeled” her as a talented/gifted student, she was regarded very highly by all of her teachers and earned top grades in all of her classes. Bridget was also regarded as bright, but she was in the second highest mathematics class. The other pair of students in the study was in the middle level mathematics class (out of five levels).

I taught the pairs of students twice per week for two to three weeks, followed by a week off, which resulted in 35 episodes for Deborah and Bridget and 33 for the other pair of students. Each teaching episode lasted approximately 30 minutes and was videotaped with two cameras. One camera focused on the interaction between the students and the teacher, and the other captured the activity of the students on paper or computer. In most teaching episodes, the students worked with a computer software program called JavaBars (Biddlecomb & Olive, 2000), which was designed to foster students' mental actions in building fraction schemes. Two witness-researchers offered feedback on teaching activities and provided triangulation of perspectives in on-going and retrospective data analysis (Steffe & Thompson, 2000).

Analytic process. On-going analysis occurred at the time of the teaching episodes. The main goal of on-going analysis is to make working models of the students' schemes and operations, and in my case, of the students' energetic responses to mathematical activity. Central activities of on-going analysis include making local conjectures about students' current schemes and designing new problems and problem sequences for the next teaching episode. In my use of the methodology, I also made conjectures about students' energetic responses. In addition, I kept a research journal of my impressions after each episode, including my assessment of fluctuations in the students' and my own subjective vitality.

Intensive retrospective analysis of videofiles and student work occurred following the last episode of the experiment. I began retrospective analysis by reviewing all videofiles and creating a chronology of the path that I took with each student through the experiment (Cobb & Whitenack, 1996). My notes focused on five categories: what students were able to do mathematically; changes in how students operated; constraints that I experienced with the students; when students indicated a reception of my mathematical care; and when students did not seem to receive mathematical care. My criteria for the latter two categories were my assessments that students appeared to be experiencing (a) maintenance or enhancement of subjective vitality; (b) some depletion that was not severe or prolonged; or (c) significant depletion. I used all five categories to identify critical segments within teaching episodes on which I would focus deeper analysis.

I used these critical segments to make analytic models of the students' schemes (the usual goal of the methodology). Then, to understand MCRs I did a second analysis from the perspective of the student-teacher pair. My goal was to uncover chains of student-teacher interactions that contributed to and detracted from the establishment of an MCR with each student. First I categorized chains of interaction according to when my actions as a teacher seemed in harmony with a student's mathematical way of operating; when my actions appeared to trigger productive mathematical activity (e.g., an accommodation); when I responded to a student in a way that did *not* seem productive; and when I recalibrated my response (intentionally or not) in an interaction with a student. For each chain of interaction I also coded the students' and teacher's body language, facial expressions, verbal expressions, and verbal tone to develop a more fine-grained picture of when the students and teacher seemed to experience enhancement or depletion of subjective vitality. I used these codes to write analytic memos that served as the foundation for telling a "story" of my MCR with each student. Finally, I consulted my research journal for my assessment of fluctuations in my own subjective vitality, and my conjectures about students' energetic responses, at the time of the experiment.

Analysis

In this section I analyze three data excerpts to demonstrate the challenge of establishing an MCR with Deborah. The excerpts show how Deborah's strong mathematical reasoning[ii] led me to undervalue her cognitive difficulties, which in turn led to a lack of harmonising with her schemes and energetic responses and, at times, the posing of inappropriate challenges for her. The excerpts also show how our work in the experiment threatened Deborah's mathematical self-concept, which could lead to sustained depletion of subjective vitality.

December: Establishing an MCR with Deborah, with a Caveat

I present a glimpse of Deborah in December for three purposes: (a) to demonstrate the nature of Deborah's mathematical power; (b) to show that Deborah and I had established an MCR early in the experiment; and (c) to give an indication of the challenges to follow.

Deborah's success in transforming fraction bars. On December 2nd, both Deborah and her partner Bridget solved problems like making a 2/2-bar into a 7/6-bar without erasing the half mark on the 2/2-bar.[iii] They did so by inserting more parts into each original part of the bar (in this case, they inserted three parts into each half, Figure 1a), and then they pulled out the number of parts needed to make the improper fraction (in this case, one part, Figure 1b). So, first they transformed halves into sixths, and then they made a bar consisting of seven sixths.

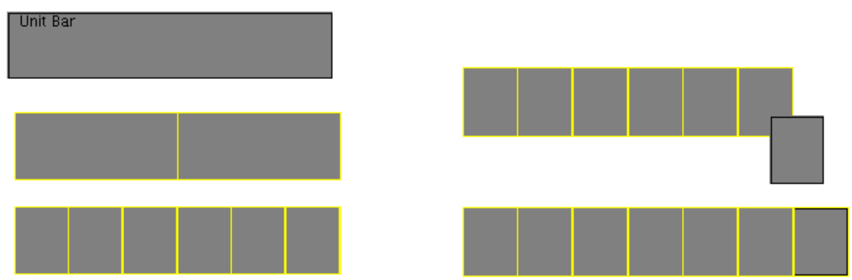

Figure 1a (left). Inserting three parts into each half of the 2/2-bar.

Figure 1b (right). Pulling out one part to join to the 6/6-bar, making a 7/6-bar.

Subsequently I began to pose problems like making a 4/4-bar into a 6/5-bar without erasing the fourths marks, which met with a good deal of concern and surprise. Deborah solved this problem by inserting more parts into the fourths to create a number of "mini-parts" (20) that she used to make fifths (Figure 2). For Deborah, making 20 mini-parts was useful because it

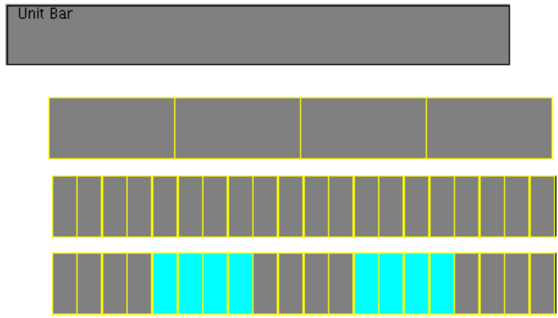

Figure 2. A 4/4-bar converted into a 5/5-bar.

allowed her to view the bar as a unit of five units, each containing four units. Then, since each fifth consisted of four mini-parts (twentieths), she pulled out four mini-parts to join onto the bar. Deborah's solution indicated that she viewed the bar as two different unit structures: as a unit of four units each containing five units, and as a unit of five units each containing four units (Hackenberg, 2005a). Although Bridget interacted with Deborah during their work on this problem, she did not produce an independent solution.

When I posed another problem similar to this one (make a 3/3-bar into a 9/8-bar without erasing the thirds marks), Deborah's immediate response was, "Oh man, you're trying to hurt our brains!" We all laughed, but I infer that her comment was significant: These problems were challenging for her and she had not yet constructed a scheme to solve them. However, she did solve this problem using the coordination of units within units as shown above. In subsequent episodes, she solved more problems like these by searching for a total number of parts that could be partitioned to show both fractions. So, she was using her multiplying scheme in service of her scheme for making fractions (Hackenberg, 2005a). In contrast, Bridget struggled with such problems during the time that we worked on them, and she appeared to experience considerable depletion of subjective vitality over the three episodes that we worked on these problems.

On December 9th I asked the girls to make an 8/8-bar into a 13/12-bar—a potentially (though not necessarily) more difficult task than previous ones since 8 and 12 are not relatively prime. After about 30 seconds of silence, Deborah had an idea for solving the problem that was similar to how she had solved prior problems, using 24 parts total. While she explained her solution, Bridget sat back with her head resting on her hand. I was growing very concerned about Bridget's inability to operate in these situations, but I myself didn't have a way of operating to help her to act in the situation. I tried to involve her by asking her why Deborah's solution would work. Although Bridget showed some understanding of what Deborah had done, she remained glum. I then asked the girls if there was *another way* to make an 8/8-bar into a 13/12-bar. Deborah suggested using 48 parts total, and so we worked on how many of those parts would need to be added to make thirteen-twelfths. This activity seemed a bit onerous for both girls, and we became quiet. Bridget said she was sleepy. I hesitated, trying to figure out what to do to help Bridget alleviate her depletion and be able to operate. Finally Deborah asked me, a little suspiciously: "Do *you* know [how many forty-eighths to add on]?" She stated that she knew.

Establishing an MCR with Deborah. By December, after the teaching experiment had been in progress for just over one month, Bridget and Deborah and I had worked on a variety of problems involving improper fractions, as well as problems that involved coordinating units within units (Hackenberg, 2005a). So, in posing problems like making a 4/4-bar into a 6/5-bar, I was testing out the extent to which Deborah and Bridget could bring together both ways of operating in solving a problem. That Deborah solved such problems during the very first episode in which we worked on them demonstrates the swift and powerful ways that she coordinated her schemes. It also shows that I was posing problems that harmonised with her schemes: These were problems for which she had ways of operating at her disposal.

Yet Deborah clearly seemed to experience the problems I posed as challenging since she repeatedly asked if I was "trying to hurt their brains". Her manner in asking this question was partly joking. Even so, it indicates that these problems provoked perturbations for her, and she had to create ideas for how to solve them (and eliminate the perturbations). When she did so, her tone became excited, and her body language demonstrated eagerness. Her excitement—a state of invigoration or increased energy—was not uncommon. Throughout the teaching experiment she often became excited about figuring out a mathematical problem or idea, thereby eliminating a perturbation. Although her excitement seemed to be part of her personality, it also seemed to occur because solving these *particular* problems was enjoyable or satisfying to her. She did not explicitly state that these problems were fun or interesting, but she communicated a sense of engaged enjoyment, perhaps partly from the tussle involved in figuring out something "hard". So she seemed to derive enhancement of subjective vitality from her activity, and thus I seemed to be establishing an MCR with Deborah in December.

Deborah's mathematical self-concept. Deborah also may have demonstrated excitement over solving these problems because doing so confirmed her mathematical self-concept as someone who could find answers to mathematical problems quickly. Deborah seemed to gain an identity-level satisfaction from her accurate and swift computational abilities. Indeed, her computational abilities with whole numbers were *not* algorithmic but quite sophisticated: She had embedded a distributive operation in her multiplying schemes with whole numbers so as to regularly engage in strategic reasoning. For example, in February Deborah reasoned as follows to compute 104 divided by 8: The result must be 13 because 10 times 8 is 80, and then there are only 24 more, and 3 times 8 is 24. So, I infer that being able to make sense of the problems in early December, and to find solutions for them when Bridget struggled, corroborated Deborah's view of herself as a mathematical knower. Furthermore, if Deborah did not understand straight away what to do to solve a problem, her response to me was “you’re confusing me”. This comment, which could be mildly to overtly accusatory, indicated to me that she expected to understand immediately and had little tolerance for not knowing what to do. Her expectation was likely based on a history of understanding mathematical work in school very quickly. Thus if she didn’t understand immediately, from her point of view it was probably the teacher’s fault.

The notion that Deborah considered a threat to her mathematical self-concept to lie with the teacher is consistent with my interpretation of our interactions on December 9th. As I have described, during that teaching episode I was in a state of perturbation with Bridget. While we worked on how to make the 8/8-bar into a 13/12-bar using forty-eighths, I was at a loss, searching for a way to reestablish an MCR with Bridget. The interaction seemed to slow down because of my uncertainty and Bridget’s feelings of depletion. I infer that Deborah began to wonder why I repeatedly asked her to explain her reasoning, and why I seemed hesitant. When Deborah asked me, “Do *you* know?”, I infer that she interpreted my hesitation as not knowing how to solve the problem. From her perspective, it was likely the job of the teacher to tell students like Bridget how to do something if they did not know how. Since I (purposefully) did not do that, I garnered Deborah’s suspicions of me as a teacher. Thus I infer that in this episode Deborah’s perception of me altered: I believe I lost some of her respect or trust. This change foreshadowed further complications in our MCR during the rest of the experiment.

February: A Rupture in Our MCR

Work on the Box Problem. From mid-January to mid-February, the girls worked on drawing pictures of quantitative situations and on notating relationships between knowns and unknowns. Deborah did not like drawing pictures, especially when she thought she already knew the answer to a problem. As she stated, "I'm good at multiplying; I'm not good at drawing." On February 18th, I posed this problem, which was complex enough to require pictures:

> *The Box Problem.* Two groups of students, the Cobras and Lizards, have a box-stacking contest. The Cobras' tower is 3/4 of a decameter tall, and that's 2/3 the height of the Lizards' tower. How tall is the Lizards' tower?

Deborah stated flatly that she couldn't figure it out, while Bridget drew a rectangle partitioned into three equal parts to represent 3/4 dm, the Cobras' tower height. I suggested to Deborah that she draw a picture, but she refused. As will be seen in the following data excerpt, over the approximately 5 minutes we spent on the problem she appeared to be in a state of perturbation that was consciously conflictive and quite uncomfortable, and she did not find a way of operating to eliminate the perturbation. Because I am focused on Deborah in this paper, in the data excerpt I omit most of the dialogue with Bridget.

> *Excerpt 1:* Stumped by the Box Problem on February 18th. [iv]
>
> [Bridget states that the Lizards' tower must be bigger because 3/4 dm is only two-thirds of the Lizards' height. Deborah moans a little and writes on her paper. Then she rubs her paper back and forth on the table. Bridget continues to confer about the problem with the teacher-researcher. She indicates she wants to partition the Cobras' height into two equal parts.]
> T [to Deborah]: Can you draw a picture, Deborah?
> D [head in hand, looking down toward her paper]: I'm thinking.
> T: Oh, you're thinking, okay.
> D: It's not working, my brain's not working.
> T: Oh, okay.
> [Bridget draws a second rectangle to represent the Lizards' height, see Figure 3, right. In her drawing, each third of this new rectangle is about the same size as half of the Cobras' height. The teacher-researcher reaches across Deborah's work space to point to Bridget's drawing, asking how much half of the Cobras' height is. Bridget articulates that half of the Cobras' height is one-third of the Lizard's height.]

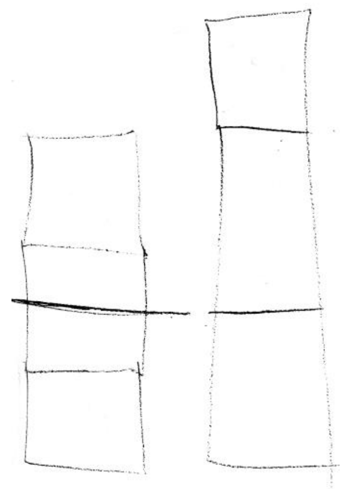

Figure 3. Bridget's drawing of two rectangles to represent the height of the Cobras' tower, 3/4 dm (left) and the unknown height of the Lizards' tower (right).

T [to Bridget]: So could you figure out how tall the Lizards' tower is?
D [answering the teacher-researcher's question]: No.
[While the teacher-researcher questions Bridget about her picture, Deborah taps her pencil. Then she drops her head into her arms and moans. She looks up at the conversation that Bridget and the teacher-researcher are having, and then drops her head again for about 5 seconds.]
B: I don't know!
T: Oh, I think you're doing well.
[Deborah moans. Then she picks up her pencil and starts to write. Meanwhile, the teacher-researcher has asked Bridget how tall half of the Cobras' height is.]
D [murmuring to herself as she writes]: This is two-thirds… [She makes a small exclamation and covers her mouth, as if she's figured something out. Then she grabs her paper and starts to write excitedly. She computes a long division problem on her paper, 8.0 divided by 3.]
D [tapping her head]: Think, think think, think.
T [to Bridget]: How can you take half of three-fourths?
[Deborah moans again. She puts her palms to her forehead, then taps her head with her palms. Then she puts her head in her hand and hits her pencil loudly on the table.]
D [leaning down, knocking her head on the table]: Oh man! [Looking up from her work] What? [She squeals as if in frustration, and hits her pencil against the table again.] Oh man. Oh man. [Muttering] I don't know how to help myself.
[Meanwhile, Bridget has drawn a rectangle to represent a full decameter partitioned into fourths. She cuts all of the fourths into two equal parts, Figure 4, leftmost rectangle.]

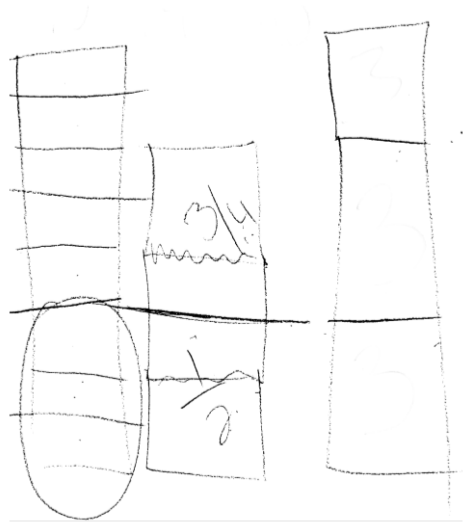

Figure 4. Bridget's drawing of the full decameter partitioned into eighths (leftmost rectangle).

B: Three eighths!
T [surprised]: Oh, awesome. So where's three eighths? [Bridget shows it on her drawing, see the circled part of the leftmost rectangle in Figure 4. Deborah puts her head down again.] Okay, really good. [To Deborah] Now what are you thinking about Deborah?
D: Nothing.
T: It's not working for you right now, huh?
D: No.
B [still looking at her paper]: Wait a minute.
T: Okay, well, I think that was a really good start. We didn't figure out how tall the tower is, though.
B [gesturing to her paper]: This is—this is nine-eighths! [The teacher-researcher requests that Bridget show Deborah what she did, and how the picture helped her solve the problem. Deborah stabs dots onto her paper with her pencil.]

As shown in the protocol, Bridget used her picture, albeit with a lot of coaching, to solve the problem. Deborah did not draw, despite my invitations and her awareness that Bridget was both drawing and making progress. In a word, she appeared to be "blocked." My basic explanation of Deborah's block rests on the nature of the reasoning that she used to produce swift answers to mathematical problems. As I have noted, throughout the experiment she demonstrated adept computational abilities that were often based on sophisticated multiplicative reasoning with whole numbers. So I infer that she had a rich set of images when operating with whole numbers, but her aversion to drawing indicates that these images were probably entirely implicit in her thinking. In particular, her response to the Box Problem indicates that she had not yet developed that level of sophistication—or private imagery—with her fraction calculation. On February 18th, being unable to generate imagery in order to find a viable computational avenue seemed to prolong her state of perturbation and block schemes she might have used to solve the problem—which was clearly very frustrating for her.

Caring mathematically for Deborah. My intent in posing the Box Problem was to care mathematically for Deborah by presenting a significant challenge. As I have noted, previously Deborah had seemed to thrive on challenges. So, I predicted that posing a difficult problem would ultimately enhance her subjective vitality, even if she also experienced some depletion in the solution process. In addition, I conjectured that the Box Problem could help Deborah confront some limitations in her tendency to use a computational approach without explicit reference to quantities. I hoped that the problem would prompt her to use a picture and, ultimately, to learn to operate more powerfully. Thus I was aiming to help her expand her mathematical world beyond what she might imagine possible, a hallmark of establishing MCRs.

However, during the episode I also acted inadequately as a mathematical carer for Deborah because her aversion to drawing and her lack of imagery in the situation left her with no way to engage with the problem. I invited her to draw, and I deliberately made space for her to experience conflict over not being able to operate. But I did not fully appreciate that Deborah *did not know how* to draw a picture in this situation. I interpreted her resistance to and suspicion about drawings mostly at an affective, not a cognitive, level. Had I understood that Deborah did not know how a drawing, or the generation of imagery, could be useful, I would have tried to tailor a sequence of problems for her that addressed that issue more purposefully—and that did not leave her experiencing sustained and significant depletion.

Mathematical self-concept threatened. My approach to helping Deborah expand her mathematical world made her feel quite vulnerable by, from her perspective, threatening her mathematical self-concept. As I have noted, self-concepts are formed in relation to others, so Bridget's relative "success" in this episode, the praise she received from me for her work, and the rarity of her solving a problem prior to Deborah, all appeared to contribute to this threat. I did not act as swiftly as I might have to try to counter Deborah's experience of frustration and tension for three reasons: (a) This level of depletion was rather rare and unexpected with her; (b) I assumed her ways of operating were powerful enough that she would be able to determine a way to operate in the situation; and (c) I sensed that she might need to feel some frustration before she might be open to drawing a picture to work on the problem.

Deborah was good humored in subsequent episodes, so I can infer that our MCR was not completely abolished on February 18th. However, during the rest of the experiment periodically she asked me whether I thought she was smart, and she continued to resist drawing pictures. So Deborah's experience of sustained depletion seemed to shake her confidence, at least with regard to the mathematical work in which we engaged, and to solidify her idea that drawing pictures to solve mathematics problems was *not* something she could do.

Deborah's activity as a perturbation for the teacher. At the time of the episode I accomplished one of my goals for Deborah, which was for her to experience the difficulty of solving a problem if she refused to draw a picture. However, upon review of the videofile of the episode three days later during on-going analysis, I felt abashed because I believed I had not supported Deborah well during her work on the Box Problem and was not in harmony with her mathematical ways of operating or her energetic responses.

During the Box Problem I did not support Deborah in at least two ways, and I have already explained one of them: I did not conceive of her difficulty in solving the problem to include a cognitive component. The second way was more problematic for me as a teacher. By February Deborah's suspicion of me as a teacher, combined with her tendency to put down Bridget when Bridget seemed to be posing a threat to Deborah's mathematical self-concept, dampened my response to Deborah's mathematical activity. I still expressed interest and enthusiasm to Deborah, and I still praised her work. But I could feel that my emotional response to her was not quite as open as my responses to the other students. I had a rationale for my response: I believed that Deborah did not need overt praise from me in the way she may have earned it from many of her teachers. Yet it was still quite uncomfortable for me since I aimed to engender a feeling of trust between us, and I felt a lack of it in my own lack of openness. Thus our interactions often produced tension in me and decreased my own subjective vitality.

So, following the February 18th episode I entered a state of perturbation regarding how to better harmonise with Deborah's ways of operating and energetic responses. I began to eliminate my perturbation by shifting problem contexts and aiming for the girls to engage in more exploratory mathematical activity so that I could revise my working models of their ways of operating. However, as I will demonstrate in the next section, doing so did not fully smooth out the challenge of establishing an MCR with Deborah.

May: Uneasy Maintenance of Our MCR

Deborah solves a "Box-like" problem. Deborah did solve problems like the Box Problem, but not until May. During the four episodes that month each girl worked on a separate laptop computer positioned side-by-side on the table. For the last two teaching episodes on May 10th and 12th, I differentiated problems that I posed to Deborah and Bridget in an attempt to adapt to my perception of significant differences in their ways of operating. On May 12th this problem was one of the first ones I posed to Deborah, about a homemade racecar contest between two teams from a science class:

> *Racecar Problem.* The Lizards' car goes 2/3 of a meter. That's 3/4 of the distance the Cobras' car traveled. Can you make the distance the Cobras' car traveled and tell how far it traveled?

Deborah worked on this problem for 1 minute and 45 seconds while I interacted with Bridget. During that time, Deborah made two-thirds of a unit meter and then partitioned each of the thirds into six equal parts. She pulled out one of those parts and used it to make a 16-part bar (Figure 5). As with Excerpt 1, in the following data excerpt I omit dialogue with Bridget.

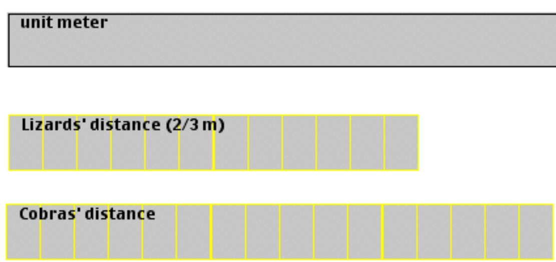

Figure 5. The Lizards' distance (2/3 meter) and Cobras' distance (16-part bar).

Excerpt 2: Deborah making the Cobras' distance on May 12th.

D [rests her forehead in her left hand while she uses the mouse with her right hand, appearing to be in deep concentration]: Okay [sits up].
T [looking at Deborah's screen]: Whoa—holy smokes! What is all that?
D [pointing to the partitioned 2/3-bar]: Since this is two-thirds of a meter, and I divided it into twelve pieces because I needed three-fourths.
T: Oh! Okay.
D: And I put sixteen twelfths here [points to the 16-part bar].
T: Sixteen-twelfths.
D: Because that, that is—'cause I knew each third is four pieces. So four times four, because you need four thirds for this one [points to the 16-part bar, the Cobras' distance].
T: Mmm [nods].
D: Er—yeah [looks away]. Well, sort of. Yeah [smiles].
T: Mmm. So how much of a meter did the Cobras go?

D [pause]: Three-fourths. That's what you said! [Deborah drags the 16-part bar below the unit meter and aligns the left ends of the bars.]
T: Well, no I only said—
D: You said this [dragging the 2/3-bar on top of the unit meter and aligning the left ends of the bars] is two-thirds. [The teacher agrees.]
T & D: And that's three-fourths [The teacher points to the 2/3-bar].
D: See, see, and that's [points to the 16-part bar], that's four-fourths.
T: Oh I see. But I wonder how much of a *meter* the Cobras went?
D: Oh. Um. [Softly] Let's see…[Deborah drags the unit meter out from underneath the 2/3-bar. For 30 seconds Deborah participates in some surrounding talk but also stares at the screen.] Sixteen-eighteenths of a meter.
T: How'd you figure that out?
D: 'Cause I looked and there could be eighteen of these little pieces [in the unit bar], and there's sixteen [in the Cobras' distance].
T: Oh. Could you figure it out without doing all that? Could you reason through why this [points to one part] would have to be an eighteenth?
[Deborah stares at the screen and then eyes the teacher.]
T: I bet you could!
[Deborah shakes her head no.]
T: See if you can.
D: I can't!
T: Oh, I know you can.
D: I can't!
[The teacher suggests that Deborah think about how she divided up her 2/3-bar.]

In solving the Racecar Problem, I infer that Deborah used her reversible fraction scheme (Hackenberg, 2005a) to make the new distance. That is, since the known distance was three-fourths of the unknown distance, Deborah aimed to partition the known distance into three parts, each of which would be one-fourth of the unknown distance. However, the known distance consisted of two equal parts, two-thirds, which she could not immediately partition into three equal parts. So, in order to achieve her goal, she used her multiplying scheme: She partitioned each part of the 2/3-bar into some number of parts that would allow her to divide the total number of parts by three. Her coordination of units within units in December was likely a constructive resource for her in solving the Racecar Problem. However, in contrast with the Box Problem in February, in solving the Racecar Problem Deborah could rely on making images in

JavaBars—she did not have to draw by hand. Making images in the computer microworld seemed both more palatable to Deborah and more generative for her.

Excerpt 2 also shows that Deborah did not determine the measure of the new quantity as an outgrowth of her ways of making it. In fact, in Excerpt 2 she made visual assessments to determine the length retrospectively. During the experiment she had demonstrated evidence of having constructed the operations necessary to determine the size of parts of parts of bars in relation to a predetermined unit (Hackenberg & Tillema, 2009; Steffe & Olive, in press). So, I conjecture that her refusal to reason through how she knew the length of the 16-part bar was sixteen-eighteenths of a meter did not involve a lack of operations, but instead had to do with a certain edginess that was linked to our MCR.

The "edgy" nature of our MCR. At this point in the experiment, both Deborah's suspicion of me as a teacher and my feeling of a lack of openness toward her had been going on for some time. Nevertheless, I had not stopped trying to care mathematically for Deborah. My goal in the May 12th episode was to open possibilities for her to construct a general scheme for solving problems like the Racecar Problem. My plan was to pose several such problems that she could solve using JavaBars and then ask her to solve a problem without using the microworld. This plan was inadequate for maintaining an MCR with Deborah because it was based on my conjectures that she was very close to constructing such a scheme and that repeated experiences in solving problems like the Racecar Problem were all she needed to complete her construction. In particular, my planned sequence of problems rested on the conjecture that Deborah's distributive operation was solidly embedded into her ways of operating *with fractions*.

As I have already noted, Deborah reasoned with distribution in sophisticated ways with whole numbers. During March, she also reasoned with distribution in making fractions of fractions (Hackenberg & Tillema, 2009). Because of her work in March, I assumed that she might consider the following ideas in solving the Racecar Problem (Figure 6): “I need to make one-third of two-thirds of a meter. To do that, I can take one-third from each of the two thirds

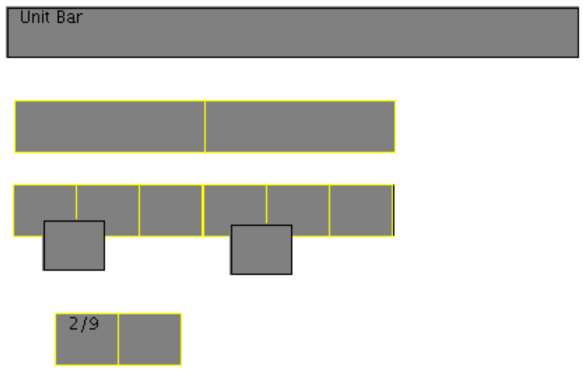

Figure 6. Making one-third of two-thirds of a meter.

of a meter (partition each third of a meter into three equal parts and take one of those parts from each third, or distribute taking one-third of two thirds across each of the two-thirds). Since one-third of one-third is one-ninth, one-third of two-thirds is two-ninths. This amount is one-fourth of the other distance, so I need four of those two-ninths to make the other distance.” Solving the Racecar Problem in this way allows the size of the unknown distance to grow directly out of making it, because the solver focuses on taking fractional parts of the thirds (Tunc-Pekkan, 2008). Although taking one-third of two-thirds is not simple (Steffe & Olive, in press), Deborah’s work in March indicated to me that all of the ideas expressed in the above solution were within her ways of operating (Hackenberg & Tillema, 2009).

However, the sequence of tasks in the May 12th episode did not open the way for her to construct the above solution or a general scheme for solving problems like the Racecar Problem because it did not help her to build her awareness of her distributive activity. In fact, by the end of the episode Deborah said that solving these problems was “stressful,” which was an unusual remark for her. I infer that she was experiencing tense tiredness (Thayer, 2001). Thus I can conclude that the sequence of problems depleted her subjective vitality, and that I was not caring well for her mathematically. I conjecture that one consequence of my lack of mathematical care for her during the episode was her resistance to my requests, as seen at the end of Excerpt 2.

The role of social comparisons in Deborah's mathematical self-concept. Another aspect that likely contributed to Deborah's depletion on May 12th involved her mathematical self-concept at this point in the experiment. I have noted that Deborah often made social comparisons about her own mathematical "progress" in relation to Bridget's mathematical activity. As seen in February, Bridget could pose a threat to Deborah's mathematical self-concept, particularly when Bridget engaged in novel or adept activity that earned her praise. However, Bridget rarely solved a problem prior to Deborah. The set of episodes from December was more typical: Bridget's ways of operating often precluded her from solving problems that Deborah could solve swiftly. In fact, Bridget had been largely depleted during the previous three episodes in May because I had posed problems that were outside of her zone of potential construction (Steffe, 1991). On May 12th, much to my relief, Bridget seemed to be finally working fluidly without depletion (Hackenberg, 2005a). Deborah's observations of her partner's relative "success," combined with work on problems that were quite challenging for her, appeared to drain her subjective vitality.

The teacher's responsiveness to Deborah in May. As I have noted, in May I continued to feel my own problematic lack of openness to Deborah. For example, in Excerpt 2 when I greeted Deborah's work on the Racecar Problem with "Whoa—holy smokes! What is all that?", my tone was rather flat—and somewhat critical. My flat tone came partly from believing that Deborah "knew" what she was doing and repeatedly resisted my requests out of stubbornness. Yet later in the episode, when I began to understand that Deborah had not constructed a scheme for solving these problems and that she found them stressful, I had to reevaluate my response. In fact, near the end of the episode I became quite concerned about her fatigue, and my lack of openness dissipated. In my journal I mused that not having a computational avenue in these problems was likely depleting for her and that she did not think of imagining her activity in the microworld when working without the computer. However, I had no way to test these conjectures—except through retrospective analysis—since the experiment ended on May 12th.

Discussion and Concluding Comments

At the beginning of this paper I asked: What are the challenges involved in establishing an MCR with a mathematically talented female student? I now respond to this question and then consider the contributions of this study of MCRs.

First, nurturing the cognitive and emotional development of strong mathematical reasoners requires taking cognitive constraints *and* energetic responses seriously. Although this finding applies to any student, adhering to it can be especially challenging with mathematically talented students because of their mathematical adroitness. Because of Deborah's strong reasoning, I did not always consider that her difficulties—often evidenced by energetic drops—had a cognitive component. Thus, I sometimes assumed that Deborah's schemes were sufficiently developed so as to allow her to (learn to) operate in any situation I posed to her. In addition, I assumed that the depletion she experienced was more personality-based, not closely tied to cognitive constraints. These assumptions meant I did not decenter enough cognitively and did not take seriously enough her drops in subjective vitality. Had I taken more seriously that Deborah *did not know how* to draw pictures or how doing so would be useful in solving mathematical problems, I would have designed a sequence of problems for her to work on this issue more deliberately. Attending more sufficiently to her depletion by taking it as a stronger indicator of cognitive constraints would have been a first step in this process.

Second, students who derive identity-level satisfaction from being perceived as top mathematical knowers may find some cognitive challenges particularly depleting. In fact, developing a mathematical self-concept that is confirmed largely via computational correctness and swiftness may be especially problematic, as it was at times for Deborah, if the teacher endeavors to help the student expand her mathematical world in ways that render swift computation ineffective. Deborah's work on the Box Problem in February is a good example of experiencing a challenge that overwhelmed her in two ways. As noted above, she did not know how to draw pictures in a way that would be helpful for her. In addition, she expected to be able to solve mathematical problems swiftly and prior to others (such as her partner). When these expectations were thwarted, the affective and energetic consequences were pronounced.

Third, sustained depletion such as Deborah experienced in working on the Box Problem in February can be a significant obstacle to establishing an MCR with a student. So, teachers who aim to establish MCRs with their students may need to be especially alert to those students who are mathematically talented and who gain an identity-level satisfaction from correctness and swiftness. Such students are often rewarded in the school system (Ingram, 2007), which may only reinforce their mathematical self-concepts and make establishing MCRs with them quite challenging. This finding may be surprising, given that these students are often ones who are motivated to do mathematics and who may enjoy it—or at least enjoy being deemed "quick" and "correct". Although making conclusive statements about mathematically talented girls based on a single case study is not possible, this study agrees with prior research that mathematically talented girls are vulnerable to a lack of confidence in their mathematical abilities (Junge & Dretzke, 1995; Pajares & Graham, 1999; Seigle & Reis, 1998).

Fourth, feeling a lack of openness toward a student like Deborah is itself a signal to the teacher to try to harmonise better with that students' ways of operating. My own lack of openness to Deborah ebbed and flowed during the experiment, but from February onward was usually present at some level—a feeling of resistance that was almost like a slight hum in the background. Yet it dissolved in the episode on May 12th once I realised that her edginess was at least partly rooted in the significant demands the problems I posed were placing on her, and that she had not constructed a scheme for solving such problems. I don't pretend that acknowledging a lack of openness as an indicator of needing to harmonise better is some kind of "quick fix": My journal entries reveal that during the 8-month experiment I repeatedly aimed to harmonise better with Deborah's schemes and energetic responses. However, because the project of understanding a student's thinking is significant work, becoming aware of indicators—like feeling a lack of openness toward a student—can be quite valuable. This finding applies to any student, but it may apply especially to mathematically talented students because teachers may assume more about their ways of operating, as I sometimes did with Deborah.

This study reveals that mathematically talented students may feel suspicious of teachers who endeavor to establish MCRs with them, because doing so can challenge students' ideas about what is entailed in being a strong mathematics student, as well as what a teacher's role should be when interacting with students (Boaler, 1998; Cobb, Gresalfi, & Hodge, 2009; Ingram, 2007). In turn, a teacher who is aiming to establish MCRs may find mathematically talented students particularly challenging because of the assumptions the teacher makes about the students' schemes and energetic responses to mathematical interaction, including the resistance these students may demonstrate toward the teacher's efforts. So, this study offers a way to interpret the lack of trust a mathematically talented student may demonstrate toward a teacher, as well as the lack of openness a teacher may feel toward a mathematically talented student. Since these tensions occur many times over in the daily lives of students and teachers, an open question regarding MCRs is how teachers can use it to sustain their work with mathematically talented (and all) students. Viewing a lack of trust and a lack of openness as resistances that detract from enhancing subjective vitality and as indicators of cognitive difficulty (rather than solely personality-based difficulty) may allow the resistances to dissolve and more productive interaction to ensue.

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[i] I use the phrase "ways of operating" to refer to a range of repeatable activity in which a person engages, such as a student regularly telling the teacher that his head hurts when a problem seems difficult, or a student consistently stating reciprocal multiplicative relationships between two quantities. I use schemes and operations in the more specific sense defined previously.

[ii] I define *reasoning* as the functioning of a person's schemes and operations in on-going interaction in her experiential world.

[iii] In JavaBars (Biddlecomb & Olive, 2000) students can draw a bar and set it as the unit bar. All subsequent bars made by copying that bar and manipulating it can be measured in relation to the unit bar. A 2/2-bar is a copy of the unit bar partitioned into halves. By December the girls had been working in JavaBars for a month, and so the activity of setting a unit bar, and the terminology of 2/2-bars, etc., was familiar to them.

[iv] In the data excerpts, D stands for Deborah, B for Bridget, and T for the teacher-researcher (the author). Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue.

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