COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XVIII, 51–63 (1965)

Some Distribution-Free k-Sample Rank Tests of Homogeneity Against Ordered Alternatives*

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1. Introduction and Summary

A problem which occurs frequently in statistical analysis is that of deciding whether several samples should be regarded as coming from the same population. This problem, usually referred to as the k-sample problem, when expressed formally is stated as follows: Let X_{ij}, $j = 1, \cdots, m_i$, $i = 1, \cdots, k$, be a set of independent random variables and let $F_i(x)$ be the probability distribution function of X_{ij}. The set of admissible hypotheses designates that each F_i belongs to some class of distribution functions Ω. The hypothesis to be tested, say H_0, specifies that F_i is an element of Ω, for each i, and that furthermore

$$F_1(x) = \cdots = F_k(x) \text{ for all real } x . \tag{1.1}$$

The class of alternatives to H_0 is considered to consist of all sets $(F_1(x), \cdots, F_k(x))$ which belong to Ω but which violate (1.1). This is the most general form of the alternative and is the basis of most of the existing work in the non-parametric theory. Reference to prior work on this problem and some of the recent work may be found in Dwass [7], Kruskal–Wallis [12], Mood [15], Terpestra [20], and the author [16].

However, in some problems, it is possible to be more precise in the specification of the alternative. When this is the case, it is advantageous to make use of this extra information to obtain more powerful tests. Thus instead of the unrestricted form of the alternative mentioned above, we shall consider in this paper the ordered alternatives

$$F_1(x) \geqq \cdots \geqq F_k(x) \tag{1.2}$$

(at least one inequality being strong).

For the case $k = 2$, the situation is met by using the single-tail test but for $k > 2$ the distinction between one- and two-tail tests is lost. The present work may therefore be regarded as generalizations of some of the single-tail non-parametric tests.

* This work represents results obtained under Sloan Foundation grant for statistics and under U.S. Navy Contract Nonr-285(38). Reproduction in whole or in part is permitted for any purpose of the United States Government.

This problem has many applications in social sciences. Jonckheere [11], for example, has mentioned an experiment to test the effect of stress on the task of manual dexterity. Here data would be obtained from groups of subjects working under high, medium, low, and minimal stress; the null hypothesis being that stress has no effect on performance, and the alternative that increasing stress produces an increasing effect. Armitage [1] discussed a similar problem in connection with $2 \times k$ contingency tables and found the applications in the medical field.

A few tests of parametric nature have been developed for this problem by Bartholomew [2], Chacko [5], Kudo [13], among others (see Bartholomew [3] for references). In non-parametric theory, attempts to meet the need for a test against ordered alternatives have only been made very recently. Jonckheere [11] discussed the one-way analysis of variance and proposed a distribution-free test which may be considered the most direct predecessor of the tests presented in this paper. Chacko [5] proposed another test similar to the one proposed by Kruskal and Wallis [12] for the unrestricted alternatives and studied its asymptotic Pitman efficiency against translation alternatives. In the present paper, we propose and develop a family V of rank tests for the equality of k probability distributions against the ordered alternatives. Limiting distributions of the proposed test statistics are derived, following the methods used in Chernoff and Savage [6] and the author [4], [16]. These results are used to derive general formulas for the asymptotic efficiencies of these tests with respect to one another and their parametric competitor, viz. the test based on the Student statistic. In some of the cases where the asymptotic efficiency cannot be used to compare the tests, the asymptotic power comparisons are made in an attempt to select the best test.

2. The Proposed Family of Tests

The over-all sample consists of $N = \sum_{i=1}^{k} m_i$ independent random variables X_{ij}, $i = 1, \cdots, k$, $j = 1, \cdots, m_i$, where the first subscript refers to the sub-sample and the second subscript indexes observations within a sub-sample. Under the null hypothesis, all the X's have the same continuous but unknown c.d.f. (cumulative distribution function) $F(x)$.

Denote by X_i the vector $(X_{i1}, \cdots, X_{im_i})$ and consider all the samples in pairs, there being $k(k-1)/2$ pairs in all. Let $\xi_\nu^{(i,j)} = 1$ if the ν-th smallest observation from the combined i-th and j-th samples is an X_i observation and, otherwise, let $\xi_\nu^{(i,j)} = 0$. Let $\eta_\nu^{(i,j)} = -1$, if the ν-th smallest observation from the combined i-th and j-th samples is an X_j observation and, otherwise, let $\eta_\nu^{(i,j)} = 0$.

Denote

$$h_{ij} = \tau_{ij}^{(i)} + \tau_{ij}^{(j)}, \tag{2.1}$$

where

$$m_i\tau_{ij}^{(i)} = \sum_{\nu=1}^{m_i+m_j} E_\nu^{(i,j)}\xi_\nu^{(i,j)} \tag{2.2}$$

and

$$m_j\tau_{ij}^{(j)} = \sum_{\nu=1}^{m_i+m_j} E_\nu^{(i,j)}\eta_\nu^{(i,j)}, \tag{2.3}$$

where the $\{E_\nu^{(i,j)}, \nu = 1, \cdots, m_i + m_j); i < j\}$ are constants satisfying certain restrictions to be stated below. Then we propose to consider the test statistics of the form

$$V = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_i m_j h_{ij} \tag{2.4}$$

for testing the null hypothesis against the alternative of ordered cumulative distribution functions.

Relationship to other tests. The V test presented here is a simple extension to several groups, of a class of procedures, which have been frequently recommended for the problem of deciding whether two samples come from the same population. For example, when $E_\nu^{(i,j)} = \nu/(m_i + m_j)$, the test described above coincides with the Jonckheere test [11] which is a direct generalization of the one-sided Wilcoxon test discussed in detail by Mann and Whitney [14]. When $k = 2$ and $E_\nu^{(i,j)}$ is the expected value of the k-th order statistic of a sample of size $(m_i + m_j)$ from the standard normal distribution function, then the V test is the same as the one-tail normal scores test (which is asymptotically equivalent to the Van der Waerden test) discussed in detail by Hoeffding [10], Terry [19], Chernoff and Savage [6], Hodges and Lehmann [9], and the author [16]. When $k = 2$ and $E_\nu^{(i,j)}$ is the expected value of the ν-th order statistic of a sample of size $(m_i + m_j)$ from the exponential distribution, then the V test reduces to the I. R. Savage test [18].

3. Assumptions and Notations

Let $X_{i1}, \cdots, X_{im_i}$ be the ordered observations of a random sample from a population with continuous c.d.f. $F_i(x)$. Let $N = \sum_{i=1}^{k} m_i$ and suppose that the m_i tend to infinity in such a way that $m_i = \rho_i \cdot N$, $N \to \infty$. Write $m_{ij} = m_i + m_j$. Let $F_{m_i}(x)$ be the sample c.d.f. of m_i observations X_i. Then

$$H_{m_{ij}}(x) = \frac{m_i}{m_{ij}} F_{m_i}(x) + \frac{m_j}{m_{ij}} F_{m_j}(x)$$

is the combined sample c.d.f. of the i-th and j-th samples. The combined population c.d.f. of the i-th and j-th samples is

$$H_{ij}(x) = \frac{m_i}{m_{ij}} F_i(x) + \frac{m_j}{m_{ij}} F_j(x) .$$

Then the following representation of h_{ij} is equivalent to (2.1):

$$h_{ij} = \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, d(F_{m_i}(x) - F_{m_j}(x)) , \tag{3.1}$$

where

$$J_{(m_{ij})}[\nu/m_{ij}] = E_\nu^{(i,j)}, \qquad \nu = 1, \cdots, m_{ij} , \qquad i < j = 1, \cdots, k .$$

While the function $J_{(m_{ij})}$ need be defined only at $1/m_{ij}, \cdots, m_{ij}/m_{ij}$, we may extend its domain of definitions to $(0, 1]$ by letting it be constant on $(\nu/m_{ij}, (\nu + 1)/m_{ij}]$. Furthermore, we make the following assumptions:[1]

Assumption 1. $\lim_{N\to\infty} J_N(u) = J(u)$ exists for $0 < u < 1$ and is not a constant.

Assumption 2.

$$\int_{I_{m_{ij}}} [J_{(m_{ij})}[H_{m_{ij}}(x)] - J[H_{m_{ij}}(x)]]\, dF_{m_i}(x) = o_p(1/\sqrt{N}) ,^2$$

where

$$I_{m_{ij}} = \{x\colon 0 < H_{m_{ij}}(x) < 1\}, \qquad (i,j) = 1, \cdots, k, \qquad i < j .$$

Assumption 3. $J_N(1) = o(N^{1/2})$.

Assumption 4. $|J^{(i)}(u)| = |d^{(i)}J/du^{(i)}| \leqq K[u(1 - u)]^{\delta - 1/2 - i}$, $i = 0, 1, 2$, for some K and some $\delta > 0$.

4. Asymptotic Normality

We shall prove the following theorem.

Theorem 4.1. *Under Assumptions* 1–4,

$$\lim_{N\to\infty} P\left[N^{-3/2}\left(\frac{V - \mu_N}{\sigma_N}\right) \leqq t\right] = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx , \tag{4.1}$$

where

$$\mu_N = \sum_{i<j}\sum m_i m_j \left[\int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_i(x) - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_j(x)\right] \tag{4.2}$$

[1] These assumptions are analogous to those of Chernoff and Savage [6], to which paper the reader is also referred to for general background.

[2] If $\{X_n\}$ is a sequence of random variables and $\{f_n\}$ a sequence of positive numbers, we write $X_n = o_p(f_n)$ if $X_n/f_n \to 0$ in probability, or equivalently, if for each $\varepsilon > 0$ there is a sequence $M_{n_\varepsilon} \to 0$ such that $P\{|x_n| > M_{n_\varepsilon} f_n\} < 1 - \varepsilon$.

and

$$\sigma_N^2 = \sum_{i<j}\sum \rho_i^2\rho_j^2\sigma_{ij}^2 + \sum_{\substack{i<j;\ r<s\\ \{i,j\neq r,s\}}}\sum\sum\sum \rho_i\rho_j\rho_r\rho_s\sigma_{ij,rs}\,, \tag{4.3}$$

σ_{ij}^2 *and* $\sigma_{ij,rs}$ *being given by* (4.12) *and* (4.13), *respectively.*

The proof of this theorem rests on the following lemma.

LEMMA 4.1. *Under Assumptions* 1–4, *the matrix with elements* $N^{1/2}(h_{ij} - \mu_{ij})$, *where*

$$\mu_{ij} = \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_i(x) - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_j(x)\,, \tag{4.4}$$

has a limiting normal distribution with zero mean and covariance matrix given by (4.12) *and* (4.13).

Proof of Lemma 4.1: We can rewrite h_{ij} (cf. [16]) as

$$\begin{aligned} h_{ij} &= \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, dF_{m_i}(x) - \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, dF_{m_j}(x) \\ &= \mu_{ij} + B_{m_{ij}} + \sum_{K=1}^{6} C_{K,m_{ij}}^{(i)} + \sum_{K=1}^{6} C_{K,m_{ij}}^{(j)}\,, \end{aligned} \tag{4.5}$$

where

$$\mu_{ij} = \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_i(x) - F_j(x))\,, \tag{4.6}$$

$$\begin{aligned} B_{m_{ij}} &= \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_{m_i}(x) - F_i(x)) \\ &\quad + \int_{-\infty}^{+\infty} [H_{m_{ij}}(x) - H_{ij}(x)]J'[H_{ij}(x)]\, dF_i(x) \\ &\quad - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_{m_j}(x) - F_j(x)) \\ &\quad - \int_{-\infty}^{+\infty} [H_{m_{ij}}(x) - H_{ij}(x)]J'[H_{ij}(x)]\, dF_j(x)\,, \end{aligned} \tag{4.7}$$

and the C-terms are all $o_p(N^{-1/2})$ (cf. [16]). The difference $N^{1/2}(h_{ij} - \mu_{ij}) - N^{1/2}B_{m_{ij}}$ tends to zero in probability and so the matrices with elements $N^{1/2}(h_{ij} - \mu_{ij})$ and $N^{1/2}B_{m_{ij}}$ possess the same limiting distribution if they have one at all. Thus, to prove this lemma, it suffices to show that for any real δ_{ij}, $i < j = 1, \cdots, k$, not all zero, $N^{1/2}\sum_{i<j}\sum \delta_{ij}B_{m_{ij}}$ has the normal distribution in the limit. Now

$B_{m_{ij}}$, after omitting straightforward but tedious computations, can be rewritten as

$$B_{m_{ij}} = \frac{1}{m_i} \sum_{\nu=1}^{m_j} [B_j(X_{i\nu}) - EB_j(X_i)] \tag{4.8}$$
$$- \frac{1}{m_j} \sum_{\nu=1}^{m_i} [B_i^*(X_{j\nu}) - EB_i^*(X_j)] ,$$

where

$$B_j(x) = \int_{x_0}^{x} J'[H_{ij}(y)]\, dF_j(y) \tag{4.9}$$

and

$$B_i^*(x) = \int_{x_0}^{x} J'[H_{ij}(y)]\, dF_i(y) \tag{4.10}$$

with x_0 determined somewhat arbitrarily, say by $H_{ij}(x_0) = \frac{1}{2}$; E represents the expectation and X_i has the F_i distribution. The rest of the proof follows by standard arguments, see for example Bhuchongkul and Puri [4].

To compute the variance-covariance matrix of $B_{m_{ij}}$, we note from (4.8) that $B_{m_{ij}}$ can be rewritten as

$$B_{m_{ij}} = \int_{-\infty}^{+\infty} B_j(x)\, d[F_{m_i}(x) - F_i(x)] - \int_{-\infty}^{+\infty} B_i^*(x)\, d[F_{m_j}(x) - F_j(x)]$$
$$= - \int_{-\infty}^{+\infty} [F_{m_i}(x) - F_i(x)] J'[H_{ij}(x)]\, dF_j(x) \tag{4.11}$$
$$+ \int_{-\infty}^{+\infty} [F_{m_j}(x) - F_j(x)] J'[H_{ij}(x)]\, dF_i(x) .$$

Since the two samples are independent and $EB_{m_{ij}} = 0$, we have

$$\sigma_{ij}^2 = \mathrm{Var}\,(B_{m_{ij}}) = E\left\{\int_{-\infty}^{+\infty} [F_{m_i}(x) - F_i(x)] J'[H_{ij}(x)]\, dF_j(x)\right\}^2$$
$$+ E\left\{\int_{-\infty}^{+\infty} [F_{m_j}(x) - F_j(x)] J'[H_{ij}(x)]\, dF_i(x)\right\}^2 .$$

This gives (after omitting the routine computations)

$$\sigma_{ij}^2 = \frac{2}{m_i} \iint_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)] J'[H_{ij}(x)] J'[H_{ij}(y)]\, dF_j(x)\, dF_j(y) \tag{4.12}$$
$$+ \frac{2}{m_j} \iint_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)] J'[H_{ij}(x)] J'[H_{ij}(y)]\, dF_i(x)\, dF_i(y) .$$

Note that the application of Fubini's theorem permits the interchange of integral and expectation.

Similarly,

$$\sigma_{ij,rs} = \operatorname{Cov}(B_{m_{ij}}, B_{m_{rs}})$$

$$(4.13)\quad \begin{cases} = 0 \quad \text{if } i, j, r, s \text{ are distinct,} \\ = \dfrac{1}{m_i}\Bigg[\displaystyle\iint_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)]J'[H_{ij}(x)]J'[H_{is}(y)]\, dF_j(x)\, dF_s(y) \\ \quad + \displaystyle\iint_{-\infty<y<x<\infty} F_i(y)[1 - F_i(x)]J'[H_{ij}(x)]J'[H_{is}(y)]\, dF_j(x)\, dF_s(y)\Bigg] \\ \qquad\qquad \text{if } i = r,\ j \neq s, \\ = \dfrac{1}{m_j}\Bigg[\displaystyle\iint_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)]J'[H_{ij}(x)]J'[H_{rj}(y)]\, dF_i(x)\, dF_r(y) \\ \quad + \displaystyle\iint_{-\infty<y<x<\infty} F_j(y)[1 - F_j(x)]J'[H_{ij}(x)]J'[H_{rj}(y)]\, dF_i(x)\, dF_r(y)\Bigg] \\ \qquad\qquad \text{if } i \neq r,\ j = s, \\ = -\dfrac{1}{m_i}\Bigg[\displaystyle\iint_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)]J'[H_{ij}(x)]J'[H_{ri}(y)]\, dF_j(x)\, dF_r(y) \\ \quad + \displaystyle\iint_{-\infty<y<x<\infty} F_i(y)[1 - F_i(x)]J'[H_{ij}(x)]J'[H_{ri}(y)]\, dF_j(x)\, dF_r(y)\Bigg] \\ \qquad\qquad \text{if } i = s,\ j \neq r, \\ = -\dfrac{1}{m_j}\Bigg[\displaystyle\iint_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)]J'[H_{ij}(x)]J'[H_{js}(y)]\, dF_i(x)\, dF_s(y) \\ \quad + \displaystyle\iint_{-\infty<y<x<\infty} F_j(y)[1 - F_j(x)]J'[H_{ij}(x)]J'[H_{js}(y)]\, dF_j(x)\, dF_s(y)\Bigg] \\ \qquad\qquad \text{if } i \neq s,\ j = r. \end{cases}$$

We have now proved that the set of random variables $\{N^{1/2}(h_{ij} - \mu_{ij})\,,\, i < j\}$ is asymptotically normally distributed. Consequently, the matrix with elements $\{N^{-3/2}U_{ij}\,,\, i < j\}$, where $U_{ij} = m_i m_j(h_{ij} - \mu_{ij})$, has a limiting normal distribution. The theorem follows.

5. Asymptotic Distribution Under Translation Alternatives and Efficiency

In this section, we shall concern ourselves with a sequence of admissible alternative hypothesis H_N^P which specifies that, for each $i = 1, \cdots, k$, $F_i(x) = F(x + \theta_i/\sqrt{N})$, with $F \in \Omega$ but not specified further, and not all the θ's being equal.

THEOREM 5.1. *For each index N, assume that $m_i = \rho_i \cdot N$, with ρ_i a positive integer and that the hypothesis H_N^P is true. Let h_{ij} be defined as in* (3.1) *with the function J satisfying the assumptions of Lemma* 7.2 *of* [16]. *Then the matrix with elements $\{N^{-3/2}U_{ij}, i < j\}$, where $U_{ij} = m_i m_j(h_{ij} - \mu_{ij})$, has a joint asymptotic normal distribution with zero mean and convariance matrix*

$$\operatorname{Var}(N^{-3/2}U_{ij}) = \rho_i\rho_j(\rho_i + \rho_j)A^2,$$

$$(5.1)\quad \operatorname{Cov}(N^{-3/2}U_{ij}, N^{-3/2}U_{rs}) \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \rho_i\rho_j\rho_s A^2 & \text{if } i = r, \ j \neq s, \\ = \rho_i\rho_j\rho_r A^2 & \text{if } i \neq r, \ j = s, \\ = -\rho_i\rho_j\rho_r A^2 & \text{if } i = s, \ j \neq r, \\ = -\rho_i\rho_j\rho_s A^2 & \text{if } i \neq s, \ j = r, \end{cases}$$

where

$$(5.2)\qquad A^2 = \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2.$$

This theorem is an immediate consequence of Theorem 4.1 and the fact that under the assumptions of Lemma 7.2 of [16],

$$\lim_{N\to\infty} N\sigma_{ij,rs} \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \dfrac{A^2}{\rho_i} & \text{if } i = r, \ j \neq s, \\ = \dfrac{A^2}{\rho_i} & \text{if } i \neq r, \ j = s, \\ = -\dfrac{A^2}{\rho_i} & \text{if } i = s, \ j \neq r, \\ = -\dfrac{A^2}{\rho_j} & \text{if } i \neq s, \ j = r, \end{cases}$$

and

$$\lim_{N\to\infty} N\sigma_{ij}^2 = (1/\rho_i + 1/\rho_j)A^2.$$

Furthermore, since under the regularity assumptions

$$N^{1/2}(\mu_{ij}(\theta) - \mu_{ij}(0)) \to (\theta_j - \theta_i)\int \{dJ[F(x)]/dx\}\,dF(x),$$

we conclude

THEOREM 5.2. *For each index N assume that $m_i = \rho_i \cdot N$, with ρ_i a positive integer, and that the hypothesis H_N^P is true. Then the statistic $N^{-3/2}V$ has a limiting distribution with mean*

$$\sum_{i<j}\sum \rho_i\rho_j(\theta_j - \theta_i)\int \{dJ[F(x)]/dx\}\,dF(x)$$

and variance

$$\frac{1}{3}\left[\left(\sum_{i=1}^{k} \rho_i\right)^3 - \sum_{i=1}^{k} \rho_i^3\right] A^2 ,$$

where A^2 is given by (5.2). *Here the function J is assumed to satisfy the regularity conditions of Lemma* 7.2 *of* [16].

We are now in a position to make large sample comparison between different members of the V test and their normal theory competitor based on Student's statistic. We shall adopt a method developed by Pitman [15a] who defined the relative asymptotic efficiency of two sequences of tests as the limiting inverse ratio of sample sizes necessary to achieve the same power against the same sequences of alternatives at the same significance level.

THEOREM 5.3. *The asymptotic efficiency of the V test relative to the normal theory test based on the statistic*

$$T = \sum_{i<j}\sum m_i m_j (X_{i\cdot} - X_{j\cdot}) ,$$

where $X_{i\cdot} = \sum_{\alpha=1}^{m_i} X_{i\alpha}/m_i$, is

$$e_{V,T}(F) = \frac{\sigma^2}{A^2}\left(\int_{-\infty}^{+\infty} \{dJ[F(x)]/dx\}\, dF(x)\right)^2 , \tag{5.3}$$

where $\sigma^2 = \mathrm{Var}\,(X_{i\alpha})$.

Proof: Let $T_{ij} = X_{i\cdot} - X_{j\cdot}$ and $V'_{ij} = N^{1/2}(T_{ij} - (\xi_i - \xi_j))$. Then the variables $\{V'_{ij},\ i<j\}$ have an asymptotic normal distribution with zero mean and covariance matrix

$$\mathrm{Var}\,(V'_i) = \sigma^2(1/\rho_i + 1/\rho_j) ,$$

$$\mathrm{Cov}\,(V'_{ij,rs}) \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \dfrac{\sigma^2}{\rho_i} & \text{if } i = r,\ j \neq s , \\ = \dfrac{\sigma^2}{\rho_j} & \text{if } i \neq r,\ j = s , \\ = -\dfrac{\sigma^2}{\rho_i} & \text{if } i = s,\ j \neq r , \\ = -\dfrac{\sigma^2}{\rho_j} & \text{if } i \neq s,\ j = r . \end{cases}$$

Hence $N^{-3/2}T$ has a limiting normal distribution with zero mean and variance (after omitting the details of computation) equal to

$$\frac{\sigma^2}{3}\left[\left(\sum_{i=1}^{k} \rho_i\right)^3 - \sum_{i=1}^{k} \rho_i^3\right] .$$

Now proceeding by the standard arguments, see for example Puri [17] and Chernoff-Savage [6], the result follows.

The relative efficiency of the V test relative to the T test is the same as found by Chernoff-Savage [6] for the corresponding procedures in the two-sample problem, and shown by the author [16] to be valid also for the multi-sample problem (unrestricted alternatives).

Special cases. (i) let J be the inverse of the rectangular distribution on $(0, 1)$, then the V test reduces to the rank-sum $V(R)$ test, better known as the Jonckheere test [11]. The efficiency (5.3) then is equal to $12\sigma^2(\int f^2(x)\,dx)^2$. This is known to satisfy $e_{V(R),T}(F) \geqq 0.864$ for all F; $e_{V(R),T}(F) = \frac{3}{\pi} \sim 0.955$ when F is normal, and $e_{V(R),T}(F) > 1$ for many non-normal distributions. (For the Gamma distribution with parameter $p = 1$, $e_{V(R),T}(F) = 3$.)

(ii) Let $J = \Phi^{-1}$, where Φ is the standard normal distribution function. The V test reduces to the normal scores $V(\Phi)$ test. The efficiency then is known to satisfy $e_{V(\Phi),T}(F) \geqq 1$ for all F and $e_{V(\Phi),T}(F) = 1$ if and only if F is normal.

Thus from the asymptotic efficiency point of view both the $V(R)$ and $V(\Phi)$ tests can appear to be advantageous compared with T test unless one can be reasonably sure of the absence of gross errors and other departures from normality. (For the asymptotic efficiency comparison of the rank-sum to the normal scores procedure, see Hodges and Lehmann [9].)

In [2], under the assumptions of normality, Bartholomew derived the likelihood ratio statistic $\bar{E}^2$ relevant to the problem treated in this paper. Chacko [5] extended the work of Bartholomew and showed that the $\bar{E}^2$ statistic has the limiting non-central chi-square distribution as $N \to \infty$. Asymptotic relative efficiency cannot be used to compare the V and $\bar{E}^2$ tests because of the fact that the forms of their limiting distributions are different. For the same reason, it is not possible to find the asymptotic relative efficiency of the V and $\mathscr{F}$ tests. However, some light can be thrown on the question of the choice between the V, $\bar{E}^2$ and $\mathscr{F}$ tests by making large sample power comparisons. Some numerical results for $V(\Phi)$, $V(R)$, $\bar{E}^2$ and $\mathscr{F}$ are given in Table 1. Some of the figures for $\bar{E}^2$ and $\mathscr{F}$ have already been given by Bartholomew [2] but they are reproduced here for ease of comparison. It must be borne in mind that they are asymptotic results and that they involve the assumptions of normality. Furthermore, it is assumed that the sample sizes m_i are all equal. Two configurations of θ's are considered and the power in each case is expressed as a function of

$$\Delta = \sqrt{\sum_{i=1}^{k} (\theta_i - \bar{\theta})^2},$$

where $\bar{\theta} = \sum_{i=1}^{k} \theta_i/k$.

TABLE 1.[3] **The asymptotic power comparisons of $V(\Phi)$, $V(R)$, $\bar{E}^2$ and $\mathscr{F}$ when $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_k$.**

		Δ				
		0	1	2	3	4
$k=3$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.218	.532	.829	.965
	$V(R)$	.050	.252	.622	.901	.988
		.050	.212	.519	.814	.959
	$\bar{E}^2$	.050	.239	.594	.885	.980
		.050	.221	.569	.872	.983
	$\mathscr{F}$	.050	0.130	0.402	0.776	.959
$k=4$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.192	.460	.749	.926
	$V(R)$	.050	0.252	0.622	0.901	0.988
		.050	.187	.448	.734	.917
	$\bar{E}^2$	.050	.239	.594	.885	.980
		.050	.202	.531	.849	.978
	$\mathscr{F}$	0.050	0.115	0.350	0.710	0.945
$k=8$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.142	.311	.532	.744
	$V(R)$	.050	.252	.622	.901	.988
		.050	.140	.303	.519	.730
	$\bar{E}^2$	.050	—	—	—	—
		.050	.191	.456	.800	.973
	$\mathscr{F}$	.050	0.090	0.249	0.535	0.853
$k=12$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.121	.258	.417	.606
	$V(R)$	.050	.252	.622	.901	.988
		.050	.120	.240	.406	.592
	$\bar{E}^2$	.050	—	—	—	— —
		.050	.0178	.423	.766	.963
	$\mathscr{F}$	0.050	0.080	0.205	0.466	0.776

(i) θ's equally spaced. Let $\theta_{i-1} - \theta_i = A^*$, $i = 2, \cdots, k$, then $\theta_j - \theta_i = (i-j)A^*$, $i < j$. The asymptotic power of the $V(R)$ test (cf. Theorem 5.2) is then

$$\beta(V(R)) = 1 - \Phi[\lambda_\alpha - \Delta\sqrt{3/\pi}]\,, \tag{5.4}$$

and the asymptotic power of the $V(\Phi)$ test is given by (cf. Theorem 5.2)

$$\beta(V(\Phi)) = 1 - \Phi[\lambda_\alpha - \Delta]\,. \tag{5.5}$$

Here

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}$$

and λ_α is the upper 100α ·/· point of $\Phi(x)$.

[3] The upper figure of each pair corresponds to equal spacing of the θ's and the lower figure to the case when all but one of the θ's are equal.

(ii) $\theta_1 \geqq \theta_2 = \cdots = \theta_k$. The asymptotic powers of the $V(R)$ and $V(\Phi)$ tests in this case are given by

$$\beta(V(R)) = 1 - \Phi\left[\lambda_\alpha - \Delta\sqrt{\frac{3}{\pi}}\sqrt{\frac{3}{k+1}}\right] \tag{5.6}$$

and

$$\beta(V(\Phi)) = 1 - \Phi\left[\lambda_\alpha - \Delta\sqrt{\frac{3}{k+1}}\right]. \tag{5.7}$$

It is clear that, unless $k = 2$, both the $V(R)$ and $V(\Phi)$ tests are more powerful in detecting a given Δ when the means are equally spaced than when all but one are equal. Furthermore, in the latter case the power of the $V(R)$ test as well as that of the $V(\Phi)$ test decreases as k increases.

The following conclusions may be drawn from these results.

(i) The $V(\Phi)$, $V(R)$ and $\bar{E}^2$ tests are always to be preferred to the classical $\mathscr{F}$ test which assumes no prior information regarding the θ's.

(ii) The powers of the $V(\Phi)$, $V(R)$ and $\bar{E}^2$ tests for the case when all but one of the θ's are equal, are lower than the powers of the corresponding tests for the case when the θ's are equally spaced.

(iii) The $V(\Phi)$ test is superior to the $V(R)$ and $\bar{E}^2$ tests when the θ's are equally spaced; it is also superior to the $V(R)$ test but inferior to the $\bar{E}^2$ test when all but one of the θ's are equal.

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Received July, 1964.