

ON THE ORDER OF MAGNITUDE OF CUMULANTS OF VON MISES FUNCTIONALS AND RELATED STATISTICS

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It is shown that under appropriate conditions the s th cumulant of a von Mises statistic or a U (or V) statistic is $O(n^{-s+1})$, $s \geq 2$, as the sample size n goes to infinity. A possible route toward the derivation of an asymptotic expansion of the characteristic function is indicated.

1. Introduction. The Edgeworth expansion of the characteristic function of a normalized sum of n independent and identically distributed (i.i.d) random variables derives from the order of magnitude $O(n^{-(s-2)/2})$ of the s th cumulant ($s \geq 2$) (See, e.g., Bhattacharya, 1977). For statistics which may be expressed as or approximated by polynomials in several average sample characteristics (e.g., (i) polynomials in sample moments and (ii) maximum likelihood estimators in the regular case), the validity of the so-called "formal Edgeworth expansion" depends crucially on the above order of magnitude of the s th cumulant ($s \geq 2$) of the normalized statistic (see Bhattacharya and Ghosh, 1978). In this note it is shown that cumulants of normalized U -statistics and von Mises functionals have the above order of magnitude, if certain conditions are satisfied. For general background on these statistics we refer to von Mises (1947) and Serfling (1980). Assuming the validity of (a) the above order of magnitude of the cumulants and (b) the Edgeworth expansion of the distribution function of a von Mises functional, Withers (1980) has given an algorithm for computing the coefficients in the asymptotic expansion. Some of the moment computations in Section 2 are similar to those in Withers (loc. cit). In Section 3 a new method of derivation of Cramér-Edgeworth expansions of characteristic functions of a class of statistics is provided.

2. Moments and cumulants. Let χ be a separable metric space (e.g., a subset of R^d), \mathcal{B}_χ its Borel sigma field, and P a given probability measure on \mathcal{B}_χ , whose support is S . Let \mathcal{P}_f denote the set of all probability measures on $\mathcal{B}_\chi \cap S$ having finite supports. Endow $\mathcal{P}_f \cup \{P\}$ with the weak-star topology. Consider for each n the product space $(\chi^n, \mathcal{B}_{\chi^n})$, and let X_1, \dots, X_n be the n coordinate random variables. Let $G^{\otimes n} = G \times G \times \dots \times G$ denote the product probability measure on \mathcal{B}_{χ^n} , where G is a probability measure on \mathcal{B}_χ . Under $G^{\otimes n}$ the random variables X_1, \dots, X_n are i.i.d. with common distribution G . We shall write E_G to denote expectation under $G^{\otimes n}$. Denote the empirical distribution of the "observations" X_1, \dots, X_n by F_n , i.e., $F_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$, where δ_x is the Dirac measure with point mass at x .

Let $h(x_1, x_2, \dots, x_r)$ be a real-valued, Borel measurable, symmetric function on χ^r , for some $r \geq 2$. Define the V -statistic (with kernel h)

$$(2.1) \quad V_n = n^{-r} \sum_{i_1=1}^n \dots \sum_{i_r=1}^n h(X_{i_1}, X_{i_2}, \dots, X_{i_r}),$$

and the U -statistic (with kernel h)

$$(2.2) \quad U_n = \binom{n}{r}^{-1} \sum h(X_{i_1}, X_{i_2}, \dots, X_{i_r})$$

where the summation is over $1 \leq i_1 < i_2 < \dots < i_r \leq n$.

Received February 1982; revised August 1982.

¹ Research supported by NSF Grant MCS 8201628.

² Research supported by Office of Naval Research under Contract N00014-77-C-0659.

AMS 1970 subject classifications. Primary 62E20; secondary 62G05, 62G10.

Key words and phrases. V -Statistics, U -statistics, Edgeworth expansion.

THEOREM 2.1. (a) *If for some integer $s \geq 3$ one has*

$$(2.3) \quad E_P |h(X_{j_1}, X_{j_2}, \dots, X_{j_r})|^s < \infty$$

for all choices of $j_1, j_2, \dots, j_r (1 \leq j_1, j_2, \dots, j_r \leq r)$, then the p th cumulant $k_{p,n}(P)$ of V_n under P is of the form

$$(2.4) \quad k_{p,n}(P) = \sum_{m=p-1}^{s-1} n^{-m} \lambda_{m,p}(P) + o(n^{-s+1}), \quad (2 \leq p \leq s).$$

The quantities $\lambda_{m,p}(P)$ are independent of n .

(b) *Suppose that, for some integer $s \geq 3$ one has*

$$(2.5) \quad E_P |h(X_1, X_2, \dots, X_r)|^s < \infty.$$

Then the cumulants of the statistic U_n also are of the form (2.4).

PROOF. (a) Write

$$(2.6) \quad V_n = \int \dots \int h(x_1, \dots, x_r) F_n(dx_1) \dots F_n(dx_r).$$

For $G = \sum_{i=1}^q \alpha_i \delta_{y_i}$ in \mathcal{P}_f , F_n may be expressed as $\sum_{i=1}^q \hat{\alpha}_i \delta_{y_i}$ (with $G^{\otimes n}$ -probability one), where $\hat{\alpha}_i$ is the proportion of y_i 's in the "sample" $\{X_1, \dots, X_n\}$. Thus V_n becomes a polyhomial in the q -variables $\hat{\alpha}_i$, $1 \leq i \leq q$. Hence by a result of James and Mayne (1962) (this may also be derived from the results of Leonov and Shiryaev, 1959), the p th cumulant of V_n under G is of the form

$$(2.7) \quad k_{p,n}(G) = \sum_{m=p-1}^{r-1} n^{-m} \lambda_{m,p}(G), \quad (p \geq 2).$$

On the other hand, for all G such that $E_G |V_n|^p < \infty$, one has (for all $n > rp$)

$$(2.8) \quad \begin{aligned} E_G V_n^p &= n^{-rp} E_G \int \dots \int (\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})) \\ &\quad \cdot (\delta_{X_1} + \delta_{X_2} + \dots + \delta_{X_n})(dx_1) \dots (\delta_{X_1} + \delta_{X_2} + \dots + \delta_{X_n})(dx_{rp}) \\ &= n^{-rp} \int \dots \int (\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})) \\ &\quad \left[\sum_{m=1}^{rp} \frac{n!}{(n-m)!} \sum_2 \sum_1 E_G \{ \delta_{X_1}(dx_{j_{11}}) \delta_{X_1}(dx_{j_{12}}) \dots \delta_{X_1}(dx_{j_{1s_1}}) \right. \\ &\quad \left. \delta_{X_2}(dx_{j_{21}}) \dots \delta_{X_2}(dx_{j_{2s_2}}) \dots \delta_{X_m}(dx_{j_{m1}}) \dots \delta_{X_m}(dx_{j_{ms_m}}) \right]. \end{aligned}$$

Here, for a given m , \sum_2 denotes summation over all collections of m positive integers $\{s_1, s_2, \dots, s_m\}$ satisfying $\sum s_i = rp$; for a given collection $\{s_1, s_2, \dots, s_m\}$, \sum_1 denotes summation over all partitions of $\{1, 2, \dots, rp\}$ into m groups of s_1, s_2, \dots, s_m elements, a typical partition being $(\{j_{11}, j_{12}, \dots, j_{1s_1}\}, \{j_{21}, j_{22}, \dots, j_{2s_2}\}, \{j_{m1}, j_{m2}, \dots, j_{ms_m}\})$. Denote by $H_s(G)$ the distribution of the s -dimensional random vector (X_1, X_1, \dots, X_1) under G , and let $H_{s_1, s_2, \dots, s_m}(G)$ stand for the measure

$$(2.9) \quad \begin{aligned} H_{s_1, s_2, \dots, s_m}(G)(dx_1 dx_2 \dots dx_{rp}) &= \sum_1 H_{s_1}(G)(dx_{j_{11}} dx_{j_{12}} \dots dx_{j_{1s_1}}) H_{s_2}(G) \\ &\quad \cdot (dx_{j_{21}} dx_{j_{22}} \dots dx_{j_{2s_2}}) \dots H_{s_m}(G)(dx_{j_{m1}} dx_{j_{m2}} \dots dx_{j_{ms_m}}). \end{aligned}$$

Also note that

$$(2.10) \quad \begin{aligned} n^{-rp} \frac{n!}{(n-m)!} &= n^{-rp} n(n-1) \dots (n-m+1) \\ &= \sum_{m'=1}^m (-1)^{m-m'} n^{-rp+m'} \theta(m-m'; m-1), \end{aligned}$$

where $\theta(i; N)$ is the sum of all products of i distinct integers taken from $\{1, 2, \dots, N\}$,

$\theta(0; N) = 1$. From (2.8)–(2.10) one obtains

$$\begin{aligned}
 E_G V_n^p &= \sum_{m=1}^{rp} \sum_{m'=1}^m (-1)^{m-m'} n^{-rp+m'} \theta(m-m'; m-1) \\
 &\quad \cdot \int \cdots \int (\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})) \sum_2 H_{s_1, s_2, \dots, s_m}(G)(dx_1 \cdots dx_{rp}) \\
 (2.11) \quad &= \sum_{j=0}^{rp-1} n^{-j} \{\sum_{m=rp-j}^{rp} (-1)^{m-rp+j} \theta(m-rp+j; m-1) \\
 &\quad \cdot \int \cdots \int (\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})) \sum_2 H_{s_1, s_2, \dots, s_m}(G)(dx_1 \cdots dx_{rp})\} \\
 &= \sum_{j=0}^{rp-1} n^{-j} \mu_{j,p}(G), \quad (1 \leq p \leq s),
 \end{aligned}$$

say. Here $\mu_{j,p}(G)$ is a linear combination (with coefficients not depending on n , G or h) of terms like

$$(2.12) \quad \int \cdots \int (\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})) H_{s_1, s_2, \dots, s_m}(G)(dx_1 \cdots dx_{rp}).$$

Using the familiar relations between moments and cumulants one has

$$(2.13) \quad k_{p,n}(G) = \sum_{j=0}^{rp-1} n^{-j} \bar{\lambda}_{j,p}(G),$$

where $\bar{\lambda}_{j,p}(G)$ is a polynomial in $\mu_{j,p'}$ ($1 \leq p' \leq p$), whose coefficients are absolute constants. Since the map $G \rightarrow H_{s_1}(G)$ is continuous in the weak-star topology, so is the map $G \rightarrow H_{s_1, s_2, \dots, s_m}(G)$. It follows that for a bounded continuous h the integral (2.12) is a weak-star continuous function of G ; this implies that the maps $G \rightarrow \mu_{j,p}$ and, therefore, $G \rightarrow \bar{\lambda}_{m,p}(G)$ are continuous. If $p \geq 2$, then $\bar{\lambda}_{m,p}(G) = 0$ for $2 \leq m < p-1$ and $G \in \mathcal{P}_f$. Also there exists $G_N \in \mathcal{P}_f$ ($N = 1, 2, \dots$) such that G_N converges to P (This is where the separability of χ is made use of; see, e.g., Parthasarathy (1967), Theorem 6.3). Therefore, one must have $\bar{\lambda}_{m,p}(P) = 0$ for $1 \leq m < p-1$. This completes the proof of (a) for bounded continuous h . Since functions of the form $\prod_{t=1}^p h(x_{r(t-1)+1}, \dots, x_{rt})$ belonging to $L^1(\chi^{rp}, H_{s_1, s_2, \dots, s_m}(P))$ may be approximated (in L^1) by continuous bounded functions of the same form, the proof is complete. Note that for this last argument (2.3) is needed.

(b) First assume (i) $h(x_1, x_2, \dots, x_r) = 0$ if $x_i = x_j$ for some i, j ($i \neq j$). Then the cumulants of U_n satisfy (2.4), since

$$U_n = \left(\frac{n!}{(n-r)!} \right)^{-1} n^r V_n = \left(\left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \cdots \left(1 - \frac{r-1}{n} \right) \right)^{-1} V_n = (1 + o(1)) V_n.$$

Next, instead of (i) assume (ii) P has no atoms. Then modify h so as to satisfy (i); this does not change U_n , except on a set of probability zero. Finally, consider an arbitrary P . Let D be its set of atoms. Let D' be a subset of reals in one-one correspondence with D . Consider the space $\chi' = (\chi \setminus D) \cup R$, with $\chi \setminus D$ and R each carrying its own topology but their union is topologically disconnected. Then P lifted to this space χ' (by placing the discrete mass on D') is a weak-star limit of nonatomic probability measures. Extend h to $(\chi')^r$ by setting it zero if any coordinate is in $R \setminus D'$. Now apply an argument entirely analogous to that in the preceding paragraph.

REMARK 2.1.1. The U_n and V_n defined above are not centered around their expectations (under P). Centering has been avoided deliberately to ensure that h does not depend on P . For the general von Mises functional considered below centering seems unavoidable; this causes some technical problems.

REMARK 2.1.2. Under the hypotheses of Theorem 2.1 the p th cumulants of the normalized statistics $\sqrt{n}(V_n - EV_n)$, $\sqrt{n}(U_n - EU_n)$ are of the order $O(n^{-(p-2)/2})$, $2 \leq p \leq s$.

Let T be a von Mises functional defined on $\mathcal{P}_f \cup \{P\}$, and let the statistic $T(F_n)$ have

the expansion

$$(2.14) \quad \begin{aligned} T(F_n) - T(G) &= \sum_{i=1}^r \int \cdots \int T^{(i)}(G; x_1, x_2, \dots, x_i) \prod_{j=1}^i (F_n - G)(dx_j) + R_n \\ &= \sum_{i=1}^r V_{n,i}(G) + R_n \quad (G \in \mathcal{P}_f \cup \{P\}), \end{aligned}$$

where $T^{(i)}$ is a real-valued, symmetric (in the arguments x_1, \dots, x_i), Borel measurable function on $\mathcal{P}_f \cup \{P\} \times \chi^i$ satisfying

$$(2.15) \quad E_P |T^{(i)}(P; X_{j_1}, X_{j_2}, \dots, X_{j_i})|^s < \infty, \quad (1 \leq i \leq r),$$

for all $1 \leq j_1, j_2, \dots, j_i \leq r$, and the ‘‘remainder term’’ R_n satisfies

$$(2.16) \quad E_P |R_n|^p = o(n^{-s+1}), \quad (1 \leq p \leq s).$$

Write

$$(2.17) \quad V_n(G) = \sum_{i=1}^r V_{n,i}(G).$$

Then

$$(2.18) \quad E_G V_n^p(G) = \sum_{\beta} V_{n,i_1}(G) V_{n,i_2}(G) \cdots V_{n,i_p}(G),$$

where \sum_{β} denotes summation over all p -tuples (i_1, i_2, \dots, i_p) such that $1 \leq i_1, \dots, i_p \leq r$. Now let $I_p = i_1 + \dots + i_p$ and write, as in (2.8),

$$(2.19) \quad \begin{aligned} E_G V_n^p(G) &= \sum_{\beta} E_G \int \cdots \int (\prod_{t=1}^p T^{(i_t)}(G; x_{I_{t-1}+1}, \dots, x_{I_t})) \\ &\cdot \left[n^{-I_p} \sum_{m=1}^{\lfloor I_p/2 \rfloor} \frac{n!}{(n-m)!} \sum_2' \sum_1' (\delta_{x_1} - G)(dx_{j_{11}}) (\delta_{x_1} - G)(dx_{j_{12}}) \right. \\ &\quad \left. \cdots (\delta_{x_1} - G)(dx_{j_{1s_1}}) \cdots (\delta_{x_m} - G)(dx_{j_{m1}}) \cdots (\delta_{x_m} - G)(dx_{j_{ms_m}}) \right]. \end{aligned}$$

Here, for a given m , \sum_2' denotes summation over all collections of m integers $\{s_1, s_2, \dots, s_m\}$ satisfying $s_i \geq 2$ and $\sum s_i = I_p$; and \sum_1' denotes, for each collection $\{s_1, s_2, \dots, s_m\}$, summation over all partitions of $\{1, 2, \dots, I_p\}$ into m subgroups of s_1, s_2, \dots, s_m elements such as $(\{j_{11}, j_{12}, \dots, j_{1s_1}\}, \dots, \{j_{m1}, j_{m2}, \dots, j_{ms_m}\})$. Note that expectations of terms involving $s_i = 1$ for some i vanish. Next let $H_q(i_1, i_2, \dots, i_\ell; G)$ denote the distribution of a q -dimensional random vector whose i_1 th, \dots , i_ℓ th coordinates are X_1 , while the remaining coordinates are i.i.d with distribution G and independent of X_1 . Write $\tilde{H}_q(G)$ for the signed measure

$$(2.20) \quad \tilde{H}_q(G) = \sum_{\ell=0}^q (-1)^{q-\ell} \sum_4 H_q(i_1, \dots, i_\ell; G),$$

where \sum_4 denotes summation over all choices $\{i_1, i_2, \dots, i_\ell\}$ of ℓ distinct integers from $\{1, 2, \dots, q\}$. Now define

$$(2.21) \quad \begin{aligned} \tilde{H}_{s_1, s_2, \dots, s_m}(G; dx_1 dx_2 \cdots dx_{I_p}) \\ = \sum_1' \tilde{H}_{s_1}(G)(dx_{j_{11}} dx_{j_{12}} \cdots dx_{j_{1s_1}}) \cdots \tilde{H}_{s_m}(G)(dx_{j_{m1}} dx_{j_{m2}} \cdots dx_{j_{ms_m}}). \end{aligned}$$

Then, as in (2.10),

$$(2.22) \quad \begin{aligned} E_G V_n^p(G) &= \sum_{\beta} \left[\sum_{j=\lfloor \frac{I_p+1}{2} \rfloor}^{I_p-1} n^{-j} \left\{ \sum_{m=I_p-j}^{\lfloor I_p/2 \rfloor} (-1)^{m-I_p+j} \theta(m - I_p + j; m - 1) \right. \right. \\ &\quad \left. \left. \cdot \int \cdots \int (\prod_{t=1}^p T^{(i_t)}(G; x_{I_{t-1}+1}, \dots, x_{I_t})) \sum_2' \tilde{H}_{s_1, \dots, s_m}(G; dx_1 \cdots dx_{I_p}) \right\} \right]. \end{aligned}$$

For $G = \sum_{i=1}^q \alpha_i \delta_{y_i}$, $V_n(G)$ is a polynomial in $\hat{\alpha}_i - \alpha_i$, so that the p th cumulant of $V_n(G)$ is of the order $O(n^{-p+1})$ under G ($2 \leq p \leq s$). In view of (2.16) and (2.22), the proof of the following theorem is now complete.

THEOREM 2.2 *Suppose that (2.14)–(2.16) hold. Assume, in addition, that there exists a sequence $\{G_N: N \geq 1\}$ having finite support such that*

$$(2.23) \quad \lim_{N \rightarrow \infty} E_{G_N}(\prod_{i=1}^p T^{(i)}(G_N; X_{i_1}, \dots, X_{i_{i_p}})) = E_P(\prod_{i=1}^p T^{(i)}(P; X_{i_1}, \dots, X_{i_{i_p}}))$$

for all $1 \leq i_1, i_2, \dots, i_p \leq r$, and all $1 \leq t_1, t_2, \dots, t_{i_p} \leq rp$ ($1 \leq t \leq p$). Then the p th cumulant of $T(F_n)$ under P is of the order $O(n^{-p+1})$ for $2 \leq p \leq s$.

REMARK 2.2.1. Notice that the statement “condition (2.23) holds for some $\{G_N: N \geq 1\} \subset \mathcal{P}_f$ ” is much weaker than the statement “condition (2.23) holds for all sequences $\{G_N: N \geq 1\}$ converging to P (weak-star)”, the latter being equivalent to saying that the integral is weak-star continuous at P (on $\mathcal{P}_f \cup \{P\}$). To illustrate this point, note that even such functionals as $T(G) = \int x^k G(dx)$, $k \geq 1$, are not weak-star continuous on $\mathcal{P}_f \cup \{P\}$, where P is a probability measure on the line having a finite k th moment. The difficulty is that one may place a mass $O(N^{-k/2})$ at $x = N$ which goes to zero to ensure weak-star convergence, but is large enough to blow up the integral as $N \rightarrow \infty$. On the other hand, one may integrate (with respect to P) a step-function approximation, $f_N(x)$ to x^k , which amounts to integrating x^k with respect to an appropriate $G_N \in \mathcal{P}_f$; and the latter integral $\int x^k G_N(dx)$ will converge to $\int x^k P(dx)$, as the intervals of constancy decrease to zero in width. These considerations apply to more general functions (see, Serfling (1980), pages 214–216, for examples).

REMARK 2.2.2. The fact that the s th cumulant of V_n (or T_n) is $O(n^{-s+1})$ when G has finite support means the vanishing of a number of polynomials in the variables $\mu_p(G)$. One should be able to prove that these polynomials are identically zero by showing that the $\mu_p(G)$ ’s assume a broad enough spectrum of values as G ranges over the set of all probability measures having finite support. This would enable one to dispense with the condition (2.23) in Theorem 2.2. However, we are unable to make this algebraic argument firm.

Finally, the method used here should be more widely applicable in deriving orders of magnitudes of cumulants.

3. A method of derivation of Edgeworth expansions of characteristic functions, and an unsolved problem. In the present section we provide a method (which appears to be new) for the derivation of Cramér-Edgeworth expansions of characteristic functions of a class of statistics T_n having zero means, finite moment generating functions (m.g.f.’s), and cumulants $\chi_{p,n}$ satisfying

$$(3.1) \quad \chi_{p,n} = n^{-(p-2)/2} \lambda_p + o(n^{-(p-2)/2}), \quad (p \geq 2), \quad \lambda_2 > 0.$$

Let

$$(3.2) \quad f_n(\xi) = E \exp\{i\xi T_n\}$$

denote the characteristic function of T_n . One may write

$$(3.3) \quad f_n(\xi) = f(i\xi, \varepsilon),$$

with $\varepsilon = n^{-1/2}$. Under the additional assumption that $f(i\xi, \varepsilon)$ has an absolutely convergent power series expansion in ξ and ε in a neighborhood of the origin $(0, 0)$, it is shown in Theorem 3.1 that $f_n(\xi)$ and its derivatives have a proper asymptotic expansion of the Cramér-Edgeworth type. The *unsolved problem* is to identify a large enough class of von Mises functions for which this analyticity holds. In particular, we do not know if the analyticity property holds for U -statistics (see (2.2)) with kernels h satisfying:

$$(3.4) \quad E \exp\{th(X_1, X_2, \dots, X_r)\} < \infty, \quad (-\infty < t < \infty).$$

In remarks following the corollaries to Theorem 3.1 it is shown that the assumption of

analyticity does hold for some special classes. We expect the moment computations of Section 2 to be crucial in resolving the problem of analyticity in the general case.

THEOREM 3.1. *Let $T_n(n = 1, 2, \dots)$ be a sequence of random variables having zero means. Assume that (i) $E \exp\{tT_n\} < \infty$ for all $t(-\infty < t < \infty)$ and n , (ii) $f(i\xi, \epsilon)$ can be extended as an analytic function $f(z, \eta)$ of the complex variables z and η in a neighborhood of the origin $(0, 0)$ in \mathbb{C}^2 , and (iii) the cumulants $\chi_{p,n}$ of T_n satisfy (3.1). Then the following results hold:*

(a) *There exist a positive constant δ_0 and polynomials P_j , whose coefficients do not depend on n , such that for all $\xi, -\delta_0\sqrt{n} < \xi < \delta_0\sqrt{n}$, one has*

$$f_n(\xi) = \exp\left\{-\frac{\lambda_2}{2} \xi^2\right\} \left(1 + \sum_{j=1}^{\infty} n^{-j/2} P_j(i\xi)\right).$$

(b) *For every pair of integers m and p satisfying $p \geq 2, 0 \leq m \leq p$, there exist positive constants δ_0, c_1, c_2 such that*

$$\begin{aligned} & \left| \frac{d^m}{d\xi^m} \left[f_n(\xi) - \exp\left\{-\frac{\lambda_2}{2} \xi^2\right\} \left(1 + \sum_{j=1}^{p-2} n^{-j/2} P_j(i\xi)\right) \right] \right| \\ & \leq \frac{c_1}{n^{(p-1)/2}} \left[|\xi|^{p+1-m} + |\xi|^{3(p-1)+m} \right] \exp\{-c_2 \xi^2\}, \quad (|\xi| < \delta_0\sqrt{n}). \end{aligned}$$

PROOF. Since $f(z, \eta)$ is analytic in a neighborhood of $(0, 0)$, and $f(0, 0) = 1, \phi(z, \eta) = \log f(z, \eta)$ (we take the principal branch of the logarithm) is defined and analytic in a neighborhood of $(0, 0)$. In view of (3.1) and the fact that $ET_n = 0$, one may express $\phi(z, \eta)$ as

$$\begin{aligned} (3.5) \quad \phi(z, \eta) &= \frac{z^2}{2!} (\lambda_2 + \sum_{j=1}^{\infty} \lambda_{2,j} \eta^j) + \dots + \frac{z^k}{k!} \eta^{k-2} (\sum_{j=0}^{\infty} \lambda_{k,j} \eta^j) + \dots \\ &= z^2 \left[\sum_{k=2}^{\infty} \frac{(\eta z)^{k-2}}{k!} (\sum_{j=0}^{\infty} \lambda_{k,j} \eta^j) \right], \quad (\lambda_{k,0} = \lambda_k). \end{aligned}$$

Since this last series is absolutely convergent in a neighborhood of $(0, 0)$, so is the series within square brackets. Let δ_1, δ_2 be two positive numbers such that this last series is absolutely convergent for $|z| = \delta_1, |\eta| = \delta_2$. Then

$$(3.6) \quad \sum_{k=2}^{\infty} \frac{(\delta_1 \delta_2)^{k-2}}{k!} \sum_{j=0}^{\infty} |\lambda_{k,j}| \delta_2^j < \infty.$$

It follows that (3.5) is absolutely convergent for $|z\eta| \leq \delta_1 \delta_2$ and $|\eta| \leq \delta_2$. Therefore, the last expression in (3.5) defines an analytic function in the region $D = \{(z, \eta) \in \mathbb{C}^2: |z| < \delta_1 \delta_2 / |\eta|, |\eta| < \delta_2\}$, and over this region $\exp\{\phi(z, \eta)\}$ defines an analytic continuation of $f(z, \eta)$. We shall refer to this extension also by $f(z, \eta)$. Since the characteristic function $\xi \rightarrow f_n(\xi)$ is entire (by assumption (i)) and since analytic continuations are unique, $f_n(\xi) = f(i\xi, n^{-1/2})$ for $-\infty < \xi < \infty$ (note that one could not assume a priori that this equality holds between f_n and the analytically extended f). In addition, on D one has

$$(3.7) \quad |f(z, \eta) - 1| < c' < 1,$$

for some constant c' , and $\phi(z, \eta)$ is the principal branch of the logarithm of $f(z, \eta)$ on D . The relations (3.5) now hold on D and one may rewrite the first relation in (3.5) as

$$(3.8) \quad \log f(z, \eta) - \frac{\lambda_2}{2!} z^2 = \sum_{j=1}^{\infty} \eta^j Q_j(z), \quad (z, \eta) \in D,$$

where Q_j is a polynomial of degree $j + 2$. Thus

$$(3.9) \quad f(z, \eta) \exp\left\{-\frac{\lambda_2}{2} z^2\right\} = \exp\left\{\sum_{j=1}^{\infty} \eta^j Q_j(z)\right\} = 1 + \sum_{j=1}^{\infty} \eta^j P_j(z), \quad (z, \eta) \in D,$$

where P_j 's are appropriate polynomials. From (3.9) one gets

$$(3.10) \quad f(z, \eta) = \exp\left\{\frac{\lambda_2}{2} z^2\right\} (1 + \sum_{j=1}^{\infty} \eta^j P_j(z)), \quad (z, \eta) \in D,$$

and, in particular (with $z = i\xi, \eta = n^{-1/2}$),

$$(3.11) \quad f_n(\xi) = \exp\left\{-\frac{\lambda_2}{2} \xi^2\right\} (1 + \sum_{j=1}^{\infty} n^{-j/2} P_j(i\xi)), \quad (-\delta_1 \delta_2 \sqrt{n} < \xi < \delta_1 \delta_2 \sqrt{n}).$$

This proves part (a). To prove part (b) one may first approximate $\log f(z, \eta)$ by

$$(3.12) \quad \phi_p(z, \eta) = z^2 \sum_{k=2}^{p+2} \frac{(\eta z)^{k-2}}{k!} (\sum_{j=0}^{\infty} \lambda_{k,j} \eta^j).$$

Writing

$$(3.13) \quad \psi(z, \eta) = \phi(z, \eta) - \frac{\lambda_2}{2} z^2, \quad \psi_p(z, \eta) = \phi_p(z, \eta) - \frac{\lambda_2}{2} z^2,$$

one has (using (3.6), or analyticity on D)

$$(3.14) \quad |\phi(z, \eta) - \phi_p(z, \eta)| = |\psi(z, \eta) - \psi_p(z, \eta)| \leq c_3 |\eta|^{p-1} |z|^{p+1}, \quad (z, \eta) \in D,$$

for an appropriate constant c_3 . By (3.6) and (3.14), if δ_1 is small, then

$$(3.15) \quad |\exp\{\psi(z, \eta)\} - \exp\{\psi_p(z, \eta)\}| \leq c_4 |\eta|^{p-1} |z|^{p+1} \exp\left\{\frac{\lambda_2 |z|^2}{4}\right\},$$

for some constant c_4 ; this may be written as

$$(3.16) \quad \left| e^{-\frac{\lambda_2}{2} z^2} [f(z, \eta) - \exp\{\phi_p(z, \eta)\}] \right| \leq c_4 |\eta|^{p-1} |z|^{p+1} \exp\left\{\frac{\lambda_2 |z|^2}{4}\right\}.$$

Letting $z = i\xi, \eta = n^{-1/2}$, (3.16) becomes

$$(3.17) \quad |f_n(\xi) - \exp\{\phi_p(i\xi, n^{-1/2})\}| \leq c_4 n^{-(p-1)/2} |\xi|^{p+1} \exp\left\{-\frac{\lambda_2}{4} \xi^2\right\}, \quad (|\xi| < \delta_1 \delta_2 \sqrt{n}).$$

The comparison of $\exp\{\phi_p(i\xi, n^{-1/2})\}$ with $\exp\left\{-\frac{\lambda_2}{2} \xi^2\right\} (1 + \sum_{j=1}^{p-2} n^{-j/2} P_j(i\xi))$ is carried out exactly as in Lemmas 9.7, 9.8 in Bhattacharya and Ranga Rao (1976). \square

COROLLARY 3.1.1. *Under the hypothesis of Theorem 3.1 one has the Berry-Esseen bound*

$$(3.18) \quad \sup_x |P(T_n \leq x) - \Phi_{\lambda_2}(x)| \leq cn^{-1/2},$$

for some constant $c > 0$. Here Φ_{λ_2} is the normal distribution function with mean zero and variance λ_2 .

PROOF. Use Theorem 3.1 (b) and Esseen's inequality (see Lemmas 12.1, 12.2 in Bhattacharya and Ranga Rao (1976)). \square

COROLLARY 3.1.2. *Assume the hypothesis of Theorem 3.1. If, for some $p \geq 2$, g is a p -times continuously differentiable function on \mathbb{R}^1 such that $\sup\{(1 + |x|^p)|g^{(m)}(x)|:$*

$x \in \mathbb{R}^1\} < \infty$ for $0 \leq m \leq p$, then

$$(3.19) \quad |Eg(T_n) - \int_{\mathbb{R}^1} g(x) \left[1 + \sum_{j=1}^{p-2} n^{-j/2} P_j \left(-\frac{d}{dx} \right) \right] \phi_{\lambda_2}(x) dx| \leq dn^{-(p-1)/2}$$

for some positive constant d .

PROOF. One may apply the method of Götze and Hipp (1978) to the estimate in Theorem 3.1 (b) to derive (3.19) directly. Alternatively, first establish (3.19) for the class of all Schwartz functions as in Bhattacharya and Ranga Rao (1976), Theorem 20.7, expressing the error estimate in terms of a Sobolev norm; then extend the result to a wider class by completion in the Sobolev norm. □

REMARK 3.1.3. Let X_1, X_2, \dots be an i.i.d. sequence having mean zero and a positive variance. The hypothesis of Theorem 3.1 is satisfied for the statistics $T_n = n^{-1/2}(X_1 + \dots + X_n)$ if the m.g.f. of X_1 is finite everywhere. Of course, in this classical case Theorem 3.1 (b) holds under less stringent assumptions (see, e.g., Bhattacharya and Ranga Rao (1976), Chapter 2). Note, however, the conclusion of part (a) of Theorem 3.1 requires stronger assumptions than finiteness of moments.

REMARK 3.1.4 Let U_n be a U -statistic with kernel h (see (2.2)). Assume, without loss of generality, that $Eh(X_1, X_2, \dots, X_r) = 0$. If $E \exp\{th(X_1, \dots, X_r)\} < \infty$ for all $t, -\infty < t < \infty$, then hypothesis (i) of Theorem 3.1 is satisfied for the statistic $T_n = \sqrt{n} U_n$ (see Serfling (1980), Lemma C, page 200). In addition, assume $E\phi^2(X_1) = \lambda_2 > 0$, where $\phi(x) = Eh(x, X_2, \dots, X_r)$. Then T_n is asymptotically normal (see Serfling (1980), Theorem A, page 192) and, by Theorem 2.1 (b), hypothesis (iii) of Theorem 3.1 also holds. It would be of great interest to see if hypothesis (ii) of Theorem 3.1 is a consequence of the above assumptions. We emphasize that this is the main unresolved problem in the context of the present article. For kernels h which are sums of products of functions of single variables, analyticity of $f(z, \eta)$ in a neighborhood of the origin in \mathbb{C}^2 has been proved by methods of statistical mechanics (see, e.g., Ruelle, 1969). However, for these special kernels an adequate theory of Edgeworth expansions has been derived in Bhattacharya and Ghosh (1978) under less stringent assumptions.

REMARK 3.1.5. Some partial expansions of characteristic functions of U -statistics have been obtained by Callaert, Janssen and Veraverbeke (1980).

Acknowledgement. The authors wish to thank the referee for suggesting that some terse comments in an earlier draft be expanded; the present Section 3 constitutes the resulting expansion. Thanks are also due to the Associate Editor for indicating that asymptotic expansions for von Mises functionals are of potential use in robust estimation. Finally, we are indebted to Charles M. Newman for pointing out that the analyticity hypothesis in Theorem 3.1 has been verified in the context of statistical mechanics for the special class of kernels mentioned at the end of Remark 3.1.4.

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