Running Head: Students' Solution of Arrangement Problems

Students' Solution of Arrangement Problems and their Connection to Cartesian Product Problems

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Abstract

Two-hour long developmental teaching interviews were conducted with each of 14 sixth grade students, ages 11-12. The purposes of the interviews were to investigate how students solved arrangement problems (APs), and how their solutions of these problems differed from their solution of Cartesian product problems (CPPs). The 14 students represented a balanced mix of students operating with each of three different multiplicative concepts that have been identified in prior research (Hackenberg, 2010). This paper reports on the 11 students who were using the first or second multiplicative concept. Students operating with different multiplicative concepts all experienced similar perturbing elements in their solution of APs relative to their solution of CPPs, but they operated differently to resolve these perturbing elements. These differences are identified and their significance discussed in relation to other research findings on students' combinatorial and multiplicative reasoning.

Introduction

Mathematics educators have provided long-standing recommendations to include enumerative combinatorics in K-12 curricula (e.g., Kapur, 1970). They have made these recommendations because combinatorics problems are closely linked to other important content domains like probability and computer science (Jones, Langrall, & Mooney, 2007; National Council of Teacher of Mathematics [NCTM], 2000; Watson & Neal, 2012), can foster students' problem solving and posing (English, 1996, 1999), and can support students in the development of justification and proof (Maher, Uptegrove & Powell, 2010). These recommendations have impacted K-12 standards (e.g., Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.; Ministry of Education, Science, and Technology, Korea, 2007; National Governors Association for Best Practices, Council of Chief State School Officers, 2010), and curricular materials worldwide (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002; McGillivray, Petosz, Evans, & Jones, 2013). Despite the increased presence in standards and curricula, there remains a relatively small body of research on how students reason about such problems. Moreover, numerous studies have documented that combinatorics problems are challenging for students (e.g., Fischbein & Gazit, 1988; Mulligan & Mitchelmore, 1997), suggesting a need for research that provides models of students' reasoning that help mathematics educators effectively support students in this domain (Lockwood & Swinyard, 2016).

One area of research where models of students' reasoning are particularly important are those that shed light on whether and how students see connections or differences among different kinds of combinatorics problems (Batanero, Navarro-Pelayo, & Godino, 1997; Lockwood, 2011). Tillema (2014) provided one example of the way three 8th grade students saw connections

and experienced differences between Cartesian product problems (CPPs) like the Outfits Problem and arrangement problems (APs) like the Flag Problem.

The Outfits Problem (CPP): You have four shirts and three pants. An outfit consists of one shirt and one pants. How many possible outfits could you make?

Flag Problem (AP): You have eight colors. How many possible two-striped flags can you make? Assume you count ordered outcomes.

Study participants in Tillema (2014) were using the third, and most advanced, of three multiplicative concepts that have been reported in earlier research (Hackenberg 2007, 2010; Hackenberg & Tillema, 2009; Steffe, 1992, 1994; Ulrich, 2015, 2016), and the findings from that study will be discussed in the Literature Review. The study reported in this paper has a similar focus, students transition from CPPs to APs, but expands the earlier study: the participants in the current study represented a balanced mix of students using all three multiplicative concepts. The participants of the current study were 14 6th grade students who were asked a sequence of combinatorics problems over the course of two, one-hour interviews, where students started by solving CPPs and then transitioned into solving APs. Given that the earlier study reported on students using the third multiplicative concept, this paper focuses on the 11 students who were using the first and second multiplicative concepts. The research questions for this paper are:

(RQ1) What novel ways of operating did students' produce in their solutions of APs relative to their solutions of CPPs?

(RQ2) What, if any, qualitative differences were there in students' solutions of APs among students using the same multiplicative concept?

(RQ3) What, if any, qualitative differences were there in students' solutions of APs among students using different multiplicative concepts?

Literature Review

Students' Solutions of CPPs and APs

To date, researchers investigating students' solutions of CPPs have largely done so with elementary grades students (ages 5-11) (e.g., Mulligan & Mitchelmore, 1997; Nunes & Bryant, 1996; Outhred, 1996), while researchers investigating students' solutions of APs have largely done so with middle grades students (ages 12-14) (e.g., Batanero, et al., 1997; Fischbein & Gazit, 1988; see Maher, et al., 2010 for a notable exception). However, one question that neither body of research answers is how students' transition between CPPs and APs. Based on when these problems typically appear in curricula (CPPs in elementary and APs in middle school), studying this transition is important because it marks a transition between combinatorics problems that are appropriate for elementary grades students and those appropriate for middle grades students.

In her investigation of students' solutions of CPPs, English (1991, 1993) identified five strategies that students aged 7-12 used. The strategies ranged from least to most systematic where the least systematic was trial and error, and the most systematic was the odometer method. The odometer method consists of a student holding an item from one set constant (e.g., red shirt), cycling through all of the items of the other set (e.g., blue pants, tan pants, black pants), then changing the item from the first set (e.g., white shirt), cycling through all the items from the second set in the *same order* (e.g., blue pants, tan pants, black pants), and repeating this process until all outcomes are created. Maher and Yankelwitz (2010) have argued that the odometer method is powerful because it can support students to know when, and justify why, they have produced all possible outcomes in their solution of CPPs (see also, Maher & Martino, 1996). For a similar reason, the odometer method is powerful in combinatorics well beyond the solution of CPPs. However, studies with older students indicate that students often do not use the odometer

method in the solution of more advanced combinatorics problems when it would be useful (Eizenberg & Zaslavsky, 2004; Halani, 2013; Lockwood & Gibson, 2016). For this reason, Tillema (2014) considered it important to investigate how three 8th grade students (aged 12-13) who used the odometer method to solve CPPs reconstructed this strategy in their solution of APs.

Tillema (2014) found that all three students initially solved APs as a sum because they did not count ordered outcomes (e.g., count the red-blue and blue-red flags as two different outcomes). He explained this finding based on three inter-connected issues: (a) the way students used the odometer method in their solution of CPPs; (b) the fact that students' interpreted APs to involve only one set (e.g., the Flag Problem involves 8 colors), while they interpreted CPPs to involve two sets (e.g., the Outfits Problem involves 4 shirts and 3 pants); and (c) the students had not constructed ordered outcomes. The third issue explains why the students in the study did not count ordered outcomes in their solution of APs, but it does not explain how the students' solution of APs as a sum was related to their solution of CPPs.

Tillema (2014) explained this relationship by analyzing the students' use of the odometer method in CPPs and accounting for how this method functioned differently in the context of an AP, which the students interpreted as involving only one set rather than two. To solve CPPs using the odometer method, students did not use the first item of one set (e.g., the first shirt) after they had paired it with all items of the other set. Students used this same way of operating to solve APs, like the Flag Problem: They took the first item from the set (e.g., first color) and paired it with all items of the set (e.g., first color, second color, …, eighth color) to create eight flags. When they got done creating the first eight flags, they did not use the first item again (e.g., first color) because that corresponded to what they had done when they used the odometer method in their solution of CPPs. Doing so left only seven colors from which to make flags, and

so students took the second item from the set (i.e., second color) and paired it with all remaining items of the set (i.e., second color, third color,…eighth color) to create seven flags. This solution led to producing a sum in the context of solving APs.

One of the three students in Tillema's (2014) study did subsequently solve APs using multiplication. To do so, she first constructed ordered pairs. Once she had constructed ordered pairs she also introduced a novel way of operating. The student who solved APs using multiplication learned to create a second set (e.g., 8 colors) from the first set (e.g., 8 colors) where she considered the second set to be *identical to* the first, and established the entire second set *prior* to making any outcomes. The fact that the two sets were identical to each other meant that, for example, in the Flag Problem, if red was the third color in the first set it was also the third color in the second set. This relationship between the two sets meant that she re-constructed the odometer method in a novel way; the order of the items she used in the second position was *the same as, and dependent on*, the order of the items she used in the first position when creating flags. Figure 1 is a list for the Flag Problem that shows this issue: the order of the colors in the second position (italicized blue letters in second position of the first column) is the same as the order of the colors in the first position (italicized red letters in first position of first the first row).[1] This feature was not present when she used the odometer method in her solution of CPPs because the order of the elements of one of the sets (e.g., shirts) was *unrelated to* the order of the elements of the other set (e.g., pants).

Tillema's (2014) findings were compatible with prior research that had identified middle grades students tended to solve combination and arrangement problems as a sum first (e.g.,

[1] "G" represents the color green; "Y" represents the color yellow; "R" represents the color red; "Pnk" represents the color pink; "Bk" represents the color black; "O" represents the color orange; "P" represents the color purple; "B" represents the color blue.

Fischbein & Gazit, 1988; Piaget & Inhelder, 1975). However, his findings extended these results by providing an explanatory model for: why students would solve APs as a sum based on how they solved CPPs; and what new ways of operating students introduced when they solved APs relative to their solutions of CPPs. Three limitations of Tillema's (2014) study were that: (a) only a single student constructed ordered pairs, which did not provide a significant amount of data on the process by which students make this construction; (b) only one of the students considered APs to involve multiplication; and (c) all three students were using the most advanced of three multiplicative concepts, as reported in prior research (Hackenberg, 2007, 2010; Steffe, 1992, 1994). The current study's intent was to address these limitations.

G, *G*	*Y*, G	*R*, G	*Pnk*, G	*Bk*, G	*O*, G	*P*, G	*B*, G
G, *Y*	Y, Y	R, Y	Pnk, Y	Bk, Y	O, Y	P, Y	B, Y
G, *R*	Y, R	R, R	Pnk, R	Bk, R	O, R	P, R	B, R
G, *Pnk*	Y, Pnk	R, Pnk	Pnk, Pnk	Bk, Pnk	O, Pnk	P, Pnk	B, Pnk
G, *Bk*	Y, Bk	R, Bk	Pnk, Bk	Bk, Bk	O, Bk	P, Bk	B, Bk
G, *O*	Y, O	R, O	Pnk, O	Bk, O	O, O	P, O	B, O
G, *P*	Y, P	R, P	Pnk, P	Bk, P	O, P	P, P	B, P
G, *B*	Y, B	R, B	Pnk, B	Bk, B	O, B	O, B	B, B

Figure 1. List for the Flag Problem

Conceptual Framework

Product of Measures Problems

Combinatorics problems (including CPPs and APs) can be considered *product of measures* problems (Vergnaud, 1983). One characteristic of these problems is the unit that gets counted, for example, an outfit, can be conceptualized as a multiplicative composition of two elementary units, one shirt and one pants (Simon & Blume, 1994).[2] Behr, et al. (1994) identified that this property means that the units in product of measures problems differ from the units in

[2] Elementary units include discrete ones that can be used to create more complicated units like outfits.

isomorphism of measures problems (Vergnaud, 1983). The Donut Problem is an isomorphism of measures problem and can be used to illustrate this difference.

The Donut Problem. You have 3 packages of 4 donuts. How many total donuts are there?

In the Donut Problem students can make a one-to-many coordination between one package and four donuts to create a package of four donuts (Figure 2a). The units (donuts) inside the containing unit (the package) are units of one. This unit structure differs from the Outfits Problem in that students need to create the outfits by composing pants and shirts. Doing so means that they might pair the first pants with the first shirt, the first pants with the second shirt, the first pants with the third shirt, and the first pants with the fourth shirt, and then make a one to many coordination to create a unit containing all outfits with the first shirt (Figure 2b). Here the four units inside the containing unit are *not* units of one but *pairs* (cf. Behr, et al., 1994).

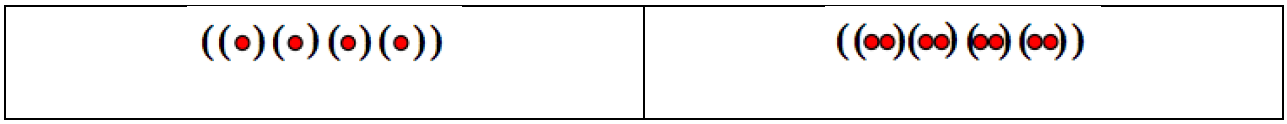


Figure 2a (left) and 2b (right): Difference in unit structures

Tillema and Gatza (2016) identified that researchers investigating combinatorial reasoning had not made models of student reasoning that take into account this difference in unit structure. They argued for the need for such models in order to understand how students conceptualized relationships among one- and two-dimensional discrete units in the context of solving combinatorics problems (cf. Battista, 2007; Kobiela & Lehrer, 2019; Smith & Barrett, 2018). Thus, one goal of Tillema's (2013, 2014, 2018) prior work has been to model whether and how students created more complex unit structures in their solutions of product of measures problems.

Operations, Schemes, Perturbations, and Concepts

Broadly speaking, to characterize students' mathematical reasoning, I use *operations*, which are mental actions, and *schemes*, which are goal-directed organizations of operations. Schemes

consist of an assimilatory mechanism, an activity, and a result (Piaget, 1970; von Glasersfeld, 1995). The assimilatory mechanism of a scheme involves a student in interpreting a problem situation; this interpretation can trigger an activity involving particular operations; implementing operations transforms a student's initial interpretation of a situation into a result. A student may experience a perturbation when one or more parts of a scheme does not match with her expectation—although perturbations often occur outside of a person's conscious awareness. Perturbations can be occasions for learning because, in response, a student may change one or more parts of her scheme. When the result of a scheme is available to a person prior to operating in a situation, then a person has constructed a concept (von Glasersfeld, 1982).

Multiplicative Concepts: MC1 and MC2 Students

Tillema's (2013, 2018) research on students' solutions of CPPs builds on Steffe's (1992, 1994) and Hackenberg's (2010) research on students' multiplicative reasoning (see, Ulrich, 2015, 2016, for a synthesis of these perspectives). Hackenberg has identified three multiplicative concepts that are based on the number of embedded units students can take as a given prior to operating. Here I discuss students operating with the first and second multiplicative concepts, MC1 and MC2 students, because these students are the focus of this paper.

MC1 students have constructed an abstract unit of one (Steffe, von Glasersfeld, Richards, & Cobb, 1983).[3] MC1 students can iterate this unit of one, say, seven times and unite these iterations into a composite unit (Ulrich, 2016). For MC1 students to establish a situation like the Donut Problem as multiplicative, they engage in a units coordination (Steffe, 1992), which involves inserting the units of one composite unit into the units of a second composite unit. For example, to solve the Donut Problem, MC1 students can insert four units (donuts) into each of

[3] An abstract unit of one is a unit that has been stripped of specific sensory material (e.g., dots that a student imagines to be in a particular configuration); the process of stripping specific sensory material is what produces a unit of one that is *abstract*.

three units (packages). Typically, MC1 students count by ones past known skip counting patterns: For example, they might know that two 4s is 8. Then they would say, "8, that is 2 packages; 9, 10, 11, 12, that is 3 packages," etc. That is, once MC1 students get beyond known facts they return to counting by ones (i.e., iterating a unit of one). The result of their activity is establishing a sequence of "coordinated units:" 4 donuts is 1 package, 8 donuts is 2 packages, etc., but this coordination is not something that they can take as a given prior to operating.

In contrast, MC2 students have interiorized these operations. Broadly speaking, a student has *interiorized* mental operations when they no longer have to implement the mental operations (activity of their scheme) in order to take the result of their scheme as a given in a problem situation. MC2 students do not need to iterate a unit of one in order to establish a composite unit (Steffe, 1994). Rather, a number word like "eight" implies iterating a unit of one eight times without having to carrying out these iterations. In addition, MC2 students can take the units coordination described for MC1 students as a given prior to activity. So, in solving the Donut Problem, MC2 students start by treating four donuts as one package, and they can operate on this structure further; for example, they might reason that four and four is eight, which is two packages, and that eight and two more is ten, and two more than that is twelve, establishing the third four as two and two, while maintaining that it is still one four. When students are able to establish a part of a composite unit as independent of, but related to, the composite unit without destroying it, Steffe (1992) considers students to be using a *disembedding* operation. This operation enables them to strategically take apart and recombine the units contained in one package to solve problems like the Donut Problem.

MC1 and MC2 Students' Solutions of CPPs

Steffe (2007) has identified these multiplicative concepts as distinct stages of reasoning that can last for two or more years. However, he has conceptualized them as dynamic concepts that students make modifications to over this period of time even if they do not necessarily transition from one multiplicative concept to the next. One modification that researchers using this framework had not addressed was how students might produce a more complex unit structure in their solution of product of measures problems, broadly, and CPPs, specifically.

To address this issue, Tillema (2013, 2018) identified two operations, pairing and ordering that students used to solve CPPs. A *pairing operation* entails establishing a correspondence between two units of one (e.g., a shirt and pants) to create a unit that contains two units, but is counted as a single unit (e.g., an outfit). He termed the units containing two units, *pairs*, in order to differentiate them from units of one. He also defined an *ordering operation* as involving a student in creating an order for the units of a composite unit: for example, a student uses an ordering operation when she creates a first unit, a second unit, and a third unit in a composite unit of three where evidence for this operation could be assigning a number to each pants as the student solves the Outfits Problem. Tillema (2013, 2018) used pairing and ordering operations along with the three operations that Steffe (1992, 1994) previously defined—units coordination, iteration, and disembedding—to characterize students' solutions of CPPs.

Tillema (2018) found that to successfully solve CPPs using the odometer method, MC1 students assimilate problems like the Outfits Problem using two composite units, for example, three and four. They then order the units of each composite unit. Once they have ordered the units they can use a pairing operation to create a pair (e.g., the first unit from each composite unit makes an outfit). Tillema (2018) identified that MC1 students create pairs as part of the activity they produce in a situation. One example Tillema (2018) used to illustrate this issue was a

student who created pairs in the context of making a tree diagram, but then needed to make the pairs again when the interviewer asked her to determine how many outcomes she had represented; for the student, the pairs were not items she could count outside of the act of making them.

MC1 students can also engage in additional operations when they create pairs; they can use their disembedding operation prior to using their pairing operation. When MC1 students use a disembedding operation, they disembed a unit of one from each composite unit. In this context, a *disembedding operation* enables students to establish a pair as *independent from, but related to* the two units of one that comprise it (Figure 3); this way of operating allows students to establish a two-dimensional discrete unit, a pair, independently from, but related to, two one-dimensional discrete units. This issue is shown in Figure 3 by having the pair represented independently from the units of one. When students operate in this way, Tillema (2018) considers them to have established a multiplicative relationship between a unit of one, a unit of one, and a pair.

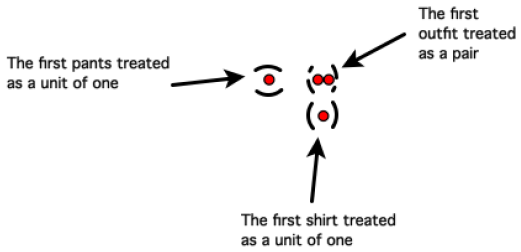


Figure 3. Creating a relationship among a unit of one, a unit of one, and a pair in activity[4]

Like MC1 students, MC2 students assimilate CPPs using two composite units, for example, three and four, and order the units of each composite unit. They then can disembed one unit from each composite unit and pair them together, creating a multiplicative relationship between a unit

[4] In figures like Figure 3, I represent units that a student creates in activity with dashed parenthesis, and units that are interiorized (i.e., not made in activity) with solid parenthesis.

of one, a unit of one, and a pair. In contrast to MC1 students, MC2 students can interiorize this multiplicative relationship. One indicator that MC2 students have interiorized the multiplicative relationship shown in Figure 3 is that they can reason with and about pairs even if they have not created them in immediate past experience. Doing so enables them to operate as if one discrete two-dimensional unit is multiplicatively composed of two discrete one-dimensional units.

Some, but not all, MC2 students engage in a units coordination when solving CPPs. In the case of CPPs, this operation involves embedding, for example, four pairs into a containing unit to create four pairs for one pants (recall Figure 2b). Tillema (2018) illustrated the difference between MC2 students who do and do not make a units coordination by examining how they respond to questions like the Extension of the Outfits Problem: Suppose you get one additional shirt (now 5 shirts rather than 4 shirts) and one additional pair of pants (now 4 pants rather than 3 pants). How many *new* outfits could you make? Figure 4 illustrates an array for this problem where the *new* outfits from the Extension of the Outfits Problem are represented as blue points above and to the right of the dashed line.

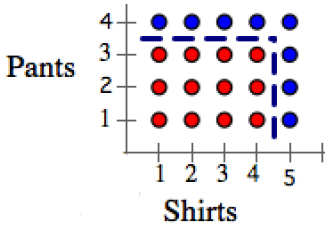


Figure 4. Array for the extension of the Outfits Problem

MC2 students who *do not* engage in a units coordination may predict that: (a) there will be five new outfits, only considering those with the new pants; or (b) there will be four new outfits, only considering those with the new shirt; or (c) there will be a total of nine new outfits, counting the outfit with the new shirt and the new pants twice (the corner point of the array). They can correct these predictions by, for example, filling in the points on their arrays, but the correction is

not straightforward. For example, one student in Tillema (2018) filled in all new outfits on her array, during which time she filled in the corner point twice, and only realized this issue once the interviewer questioned her. The reason Tillema (2018) considered these students not to have engaged in a units coordination is they were not yet reasoning with the four pairs (outfits with new shirt) and the five pairs (outfits with new pants) by comparing what was common to each.

MC2 students who *do engage* in a units coordination establish, for example, a multiplicative relationship among a unit of one, a unit of five ordered units, and a unit of five pairs in activity (Figure 5). Establishing this relationship is the basis for seeing a "row" or a "column" of an array multiplicatively where one one-dimensional unit times five one-dimensional units are equal to five two-dimensional units. Creating this relationship enables MC2 students to reason that the outfit with the new shirt and new pants (the corner point) is part of both the outfits with the new shirt and the outfits with the new pants. Therefore, they recognize that it does not need to be counted twice in the absence of working directly with a written list, tree diagram, or array.

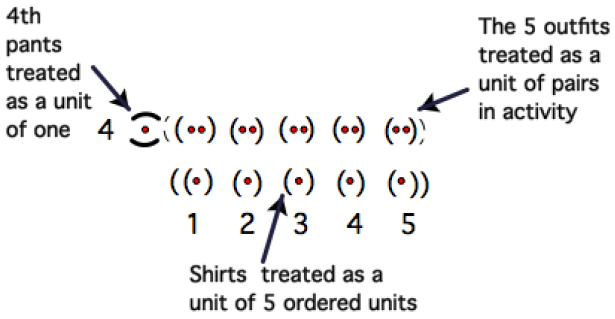


Figure 5. A relationship between a unit of one, a unit of units, and a unit of pairs in activity

Methods

Interview Study Methodology

Like teaching and design experiment methodology (Confrey & LaChance, 2000; Lobato, 2008), a developmental teaching interview involves a researcher using problems as a context for

exploring conjectures about how students might reason (cf. Chini, Carmichael, Rebellow, & Puntambekar, 2009). To this end, an interview protocol is developed with conjectures in mind that are framed by current research and a researcher's prior interactions with students. While these conjectures guide the design of the study, a researcher may make modifications to problems in the context of interviewing students in order to test on-the-spot conjectures (Steffe & Thompson, 2000). These on-the-spot conjectures allow a researcher to be responsive to unexpected ways of operating that students produce. Unlike teaching and design experiments, developmental teaching interviews provide only a snapshot of learning from a small number of interactions rather than information about how students change over an extended period of time.

Data Collection and Participants

The school where the study took place was a science magnet school for 6th-12th grade students in a large urban district in the United States. The school population was approximately: 70% African American, 19% Latinx, 7% White, and 4% Multiracial; 65% girls and 35% boys; with 77% of students receiving free or reduced lunch. The district had a school choice model, which meant that to attend the school students from the entire district applied and were selected. To maintain enrollment at the school, students needed to maintain good academic standing and attendance. All 6th grade students in the school enrolled in a course that was designed to prepare them to be ready to take algebra in the 8th grade. Approximately 10% of the student population enrolled in a college level calculus course by the time they were in 12th grade.

The research team consisted of a White male mathematics education researcher, an Asian American male graduate student, and two White female undergraduate researchers. For the first four months of the school year, members of the research team worked one day a week in a 6th grade classroom to familiarize themselves with the students and teacher. This included working

342 individually and in small groups with students, co-teaching or teaching the class, and co-
343 planning with the teacher. The classroom teacher, a Jamaican American female, was interested in
344 supporting student thinking but was inexperienced at making this central to her instruction. Thus
345 much of her instruction focused on mathematical procedures, but she was actively interested in
346 suggestions about how to support student thinking in her instruction.

347 After the first four months of the school year, 16 students volunteered to participate in a
348 selection interview. The selection interview consisted of an un-recorded hour-long interview
349 involving non-combinatorial problems with the goal of identifying students' multiplicative
350 concepts (Appendix A). From the 16 students 14 were asked to participate in the study: three
351 were MC1 (21.4% of the sample), eight were MC2 (57.2% of the sample), and three were MC3
352 (21.4 % of the sample, not reported on in this paper). Of the 16 students, two were not asked to
353 participate because they were also MC2 students; given that the goal was to have a balanced mix
354 of students the research team decided not to include any more MC2 students. Steffe (2007) has
355 estimated that in the general 6th grade population 30% of students are MC1, 30% are MC2, and
356 40% are MC3. Therefore, the percentage of students' in the study under represented MC1 and
357 MC3 students, and over represented MC2 students. However, the students who participated in
358 the study were representative of the racial and gender demographics of the school.

359 During selection interviews, the researcher posed problems that can involve embedded units
360 to make a determination of multiplicative concept (e.g., a package contains 8 candies; a box
361 contains 6 packages; a crate contains 4 boxes; a truck contains 5 crates) (Hackenberg, 2007).
362 Within a given problem a student is asked about relationships they see among units (e.g., How
363 many candies are in a truck?). Questions are developed to be responsive to the student and test
364 conjectures about whether and how students are embedding units within each other. A

comparison of student responses across all problems is used to infer multiplicative concept. After the selection interview, the research team conducted two video-recorded, hour-long interviews[5] involving combinatorics problems (Appendix B). Prior to these interviews, the students had no experience with combinatorics problems during their 6th grade.

The first interview was structured to consider ordered outcomes in their solution of APs. To accomplish this goal, students were presented first with CPPs whose statement refers to two sets (e.g., three pants and four shirts) where the qualitative referent for the sets was different (e.g., shirts and pants). Once they were familiar with this type of problem the interviewer presented CPPs whose statement referred to two sets where the difference in the qualitative referent for the sets was less clear (e.g., You have eight spade cards. I have eight heart cards. How many two-card hands can we make that have one spade and one heart?). The purpose of problems like the Card Problem was that both people had cards so there was the opportunity for students to introduce a distinction between the suit of the cards. By doing so, they could consider how the outcome 3 of hearts-8 of spades was different from the outcome 8 of hearts-3 of spades.

The students were then asked an AP involving a small number of outcomes (flipping one coin twice) where they had the opportunity to create most or all possible outcomes from their activity of flipping a coin. This provided the opportunity to experientially differentiate between the T-H outcome and the H-T outcome. Then they were asked The Flag Problem, which refers to only one set. To solve the Flag Problem, the students were given eight colored pencils (red, orange, yellow, green, blue, purple, pink, and black) and a handout that had the outline of nine two striped flags so that they could color in some example flags. While they colored in example flags, they were asked questions like, "Could you have a flag that has red in the top stripe and

[5] Given school schedules, five students participated in three forty minute interviews.

red in the bottom stripe?" and "What is the difference between the red-yellow flag you made and the yellow-red flag you made?" Once students had colored in flags on their handout, considering what they might count, they were asked to predict how many possible flags they could make. After their prediction, students created a list, a tree diagram, or an array to check predictions. All students solved the Coin and Flag Problem, and students who moved quickly solved more APs.

The research team anticipated that a likely progression for representations of the problems would be from a picture, to a list, to a tree diagram, and finally to an array, but they also wanted to provide space for students to make sensible decisions about what representation they used. So, for example, the interviewer would ask a student who solved the first problem using a picture questions to support the student to create a list starting with relatively open questions (e.g., I: Can you think of another way to represent the problem?) and moving to more guided questions (e.g., I: What outfit does this picture represent? S: Red shirt and blue pants. I: Can you represent that outfit using letters? S: [writes "RB" or "Red-Blue"] I: Can you represent all of the outfits that way?). Throughout working with students on making representations, the interviewer was careful to observe how students produced representations, to ask probing questions to determine student interpretations of representations, and to have students use multiple representations.

Method of Data Analysis

Data analysis occurred in two phases—ongoing and retrospective analysis (Steffe & Thompson, 2000). On-going analysis included regular meetings of the research team to discuss each interview where they reviewed notes taken during the interviews and watched video of interviews to examine key moments in them. The primary purpose of this phase of analysis was to identify current conjectures about students' reasoning and possible ways to test these conjectures. Retrospective analysis occurred after the interviews were done. The mathematics

education researcher transcribed the interview data, wrote low-inference data summaries, and memos that contained conjectures about the data (Saldaña, 2013). The research team then met to discuss the transcripts, data summaries, and memos to triangulate interpretations (Mathison, 1988). This process was iterative: the researcher cycled through the data, discussing it with the research team until a stable account of each student's reasoning was outlined.

The development of a stable account of students' reasoning involved examining the coherence of an account for a particular student, and comparing accounts across all students. As part of this process, a researcher does not use the multiplicative concepts to pre-determine or limit what an explanatory account will be. Rather the researcher actively engages with the data to look for evidence that confirms or refutes both his or her prior and current interpretations of student reasoning (Steffe & Thompson, 2000). The goal of this process is to make a second-order model of students' reasoning—a researcher's account of the ways and means of operating of a student and any changes to that reasoning (Steffe, von Glasersfeld, Richards & Cobb, 1983).

The primary explanatory tools used to make second order models are schemes and operations. To make an inference that a student is using a particular scheme or mental operation, the researcher use student's observable behaviors (e.g., verbal statements, ways of producing a representation, gestures, etc.) for indication and counter-indication of inferring a student's operations and schemes. For example, a common indicator that a student is using a pairing operation is that they can answer questions about a property of a pair or a set of pairs (e.g., Where are the flags that have red in the bottom stripe?). Observable behaviors, however, are not in one-to-one correspondence with mental operations and so a researcher looks across all student-researcher interactions to make inferences about the students' schemes and operations.

For this paper, the researcher identified that all students in the study experienced a perturbation when they transitioned from solving CPPs to solving APs. One marker that students experienced a perturbation was that all 11 students could successfully solve CPPs, but none of the 11 students accurately predicted the total number of flags that they could create in the Flag Problem. The research team examined student predictions in order to find common themes. One theme that arose from this analysis was that a majority of students' predictions indicated an uncertainty about how to solve a problem that involved a single set. For example, one student explained his prediction of sixteen flags as follows, "I split the eight colors into two groups (of four)….(it'd be) sixteen flags because four colors could be paired with four colors." In examining how students resolved this perturbation, the research team identified two additional issues: (a) that MC1 and MC2 students did not reconstruct the odometer method in the same way as each other, and in addition did not reconstruct the odometer method in the way that MC3 students had done as reported in the earlier study (Tillema, 2014); and (b) that there were common criteria that could be used to infer that MC1 and MC2 students had constructed ordered pairs, but that MC1 and MC2 students used ordered pairs in different ways to structure sets of outcomes. Together these three issues—(a) solving a problem with a single set, (b) the odometer method, and (c) construction and use of ordered pairs—frame the presentation of results.

Results

MC1 Students' Solutions of APs

Table 1 provides a summary for the three MC1 students on each of the three issues that I analyze. To illustrate each, I use data from Carlos's solution of the Flag Problem. As I discuss his solution, I provide comparison and contrast of his solution to other MC1 students.

Table 1. Summary of MC1 students' solutions of APs

	Odometer Method	Operating with a Single Set	Ordered Pairs
Carlos	Did not reconstruct odometer method	Separated single composite unit into two parts	Constructed ordered pairs; found ordered pairs in pairs
Darryl			Constructed ordered pairs; found ordered pairs in pairs
Alana			Did not construct ordered pairs

Data for issue 1: The odometer method. To start the Flag Problem, Carlos colored in six flags on his handout (blue-green; red-purple; yellow-black; orange-pink; purple-blue; and green-blue). The interviewer asked him to predict the total number of flags he could make to which he responded that there were "nine" flags because that was the number of flags on the handout, then created a list for the six flags, saying "you could make three more." (Figure 6).[6] The interviewer

Figure 6. Carlos lists the six flags he has colored in on his handout

pressed Carlos about whether this actually counted *all possible* flags, and he began to list flags that had blue in the first position, excluding the blue-blue flag (Figure 7). As he created Figure 7,

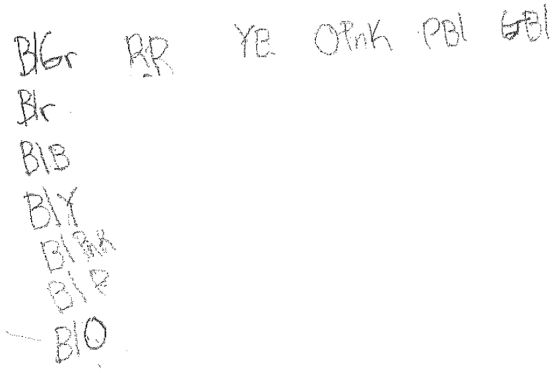


Figure 7. Carlos lists the flags with blue in the first position

[6] He used the letters "Bl" for blue, "B" for black, "P" for purple, "Pnk" for pink, "G" or "Gr" for green, "R" or "r" for red, "Y" for yellow, and "O" for orange.

he visually scanned the colored pencils to determine that he had used all of the colors except blue in the second position. He then began the second column of his list, which represented all flags that had red in the first position. To create the second column, he used the order of the colors in the first column to determine which colors to pair with the color red (Figure 8 shown in red and italicized).

He encountered two issues in using the order of the colors from the first column to establish an order for the colors in the second column. The first was that the color in the second position of the first flag in each column was not the same (Figure 8 shown in blue as the first flag in the first and second column), which meant that he listed the red purple flag twice in the second column ("RP") and did not list the red-green flag. The second issue was that the color blue did not appear as a color in the second position in the first column of his list because he had not included the blue-blue flag. He did, however, remember to include the red-blue flag in the second column—he recorded it as the second flag.

BlGr	RP	YB	OPnk	PBl	GBl
Blr	RBl				
Bl*B*	R*B*				
Bl*Y*	R*Y*				
Bl*Pnk*	R*Pnk*				
Bl*P*	R*P*				
Bl*O*	R*O*				

Figure 8. Carlos writes the second column of his list

To create the third column of his list, he again relied on the prior column to guide the order of the colors for the second position. As he did, he independently noticed both issues and found a way to address them: He skipped the color black when he saw it in the second column of his list since he had already recorded the yellow-black flag as the first flag in the third column (Figure 9, shown in red and italicized); he decided to include the yellow-yellow flag because then he did

not have to skip the color yellow when he saw it in the second column (Figure 9, shown in blue with the arrow); and he included the yellow-purple flag, recorded as "YP", as the last flag in the

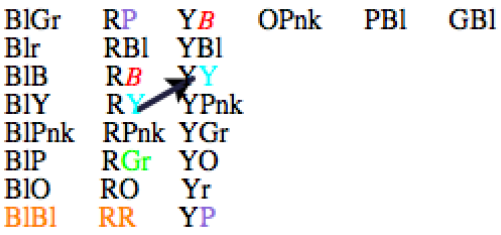


Figure 9. Carlos finishes the third column of his list

third column because the color purple was in the first flag of the second column and so he knew he had not used purple yet in the third column (Figure 9, shown in purple). He continued to use this pattern over the course of creating the rest of the columns of his list. Before finishing the third column, he also adjusted the first column to include the blue-blue flag, adjusted the second column to include the red-red flag (Figure 9, shown in orange as the last flag in the first and second column), and replaced the red purple flag that he had written twice in the second column with the red green flag (Figure 9, shown in green third from the bottom in the second column).

Analysis of Issue 1: The odometer method. Carlos's actions indicate that initially he intended to pair, for example blue, with all colors except blue, and subsequently that he did include flags like the blue-blue flag, which he did largely to ensure that he could use his prior column (e.g., second column) to keep track of which colors he had used in the second position of his current column (e.g., third column). He did not, however, follow a fully consistent order for the colors in the second position from column to column. Ultimately, he produced a consistent *pattern* to ensure that he used all colors in the second position in each column, but not the same *order* for the colors, indicating that he did not re-construct the odometer method. One reason Carlos did not re-construct the odometer method is that he included his initial list of six flags (Figure 6) as the first flag of each column of his final list (Figure 7). The fact that he included his

initial list of six flags in his final list added a layer of difficulty to his successful re-construction of the odometer method. However, this issue did not seem to be the only factor that contributed to his lack of re-construction of the odometer method; throughout his solution he remained reliant on using either the colored pencils or his previous column of notation to determine that he had used all possible colors in the second position. This reliance was common across the MC1 students. Unlike Carlos, the other MC1 students did not include an initial list in their final list, but they too did not re-construct the odometer method. They did not re-construct the odometer method for two reasons: (a) they changed the order of the colored pencils on the table in between writing columns of notation in their lists; or (b) they did not count same color flags like the blue-blue flag, which meant that their prior column of notation did not contain all eight colors for them to use in creating their next column of notation. These factors meant MC1 students used a consistent pattern for listing the colors in the second position, but not the same order.

Data for Issue 2: Operating with a single set. After Carlos finished his third column of notation, the interviewer asked him how many flags he would list in the fourth column prior to his actually listing them. Carlos responded by saying, "I think the same thing as these," pointing to the first three columns. The interviewer asked him how many flags were in each of the first three columns. Recall that at this point there were eight flags in each column. He indicated he was uncertain and counted the flags in the first column by one, skipping the first flag in the column, and said there were "seven" flags. He then counted the flags in the third column, again skipping the first flag, and said "yeah, seven." The interviewer asked him to recount the flags. This time he counted the first flag in the third column, and said, "Oh, there is eight." He continued his list, and on several occasions he counted the number of flags in a column, getting eight each time. When he finished his list, the interviewer asked him the following.

Data Excerpt 1: Carlos considers how many flags he could make with the color blue[7]

I: With the blue one, how do you know how many flags you can make with the blue?

C: You just do all the colors that you could do. Like blue and yellow, blue and black, blue and red, blue and purple. You are going to use all the colors.

I: All the colors. Is there a way you could figure out how many colors there are to use with blue? In other words, how many you are going to make without writing them all out?

C: You could count up the colors there are and so there is seven colors. [He does not count the colored pencils].

I [Surprised because this response does not match with the list he has created]: There are seven colors you could use with blue? [Carlos nods his head "yes." The interviewer continues to question him, and, after several questions, Carlos explains why there could be eight flags with the color blue.]

Analysis of Issue 2: Operating with a single set. Carlos's response that there were "seven colors" that he could use with the color blue was surprising. One interpretation of his response is that he simply thought there were a total of seven colored pencils, including the color blue, rather than eight. I don't make this interpretation given features of his solution, together with the way that other MC1 students responded to solving the Flag Problem. For example, Darryl, another MC1 student, responded to a similar question by saying, "…so if you have orange on top and you have eight colors you have to subtract it from one is seven." These types of responses were typical for MC1 students and explain why they did not, at least initially, count flags like the blue-blue flag; they thought about removing one color from the total number of colors to put in the

[7] In the Data Excerpts, I is for the interview-researcher,W is for a witness-researcher, and all other letters stand for the student. Comments enclosed in brackets [] describe students' nonverbal action or interaction from the interviewer's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off; four periods (….) denote omitted dialogue or interaction; and parenthesis ( ) are the reseracher's interpretations of a word meaning.

first position, and then had seven other colors remaining. Carlos's actions and response illuminate that this issue was not a superficial one because even after he included flags like the blue-blue flag, he still seemed to interpret the situation as involving one color, the color blue, and seven colors that could go with the color blue. He did eventually identify that eight colors could go with the color blue, but he did so in response to multiple questions from the interviewer.

My explanation for this way of operating is that MC1 students assimilated APs using a single composite unit (Figure 10a). Carlos, then had to figure out how to operate on the single composite unit in order to create pairs; he operated on it by separating it into two parts, one unit (the color blue) and seven units (the seven remaining colors) (Figure 10b). I make the inference

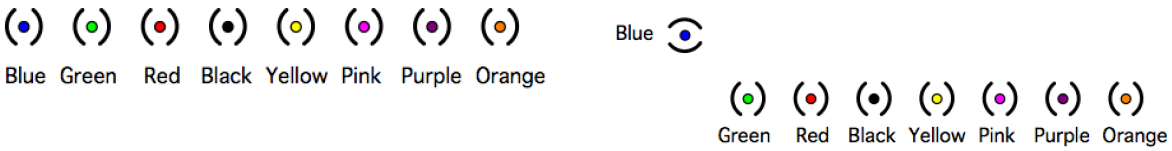


Figure 10a (left),10b (right). Carlos assimilates with one composite unit and separates it

that he separated the composite unit into two parts rather than *disembedded* a unit of one from the composite unit. The key difference in these two interpretations is that if he had used a disembedding operation he would have established the unit of one as *independent from, but related to* the composite unit of eight; it would have been one of eight colors that could be used to create flags even if ultimately he decided not to use the color blue to create a blue-blue flag. The language that MC1 students used (e.g., Darryl's language "you have to subtract" the one color) was more consistent with separating the original colors into one color for the first position and seven colors for the second position. This interpretation means there is a need to explain Carlos's inclusion of flags like the blue-blue flag; I interpret his inclusion of this kind of flag as an outgrowth of his using the prior column in his list to keep track of what colors he had already

used in the current column. However, I don't infer that including this kind of flag indicated that he changed how he was operating, an inference that is supported by his statement after he finished his list that seven colors could go with the color blue.

Once Carlos separated the composite unit I infer he used his pairing operation. As he made pairs, a key question was whether he retained a composite unit of *seven* as part of the situation. I infer that he could *recall* that there were seven colors for the second position after he made flags in a single column, but as he was making the flags the seven colors were "exhausted." I base this inference on the fact that when he was asked how many flags were in each of the first three columns he expressed uncertainty, and then counted the number of flags in the first and third column. Had he retained the composite unit of seven as he was making the pairs I infer that he would have likely just stated that there would be seven flags in each column prior to counting. Figure 11 shows this issue through the removal of the first unit, the color green, from the composite unit once the first pair, the blue-green flag, is created. From the perspective of operations, Figure 11 is intended to show that he *did not* use a disembedding operation as he was creating pairs. Using a disembedding operation would mean that he produced pairs as independent from, but related to, the two units of one that comprised them.

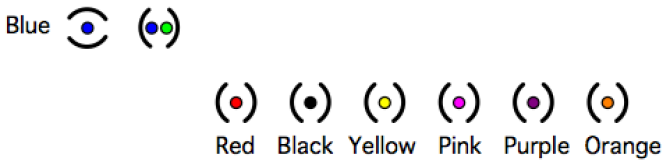


Figure 11. Carlos creates pairs

It is interesting that when he did count the total number of flags in the first and third columns he initially counted only seven flags; he counted all eight only after interviewer prompting. Doing so provides supporting evidence for my interpretation that outside the immediate experience of making flags he could *recall* that he separated the composite unit of eight into one

and seven. It supports this interpretation because I infer he could recall separating or actually did separate the composite unit again prior to counting, which provided him with some expectation that he would create seven flags in the first and third column prior to his counting. This expectation led to his counting only seven flags in each column. However, I do not interpret his counting seven flags to mean that he maintained the seven colors while he was actually making flags because he expressed uncertainty about the number of flags in each column.

This interpretation means I considered Carlos to be creating pairs in activity—where the pairs he produced constituted his "mathematical world," *but not* to be producing a multiplicative relationship among a unit of one, a unit of one, and a pair where he established the pair as independent from, but related to the units of one that comprised them. Carlos's way of operating was representative of the other MC1 students. However, it was the most complicated response among them because of his inclusion of flags like the blue-blue flag. The other MC1 students simply did not consider these as possible flags to be counted, and in Darryl's case stated that they were "removing" one color from the eight colors to establish one color and seven colors.

Data for issue 3: Construction and use of ordered pairs. Carlos's solution also highlights the criteria I used for identifying students' initial construction of ordered pairs.

Data Excerpt 2: Carlos establishes ordered pairs in the Flag Problem[8]

I [Carlos has colored in five flags on his handout—blue-green; red-purple; yellow-black; orange-pink; purple-black. The fifth flag has purple in the bottom stripe and the first flag has purple in the top stripe]: So here ah purple is in the top stripe and here purple is in the bottom stripe. Does that make any difference?

[8] Portions of this data excerpt occurred prior to the data that was shared in analyzing issue 1 and issue 2.

C: No. [Carlos pauses for three seconds.] It would make a difference if I would have green and blue. [Carlos points to the blue-green flag he has colored on the handout].

I: Oh, would you, could you actually make that flag?

C: Yeah [colors in a green-blue flag on the handout.]

I: Okay so if we were going to count all of the different flags that you could make if you just kept on doing all the flags. Do you want to count these two as different or the same [points to the blue-green and green-blue flag]?

C: Well they would be different because in one the green stripe is here at the bottom instead of the top.

….

I [Carlos has completed his list of all possible flags]: How can you tell the difference between these when you wrote them down [points to "BlG", and "GBl" in his list]?

C: Because blue comes first in this one and blue comes last in this one.

….

I: So if you have the blue, well first of all, remind me again which is the top stripe and which is the bottom stripe. Would you label that in your picture (meaning in his list)?

C: Yeah I am using like this example. [He writes "BlG", and then writes top under "Bl" and bottom under "Gr"]. Like the second one would be the bottom.

Analysis of issue 3: Construction and use of ordered pairs. Carlos introduced the difference between a green-blue and a blue-green flag in response to the interviewer's question about whether it mattered if the color purple was located in the top or bottom stripe, and decided that he would count these two flags as different because the "green stripe is here at the bottom instead of the top." Doing so was necessary, but not sufficient, evidence to conclude that he had

established ordered pairs; it was necessary because it meant he established that the two flags could be counted as different, but it was not sufficient because the difference in the two flags could have simply been based on the fact that they looked different visually. The second criteria that I considered essential to infer the construction of ordered pairs was a student differentiated between the position each color appeared in a pair. Carlos gave indication of this criteria, saying, "because blue comes first in this one and blue comes last in this one," referring to his written notation for the two pairs. The third criteria I considered essential to infer the construction of ordered pairs was that a student connected each position in the pair to one of the stripes in the flag. Carlos provided indication of this criteria when he said, "like the second one (the color in the second position) would be the bottom (stripe)." I considered all three criteria essential to infer the construction of ordered pairs to students because it indicated that they considered two flags with the same colors to be different; they related this difference to the position in which a color appeared in the pair; and they related the position of the color back to the original difference that they established (first position means top stripe and second position means bottom stripe).

Of the three MC1 students, two provided evidence of all criteria essential to the construction of ordered pairs. The third MC1 student considered flags like the blue-green and green-blue flag to be different from one another based on the colors being in the top or bottom stripe (first criteria), but did not provide evidence of establishing the second or third criteria.

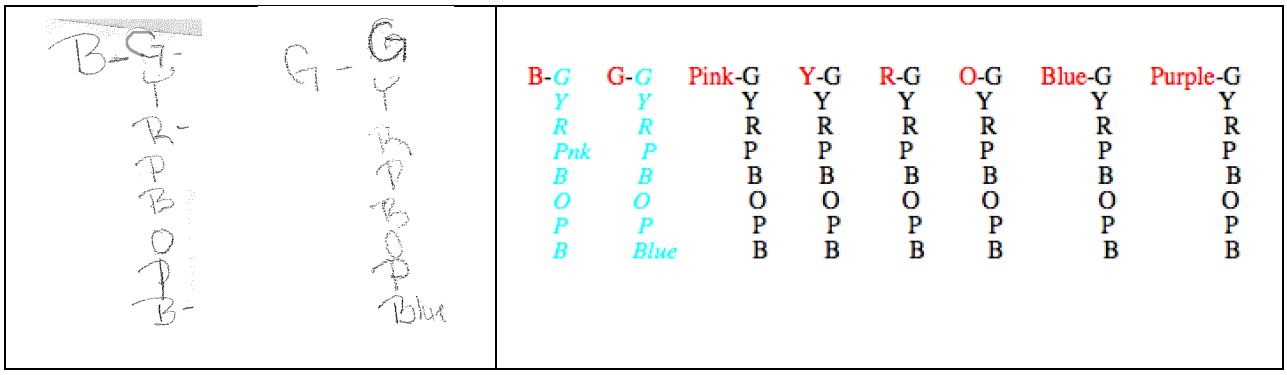
MC2 Students' Solutions of APs

Table 2 provides a brief summary for the eight MC2 students on each of the three issues that I analyze. To illustrate each issue, I use data from three students' solutions of the Flag Problem. As I discuss each issue, I provide comparison and contrast for all MC2 students on that issue.

Table 2. Summary of MC2 students' solutions of APs

	Odometer Method	Operating with a Single Set	Ordered Pairs
Shonda	Reconstructed the odometer method.	Disembedded a unit of one from the composite unit, disembedded pairs, evidence of creating a multiplicative relationship among a unit of one, a unit of eight units, and a unit of eight pairs.	Constructed ordered pairs; used ordered pairs to find a row and column in list/array.
Terionna			
Tiana			
Jose		Disembedded a unit of one from the composite unit, disembedded pairs, evidence of creating a multiplicative relationship among a unit of one, a unit of one, and a pair.	Constructed ordered pairs; found ordered pairs in pairs.
Deondria			
Jada			
Bernice	Did not reconstruct the odometer method.		Counted green-blue and blue-green flags, but did not construct ordered pairs.
Kai	N/A did not solve an AP successfully	N/A, did not solve an AP successfully.	N/A, did not solve an AP successfully.

Data for issue 1: The odometer method. Jada began the Flag Problem by coloring in all nine flags on her handout. The interviewer asked her to predict the total number of flags to which she responded, "there'd be a lot." The interviewer asked her to make a list, a tree diagram, or an array. Figure 12a shows the beginning of her list. As she created the second column of her list, she did not give indication she was using the first column of her list (e.g., point back, run her finger along, visually scan, etc.). After she completed the second column, the interviewer asked her, "Did you use the same order here [pointing to the colors in the second position of the first column] and here [pointing to the colors in the second position of the second column]? Jada said "yes," stating that the order of the colors in the second position was "green, yellow, red, pink, black, orange, purple, and blue." She then completed her list (Figure 12b).



B-*G*	G-*G*	Pink-G	Y-G	R-G	O-G	Blue-G	Purple-G
Y	*Y*	Y	Y	Y	Y	Y	Y
R	*R*	R	R	R	R	R	R
Pnk	*P*	P	P	P	P	P	P
B	*B*	B	B	B	B	B	B
O	*O*	O	O	O	O	O	O
P	*P*	P	P	P	P	P	P
B	*Blue*	B	B	B	B	B	B

Figure 12a (left) and 12b (right). Jada's list

Analysis of issue 1: Odometer method. Jada's list and statement indicates that she used the same order for the colors in the second position from column to column (see Figure 12b in blue and italicized). Her creation of an order for the colors in the second position differed from the MC1 students in that once she created an order for the colors in the second position of the first column she did not give indication that she was reliant on either the colored pencils or using her first column of notation to produce the same order for the subsequent columns in her list.

She, however, did not use the same order for the colors in the first position as she had for the colors in the second position. That is, the order of the colors going across (Figure 12b in red) was different from the order of the colors going down (Figure 12b in blue and italicized). This way of operating indicated that Jada treated the order of the colors in the first and second position as *independent* of one another. Six of the eight MC2 students operated similarly to Jada when they produced lists or tree diagrams in their solution of APs; they used the same order for the colors in the second position from column to column, but this order was not the same as the order for the colors they used in the first position. One of the other two MC2 students was like the MC1 students in that she did not use the same order for the colors in the second position from column to column, and the other MC2 student did not successfully solve an AP.

Data for issue 2: Operating with a single set. Shonda's solution of the Flag Problem illustrates well how MC2 students operated when they interpreted APs as involving only a single

set out of which to make outcomes. Her solution is also a good contrast to the MC1 students because she initially seemed to operate similarly to the MC1 students, but then changed how she was operating. To solve the Flag Problem, Shonda simultaneously made a tree diagram and a list.

Data Excerpt 3: Shonda's list and tree diagram for the Flag Problem

S [After coloring in six flags on her handout, the interviewer asked Shonda to predict the total number of flags. She said she was uncertain, and so the interviewer asked her what she would do to figure out the total number of possible flags]: Like blue, blue could be matched up with [counts the first four colored pencils on the table quietly] five, six, seven different colors. And that'd be orange, pink, purple, green, yellow, black, red. [Shonda makes a tree diagram first (Figure 13a). To make her tree diagram she uses the "B" in the center to mean blue, and the "B" on the leaf to mean black. She then uses Figure 13a to make Figure 13b, eliminating the "BB" flag when she makes Figure 13b presumably because she thinks it is the blue-blue flag.]

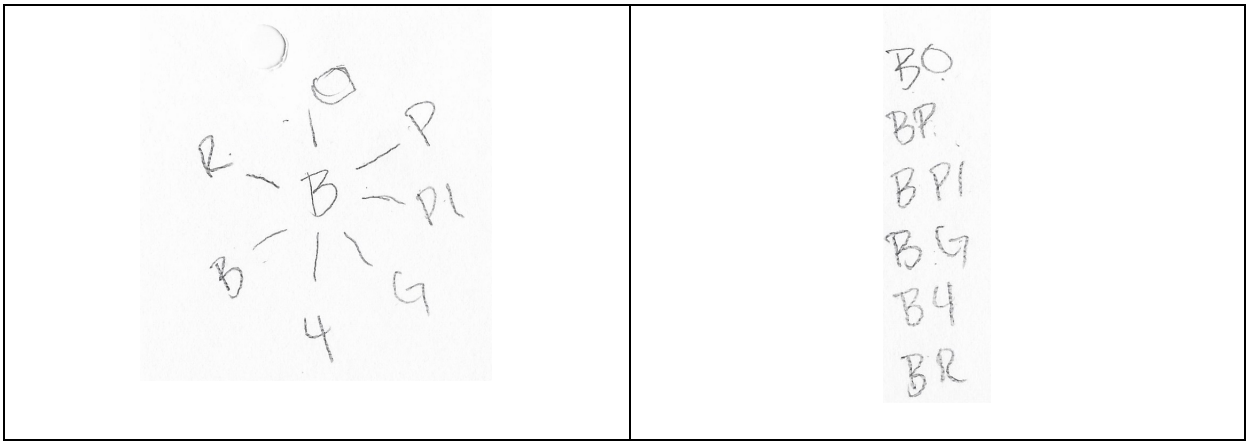


Figure 13a (left) and Figure 13b (right). The beginning of Shonda's tree diagram and list.

....

S [sees the "B-B" flag in her tree diagram]: Wait, but then there could be a blue-blue (flag).

I: A blue-blue (flag)? How many did you make in this one [points to her tree diagram (Figure 13a)]? Is this a blue-blue (flag) [points to Figure 13a where the B-B combination is recorded]?

S: Is it? Wait, that is black. Okay. My bad. So that is black [writes "k" after the B in Figure 13a. She then adds the blue-blue flag to her tree diagram (Figure 14a) and the blue-blue and blue-black flag to her list (Figure 14b).]

….

I: And how many did you make [points at Figure 14a] on the tree?

S [counts the number of flags in Figure 14a]: Eight, wait…[She appears surprised. She re-counts the number of flags in Figure 14a. She says confidently,] Okay so it is eight combinations you could make. [Prior to continuing her list and tree diagram, she adds a "B" to the upper left corner of her list (Figure 15, underlined). Each time she starts a new column in her list she writes a letter like the underlined "B" in Figure 15.]

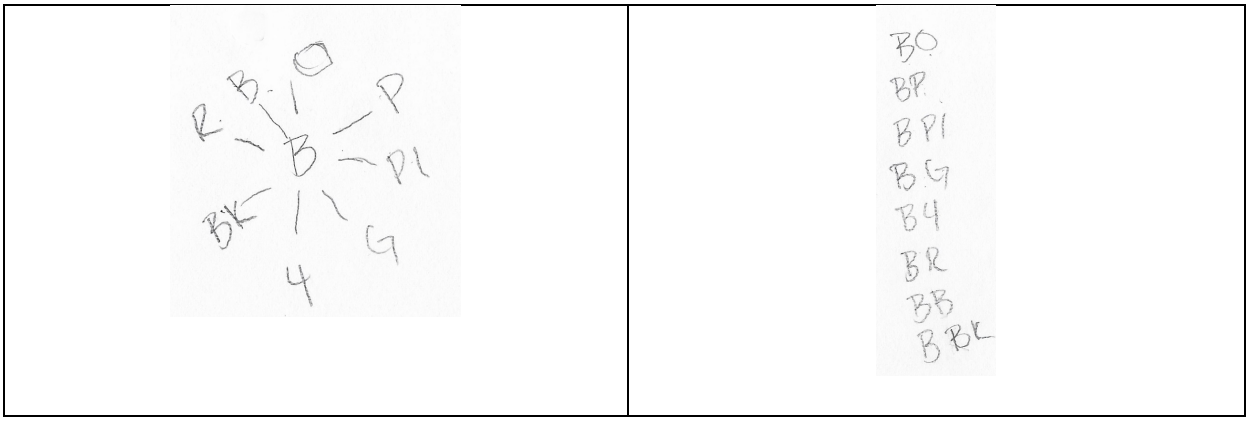


Figure 14a (left) and Figure 14b (right): Shonda adjusts her tree diagram and list

….

I [Shonda has recorded all flags except those that have black in the first position. She has not counted the number of flags in a column since her first column. The interviewer starts to ask how many flags she can make with black]: Do you have a prediction, how many you think…

S [cutting the interviewer off]: Eight. [Both she and the interviewer laugh].

I: You don't want to make it?

S: Well I can. [Shonda indicates she considers listing the final flags unnecessary.]

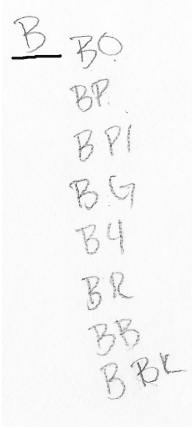


Figure 15. Shonda adjusts her list to show that this column uses the color blue

Analysis of issue 2: Operating with a single set. To start her solution of the Flag Problem, I infer Shonda operated similarly to MC1 students. She referenced that there would be seven colors that could go with the color blue, indicating that she assimilated the situation using a single composite unit of eight and separated the composite unit into one and seven. However, there was evidence that she changed how she operated as she made her tree diagram and list. One piece of evidence was her inclusion of the blue-blue flag, which was, in part, a happy accident in that she momentarily interpreted the "B-B" flag in her tree diagram as the blue-blue flag, even though she initially seemed to intend it to be the blue-black flag. Her inclusion of the blue-blue flag differed from Carlos; Carlos had primarily included flags like the blue-blue flag in order to ensure that he could use the previous column of his list to help him keep track of what flags to make in the current column. Shonda, however, seemed to include it because she considered it to be no different than any of the other flags that had blue in the first position, stating, "wait, but then there could be a blue-blue (flag)." She also differed from Carlos in that it seemed to make sense to her that if she included blue as one of the colors for the second position she would get eight flags when blue was in the first position, stating, "Okay so it is eight combinations you could make." Some MC2 students, for example, Jose expressed this relationship verbally as, "So

(it'd be) eight (flags for one color) because one times eight equals eight," a multiplicative statement that referred to one color, eight colors, and eight flags.

These differences led me to a different interpretation of the operations Shonda, and the other MC2 students, produced to solve APs. I infer that Shonda assimilated the situation with a single composite unit of eight (Figure 16a), and disembedded a unit of one from the composite unit (Figure 16b). Using a disembedding operation established a unit of one, independent from, but related to the composite unit of eight. This interpretation is consistent with her inclusion of the underlined letter "B" in Figure 15; she explicitly recorded the one color that she was using to make all of the flags in that column, independently of the flags she listed. I infer she, then, used

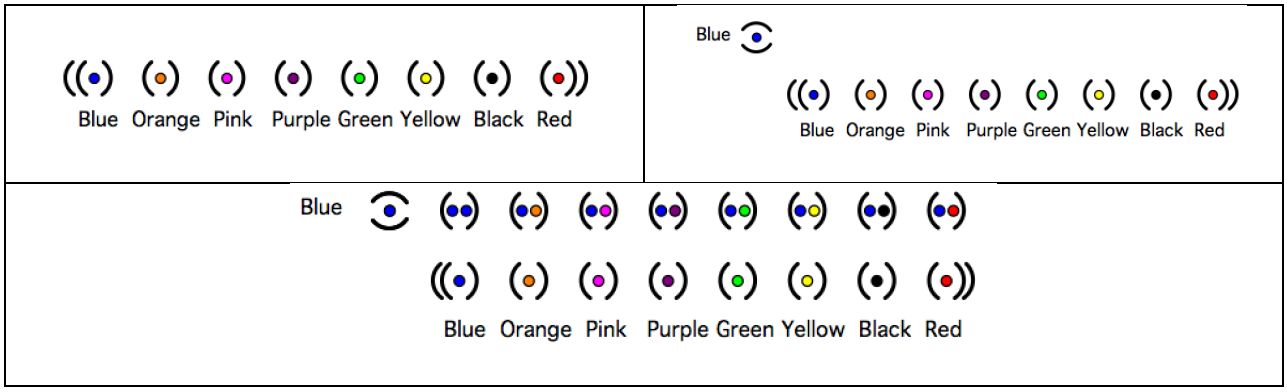


Figure 16a (top left), 16b (top right), and 16c (bottom). Shonda's activity

her disembedding operation again in concert with her pairing operation to establish the pairs independently from, but related to the composite unit of eight (Figure 16c). One piece of evidence that she established the pairs independently from, but related to, the one color and eight colors was her statement that she would make eight flags with the color black prior to actually listing them. This statement indicated that she had adjusted her expectation of how many flags she could make with each individual color, even though she had not counted the number of flags she could make with any of the colors except blue. Another piece of evidence were statements

like Jose's that "one times eight was equal to eight," indicating that the one color, the eight colors, and the eight flags were all independent of each other.

Shonda's way of operating was representative of seven of the eight MC2 students; the eighth student did not correctly solve an AP. The seven students who did operate in this way provided evidence that not all of them established the same multiplicative relationships in these situations. This difference is illustrated through differences in how they constructed and used ordered pairs to structure the outcomes, the final issue that I analyze.

Data for issue 3: Construction and use of ordered pairs. Recall the criteria I considered essential to infer the construction of ordered pairs: (a) a student considered flags like the blue-green and green-blue flag to be different; (b) related this difference to the position in which a color appeared in a pair (e.g., green is in the first position); and (c) related the position back to the original difference that they established (e.g., first position means top stripe). I could attribute these three criteria to six of the eight MC2 students; the other two established the first criteria, but did not establish the second two criteria. For the MC2 students who constructed ordered pairs, I was interested in determining whether they could also see that all ordered pairs with a particular color (e.g., black) made a row and column in their lists or arrays (Figure 17 in red and

G,G	Y,G	R,G	Pnk,G	*Bk,G*	O,G	P,G	B,G
G,Y		...		*Bk,Y*		...	
G,R				*Bk,R*			
G,Pnk		...		*Bk, Pnk*		...	
G,Bk	*Y,Bk*	*R,Bk*	*Pnk,Bk*	*Bk,Bk*	*O,Bk*	*P,Bk*	*B,Bk*
G,O		...		*Bk,O*		...	
G,P				*Bk,P*			
G,B		...		*Bk,B*		...	

Figure 17. A list showing all flags with black in either the first or second position or both

779 italicized).[9] The following data excerpt with Terionna illustrates this issue.

780 Data Excerpt 4: Terionna uses ordered pairs to structure the outcomes[10]

781 I [Terionna has represented the flags using a list, tree diagram, and array. The interviewer
782 covers all representations]: Can I ask you a challenge question? [Terionna nods her head
783 "yes"]. Do you know how many total flags you could make that would have black in
784 them in some way? Either in the top stripe or the bottom or in both?
785 T: Maybe eight. Eight different kinds of combinations. [The interviewer asks her to
786 verbally list the flags. She does starting with red-black and ending with black-black flag.]
787 ….
788 T: Wait, you ["you" refers to herself not to the interviewer] are just making the black on
789 the bottom, and then the other colors on top.
790 I: Yeah that is right. So how many total would have black in either stripe?
791 T: Sixteen. Yeah, sixteen.
792 I: Okay so go ahead and tell me why it is sixteen. That is good.
793 T: Because, well, you said how many could have either black on top or black on bottom.
794 There are eight and you could put it on, put the eight different colors on top of the black,
795 and then you could put it on the bottom, if you switched it around, and if you put it
796 altogether, it would be sixteen different groups of the black first and color, and then
797 color and then black second. [The interviewer asks her to verbally list the sixteen flags.
798 She lists the first eight flags, saying the black-black flag as the eighth flag. She then

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[9] This problem has similarities to the Extension of the Outfits Problem outlined in the Theoretical Framework. However, this problem is different in two ways: (a) no new colors are added; and (b) a single color, rather than one shirt and one pants, creates the row and column structure. Recall that some MC2 students could solve the Extension of the Outfits Problem in the absence of using a list or an array while others could not.

[10] We note that Terionna uses an array in her solution to the problem. Students' transition to arrays entailed some changes in their reasoning. We do not analyze those changes here because the reasoning that Terionna produced largely occurred prior to her consideration of the implications it had for her array representation.

pauses for ten seconds, and says]: So it wouldn't be [Terionna means it would not be sixteen]. So black and black wouldn't be one group again plus another group because you can't switch black and black around, like it's the same color. So it would be eight and seven. [Terionna circles these on her array, circling the black-black flag first, and then the seven flags horizontally followed by the seven flags vertically (Figure 18).]

Analysis of issue 3: Use of ordered pairs. Terionna established that there would be 15 flags with black in either the first or second position or both positions. She did so in the absence of looking at her written list, tree diagram, or array, which meant that she could not simply locate and count the ordered pairs in one of those representations. To start her solution, she first stated there would be eight flags with black in them. At the request of the interviewer, she generated a

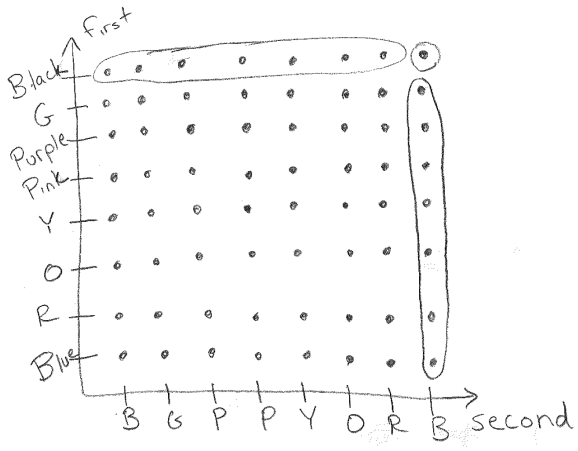


Figure 18. Terionna's array[11]

verbal list, saying the black-black flag last. After saying the black-black flag, she said, "wait, you are ("you are" meaning "she was") just making black on the bottom and the other colors on top," and adjusted her answer to 16 flags. I infer her adjustment from 8 to 16 flags came from verbally listing the black-black flag, which helped her realize that black could also be used as a color in

[11] This array is a replica of her array. Her array had too many markings on it to reproduce it. Note that she labeled the vertical axis as representing the color in the first position in a pair and the horizontal axis as representing the color in the second position in the pair, which reverses the conventional way that ordered pairs are read.

the top stripe. The interviewer asked her again to verbally list the 16 flags she was counting; she did, but stopped after she said the black-black flag, which was only the eighth flag. Here she realized that for the black-black flag she could not switch the color in the first position with the color in the second position to produce a new flag, saying "you can't switch black and black around." Her realization enabled her to adjust the total number of flags again from 16 to 15.

To solve this problem, I infer Terionna used her concept of *ordered pairs*. I base this inference on her explanation of why she thought there would be sixteen flags. During this explanation, she said, "sixteen different groups of the black first and color, and then color and then black second." This explanation indicated that she was not just envisioning switching the colors around in the flag; she was also relying on which position black appeared in an ordered pair, a key criterion for the inference that a student was using ordered pairs. Moreover, I infer that Terionna engaged in a units coordination, which established a multiplicative relationship among a unit of one, a unit of eight units, and a unit of eight ordered pairs *in activity* (Figure 19). I make this inference because Terionna could reason about features of the eight ordered pairs without *always* having to create them individually. However, I considered her to produce this relationship *in activity* because she needed to produce *some* of the ordered pairs in order to establish the relationships. Establishing this relationship enabled her to reason that the black-

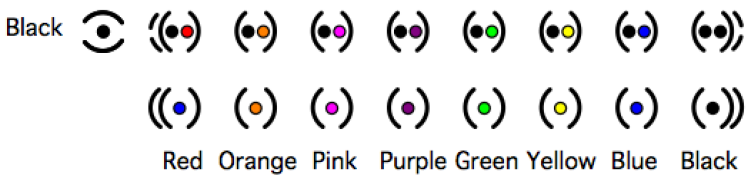


Figure 19. Multiplicative relationship among a unit of one, a unit of eight units, and a unit of eight pairs in activity

black flag was contained in the flags that had black in the second position even in the absence of producing all of these flags. However, to consider this issue she produced all flags with black in the first position. Figure 19 shows that she produced this multiplicative relationship in activity by using dotted parenthesis around the pairs (compare with Figure 16c).

Recall that only six of the MC2 students constructed ordered pairs. Of these students, I could attribute Terionna's way of operating to three of them. The other three MC2 students did *not* produce this way of operating. These students could locate ordered pairs in *pairs*, anticipating that for any pair they were given there was exactly one other pair that had the same colors in it. I considered this anticipation along with the operations I could infer about how they operated on a single composite unit to be sufficient to conclude that these MC2 students established a multiplicative relationship among a unit of one, a unit of one, and an ordered pair.

Discussion

Response to Research Questions: Identifying Domain Specific Contributions of the Study

Table 3 provides a summary of MC1 and MC2 students' solutions of APs from the Results, and it compares this information to MC1 and MC2 students' solutions of CPPs. The individual cells of Table 3 provide responses to the first and second research questions, and a comparison of the rows in Table 3 provide a response to the third research question. I amplify these responses through a discussion of the three issues that I used to structure the Results section. Through this discussion I identify contributions of the study to the literature on combinatorial reasoning.

The findings from this study about the odometer method, coupled with Tillema's (2014) earlier findings, provides a trajectory for how students can re-construct this method. As students transition from solving CPPs to APs, students may move from: (a) using a consistent pattern for the items in the second position from column to column even though they do not use the *same*

858 Table 3. Comparison of Students' Schemes for Solving APs and CPPs

	Differences in assimilatory structure of the scheme	Differences in the activity of the scheme	Difference in the results of the scheme
MC1 students	All three MC1 students assimilated APs using a single composite unit. In contrast, they assimilated CPPs using two composite units.	• For APs, all three MC1 students operated on the single composite unit by separating it into two parts (e.g., one and seven). They then used a pairing operation to produce pairs. • For CPPs, two of the MC1 students used a *disembedding* and pairing operation. By doing so, they established pairs as independent from, but related to the units of one they used to create them. • For APs, they did not use the odometer method to organize producing pairs. • For CPPs, they did use the odometer method.	• For APs, two of the three students established ordered pairs. • For CPPs, they did not establish ordered pairs.
MC2 students	All eight MC2 students assimilated APs using a single composite unit. In contrast, they assimilated CPPs with two composite units.	• For APs, seven of the eight MC2 students *recursively* used their disembedding operation to solve APs: they disembedded a singleton unit from the composite unit they used in assimilation; and then they disembedded the singleton unit and first unit from the composite unit to create pairs. • For CPPs MC2 students used their disembedding operation, but did not use it *recursively* because they assimilated situations with two composite units. • For APs, six of the MC2 students used the odometer method to structure producing pairs. • For CPPs, all eight students used the odometer method.	• For APs, six of the eight MC2 students constructed ordered pairs. • For CPPs, they did not establish ordered pairs. • For APs, of the six who constructed ordered pairs, three established a multiplicative relationship between a unit of one, a unit of one, and an ordered pair. These three students could anticipate finding ordered pairs in pairs in their lists, tree diagrams, or arrays. • For APs, the other three MC2 who constructed ordered pairs established a multiplicative relationship among a unit of one, a composite unit, and a unit of ordered pairs in activity. These three students were able to use ordered pairs to structure all flags with a particular color in the first or second position as a row and column of an array.

order for the items in the second position (what MC1 students did); to (b) using the same order for items in the second position from column to column, but not using the same order for the items in the first and second position (what the majority of MC2 students did); to (c) using the same order for the items in the second position from column to column *and* using the same order for items in the first and second position (what MC3 students did in Tillema, 2014). Although there were differences according to multiplicative concept in how students re-constructed the odometer method, I conjecture that this learning is accessible to students using any of the multiplicative concepts, and that the identified differences were tied to the duration of the study. I make this conjecture because the odometer method is tied more strongly to how a student organizes producing outcomes, and tied less strongly to the complexity of the unit structure a student produces in a situation. More broadly, this trajectory supports the idea that students may re-construct the odometer method multiple times prior to constructing it as a generalized method (Halani, 2012) that they can use to justify producing all possible outcomes (Maher, et al., 2010).

The finding that MC1 and MC2 students in this study assimilated APs using a single composite unit, and as a result experienced a perturbation, confirmed Tillema's (2014) earlier results with MC3 students. The prevalence of the perturbation among the students in the two studies helps to identify a key way that students experience APs as different from CPPs. One way to interpret this particular difference is that it is "in the problems;" APs refer to only a single set when they are stated whereas CPPs refer to two sets when they are stated. I do not interpret the difference in this way because it is possible to interpret an AP like the Flag Problem as involving two composite units; a student could anticipate, for example, that there are eight possible colors for the top stripe and eight possible colors for the bottom stripe prior to producing any flags. In fact, one MC3 student in Tillema (2014) learned to interpret APs in this way and

one MC3 student from this study entered the study interpreting APs in this way. In the Implications Section, I provide an analysis of why interpreting APs in this way can be mathematically significant. This analysis provides additional rationale for why it is important to understand how students resolve this commonly experienced perturbation.

One interesting difference between this study and Tillema's (2014) earlier study was that in this study there was a more explicit design to promote the construction of ordered pairs, which allowed for identifying three criteria for defining when MC1 and MC2 students had constructed ordered pairs. Establishing such criteria is significant given that researchers have found: (a) that students often do not consider counting ordered pairs in combinatorial contexts (e.g., Fischbein & Gazit, 1988; Piaget & Inhelder, 1975;), and (b) that students' lack of construction of ordered pairs adversely impacts how they reason about probabilistic situations (Morsanyi, Handley, & Serpell, 2013; Rubel, 2007). The findings from these studies suggest that even when students do consider two outcomes to be different (e.g., the red-blue and blue-red flag) this alone is not sufficient for students to reason with ordered pairs in normative ways, adding credence to making this a necessary, but not sufficient criterion to infer the construction of ordered pairs.

Response to Research Questions: Identifying Theoretical Contributions of the Study

Current models of student reasoning have not yet carefully traced out whether and when students establish relationships among one and two-dimensional units (Battista, 2007; Smith & Barrett, 2018; Tillema & Gatza, 2016). The findings from this study provide one example of models of student reasoning that explicitly attend to this issue in combinatorial contexts (see Kobiela & Lehrer, 2019, for an example in the context of area). Doing so provides more nuanced psychological models of students' multiplicative reasoning. These models extend earlier work that has attended to unit structures (e.g., Steffe, 1992, 1994; Hackenberg, 2007, 2010), but which

has not explicitly modeled when students establish relationships among one and two-dimensional units. A discussion of the differences in operations that students used to solve CPPs and APs provides insight into how students established and conceptualized these relationships differently.

To solve CPPs, MC1 students used a disembedding operation in concert with their pairing operation. Using this combination of operations enables a student to establish one two-dimensional unit as independent of, but related to, the two one-dimensional units that comprise it (Tillema, 2018). MC1 students' operated differently in their solution of APs; they used a pairing operation, but *not* a disembedding operation. Operating in this way meant that in their solution of APs they did establish a two-dimensional unit (i.e., a pair), but they did not retain the two one-dimensional units that comprised the pair. This change in how they operated was in part a result of assimilating APs using a single composite unit, which meant to retain the two one-dimensional units would involve a recursive use of their disembedding operation. There was not sufficient evidence to conclude that MC1 students operated in this way. Providing models of student reasoning that account for when they establish this relationship is important given researchers' findings that students do not easily establish two-dimensional units as the product of two one-dimensional units (Nunes, Light, & Mason, 1993; Outhred & Mitchelmore, 2000).

MC2 students established similar multiplicative relationships in their solution of APs and CPPs; three MC2 students had interiorized a multiplicative relationship among a unit of one, a unit of one, and an ordered pair and three MC2 students produced a multiplicative relationship among a unit of one, a unit of eight units, and a unit of eight ordered pairs in activity. The primary difference was that in APs to establish these multiplicative relationships they used their disembedding operation recursively. One way to think about the importance of using one's disembedding operation recursively is that it means that a student creates the one-dimensional

units from the same material rather than conceptualizing the one-dimensional units as qualitatively distinct (e.g., as shirts and pants). When students operate in this way in their solution of APs, they can make a qualitative distinction between the one-dimensional units (e.g. colors for the first stripe and colors for the second stripe), but the qualitative distinction results from them imposing it on what they conceive of as one-dimensional units that do not initially have a qualitative distinction. Providing models that allow researchers to make this kind of inference is important given that students frequently treat qualitative labels (e.g., "length" and "width") as if they have an impact on the quantitative units of measure (Battista, 2007).

Limitations of the Study

One of the central limitations of the current study is that it was conducted over a short period of time. Therefore, it gives a small snapshot about what students operating with different multiplicative concepts could do, but does not provide information about what learning is possible over an extended period of time. Maher & colleagues (2010) have provided one of the few longitudinal studies in the domain of combinatorics. Other longitudinal studies are needed. A second limitation of the study is that the results are tentative in the sense that researchers have not made explicit models of the relationships students establish among one- and two-dimensional units in combinatorial contexts. The process of making these models is in its early stages, and continued work is needed to clarify and refine the models, while also considering how they could be integrated into what we already know about students' multiplicative (e.g., Greer, 1992; Verschaffel, Greer, & De Corte, 2007), geometric (e.g., Battista, 2007), and measurement based reasoning (e.g., Smith & Barrett, 2018). Both limitations point to directions for future research.

Implications and Directions for Future Research

One primary reason for attending to students' transition between CPPs and APs has to do with a mathematical difference between these problems. The mathematical difference stems from Vergnaud's (1983) characterization of product of measures problems as involving a double proportion. In the case of CPPs, Vergnaud's double proportion means that, for example, the total number of outfits in the Outfits Problem is proportional to both the number of shirts and the number of pants. Therefore, if the number of pants is tripled, then the total number of outfits is tripled (Figure 20a), and similarly if the number of shirts is tripled, then the total number of outfits is tripled (Figure 20b). In the case of CPPs, these two proportional relationships can be considered independent of each other—changing the number of pants can be done independently from changing the number of shirts. On the other hand, APs like the Flag Problem can be seen as a special case of the double proportion property (Tillema, 2013); the number of colors for the second stripe can be interpreted as *dependent* on the number of colors for the first stripe. So, for example, doubling the number of colors means that the number of colors for the first stripe is

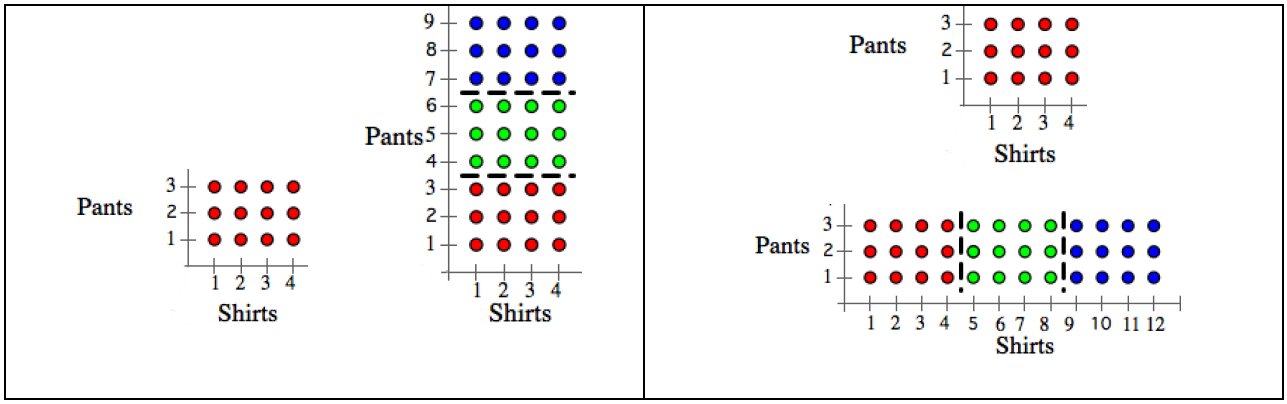


Figure 20a (left) and 20b (right): Arrays to represent the multiplicative relationships

doubled *and* the number of colors for the second stripe is doubled, which produces *four* times the total number of flags (Figure 21). This lack of independence of the proportions means that the double proportion functions as a direct *square* proportion: a change in the number of colors is related to a change in the number of flags through a quadratic relationship.

This mathematical distinction suggests that the way students understand the relationship in APs between, for example, the number of colors for the first stripe and the number of colors for the second stripe has important mathematical implications; this relationship is the basis for how students might establish quadratic relationships in their solution of APs. The relationship students establish is a function of what operations they use to resolve the perturbation they experience when they assimilate APs using a single composite unit. Thus, one implication of this study, and direction for future research, is to understand how solving APs might support students to establish quadratic relationships. Understanding how students establish quadratic relationships is critical because of how pervasive it is for students to reason as if *all* situations involve linear relationships (e.g., Greer, 2010; Van Dooren, De Bock, Janssens, & Verschaffel 2008).

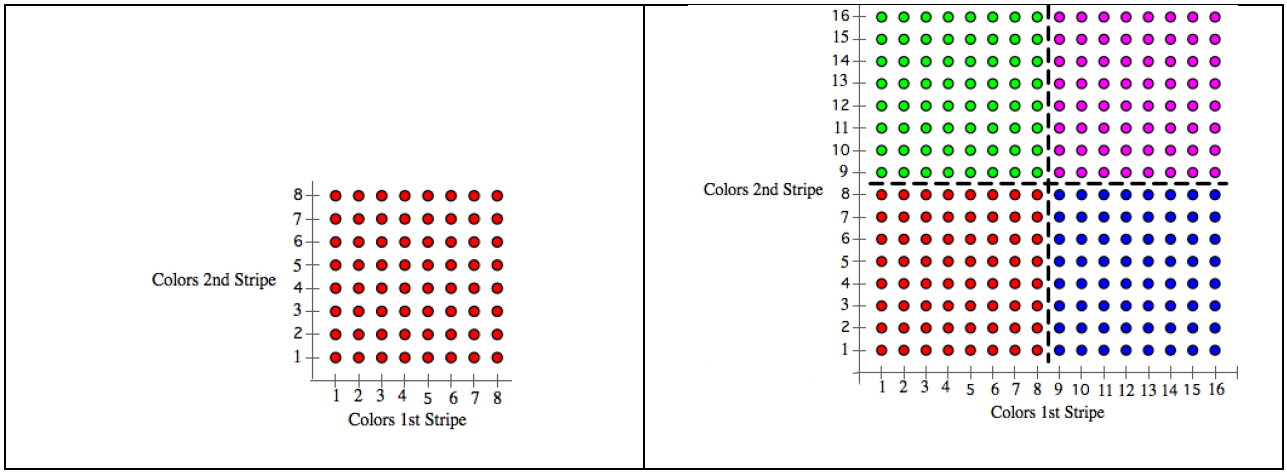


Figure 21. Array for the Flag Problem

Conclusion

The increased focus in standards and curricula on combinatorics, accompanied with research indicating that this domain is quite challenging for students (e.g., Fischbein & Gazit, 1988; Halani, 2012), suggests a need for researchers to continue to work on developing models of student reasoning that can be used to support instructional decision making (e.g., English, 1991, 1993; Lockwood & Swinyard, 2016). The findings from this article provide one example of this

kind of model. As these models are developed, refined, and extended, they can serve as powerful instruments to support teachers and researchers to engage students in productive interactions (Ulrich, Tillema, Hackenberg, & Norton, 2014). Moreover, attending to how students who use different multiplicative concepts reason about similar situations can serve as a basis for mathematics educators to differentiate instruction in ways that are rooted in differences in student reasoning (Hackenberg, Creager, Eker, & Lee, 2016). Differentiation of instruction helps mathematics educators make informed decisions about how to support students who bring different resources to the classroom.

993 References

994 Australian Curriculum, Assessment and Reporting Authority [ACARA], (n.d.). Retrieved from
995 https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/
996 Batanero, C., Navarro-Pelayo, V., & Godino, J. D. (1997). Effect of the implicit combinatorial model on
997 combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32*(2),
998 181-199.
999 Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester, Jr. (Ed.),
1000 *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 843-908).
1001 Charlotte, NC: Information Age Publishing.
1002 Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity. In G. Harel & J. Confrey (Eds.), *The*
1003 *development of multiplicative reasoning in the learning of mathematics* (pp. 121-176). Albany,
1004 New York: State University of New York Press.
1005 Chini, J. J., Carmichael, A., Rebello, N. S., & Puntambekar, S. (2009, November). Does the
1006 teaching/learning interview provide an accurate snapshot of classroom learning?. In *AIP*
1007 *Conference Proceedings* (Vol. 1179, No. 1, pp. 113-116). AIP.
1008 Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven
1009 research design. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics*
1010 *and science education* (pp. 231-265). Mahwah: NJ: Lawrence Erlbaum Associates.
1011 Eizenberg, M. M., & Zaslavsky, O. (2004). Students' verification strategies for combinatorial
1012 problems. *Mathematical Thinking and Learning, 6*(1), 15-36.
1013 English, L. D. (1991). Young children's combinatoric strategies. *Educational Studies in Mathematics,*
1014 *22*, 451-474.
1015 English, L. D. (1993). Children's strategies for solving two- and three-dimensional combinatorial

problems. *Journal for Research in Mathematics Education, 24*, 255-273.

English, L. D. (1996). Children's construction of mathematical knowledge in solving novel isomorphic problems in concrete and written form. *Journal of Mathematical Behavior, 15*, 81-112.

English, L. D. (1999). Assessing for structural understanding in children's combinatorial problem solving. *Focus on Learning Problems in Mathematics, 21*(4), 63-83.

Fischbein, E., & Gazit, A. (1988). The combinatorial solving capacity in children and adolescents. *Zentralblatt fur Didaktik der Mathematik, 5*, 193-197.

Greer, B. (1992) Multiplication and division as models of situations. In Grouws, D.A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*, pp. 276-295. New York: Macmillan.

Greer, B. (2010). Overview of papers: Why is linear thinking so dominant? *Mathematical Thinking and Learning, 12*(1), 109-115.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior, 26*, 27-47.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 1-50.

Hackenberg, A. J., Creager, M., Eker, A., & Lee, M. L. (2016, April). *Understanding how to differentiate instruction for middle school students.* Paper presentation at the Research Conference of the NCTM, San Francisco, CA.

Hackenberg, A.J., & Tillema, E.S. (2009). Students' fraction composition schemes. *Journal of Mathematical Behavior*, 28(*1*), 1-18.

Halani, A. (2012). Students' ways of thinking about enumerative combinatorics solution sets: The odometer category. In the *Electronic Proceedings for the Fifteenth Special Interest*

Group of the MAA on Research on Undergraduate Mathematics Education. (pp. 59-68) Portland, OR: Portland State University.

Kapur, J. N. (1970). Combinatorial analysis and school mathematics. *Educational Studies in Mathematics, 3*(1), 111-127.

Kobiela, M. & Lehrer, R. (2019). Supporting dynamic conceptions of area and its measure. *Mathematical Thinking and Learning*, *21*(1), 1-29.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 909-955). Charlotte, NC: Information Age Publishing.

Lappan, G., Fey, J.T., Fitzgerald, W.M., Friel, S.N., & Phillips, E.D. (2002). *Clever counting*. Upper Saddle River, NJ: Prentice Hill.

Lobato, J. (2008). Research methods for alternative approaches to transfer: Implications for design experiments. *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, 167-194.

Lockwood, E. (2011). Student connections among counting problems: An exploration using actor-oriented transfer. *Educational Studies in Mathematics, 78*(3), 307-322.

Lockwood, E., & Gibson, B. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Education Studies in Mathematics, 91*(2), 247-270.

Lockwood, E. & Swinyard, C.A. (2016). An introductory set of activities designed to facilitate successful combinatorial enumeration for undergraduate students. *Problems, resources, and issues in mathematics undergraduate studies*, *26*(10), 889-904.

Maher, C.A., & Martino, A. (1996). The development of the idea of mathematical proof. *Journal for*

Research in Mathematics Education, 27(2), 194-214.

Maher, C. A., Powell, A. B., & Uptegrove, E. B. (Eds.). (2010). *Combinatorics and reasoning: Representing, justifying and building isomorphisms* (Vol. 47). Springer Science & Business Media.

Maher, C.A., & Yankelewitz, D. (2010). Representations as tools for building arguments. In C.A. Maher, A.B. Powell, & E.B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying, and building isomorphisms* (pp. 17-25). New York: Springer.

Mathison, S. (1988). Why triangulate? *Educational Researcher, 17*(2), 13-17.

Mcgillivray, H., Petosz, P., Evans, M., & Jones, P. (2013). *Statistics and probability for the Australian curriculum*. Sydney: Cambridge University Press

Ministry of Education, Science, and Technology, Korea (2007). *The school of curriculum of the Republic of Korea: General guidelines (Notification No. 2008-160).* Seoul: Author.

Morsanyi, K., Handley, S. J., & Serpell, S. (2013). Making heads or tails of probability: An experiment with random generators. *British Journal of Educational Psychological Society, 83*, 379-395.

Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309-330.

National Council of Teachers of Mathematics (2000). *Principles and standards of school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association for Best Practices, Council of Chief State School Officers (2010). *Common core state standards mathematics*. Washington D.C.: National Governors Association for Best Practices, Council of Chief State School Officers.

Nunes, T., & Bryant, P. (1996). The progress to multiplication and division. *Children doing*

mathematics (pp. 142-201). Oxford, UK: Blackwell Publishers.

Nunes, T., Light, P., Mason, J. (1993) Tools for thought: The measurement of length and area. *Learning and Instruction*, 3, 39-54.

Outhred, L. (1996). Children's drawings of multiplicative structures: Cartesian products and area. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 185-204). Adelaide, Australia: Australian Association of Mathematics Teachers for the Mathematics Education Research Group of Australasia.

Outhred, L. & Mitchelmore, M. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal of Research in Mathematics Education, 31*(2), 144-167.

Piaget, J. (1970). *Genetic Epistemology* (E. Duckworth, Trans.). New York: W.W. Norton & Company Inc.

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children* (L. J. Leake, P. Burrell & H. D. Fishbein, Trans.). New York: W.W. Norton & Company.

Rubel, L. H., (2007). Middle school and high school students' probabilistic reasoning on coin tasks. *Journal for Research in Mathematics Education, 38*(5), 531-556.

Saldana, J. (2013). *The Coding Manual for Qualitative Researchers.* London: Sage Publications Ltd.

Simon, M. A., & Blume, G.W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, *25*(5), 472-494.

Smith, J.P., III, & Barrett, J.E. (2018). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 355-385). Reston VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and*

individual difference, 4(3), 259-309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 1-41). Albany, New York: State University of New York Press.

Steffe, L. P. (2007, April). *Problems in mathematics education*. Paper presented for the Senior Scholar Award of the Special Interest Group for Research in Mathematics Education (SIG-RME) at the annual conference of the American Educational Research Association in Chicago, Illinois.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York City: Praeger Publishers.

Tillema, E.S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior*, 32, 331-352.

Tillema, E.S. **(**2014). Students' coordination of lower and higher dimensional units in the context of evaluating sums of consecutive whole numbers. *Journal of Mathematical Behavior,* 36, 51-72.

Tillema, E.S. (2018). An investigation of 6th graders' solutions of combinatorics problems and representation of these problems using arrays. *Journal of Mathematical Behavior*, 52, 1-20.

Tillema, E.S. & Gatza, A. (2016). A quantitative and combinatorial approach to non-linear meanings of multiplication. *For the Learning of Mathematics,* 36(*2*), 26-33.

Ulrich, C. (2015). Stages in Constructing and Coordinating Units Additively and Multiplicatively (Part 1). *For the Learning of Mathematics*, *35*(3), 2-7.

Ulrich, C. (2016). Stages in Constructing and Coordinating Units Additively and Multiplicatively (Part 2). *For the Learning of Mathematics*, *36*(1), 34-39.

Ulrich, K., Tillema, E.S., Hackenberg, A.J., Norton, A.N. (2014). Units coordination: An example of the utility of radical constructivist thought in education. *Constructivist Foundations,* 9(*3*), 328-339.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' overuse of linearity. *Journal for Research in Mathematics Education, 39*(3), 311-342.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquistion of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 557-628). Charlotte, NC: Information Age Publishing.

Von Glasersfeld, E. (1982). Subitizing: The role of figural patterns in the development of numerical concepts. *Archives de Psychologie, 50*, 191-218.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning* (Vol. 6). Washington D.C.: The Falmer Press.

Watson, J. & Neal, D. (2012). Preparing students for decision-making in the 21st Century—statistics and probability in the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon, (Eds.). *Engaging the Australian national curriculum: Mathematics – perspectives from the field* (pp. 89-113). Online Publication: Mathematics Education Research Group of Australasia.

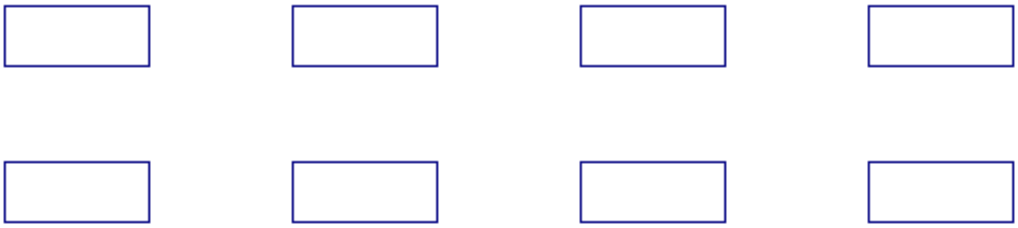
Appendix A: Selection Interview Tasks

1)
   a. The 7th grade classrooms has 4 rows of desks with 14 desks in each row. How many desks are in a 7th grade classroom?

   b. Suppose that 28 desks and 1 row are added to the classroom. How many rows and desk would there be?

2) You and 11 friends go to the movies (12 people total). You and your friends fill the first 3 rows of the theater. Determine the number of seats in the theater if there are 13 rows. (Again all rows have the same number of seats.)

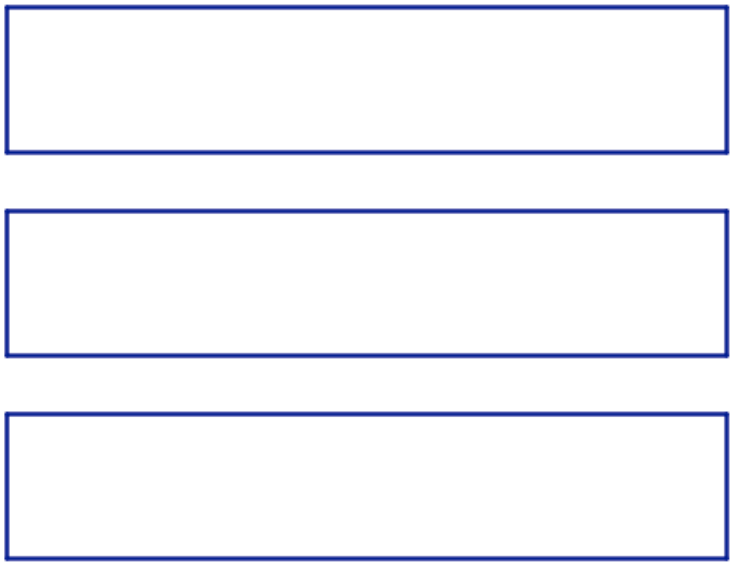
3) The 8th grade is taking 6 buses on a field trip. There are 96 students in the 8th grade. Use the picture below to show how many students will be on each bus (assume there are an equal number of students on each bus).



4)
   a. A candy factory puts 10 candies in each package and puts 10 packages in each box. The factory currently has 3 boxes, 4 packages, and 7 candies. How many total candies do they have?

   b. You buy 534 candies from the candy factory. How many packages of candy do you get?

5) Five people want to share the three identical candy bars. Use the bars below to show how you would accomplish this goal.



6) Jose buys some rope that is 65 centimeters long. Jose's rope is 5 times the size of the piece of rope you bought. Below is a picture of Jose's rope. Use the picture to help you determine the length of your rope.

7) Stephanie checks out 14 books on spiders from the library to prepare for her science fair project. The number of books Stephanie checked out is 2/7 of the total number of books the library owns on spiders. How many total books does the library own on spiders?

Appendix B: Interview 1 Tasks

1) You have 4 shirts and 3 pairs of pants. An outfit is 1 shirt and 1 pair of pants. Draw a picture to help you determine how many total outfits you can make.

2) Subway has 6 kinds of bread and 7 different kinds of meat. You can make a sandwich by choosing one kind of meat and one kind of bread. Draw a picture to help you determine how many total sandwiches you can order.

3) You have the Ace thru 7 of spades and your friend has the Ace thru 7 of hearts. A two-card hand consists of one card from your friend and one card from you (e.g., the two of spades and the two of hearts would be 1 two-card hand). How many different two-card hands could you make?

4) You and a partner each flip a coin and record the result of each of your coin flips (e.g., tails-tails would mean you each got tails on a flip). How many different outcomes could you record?

5) You are designing a flag for a recently discovered country. The flag has two stripes, each of which can be filled with a color. You have 8 different colors to choose from. Illustrate the total number of flags you could make.

6) A volleyball league has 10 teams. Each team wants to play each other once at home and once away. How many total games would there be?

7) You have the ace thru king of spades. To create a two card hand, you draw a card, replace it, and then draw a second card. How many possible two card hands could you make?

8) You are creating a two-character password from the letters A through N. How many different passwords could you create?