January 16, 2017

Local Martingale Solutions to the Stochastic Two Layer Shallow Water Equations with Multiplicative White Noise

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Abstract

We consider the two layers shallow water equations on a bounded domain $\mathcal{M} \subset \mathbb{R}^2$ forced by a multiplicative white noise, and obtain the existence and uniqueness of a maximal pathwise solution for a limited period of time. The proof relies on the Skorohod representation theorem, the Gyöngy-Krylov theorem, stopping time arguments, and isotropic estimates

Contents

1 Introduction

In this article, we consider a shallow water equation model that, to the best of our knowledge, has not been studied before. In the deterministic problem, which has been studied extensively, one must assume that the initial data is small or, otherwise, the solution is only known to exist for a short period of time. In the stochastic context we consider the shallow water equations forced by a multiplicative white noise representing e.g. random wind perturbations at the surface and we opt to focus on the latter situation that is we will seek a solution up to a small stopping time. The preceding paper [LNT16] addressed the single layer model proposed by Orenga et al in [DMCO99] and [Ore95]. Orenga's model omits the Coriolis term and assumes the external force is zero in the momentum equation. In the deterministic context see e.g. [DMCO99] and [Ore95], who omit the Coriolis term and assume the external force to be zero in the momentum equation. In [CMZ08] and [WX05] the model is similar to that of Orenga et al., but it has an additional term with $\frac{1}{h}$ in the momentum equation. The model most closely related to the present article can be found in [Sun98]. It does include a Coriolis term, but it still assumes no external forcing and it contains the $\frac{1}{h}$ term. For convenience, we choose a model which omits the $\frac{1}{h}$ term and adds the term $-\delta\Delta h$ to the continuity equation in order to absorb some of the terms involving the gradient of the height of the water. Because of this extra viscous term, we require boundary conditions on h, that are specified below. We also choose to include an external force that is independent of the solution. A realistic formulation of the external force can be found in e.g. [TT11], but this adds more unnecessary difficulties to the problem. For more about the physical derivation of these equations, see e.g. [TM05].

For the two layer model, we began with the models proposed in [Orenga05] and [Mac99] but ultimately decided on one more similar to [Simon1998] (see Appendix C). As in the single layer problem, the momentum equations lack any external force. We choose to include the external forces F and G which are both independent of the solution. We also include the extra terms $\delta_1\Delta h_1$ and $\delta_2\Delta h_2$ to the continuity equations for simpler calculations and to absorb terms with the gradients of the heights of each layer of water.

One of the major difficulties inherent to this problem is that we do not have the cancellation property for the nonlinear term, as is the case in e.g. [Ben95], [GHT11], [GHZ08], and [GHZ09]. We also do not have the assumption that $\mathbf{v}_1$ or $\mathbf{v}_2$ is divergence free, as in the Navier-Stokes system (see e.g. [BF00]). In the deterministic case, this implies that, in general, one can only obtain local in time a priori estimates for the solution, and hence local in time existence of solutions. As we will see below, the same holds in the stochastic context. We know of very few results of local in time existence of solutions of stochastic partial differential equations. Local in time solutions of the Navier-Stokes equations have been obtained in [Ben95].In a second step the mapping defining the solutions is "randomized" to account for a white noise forcing. In partly related directions, we would like to mention the lecture notes [Fla11] in which the author studies the role that white noises may have in preventing blow up. See also [Cho09], [Cho11] in which the author derives results of blow-up in finite time for solutions of stochastic pdes. See also [CDS01] in which the authors study the two layer quasi-geostrophic equation; these equations have some similarity with the shallow water equations but, unlike the shallow water equations that we consider, well-posedness is granted for all time in the deterministic context and then in the stochastic context.

Unlike in [BF00], the point of view that we adopt in this article for the shallow water equations is to leave the equations unchanged and to obtain the stochastic solution by deriving the suitable a priori estimates and exploiting them in the context of the Galerkin method using the usual compactness arguments.

In the theory of stochastic evolution equations two notions of solutions are typically considered namely pathwise (or strong) solutions and martingale (or weak) solutions. In the former notion the driving noise is fixed in advance while in the later case these underlying stochastic elements enter as an unknown in the problem. For more details about the two type solutions, we refer the reader to e.g. [DPZ92], [Fla08], [FG95] and [Øks03]. In the study of deterministic nonlinear evolutionary partial differential equations, when the L^p bounds on the time approximation solutions are obtained, the Galerkin approximation scheme will provide us estimates on the time derivatives, and the classical compactness results, such as Aubin-Lions or Arzèla-Ascoli can be applied to help us pass to the limit in the nonlinear terms. However, the classical compactness results cannot extend to the stochastic setting due to the lack of differentiation in time of the solutions. We will utilize a different compactness result based on fractional Sobolev spaces that allows us to treat nonlinear stochastic equations in a way similar to the deterministic case; see [FG95], [Tem95]. Proofs of other compactness embedding theorems can be found in [Bil95], [CF88], [Ros96], and [Tem01].

In this work, we will use the same approach as in [FG95] and [DGHT11] to establish the existence of both martingale and pathwise solutions but our results will provide finite time existence only. We first derive a formal a priori estimate for the original stochastic system assuming that the solutions are sufficiently regular and then we obtain uniform bounds for solutions up to a stopping time. However, in contrast to the deterministic setting, the positiveness of random stopping times are not granted. The absence of lower bound on the stopping times leads to further difficulties later on when deriving the compactness result and passing to the limit. To circumvent these difficulties, we introduce the modified system which truncates the nonlinear terms. We then derive the existence of global martingale solutions for this system by using the Prokrohov theorem, which is used to obtain the compactness results for the sequence of probability measures associated with the approximate solutions. We then upgrade convergence in distributions to almost sure convergences relative to the new underlying stochastic basis by employing the Skorohod embedding theorem. To deduce the existence of global pathwise solutions for the system relative to the initial stochastic basis, we will employ the Gyöngy-Krylov theorem which is the infinite dimensional vesion of the classical Yamada-Watanabe theorem (see e.g.[PR07]). Consequently, we derive the existence of both local martingale and pathwise solutions for the original stochastic system by introducing an appropriate positive stopping time afterward.

This article is organized as follows: In Section 2, we introduce the function spaces (Section 2.1) as well as the deterministic and stochastic frameworks (Section 2.2). In Section 3, we provide formal a priori estimates on the original system. In Section 4, we introduce a Galerkin scheme for the modified system, and by making use of an appropriate cut-off function, we are able to establish uniform a priori estimates for the corresponding sequence of approximate solution.

These estimates are used to develop the compactness argument, then with an application of the Skorokhod embedding theorem, which leads to strong convergence of some subsequence, we obtain the global existence of martingale solutions to the modified system. In this section, we also prove the pathwise uniqueness of global martingale solutions and by an application of the Gyöngy-Krylov Theorem we deduce the global existence of pathwise solutions. In Section 5, we establish the existence of both local martingale solutions pathwise solutions and maximal pathwise solution by defining an appropriate stopping time. Finally in the Appendices, we present some measurability results and the adapted stochastic Gronwall lemma, among the other existing results used in the article.

For the 2-layer model, we consider two fluids with different densities so that no mixing occurs. As in [Orenga05], we have a bounded open domain $\mathcal{M} \subset \mathbb{R}^2$ with smooth boundary $\partial\mathcal{M}$ and $T \in (0, \infty)$. In the equations below, $U := (\mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$.

$$d\mathbf{v}_1 - \nu_1 \Delta \mathbf{v}_1 dt + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 dt + g\nabla h_1 dt + g\frac{\rho_2}{\rho_1}\nabla h_2 dt + f\mathbf{k}\times\mathbf{v}_1 dt = F dt + \sigma_1(U) dW_1 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1a}$$

$$d\mathbf{v}_2 - \nu_2 \Delta \mathbf{v}_2 dt + (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 dt + g\nabla h_2 dt + g\nabla h_1 + f\mathbf{k}\times\mathbf{v}_2 dt = G dt + \sigma_2(U) dW_2 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1b}$$

$$dh_1 + \nabla\cdot(h_1\mathbf{v}_1)dt - \delta_1\Delta h_1 dt = \sigma_3(U)dW_3 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1c}$$

$$dh_2 + \nabla\cdot(h_2\mathbf{v}_2)dt - \delta_2\Delta h_2 dt = \sigma_4(U)dW_4 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1d}$$

with initial conditions:

$$\mathbf{v}_1(t=0) = \mathbf{v}_1^0(x,y) \qquad \text{in } \mathcal{M}, \tag{1.1e}$$

$$\mathbf{v}_2(t=0) = \mathbf{v}_2^0(x,y) \qquad \text{in } \mathcal{M}, \tag{1.1f}$$

$$h_1(t=0) = h_1^0(x,y) > 0 \qquad \text{in } \mathcal{M}, \tag{1.1g}$$

$$h_2(t=0) = h_2^0(x,y) > 0 \qquad \text{in } \mathcal{M}, \tag{1.1h}$$

And we assume Dirichlet boundary conditions

$$\mathbf{v}_1 = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T), \tag{1.1i}$$

$$\mathbf{v}_2 = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T) \tag{1.1j}$$

$$h_1 = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T) \tag{1.1k}$$

$$h_2 = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T) \tag{1.1l}$$

Here, $h_1 := h_1(x,y,t)$ is the depth of the top layer of water and $h_2 := h_2(x,y,t)$ is the depth of the bottom layer of water. We will assume for $i = 1, 2$ that $h_i = H_i + \mathring{h}_i$, where H_i is the average depth of the water, a constant, and $\mathring{h}_i$ is how much the height of the water deviates from its average depth. Additionally, $\mathbf{v}_1 = (u_1, v_1)$ where $u_1 := u_1(x, y, t)$ is the velocity of the top layer of water in the x direction and $v_1 := v_1(x, y, t)$ is the velocity of the top layer of water in the y direction, and $\mathbf{v}_2 = (u_2, v_2)$ where $u_2 := u_2(x, y, t)$ is the velocity of the bottom layer of water in the x direction and $v_2 := v_2(x, y, t)$ is the velocity of the bottom layer of water in the y direction. Also, ν_1 and ν_2 are the viscosities, δ_1, and δ_2 are given positive constants, g is the gravitational constant, f is the Coriolis parameter assumed to be constant, ρ_1 is the density of the top fluid, and ρ_2 is the density of the bottom fluid. Finally, $F := F(x, y, t)$, $G := G(x, y, t)$, $\mathbf{v}_1^0(x, y)$, $\mathbf{v}_2^0(x, y)$, $h_1^0(x, y)$ and $h_2^0(x, y)$ are given. As in the single layer model, we will impose certain Lipschitz conditions on σ_i, $i = 1, 2, 3, 4$, which are all given, and $W_i, i = 1, 2, 3, 4$, are cylindrical Brownian motions or Wiener processes specified in section 2.2 below.

Remark 1.1. *Different boundary conditions on $\mathbf{v}_i$ and h_i for $i = 1, 2$ appear also in the literature, such as:*

$$\mathbf{v}_i \cdot n = 0 \qquad \text{and} \qquad \operatorname{curl}(\mathbf{v}_i) = 0 \qquad \text{on } \partial\mathcal{M} \times (0, T), \tag{1.2}$$

$$\nabla \mathring{h}_i \cdot n = 0 \qquad \text{on } \partial\mathcal{M} \times (0, T). \tag{1.3}$$

This set of boundary conditions yields the same type of results but it requires more technical work.

Remark 1.2. *Physically the depths of each layer of water are necessarily positive. For a proof of the positivity of h_i, see Appendix B.*

2 Analytic Tools

In this section, we collect and define the deterministic and stochastic tools needed throughout this article.

2.1 Function Spaces

We will use in the spaces $H = H_1 \times H_1 \times H_2 \times H_2, V = V_1 \times V_1 \times V_2 \times V_2$ where

$$H_1 := L^2(\mathcal{M})^2,\ V_1 := (H_0^1(\mathcal{M}))^2,\ H_2 := L^2(\mathcal{M}),\ V_2 := H_0^1(\mathcal{M}). \tag{2.1}$$

On H_2 and H_1, we will use the typical inner product and norm denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively, while on V_1 and V_2, we will use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, which are the usual inner product and norm of the gradients.

We also consider fractional powers of the $(-\Delta)$ operator with the boundary conditions (1.1i),(1.1j),(1.1k) and (1.1l). By the classical theory, there is an orthonormal basis $\{\psi_k\}_{k\geq 1}$ of H with an unbounded increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ such that $-\Delta\psi_k = \lambda_k\psi_k$. We define $D(-\Delta) = V \cap (H^2(\mathcal{M}))^3$ and for $\alpha \geq 0$ we take:

$$D((-\Delta)^\alpha) = \Big\{ u \in H_1 : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < \infty \Big\}, \tag{2.2}$$

endowed with the Hilbertian norm

$$|u|_\alpha := |(-\Delta)^\alpha u| = \left(\sum_{k=1}^{\infty} \lambda_k^{2\alpha}|u_k|^2\right)^{1/2}. \tag{2.3}$$

Here, $u = \sum_{k=1}^{\infty} u_k\psi_k$ with $|u|^2 = \sum_{k=1}^{\infty} |u_k|^2 < \infty$.

For the Galerkin scheme below, we introduce the finite dimensional spaces $H_n = \text{span}\{\psi_1, \dots, \psi_n\}$ and let $P_n, Q_n = I - P_n$ be the projection operators onto H_n and onto its orthogonal complement. By abuse of notation we will also use the operator P_n to denote $P_n\mathbf{v} = P_n(\mathbf{v}, 0)$ and $P_n h = P_n(\mathbf{0}, h)$. We have the generalized and reverse Poincaré inequalities which hold for any $\alpha_1 < \alpha_2$:

$$|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2-\alpha_1}|P_n u|_{\alpha_1} \quad \text{and} \quad |Q_n u|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2-\alpha_1}}|Q_n u|_{\alpha_2}. \tag{2.4}$$

2.2 Stochastic Framework

In order to define the stochastic terms in (1.1a), (1.1b),(1.1c) and (1.1d) that is $\sigma_1(U)dW_1$, $\sigma_2(U)dW_2$, $\sigma_3(U)dW_3$ and $\sigma_4(U)dW_4$, we will recall some basic notations and notions from stochastic analysis. For an extended treatment of this topic, we refer to [DPZ92].

To begin with, we define a stochastic basis $\mathcal{S} := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W_i^k\}_{k\geq 1}\right)$ that is a filtered probability space and $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete right continuous filtration, and for $i = 1, 2, 3, 4$, $\{W_i^k\}_{k\geq 1}$ is a sequence of independent one-dimensional Brownian motions adapted to $\mathcal{F}_t$.

Let $\mathfrak{U}$ be an auxiliary separable real Hilbert space endowed with a Hilbert basis $\{e_j\}_{j\geq 1}$. We then consider $W_i(t, \cdot, \omega)$ the $\mathfrak{U}-$valued stochastic processes, formally represented for the moment, as the following series:

$$W_i(t, \cdot, \omega) = \sum_{\ell=1}^{\infty} W_i^\ell(t, \omega)e_\ell(\cdot). \tag{2.5}$$

This expression makes each W_i a cylindrical Brownian motion evolving over a separable space $\mathfrak{U}$ with orthogonal basis e_k.

We next recall some basic definitions and properties of spaces of Hilbert-Schmidt operators. To this end, we suppose that X and Y are two separable Hilbert spaces with the associated norms and inner products given by $|\cdot|_X, |\cdot|_Y$ and $\langle\cdot,\cdot\rangle_X, \langle\cdot,\cdot\rangle_Y$, respectively.

We denote by $L_2(\mathfrak{U}, H) := \{R \in \mathcal{L}(\mathfrak{U}, H) : \sum_{k=1}^{\infty} |Re_k|_X^2 < \infty\}$ the collection of Hilbert-Schmidt operators mapping from $\mathfrak{U}$ into X. This space $L_2(\mathfrak{U}, H)$ is a Hilbert space endowed with the following inner product and norm

$$\langle R, S\rangle_{L_2(\mathfrak{U},H)} = \sum_{k=1}^{\infty}\langle Re_k, Se_k\rangle_H \text{ and } \|R\|_{L^2(\mathfrak{U},H)}^2 = \sum_{k=1}^{\infty} |Re_k|_H^2.$$

One may show that if $R^1 \in L_2(\mathfrak{U}, H)$ and $R^2 \in \mathcal{L}(\mathfrak{U}, H)$ then R^2R^1 and $R^1R^2 \in L_2(\mathfrak{U}, H)$.

We also define another auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ as

$$\mathfrak{U}_0 := \left\{ u = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$|v|^2_{\mathfrak{U}_0} := \sum_{k=1}^{\infty} \frac{a_k^2}{k^2}, \text{ for } v = \sum_{k=1}^{\infty} a_k e_k.$$

Note that the embedding of $\mathfrak{U} \subset \mathfrak{U}_0$ is Hilbert-Schmidt.

Assumptions on $\sigma_i, i = 1, 2, 3, 4$ For simplicity, we set $U = (\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, t, \omega)$ and $\sigma_i(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, t, \omega) = \sigma_i(U)$. We shall assume throughout this work that

$$\sigma_i := \sigma_i(U) : H_1 \times H_1 \times H_2 \times H_2 \times [0, T] \times \Omega \to L_2(\mathfrak{U}, H_i), \tag{2.6}$$

$$\sigma_i := \sigma_i(U) : V_1 \times V_1 \times V_2 \times V_2 \times [0, T] \times \Omega \to L_2(\mathfrak{U}, V_i), \tag{2.7}$$

are measurable, essentially bounded in time and L^2 in Ω, $\{\mathcal{F}_t\}_{t \geq 0}$ adapted and satisfying the following conditions for a.e. $t \in [0, T]$ and a.s:

$$\|\sigma_i(\mathbf{v}, h, t, \omega)\|^2_{L_2(\mathfrak{U}, V_i)} \leq K_V (1 + \|\mathbf{v}\|^2 + \|h\|^2) \tag{2.8a}$$

$$\|\sigma_i(\mathbf{v}, h, t, \omega)\|^2_{L_2(\mathfrak{U}, H_i)} \leq K_H (1 + |\mathbf{v}|^2 + |h|^2) \tag{2.8b}$$

$$\|\sigma_i(\mathbf{v}_1, h_1, t, \omega) - \sigma_i(\mathbf{v}_2, h_2, t, \omega)\|^2_{L_2(\mathfrak{U}, V_i)} \leq K_V (1 + \|\mathbf{v}_1 - \mathbf{v}_2\|^2 + \|h_1 - h_2\|^2) \tag{2.8c}$$

$$\|\sigma_i(\mathbf{v}_1, h_1, t, \omega) - \sigma_i(\mathbf{v}_2, h_2, t, \omega)\|^2_{L_2(\mathfrak{U}, H_i)} \leq K_H (1 + |\mathbf{v}_1 - \mathbf{v}_2|^2 + |h_1 - h_2|^2). \tag{2.8d}$$

Finally, given an $X-$valued predictable process $G \in L^2(\Omega; L^2_{loc}([0, \infty); L_2(\mathfrak{U}, X)))$ one may define the (Itô) stochastic integral

$$M_t := \int_0^t G dW, \tag{2.9}$$

which belongs to $\mathcal{M}^2_X$, the space of all $X-$valued square integrable martingales (see e.g. [PR07]). For a.e. t and a.s., $G \in L_2(\mathfrak{U}, H)$ so that $G_k = G \cdot e_k \in H$, where $\{e_k\}$ is the basis of $\mathfrak{U}$. Then (2.9) can be represented as

$$M_t = \sum_k \int_0^t G_k dW^k.$$

The martingale $\{M_t\}_{t \geq 0}$ has many desirable properties. Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form,

$$\mathbb{E} \left(\sup_{t \in [0,T]} \left| \int_0^t G dW \right|_X^r \right) \leq C_1 \mathbb{E} \left(\int_0^T \|G\|^2_{L_2(\mathfrak{U}, X)} dt \right)^{\frac{r}{2}}, \tag{2.10}$$

valid for $r \geq 1$. With $G_k = G \cdot e_k$, (2.10) can be rewritten as

$$\mathbb{E} \left(\sup_{t \in [0,T]} \left| \int_0^t \sum_{k=1}^{\infty} G \cdot e_k dW^k \right|_X^r \right) \leq C_1 \mathbb{E} \left(\int_0^T \sum_{k=1}^{\infty} \|G \cdot e_k\|_X^2 dt \right)^{\frac{r}{2}}. \tag{2.11}$$

Here C_1 is an absolute constant depending on r. We shall also make use of a variation of inequality (2.10), which applies to fractional derivatives of M_t. For $p \geq 2$ and $\alpha \in [0, 1/2)$ we have

$$\mathbb{E} \left(\sup_{t \in [0,T]} \left| \int_0^t G dW \right|^p_{W^{\alpha,p}([0,T];X)} \right) \leq C \, \mathbb{E} \left(\int_0^T \|G\|^p_{L_2(\mathfrak{U}, X)} dt \right), \tag{2.12}$$

which holds for all $X-$valued predictable $G \in L^2(\Omega; L^p_{loc}([0,\infty); L_2(\mathfrak{U}, X)))$.

For the convenience of the reader, we shall recall the definition of the spaces $W^{\alpha,p}([0,T];X)$ in Section 6 below.

We can express (2.12) in a similar form as in (2.11) as

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\sum_k \int_0^t Ge_k dW^k\right|^p_{W^{\alpha,p}([0,T];X)}\right) \le C\,\mathbb{E}\left(\int_0^T \sum_k |Ge_k|^p_X\, dt\right). \tag{2.13}$$

We will also make use of the decomposition $\mathbf{u} = \sum_{j=1}^{\infty} \xi_j \phi_j$ where $\xi_j = \xi_j(t,\omega)$ and the ϕ_j are the eigenvalues of $A = -\Delta$ in $D(A) \subset H$ so that $A\mathbf{u}$ becomes $\sum_{j=1}^{\infty} \xi_j \lambda_j \phi_j$; and if $b \in H$, $b = \sum_{j=1}^{\infty} b_j \phi_j$ with $b_j = (b, \phi_j)$.

Throughout the article, we denote by C the generic constant and we frequently use the notation $\preceq$ to mean up to the multiplicative constant.

Remark 2.1 (Notation)**.** *For $i = 1,2,3,4$, $\sigma_i(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, t, \omega) = \sigma_i(U)$ as is (2.6) and $W_i = \sum_{k=1}^{\infty} e_k W_i^k$, we have:*

$$\begin{aligned}\sigma_i(U)dW_i &= \sum_{k=1}^{\infty} \sigma_i^k(U)\cdot e_k dW_i^k = \sum_{k,\ell=1}^{\infty} \langle \sigma_i(U)e_k, \phi_\ell\rangle \phi_\ell dW_i^k \\ &= \sum_{k,\ell=1}^{\infty} \sigma_i^{k\ell}\phi_\ell dW_i^k,\end{aligned} \tag{2.14}$$

where

$$\sigma_i(U)\cdot e_k = \sum_\ell \sigma_i^{k\ell}\phi_\ell,\ \sigma_i^{k\ell} = \langle \sigma_i(U)\cdot e_k, \phi_\ell\rangle,$$

which makes sense since $\sigma_i(U)\cdot e_k \in H$ and $\{\phi_\ell\}$ is a Hilbert basis of H.

We shall assume furthermore that if $\tilde{\mathbf{v}} : [0,T]\times\Omega \to H_1$, $\tilde{h} : [0,T]\times\Omega \to H_2$ are predictable, then so is $\sigma_i(U)$. Given an $H-$valued predictable process $\tilde{\mathbf{v}} \in L^2(\Omega; L^2(0,T;H_1))$, $\tilde{h} \in L^2(\Omega; L^2(0,T;H_2))$ the series expansion (2.14) can be shown to be well defined as a stochastic integral and

$$\begin{aligned}\left\langle \int_0^r \sigma_1(U)dW_1, \tilde{\mathbf{v}}\right\rangle &= \left\langle \sum_k \int_0^r \sigma_1^k(U)dW_1^k, \tilde{\mathbf{v}}\right\rangle = \sum_k \int_0^r \langle \sigma_1^k(U), \tilde{\mathbf{v}}\rangle dW_1^k, \\ \left\langle \int_0^r \sigma_2(U)dW_2, \tilde{\mathbf{v}}\right\rangle &= \left\langle \sum_k \int_0^r \sigma_2^k(U)dW_2^k, \tilde{\mathbf{v}}\right\rangle = \sum_k \int_0^r \langle \sigma_1^k(U), \tilde{\mathbf{v}}\rangle dW_2^k \\ \left\langle \int_0^r \sigma_3(U)dW_3, \tilde{h}\right\rangle &= \left\langle \sum_k \int_0^r \sigma_2^k(U)dW_3^k, \tilde{h}\right\rangle = \sum_k \int_0^r \langle \sigma_2^k(U), \tilde{h}\rangle dW_2^k, \\ \left\langle \int_0^r \sigma_4(U)dW_4, \tilde{\mathbf{v}}\right\rangle &= \left\langle \sum_k \int_0^r \sigma_4^k(U)dW_4^k, \tilde{\mathbf{v}}\right\rangle = \sum_k \int_0^r \langle \sigma_1^k(U), \tilde{\mathbf{v}}\rangle dW_4^k\end{aligned} \tag{2.15}$$

for all $\tilde{\mathbf{v}} \in H_1$, $\tilde{h} \in H_2$ and stopping time τ. In this context, the four equations from (1.1a) to(1.1d) fully make sense as Itô integrals with values in the spaces V_1', V_2' after (Itô) integration from 0 to t, for a.e. $t \in [0,T]$ We will frequently apply the following applications of the infinite dimensional version of the Itô Formula for $p = 2$ or $p \geq 4$:

$$\begin{aligned} &d||\mathbf{v}_1||^p + p\nu_1|\Delta\mathbf{v}_1|^2||\mathbf{v}_1||^{p-2}dt = p\langle F, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2}dt - p\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2}dt - pg\langle\nabla h_1, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2}dt \\ &- pg\frac{\rho_2}{\rho_1}\langle\nabla h_2, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2} - p\langle f\mathbf{k}\times\mathbf{v}_1, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2}dt + \frac{p}{2}||\sigma_1(U)||^2_{L_2(\mathfrak{U},V)}||\mathbf{v}_1||^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k, \Delta\mathbf{v}_1\rangle^2||\mathbf{v}_1||^{p-4}dt + \sum_{k=1}^{\infty}p\langle\sigma_1(U)e_k, \Delta\mathbf{v}_1\rangle||\mathbf{v}_1||^{p-2}dW_1^k, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &d||\mathbf{v}_2||^p + p\nu_2|\Delta\mathbf{v}_2|^2||\mathbf{v}_2||^{p-2}dt = p\langle G, \Delta\mathbf{v}_2\rangle||\mathbf{v}_2||^{p-2}dt - p\langle(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2, \Delta\mathbf{v}_2\rangle||\mathbf{v}_2||^{p-2}dt - pg\langle\nabla h_2, \Delta\mathbf{v}_2\rangle||\mathbf{v}_2||^{p-2}dt \\ &- pg\langle\nabla h_1, \Delta\mathbf{v}_2\rangle||\mathbf{v}_2||^{p-2} - p\langle f\mathbf{k}\times\mathbf{v}_2, \Delta\mathbf{v}_2\rangle||\mathbf{v}_2||^{p-2}dt + \frac{p}{2}||\sigma_2(U)||^2_{L_2(\mathfrak{U},V)}||\mathbf{v}_2||^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k, \mathbf{v}_2\rangle^2||\mathbf{v}_2||^{p-4}dt + p\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k, \Delta\mathbf{v}_2\rangle|\mathbf{v}_2|^{p-2}dW_2^k, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &d||h_1||^p + p\delta_1|\Delta h_1|^2||h_1||^{p-2}dt = -p\langle\nabla\cdot(h_1\mathbf{v}_1), \Delta h_1\rangle||h_1||^{p-2}dt + \frac{p}{2}||\sigma_3(U)||^2_{L_2(\mathfrak{U},V)}||h_1||^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k, \Delta h_1\rangle^2||h_1||^{p-4}dt + p\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k, \Delta h_1\rangle||h_1||^{p-2}dW_3^k, \end{aligned} \tag{2.18}$$

and
$$\begin{aligned} &d||h_2||^p + p\delta_2|\Delta h_2|^2||h_2||^{p-2}dt = -p\langle\nabla\cdot(h_2\mathbf{v}_2), \Delta h_2\rangle|h_2|^{p-2}dt + \frac{p}{2}||\sigma_4(U)||^2_{L_2(\mathfrak{U},H)}||h_2||^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k, \Delta h_2\rangle^2||h_2||^{p-4}dt + p\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k, \Delta h_2\rangle||h_2||^{p-2}dW_4^k. \end{aligned} \tag{2.19}$$

Remark 2.2. *In (2.16), (2.17), (2.18) and (2.19), $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V_1 and V_2' relative to H_1, and that between V_2 and $(V_2)'$ relative to H_2.*

2.3 Definitions of Solutions

We now need to define what exactly we mean by a solution to problem (1.1) as in [GHZ09]. First, we recall what it means for a stochastic process to be predictable:

Definition 2.1. *For a given stochastic basis $\mathcal{S}$, let $\Phi = \Omega\times[0,\infty)$ and take $\mathcal{G}$ to be the σ-algebra generated by sets of the form*

$$(s,t]\times\mathfrak{F}, \quad 0\leq s<t<\infty, \quad \mathfrak{F}\in\mathcal{F}_s; \qquad \{0\}\times\mathfrak{F}, \quad \mathfrak{F}\in\mathcal{F}_0. \tag{2.20}$$

An X-valued process U is called predictable *w.r.t. $\mathcal{S}$ if it is measurable from $(\Phi,\mathcal{G})$ into $(X,\mathcal{B}(X))$ where $\mathcal{B}(X)$ is the family of Borel sets of X.*

We next give the definitions of local and global solutions of (1.1) for both martingale and pathwise solutions. Before that, we make some assumptions for the initial condition $(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0))$, which may be random in general. For the case of martingale solutions, since the stochastic basis is unknown, we are only able to specify $(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0))$ as an initial probability measure μ_0 on $V_1 \times V_1 \times V_2 \times V_2$. For the case of pathwise solutions where the stochastic basis $\mathcal{S}$ is fixed in advance, we assume that relative to this basis $(\mathbf{v}_0, h_0)$ is a $V_1 \times V_1 \times V_2 \times V_2$ valued random variable such that $(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0)) \in L^2(\Omega, V_1 \times V_1 \times V_2 \times V_2)$ and is $\mathcal{F}_0$-measurable:

$$\mathbf{v}_i(0) \in L^p(\Omega, H_1), h_i(0) \in L^p(\Omega, H_2) \text{ and are } \mathcal{F}_0\text{-measurable for } i = 1, 2. \tag{2.21}$$

Definition 2.2 (Local and global martingale solutions). *Suppose that μ_0 is a probability measure on $V_1 \times V_2$ and suppose $F \in L^2(0, T; H_1)$ and $G \in L^2(0, T; H_2)$* [1]*, and for $i = 1, 2, 3, 4$, $\sigma_i(U)$ satisfies the Lipschitz conditions in (2.8), is predictable, and $\mathcal{F}_t$-adapted. Then we say a $\left(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tilde{\tau}\right)$ is a* local Martingale solution *of problem (1.1) if $\tilde{\mathcal{S}} := \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t\right\}_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4\right)$ is a stochastic basis, $\tilde{\tau}$ is a strictly positive stopping time (i.e. $\tilde{\tau} > 0$ almost surely) relative to $\tilde{\mathcal{F}}_t$, and for $i = 1, 2$ $\tilde{\mathbf{v}}_i(\cdot \wedge \tilde{\tau}), \tilde{h}_i(\cdot \wedge \tilde{\tau})$ are $\tilde{\mathcal{F}}_t$-adapted processes in H_1, H_2, respectively, so that*

$$\tilde{\mathbf{v}}_i(\cdot \wedge \tilde{\tau}) \in L^2(\tilde{\Omega}; L^\infty([0, T]; V_1)), \tag{2.22a}$$
$$\tilde{h}_i(\cdot \wedge \tilde{\tau}) \in L^2(\tilde{\Omega}; L^\infty([0, T]; V_2)), \tag{2.22b}$$
$$\tilde{\mathbf{v}}_i(t)\mathbb{1}_{t\leq\tilde{\tau}} \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)), \tag{2.22c}$$
$$\tilde{h}_i(t)\mathbb{1}_{t\leq\tilde{\tau}} \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{2.22d}$$

Furthermore, the law of $\left(\tilde{\mathbf{v}}_1(0), \tilde{\mathbf{v}}_2(0), \tilde{h}_1(0), \tilde{h}_2(0)\right)$ is μ_0, i.e. $\mu_0(E) = \mathbb{P}\left(\left(\tilde{\mathbf{v}}_1(0), \tilde{\mathbf{v}}_2(0), \tilde{h}_1(0), \tilde{h}_2(0)\right) \in E\right)$ for all Borel subsets $E \subset H_1 \times H_1 \times H_2 \times H_2$, and $\left(\tilde{\mathbf{v}}, \tilde{h}\right)$ must satisfy almost surely for every $t \geq 0$, every $v \in H_1$, every $\eta \in H_2$ and for $i = 1, 2$

$$\begin{aligned}(\tilde{\mathbf{v}}_1(t \wedge \tilde{\tau}), v) + \int_0^{t\wedge\tilde{\tau}} \left(-\nu_1\Delta\tilde{\mathbf{v}}_1 + (\tilde{\mathbf{v}}_1 \cdot \nabla)\tilde{\mathbf{v}}_1 + g\nabla\tilde{h}_1 + g\frac{\rho_2}{\rho_1}\nabla\tilde{h}_2 + f\mathbf{k} \times \tilde{\mathbf{v}}_1 - F, v\right) ds \\ = (\tilde{\mathbf{v}}_1(0), v) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_1^k(U), v)d\tilde{W}_1^k,\end{aligned} \tag{2.23}$$

$$\begin{aligned}(\tilde{\mathbf{v}}_2(t \wedge \tilde{\tau}), v) + \int_0^{t\wedge\tilde{\tau}} \left(-\nu_2\Delta\tilde{\mathbf{v}}_2 + (\tilde{\mathbf{v}}_2 \cdot \nabla)\tilde{\mathbf{v}}_2 + g\nabla\tilde{h}_1 + g\nabla\tilde{h}_2 + f\mathbf{k} \times \tilde{\mathbf{v}}_2 - G, v\right) ds \\ = (\tilde{\mathbf{v}}_2(0), v) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_1^k(U), v)d\tilde{W}_1^k,\end{aligned} \tag{2.24}$$

$$\left(\tilde{h}_1(t \wedge \tilde{\tau}), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \left(\nabla \cdot \left(\tilde{h}_1\tilde{\mathbf{v}}_1\right) - \delta\Delta\tilde{h}_1, \eta\right) ds = \left(\tilde{h}_1(0), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_3^k(U), \eta)d\tilde{W}_3^k. \tag{2.25}$$

$$\left(\tilde{h}_2(t \wedge \tilde{\tau}), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \left(\nabla \cdot \left(\tilde{h}_2\tilde{\mathbf{v}}_2\right) - \delta\Delta\tilde{h}_2, \eta\right) ds = \left(\tilde{h}_2(0), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_4^k(U), \eta)d\tilde{W}_4^k. \tag{2.26}$$

We say that the martingale solution $\left(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}, \tilde{h}, \tilde{\tau}\right)$ is global if $\tilde{\tau} = \infty$ a.s.

[1]One can also assume F and G to be random,but we choose F and G to be deterministic here, or else it will be unnecessarily tricky for the proof of the existence of the martingale solutions later on.

Definition 2.3 (Local, maximal and global pathwise solutions)**.** *Suppose that* $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W)$ *is a fixed stochastic basis and that* $(\mathbf{v}_0, h_0)$ *is a* $V_1 \times V_2$ *valued random variable (relative to* $\mathcal{S}$*) satisfying* (2.22) *and the same conditions hold for* F *and* σ_i*,* $i = 1, 2$*.*

(i) *A quintuplets* $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \tau)$ *is a local pathwise solution to* (1.1) *if* τ *is a strictly positive stopping time,* $\mathbf{v}_1(\cdot \wedge \tau)$ *,*$\mathbf{v}_2(\cdot \wedge \tau)$ $\mathcal{F}_t$*-adapted processes in* V_1*, and* $h_1(\cdot \wedge \tau)$*,*$h_2(\cdot \wedge \tau)$ $\mathcal{F}_t$*-adapted processes in* V_2 *(relative to the fixed basis* $\mathcal{S}$*) such that* (2.22)*–*(2.24) *hold.*

(ii) *Pathwise solutions of* (1.1) *are said to be* unique up to a stopping time $\tau > 0$ *if given any pair of pathwise solutions* $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \tau)$ *and* $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tau)$ *which coincide at* $t = 0$ *on a subset* Ω_0 *of* Ω*:*

$$\Omega_0 := \left\{\mathbf{v}_1(0) = \tilde{\mathbf{v}}_1(0), \mathbf{v}_1(0) = \tilde{\mathbf{v}}_1(0), h_1(0) = \tilde{h}_1(0), h_2(0) = \tilde{h}_2(0)\right\} \subset \Omega, \tag{2.27}$$

then

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}(\mathbf{v}_1(t\wedge\tau) - \tilde{\mathbf{v}}_1(t\wedge\tau)) = 0, \forall t \geq 0\right) = 1, \tag{2.28}$$

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}(\mathbf{v}_2(t\wedge\tau) - \tilde{\mathbf{v}}_2(t\wedge\tau)) = 0, \forall t \geq 0\right) = 1, \tag{2.29}$$

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}\left(h_1(t\wedge\tau) - \tilde{h}_1(t\wedge\tau)\right) = 0, \forall t \geq 0\right) = 1. \tag{2.30}$$

$$\text{and} \qquad \mathbb{P}\left(\mathbb{1}_{\Omega_0}\left(h_2(t\wedge\tau) - \tilde{h}_2(t\wedge\tau)\right) = 0, \forall t \geq 0\right) = 1. \tag{2.31}$$

(iii) *Suppose we have* $\{\tau_n\}_{n\geq 1}$*, a strictly increasing sequence of stopping times that converge to a stopping time* ξ*, and assume that* $\mathbf{v}_1$*,*$\mathbf{v}_2$*,* h_1 *and* h_2 *are predictable continuous* $\mathcal{F}_t$*-adapted processes in* H_1 *and* H_2*, respectively. We say that* $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi) := (\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi, \{\tau_n\}_{n\geq 1})$ *is a* maximal pathwise solution *if* $(\mathbf{v}, \mathbf{v}_2, h_1, h_2, \tau_n)$ *is a local pathwise solution for each* n *and*

$$\begin{aligned}\sup_{t\in[0,\xi]} ||\mathbf{v}_1||^2 + \sup_{t\in[0,\xi]} ||\mathbf{v}_2||^2 + \int_0^\xi |\Delta\mathbf{v}_1|^2 ds + \int_0^\xi |\Delta\mathbf{v}_2|^2 ds + \\ \sup_{t\in[0,\xi]} ||h_1||^2 + \sup_{t\in[0,\xi]} ||h_2||^2 + \int_0^\xi |\Delta h|^2 ds + \int_0^\xi |\Delta h_2|^2 ds = \infty,\end{aligned} \tag{2.32}$$

a.s. on the set $\{\xi < \infty\}$*. If we have*

$$\begin{aligned}\sup_{t\in[0,\xi]} ||\mathbf{v}_1||^2 + \sup_{t\in[0,\xi]} ||\mathbf{v}_2||^2 + \int_0^\xi |\Delta\mathbf{v}_1|^2 ds + \int_0^\xi |\Delta\mathbf{v}_2|^2 ds + \\ \sup_{t\in[0,\xi]} ||h_1||^2 + \sup_{t\in[0,\xi]} ||h_2||^2 + \int_0^\xi |\Delta h|^2 ds + \int_0^\xi |\Delta h_2|^2 ds = n,\end{aligned} \tag{2.33}$$

for almost every $\omega \in \{\xi < \infty\}$*, then the sequence* τ_n announces *a finite blow-up time.*

(iv) *If* $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi)$ *is a maximal pathwise solution and* $\xi = \infty$ *almost surely, then we say that the solution is* global.

We now state the main results in this work:

Theorem 2.1. *We are given* μ_0 *as a probability measure on* $H_1 \times H_1 \times H_2 \times H_2$*,* $F \in L^2(0, T; H_1)$*,* $G \in L^2(0, T; H_2)$ *and* $\sigma_i(U)$*,* $i = 1, 2, 3, 4$ *satisfying the Lipschitz conditions* (2.8)*, predictable, and* $\mathcal{F}_t$*-adapted. Then there exists a local martingale solution* $(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tau)$ *to* (1.1)*.*

Theorem 2.2. *Assume we are working relative to a given fixed stochastic basis. Suppose furthermore that* (2.22) *also holds. Then there exists a unique, maximal pathwise solution* $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi, (\tau_n)_{n\geq 1})$ *to* (1.1).

The strategy to prove both of these theorems is that we will consider a modified system with a cut-off function damping the non-linear term, so that we have global existence of martingale solutions and pathwise solutions for this modified system using the Galerkin approximation. We then return to the original system (1.1) by introducing a stopping time which will be proven to be positive almost surely.

3 Formal a priori Estimates

First, we prove some a priori estimates on this system, assuming all of the solutions are smooth Adding Itô's formulas from (2.16) to (2.19) with $p = 2$ for $\mathbf{v}_1, \mathbf{v}_2, h_1, h_2$, and adding the corresponding relations yield:

$$
\begin{aligned}
&d\|\mathbf{v}_1\|^2 + 2\nu_1 \left|\Delta \mathbf{v}_2\right|^2 dt + d\|\mathbf{v}_2\|^2 + 2\nu_2 \left|\Delta \mathbf{v}_2\right|^2 dt + d\|h_1\|^2 + 2\delta_1 \left|\Delta h_1\right|^2 dt + d\|h_2\|^2 + 2\delta_2 \left|\Delta h_2\right|^2 dt \\
&= 2\langle F, \Delta \mathbf{v}_1\rangle dt + 2\langle G, \Delta \mathbf{v}_2\rangle dt - 2\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1, \Delta \mathbf{v}_1\rangle dt - 2g\langle \nabla h_1, \Delta \mathbf{v}_1\rangle dt - 2g\frac{\rho_2}{\rho_1}\langle \nabla h_2, \Delta \mathbf{v}_1\rangle dt \\
&- 2\langle f\mathbf{k} \times \mathbf{v}_1, \Delta \mathbf{v}_1\rangle dt + \sum_{k=1}^{\infty} ||\sigma_1(u)e_k||_{V_1}^2 dt + 2\sum_{k=1}^{\infty}\langle \sigma_1(u)e_k, \Delta \mathbf{v}_1\rangle dW_1^k - 2\langle (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2, \Delta \mathbf{v}_2\rangle dt \\
&-2g\langle \nabla h_2, \Delta \mathbf{v}_2\rangle dt - 2g\langle \nabla h_1, \Delta \mathbf{v}_2\rangle dt - 2\langle f\mathbf{k} \times \mathbf{v}_2, \Delta \mathbf{v}_2\rangle dt + \sum_{k=1}^{\infty} ||\sigma_2(u)e_k||_{V_2}^2 dt + 2\sum_{k=1}^{\infty}\langle \sigma_2(u)e_k, \Delta \mathbf{v}_2\rangle dW_2^k \\
&- 2\langle \nabla \cdot (h_1\mathbf{v}_1), \Delta h_1\rangle dt + \sum_{k=1}^{\infty} ||\sigma_3(u)e_k||_{V_2}^2 dt + 2\sum_{k=1}^{\infty}\langle \sigma_3(u)e_k, \Delta h_1\rangle dW_3^k - 2\langle \nabla \cdot (h_2\mathbf{v}_2), \Delta h_2\rangle dt \\
&+ \sum_{k=1}^{\infty} ||\sigma_4(u)e_k||_{V_2}^2 dt + \sum_{k=1}^{\infty}\langle \sigma_4(\mathbf{v}, h)e_k, \Delta h_2\rangle dW_4^k. \qquad (3.1)
\end{aligned}
$$

We integrate (3.1) in time over $[0, r]$, for $0 \leq r \leq s \leq T$, take the supremum in r over $[0, s]$; we deduce that

$$
\begin{aligned}
&\sup_{0\le r\le s}\|\mathbf{v}_1(r)\|^2dr+2\nu_1\int_0^s|\Delta\mathbf{v}_1(r)|^2\,dr+\sup_{0\le r\le s}\|\mathbf{v}_2(r)\|^2+2\nu_2\int_0^s|\Delta\mathbf{v}_2(t)|^2\,dt\\
&\quad+\sup_{0\le r\le s}\|h_1(r)\|^2+2\delta_1\int_0^s|\Delta h_1(t)|^2\,dt+\sup_{0\le r\le s}\|h_2(r)\|^2+2\delta_2\int_0^s|\Delta h_2(t)|^2\,dt\\
&\le 8\|\mathbf{v}_1(0)\|^2+8\|\mathbf{v}_2(0)\|^2+8\|h_1(0)\|^2+8\|h_2(0)\|^2+16\int_0^s|\langle F,\Delta\mathbf{v}_1\rangle|\,dt+16\int_0^s|\langle G,\Delta\mathbf{v}_2\rangle|\,dt\\
&\quad+16\int_0^s|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1,\Delta\mathbf{v}_1\rangle|\,dt+16g\int_0^s|\langle\nabla h_1,\Delta\mathbf{v}_1\rangle|\,dt+16g\frac{\rho_2}{\rho_1}\int_0^s|\langle\nabla h_2,\Delta\mathbf{v}_1\rangle|\,dt\\
&\quad+16\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_1,\Delta\mathbf{v}_1\rangle|\,dt+16\int_0^s|\langle(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2,\Delta\mathbf{v}_2\rangle|\,dt+16g\int_0^s|\langle\nabla h_2,\Delta\mathbf{v}_2\rangle|\,dt\\
&\quad+16g\int_0^s|\langle\nabla h_1,\Delta\mathbf{v}_2\rangle|\,dt+16\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_2,\Delta\mathbf{v}_2\rangle|\,dt+16\int_0^s|\langle\nabla\cdot(h_1\mathbf{v}_1),\Delta h_1\rangle|\,dt\\
&\quad+16\int_0^s|\langle\nabla\cdot(h_2\mathbf{v}_2),\Delta h_2\rangle|\,dt+16\sup_{r\in[0,s]}\left|\int_0^r\sum_{k=1}^\infty\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle dW_1^k\right|\\
&\quad+16\sup_{r\in[0,s]}\left|\int_0^r\sum_{k=1}^\infty\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle dW_2^k\right|+16\sup_{r\in[0,s]}\left|\int_0^r\sum_{k=1}^\infty\langle\sigma_3(U)e_k,\Delta h_1\rangle dW_3^k\right|\\
&\quad+16\sup_{r\in[0,s]}\left|\int_0^r\sum_{k=1}^\infty\langle\sigma_4(U)e_k,\Delta h_2\rangle dW_4^k\right|+8\int_0^s\sum_{k=1}^\infty\|\sigma_1(U)e_k\|^2dt\\
&\quad+8\int_0^s\sum_{k=1}^\infty\|\sigma_2(U)e_k\|^2dt+8\int_0^s\sum_{k=1}^\infty\|\sigma_3(U)e_k\|^2dt+8\int_0^s\sum_{k=1}^\infty\|\sigma_4(U)e_k\|^2dt\\
&\le 8\|\mathbf{v}_1(0)\|^2+8\|\mathbf{v}_2(0)\|^2+8\|h_1(0)\|^2+8\|h_2(0)\|^2+\sum_{i=1}^{20}J_i.
\end{aligned}
\tag{3.2}
$$
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J_1 and J_2 are estimated by simply using Cauchy-Schwarz inequality:

$$J_1=16\int_0^s|\langle F,\Delta\mathbf{v}_1\rangle|\,dt\le\frac{48}{\nu_1}\int_0^s|F|^2\,dt+\frac{\nu_1}{6}\int_0^s|\Delta\mathbf{v}_1|^2\,dt. \tag{3.3}$$
(2)

$$J_2=16\int_0^s|\langle G,\Delta\mathbf{v}_2\rangle|\,dt\le\frac{48}{\nu_2}\int_0^s|G|^2\,dt+\frac{\nu_2}{6}\int_0^s|\Delta\mathbf{v}_2|^2\,dt. \tag{3.4}$$
(3)

By Cauchy-Schwarz inequality and Poincaré inequality , we obtain the following estimates:

$$J_3=16g\int_0^s|\langle\nabla h_1,\Delta\mathbf{v}_1\rangle|\,dt\le\frac{48g^2}{\nu_1}\int_0^s\|h_1(t)\|^2dt+\frac{\nu_1}{6}\int_0^s|\Delta\mathbf{v}_1|^2\,dt, \tag{3.5}$$
(4)

$$J_4=16g\int_0^s|\langle\nabla h_2,\Delta\mathbf{v}_2\rangle|\,dt\le\frac{48g^2}{\nu_2}\int_0^s\|h_2(t)\|^2dt+\frac{\nu_2}{6}\int_0^s|\Delta\mathbf{v}_2|^2\,dt, \tag{3.6}$$
(5)

$$J_5=16g\frac{\rho_2}{\rho_1}\int_0^s|\langle\nabla h_2,\Delta\mathbf{v}_1\rangle|\,dt\le\frac{48\rho_2g^2}{\rho_1\nu_1}\int_0^s\|h_2(t)\|^2dt+\frac{\nu_1}{6}\int_0^s|\Delta\mathbf{v}_1|^2\,dt \tag{3.7}$$
(6)

$$J_6=16g\int_0^s|\langle\nabla h_1,\Delta\mathbf{v}_2\rangle|\,dt\le\frac{48g^2}{\nu_1}\int_0^s\|h_1(t)\|^2dt+\frac{\nu_1}{6}\int_0^s|\Delta\mathbf{v}_2|^2\,dt \tag{3.8}$$
(7)

$$J_7:=16\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_1,\Delta\mathbf{v}_1\rangle|\,dt\le\frac{48g^2}{\nu_1}\int_0^s\|\mathbf{v}_1(t)\|^2dt+\frac{\nu_1}{6}\int_0^s|\Delta\mathbf{v}_1|^2\,dt \tag{3.9}$$
(8)

$$J_8:=16\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_2,\Delta\mathbf{v}_2\rangle|\,dt\le\frac{48g^2}{\nu_2}\int_0^s\|\mathbf{v}_2(t)\|^2dt+\frac{\nu_2}{6}\int_0^s|\Delta\mathbf{v}_2|^2\,dt \tag{3.10}$$
(9)

J_9 and J_{10} are evaluated based on Hölder's inequality, Agmon's inequality, Young's inequality, Poincaré inequality and the classical interpolation Sobolev embedding. More precisely,

$$\begin{aligned} J_9 &= 16\int_0^s |\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1, \Delta\mathbf{v}_1\rangle|\,dt \le 16\int_0^s |\mathbf{v}_1|_{L^\infty(\mathcal{M})^2}\,|\nabla\mathbf{v}_1|_{L^2(\mathcal{M})^2}\,|\Delta\mathbf{v}_1|_{L^2(\mathcal{M})^2}\,dt \\ &\le C\int_0^s |\Delta\mathbf{v}_1|^{\frac{1}{2}}_{L^2(\mathcal{M})^2}\,\|\mathbf{v}_1\|^{\frac{1}{2}}\,|\nabla\mathbf{v}_1|_{L^2(\mathcal{M})^2}\,|\Delta\mathbf{v}_1|_{L^2(\mathcal{M})^2}\,dt = C\int_0^s |\Delta\mathbf{v}_1|^{\frac{3}{2}}_{L^2(\mathcal{M})^2}\,\|\mathbf{v}_1\|^{\frac{3}{2}}dt \\ &\le \frac{\nu_1}{6}\int_0^s |\Delta\mathbf{v}_1|^2\,dt + C\int_0^s \|\mathbf{v}_1\|^6 dt. \end{aligned} \tag{3.11}$$

$$\begin{aligned} J_{10} &= 16\int_0^s |\langle(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2, \Delta\mathbf{v}_2\rangle|\,dt \le 16\int_0^s |\mathbf{v}_2|_{L^\infty(\mathcal{M})^2}\,|\nabla\mathbf{v}_2|_{L^2(\mathcal{M})^2}\,|\Delta\mathbf{v}_1|_{L^2(\mathcal{M})^2}\,dt \\ &\le C\int_0^s |\Delta\mathbf{v}_2|^{\frac{1}{2}}_{L^2(\mathcal{M})^2}\,\|\mathbf{v}_2\|^{\frac{1}{2}}\,|\nabla\mathbf{v}_2|_{L^2(\mathcal{M})^2}\,|\Delta\mathbf{v}_2|_{L^2(\mathcal{M})^2}\,dt = C\int_0^s |\Delta\mathbf{v}_2|^{\frac{3}{2}}_{L^2(\mathcal{M})^2}\,\|\mathbf{v}_2\|^{\frac{3}{2}}dt \\ &\le \frac{\nu_2}{6}\int_0^s |\Delta\mathbf{v}_2|^2\,dt + C\int_0^s \|\mathbf{v}_2\|^6 dt. \end{aligned} \tag{3.12}$$

By adopting Ladyzhenskaya's inequality, Young's inequality and Sobolev embeddings, J_{11} and J_{12} are estimated as follows

$$\begin{aligned} J_{11} &= 16\int_0^s |\langle\nabla\cdot(h_1\mathbf{v}_1), \Delta h_1\rangle|\,dt \le 16\int_0^s |\langle(\nabla h_1\mathbf{v}_1, \Delta h_1\rangle|\,dt + 16\int_0^s |\langle h_1\nabla\cdot\mathbf{v}_1, \Delta h_1\rangle|\,dt \\ &\le 16\int_0^s |\nabla\mathbf{v}_1|_{L^4(\mathcal{M})}\,|\mathbf{v}_1|_{L^4(\mathcal{M})}\,|\Delta h_1|_{L^2(\mathcal{M})^2}\,dt + 16\int_0^s |h_1|_{L^4(\mathcal{M})}\,|\nabla v_1|_{L^4(\mathcal{M})^2}\,|\Delta h_1|_{L^2(\mathcal{M})^2}\,dt \\ &\le C\int_0^s \Big(|\nabla h_1|^{\frac{1}{2}}_{L^2}\,|\Delta h_1|^{\frac{1}{2}}_{L^2}\,\|\mathbf{v}\|^{\frac{1}{2}}\,|\Delta\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta h_1|_{L^2} + |\nabla h_1|_{L^2}\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta h_1|_{L^2}\Big)\,dt \\ &\le C\int_0^s \|h_1\|^{\frac{1}{2}}_{L^2}\,|\Delta h_1|^{\frac{3}{2}}_{L^2}\,\|\mathbf{v}_1\|dt + C\int_0^s \|h_1\|\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta h_1|_{L^2}\,dt \\ &\le \delta_1\int_0^s |\Delta h_1(t)|^2\,dt + C\int_0^s \Big(\|\mathbf{v}_1(t)\|^6 + \|h_1(t)\|^6\Big)\,dt + \frac{\nu_1}{6}\int_0^s |\Delta\mathbf{v}_1(t)|^2_{L^2}\,dt. \end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned} J_{12} &= 16\int_0^s |\langle\nabla\cdot(h_2\mathbf{v}_2), \Delta h_2\rangle|\,dt \le 16\int_0^s |\langle(\nabla h_2\mathbf{v}_2, \Delta h_2\rangle|\,dt + 8\int_0^s |\langle h_2\nabla\cdot\mathbf{v}_2, \Delta h_2\rangle|\,dt \\ 16 &\le \int_0^s |\nabla\mathbf{v}_2|_{L^4(\mathcal{M})}\,|\mathbf{v}_2|_{L^4(\mathcal{M})}\,|\Delta h_2|_{L^2(\mathcal{M})^2}\,dt + 16\int_0^s |h_2|_{L^4(\mathcal{M})}\,|\nabla v_2|_{L^4(\mathcal{M})^2}\,|\Delta h_2|_{L^2(\mathcal{M})^2}\,dt \\ &\le C\int_0^s \Big(|\nabla h_2|^{\frac{1}{2}}_{L^2}\,|\Delta h_2|^{\frac{1}{2}}_{L^2}\,\|\mathbf{v}\|^{\frac{1}{2}}\,|\Delta\mathbf{v}_2|^{\frac{1}{2}}_{L^2}\,|\Delta h_2|_{L^2} + |\nabla h_2|_{L^2}\,|\nabla\mathbf{v}_2|^{\frac{1}{2}}_{L^2}\,|\Delta\mathbf{v}_1|^{\frac{1}{2}}_{L^2}\,|\Delta h_2|_{L^2}\Big)\,dt \\ &\le C\int_0^s \|h_2\|^{\frac{1}{2}}_{L^2}\,|\Delta h_2|^{\frac{3}{2}}_{L^2}\,\|\mathbf{v}_1\|dt + C\int_0^s \|h_2\|\,|\nabla\mathbf{v}_2|^{\frac{1}{2}}_{L^2}\,|\Delta\mathbf{v}_2|^{\frac{1}{2}}_{L^2}\,|\Delta h_2|_{L^2}\,dt \\ &\le \delta_2\int_0^s |\Delta h_2(t)|^2\,dt + C\int_0^s \Big(\|\mathbf{v}_2(t)\|^6 + \|h_2(t)\|^6\Big)\,dt + \frac{\nu_2}{6}\int_0^s |\Delta\mathbf{v}_2|^2_{L^2}\,dt. \end{aligned} \tag{3.14}$$

By utilizing the Lipschitz assumptions (2.8), we find

$$\begin{aligned} J_{13}+J_{14}+J_{15}+J_{16} &= 32\int_0^s \sum_{k=1}^{\infty}\Big(\|\sigma_1(U)e_k\|^2 + \|\sigma_2(U)e_k\|^2 + \|\sigma_3(U)e_k\|^2 + \|\sigma_4(U)e_k\|^2 dt\Big) \\ &\le 32K_V\int_0^s (1 + \|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)dt. \end{aligned} \tag{3.15}$$

Combing (3.3)-(3.15) and taking the mathematical expectation on both sides, we find:

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s]}(\|\mathbf{v}_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_1(r)\|^2+\|h_2(r)\|^2)+\nu_1\int_0^s|\Delta\mathbf{v}_1|^2\,dt+\nu_2\int_0^s|\Delta\mathbf{v}_2|^2\,dt+\delta_1\int_0^s|\Delta h_1|^2\,dt\\
&+\delta_2\int_0^s|\Delta h_2|^2\,dt\Bigg)\leq\mathbb{E}\Bigg(8(\|\mathbf{v}_1(0)\|^2+\|h_1(0)\|^2+\|\mathbf{v}_2(0)\|^2+\|h_2(0)\|^2)\\
&+\int_0^s(32K_V+\frac{48}{\nu_1}|F|^2+\frac{48}{\nu_2}|G|^2)dt+\left(\frac{144g^2}{\nu_1}+\frac{96g^2}{\nu_2}+\frac{48\rho_2 g^2}{\rho_1\nu_1}\right)\int_0^s(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2)dr\\
&C\int_0^s(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2)^3dr\Bigg)\\
&+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}\rangle dW_1^k\right|\Bigg)+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle dW_1^k\right|\Bigg)\\
&\quad+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle dW_2^k\right|+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k,\Delta h_1\rangle dW_3^k\right|\Bigg)\\
&\quad+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k,\Delta h_2\rangle dW_4^k\right|\Bigg).
\end{aligned}
\tag{3.16}
$$

After simplifying and noticing that both F and G are deterministic, we obtain

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s]}(\|\mathbf{v}_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_1(r)\|^2+\|h_2(r)\|^2)+\nu_1\int_0^s|\Delta\mathbf{v}_1|^2\,dt+\nu_2\int_0^s|\Delta\mathbf{v}_2|^2\,dt+\delta_1\int_0^s|\Delta h_1|^2\,dt\\
&+\delta_2\int_0^s|\Delta h_2|^2\,dt\Bigg)\leq\mathbb{E}\Bigg(8(\|\mathbf{v}_1(0)\|^2+\|h_1(0)\|^2+\|\mathbf{v}_2(0)\|^2+\|h_2(0)\|^2)+\mathcal{K}_0\\
&+\int_0^s\sup_{0\leq r\leq t}(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2)[(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2\\
&+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2]^2+\mathcal{K}_1]dt\Bigg)+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle dW_1^k\right|\Bigg)\\
&\quad+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle dW_2^k\right|+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k,\Delta h_1\rangle dW_3^k\right|\Bigg)
\end{aligned}
\tag{3.17}
$$

$$
\quad+16\mathbb{E}\Bigg(\sup_{0\leq r\leq s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k,\Delta h_2\rangle dW_4^k\right|\Bigg)\Bigg).
\tag{3.18}
$$

where

$$
\mathcal{K}_0=\int_0^s(32K_V+\frac{48}{\nu_1}|F|^2+\frac{48}{\nu_2}|G|^2)dt
\tag{3.19}
$$

and

$$
\mathcal{K}_1=\frac{144g^2}{\nu_1}+\frac{96g^2}{\nu_2}+\frac{48\rho_2 g^2}{\rho_1\nu_1}
\tag{3.20}
$$

We observe that since $V=H_0^1(\mathcal{M})^6$, we have for $\sigma_i(\mathbf{v},h)e_k\in V$, $i=1,2$, $\Delta u\in H$,

$$
\langle\sigma_i(\mathbf{v},h)e_k,\Delta u\rangle=\int_{\mathcal{M}}\sigma_i(\mathbf{v},h)e_k\cdot\Delta u d\mathcal{M}.
\tag{3.21}
$$

By integrating by parts, this is equal to

$$-\int_{\mathcal{M}} \nabla\sigma_i(U)e_k \cdot \nabla u d\mathcal{M} + \int_{\partial\mathcal{M}} \sigma_i(U)e_k(\nabla u\cdot n)dS. = -\int_{\mathcal{M}} \nabla\sigma_i(U)e_k \cdot \nabla u d\mathcal{M}.$$

By the above expression, the Burkholder- Davis-Gundy inequality and by the Lipschitz assumptions (2.8), the first stochastic term is bounded as follows:

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le r\le s}\bigg|\int_0^r \sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle dW_1^k\bigg|\bigg) \le (\text{with } Ge_k = G_k = \langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle)\\
&\le C_1\mathbb{E}\bigg(\int_0^s \sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle^2 dt\bigg)^{\frac{1}{2}} \le C_1\mathbb{E}\bigg(\int_0^s \sum_{k=1}^{\infty}\langle\nabla\sigma_1(U)e_k,\nabla\mathbf{v}\rangle^2 dt\bigg)^{\frac{1}{2}}\\
&\le C_1\mathbb{E}\bigg(\int_0^s \sum_{k=1}^{\infty}|\nabla\sigma_1(U)e_k|_{H_1}^2\,|\nabla\mathbf{v}_1|_{H_1}^2\,dt\bigg)^{\frac{1}{2}} \le C_1\mathbb{E}\bigg(\sup_{0\le r\le s}\|\mathbf{v}_1(r)\|^2\int_0^s\sum_{k=1}^{\infty}\|\sigma_1(U)e_k\|^2 dt\bigg)^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le r\le s}\|\mathbf{v}_1(r)\|^2\bigg) + C_1\mathbb{E}\bigg(\int_0^s\sum_{k=1}^{\infty}\|\sigma_1(U)e_k\|^2 dt\bigg)\\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le r\le s}\|\mathbf{v}_1(r)\|^2\bigg) + C\mathbb{E}\bigg(\int_0^s(1+\|\mathbf{v}_1\|^2+\|h_1\|^2+\|\mathbf{v}_2\|^2+\|h_2\|^2)dt\bigg)
\end{aligned}\tag{3.22}$$

We obtain the similar bounds for the last three stochastic terms

$$\mathbb{E}\bigg(\sup_{0\le r\le s}\int_0^r\bigg|\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle dW_2^k\bigg|\bigg) \le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le r\le s}\|\mathbf{v}_2(r)\|^2\bigg) + C\mathbb{E}\bigg(\int_0^s(1+\|\mathbf{v}_1\|^2+\|h_1\|^2+\|\mathbf{v}_2\|^2+\|h_2\|^2)dt\bigg). \tag{3.23}$$

$$\mathbb{E}\bigg(\sup_{0\le r\le s}\int_0^r\bigg|\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k,\Delta h_1\rangle dW_3^k\bigg|\bigg) \le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le r\le s}\|h_1(r)\|^2\bigg) + C\mathbb{E}\bigg(\int_0^s(1+\|h_1\|^2+\|h_1\|^2+\|\mathbf{v}_2\|^2+\|h_2\|^2)dt\bigg). \tag{3.24}$$

$$\mathbb{E}\bigg(\sup_{0\le r\le s}\int_0^r\bigg|\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k,\Delta h_2\rangle dW_4^k\bigg|\bigg) \le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le r\le s}\|h_2(r)\|^2\bigg) + C\mathbb{E}\bigg(\int_0^s(1+\|h_2\|^2+\|h_1\|^2+\|\mathbf{v}_2\|^2+\|h_2\|^2)dt\bigg). \tag{3.25}$$

Rearranging (3.31)–(3.25) and multiplying by 2, we arrive at

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{r\in[0,s]}(\|\mathbf{v}_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_1(r)\|^2+\|h_2(r)\|^2) + 2\nu_1\int_0^s|\Delta\mathbf{v}_1|^2\,dt + 2\nu_2\int_0^s|\Delta\mathbf{v}_2|^2\,dt + 2\delta_1\int_0^s|\Delta h_1|^2\,dt\\
&+ 2\delta_2\int_0^s|\Delta h_2|^2\,dt\bigg) \le \mathbb{E}\bigg(16(\|\mathbf{v}_1(0)\|^2+\|h_1(0)\|^2+\|\mathbf{v}_2(0)\|^2+\|h_2(0)\|^2+\mathcal{K}_0)\bigg)\\
&\qquad + C\mathbb{E}\bigg(\int_0^s\sup_{0\le r\le t}(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2)\\
&\qquad [(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2]^2+\mathcal{K}_1+Cdt\bigg) + CT\\
&\le \mathbb{E}\bigg(16(\|\mathbf{v}_1(0)\|^2+\|h_1(0)\|^2+\|\mathbf{v}_2(0)\|^2+\|h_2(0)\|^2+\mathcal{K}_0)\bigg)\\
&\qquad + C\mathbb{E}\bigg(\int_0^s\sup_{0\le r\le t}(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2)[(\|\mathbf{v}_1(r)\|^2+\|h_1(r)\|^2+\|\mathbf{v}_2(r)\|^2+\|h_2(r)\|^2]^2dt\bigg)\\
&+ CT\mathcal{K}_1.
\end{aligned}\tag{3.26}$$

Now, we assume that $M > 1$ and consider the stopping time

$$\tau = \tau_M := \inf_{s \geq 0} \left\{ \sup_{r \in [0,s]} \left(\|\mathbf{v}_1(r)\|^2 + \|h_2(r)\|^2 + \|\mathbf{v}_2(r))\|^2 + \|h_2(r))\|^2 \right) > M \right\} \tag{3.27}$$

Replacing s by $s \wedge \tau$ in (3.26) yields

$$\mathbb{E}\Bigg(\sup_{r \in [0, s\wedge\tau]} \left(\|\mathbf{v}_1(r)\|^2 + \|h_1(r)\|^2 + \|\mathbf{v}_2(r)\|^2 + \|h_2(r)\|^2 \right) + 2\nu_1 \int_0^{s\wedge\tau} |\Delta \mathbf{v}_1|^2 \, dt + 2\delta_1 \int_0^{s\wedge\tau} |\Delta h_1|^2 \, dr +$$
$$2\nu_2 \int_0^{s\wedge\tau} \|\mathbf{v}_2(r)\|^2 dr + 2\delta_2 \int_0^{s\wedge\tau} \|h_2(r)\|^2 dr \Bigg) \leq \mathbb{E}\Bigg(16(\|\mathbf{v}_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_1(0)\|^2 + \|h_2(0)\|^2) + \mathcal{K}_0 \Bigg)$$
$$+ CM^2 \mathbb{E}\Bigg(\int_0^{s\wedge\tau} \sup_{0 \leq r \leq t} (\|\mathbf{v}_1(r)\|^2 + \|h_1(r)\|^2 + \|\mathbf{v}_2(r)\|^2 + \|h_2(r)\|^2) dt \Bigg) + CT\mathcal{K}_1. \tag{3.28}$$

Now, we define

$$\mathcal{Y}(t) := \mathbb{E}\Bigg(\int_0^{t\wedge\tau} \sup_{0 \leq r \leq s} (\|\mathbf{v}_1(r)\|^2 + \|h_1(r)\|^2 + \|\mathbf{v}_2(r)\|^2 + \|h_2(r)\|^2) ds \Bigg), \tag{3.29}$$

and

$$\mathcal{K}_2 := \mathbb{E}\Bigg(16(\|\mathbf{v}_1(0)\|^2 + \|h_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_2(0)\|^2) \Bigg) + \mathcal{K}_0 + CT\mathcal{K}_1. \tag{3.30}$$

From (3.28)and (3.29), we obtain:

$$\mathcal{Y}'(s) \leq \mathcal{K}_2 + CM^2 \mathcal{Y}(s).$$

This gives

$$\mathcal{Y}(s) \leq \frac{\mathcal{K}_2}{CM^2} \left(e^{CM^2 s} - 1 \right). \tag{3.31}$$

Along with (3.28) and (3.29), we deduce that

$$\mathbb{E}\Bigg(\sup_{r \in [0, s\wedge\tau]} (\|\mathbf{v}_1(r)\|^2 + \|h_1(r)\|^2 + \|\mathbf{v}_2(r)\|^2 + \|h_2(r)\|^2) + \nu_1 \int_0^{s\wedge\tau} |\Delta \mathbf{v}_1|^2 \, dt + \nu_2 \int_0^{s\wedge\tau} |\Delta \mathbf{v}_2|^2 \, dt$$
$$+ \delta_1 \int_0^{s\wedge\tau} |\Delta h_1|^2 \, dt + \delta_2 \int_0^{s\wedge\tau} |\Delta h_2|^2 \, dt \Bigg) \leq \mathcal{K}_2 + \frac{\mathcal{K}_2}{CM^2} e^{CM^2 s} CM^2$$
$$\leq \mathcal{K}_2 + \mathcal{K}_2 e^{CM^2 s} \tag{3.32}$$

The right hand side of (3.28) is bounded by M if

$$t \leq \frac{\mathcal{K}_2}{CM^2} \log \frac{M - \mathcal{K}_2}{\mathcal{K}_2} := s_M.$$

As long as M is large enough such that $M - \mathcal{K}_2 > \mathcal{K}_2$ or $M > 2\mathcal{K}_2$, we find

$$0 < s < s_M$$

which yields the local existence in time.

Hence

$$\mathbb{E}\bigg(\sup_{r\in[0,s\wedge\tau]} \|\mathbf{v}_1(r)\|^2 + \sup_{r\in[0,s\wedge\tau]} \|\mathbf{v}_2(r)\|^2 + \sup_{r\in[0,s\wedge\tau]} \|h(r)\|^2 + \sup_{r\in[0,s\wedge\tau]} \|h(r)\|^2$$
$$+ \nu_1 \int_0^{s\wedge\tau} |\Delta \mathbf{v}_1|^2\,dt + \nu_2 \int_0^{s\wedge\tau} |\Delta \mathbf{v}_2|^2\,dt + \delta_1 \int_0^{s\wedge\tau} |\Delta h|^2\,dt + \delta_2 \int_0^{s\wedge\tau} |\Delta h|^2\,dt \bigg) \tag{3.33}$$

is bounded by M for $0 < s < \dfrac{\mathcal{K}_2}{CM^2} \log \dfrac{M-\mathcal{K}_2}{\mathcal{K}_2}$, with $M > 2\mathcal{K}_2$ where $\mathcal{K}_1$ and $\mathcal{K}_2$ are at (3.20) and (3.30).

4 The modified system with a cut-off function

This section is aimed to study the martingale solutions of the following modified system

$$d\mathbf{v}_1 + \Big(-\nu_1\Delta\mathbf{v}_1 + \theta(\|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2) g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2 + f\mathbf{k}\times\mathbf{v}_1\Big)\,dt = F + \sum_{k=1}^{\infty} \sigma_1(U) e_k dW_1^k, \tag{4.1a}$$

$$d\mathbf{v}_2 + \Big(-\nu_2\Delta\mathbf{v}_2 + \theta(\|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2) g\nabla h_2 + g\nabla h_2 + f\mathbf{k}\times\mathbf{v}_2\Big)\,dt = G + \sum_{k=1}^{\infty} \sigma_2(U) e_k dW_2^k, \tag{4.1b}$$

$$dh_1 + \Big(-\delta_1\Delta h_1 + \theta(\|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)\nabla(h_1\mathbf{v}_1)\Big)\,dt = \sum_{k=1}^{\infty} \sigma_3(U) e_k dW_3^k, \tag{4.1c}$$

$$dh_2 + \Big(-\delta_2\Delta h_2 + \theta(\|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)\nabla(h_2\mathbf{v}_2)\Big)\,dt = \sum_{k=1}^{\infty} \sigma_4(U) e_k dW_4^k. \tag{4.1d}$$

To simplify writing, we denote $\theta(\|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)$ by $\theta(\|\mathbf{u}\|)$ for $\|\mathbf{u}\| = \|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2$.

Here $\theta : \mathbb{R} \to [0,1]$ is a $\mathcal{C}^\infty$ cut-off function satisfies

$$\theta(\epsilon) = \begin{cases} 1 \text{ if } |\epsilon| \le \mathcal{K}, \\ 0 \text{ if } |\epsilon| \ge 2\mathcal{K}. \end{cases}$$

where $\mathcal{K}$ is any positive number and is independent of n. The particular choice for $\mathcal{K}$ will be made more evidently in the next section.

Theorem 4.1 (Global existence of martingale solutions to the modified system)**.** *With the same assumptions as in Theorem 2.1, there exists a global martingale solution to (4.1).*

Theorem 4.2 (Global existence of pathwise solutions to the modified system)**.** *Under the same assumptions as in Theorem 2.2, there exists a global pathwise solution to (4.1) relative to given the stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$.*

4.1 The Galerkin Scheme

Considering the projection P_n defined as in (2.4), we introduce the Galerkin approximation $U^n := (\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n)$ of (4.1), with $\mathbf{v}_i^n$ and h_i^n functions from some interval $(0, \tau_n)$ into $P_n(V_1 \times$

$V_1\times V_2\times V_2$), namely,

$$d\mathbf{v}_1^n-\nu_1\Delta\mathbf{v}_1^n dt+P_n\Big[\theta\left(\|\mathbf{u}^n\|\right)(\mathbf{v}_1^n\cdot\nabla)\mathbf{v}_1^n+\mathrm{g}\nabla\mathrm{h}_1^n+g\frac{\rho_2}{\rho_1}\nabla\mathrm{h}_2^n+\ textitf\ \mathbf{k}^n\times\mathbf{v}_1^n\Big]dt$$
$$=P_n\mathrm{F}dt+\sum_{k=1}^{\infty}P_n\sigma_1(U^n)e_k dW_1^k, \tag{4.2a}$$

$$d\mathbf{v}_2^n-\nu_2\Delta\mathbf{v}_2^n dt+P_n\Big[\theta\left(\|\mathbf{u}^n\|\right)(\mathbf{v}_2^n\cdot\nabla)\mathbf{v}_2^n+\mathrm{g}\nabla\mathrm{h}_2^n+g\nabla\mathrm{h}_1^n+\mathrm{f}\ \mathbf{k}^n\times\mathbf{v}_2^n\Big]dt$$
$$=P_n\mathrm{G}dt+\sum_{k=1}^{\infty}P_n\sigma_2(U^n)e_k dW_2^k, \tag{4.2b}$$

$$d\mathrm{h}_1^n+P_n\Big[-\delta_1\Delta\mathrm{h}_1^n+\theta\left(\|\mathbf{u}^n\|\right)\nabla\cdot(\mathrm{h}_1^n\mathbf{v}_1^n)\Big]dt=\sum_{k=1}^{\infty}P_n\sigma_3(U^n)e_k dW_3^k, \tag{4.2c}$$

$$d\mathrm{h}_2^n+P_n\Big[-\delta_2\Delta\mathrm{h}_2^n+\theta\left(\|\mathbf{u}^n\|\right)\nabla\cdot(\mathrm{h}_2^n\mathbf{v}_2^n)\Big]dt=\sum_{k=1}^{\infty}P_n\sigma_4(U^n)e_k dW_4^k, \tag{4.2d}$$

$$\mathbf{v}_i^n(0)=\mathbf{v}_{i0}^n=P_n\mathbf{v}_{i0},\qquad h_i^n(0)=h_{i0}^n=P_nh_{0_i},\ \text{for } i=1,2. \tag{4.2e}$$

4.2 Uniform Estimates for the Galerkin System

The essential estimate for our study below is the following:

Lemma 4.1. *Let $\mathbf{v}_1^n,\mathbf{v}_2^n,h_1^n$ and h_2^n be the solutions of (4.1) and assume that $\mathbf{v}_1(0),\mathbf{v}_2(0)\in L^p(\Omega,\mathcal{F}_0,H_1), h_1(0),h_2(0)\in L^p(\Omega,\mathcal{F}_0,H_2), F,G\in L^p(\Omega\times[0,T],H)$ for some $p\geq 2$, we obtain the following estimates*

$$\mathbb{E}\left(\sup_{0\leq r\leq T}\|\mathbf{v}_1^n(r)\|^p+\sup_{0\leq r\leq T}\|h_1^n(r)\|^p+\sup_{0\leq r\leq T}\|\mathbf{v}_2^n(r)\|^p+\sup_{0\leq r\leq T}\|h_2^n(r)\|^p\right)\leq\mathcal{K}_1, \tag{4.3a}$$

and

$$\mathbb{E}\left(\int_0^T|\Delta\mathbf{v}_1^n|^2\,dt+\int_0^T|\Delta h_1^n|^2+|\Delta\mathbf{v}_2^n|^2\,dt+\int_0^T|\Delta h_2^n|^2\,dt\right)\leq\mathcal{K}_2, \tag{4.3b}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ depend only on the data and are independent of n.

Proof of Lemma 4.1: We apply the Itô formulas from (2.16) to (2.19) for $\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n$ and h_2^n and add the corresponding relations, which yields:

$$
\begin{aligned}
& d\|\mathbf{v}_1^n\|^p + p\nu_1|\Delta \mathbf{v}^n|^p dt + d\|\mathbf{v}_2^n\|^2 + p\nu_2|\Delta \mathbf{v}_2^n|^p dt + d\|h_1^n\|^2 + p\delta_1 \left|\Delta h_1^n\right|^2 dt + d\|h_2^n\|^p + p\delta_2 \left|\Delta h_2^n\right|^2 dt \\
& = p\langle P_n F, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt - pg\langle \nabla h_1^n, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt - p\langle P_n[\theta(\|\mathbf{u}^n\|)(\mathbf{v}_1^n\cdot\nabla)\mathbf{v}_1^n], \Delta\mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt \\
& \quad - p\langle P_n G, \Delta \mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dt - pg\langle \nabla h_1^n, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt - pg\frac{\rho_1}{\rho_2}\langle \nabla h_2^n, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt - p\langle f\mathbf{k}\times \mathbf{v}_1^n, \Delta\mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dt \\
& \quad + \frac{p}{2}\sum_{k=1}^{\infty} ||P_n\sigma_1(U^n)e_k||^2 \|\mathbf{v}_1^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\langle P_n\sigma_1(U^n)e_k, \Delta\mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dW_1^k \\
& \quad - p\langle P_n[\theta(\|\mathbf{u}^n\|)(\mathbf{v}_2^n\cdot\nabla)\mathbf{v}_2^n], \Delta\mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dt - pg\langle \nabla h_2^n, \Delta\mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dt - pg\langle \nabla h_1^n, \Delta\mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dt \\
& \quad - p\langle f\mathbf{k}\times\mathbf{v}_2^n, \Delta\mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dt + p\sum_{k=1}^{\infty} ||P_n\sigma_2(U^n)e_k||^2 \|\mathbf{v}_2^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\langle P_n\sigma_2(U^n)e_k, \Delta\mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dW_2^k \\
& \quad - p\langle P_n[\theta(\|\mathbf{u}^n\|)\nabla(h_1^n\mathbf{v}_1^n)], \Delta h_1^n\rangle \|h_1^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\|P_n\sigma_3(U^n)e_k\|^2 \|h_1^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\langle P_n\sigma_3(U^n), \Delta h_1^n\rangle \|h_1^n\|^{p-2} dW_3^k \\
& \quad - p\langle P_n[\theta(\|\mathbf{u}^n\|)\nabla(h_2^n\mathbf{v}_2^n)], \Delta h_1^n\rangle \|h_1^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\|\sigma_4(U^n)e_k\|^2 \|h_2^n\|^{p-2} dt + p\sum_{k=1}^{\infty}\langle P_n\sigma_4(U^n)e_k, \Delta h_2^n\rangle \|h_2^n\|^{p-2} dW_4^k. \\
& \quad + \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle \sigma_1(U)e_k, \Delta\mathbf{v}_1\rangle^2 ||\mathbf{v}_1||^{p-4} dt + \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle \sigma_2(U)e_k, \Delta\mathbf{v}_2\rangle^2 ||\mathbf{v}_2||^{p-4} dt \\
& \quad \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle \sigma_3(U)e_k, \Delta h_1\rangle^2 ||h_1||^{p-4} dt + \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle \sigma_4(U)e_k, \Delta h_2\rangle^2 ||h_2||^{p-4} dt \qquad (4.4)
\end{aligned}
$$

We integrate (4.4) in time over $[0, r]$ for $0 \le r \le s \le T$, take the supremum in r over $[0, s]$; we deduce that:

$$\sup_{0\le t\le s}\Big[\|\mathbf{v}_1^n\|^p+\|\mathbf{v}_2^n\|^p+\|h_1^n\|^p+\|h_2^n\|^p\Big]+p\nu_1\int_0^s|\Delta\mathbf{v}_1^n|^2\|\mathbf{v}_1^n\|^{p-2}dt+p\nu_2\int_0^s|\Delta\mathbf{v}_2^n|^2\|\mathbf{v}_2^n\|^{p-2}dt+p\delta_1\int_0^s|\Delta h_1^n|^2\,\|h_1^n\|^{p-2}dt$$

$$+p\delta_2\int_0^s|\Delta h_2^n|^2\,\|h_2^n\|^{p-2}dt\le 8(\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p)+8p\int_0^s|\langle P_nF,\Delta\mathbf{v}_1^n\rangle|\|\mathbf{v}_1^n\|^{p-2}dt$$

$$+8pg\int_0^s|\langle\nabla h_1^n,\Delta\mathbf{v}_1^n\rangle|\|\mathbf{v}_1^n\|^{p-2}dt+8p\int_0^s\langle P_n[\theta(\|\mathbf{u}^n\|)(\mathbf{v}_1^n\cdot\nabla)\mathbf{v}_1^n],\Delta\mathbf{v}_1^n\rangle\|\mathbf{v}_1^n\|^{p-2}dt$$

$$+8p\int_0^s|\langle P_nG,\Delta\mathbf{v}_2^n\rangle|\|\mathbf{v}_2^n\|^{p-2}dt+8pg\int_0^s|\langle\nabla h_1^n,\Delta\mathbf{v}_1^n\rangle|\|\mathbf{v}_1^n\|^{p-2}dt+8pg\int_0^s\frac{\rho_1}{\rho_2}\langle\nabla h_2^n,\Delta\mathbf{v}_1^n\rangle\|\mathbf{v}_1^n\|^{p-2}dt$$

$$+8p\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_1^n,\Delta\mathbf{v}_1^n\rangle|\|\mathbf{v}_1^n\|^{p-2}dt+8p\int_0^s\sum_{k=1}^\infty||P_n\sigma_1(U^n)e_k||^2\|\mathbf{v}_1^n\|^{p-2}dt$$

$$+8p\int_0^s\sum_{k=1}^\infty|\langle P_n\sigma_1(U^n)e_k,\Delta\mathbf{v}_1^n\rangle|\|\mathbf{v}_1^n\|^{p-2}dW_1^k+8p\int_0^s|\langle P_n[\theta(\|\mathbf{u}^n\|)(\mathbf{v}_2^n\cdot\nabla)\mathbf{v}_2^n],\Delta\mathbf{v}_2^n\rangle|\|\mathbf{v}_2^n\|^{p-2}dt$$

$$+8pg\int_0^s|\langle\nabla h_2^n,\Delta\mathbf{v}_2^n\rangle|\|\mathbf{v}_2^n\|^{p-2}dt+8pg\int_0^s|\langle\nabla h_1^n,\Delta\mathbf{v}_2^n\rangle|\|\mathbf{v}_2^n\|^{p-2}dt$$

$$+8p\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_2^n,\Delta\mathbf{v}_2^n\rangle|\|\mathbf{v}_2^n\|^{p-2}dt+8p\int_0^s\sum_{k=1}^\infty||P_n\sigma_2(U^n)e_k||^2\|\mathbf{v}_2^n\|^{p-2}dt$$

$$+8p\int_0^s\sum_{k=1}^\infty\langle P_n\sigma_2(U^n)e_k,\Delta\mathbf{v}_2^n\rangle\|\mathbf{v}_2^n\|^{p-2}dW_2^k+8p\int_0^s\langle P_n[\theta(\|\mathbf{u}^n\|)\nabla(h_1^n\mathbf{v}_1^n)],\Delta h_1^n\rangle\|h_1^n\|^{p-2}dt$$

$$+8p\int_0^s\sum_{k=1}^\infty\|P_n\sigma_3(U^n)e_k\|^2\|h_1^n\|^{p-2}dt+8p\int_0^s\sum_{k=1}^\infty\langle\sigma_3(U^n),\Delta h_1^n\rangle\|h_1^n\|^{p-2}dW_3^k$$

$$+8p\int_0^s\langle P_n[\theta(\|\mathbf{u}^n\|)\nabla(h_2^n\mathbf{v}_2^n)],\Delta h_1^n\rangle\|h_1^n\|^{p-2}dt+8p\int_0^s\sum_{k=1}^\infty\|\sigma_4(U^n)e_k\|^2\|h_2^n\|^{p-2}dt$$

$$8p\int_0^s\sum_{k=1}^\infty\langle\sigma_4(U^n)e_k,\Delta h_2^n\rangle\|h_2^n\|^{p-2}dW_4^k.$$

$$+4p(p-2)\Big[\int_0^s\sum_{k=1}^\infty\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle^2||\mathbf{v}_1||^{p-4}dt+\int_0^s\sum_{k=1}^\infty\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle^2||\mathbf{v}_2||^{p-4}dt\Big]$$

$$4p(p-2)\Big[\int_0^s\sum_{k=1}^\infty\langle\sigma_3(U)e_k,\Delta\mathbf{v}_1\rangle^2||h_1||^{p-4}dt+\sum_{k=1}^\infty\langle\sigma_4(U)e_k,\Delta\mathbf{v}_1\rangle^2||h_2||^{p-4}dt\Big].$$

$$\preceq(\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p)+\sum_{i=1}^{20}J_i.$$

Note that we can drop P_n in all of terms since P_n is self-adjoint and $P_n\Delta\mathbf{v}_i^n=\Delta\mathbf{v}_i^n,\ P_n\Delta h_i^n=\Delta h_i^n$, for $i=1,2$.

All the linear terms are treated using Cauchy-Schwarz inequality and Young's inequality

$$\begin{aligned}J_1:=\int_0^s|\langle P_nF,\Delta\mathbf{v}_1^n\rangle|\,\|\mathbf{v}_1^n\|^{p-2}dt&=\int_0^s|\langle F,\Delta\mathbf{v}_1^n\rangle|\,\|\mathbf{v}_1^n\|^{p-2}dt\le\int_0^s|F|\,|\Delta\mathbf{v}_1^n|\,\|\mathbf{v}_1^n\|^{p-2}dt\\&\le C(p,\nu_1)\int_0^s|F|^2\|\mathbf{v}_1^n\|^{p-2}+\frac{p\nu_1}{5}\int_0^s|\Delta\mathbf{v}_1^n|^2\,\|\mathbf{v}_1^n\|^2dt\qquad(4.5)\\&\le C(p,\nu_1)\int_0^s|F|^pdt+C(p,\nu_1)\int_0^s\|\mathbf{v}_1^n\|^pdt+\frac{p\nu_1}{5}\int_0^s|\Delta\mathbf{v}_1^n|^2\,\|\mathbf{v}_1^n\|^2dt\end{aligned}$$

$$
\begin{aligned}
J_2 := \int_0^s |\langle P_n G, \Delta \mathbf{v}_2^n \rangle| \, \|\mathbf{v}_2^n\|^{p-2} dt = \int_0^s |\langle G, \Delta \mathbf{v}_2^n \rangle| \, \|\mathbf{v}_2^n\|^{p-2} dt \leq \int_0^s |G| \, |\Delta \mathbf{v}_1^n| \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\nu_2) \int_0^s |G|^2 \|\mathbf{v}_2^n\|^{p-2} + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^2 dt \\
\leq C(p,\nu_2) \int_0^s |G|^p dt + C(p,\nu_2) \int_0^s \|\mathbf{v}_2^n\|^p dt + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^2 dt
\end{aligned} \tag{4.6}
$$ 12

$$
\begin{aligned}
J_3 := \int_0^s |\langle \nabla h_1^n, \Delta \mathbf{v}_1^n \rangle| \, \|\mathbf{v}_1^n\|^{p-2} dt \leq \int_0^s |\nabla h_1^n| \, |\Delta \mathbf{v}_1^n| \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\nu_1) \int_0^s |\nabla h_1^n|^p dt + C(p) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{4} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^2 dt \\
\leq C(p,\nu_1) \int_0^s \|h_1^n\|^p dt + C(p,\nu_1) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{5} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^2 dt
\end{aligned} \tag{4.7}
$$ 13

$$
\begin{aligned}
J_4 := \int_0^s |\langle \nabla h_2^n, \Delta \mathbf{v}_1^n \rangle| \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\nu_1) \int_0^s \|h_2^n\|^p dt + C(p,\nu_1) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{5} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^2 dt
\end{aligned} \tag{4.8}
$$ 14

$$
\begin{aligned}
J_5 := \int_0^s |\langle \nabla h_2^n, \Delta \mathbf{v}_2^n \rangle| \, \|h_1^n\|^{p-2} dt \\
\leq C(p,\nu_2) \int_0^s \|h_2^n\|^p dt + C(p,\nu_2) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^2 dt
\end{aligned} \tag{4.9}
$$ 15

$$
\begin{aligned}
J_6 := \int_0^s |\langle \nabla h_1^n, \Delta \mathbf{v}_2^n \rangle| \, \|\mathbf{v}_2^n\|^{p-2} dt \\
\leq C(p,\nu_2) \int_0^s |\nabla h_1^n|^p dt + C(p,\nu_2) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^2 dt
\end{aligned} \tag{4.10}
$$ 16

By Young's inequality and Poincaré's inequality, the next two terms are bounded as follows

$$
\begin{aligned}
J_7 := \int_0^s |\langle f\mathbf{k} \times \mathbf{v}_1^n, \Delta \mathbf{v}_1^n \rangle| \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\nu_1,\lambda_1) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{5} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^{p-2} dt
\end{aligned} \tag{4.11}
$$ 17

$$
\begin{aligned}
J_8 := \int_0^s |\langle f\mathbf{k} \times \mathbf{v}_2^n, \Delta \mathbf{v}_2^n \rangle| \, \|\mathbf{v}_2^n\|^{p-2} dt \\
\leq C(p,\nu_2,\lambda_1) \int_0^s \|\mathbf{v}_2^n\|^p dt + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^{p-2} dt
\end{aligned} \tag{4.12}
$$ 18

We obtain the bounds for non-linear terms by employing Hölder's inequality, Cauchy Schwarz inequality and Young's inequality along with (4),

$$
\begin{aligned}
J_9 := \int_0^s |\langle \theta(\|\mathbf{u}^n\|)(\mathbf{v}_1^n \cdot \nabla)\mathbf{v}_1^n, \Delta \mathbf{v}_1^n \rangle| \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq \int_0^s |\theta(\|\mathbf{u}^n\|)\mathbf{v}_1^n|_{L^\infty(\mathcal{M})^2} \, |\nabla \mathbf{v}_1^n|_{L^2(\mathcal{M})^2} \, |\Delta \mathbf{v}_1^n|_{L^2(\mathcal{M})^2} \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\mathcal{K},\nu_1) \int_0^s |\nabla \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^{p-2} dt + \frac{p\nu_1}{5} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^{p-2} dt \\
\leq C(p,\mathcal{K},\nu_1) \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{5} \int_0^s |\Delta \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^{p-2} dt
\end{aligned} \tag{4.13}
$$ 19

Similarly,

$$J_{10} := \int_0^s |\langle \theta(\|\mathbf{u}^n\|)(\mathbf{v}_2^n \cdot \nabla)\mathbf{v}_2^n, \Delta \mathbf{v}_2^n\rangle| \, \|\mathbf{v}_2^n\|^{p-2} dt \leq C(p, \mathcal{K}, \nu_2) \int_0^s \|\mathbf{v}_2^n\|^p dt + \frac{p\nu_2}{5} \int_0^s |\Delta \mathbf{v}_2^n|^2 \, \|\mathbf{v}_2^n\|^{p-2} dt \tag{4.14}$$

$$\begin{aligned} J_{11} &:= \int_0^s |\langle \theta(\|\mathbf{u}^n\|)\nabla \cdot (h_1^n \mathbf{v}_1^n), \Delta h_1^n\rangle| \, \|h_1^n\|^{p-2} dt = \int_0^s \Big[\langle \theta(\|\mathbf{u}^n\|)\nabla \cdot \mathbf{v}_1^n h_1^n, \Delta h_1^n\rangle \|h_1^n\|^{p-2} dt + \\ &\qquad + \langle \theta(\|\mathbf{u}^n\|)\nabla h_1^n \mathbf{v}_1^n, \Delta h_1^n\rangle \|h_1^n\|^{p-2} dt \Big] \\ &\leq \Big[\int_0^s |\theta(\|\mathbf{u}^n\|) h_1^n|_{L^\infty} \, |\nabla \mathbf{v}_1^n|_{L^2} \, |\Delta h_1^n| \, \|h_1^n\|^{p-2} dt + \int_0^s |\theta(\|\mathbf{u}^n\|)\mathbf{v}_1^n|_{L^\infty} \, |\nabla h_1^n|_{L^2} \, |\Delta h_1^n| \, \|h_1^n\|^{p-2} dt \Big] \\ &\qquad \Big[C(p, \delta_1, \mathcal{K}_1) \int_0^s \|\mathbf{v}_1^n\|^p + \int_0^s \|h_1^n\|^p \Big] + \frac{p\delta_1}{2} \int_0^s |\Delta h_1^n|^2 \, \|h_1^n\|^{p-2} dt. \end{aligned} \tag{4.15}$$

$$\begin{aligned} J_{12} &:= \int_0^s |\langle \theta(\|\mathbf{u}^n\|)\nabla \cdot (h_2^n \mathbf{v}_2^n), \Delta h_2^n\rangle| \, dt \\ &\qquad \Big[C(p, \delta_2, \mathcal{K}_1) \int_0^s \|\mathbf{v}_2^n\|^p + \int_0^s \|h_2^n\|^p \Big] + \frac{p\delta_2}{2} \int_0^s |\Delta h_2^n|^2 \, \|h_2^n\|^{p-2} dt \end{aligned} \tag{4.16}$$

By adopting Lipschitz assumptions (2.8) and Young's inequality, we obtain:

$$\begin{aligned} J_{13} + J_{14} + J_{15} + J_{16} &= 8p\Big[\int_0^s \sum_{k=1}^\infty \|P_n \sigma_1(U^n) e_k\|^2 \|\mathbf{v}_1^n\|^{p-2} dt + \int_0^s \sum_{k=1}^\infty \|P_n \sigma_2(U^n) e_k\|^2 \|\mathbf{v}_2^n\|^{p-2} dt \\ &\qquad + \int_0^s \sum_{k=1}^\infty \|P_n \sigma_3(U^n) e_k\|^2 \|h_1^n\|^{p-2} dt + \int_0^s \sum_{k=1}^\infty \|P_n \sigma_4(U^n) e_k\|^2 \|h_2^n\|^{p-2} dt \Big] \\ &\leq 8K_V p \int_0^s (1 + \|\mathbf{v}_1^n\|^2 + \|\mathbf{v}_2^n\|^2 + \|h_1^n\|^2 + \|h_2^n\|^2)(\|\mathbf{v}_1^n\|^{p-2} + \|\mathbf{v}_1^n\|^{p-2} + \|\mathbf{v}_2^n\|^{p-2} + \|h_1^n\|^{p-2} + \|h_2^n\|^{p-2}) \\ &\leq C(K_V, p) \int_0^s (\|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) dt + CT. \end{aligned} \tag{4.17}$$

By Lipschitz assumptions (2.8) and integration by parts, we obtain

$$\begin{aligned} J_{17} + J_{18} + J_{19} + J_{20} &= 4p(p-2)\Big[\int_0^s \sum_{k=1}^\infty \langle \sigma_1(U)e_k, \Delta \mathbf{v}_1\rangle^2 ||\mathbf{v}_1||^{p-4} dt + \int_0^s \sum_{k=1}^\infty \langle \sigma_2(U)e_k, \Delta \mathbf{v}_2\rangle^2 ||\mathbf{v}_2||^{p-4} dt \\ &\qquad \int_0^s \sum_{k=1}^\infty \langle \sigma_3(U)e_k, \Delta h_1\rangle^2 ||h_1||^{p-4} dt + \int_0^s \sum_{k=1}^\infty \langle \sigma_4(U)e_k, \Delta h_2\rangle^2 ||h_2||^{p-4} dt \Big] \\ &= 4p(p-2)\Big[\int_0^s \sum_{k=1}^\infty \langle \nabla \sigma_1(U)e_k, \nabla \mathbf{v}_1\rangle^2 ||\mathbf{v}_1||^{p-4} dt + \int_0^s \sum_{k=1}^\infty \langle \nabla \sigma_2(U)e_k, \nabla \mathbf{v}_2\rangle^2 ||\mathbf{v}_2||^{p-4} dt \\ &\qquad + \int_0^s \sum_{k=1}^\infty \langle \nabla \sigma_3(U)e_k, \nabla h_1\rangle^2 ||h_1||^{p-4} dt + \int_0^s \sum_{k=1}^\infty \langle \nabla \sigma_4(U)e_k, \nabla h_2\rangle^2 ||h_2||^{p-4} dt \Big] \\ &\leq 16p(p-2)K_V \int_0^s (1 + \|\mathbf{v}_1^n\|^2 + \|\mathbf{v}_2^n\|^2 + \|h_1^n\|^2 + \|h_1^n\|^2)(\|\mathbf{v}_1^n\|^{p-2} + \|\mathbf{v}_2^n\|^{p-2} + \|h_1^n\|^{p-2} + \|h_2^n\|^{p-2}) \\ &\leq C(K_V, p) \int_0^s (\|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) + C(K_V, T, p) \end{aligned} \tag{4.18}$$

Combining (4.5)–(4.18), multiplying by 2, and taking the mathematical expectation on both sides, we obtain

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{0\le r\le s} \|\mathbf{v}_1^n(r)\|^p + \sup_{0\le r\le s} \|\mathbf{v}_2^n(r)\|^p + \sup_{0\le r\le s} \|h_1^n(s)\|^p + \sup_{0\le r\le s} \|h_2^n(s)\|^p + \nu_1 \int_0^s |\Delta \mathbf{v}_1^n(t)|^2 \, \|\mathbf{v}_1^n\|^{p-2} dt \qquad (4.19)\\
&\quad + \nu_2 \int_0^s |\Delta \mathbf{v}_2^n(t)|^2 \, \|\mathbf{v}_2^n\|^{p-2} dt + \delta_1 \int_0^s |\Delta h_1^n(t)|^2 \, \|h_1^n\|^{p-2} dt + \delta_2 \int_0^s |\Delta h_2^n(t)|^2 \, \|h_2^n\|^{p-2} dt \Bigg)\\
&\preceq \mathbb{E}\left(\|\mathbf{v}_1^n(0)\|^p + \|\mathbf{v}_2^n(0)\|^p + \|h_1^n(0)\|^p + \|h_2^n(0)\|^p \right) + \mathbb{E}\left(\int_0^s (\|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) dt \right)\\
&\quad + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle P_n\sigma_1(U^n)e_k, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dW_1^k \right| \right) + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle P_n\sigma_2(U^n)e_k, \Delta \mathbf{v}_2^n\rangle \|\mathbf{v}_2^n\|^{p-2} dW_2^k \right| \right)\\
&\quad + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle P_n\sigma_3(U^n)e_k, \Delta h_1^n\rangle \|h_1^n\|^{p-2} dW_3^k \right| \right) + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle P_n\sigma_4(U^n)e_k, \Delta h_2^n\rangle \|h_2^n\|^{p-2} \|\mathbf{v}_1^n\|^{p-2} dW_4^k \right| \right),\\
&\quad + C(p,\nu_1,\nu_2,\delta_1,\delta_2,\mathcal{K}_1) \qquad (4.20)
\end{aligned}
$$

By utilizing integration by part, BDG's inequality, and Young's inequality, two stochastic terms are bounded as follows:

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s]} \Bigg| \int_0^r \sum_{k=1}^\infty \langle \sigma_1(U^n)e_k, \Delta \mathbf{v}_1^n\rangle \|\mathbf{v}_1^n\|^{p-2} dW_1^k \Bigg| \Bigg) \le C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty \langle \sigma_1(U^n)e_k, \Delta \mathbf{v}_1^n\rangle^2 \|\mathbf{v}_1^n\|^{2(p-2)} dt \Bigg)^{\frac{1}{2}}\\
&= C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty \langle \nabla\sigma_1(U^n)e_k, \nabla \mathbf{v}_1^n\rangle^2 \|\mathbf{v}_1^n\|^{2(p-2)} dt \Bigg)^{\frac{1}{2}} \le C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_1(U^n)e_k|^2 \, |\nabla \mathbf{v}_1^n|^2 \, \|\mathbf{v}_1^n\|^{2(p-2)} dt \Bigg)^{\frac{1}{2}}\\
&\le C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_1(U^n)e_k|^2 \, \|\mathbf{v}_1^n\|^{2(p-1)} dt \Bigg)^{\frac{1}{2}} \le C\mathbb{E}\Bigg(\sup_{r\in[0,s]} \|\mathbf{v}_1^n\|^{p-1} \int_0^s \sum_{k=1}^\infty |\nabla\sigma_1(U^n)e_k|^2 \, dt \Bigg)^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E} \sup_{r\in[0,s]} \|\mathbf{v}_1^n\|^p + C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_1(U^n)e_k|^2 \, dt \Bigg)^{\frac{p}{2}}\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,s]} \|\mathbf{v}_1^n(r)\|^p \right) + C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty (1 + \|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) dt \Bigg), \qquad (4.21)
\end{aligned}
$$

The last line hold true due to the Lipschitz assumptions (2.8) and Hölder's inequality.
We also obtain

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s]} \Bigg| \int_0^r \sum_{k=1}^\infty \langle \sigma_2(U^n)e_k, \Delta \mathbf{v}_2^n\rangle dW_2^k \Bigg| \Bigg)\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,s]} \|\mathbf{v}_2^n(r)\|^p \right) + C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty (1 + \|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) dt \Bigg), \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s]} \Bigg| \int_0^r \sum_{k=1}^\infty \langle \sigma_3(U^n)e_k, \Delta h_1^n\rangle dW_3^k \Bigg| \Bigg)\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,s]} \|h_1^n(r)\|^p \right) + C\mathbb{E}\Bigg(\int_0^s \sum_{k=1}^\infty (1 + \|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^2) dt \Bigg), \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{r\in[0,s]}\Big|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_4(U^n)e_k,\Delta h_2^n\rangle dW_4^k\Big|\Big)\\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,s]}\|h_2^n(r)\|^p\right)+C\mathbb{E}\Big(\int_0^s\sum_{k=1}^{\infty}(1+\|\mathbf{v}_1^n\|^p+\|\mathbf{v}_2^n\|^p+\|h_1^n\|^p+\|h_2^n\|^p)dt\Big).
\end{aligned}
\tag{4.24}
$$

Rearranging (4.18)–(4.24), and multiplying by 2 we obtain:

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq r\leq s}\|\mathbf{v}_1^n(r)\|^p+\sup_{0\leq r\leq s}\|\mathbf{v}_2^n(r)\|^p+\sup_{0\leq r\leq s}\|h_1^n(s)\|^p+\sup_{0\leq r\leq s}\|h_2^n(s)\|^p\\
&\qquad+\nu_1\int_0^s|\Delta\mathbf{v}_1^n(t)|^2\,dt+\nu_2\int_0^s|\Delta\mathbf{v}_2^n(t)|^2\,dt+\delta_1\int_0^s|\Delta h_1^n(t)|^2\,dt+\delta_2\int_0^s|\Delta h_2^n(t)|^2\,dt\Big)\\
&\quad\preceq \mathbb{E}\left(\|\mathbf{v}_1^n(0)\|^p+\|\mathbf{v}_2^n(0)\|^p+\|h_1^n(0)\|^p+\|h_2^n(0)\|^p\right)+\mathbb{E}\left(\int_0^s(|F|^p+|G|^p)dt\right)\\
&\qquad\qquad\qquad+\mathbb{E}\left(\int_0^s(\|\mathbf{v}_1^n\|^p+\|\mathbf{v}_2^n\|^p+\|h_1^n\|^p+\|h_2^n\|^p)dt\right)\\
&\preceq \mathbb{E}\left(\|\mathbf{v}_1^n(0)\|^p+\|\mathbf{v}_2^n(0)\|^p+\|h_1^n(0)\|^p+\|h_2^n(0)\|^p\right)+\mathbb{E}\left(\int_0^s(|F|^p+|G|^p)dt\right)+C\\
&\quad+\mathbb{E}\left(\int_0^s\left[\sup_{0\leq r\leq t}\|\mathbf{v}_1^n(r)\|^p+\sup_{0\leq r\leq t}\|\mathbf{v}_2^n(r)\|^p+\sup_{0\leq r\leq t}\|h_1^n(r)\|^p+\sup_{0\leq r\leq t}\|h_2^n(r)\|^p\right]dt\right)+C.,
\end{aligned}
\tag{4.25}
$$

where $\preceq$ means $\leq$ up to a multiplicative constant.

By applying the deterministic Gronwall inequality to

$$
Y(s)=E\Big(\sup_{r\in[0,s]}\|\mathbf{v}_1^n(r)\|^p+\sup_{r\in[0,s]}\|\mathbf{v}_2^n(r)\|^p+\sup_{s\in[0,s]}\|h_1^n(r)\|^p+\sup_{r\in[0,s]}\|h_2^n(r)\|^p\Big),
$$

we obtain:

$$
\begin{aligned}
&E\Big(\sup_{r\in[0,s]}\|\mathbf{v}_1^n(r)\|^p+\sup_{r\in[0,s]}\|\mathbf{v}_2^n(r)\|^p+\sup_{s\in[0,s]}\|h_1^n(r)\|^p+\sup_{r\in[0,s]}\|h_2^n(r)\|^p\Big)\\
&\qquad\qquad\preceq\left(\|\mathbf{v}_1^n(0)\|^p+\|\mathbf{v}_2^n(0)\|^p+\|h_1^n(0)\|^p+\|h_2^n(0)\|^p\right)+\mathbb{E}\left(\int_0^s|F|^p+|G|^p\,dt\right).
\end{aligned}
\tag{4.26}
$$

From (4.25) and (4.26), the lemma is proven. □

Lemma 4.2 (Estimates in Fractional Sobolev spaces)**.** *Under the same assumptions as in Theorem 2.1, we consider the associated sequence of solutions* $\{(\mathbf{v}_1^n,\mathbf{v}_2^n,h_1^n,h_2^n)\}_{n\geq1}$ *of the Galerkin system* (4.1)*. Let* $p>2$ *and* $\mathbb{E}(\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p)<\infty$*. Then there exists a finite number* $\mathcal{K}>0$ *(depending only on the data) such that*

$$
\mathbb{E}\left(\left|\int_0^t\sum_{k=1}^{\infty}P_n\sigma_i(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0.T];H_1)}\right)\leq\mathcal{K},\ \textit{for } i=1,2,
\tag{4.27a}
$$

$$
\mathbb{E}\left(\left|\int_0^t\sum_{k=1}^{\infty}P_n\sigma_i^k(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0.T];H_2)}\right)\leq\mathcal{K},,\ \textit{for } i=3,4,
\tag{4.27b}
$$

$$
\mathbb{E}\left(\left|\mathbf{v}_i^n(t)-\int_0^t\sum_{k=1}^{\infty}P_n\sigma_i(\mathbf{v}^n,h^n)dW_i^k\right|^2_{W^{1,2}([0.T];H_1)}\right)\leq\mathcal{K},\ \textit{for } i=1,2,
\tag{4.27c}
$$

$$
\mathbb{E}\left(\left|h_i^n(t)-\int_0^t\sum_{k=1}^{\infty}P_n\sigma_j^k(\mathbf{v}^n,h^n)dW_j^k\right|^2_{W^{1,2}([0.T];H_2)}\right)\leq\mathcal{K},\ \textit{for } i=1,2,\ j=3,4.
\tag{4.27d}
$$

Proof of Lemma 4.2: Proof of (4.27a). We have for $Ge_k = \sigma_1(U^n)e_k = \sigma_i(U^n)e_k$

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t \sum_{k=1}^{\infty} P_n\sigma_i(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0,T];H_1)}\right) \leq \mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t \sum_{k=1}^{\infty} P_n\sigma_i(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0,T];H_1)}\right)\\
&\leq c\mathbb{E}\left(\int_0^T \sum_{k=1}^{\infty}|\sigma_i(U^n)e_k|^p_{H_1}\,dt\right) \leq c\mathbb{E}\left(\int_0^T \|\sigma_i(U^n)\|^p_{L_2(\mathfrak{U},H_1)}dt\right)\\
&\leq c\mathbb{E}\left(\int_0^T (1+|\mathbf{v}_1^n|^2+|\mathbf{v}_2^n|^2+|h_1^n|^2+|h_1^n|^2)dt\right)^{\frac{p}{2}}\\
&\leq c\mathbb{E}\left(\int_0^T (1+|\mathbf{v}_1^n|^p+|\mathbf{v}_2^n|^p+|h_1^n|^p+|h_2^n|^p)dt\right)\\
&\leq c\mathbb{E}\left(\int_0^T (1+\|\mathbf{v}_1^n\|^p+\|\mathbf{v}_2^n\|^p+\|h_1^n\|^p+\|h_2^n\|^p)dt\right)\\
&\leq c\mathbb{E}\left(\int_0^T \sup_{t\in[0,T]}(1+\|\mathbf{v}_1^n\|^p+\|\mathbf{v}_2^n\|^p+\|h_1^n\|^p+\|h_2^n\|^p)dt\right)\\
&\leq \mathcal{K}\ (\text{a.s. by Lemma 4.1}).
\end{aligned}
$$

The proof of (4.27b) follows analogously.

Proof of (4.27c) for $i=1$: integrating (4.2a)on $(0,t)$ $0\leq t\leq T$, we obtain

$$
\begin{aligned}
\mathbf{v}_1^n - \nu_1\int_0^t \Delta\mathbf{v}_1^n ds + \int_0^t P_n[\theta(U^n)(\mathbf{v}_1^n\cdot\nabla)\mathbf{v}_1^n + g\nabla h_1^n + g\frac{\rho_2}{\rho_1}\nabla h_2^n + f\mathbf{k}\times\mathbf{v}_1^n]ds&\\
= \int_0^t P_nF ds + \int_0^t\sum_{k=1}^{\infty} P_n\sigma_1 e_k(U^n)dW_1^k + \mathbf{v}_1^n(0).&
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}\left(\left|\mathbf{v}_1^n - \int_0^t\sum_{k=1}^{\infty}P_n\sigma_1(U^n)e_k dW_1^k\right|^2_{W^{1,2}([0,T];H_1)}\right)\\
&= \mathbb{E}\left(\left|\mathbf{v}_1^n - \int_0^t\sum_{k=1}^{\infty}P_n\sigma_1(U^n)e_k dW_1^k\right|^2_{L^2([0,T];H_1)} + \frac{d}{dt}\left|\mathbf{v}_1^n - \int_0^t\sum_{k=1}^{\infty}P_n\sigma_1(U^n)e_k dW_1^k\right|^2_{L^2([0,T];H_1)}\right)
\end{aligned}
$$

using $\left|\theta^{\frac{1}{2}}(\|\mathbf{u}^n\|)\mathbf{v}_1^n\right|_{L^2(\mathcal{M})}\leq 1$ and $\left|\theta^{\frac{1}{2}}(\|\mathbf{u}^n\|)\nabla\mathbf{v}_1^n\right|_{L^2(\mathcal{M})}\leq 1$, we find

$$
\begin{aligned}
&\mathbb{E}\left(\left|\mathbf{v}_1^n - \int_0^t\sum_{k=1}^{\infty}P_n\sigma_1(U^n)e_k dW_1^k\right|^2_{W^{1,2}([0,T];H_1)}\right)\\
&\leq c\mathbb{E}\left(\|\mathbf{v}_1(0)\|^2 + \nu\int_0^T|\Delta\mathbf{v}_1^n(t)|^2\,dt + g\int_0^T\|\mathbf{v}_1^n(t)\|^2dt + g\frac{\rho_2}{\rho_1}\|h_2^n\|^2dt + (1+|F|^2)dt\right)\\
&\leq c\mathbb{E}\Bigg(\|\mathbf{v}_1(0)\|^2 + \nu\int_0^T|\Delta\mathbf{v}_1^n(t)|^2\,dt + g\sup_{0\leq r\leq T}\|\mathbf{v}_1^n(t)\|^2 + \frac{\rho_2}{\rho_1}\sup_{0\leq r\leq T}\|h_2^n(r)\|^2 + \int_0^T(1+|F|^2)dt\Bigg)\\
&\leq \mathcal{K}\ (\text{by Lemma 4.1}).
\end{aligned}
\tag{4.28}
$$

The proof of (4.27d) can be derived in the same manner. □

4.3 Compactness Arguments

We fix a stochastic basis, $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P}, W_1, W_2, W_3, W_4)$, and given initial distribution μ_0 on $H_1 \times H_2$. We define $(\mathbf{v}_1^0, \mathbf{v}_2^0, h_1^0, h_2^0)$ which is $\mathcal{F}_0$-measurable and has distribution μ_0. Then we go back to the finite dimensional approximations relative to $\mathcal{S}$ and $(\mathbf{v}_1^0, \mathbf{v}_2^0, h_1^0, h_2^0)$. We define the phase space

$$\mathcal{X} = \mathcal{X}_{\mathbf{v}_1} \times \mathcal{X}_{\mathbf{v}_2} \times \mathcal{X}_{h_1} \times \mathcal{X}_{h_2} \times \prod_{i=1}^{4} \mathcal{X}_{W_i},$$

where

$$\begin{aligned} \mathcal{X}_{\mathbf{v}_1} &= \mathcal{X}_{\mathbf{v}_1} = L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1'), \\ \mathcal{X}_{h_1} &= \mathcal{X}_{h_2} = L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2'), \\ \mathcal{X}_{W_1} &= \mathcal{X}_{W_2} = \mathcal{X}_{W_3} = \mathcal{X}_{W_4} = \mathcal{C}([0,T];\mathfrak{U}_0). \end{aligned} \tag{4.29}$$

We consider the probability measures

$$\mu^n_{\mathbf{v}_1}(\cdot) = \mu^n_{\mathbf{v}_2}(\cdot) = \mathbb{P}\left(\mathbf{v}^n \in \cdot\right) \in Pr(L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1')), \tag{4.30}$$

$$\mu^n_{h_1}(\cdot) = \mu^n_{h_2}(\cdot) = \mathbb{P}\left(h^n \in \cdot\right) \in Pr(L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2')), \tag{4.31}$$

and $$\mu_{W_i}(\cdot) = \mu^n_{W_i}(\cdot) = \mathbb{P}(W_i \in \cdot) \in Pr(\mathcal{C}([0,T];\mathfrak{U}_0)), \text{ for } i = 1,2,3,4. \tag{4.32}$$

This defines a sequence of probability measures $\mu^n = \mu^n_{\mathbf{v}_1} \times = \mu^n_{\mathbf{v}_2} \times \mu^n_{h_1} \times \mu^n_{h_2} \times \prod_{i=1}^4 \mu^n_{W_i}$ on $\mathcal{X}$. Then we have the following tightness result:

Lemma 4.3. *Consider the measure μ^n on $\mathcal{X}$ defined as above in (4.30)–(4.32). Then the sequence $\{\mu^n\}_{n\geq 1}$ is tight and therefore weakly compact on phase space $\mathcal{X}$.*

Proof: By applying Lemma 6.1 with $\mathcal{E}_0 = D(-\Delta)$, $\mathcal{E}_i = V_i$, $p = 2, p = 1/5$, for $i = 1,2$, we deduce that

$$L^2(0,T;D(-\Delta) \cap W^{1/5,2}(0,T;H_i) \subset\subset L^2(0,T;V_i),$$

For $R > 0$, and $i = 1,2$, we define the sets

$$B_R^i = \left\{U \in L^2(0,T;D(-\Delta) \cap W^{1/3,2}(0,T;H_i) : |U|^2_{L^2(0,T;D(-\Delta))} + |U|^2_{W^{1/3,2}(0,T;H_i)} \leq R^2\right\},$$

which are compact in $L^2(0,T;V_i)$. Thanks to the Chebyshev inequality, the uniform estimates (4.27), and Lemma 4.1, we obtain:

$$\begin{aligned} \mu^n_{\mathbf{v}_i}((B_R^i)^C) &= \mathbb{P}\left(\|\mathbf{v}_i^n\|^2_{L^2(0,T;D(-\Delta)} + \|\mathbf{v}_i^n\|^2_{W^{1/5,2}(0,T;H_1)} \geq R^2\right) \\ &\leq \mathbb{P}\Big(\|\mathbf{v}_i^n\|^2_{L^2(0,T;D(-\Delta))} \geq \frac{R^2}{2}\Big) + \mathbb{P}\Big(\|\mathbf{v}_i^n\|^2_{W^{1/5,2}(0,T;H_1)} \geq \frac{R^2}{2}\Big) \\ &\leq \frac{2}{R^2}\mathbb{E}\Big(\int_0^T |\Delta \mathbf{v}_i^n|^2\,dt + |\mathbf{v}_i^n|^2_{W^{1/5,2}(0,T;H_1)}\Big) \\ &\leq \frac{C}{R^2}, \end{aligned} \tag{4.33}$$

Similarly, due to Chebyshev inequality, using (4.27b), (4.27d), we estimate for $i = 1, 2$

$$\begin{aligned}
\mu^n_{h_i}((B^i_R)^C) &= \mathbb{P}\left(\|h^n_i\|^2_{L^2(0,T;D(-\Delta)} + \|h^n_i\|^2_{W^{1/5,2}(0,T;H_2)} \ge R^2\right) \\
&\le \mathbb{P}\bigg(\|h^n_i\|^2_{L^2(0,T;D(-\Delta))} \ge \frac{R^2}{2}\bigg) + \mathbb{P}\bigg(\|h^n_i\|^2_{W^{1/5,2}(0,T;H_2)} \ge \frac{R^2}{2}\bigg) \\
&\le \frac{2}{R^2}\mathbb{E}\bigg(\int_0^T |\Delta h^n_i|^2\,dt + |\mathbf{v}^n_i|^2_{W^{1/5,2}(0,T;H_2)}\bigg) \\
&\le \frac{C}{R^2},
\end{aligned} \tag{4.34}$$

For $p > 2$ we choose α such that $\alpha p > 1$, as in Lemma 6.2, we infer the following compact embeddings: for $\mathcal{E}_0 = H_i, \mathcal{E} = V_i'$, $i = 1, 2$

$$W^{1,2}(0,T;H_i) \subset\subset \mathcal{C}([0,T];V_i'), \text{ and } \quad W^{\alpha,p}(0,T;H_i) \subset\subset \mathcal{C}([0,T];V_i') \text{ for } i = 1,2.$$

Let $B^{3,1}_R$ and $B^{3,2}_R$ be the balls of radius R in $W^{1,2}(0,T;H_1)$ and $W^{\alpha,p}(0,T;H_1)$, respectively. It follows that

$$B^3_R = B^{3,1}_R + B^{3,2}_R \text{ is compact in } \mathcal{C}([0,T];V_1').$$

Similarly, Let $B^{4,1}_R$ and $B^{4,2}_R$ be the balls of radius R in $W^{1,2}(0,T;H_2)$ and $W^{\alpha,p}(0,T;H_2)$, respectively. It also follows that

$$B^4_R = B^{4,1}_R + B^{4,2}_R \text{ is compact in } \mathcal{C}([0,T];(V_2)').$$

Observe that for $i = 1, 2$

$$\{(\mathbf{v}^n_i) \subset B^3_R\} \supset \left\{\mathbf{v}^n_i - \int_0^t \sum_k P_n\sigma_i(U^n)e_k dW^k_i \in B^{3,1}_R\right\} \cap \left\{\int_0^t \sum_k P_n\sigma_i(U^n)e_k dW^k_i \in B^{3,2}_R\right\},$$

and for $(i,j) = (1,3), (2,4)$

$$\{(h^n_i) \subset B^4_R\} \supset \left\{h^n_i - \int_0^t \sum_k P_n\sigma_j(\mathbf{v}^n, h^n)e_k dW^k_j \in B^{4,1}_R\right\} \cap \left\{\int_0^t \sum_k P_n\sigma_j(U^n)e_k dW^k_j \in B^{4,2}_R\right\}.$$

By the Chebyshev inequality, the uniform bound, and the uniform estimates in (4.27a)-(4.27d), we infer for $i = 1, 2$ that

$$\begin{aligned}
\mu^n_{\mathbf{v}_i}((B^3_R)^C) &\le \mathbb{P}\left(\left|\mathbf{v}^n_i - \int_0^t \sum_{k=1}^\infty P_n\sigma_i(U^n)e_k dW^k_i\right|^2_{W^{1,2}(0,T;H_1)} \ge R^2\right) \\
&\quad + \mathbb{P}\left(\left|\int_0^t \sum_{k=1}^\infty P_n\sigma_i(U^n)e_k dW^k_i\right|^p_{W^{\alpha,p}(0,T;H_1)} \ge R^p\right) \\
&\le \frac{C}{R^2},
\end{aligned} \tag{4.35}$$

and for $(i,j) = (1,3),(2,4)$

$$\begin{aligned}
\mu^n_{h_i}((B^4_R)^C) &\le \mathbb{P}\left(\left| h^n_i - \int_0^t \sum_{k=1}^{\infty} P_n\sigma_j(U^n)e_k dW^k_j \right|^2_{W^{1,2}(0,T;H_2)} \ge R^2\right) \\
&\quad + \mathbb{P}\left(\left| \int_0^t \sum_{k=1}^{\infty} P_n\sigma_j(U^n)e_k dW^k_j \right|^p_{W^{\alpha,p}(0,T;H_2)} \ge R^p\right) \\
&\le \frac{C}{R^2}.
\end{aligned} \tag{4.36}$$

Since $B^1_R \cap B^3_R$ is compact in $L^2(0,T;V_1)\cap\mathcal{C}([0,T];V_1')$ and $B^2_R\cap B^4_R$ is compact in $L^2(0,T;V_2)\cap\mathcal{C}([0,T];V_2')$, from (4.33)–(4.36), we have for every $R>0$:

$$\mu^n_{\mathbf{v}_i}((B^1_R\cap B^3_R)^C) \le \mu^n_{\mathbf{v}_i}((B^1_R)^C) + \mu^n_{\mathbf{v}_i}((B^3_R)^C) \le \frac{C}{R^2}, \tag{4.37}$$

$$\text{and} \qquad \mu^n_{h_i}((B^2_R\cap B^4_R)^C) \le \mu^n_{h_i}((B^2_R)^C) + \mu^n_{h_i}((B^4_R)^C) \le \frac{C}{R^2}. \tag{4.38}$$

Then we take $A^{\mathbf{v}_i}_\epsilon := B^1_{\sqrt{8C/\epsilon}} \cap B^3_{\sqrt{8C/\epsilon}}$ and $A^{h_i}_\epsilon := B^2_{\sqrt{8C/\epsilon}} \cap B^4_{\sqrt{8C/\epsilon}}$. This gives us for any $\epsilon>0$, for every n, and $i=1,2$

$$\mu^n_v(A^{\mathbf{v}_i}_\epsilon) \ge 1-\frac{\epsilon}{8} \qquad \text{and} \qquad \mu^n_h(A^{h_i}_\epsilon) \ge 1-\frac{\epsilon}{8}. \tag{4.39}$$

For the constant sequences $\{\mu^n_{W_i}\}$, $i=1,2,3,4$, which are weakly compact, we see that they are also tight by Proposition 6.1, and hence there exist compact sets $\tilde{A}^i_\epsilon \subset \mathcal{C}([0,T];\mathfrak{U}_0)$ such that for each n,

$$\mu^n_{W_i}\left(\tilde{A}^i_\epsilon\right) \ge 1-\frac{\epsilon}{8}. \tag{4.40}$$

Now, for any $\epsilon>0$, we define $\mathcal{K}_\epsilon := A^{\mathbf{v}_1}_\epsilon \times A^{\mathbf{v}_2}_\epsilon \times A^{h_1}_\epsilon \times A^{h_2}_\epsilon \times \prod_{i=1}^4 \tilde{A}^i_\epsilon$ which are compact in $\mathcal{X}$. We see that for every n

$$\mu^n(\mathcal{K}_\epsilon) \ge \left(1-\frac{\epsilon}{8}\right)^8 \ge 1-\epsilon. \tag{4.41}$$

This proves that the sequence $\{\mu^n\}$ is tight in $\mathcal{X}$ and by Prohorod Theorem (Proposition 6.1), μ^n is weakly compact on $\mathcal{X}$. □

4.4 Passage to the Limit

Suppose μ_0 is a probaility measure on $H_1\times H_1\times H_2\times H_2$ satisfying

$$\int_{H_1\times H_1\times H_2\times H_2} |\mathbf{u}_0|^2\,\mu_0(d\mu) < \infty \tag{4.42}$$

where $\mathbf{u}_0 = (\mathbf{u}^0_1,\mathbf{u}^0_2,\mathbf{u}^0_1)$; by the previous lemma, we have shown that the sequence of measures $\{\mu^n\}_{n\ge1}$ associated the Galerkin sequence $\{\mathbf{v}^n_1,\mathbf{v}^n_2,h^n_1,h^n_2,W_1,W_2,W_3,W_4\}$ are weakly compact in χ. This implies the existence of convergent subsequence μ^{n_j} and to simplify writing, we write j for n_j. We now apply the Skorohod embedding theorem to infer the following theorem.

Theorem 4.3. *Given s stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$ and μ_0 is a probability measure on $H_1 \times H_2$ satisfying (4.42). Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of $\mathcal{X}$-valued random variables $(\tilde{\mathbf{v}}_1^j, \tilde{\mathbf{v}}_1^j, \tilde{h}_1^j, \tilde{h}_2^j, \tilde{W}_1^j, \tilde{W}_2^j, \tilde{W}_3^j, \tilde{W}_4^j)$, such that*

(i) $(\tilde{\mathbf{v}}_1^j, \tilde{\mathbf{v}}_1^j, \tilde{h}_1^j, \tilde{h}_2^j, \tilde{W}_1^j, \tilde{W}_2^j, \tilde{W}_3^j, \tilde{W}_4^j)$ has the same law with $(\mathbf{v}^j, h^j, W_1, W_2)$

(ii) $(\tilde{\mathbf{v}}_1^j, \tilde{\mathbf{v}}_1^j, \tilde{h}_1^j, \tilde{h}_2^j, \tilde{W}_1^j, \tilde{W}_2^j, \tilde{W}_3^j, \tilde{W}_4^j)$ converges almost surely in the topology of $\mathcal{X}$ to an element $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tilde{W}_1, \tilde{W}_2 \tilde{W}_3, \tilde{W}_4)$ i.e.

$$\tilde{\mathbf{v}}_i^j \to \tilde{\mathbf{v}}_i \ \text{in } L^2(0,T;V_1) \cap \mathcal{C}([0,T];H_1) \ i = 1,2 \ a.s, \tag{4.43a}$$

$$\tilde{h}_i^j \to \tilde{h}_i \ \text{in } L^2(0,T;V_2) \cap \mathcal{C}([0,T];H_2) \ i = 1,2 \ a.s, \tag{4.43b}$$

$$\tilde{W}_i^j \to \tilde{W}_i \ \text{in } \mathcal{C}([0,T];\mathfrak{U}_0), \ i = 1,2,3,4 \ a.s. \tag{4.43c}$$

(iii) Each $\tilde{W}_i^j$, $i = 1,2,3,4$ is a cylindrical Wiener process relative to the filtration $\tilde{F}_t^j$ given by

$$\tilde{\mathcal{F}}_t^j := \sigma(\tilde{W}_1^j(s), \tilde{W}_2^j(s), \tilde{\mathbf{v}}^j(s), \tilde{h}^j(s), s \le t).$$

(iv) Each $(\tilde{\mathbf{v}}_1^j, \tilde{\mathbf{v}}_1^j, \tilde{h}_1^j, \tilde{h}_2^j, \tilde{W}_1^j, \tilde{W}_2^j, \tilde{W}_3^j, \tilde{W}_4^j)$ satisfies:

$$\begin{aligned} d\tilde{\mathbf{v}}_1^j - \nu_1 \Delta \tilde{\mathbf{v}}_1^j dt + P_n[\theta\left(\|\tilde{\mathbf{v}}_1^j\|^2 + \|\tilde{\mathbf{v}}_2^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2\right)(\tilde{\mathbf{v}}_1^j \cdot \Delta)\tilde{\mathbf{v}}_1^j \\ + g\nabla \tilde{h_1}^j + f\mathbf{k} \times \tilde{\mathbf{v}}_1^j]dt = P_n F dt + \sum_{k=1}^{\infty} P_n \sigma_1(\tilde{U}^j) e_k d\tilde{W}_1^{kj}, \end{aligned} \tag{4.44}$$

$$\begin{aligned} d\tilde{\mathbf{v}}_2^j - \nu_2 \Delta \tilde{\mathbf{v}}_2^j dt + P_n[\theta\left(\|\tilde{\mathbf{v}}_1^j\|^2 + \|\tilde{\mathbf{v}}_2^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2\right)(\tilde{\mathbf{v}}_2^j \cdot \Delta)\tilde{\mathbf{v}}_2^j \\ + g\nabla \tilde{h_2}^j + f\mathbf{k} \times \tilde{\mathbf{v}}_2^j]dt = P_n G dt + \sum_{k=1}^{\infty} P_n \sigma_2(\tilde{U}^j) e_k d\tilde{W}_2^{k,j}, \end{aligned} \tag{4.45}$$

$$\begin{aligned} d\tilde{h}_1^j + P_n[\theta\left(\|\tilde{\mathbf{v}}_1^j\|^2 + \|\tilde{\mathbf{v}}_2^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2\right)\Delta(\tilde{h}_1^j \tilde{\mathbf{v}}_1^j) - \delta_1 \Delta \tilde{h}_1^j]dt \\ = \sum_{k=1}^{\infty} P_n \sigma_3(\tilde{U}^j) d\tilde{W}_3^{kj}, \end{aligned} \tag{4.46}$$

$$\begin{aligned} d\tilde{h}_2^j + P_n[\theta\left(\|\tilde{\mathbf{v}}_1^j\|^2 + \|\tilde{\mathbf{v}}_2^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2\right)\Delta(\tilde{h}_2^j \tilde{\mathbf{v}}_2^j) - \delta_2 \Delta \tilde{h}_2^j]dt \\ = \sum_{k=1}^{\infty} P_n \sigma_4(\tilde{U}^j) d\tilde{W}_4^{kj}, \end{aligned} \tag{4.47}$$

$$\tilde{\mathbf{v}}_i^j(0) = P_j(\mathbf{v}_i^0) = \mathbf{v}_i^{0j}, \ \tilde{h}_i^j(0) = P_j(h_i^0) = h_i^{0j} > 0, \ i = 1,2. \tag{4.48}$$

Let $\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \ge 0}, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4)$, where

$$\tilde{\mathcal{F}}_t = \sigma(\tilde{W}_1(s), \tilde{W}_2(s), \tilde{W}_3(s), \tilde{W}_4(s), \tilde{\mathbf{v}}_1(s), \tilde{\mathbf{v}}_2(s), \tilde{h}_1(s), , \tilde{h}_2(s) s \le t).$$

Then $(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1 \tilde{h}_2)$ is a global martingale solution in the sense of Definition 2.2.

Proof: The proofs of (i) and (ii) can be deduced from the Skorohod embedding theorem. In order to prove (iii), it suffices to show that for $i = 1,2$:

$(\text{iii})_A$: $\tilde{W}_i^j(t)$ is measurable with $\tilde{\mathcal{F}}_t^j$, .

$(\text{iii})_B$: $\tilde{W}_i^j(t) - \tilde{W}_i^j(s)$ is independent of $\tilde{\mathcal{F}}_t^j$.

Proof of (iii)$_A$: is trivial from the definition of $\tilde{\mathcal{F}}_t^j$.

Proof of (iii)$_B$: Since $\mathbf{v}_1^j(s), \mathbf{v}_2^j(s), h_1^j(s), h_2^j(s)$ and $W_1^j(s), W_2^j(s)$ are adapted to $\mathcal{F}_s$, we obtain

$$\sigma\{\mathbf{v}_1^j(r), \mathbf{v}_2^j(r), h_1^j(r), h_2^j(r), W_1^j(r), W_2^j(r), W_3^j(r), W_4^j(r), r \le s\} \subset \mathcal{F}_s,$$

since $W_1(t) - W_1(s), W_2(t) - W_2(s), W_3(t) - W_3(s), W_4(t) - W_4(s)$ are independent of $\mathcal{F}_s$. Hence $W_1(t) - W_1(s), W_2(t) - W_2(s), W_3(t) - W_3(s), W_4(t) - W_4(s)$ are independent of

$$\sigma\{\mathbf{v}_1^j(r), \mathbf{v}_2^j(r), h_1^j(r), h_2^j(r), W_1^j(r), W_2^j(r), W_3^j(r), W_4^j(r), r \le s\}.$$

By (i), it implies that $\tilde{W}_1(t)-\tilde{W}_1(s), \tilde{W}_2(t)-\tilde{W}_2(s), \tilde{W}_3(t)-\tilde{W}_3(s), \tilde{W}_4(t)-\tilde{W}_4(s)$ are independent of

$$\sigma\{\tilde{\mathbf{v}}_1^j(r), \tilde{\mathbf{v}}_2^j(r), \tilde{h}_1^j(r), \tilde{h}_2^j(r), \tilde{W}_1^j(r), \tilde{W}_2^j(r), \tilde{W}_3^j(r), \tilde{W}_4^j(r), r \le s\}$$

as desired.

To show (iv), it is easy to see all the statistical estimates for $\mathbf{v}_i^n$ and h_i^n, $i = 1, 2$ are valid for $\tilde{\mathbf{v}}_i^j$ and $\tilde{h}_i^j$. Hence $(\tilde{\mathbf{v}}_i^j)$, $i = 1, 2$ belong to a bounded set of $L^2(\tilde{\Omega}; L^\infty(0, T; V_1) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))$, there are $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ in this intersection space such that

$$\tilde{\mathbf{v}}_i^j \rightharpoonup \tilde{\mathbf{v}}_i \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_1)), \tag{4.49}$$

$$\text{and } \tilde{\mathbf{v}}_i^j \rightharpoonup \tilde{\mathbf{v}}_i \text{ weakly in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{4.50}$$

Similarly, there exist an $\tilde{h}_1, \tilde{h}_2$ in $L^2(\tilde{\Omega}; L^\infty(0, T; V_2) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))$ such that

$$\tilde{h}_i^j \rightharpoonup \tilde{h}_i \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_2)), \tag{4.51}$$

$$\text{weakly and } \tilde{h}_i^j \rightharpoonup \tilde{h}_i \text{ in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{4.52}$$

Combining with (4.43a), (4.43b), by applying the Vitali convergence theorem, we obtain

Since $\mathbf{v}_i^{n_j} \to \mathbf{v}_i$ in $\mathcal{C}([0, T], H_1)$ a.s., we can deduce the existence of sets $\Omega_i, i = 1, 2$ such that $\tilde{\mathbb{P}}(\Omega_i) = 1$ and on these sets, the below convergence hold

$$\lim_{j\to\infty} \langle \mathbf{v}_i^{n_j}(0) - \tilde{\mathbf{v}}_i(0), \psi \rangle_{L^2} = 0, \ i = 1, 2. \tag{4.53}$$

Similarly, there exist two sets $\Omega_j \subset \tilde{\Omega}, i = 3, 4$ of full measure such that

$$\lim_{j\to\infty} \langle h_i^{n_j}(0) - \tilde{h}_i(0), \psi \rangle_{L^2} = 0, \ i = 3, 4 \tag{4.54}$$

Set $\bar{\Omega} = \tilde{\Omega} \backslash \bigcup_{i=1}^4 \Omega$ and we now show that the convergence of the other terms hold in $L^2(\bar{\Omega} \times [0, T]$. Due to the strong convergence in (4.43a) and the estimates for $\mathbf{v}_i^{n_j}, i = 1, 2$ by using the Vitali Convergence Theorem, we imply that $\mathbf{v}_i^{n_j}$ converge to $\tilde{\mathbf{v}}_i$ in $L^2(\bar{\Omega}, L^2(0, T, V_1))$ and $h_i^{n_j}$ converges to $\tilde{h}_i$ in $L^2(\bar{\Omega}, L^2(0, T, V_2))$, for $i = 1, 2$. Hence, by extracting some subsequences, we deduce that $\mathbf{v}_i^{n_j} \to \tilde{\mathbf{v}}_i$ a.e and $\tilde{\mathbb{P}}$- a.s.in V_1 and $h_i^{n_j} \to \tilde{h}_i$ a.e and $\tilde{\mathbb{P}}$-a.s. in V_2, that is, there exist $\Omega_T^i \subset \bar{\Omega} \times [0, T]$, for $i = 1, 2, 3, 4$ with full measure such that $\forall (\omega, t) \in \Omega_T^1, \Omega_T^2$

$$\lim_{j\to\infty} \|\mathbf{v}_i^{n_j} - \tilde{\mathbf{v}}_i\|_{V_1} = 0 \tag{4.55}$$

Analogously, $\forall (\omega, t) \in \Omega_T^3, \Omega_T^4$

$$\lim_{j\to\infty} \|h_i^{n_j} - \tilde{h}_i\|_{V_2} = 0 \tag{4.56}$$

From which, we imply that

$$\lim_{j\to\infty}\langle \mathbf{v}_i^{n_j}(t)-\tilde{\mathbf{v}}_i(t),\psi\rangle = 0, \text{ and } \lim_{j\to\infty}\langle h_i^{n_j}(t)-\tilde{h}_i(t),\psi\rangle = 0 \tag{4.57}$$ 70

The convergence for the linear terms are straight forward. Indeed, due to (4.55) and (4.92), there exist sets $\Omega_T^i, i=5,...,15$ of full measure w.r.t $d\tilde{\mathbb{P}}\otimes dt$ and some extracted subsequences still denoted by $\mathbf{v}_i^{n_j}, h^{n_j}$ such that for all $(\omega,t)\in\Omega_T^i, i=5,...,14$, the following convergences hold as
71 $j\to\infty$,

$$\left|\int_0^t \nu_i\langle\Delta(\tilde{\mathbf{v}}_i^j-\tilde{\mathbf{v}}_i),\psi\rangle ds\right| \preceq \|\psi\|\left(\int_0^T\|\tilde{\mathbf{v}}_i^j-\tilde{\mathbf{v}}_i\|^2 ds\right)^{\frac{1}{2}}\to 0, \tag{4.58a}$$

$$\left|\int_0^t g\langle\nabla(\tilde{h}_i^j-\tilde{h}_i),\psi\rangle ds\right| \preceq \|\psi\|\left(\int_0^T\|\tilde{h}_i^j-\tilde{h}_i\|^2 ds\right)^{\frac{1}{2}}\to 0, \tag{4.58b}$$

$$\left|\int_0^t \langle g\frac{\rho_2}{\rho_1}\nabla(\tilde{h}_2^j-\tilde{h}_2),\mathbf{v}\rangle ds\right| \le C\sup_{0\le r\le T}\|\mathbf{v}(r)\|\left(\int_0^T\|\tilde{h}_i^j-\tilde{h}_i\|^2 ds\right)^{\frac{1}{2}}\to 0, \tag{4.58c}$$

$$\left|\int_0^t \delta_i\langle\Delta(\tilde{h}_i^j-\tilde{h}_i),\psi\rangle ds\right| \preceq \|\psi\|\left(\int_0^T\|\tilde{h}_i^j-\tilde{h}_i\|^2 ds\right)^{\frac{1}{2}}\to 0, \tag{4.58d}$$

$$\left|\int_0^t \langle f\mathbf{k}\times(\tilde{\mathbf{v}}_i^j-\tilde{\mathbf{v}}_i),\psi\rangle ds\right| \preceq \|\psi\|\left(\int_0^T\|\tilde{\mathbf{v}}_i^j-\tilde{\mathbf{v}}_i\|^2 ds\right)^{\frac{1}{2}}\to 0. \tag{4.58e}$$

71 Furthermore, in virtue of Lemma 4.1, the following estimates can be easily obtained

$$\mathbb{E}\int_0^t\left|\int_0^t \nu_i\langle\Delta\mathbf{v}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\mathbb{E}\Big(\sup_{0\le t\le T}\|\mathbf{v}_i^{n_j}\|^2\Big)\le\kappa. \tag{4.59a}$$

$$\mathbb{E}\int_0^T\left|\int_0^t g\langle\nabla(\tilde{h}_i^{n_j}),\psi\rangle ds\right|^2 dt \preceq \|\psi\|\mathbb{E}\Big(\sup_{0\le t\le T}\|h_i^{n_j}\|^2\Big)\le\kappa. \tag{4.59b}$$

$$\mathbb{E}\int_0^T\left|\int_0^t g\frac{\rho_2}{\rho_1}\langle\nabla\tilde{h}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\mathbb{E}\Big(\sup_{0\le t\le T}\|h_2^{n_j}\|^2\Big)\le\kappa. \tag{4.59c}$$

$$\mathbb{E}\int_0^T\left|\int_0^t \delta_i\langle\Delta\tilde{h}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\mathbb{E}\Big(\sup_{0\le t\le T}\|h_i^{n_j}\|^2\Big)\le\kappa. \tag{4.59d}$$

$$\mathbb{E}\int_0^T\left|\int_0^t \langle f\mathbf{k}\times\mathbf{v}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\mathbb{E}\Big(\sup_{0\le t\le T}\|\mathbf{v}_i^{n_j}\|^2\Big)\le\kappa. \tag{4.59e}$$

Collecting all above estimates and by Lebesgue Dominated Convergence Theorem, we conclude
72 that

$$\lim_{j\to\infty}\|\mu_i\int_0^t\langle\Delta\mathbf{v}_i^{n_j}-\Delta\tilde{\mathbf{v}}_i,\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T]}=0 \tag{4.60a}$$

$$\lim_{j\to\infty}\|g\int_0^t\frac{\rho_2}{\rho_1}\langle\nabla h_2^{n_j}-\nabla\tilde{h}_2,\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T]}=0 \tag{4.60b}$$

$$\lim_{j\to\infty}\|g\int_0^t\langle\nabla h_i^{n_j}-\nabla\tilde{h}_i,\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T]}=0 \tag{4.60c}$$

$$\lim_{j\to\infty}\|\delta_i\int_0^t\langle\Delta h_i^{n_j}-\Delta\tilde{h}_i,\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T]}=0 \tag{4.60d}$$

$$\lim_{j\to\infty}\|\int_0^t \langle f\mathbf{k}\times(\mathbf{v}_i^{n_j}-\tilde{\mathbf{v}}_i),\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T]}=0 \tag{4.60e}$$

Along subsequences, there exist subsets $\Omega_T^i, i=15,..,23$ with full measure such that on these sets, the convergence in (4.95) hold pointwise.
Now for the nonlinear terms, we first denote $\theta(\|\mathbf{v}_1^{n_j}\|^2+\|\mathbf{v}_2^{n_j}\|^2+\|h_1^{n_j}\|^2+\|h_2^{n_j}\|^2)$ by $\theta(\|\mathbf{v}_j\|)$ and $\theta(\|\tilde{\mathbf{v}}_1\|^2+\|\tilde{\mathbf{v}}_2\|^2+\|\tilde{h}_1\|^2+\|\tilde{h}_2\|^2)$ by $\theta(\|\tilde{\mathbf{v}}\|)$ to simplify the exposition.
Next, for $i=1,2$, we have:

$$\begin{aligned}
&\Big|\int_0^t \langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{j}]-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds\Big|\\
&\le \int_0^t |\langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}]-P_n[\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|+\int_0^t|\langle Q_n\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|.\\
&\le \int_0^t |\langle \theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds|+\int_0^t|\langle Q_n\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|.\\
&=: I_1+I_2.
\end{aligned}$$

Thanks to (4.92) and (4.93), we see for all $(x,\omega,t)\in\mathcal{M}\times\Omega_T^3$ and $(x,\omega,t)\in\mathcal{M}\times\Omega_T^4$

$$\lim_{j\to\infty}\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}=\theta(\|\tilde{\mathbf{v}}_i\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i. \tag{4.61}$$ 73

74 Next, due to Lemma 4.1 and Hölder's inequality, we are able to derive the following bounds

$$\int_0^t |\langle\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}^{n_j},\psi\rangle|ds\le C\int_0^t|\theta(\|\mathbf{v}_j\|)^{\frac12}\nabla\cdot\mathbf{v}_i^{n_j}|_{L^\infty}|\theta(\|\mathbf{v}_j\|)^{\frac12}\mathbf{v}_i^{n_j}|_{L^2}\|\psi\|\preceq\|\psi\|. \tag{4.62a}$$

And

$$\mathbb{E}\int_0^T\Big|\int_0^t|\langle\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}^{n_j},\psi\rangle ds\Big|^2dt\preceq\|\psi\|^2\mathbb{E}\Big(\sup_{0\le t\le T}\|\mathbf{v}_i^{n_j}\|^2\Big)\le C. \tag{4.62b}$$

I_2 is evaluated in the same way as I_1. More precisely, we infer from Lemma 4.1 and Hôlder's inequality that

$$\begin{aligned}
I_2&:=|\langle Q_{n_j}\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot)\tilde{\mathbf{v}}_i,\psi\rangle|\le|Q_{n_j}\theta(\|\tilde{\mathbf{v}}\|)^{\frac12}\tilde{\mathbf{v}}_i|_{L^2}|\theta(\|\tilde{\mathbf{v}}\|)^{\frac12}\nabla\tilde{\mathbf{v}}_i|_{L^\infty}^{\frac12}ds\\
&\le\|\psi\|\frac{1}{\lambda_{n_j}^{\frac12}}|Q_{n_j}\theta(\|\tilde{\mathbf{v}}\|)^{\frac12}\tilde{\mathbf{v}}_i|_{H^1}|\theta(\|\tilde{\mathbf{v}}\|)^{\frac12}\nabla\tilde{\mathbf{v}}_i|_{L^\infty}^{\frac12}ds\preceq\|\psi\|\frac{1}{\lambda_{n_j}^{\frac12}}\to0\text{ as }j\to\infty.
\end{aligned} \tag{4.63}$$ 75

76 We can deduce the similar estimates as in (4.98)

$$\int_0^t|\langle Q_{n_j}\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot)\tilde{\mathbf{v}}_i,\psi\rangle|ds\preceq\|\psi\| \tag{4.64a}$$

$$\mathbb{E}\int_0^T\Big|\int_0^t|\langle Q_{n_j}\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds\Big|^2dt\preceq\mathbb{E}\Big(\sup_{0\le t\le T}\|\tilde{\mathbf{v}}_i\|^2\Big)\le C. \tag{4.64b}$$

From (4.61) to (4.101) and with the Lebesgue Dominated Convergence Theorem, we imply that

$$\lim_{j\to\infty}\|\int_0^t\langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{j}]-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.65}$$ 77

By extracting subsequences, we imply there exist Ω_T^{24} and Ω_T^{25} such that for all $(\omega,t)\in\Omega_T^{24}$ so that

$$\lim_{j\to\infty}\int_0^t\langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_1^{n_j}\cdot\nabla)\mathbf{v}_1^{j}]-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1,\psi\rangle ds=0 \tag{4.66}$$ 78

And for all $(\omega, t) \in \Omega_T^{25}$, the below convergence holds

$$\lim_{j\to\infty}\int_0^t \langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_2^{n_j}\cdot\nabla)\mathbf{v}_2^j] - \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_2\cdot\nabla)\tilde{\mathbf{v}}_2, \psi\rangle ds = 0 \tag{4.67}$$

It is not difficult to deduce the following convergence

$$\lim_{j\to\infty}\|\int_0^t \langle P_n[\theta(\|\tilde{\mathbf{u}}^j\|)\nabla(\tilde{h}_i^j\tilde{\mathbf{v}}_i^{n_j})] - \theta(\|\tilde{\mathbf{u}}\|)\nabla(\tilde{h}_i\tilde{\mathbf{v}}_i), \psi\rangle ds\|_{L^2(\bar{\Omega}\times[0,T])} = 0. \tag{4.68}$$

We can extract two sets Ω_T^{26} and Ω_T^{27} and extracted subsequences still denoted by $\mathbf{v}_1^{n_j}, h_1^{n_j}$ and $\mathbf{v}_2^{n_j}, h_2^{n_j}$ such that on these set, the following convergence hold respectively

$$\lim_{j\to\infty}\int_0^t \langle P_n[\theta(\|\tilde{\mathbf{u}}^j\|)\nabla(\tilde{h}_i^j\tilde{\mathbf{v}}_i^{n_j})] - \theta(\|\tilde{\mathbf{u}}\|)\nabla(\tilde{h}_i\tilde{\mathbf{v}}_i), \psi\rangle ds = 0. \tag{4.69a}$$

We address the stochastic term by using Lemma **??**. We first simplify the expositions by introducing $U^{n_j} = (\tilde{\mathbf{v}}_1^{n_j}, \tilde{\mathbf{v}}_2^{n_j}, \tilde{h}_1^{n_j}, \tilde{h}_2^{n_j})$ and $U = (\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2)$.
From (**??**), we know that $\tilde{W}_i^{n_j} \to \tilde{W}_i, \forall i = 1,2,3,4$ in probability in $\mathcal{C}(0,T;\mathfrak{U}_0)$ and thus it suffices to show that $P_{n_j}\sigma_i(U^{n_j}) \to \sigma_i(U)$ in $L^2(0,T;L_2(\mathfrak{U},V))$ except on a set of measure zero of $\bar{\Omega}$ and hence in probability. We utilize the Poincaré inequality, the hypothesis (**??**), (2.4) and (**??**) and we estimate:

$$\begin{aligned}\|P_{n_j}\sigma_i(U^{n_j}) - \sigma_i(U)\|^2_{L_2(\mathfrak{U},V)} &\le \|P_{n_j}\sigma(U^{n_j}) - P_{n_j}\sigma(U)\|^2_{L_2(\mathfrak{U},V)} + \|Q_{n_j}\sigma_i(U)\|^2_{L_2(\mathfrak{U},V)} \\ &\preceq \|U^{n_j} - U\|_V^2 + \frac{1}{\lambda_{n_j}}(1+\|U\|^2) \longrightarrow 0 \text{ as } n_j \to \infty.\end{aligned}$$

Thus, we conclude that $\|P_j\sigma_1(U^j) - \sigma_1(U)\|_{L_2(\mathfrak{U},V_1)} \to 0, \forall(\omega,t) \in \Omega_T^0$, as $j \to \infty$.
On the other hand, we observe that due to (**??**) and (4.1),

$$\tilde{\mathbb{E}}\Big(\int_0^T P_j\|\sigma_1(\tilde{\mathbf{v}}^j)\|^2_{L_2(\mathfrak{U},V_1)}dt\Big) \le C\mathbb{E}\Big(\int_0^T (1+\|\tilde{\mathbf{v}}^j\|^2\Big) \le C. \tag{4.70}$$

With (**??**), (4.70) in hand and the Lebesgue Dominated Convergence Theorem, we infer that

$$P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j}) \to \sigma(\tilde{\mathbf{u}}) \text{ in } L^2(\bar{\Omega}; L^2([0,T], L_2(\mathfrak{U},V)). \tag{4.71}$$

This implies that the following convergence holds almost surely and in particular, it holds in probability:

$$P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j}) \to \sigma(\tilde{\mathbf{u}}) \text{ in } L^2([0,T], L_2(\mathfrak{U},V)). \tag{4.72}$$

Combining with (**??**), Lemma **??** is applied and we infer that

$$\int_0^t P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j})d\tilde{W}^{n_j} \to \int_0^t \sigma(\tilde{\mathbf{u}})d\tilde{W} \text{ in } L^2([0,T],V). \tag{4.73}$$

By making use of the Burkholder- Davis-Gundy inequality and the bound in (4.1), we can easily obtain the following estimate:

$$\begin{aligned}\mathbb{E}\Big(\Big\|\int_0^t P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j})d\tilde{W}^{n_j}\Big\|_V^2\Big) &\le \mathbb{E}\Big(\sup_{0\le t\le T}\Big\|\int_0^t P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j})d\tilde{W}^{n_j}\Big\|_V^2\Big) \\ &\le C\mathbb{E}\Big(\int_0^T \|P_{n_j}\sigma(\tilde{\mathbf{u}}^{n_j})\|^2_{L_2(\mathfrak{U},V)}dt\Big) \le C\mathbb{E}\Big(\int_0^T \|\sigma(\tilde{\mathbf{u}}^{n_j})\|^2_{L_2(\mathfrak{U},V)}dt\Big) \\ &\le C\mathbb{E}\Big(\int_0^T (1+\|\tilde{\mathbf{u}}^{n_j}\|^2\Big)dt \le C.\end{aligned} \tag{4.74}$$

By utilizing the Lebesgue Dominated Convergence Theorem one more time, we obtain that the convergence in (4.71) holds further in $L^2(\bar{\Omega}; L^2([0,T], L_2(\mathfrak{U}, V))$. Hence, by the stochastic Fubini theorem, we can extract a subsequence and we find a set of full measure $\Omega_T^6 \subset \bar{\Omega} \times [0,T]$ such that the convergence of the stochastic term holds for all $(\omega, t) \in \Omega_T^6$.

4.5 Global Pathwise Uniqueness

Now we prove that the global martingale solution for the modified system is pathwise unique.

Proposition 4.1. *Suppose that* $(\mathcal{S}, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{h}_1, \hat{h}_2)$ *and* $(\mathcal{S}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2)$ *are two global Martingale solutions of* (1.1) *relative to the same stochastic basis* $\mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W_1, W_2, W_3, W_4)$. *Pathwise uniqueness means that if we define* $\Omega_0 := \{\hat{\mathbf{v}}_1(0) = \tilde{\mathbf{v}}_1(0), \hat{\mathbf{v}}_2(0) = \tilde{\mathbf{v}}_2(0), \hat{h}_1(0) = \tilde{h}_1(0), \hat{h}_2(0) = \tilde{h}_2(0)\}$, *then* $(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{h}_1, \hat{h}_2)$ *and* $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2))$ *are indistinguishable on* Ω_0 *in the sense that*

$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{\mathbf{v}}_1(t) - \tilde{\mathbf{v}}_1(t)) = 0,\ \forall t \geq 0) = 1,$$
$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{\mathbf{v}}_2(t) - \tilde{\mathbf{v}}_2(t)) = 0,\ \forall t \geq 0) = 1,$$
$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{h}_1(t) - \tilde{h}_1) = 0,\ \forall t \geq 0) = 1,$$

And

$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{h}_2(t) - \tilde{h}_2(t)) = 0,\ \forall t \geq 0) = 1.$$

Proof: For $i = 1, 2$ we will let $\mathbf{v}_i = \hat{\mathbf{v}}_i - \tilde{\mathbf{v}}_i, h_i = \hat{h}_i - \tilde{h}_i, \bar{\mathbf{v}}_i = \mathbb{1}_{\Omega_0}\mathbf{v}_i$ and $\bar{h_i} = \mathbb{1}_{\Omega_0}h_i$.

We will also need the following stopping times

$$\tau^{(n)} := \inf_{t\geq 0}\left\{\int_0^t |\Delta\mathbf{v}_1|^2||\mathbf{v}_1||^2 + |\mathbf{v}_2|^2||\Delta\mathbf{v}_2||^2 ds \geq n\right\}. \tag{4.75}$$

Substituting $\mathbf{v}_1$ and $\mathbf{v}_2$ into (1.1) and taking the difference between these equations, we arrive at the following equations:

$$\begin{aligned} d\mathbf{v}_1 - \nu_1\Delta_1\mathbf{v}_1 dt + f\mathbf{k}\times\mathbf{v}_1 dt + g\nabla h_1 dt - g\frac{\rho_2}{\rho_1}\nabla h_2 dt - \theta(\hat{U})(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1 dt + \theta(\tilde{U})(\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1 dt \\ = \sum_{k=1}^{\infty}\sigma_1^k(\tilde{U})e_k dW_1^k - \sum_{k=1}^{\infty}\sigma_1^k(\hat{U})e_k dW_1^k, \\ \mathbf{v}(0) = \mathbf{v}_1(0) - \mathbf{v}_2(0) \end{aligned} \tag{4.76}$$

$$\begin{aligned} d\mathbf{v}_2 - \nu_1\Delta_2\mathbf{v}_2 dt + f\mathbf{k}\times\mathbf{v}_2 dt = -g\nabla h_2 dt - g\nabla h_1 dt + \theta(\hat{U})(\hat{\mathbf{v}}_2\cdot\nabla)\hat{\mathbf{v}}_2 dt - \theta(\tilde{U})(\tilde{\mathbf{v}}_2\cdot\nabla)\tilde{\mathbf{v}}_2 dt \\ = \sum_{k=1}^{\infty}\sigma_2^k(\tilde{U})e_k dW_2^k - \sum_{k=1}^{\infty}\sigma_2^k(\hat{U})e_k dW_2^k, \\ \mathbf{v}(0) = \mathbf{v}_1(0) - \mathbf{v}_2(0) \end{aligned} \tag{4.77}$$

$$dh_1 - \delta_1\Delta h_1 dt = \theta(\hat{U})\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1)dt - \theta(\tilde{U})\nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1)dt + \sum_{k=1}^{\infty}\sigma_3^k(\tilde{U})dW_2^k - \sum_{k=1}^{\infty}\sigma_3^k(\hat{U})dW_3^k. \tag{4.78}$$

$$dh_2 - \delta_1 \Delta h_2 dt = \theta(\hat{U})\nabla\cdot(\hat{h}_2\hat{\mathbf{v}}_1)dt - \theta(\tilde{U})\nabla\cdot(\tilde{h}_2\tilde{\mathbf{v}}_2)dt + \sum_{k=1}^{\infty}\sigma_4^k(\tilde{U})dW_4^k - \sum_{k=1}^{\infty}\sigma_4^k(\hat{U})dW_4^k. \tag{4.79}$$

diff2

Applying the Itō formulas in (**??**) and (**??**) with $p = 2$ to both equations (4.77) and (4.79) and adding the corresponding relations together yields

$$\begin{aligned}
&d\|\mathbf{v}_1\|^2 + 2\nu_1 |\Delta\mathbf{v}_1|^2 dt + d\|\mathbf{v}_2\|^2 + 2\nu_2 |\Delta\mathbf{v}_2|^2 dt + d\|h_1\|^2 + 2\delta_1 |\Delta h_1|^2 dt + d\|h_2\|^2 + 2\delta_2 |\Delta h_2|^2 dt \\
&- 2g\langle\nabla h_1, \Delta\mathbf{v}_1\rangle dt - 2g\frac{\rho_2}{\rho_1}\langle\nabla h_2, \Delta\mathbf{v}_1\rangle - 2g\langle\nabla h_2, \Delta\mathbf{v}_2\rangle dt - 2g\langle\nabla h_1, \Delta\mathbf{v}_2\rangle dt - 2\langle\theta(\hat{U})(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1, \Delta\mathbf{v}_1\rangle \\
&- 2\langle\theta(\tilde{U})(\tilde{\mathbf{v}}_2\cdot\nabla)\tilde{\mathbf{v}}_2, \Delta\mathbf{v}_2\rangle + 2\langle\theta(\tilde{U})(\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1, \Delta\mathbf{v}_1\rangle + 2\langle\theta(\hat{U})(\hat{\mathbf{v}}_2\cdot\nabla)\hat{\mathbf{v}}_2, \Delta\mathbf{v}_2\rangle - 2\langle\theta(\tilde{U})\nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1), \Delta h_1\rangle dt \\
&2\langle\theta(\hat{U})\nabla\cdot(\hat{h}_2\hat{\mathbf{v}}_2), \Delta h_1\rangle d - \sum_{k=1}^{\infty}||\sigma_1^k(U) - \sigma_1^k(\mathbf{v}_2, h_2)||^2 dt + \sum_{k=1}^{\infty}||\sigma_2^k(U) - \sigma_2^k(U)||^2 dt \\
&+ \sum_{k=1}^{\infty}||\sigma_3^k(U) - \sigma_3^k(\mathbf{v}_2, h_2)||^2 dt + \sum_{k=1}^{\infty}||\sigma_4^k(U) - \sigma_4^k(U)||^2 dt \\
&+ 2\sum_{k=1}^{\infty}\langle\sigma_1^k(U) - \sigma_1(U), \Delta\mathbf{v}_1\rangle dW_1^k + 2\sum_{k=1}^{\infty}\langle\sigma_2^k(U) - \sigma_2(U), \Delta\mathbf{v}_2\rangle dW_2^k + 2\sum_{k=1}^{\infty}\langle\sigma_3^k(U) - \sigma_3(U), \Delta h_1\rangle dW_3^k \\
&+ 2\sum_{k=1}^{\infty}\langle\sigma_4^k(U) - \sigma_4(U), \Delta h_2\rangle dW_1^k + 2\langle f\mathbf{k}\times\mathbf{v}_1, \Delta\mathbf{v}_1\rangle dt + \langle f\mathbf{k}\times\mathbf{v}_2, \Delta\mathbf{v}_2\rangle dt
\end{aligned} \tag{4.80}$$

ito difference

Now we fix n and stopping times τ_a, τ_b such that $0 \le \tau_a \le \tau_b \le \tau^{(n)}$, and we integrate (4.80) in time, multiply by $\mathbb{1}_{\Omega_0}$, and take the expected value to obtain

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}\|^2 + 2\nu\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}|^2 dt + \sup_{t\in[\tau_a,\tau_b]}\|\bar{h}\|^2 + 2\delta\int_{\tau_a}^{\tau_b}\left|\Delta\bar{h}\right|^2 dt\bigg) \\
&\le \mathbb{E}\Big(\|\bar{\mathbf{v}}(\tau_a)\|^2 + \|\bar{h}(\tau_a)\|^2\Big) + \sum_{i=1}^{10} J_i,
\end{aligned} \tag{4.81}$$

67

where

$$\sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}\|^2 = \sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}_1\|^2 + \sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}_2\|^2.$$

$$\sup_{t\in[\tau_a,\tau_b]}\|\bar{h}\|^2 = \sup_{t\in[\tau_a,\tau_b]}\|\bar{h_1}\|^2 + \sup_{t\in[\tau_a,\tau_b]}\|\bar{h}_2\|^2. \tag{4.82}$$

$$2\nu\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}|^2 dt = 2\nu_1\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}_1|^2 dt + 2\nu_2\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}_2|^2 dt \tag{4.83}$$

$$2\delta\int_{\tau_a}^{\tau_b}\left|\Delta\bar{h}\right|^2 dt2\delta_1\int_{\tau_a}^{\tau_b}\left|\Delta\bar{h}_1\right|^2 dt + 2\delta_2\int_{\tau_a}^{\tau_b}\left|\Delta\bar{h}_2\right|^2 dt \tag{4.84}$$

We treat all the nonlinear terms by simply using Cauchy-Swarz inequality as follows:

$$J_1 := 2g\mathbb{E}\bigg(\int_{\tau_a}^{\tau_b}|\langle\nabla\bar{h_1}, \bar{\mathbf{v}}_1\rangle|dt\bigg) \le C\mathbb{E}\bigg(\int_{\tau_a}^{\tau_b}\|\bar{h_1}\|^2 dt\bigg) + \frac{\nu_1}{10}\mathbb{E}\bigg(\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}_1|^2 dt\bigg). \tag{4.85}$$

$$J_2 := 2g\frac{\rho_2}{\rho_1}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle \nabla \bar{h}_2, \bar{\mathbf{v}}_1\rangle| dt\Big) \le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \|\bar{h}_2\|^2 dt\Big) + \frac{\nu_1}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_1|^2\, dt\Big). \tag{4.86}$$

$$J_3 := 2g\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle \nabla \bar{h}_1, \bar{\mathbf{v}}_2\rangle| dt\Big) \le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \|\bar{h}_1\|^2 dt\Big) + \frac{\nu_2}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_2|^2\, dt\Big). \tag{4.87}$$

$$J_4 := 2g\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle \nabla \bar{h}_2, \bar{\mathbf{v}}_2\rangle| dt\Big) \le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \|\bar{h}_2\|^2 dt\Big) + \frac{\nu_2}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_2|^2\, dt\Big). \tag{4.88}$$

$$J_5 := 2g\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle f\mathbf{k}\times \mathbf{v}_1, \Delta \mathbf{v}_1\rangle| dt\Big) \le \frac{1}{4}\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]} \|\bar{h}_1\|^2 dt\Big) + \frac{\nu_1}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_1|^2\, dt\Big). \tag{4.89}$$

And

$$J_6 := 2g\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle f\mathbf{k}\times \mathbf{v}_2, \Delta \mathbf{v}_2\rangle| dt\Big) \le \frac{1}{4}\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]} \|\bar{h}_1\|^2 dt\Big) + \frac{\nu_2}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_2|^2\, dt\Big). \tag{4.90}$$

Next,

$$\begin{aligned}
J_7 + J_8 &:= 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle \theta(\hat{U})(\hat{\mathbf{v}}_1\cdot \mathbf{v}_1)\hat{\mathbf{v}}_1 - \theta(\tilde{U})(\tilde{\mathbf{v}}_1\cdot \mathbf{v}_1)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle| dt\Big)\\
&\le \mathbb{E}\int_{\tau_a}^{\tau_b} |\langle (\theta(\hat{U}) - \theta(\tilde{U}))(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle| dt + \mathbb{E}\int_{\tau_a}^{\tau_b} |\langle (\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle| dt\\
&= J_{78}^1 + J_{78}^2.
\end{aligned} \tag{4.91}$$

J_{78}^1 and J_{78}^2 are majored by Hölder's inequality, Agmon's inequality, classical Sobolev interpolation; J_{78}^1 is treated as follows

$$\begin{aligned}
J_{78}^1 &\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (\|\bar{\mathbf{v}}_1\| + \|\bar{h}_1\|)\,|\hat{\mathbf{v}}_1|_{L^\infty(\mathcal{M})^2}\,|\nabla\hat{\mathbf{v}}_1|_{L^2(\mathcal{M})^2}\,|\Delta\bar{\mathbf{v}}_1|_{L^2(\mathcal{M})^2}\ dt\Big)\\
&\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (\|\bar{\mathbf{v}}_1\| + \|\bar{h}_1\|)\,|\Delta\hat{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2(\mathcal{M})^2}\,|\nabla\hat{\mathbf{v}}_1|^{\frac{3}{2}}_{L^2(\mathcal{M}^2)}\,|\Delta\bar{\mathbf{v}}_1|_{L^2(\mathcal{M})^2}\ dt\Big)\\
&\le \frac{\nu_1}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta\bar{\mathbf{v}}_1|^2\, dt\Big) + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (\|\bar{\mathbf{v}}_1\|^2 + \|\bar{h}_1\|^2)\,|\Delta\hat{\mathbf{v}}_1|^2_{L^2(\mathcal{M})^2}\,\|\hat{\mathbf{v}}_1\|^2_{H^1(\mathcal{M})^2}\, dt\Big).
\end{aligned}$$

Next we give the estimate for J_{23}^2:

$$\begin{aligned}
J_{78}^2 &\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle (\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle|\, dt\Big)\\
&\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle [(\hat{\mathbf{v}}_1 - \tilde{\mathbf{v}}_1)\cdot\nabla]\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\nabla)(\tilde{\mathbf{v}}_1 - \hat{\mathbf{v}}_1), \Delta\bar{\mathbf{v}}_1\rangle|\, dt\Big)\\
&= C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle [\bar{\mathbf{v}}_1\cdot\nabla]\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\nabla)\bar{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle|\, dt\Big)\\
&\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (|\bar{\mathbf{v}}_1|_{L^4}\,|\nabla\tilde{\mathbf{v}}_1|_{L^4} + |\mathbf{v}_2|_{L^4}\,|\nabla\bar{\mathbf{v}}_1|_{L^4})\,|\Delta\bar{\mathbf{v}}_1|_{L^2}\, dt\Big)\\
&\le C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \Big(|\bar{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2}\,|\nabla\bar{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2}\,|\nabla\tilde{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2}\,|\Delta\tilde{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2} + |\tilde{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2}\,|\Delta\tilde{\mathbf{v}}_1|^{\frac{1}{2}}_{L^2}\,|\nabla\bar{\mathbf{v}}_1|_{L^2}\Big)\,|\Delta\bar{\mathbf{v}}_1|_{L^2}\, dt\Big)\\
&\le \frac{\nu_1}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta\bar{\mathbf{v}}_1|^2_{L^2}\, dt\Big) + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \|\bar{\mathbf{v}}_1\|^2\,\Big(\|\hat{\mathbf{v}}_1\|^2\,|\Delta\tilde{\mathbf{v}}_1|^2_{L^2} + \|\tilde{\mathbf{v}}_1\|^2\,|\Delta\tilde{\mathbf{v}}_1|^2_{L^2}\Big)\, dt\Big)
\end{aligned} \tag{4.92}$$

Similarly,

$$
\begin{aligned}
J_9 + J_{10} &:= 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\langle \theta(\hat{U})(\hat{\mathbf{v}}_2 \cdot \hat{\mathbf{v}}_2)\hat{\mathbf{v}}_2 - \theta(\tilde{U})(\tilde{\mathbf{v}}_2 \cdot \tilde{\mathbf{v}}_2)\tilde{\mathbf{v}}_2, \Delta \bar{\mathbf{v}}_2 \rangle| dt \Big) \\
&\leq \frac{\nu_2}{10}\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} |\Delta \bar{\mathbf{v}}_2|_{L^2}^2 \, dt \Big) + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \|\bar{\mathbf{v}}_2\|^2 \left(\|\hat{\mathbf{v}}_2\|^2 |\Delta \tilde{\mathbf{v}}_2|_{L^2}^2 + \|\tilde{\mathbf{v}}_1\|^2 |\Delta \tilde{\mathbf{v}}_2|_{L^2}^2 \right) dt \Big)
\end{aligned} \tag{4.93}
$$

The next two deterministic terms are handled as follows:

$$
\begin{aligned}
J_{11} + J_{12} &:= 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \langle \theta(\hat{U})\nabla \cdot (\hat{h}_1 \hat{\mathbf{v}}_1) - \theta(\tilde{U})\nabla \cdot (\tilde{h}_1 \tilde{\mathbf{v}}_1), \Delta \bar{h_1} \rangle \right| dt \Big) \\
&\leq 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \theta(\hat{U})^{\frac{1}{2}} \nabla \hat{h}_1 \right|_{L^\infty(\mathcal{M})} \left| \theta(\hat{U})^{\frac{1}{2}} \hat{\mathbf{v}}_1 \right|_{L^2(\mathcal{M})} \left| \Delta \bar{h_1} \right| dt \Big) \\
&\quad + 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \theta(\tilde{U})^{\frac{1}{2}} \nabla \tilde{h}_1 \right|_{L^\infty(\mathcal{M})} \left| \theta(\tilde{\mathbf{v}}_1, h_1)^{\frac{1}{2}} \tilde{\mathbf{v}}_1 \right|_{L^2(\mathcal{M})} \left| \Delta \bar{h_1} \right| dt \Big) \\
&\leq 4\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \left| \Delta \bar{h_1} \right| dt \right) \leq \frac{\delta_1}{2}\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \left| \bar{\Delta} h_1 \right|^2 dt \right) + C.
\end{aligned} \tag{4.94}
$$

Analogously,

$$
\begin{aligned}
J_{13} + J_{14} &:= 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \langle \theta(\hat{U})\nabla \cdot (\hat{h}_2 \hat{\mathbf{v}}_2) - \theta(\tilde{U})\nabla \cdot (\tilde{h}_2 \tilde{\mathbf{v}}_2), \Delta \bar{h_2} \rangle \right| dt \Big) \\
&\leq 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \theta(\hat{U})^{\frac{1}{2}} \nabla \hat{h}_2 \right|_{L^\infty(\mathcal{M})} \left| \theta(\hat{U})^{\frac{1}{2}} \hat{\mathbf{v}}_2 \right|_{L^2(\mathcal{M})} \left| \Delta \bar{h_2} \right| dt \Big) \\
&\quad + 2\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} \left| \theta(\tilde{U})^{\frac{1}{2}} \nabla \tilde{h}_2 \right|_{L^\infty(\mathcal{M})} \left| \theta(\tilde{U})^{\frac{1}{2}} \tilde{\mathbf{v}}_2 \right|_{L^2(\mathcal{M})} \left| \Delta \bar{h_2} \right| dt \Big) \\
&\leq 4\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \left| \Delta \bar{h_2} \right| dt \right) \leq \frac{\delta_2}{2}\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \left| \bar{\Delta} h_2 \right|^2 dt \right) + C.
\end{aligned} \tag{4.95}
$$

For the next four terms, we simply use the Lipschitz assumption (2.8)

$$
J_{15}+J_{16}+J_{17}+J_{18} := \sum_{i=1}^{4} \mathbb{E}\left(\int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \mathbb{1}_{\Omega_0} \|\sigma_i(\hat{U})e_k - \sigma_i(\tilde{U})e_k\|^2 dt \right) \leq C\mathbb{E}\left(\int_{\tau_a}^{\tau_b} (\|\bar{\mathbf{v}}\|^2 + \|\bar{h}\|^2) dt \right). \tag{4.96}
$$

We obtain the estimates for the last stochastic terms by using the BDG inequality and the results are similar to (3.22) and (3.12)

$$
\begin{aligned}
J_{19} :&= 2\mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \langle \sigma_1^k(\hat{U}) - \sigma_1^k(\tilde{U}), \Delta \bar{\mathbf{v}}_1 \rangle dW_1^k \right| \Big) \\
&\leq \frac{1}{4}\mathbb{E}\left(\sup_{t \in [\tau_a, \tau_b]} \|\bar{\mathbf{v}}_1\|^2 \right) + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (\|\mathbf{v}\|^2 + \|\bar{h}\|^2) dt \Big),
\end{aligned} \tag{4.97}
$$

$$
\begin{aligned}
J_{20} :&= 2\mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \langle \sigma_2^k(\hat{U}) - \sigma_2^k(\tilde{U}), \Delta \bar{\mathbf{v}}_2 \rangle dW_1^k \right| \Big) \\
&\leq \frac{1}{4}\mathbb{E}\left(\sup_{t \in [\tau_a, \tau_b]} \|\bar{\mathbf{v}}_2\|^2 \right) + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (\|\mathbf{v}\|^2 + \|\bar{h}\|^2) dt \Big),
\end{aligned} \tag{4.98}
$$

Combining the last two term, we obtain the estimate:

$$J_{19}+J_{20}:=2\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]}\Big|\int_{\tau_a}^{\tau_b}\sum_{k=1}^{\infty}\langle\sigma_1^k(\hat{U})-\sigma_1^k(\tilde{U}),\Delta\bar{\mathbf{v}}_1\rangle dW_1^k\Big|\Big)+2\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]}\Big|\int_{\tau_a}^{\tau_b}\sum_{k=1}^{\infty}\langle\sigma_1^k(\hat{U})-\sigma_1^k(\tilde{U}),\Delta\bar{\mathbf{v}}_1\rangle dW_1^k\Big|\Big) \tag{4.99}$$

$$\leq\frac{1}{4}\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}_1\|^2\right)+C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b}(\|\mathbf{v}\|^2+\|\bar{h}\|^2)dt\Big)+\frac{1}{4}\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}_1\|^2\right) \tag{4.100}$$

Rearranging (4.81)-(4.98) and multiplying by two, we obtain

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\|\bar{\mathbf{v}}\|^2+\nu\int_{\tau_a}^{\tau_b}|\Delta\bar{\mathbf{v}}|^2\,dt+\sup_{t\in[\tau_a,\tau_b]}\|\bar{h}\|^2+\delta\int_{\tau_a}^{\tau_b}\left|\Delta\bar{h}\right|^2dt\right)\\ &\leq 2\mathbb{E}\left(\|\bar{\mathbf{v}}(\tau_a)\|^2+\|\bar{h}(\tau_a)\|^2\right)\\ &\quad+C\mathbb{E}\left(\int_{\tau_a}^{\tau_b}(\|\bar{\mathbf{v}}\|^2+\|\bar{h}\|^2)(1+|\Delta\hat{\mathbf{v}}_1|^2\|\hat{\mathbf{v}}_1\|^2+|\Delta\tilde{\mathbf{v}}_1|^2\|\tilde{\mathbf{v}}_1\|^2)+|\Delta\hat{\mathbf{v}}_2|^2\|\hat{\mathbf{v}}_2\|^2+|\Delta\tilde{\mathbf{v}}_2|^2\|\tilde{\mathbf{v}}_2\|^2)\right)dt.\end{aligned} \tag{4.101}$$

4.6 Compactness Revisited

Having established the existence of martingale solutions and pathwise uniqueness for the modified system (4.1), we may apply the Gyöngy-Krylov theorem (see [GK96]), which is the infinite dimensional extension of the Yamada-Watanabe Theorem (see [YW71]), to infer the existence of a local pathwise solution $(\mathbf{v},h)$. To do so, we return to the sequence $\{(\mathbf{v}_1^n,\mathbf{v}_2^n,h_1^n,h_2^n)\}$ of Galerkin solutions relative to the given stochastic basis $\mathcal{S}$. We argue in a similar manner to [DGHT11] by considering the collections of joint distributions $\mu_v^{m,n}:=\mu_{v1}^m\times\mu_{v2}^m\times\mu_{v1}^n\times\mu_{v2}^n$ and $\mu_h^{m,n}:=\mu_{h_1}^m\times\mu_{h_2}^m\times\mu_{h_1}^n\times\mu_{h_2}^n$. We define the extended phase spaces

$$\begin{aligned}\mathcal{X}^J&:=\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_h\times\mathcal{X}_h\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_h\times\mathcal{X}_h\times\mathcal{X}_{W_1}\times\mathcal{X}_{W_2}\times\mathcal{X}_{W_3}\times\mathcal{X}_{W_4},\\ \mathcal{X}^J_{v,h}&:=\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_{\mathbf{v}}\times\mathcal{X}_h\times\mathcal{X}_h\times\mathcal{X}_h\times\mathcal{X}_h.\end{aligned} \tag{4.102}$$

As above in (4.30) and (4.31) we let for $i=1,2$ and $j=1,2,3,4$

$$\begin{aligned}&\mu_{v_i}^n(E)=Pr(\mathbf{v}_i^n\in E)\text{ for }E\in\mathcal{X}_{\mathbf{v}},\\ &\mu_{h_i}^n(E)=Pr(h_i^n\in E)\text{ for }E\in\mathcal{X}_h,\\ &\mu_{W_j}(E)=Pr(W_j\in E)\text{ for }E\in\mathcal{X}_{W_j}.\end{aligned}$$

We then take

$$\nu^{m,n}=\mu_{v_1}^m\times\mu_{v_2}^m\times\mu_{v_1}^n\times\mu_{v_2}^n\times\mu_{h_1}^m\times\mu_{h_2}^m\times\mu_{h_1}^n\times\mu_{h_2}^n\times\mu_{W_1}\times\mu_{W_2}\times\mu_{W_3}\times\mu_{W_4}$$

Lemma 4.4. *The collection $\{\nu^{m,n}\}$ is tight and hence compact on $\mathcal{X}^J$.*

Proof: The proof follows exactly that of Lemma 4.3. We take B_R^1, $B_R^2 B_R^3$ and B_R^4 as in Lemma 4.3. We can therefore choose $A_\epsilon^{h_1} A_\epsilon^{h_2}, A_\epsilon^{v_1}, A_\epsilon^{v_2} \tilde{A}_\epsilon^1, \tilde{A}_\epsilon^2, \tilde{A}_\epsilon^3, \tilde{A}_\epsilon^4$ compact in $\mathcal{X}_h, \mathcal{X}_u, \mathcal{X}_{W_1}, \mathcal{X}_{W_2}, \mathcal{X}_{W_3}, \mathcal{X}_{W_4}$, respectively, so that

$$\mu_v^n(A_\epsilon^{v_i}) \geq 1 - \frac{\epsilon}{8},\ \mu_h^n(A_\epsilon^{h_i}) \geq 1 - \frac{\epsilon}{8}, \mu_{W_j}^n(\tilde{A}_\epsilon) - \geq 1 - \frac{\epsilon}{2}. \tag{4.103}$$

We take $\mathcal{A}_\epsilon = A_\epsilon^{v_1} \times A_\epsilon^{v_2} \times A_\epsilon^{h_1} \times A_\epsilon^{h_2} \times \tilde{A}_\epsilon^1 \times \tilde{A}_\epsilon^2 \times \tilde{A}_\epsilon^3 \times \tilde{A}_\epsilon^4$ which is compact in $\mathcal{X}^J$, and by (4.103) we see that

$$\nu^{m,n}(\mathcal{A}_\epsilon) \geq (\left(1 - \frac{\epsilon}{8}\right)^4 \left(1 - \frac{\epsilon}{2}\right)^2 \geq 1 - \epsilon,$$

which holds for all $0 < \epsilon < 1$. The proof of the lemma is complete.

□

We now suppose that $\{\nu^{m_k,n_k}\}_{k \geq 0}$ is any subsequence. By the above lemma, this sequence is tight and hence by Proposition 6.1 (Prohorov's Theorem), we may choose the subsequence k' so that $\nu^{m'_k,n'_k}$ converges weakly to an element ν'. Then by Proposition 6.2 (Skorohod's Theorem), we infer the existence of a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ upon which is defined a sequence of random elements $\left(\tilde{\mathbf{v}}_1{}^{m'_k}, \tilde{\mathbf{v}}_2{}^{m'_k}, \tilde{\tilde{\mathbf{v}}}_1{}^{n'_k}, \tilde{\tilde{\mathbf{v}}}_2{}^{n'_k}, \tilde{h_1}^{m'_k}, \tilde{h_2}^{m'_k}, \tilde{\tilde{h}}_1{}^{n'_k}, \tilde{\tilde{h}}_2{}^{n'_k}, \tilde{W}_1^{k'}, \tilde{W}_2^{k'}, \tilde{W}_3^{k'}, \tilde{W}_4^{k'}\right)$ converging a.s. in $\mathcal{X}^J$ to an element $\left(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\tilde{\mathbf{v}}}_1, \tilde{\tilde{\mathbf{v}}}_2, \tilde{h}_1 \tilde{h}_2, \tilde{\tilde{h}}_1, \tilde{\tilde{h}}_2, \tilde{W}_1, \tilde{W}_2\right)$ in such a way that $\tilde{\mathbb{P}}\left(\left(\tilde{\mathbf{v}}^{m'_k}, \tilde{\tilde{\mathbf{v}}}^{n'_k}, \tilde{h}^{m'_k}, \tilde{\tilde{h}}^{n'_k}, \tilde{W}_1^{k'}, \tilde{W}_2^{k'}\right) \in \cdot\right)$
$= \nu^{m'_k,n'_k}(\cdot)$, and

$$\tilde{\mathbb{P}}\left(\left(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\tilde{\mathbf{v}}}_1, \tilde{\tilde{\mathbf{v}}}_2, \tilde{h}, \tilde{\tilde{h}}, \tilde{W}_1, \tilde{W}_2\right) \in \cdot\right) = \nu'(\cdot). \tag{4.104}$$

Note that in particular $\mu_v^{m_k,n_k}$ and $\mu_h^{m_k,n_k}$ converge weakly to μ_v and μ_h, respectively, defined by

$$\mu_v(\cdot) := \tilde{\mathbb{P}}\left((\tilde{\mathbf{v}}, \tilde{\tilde{\mathbf{v}}}) \in \cdot\right) \quad \text{and} \quad \mu_h(\cdot) := \tilde{\mathbb{P}}\left(\left(\tilde{h}, \tilde{\tilde{h}}\right) \in \cdot\right). \tag{4.105}$$

We then infer exactly as in the preceding section that both $\left(\tilde{\mathbf{v}}, \tilde{h}\right)$ and $\left(\tilde{\tilde{\mathbf{v}}}, \tilde{\tilde{h}}\right)$ are martingale solutions over the same stochastic basis. One can easily prove that these solutions agree with each other at time $t = 0$ a.s., and hence, by uniqueness, we see that $\left(\tilde{\mathbf{v}}, \tilde{h}\right) = \left(\tilde{\tilde{\mathbf{v}}}, \tilde{\tilde{h}}\right)$ in $\mathcal{X}_{\mathbf{v}} \times \mathcal{X}_h$ a.s. In other words,

$$\nu'\left(\left\{(x_1, x_2, y_1, y_2) \in \mathcal{X}_{v,h}^J : x_1 = x_2, y_1 = y_2\right\}\right) = \tilde{\mathbb{P}}\left(\left(\tilde{\mathbf{v}}, \tilde{h}\right) = \left(\tilde{\tilde{\mathbf{v}}}, \tilde{\tilde{h}}\right) \text{ in } \mathcal{X}_{\mathbf{v}} \times \mathcal{X}_h\right) = 1. \tag{4.106}$$

This implies, by Proposition 6.3, that the original sequence $(\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n)$ defined on the initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$ converges to an element $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2) := \left(\tilde{\mathbf{v}} \tilde{\mathbf{v}}_2 \tilde{h_1}, \tilde{h_2}\right)$, in the topology of $\mathcal{X}$, i.e.for $i = 1, 2$,

$$\mathbf{v}_i^n \to \mathbf{v}_i \qquad \text{a.s. in } L^2(0,T;V_1) \cap \mathcal{C}([0,T];H_1), \tag{4.107}$$

$$h_i^n \to h_i \qquad \text{a.s. in } L^2(0,T;V_2) \cap \mathcal{C}([0,T];H_2). \tag{4.108}$$

By applying Section 4.5, we may conclude that $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$ is a global pathwise solution of (4.1).

5 Existence and Uniqueness of Solutions for the Original System

5.1 Local Martingale Solutions

Theorem 4.3 already shows that $(\mathcal{S}, \mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$ is a global Martingale solution for (4.1). Now for $M > 0$ we set

$$\begin{aligned}\tau := \inf_{t\geq 0}\Big\{ \sup_{0\leq r\leq t} (\|\mathbf{v}_1(r)\|^2 + \|\mathbf{v}_2(r)\|^2 + \|h_1(r)\|^2 + \|h_2(r)\|^2) > M + \\ \|\mathbf{v}_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_1(0)\|^2 + \|h_2(0)\|^2\Big\} \wedge T\end{aligned} \tag{5.1}$$

By the following lemma, τ is strictly positive, and we observe that for $i = 1, 2$

$$\int_0^{t\wedge\tau} \theta(\mathbf{v}_i, h_i)\nabla(h_i\mathbf{v}_i)ds = \int_0^{t\wedge\tau} \nabla(h_i\mathbf{v}_i)ds \text{ and } \int_0^{t\wedge\tau} \theta(\mathbf{v}_i, h_i)(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i ds = \int_0^{t\wedge\tau} (\mathbf{v}_i\cdot\nabla)\mathbf{v}_i ds.$$

We obtain that $(\mathcal{S}, \mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \tau)$ is a local martingale solution. The proof of Theorem 2.1 is complete.

Lemma 5.1. *The stopping time τ defined in (5.1) is strictly positive.*

Proof. We integrate (**??**) from 0 to s and take the supremum for s over $[0, \tau\wedge\epsilon]$ to obtain

$$\begin{aligned}\sup_{s\in[0,\tau\wedge\epsilon]} (\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2) + \nu_1\int_0^{\tau\wedge\epsilon}\|\mathbf{v}_1\|^2 dt + \nu_2\int_0^{\tau\wedge\epsilon}\|\mathbf{v}_2\|^2 dt + \sup_{s\in[0,\tau\wedge\epsilon]} (\|h_1\|^2 + \|h_2\|^2) + \\ \delta_1\int_0^{\tau\wedge\epsilon}\|h_1\|^2 dt + \delta_2\int_0^{\tau\wedge\epsilon}\|h_2\|^2 dt \leq \|\mathbf{v}_1(0)\|^2 + \|h_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_2(0)\|^2 \\ C\int_0^{\tau\wedge\epsilon}(1 + |F|^2 + |G|^2 + \|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)dt \\ + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_1^k(U), \Delta\mathbf{v}\rangle dW_1^k\right| + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_2^k(U), \Delta h\rangle dW_2^k\right|, \\ + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_3^k(U), \Delta\mathbf{v}\rangle dW_3^k\right| + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_4^k(U), \Delta h\rangle dW_4^k\right|,\end{aligned} \tag{5.2}$$

which yields

$$\begin{aligned}&\sup_{s\in[0,\tau\wedge\epsilon]}\Big(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_1\|^2 + \|h_2\|^2\Big) - \Big(\|\mathbf{v}_1(0)\|^2 + \|h_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_2(0)\|^2\Big) \\ &\leq C\int_0^{\tau\wedge\epsilon}(1 + |F|^2 + |G|^2 + \|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)dt \\ &\quad + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_1^k(U), \Delta\mathbf{v}_1\rangle dW_1^k\right| + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_2^k(U), \Delta h_2\rangle dW_2^k\right|. \\ &\quad + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_3^k(U), \Delta\mathbf{v}_2\rangle dW_3^k\right| + 2\sup_{s\in[0,\tau\wedge\epsilon]}\left|\int_0^s\sum_{k=1}^\infty\langle\sigma_4^k(U), \Delta h_2\rangle dW_4^k\right|. \\ &\leq C\int_0^{\tau\wedge\epsilon}(1 + |F|^2 + |G|^2 + \|\mathbf{v}_1\|^2 + \|h_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_2\|^2)dt + \sum_{i=1}^4 I_i\end{aligned} \tag{5.3}$$

All I's terms can be treated by employing the BDG's inequality as we carried out the uniform estimates for $\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n$ and we arrive at the conclusion

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,\tau\wedge\epsilon]} \Big(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_1\|^2 + \|h_2\|^2 \Big) - \mathbb{E}\Big(\|\mathbf{v}_1(0)\|^2 + \|h_1(0)\|^2 + \|\mathbf{v}_2(0)\|^2 + \|h_2(0)\|^2 \Big) \\
&\preceq \mathbb{E}\Big(\int_0^{\tau\wedge\epsilon} (1 + |F|^2 + |G|^2 \Big) dt \preceq \limsup_{\epsilon\to 0} \epsilon(|F|^2_{L^\infty} + |G|^2_{L^\infty})
\end{aligned}
\tag{5.4}
$$

□

5.2 Local Pathwise Solutions

We let τ be as in (5.1), and use an identical argument to Section 4.6 to conclude that $(\mathbf{v}, h, \tau)$ is a local pathwise solution of (1.1).

5.3 Maximal Pathwise Solutions

We also see that the local solution can be extended in time to be a maximal solution.

Proposition 5.1. *There exists a unique maximal solution* $(\mathbf{v}, h, \xi)$ *and a sequence* ρ_R *announcing* ξ.

Proof. With the uniqueness already proved, we consider the set $\mathcal{L}$ of all stopping times such that $\tau \in \mathcal{L}$ if and only if there exist processes $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$ s.t. $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \tau)$ is a local pathwise solution. Clearly if two stopping times are in $\mathcal{L}$, then so is their maximum and if $\sigma \in \mathcal{L}$, then so is $\rho \wedge \sigma$ where ρ is any stopping time. Let $\xi = \sup \mathcal{L}$ and choose an increasing sequence $\tau_k \in \mathcal{L}$ such that $\tau_k \to \xi$ a.s.

For each τ_k, denote by $(\mathbf{v}_{1k}, \mathbf{v}_{2k}, h_{1k}, h_{2k})$ the corresponding process that makes $(\mathbf{v}_{1k}, \mathbf{v}_{2k}, h_{1k}, h_{2k}, \tau_k)$ a local pathwise solution. Let

$$
\begin{aligned}
\Omega_{k,k'} = \{&\mathbf{v}_{1_k}(t\wedge\tau_k\wedge\tau_{k'}) = \mathbf{v}_{1'_k}(t\wedge\tau_k\wedge\tau_{k'}), h_{1_k}(t\wedge\tau_k\wedge\tau_{k'}) = h_{1'_k}(t\wedge\tau_k\wedge\tau_{k'}), \\
&\mathbf{v}_{2_k}(t\wedge\tau_k\wedge\tau_{k'}) = \mathbf{v}_{2_{k'}}(t\wedge\tau_k\wedge\tau_{k'}), h_{2_k}(t\wedge\tau_k\wedge\tau_{k'}) = h_{2_{k'}}(t\wedge\tau_k\wedge\tau_{k'}); t \geq 0\}
\end{aligned}
\tag{5.5}
$$

Then, by uniqueness, we see that $\tilde{\Omega} = \cap_{k,k'}\Omega_{k,k'}$ is a set of full measure. For ω on this set and every $t > 0$, the sequence $\{\mathbf{v}_{1_k(t\wedge\tau_k)}\mathbb{1}_{t<\xi}, \{\mathbf{v}_{2_k(t\wedge\tau_k)}\mathbb{1}_{t<\xi}, h_{1_k}(t\wedge\tau_k)\mathbb{1}_{t<\xi}, h_{2_k}(t\wedge\tau_k)\mathbb{1}_{t<\xi}$ is Cauchy in $H_1 \times H_1 \times H_2 \times H_2$. Let

$$
\tilde{\mathbf{v}}_i(t) = \lim_{k\to\infty} \mathbf{v}_{i_k(t\wedge\tau_k)}\mathbb{1}_{t<\xi} \qquad \text{and} \qquad \tilde{h_i}(t) = \lim_{k\to\infty} h_{i_k(t\wedge\tau_k)}\mathbb{1}_{t<\xi} \qquad \text{a.s} \qquad \text{for } i = 1, 2. \tag{5.6}
$$

Then for any $T > 0$, we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,\xi\wedge T]} \|\tilde{\mathbf{v}_1}\|^2 + \sup_{t\in[0,\xi\wedge T]} \|\tilde{\mathbf{v}}_2\|^2 + \sup_{t\in[0,\xi\wedge T]} \|\tilde{h_1}\|^2 + \sup_{t\in[0,\xi\wedge T]} \|\tilde{h_2}\|^2 + \\
&\int_0^{\xi\wedge T} |\Delta\tilde{\mathbf{v}}_1|^2\, dt + \int_0^{\xi\wedge T} |\Delta\tilde{\mathbf{v}}_2|^2\, dt + \int_0^{\xi\wedge T} \left|\Delta\tilde{h_1}\right|^2 dt + \int_0^{\xi\wedge T} \left|\Delta\tilde{h_2}\right|^2 dt \Big) < \infty.
\end{aligned}
\tag{5.7}
$$

We may then define for $i = 1, 2$,$\mathbf{v}_i(t) \in H_1, h_i(t) \in H_2$ by:

$$\begin{aligned}(\mathbf{v}_1(t), v) + \int_0^{t\wedge\xi} (-\nu_1 \Delta \tilde{\mathbf{v}_1} + (\tilde{\mathbf{v}_1} \cdot \nabla)\tilde{\mathbf{v}} + g\nabla \tilde{h_1} + g\frac{\rho_2}{\rho_1}\nabla \tilde{h_2} + f\mathbf{k} \times \tilde{\mathbf{v}_1}, v)ds \\ = (\mathbf{v}_1(0), v) + \int_0^{t\wedge\xi} (F, v)ds + \int_0^{t\wedge\xi} \sum_k \langle \sigma_1(U)e_k, v\rangle dW_1^k,\end{aligned} \tag{5.8}$$

$$\begin{aligned}(\mathbf{v}_1(t), v) + \int_0^{t\wedge\xi} (-\nu_2 \Delta \tilde{\mathbf{v}_2} + (\tilde{\mathbf{v}_2} \cdot \nabla)\tilde{\mathbf{v}_2} + g\nabla \tilde{h_1} + g\nabla \tilde{h_2} + f\mathbf{k} \times \tilde{\mathbf{v}_2}, v)ds \\ = (\mathbf{v}_2(0), v) + \int_0^{t\wedge\xi} (F, v)ds + \int_0^{t\wedge\xi} \sum_k \langle \sigma_2(U)e_k, v\rangle dW_2^k,\end{aligned} \tag{5.9}$$

$$(h(t), \eta) + \int_0^{t\wedge\xi} (-\delta\Delta \tilde{h_1} + \nabla \cdot (\tilde{h_1}\tilde{\mathbf{v}_1}), \eta)dt = (h_1(0), \eta) + \int_0^{t\wedge\xi} \sum_k \langle \sigma_3(U)e_k, \eta\rangle dW_3^k, \tag{5.10}$$

$$(h(t), \eta) + \int_0^{t\wedge\xi} (-\delta\Delta \tilde{h} + \nabla \cdot (\tilde{h}\tilde{\mathbf{v}}), \eta)dt = (h_2(0), \eta) + \int_0^{t\wedge\xi} \sum_k \langle \sigma_4(U)e_k, \eta\rangle dW_4^k, \tag{5.11}$$

for any $t > 0, v \in H_1, \eta \in H_2$. Clearly for $t < \xi(\omega)$, $\mathbf{v}(t,\omega) = \tilde{\mathbf{v}}(t,\omega)$, $h(t,\omega) = \tilde{h}(t,\omega)$, and $\mathbf{v}$ and h are weakly continuous a.s. in H_1 and H_2, respectively. Thus, $(\mathbf{v}, h, \xi)$ is a local pathwise solution.
For $R > 0$, define the stopping time

$$\begin{aligned}\rho_R := \inf_{t\geq 0} \Bigg\{ \sup_{s\in[0,t]} \|\mathbf{v}_1\|^2 + \sup_{s\in[0,t]} \|\mathbf{v}_2\|^2 + \int_0^t |\Delta \mathbf{v}_1|^2 \, ds + \int_0^t |\Delta \mathbf{v}_2|^2 \, ds + \\ \sup_{s\in[0,t]} \|h_1\|^2 + \sup_{s\in[0,t]} \|h_2\|^2 + \int_0^t |\Delta h_1|^2 \, ds + \int_0^t |\Delta h_2|^2 \, ds > R \Bigg\} \wedge \xi.\end{aligned} \tag{5.12}$$

+ Then $(\mathbf{v}, h, \rho_R)$ is a local pathwise solution for any $R > 0$ and $\{\rho_R\}_{R\geq 0}$ announces ξ.

□

To conclude, we have completed the proof of Theorem 2.2. maintheory2

6 Appendices

Appendices

Appendix A

Now, suppose that $\tilde{H}$ is a separable Hilbert space. Given $p \geq 2, \alpha \in (0,1)$, we define the fractional derivative space $W^{\alpha,p}(0,T;\tilde{H})$ as the Sobolev space of all $u \in L^p(0,T;\tilde{H})$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|_{\tilde{H}}^p}{|t-s|^{1+\alpha p}} dtds < \infty, \tag{6.1}$$

endowed with the norm

$$|u|^p_{W^{\alpha,p}(0,T;\tilde{H})} = \int_0^T |u(t)|^p_{\tilde{H}} dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_{\tilde{H}}^p}{|t-s|^{1+\alpha p}} dtds. \tag{6.2}$$

We have applied the following lemmas, the proofs of which can be found in e.g. [FG95]:

Lemma 6.1. *Let $\mathcal{E}_0 \subset\subset \mathcal{E} \subset \mathcal{E}_1$ be Banach spaces with the injections being continuous and $\mathcal{E}_0, \mathcal{E}_1$ reflexive. For $p \in (1, \infty), \alpha \in (0, 1)$, we have*

$$L^p(0, T; \mathcal{E}_0) \cap W^{\alpha,p}(0, T; \mathcal{E}_1) \subset\subset L^p(0, T; \mathcal{E}). \tag{6.3}$$

Lemma 6.2. *If $\mathcal{E} \subset\subset \bar{\mathcal{E}}$ are Banach spaces and $p \in (1, \infty), \alpha \in (0, 1]$ are such that $\alpha p > 1$, then*

$$W^{\alpha,p}(0, T; \mathcal{E}) \subset\subset C\left([0, T]; \bar{\mathcal{E}}\right). \tag{6.4}$$

We additionally often use the following stochastic version of the Gronwall lemma (see e.g. [GHZ09]):

Lemma 6.3. *Fix $T > 0$ and assume that $X, Y, Z, R : \Omega \times [0, T) \to \mathbb{R}$ are non-negative stochastic processes. Let $\tau < T$ be a stopping time such that*

$$\mathbb{E}\left(\int_0^\tau (RX + Z) ds\right) < \infty \qquad \text{and} \qquad \int_0^\tau R ds < \kappa, \qquad a.s.$$

Suppose that for all stopping times $0 \le \tau_a \le \tau_b \le \tau$

$$\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds\right) \le C_0 \mathbb{E}\left(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds\right),$$

where C_0 is independent of τ_a and τ_b. Then

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} X + \int_0^\tau Y ds\right) \le C\mathbb{E}\left(X(0) + \int_0^\tau Z ds\right),$$

where C is a constant depending only on C_0, T, and κ.

Finally, we require the Vitali convergence theorem (see e.g. [Fol99]):

Theorem 6.1. *Suppose that a sequence of functions $\{f_n\}$ are L^p integrable on a finite measure space, where $1 \le p < \infty$. Then this sequence converges in L^p to a measurable function f if the following conditions are satisfied:*

(i) $\{f_n\}$ converges to f in measure; and

(ii) the functions $\{|f_n|^p\}$ are uniformly integrable.

Remark 6.1. *One can easily prove for $p > 1$ and a nonempty family $\mathcal{X}$ of random variables bounded in L^p that if $\sup_{X\in\mathcal{X}} ||X||_{L^p} < \infty$, then $\mathcal{X}$ is uniformly integrable.*

Appendix B

Definition 6.1. *Suppose (X, d) is a complete separable metric space with $\mathcal{B}(X)$ its associated Borel σ-algebra. Let $C_b(X)$ be the set of all real-valued continuous bounded functions on X, and $Pr(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$. A collection $\Lambda \subset Pr(X)$ is* tight *if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ s.t.*

$$\mu(K_\epsilon) \ge 1 - \epsilon \qquad \forall \mu \in \Lambda. \tag{6.5}$$

A sequence $\{\mu_n\}_{n\geq 0} \subset Pr(X)$ converges weakly *to a probability measure* μ *if*

$$\int f d\mu_n \to \int f d\mu \qquad \forall f \in C_b(X). \tag{6.6}$$

The proofs of the following results can be found in e.g. [DPZ92].

Proposition 6.1 (Prohorov's Theorem)**.** *A collection* $\Lambda \subset Pr(X)$ *is weakly compact if and only if it is tight.*

Proposition 6.2 (Skorohod Representation Theorem)**.** *Suppose that a sequence* $\{\mu_n\}_{n\geq 0}$ *converges weakly to a measure* μ*. Then there exists a probability space* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ *and a sequence of* X*-valued random variables* $\left\{\tilde{Y}_n\right\}_{n\geq 0}$ *relative to this space such that* $\tilde{Y}_n$ *converges a.s. to the random variable* $\tilde{Y}$ *and such that the laws of* $\tilde{Y}_n$ *and* $\tilde{Y}$ *are* μ_n *and* μ*, respectively, i.e.* $\mu_n(E) = \mathbb{P}\left(\tilde{Y}_n \in E\right), \mu(E) = \mathbb{P}\left(\tilde{Y} \in E\right), \forall E \in \mathcal{B}(X)$*.*

Finally, we suppose that $\{Y_n\}_{n\geq 0}$ is a sequence of X-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mu_{m,n}\}_{m,n\geq 0}$ be the collection of joint laws of $\{Y_n\}_{n\geq 0}$, i.e.

$$\mu_{m,n}(E) := \mathbb{P}((Y_m, Y_n) \in E), \qquad \forall E \in \mathcal{B}(X \times X). \tag{6.7}$$

We also need this result from [GK96]:

Proposition 6.3 (Gyöngy-Krylov Theorem)**.** *A sequence of* X*-valued random variables* $\{Y_n\}_{n\geq 0}$ *converges in probability if and only if for every subsequence of joint probability laws,* $\{\mu_{m_k,n_k}\}_{k\geq 0}$ *there exists a further subsequence which converges weakly to a probability measure* μ *s.t.*

$$\mu(\{(x,y) \in X \times X : x = y\}) = 1. \tag{6.8}$$

Lemma 6.4. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a fixed probability space,* X *a separable Hilbert space. Consider a sequence of stochastic bases* $\mathcal{S}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t\geq 0}, \mathbb{P}, W_1^n, W_2^n)$*, where each* W_i^n *is a cylindrical Brownian motion over* $\mathfrak{U}$ *with respect to* $\mathcal{F}_t^n$*. Assume that* $\{G^n\}_{n\geq 0}$ *are a collection of* X*-valued* $\mathcal{F}_t^n$ *predictable processes such that* $G^n \in L^2(0,T; L_2(\mathfrak{U}, X))$ *a.s. Finally, consider* $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W_1, W_2)$ *and* $G \in L^2(0,T; L_2(\mathfrak{U}, X))$ *a.s., which is* $\mathcal{F}_t$ *predictable. If*

$$G^n \to G \qquad \text{in probability in } L^2(0,T; L_2(\mathfrak{U}, H)), \tag{6.9}$$

$$W^n \to W \qquad \text{in probability in } C([0,T]; \mathfrak{U}_0), \tag{6.10}$$

then

$$\int_0^t G^n dW^n \to \int_0^t G dW \qquad \text{in probability in } L^2(0,T; X). \tag{6.11}$$

Appendix C

In this appendix, we consider the deterministic problem $\left(\sigma_i \equiv 0 \text{ for } i = 1,2\right)$. Our goal is to show why we use these particular boundary conditions and to derive the variational formulation which is used in the Itō formula. We begin with (??) which we multiply by $\tilde{v} \in V$ and integrate over $\mathcal{M}$:

$$\left(\frac{\partial \mathbf{v}}{\partial t}, \tilde{v}\right) - \nu(\Delta \mathbf{v}, \tilde{v}) + ((\mathbf{v} \cdot \nabla)\mathbf{v}, \tilde{v}) + g(\nabla h, \tilde{v}) + (f\mathbf{k} \times \mathbf{v}, \tilde{v}) = (F, \tilde{v}). \tag{6.12}$$

variational

The second term can be calculated as:

$$\begin{aligned} -\nu(\Delta \mathbf{v}, \tilde{v}) &= \nu(\mathrm{curl}(\mathrm{curl}(\mathbf{v})), \tilde{v}) - \nu(\nabla(\nabla \cdot \mathbf{v}), \tilde{v}) \\ &= \nu(\mathrm{curl}(\mathbf{v}), \mathrm{curl}(\tilde{v})) - \nu \int_{\partial\mathcal{M}} \mathrm{curl}(\mathbf{v}) \cdot (\tilde{v} \wedge n) dS \\ &\quad + \nu(\nabla \cdot \mathbf{v}, \nabla \cdot \tilde{v}) - \nu \int_{\partial\mathcal{M}} (\nabla \cdot \mathbf{v})(\tilde{v} \cdot n) dS. \end{aligned} \tag{6.13}$$

Since $\tilde{v} \in V$, we need only assume that $\mathrm{curl}(\mathbf{v}) = 0$ on $\partial\mathcal{M}$ to get rid of the boundary terms. With this assumption, we see that

$$-\nu(\Delta \mathbf{v}, \tilde{v}) = \nu((\mathbf{v}, \tilde{v})), \tag{6.14}$$

where $((\cdot,\cdot))$ denotes the inner product on V. For the nonlinear term in (6.12), we obtain:

$$\begin{aligned} ((\mathbf{v} \cdot \nabla)\mathbf{v}, \tilde{v}) &= \frac{1}{2}(\nabla \mathbf{v}^2, \tilde{v}) + (\mathrm{curl}(\mathbf{v})\alpha(\mathbf{v}), \tilde{v}) \\ &= -\frac{1}{2}(\mathbf{v}^2, \nabla \cdot \tilde{v}) + (\mathrm{curl}(\mathbf{v})\alpha(\mathbf{v}), \tilde{v}) + \frac{1}{2}\int_{\partial\mathcal{M}} |\mathbf{v}|^2 (\tilde{v} \cdot n) dS, \end{aligned} \tag{6.15}$$

Gamma nonlinear v

where again the boundary term vanishes since $\tilde{v} \in V$. Another application of the divergence theorem yields:

$$g(\nabla h, \tilde{v}) = -g(h, \nabla \cdot \tilde{v}) + g \int_{\partial\mathcal{M}} h(\tilde{v} \cdot n) dS. \tag{6.16}$$

Now we take (??), multiply it by $\tilde{h} \in V_2$, and integrate over $\mathcal{M}$:

$$\left(\frac{\partial h}{\partial t}, \tilde{h}\right) + (\nabla \cdot (h\mathbf{v}), \tilde{h}) - \delta(\Delta h, \tilde{h}) = 0. \tag{6.17}$$

variational h

The second term can be rewritten as:

$$(\nabla \cdot (h\mathbf{v}), \tilde{h}) = -(h\mathbf{v}, \nabla \tilde{h}) + \int_{\partial\mathcal{M}} h\tilde{h}(\mathbf{v} \cdot n) dS. \tag{6.18}$$

Gamma nonlinear h

If we assume that $\mathbf{v} \cdot n = 0$ on $\partial\mathcal{M}$, then the boundary term vanishes. For the third term in (6.17), we have

$$-\delta(\Delta h, \tilde{h}) = \delta((h, \tilde{h})) - \delta \int_{\partial\mathcal{M}} \tilde{h}(\nabla h \cdot n) dS = \delta((h, \tilde{h})) - \delta \int_{\partial\mathcal{M}} \tilde{h}(\nabla \mathring{h} \cdot n) dS. \tag{6.19}$$

To get rid of the boundary term, we must assume that $\nabla\mathring{h}\cdot n = 0$ or $\tilde{h} = 0$ on $\partial\mathcal{M}$. We see that with these boundary conditions (or Dirichlet boundary conditions on $\mathbf{v}$ and h):

$$\mathbf{v}\cdot n = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T), \tag{6.20}$$

$$\text{curl}(\mathbf{v}) = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T), \tag{6.21}$$

$$\text{and} \quad \nabla\mathring{h}\cdot n = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T), \tag{6.22}$$

the original problem is equivalent to the variational problem:

Find $\mathbf{v}\in L^2(0,T;V)\cap L^\infty(0,T;H), h\in L^2(0,T;V_2)\cap L^\infty(0,T;H_2)$ s.t.

$$\left(\frac{\partial\mathbf{v}}{\partial t},\tilde{v}\right) + \nu((\mathbf{v},\tilde{v})) - \frac{1}{2}(\mathbf{v}^2,\nabla\cdot\tilde{v}) + (\text{curl}(\mathbf{v})\alpha(\mathbf{v}),\tilde{v}) - g(h,\nabla\cdot\tilde{v}) + (f\mathbf{k}\times\mathbf{v},\tilde{v}) = (F,\tilde{v}) \quad \forall\tilde{v}\in V, \tag{6.23}$$

$$\left(\frac{\partial h}{\partial t},\tilde{h}\right) - (h\mathbf{v},\nabla\tilde{h}) + \delta((h,\tilde{h})) = 0 \quad \forall\tilde{h}\in V_2, \tag{6.24}$$

$$\mathbf{v}(t=0) = \mathbf{v}^0(x,y), \quad \text{and} \quad h(t=0) = h^0(x,y) > 0 \qquad \text{in } \mathcal{M}. \tag{6.25}$$

Appendix D

In this section, we prove the positivity of h. Recall that $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ is the total derivative of $f(x,y,t)$. Then we can rewrite (continuity equation **??**) as

$$\frac{d(ln(h))}{dt} = -\nabla\cdot\mathbf{v} + \frac{\delta}{h}\Delta\mathbf{v} + \frac{1}{h}\sigma_2(\mathbf{v},h)\frac{dW_2}{dt}, \tag{6.26}$$

along the characteristics $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$. Hence,

$$h(t) = h_0\cdot\exp\left\{\int_0^t -\nabla\cdot\mathbf{v} + \frac{\delta}{h}\Delta\mathbf{v}ds + \int_0^t \frac{1}{h}\sigma_2(\mathbf{v},h)dW_2\right\}. \tag{6.27}$$

If we assume that $\mathbf{v}\in C^1$, then the characteristics cover all of $\mathcal{M}$ and we see that the exponent above is bounded. This implies that h is strictly positive due to the positivity assumption on h_0 in (postive h_0 **??**).

Acknowledgement.

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