

Radial squeezed states and Rydberg wave packets

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We outline an analytical framework for the treatment of radial Rydberg wave packets produced by short laser pulses in the absence of external electric and magnetic fields. Wave packets of this type are localized in the radial coordinates and have p -state angular distributions. We argue that they can be described by a particular analytical class of squeezed states, called radial squeezed states. For hydrogenic Rydberg atoms, we discuss the time evolution of the corresponding hydrogenic radial squeezed states. They are found to undergo decoherence and collapse, followed by fractional and full revivals. We also present their uncertainty product and uncertainty ratio as functions of time. Our results show that hydrogenic radial squeezed states provide a suitable analytical description of hydrogenic Rydberg atoms excited by short-pulsed laser fields.

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Excitations of Rydberg atoms by short-pulsed laser fields produce wave packets that are localized in the radial coordinates [1,2]. These wave packets initially oscillate with the classical Keplerian period between inner and outer apsidal points corresponding to those of the classical Keplerian orbit [3]. They subsequently collapse and then eventually revive almost to their original shape [4]. In the interval between collapse and full revival, subsidiary wave packets form. These are known as fractional revivals, and their orbital periods are rational fractions of the classical period [5].

Depending on the excitation process and whether external fields are present, several different electronic orbital geometries are possible for the quantum-mechanical motion. For single-photon excitations without external fields, the radial motion is localized but the angular distribution is that of a p state. For multiphoton excitations, the angular distribution can be that of a d or higher state. If an external electric field is present at the time of excitation, a parabolic wave packet is formed. This exhibits beats in the angular momentum [6]. If instead the atomic excitation is in the presence of a strong radio-frequency field or crossed electric and magnetic fields, circular atoms may be produced [7]. These are Rydberg atoms with high angular momentum that are localized in the angular coordinates.

Since Rydberg wave packets are localized and exhibit some features of the classical motion, a description involving some type of coherent state [8] would seem appropriate. This possibility has been realized in a number of cases for circular and elliptical quantum geometries [9,10], suitable for characterizing the motion of a hydrogenic Rydberg atom excited by a short laser pulse in the presence of external fields. These coherent states are either a superposition of angular-momentum eigenstates or eigenstates with a large value for the angular momentum. However, when a wave packet is produced by single-photon excitation in the absence of external fields, a p -state eigenfunction with $l=1$ results.

In this Rapid Communication, we show that a particu-

lar kind of analytical squeezed state, called a *hydrogenic radial squeezed state*, can be used to describe aspects of a wave packet generated by exciting a hydrogenic Rydberg atom with a short laser pulse in the absence of external fields. Like coherent states, squeezed states [11] for a given system are quantum wave functions minimizing an operator uncertainty relation. However, for squeezed states, the ratio of the operator uncertainties typically has a different value than that of the ground state. This means that the squeezed-state uncertainty product and ratio display a distinct time dependence.

Direct attempts to obtain analytical squeezed states in the variables r and p_r meet with a variety of obstacles. Instead, we have adopted an indirect method suggested in Ref. [10], in which a change of variables is made from r and p_r to a new set, R and P , chosen to have certain properties of harmonic-oscillator variables and hence more amenable to analysis. A complete derivation of the analytical form of a general class of radial squeezed states, also valid for Rydberg atoms other than hydrogen (in particular the alkali-metal atoms used in experiments), is somewhat lengthy and is given elsewhere [12]. We restrict ourselves here to summarizing the results for hydrogenic p -state excitations. In what follows, we work in atomic units with $\hbar=e=m_e=1$.

The effective radial potential for a hydrogenic Rydberg atom with angular momentum $l=1$ is $V_{\text{eff}}(r)=(1-2r)/2r^2$. For this case, it turns out that the appropriate new quantum operator R replacing the usual r is $R \equiv (2-r)/2r$, while $P \equiv p_r$ remains unchanged. The associated commutation relation is $[R,P]=-ir^{-2}$, leading to the uncertainty relation $\Delta R \Delta P \geq \frac{1}{2} \langle r^{-2} \rangle$. We take the hydrogenic radial squeezed states as the wave functions minimizing this uncertainty relation. They are given by

$$\psi(r) = N r^\alpha e^{-\gamma_0 r} e^{-i\gamma_1 r}, \quad (1)$$

where $\alpha, \gamma_0, \gamma_1$ are three real parameters and N is a normalization constant [13].

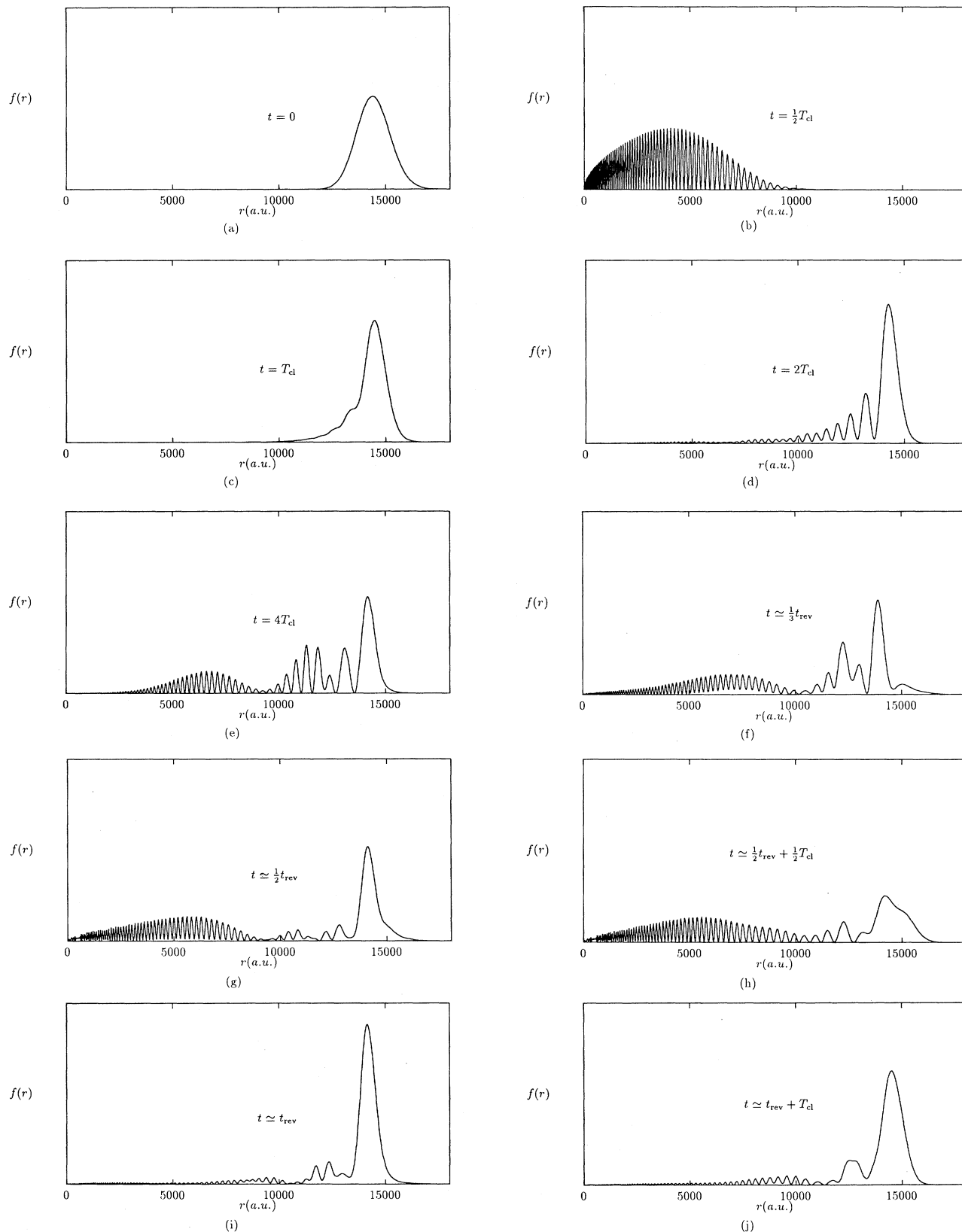


FIG. 1. Radial squeezed states of hydrogen with $\bar{n} = 85$. The unnormalized radial probability density is plotted as a function of the radial coordinate r in a.u. at times (a) $t = 0$, (b) $t = \frac{1}{2} T_{cl}$, (c) $t = T_{cl}$, (d) $t = 2T_{cl}$, (e) $t = 4T_{cl}$, (f) $t \approx \frac{1}{3} t_{rev}$, (g) $t \approx \frac{1}{2} t_{rev}$, (h) $t \approx \frac{1}{2} t_{rev} + \frac{1}{2} T_{cl}$, (i) $t \approx t_{rev}$, (j) $t \approx t_{rev} + T_{cl}$, where $T_{cl} = 93.3$ psec and $t_{rev} \approx 2.6$ nsec.

We fix the three parameters α , γ_0 , γ_1 for a particular hydrogenic radial squeezed state at time $t=0$ by matching to the corresponding hydrogenic Rydberg wave packet at its point of closest-to-minimum uncertainty. This occurs at $r=r_{\text{out}}$, representing the outer apsidal point [1]. In the present case, we assume that the laser excites a range of energy states with principal quantum numbers centered on the value \bar{n} and that contributions from continuum states are negligible. We take the outer apsidal point to be $r_{\text{out}}=\bar{n}^2+\bar{n}\sqrt{\bar{n}^2-2}$. This point is near the outer classical apsidal point given by $r_1=\bar{n}^2(1+e)$, where the eccentricity e is $e=\sqrt{1-1/\bar{n}^2}$. We perform the match by imposing the three conditions

$$\langle p_r \rangle = 0, \quad \langle r \rangle = r_{\text{out}}, \quad \langle H \rangle = E_{\bar{n}}. \quad (2)$$

Here, $E_{\bar{n}} = -1/2\bar{n}^2$ is the energy corresponding to \bar{n} . Roughly, these conditions mean that the hydrogenic radial squeezed state at $t=0$ is located at the outer turning point, has no radial velocity, and has specified energy. The conditions (2) determine the three parameters α , γ_0 , γ_1 in terms of \bar{n} and $l \equiv 1$. Our full three-dimensional wave packet at $t=0$ is then given by $Y_{10}(\theta, \phi)\psi_{\bar{n}, l=1}(r)$.

The time evolution of hydrogenic Rydberg wave packets has been previously studied [5]. After its formation, a packet is expected to execute radial oscillations with periodicity equal to the classical orbital period $T_{\text{cl}} = 2\pi\bar{n}^3$. Significant self-interference due to the gradual decoherence of the packet is anticipated at a time $t_{\text{int}} \sim \bar{n}T_{\text{cl}}/3\delta n$, where δn measures the range of dominant principal quantum numbers in the initial packet. Later, the packet should reconstitute itself almost to its original shape. This full revival is expected to occur at a time $t_{\text{rev}} = \frac{1}{3}T_{\text{cl}}$. It should persist for several orbits, performing oscillations with period T_{cl} . Between t_{int} and t_{rev} , there should be times t_r when the packet is gathered into r spatially separated pieces, oscillating with wavefunction periodicity $T_r = T_{\text{cl}}/r$. Among the values of t_r are the ones we discuss below at $t_r = t_{\text{rev}}/r$.

The analytical form of the hydrogenic radial squeezed states permits relatively simple expressions to be found for the time evolution. As these are somewhat lengthy, we do not provide them here [12]. Instead, we illustrate key features of the time evolution of the hydrogenic radial squeezed states by focusing on a particular example: radial p -state excitations of hydrogen centered on $\bar{n} = 85$. The associated classical orbit is one of high eccentricity, $e \approx 1$, and the period is $T_{\text{cl}} = 93.3$ psec. The outer apsidal point is $r_{\text{out}} \approx 2\bar{n}^2 \approx 14450$ a.u. Together with the conditions (2), this fixes the parameters of the radial squeezed state as $\alpha = 168.225$, $\gamma_0 = 0.0117465$, and $\gamma_1 = 0$.

Figure 1 shows the radial probability density $f(r) = r^2|\psi_{\bar{n}, l=1}(r)|^2$ for this radial squeezed state at various times. The initial ($t=0$) form of the state is displayed in Fig. 1(a). The packet is located around r_{out} . Its smooth, peaked shape reflects a small uncertainty product $\Delta r \Delta p_r \approx 0.5015$. The initial motion of the radial squeezed state is towards the inner turning point, near the origin.

Figure 1(b) shows the packet halfway through its first

orbit, $t = \frac{1}{2}T_{\text{cl}}$. The oscillatory nature of the probability distribution reflects the quantum nature of the packet near the core. Intuitively, the electron moves faster there and so has greater momentum, which in turn is reflected in the greater local curvature of the wave function. The uncertainty product there is $\Delta r \Delta p_r \approx 59.5$.

Figures 1(c) and 1(d) show the radial squeezed state after one and two full orbits, at $t = T_{\text{cl}}$ and $t = 2T_{\text{cl}}$, respectively. The packet is evidently repeatedly returning to the outer turning point, but after each successive orbit more decoherence appears and the quantum nature of the object becomes more apparent. This is reflected in the uncertainty product, which grows from $\Delta r \Delta p_r \approx 1.6$ at $t = T_{\text{cl}}$ to $\Delta r \Delta p_r \approx 8.0$ at $t = 2T_{\text{cl}}$.

At the time t_{int} , which is near $\sim 4T_{\text{cl}}$ in this example, it is no longer possible to attribute a single peak to the radial probability distribution. This situation is shown in Fig. 1(e). The remnants of the original peak are still visible but there are many oscillations and other subsidiary peaks present. The decoherence of the packet is reflected in the uncertainty product, which at $\Delta r \Delta p_r \approx 45.4$ is comparable to that of the highly quantum distribution in Fig. 1(b) appearing at $t = \frac{1}{2}T_{\text{cl}}$.

At the time $t_r \approx \frac{1}{3}t_{\text{rev}}$, a fractional revival is expected to appear. The corresponding distribution for the radial squeezed state is shown in Fig. 1(f). Three spatially separated packets are indeed visible. Similarly, a fractional revival containing two spatially separated packets appears at $t_2 \approx \frac{1}{2}t_{\text{rev}}$. Its forms are shown at that time and at half a classical period later, at $t \approx \frac{1}{2}t_{\text{rev}} + \frac{1}{2}T_{\text{cl}}$, in Figs. 1(g) and 1(h), respectively. These figures demon-

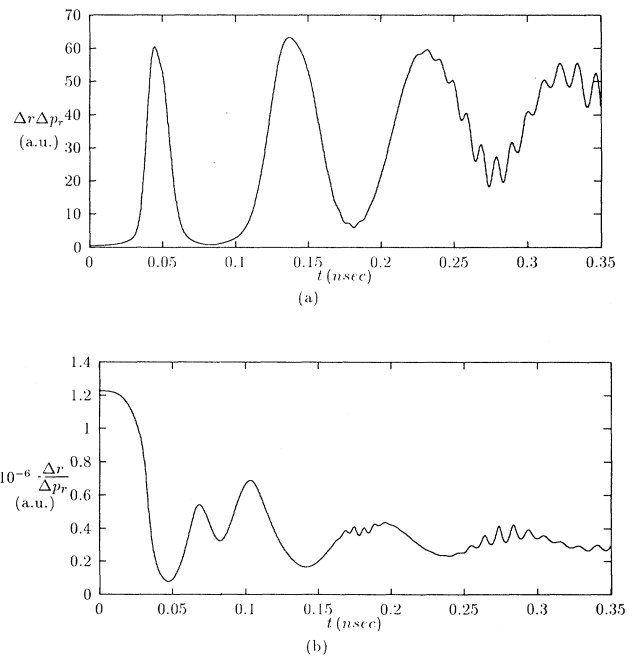


FIG. 2. (a) The uncertainty product $\Delta r \Delta p_r$, as a function of time in nanoseconds for a radial squeezed state of hydrogen with $\bar{n} = 85$. (b) The ratio of the uncertainties $\Delta r / \Delta p_r$, in units of 10^6 a.u. as a function of time in nanoseconds for a radial squeezed state of hydrogen with $\bar{n} = 85$.

strate that the periodicity of the distribution in time is indeed $\frac{1}{2}T_{\text{cl}}$, in agreement with expectation.

Figures 1(i) and 1(j) display the radial squeezed state near the revival time, $t \simeq t_{\text{rev}} \simeq 2.6$ nsec, and one classical period later. As expected, the distribution is once again similar to that of a single peak located at the outer turning point, evolving with the classical periodicity.

Figure 2(a) shows the uncertainty product $\Delta r \Delta p_r$ for the radial squeezed states as a function of time. Again, this plot is for hydrogen with $\bar{n} = 85$. The smallest value of the uncertainty product in r and p_r is close to $\frac{1}{2}$ at the initialization time $t = 0$, when by construction the uncertainty product in R and P is minimized. The cyclic behavior of the uncertainty product as a function of time, expected for a squeezed state, is revealed. The classical orbital periodicity is again manifest during the first few orbits, until the wave function begins to decohere. The product $\Delta r \Delta p_r$ then remains large until the revivals form.

Figure 2(b) displays the uncertainty ratio $\Delta r / \Delta p_r$ as a function of time for the same hydrogenic radial squeezed state. This provides a measure of the relative amount of

squeezing in the two coordinates. Overall, it is apparent that the squeezing in p_r is greater than that in r by five or six orders of magnitude. In fact, initially $\Delta r / \Delta p_r \simeq 1.2 \times 10^6$. Halfway through the first orbit, at $t = \frac{1}{2}T_{\text{cl}}$, the relative squeezing reaches its minimum of $\Delta r / \Delta p_r \simeq 8.0 \times 10^4$. The uncertainty ratio continues to change with time, but remains large so that p_r remains squeezed.

In summary, the characteristic features of the hydrogenic radial squeezed states are similar to those of Rydberg wave packets in atoms excited by a short laser pulse without external fields. Further details of our analysis, including in particular the generalization of our analytical framework via supersymmetry-based quantum defect theory [14] to the case of alkali-metal atoms, is presented elsewhere [12].

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