Complex Multiplication Symmetry of Black Hole Attractors

Lynker, M., Periwal, V., & Schimmrigk, R.

To cite this manuscript: Lynker, M., Periwal, V., & Schimmrigk, R. (2003). Complex multiplication symmetry of black hole attractors. *Nucl.Phys.B*, *667*, 484–504. https://doi.org/10.1016/S0550-3213(03)00454-1

Complex Multiplication Symmetry of Black Hole Attractors

Monika Lynker$^{\star 1}$, Vipul Periwal$^{\diamond 2}$ and Rolf Schimmrigk$^{\dagger 3}$

[1] *Indiana University South Bend, South Bend, IN 46634*

[2] *Gene Network Sciences, Ithaca, NY 14850*

[3] *Kennesaw State University, Kennesaw, GA 30144*

Abstract

We show how Moore's observation, in the context of toroidal compactifications in type IIB string theory, concerning the complex multiplication structure of black hole attractor varieties, can be generalized to Calabi-Yau compactifications with finite fundamental groups. This generalization leads to an alternative general framework in terms of motives associated to a Calabi-Yau variety in which it is possible to address the arithmetic nature of the attractor varieties in a universal way via Deligne's period conjecture.

*Email: mlynker@iusb.edu
$^{\diamond}$Email: vipul@gnsbiotech.com
†Email: netahu@yahoo.com, rschimmr@kennesaw.edu

1 Introduction

During the past few years number theoretic considerations have become useful in string theory in addressing a variety of problems in string theory, such as the understanding of the underlying conformal field theory of Calabi-Yau manifolds [1, 2, 3, 4], the nature of black hole attractor varieties [5, 6], and the behavior of periods under reduction to finite fields [7]. Our aim in the present paper is to further develop and generalize some of the observations made by Moore in his analysis of the arithmetic nature of the so-called black hole attractor varieties. The attractor mechanism [8, 9, 10, 11] describes the radial evolution of vector multiplet scalars of spherical dyonic black hole solutions in $N = 2$ supergravity coupled to abelian vector multiplets. Under particular regularity conditions the vector scalars flow to a fixed point in their target space. This fixed point is determined by the charge of a black hole, described by a vector ω in the lattice Λ of electric and magnetic charges of the $N = 2$ abelian gauge theory. If the $N = 2$ supergravity theory is derived from a type IIB string theory compactified on a Calabi-Yau space, the vector multiplet moduli space is described by the moduli space $\mathcal{M}$ of complex structures of X, and the dyonic charge vector takes values in the lattice $\Lambda = \mathrm{H}^3(X,\mathbb{Z})$.

Moore observed in the context of simple toroidal product varieties, such as the triple product of elliptic curves E^3, or the product K3$\times$E of a K3 surface and an elliptic curve, that the attractor condition determines the complex moduli τ of the tori to be given by algebraic numbers in a quadratic imaginary field $\mathbb{Q}(\sqrt{D})$, obtained by adjoining to the rational numbers $\mathbb{Q}$ an imaginary number $\sqrt{D}$, where $D < 0$. This is of interest because for particular points in the moduli space elliptic curves exhibit additional symmetries, they admit so-called complex multiplication (CM). For compactifications with toroidal factors Moore's analysis then appears to indicate an interesting link between the 'attractiveness' of varieties in string theory and their complex multiplication properties.

Calabi-Yau varieties with elliptic factors are very special because they have infinite fundamental group, a property not shared by Calabi-Yau manifolds in general. Other special features of elliptic curves are not present in general either. In particular Calabi-Yau spaces are not abelian varieties and they do not, in any obvious fashion, admit complex multiplication symmetries. Hence it is not clear how Moore's observations can be generalized. It is this problem which we wish to address in the present paper. In order to do so we adopt a cohomological approach and view the modular parameter of the elliptic curve as part of the primitive cohomology. In the case of elliptic curves E this is simply a choice of view because there exists an isomorphism between the curve itself and its Jacobian defined by $J(\mathrm{E}) = \mathrm{H}^1(\mathrm{E},\mathbb{C})/\mathrm{H}^1(\mathrm{E},\mathbb{Z})$ described by the Abel-Jacobi map $j : \mathrm{E} \rightarrow J(\mathrm{E})$. These varieties are abelian.

The Jacobian variety of an elliptic (or more general) curve has a natural generalization to higher dimensional varieties, defined by the intermediate Jacobian of Griffiths. It would be natural to use Griffiths' construction in an attempt to generalize the elliptic results described above. In general, however, the intermediate Jacobian is not an abelian variety and does not admit complex multiplication. For this and other reasons we will proceed differently by constructing a decomposition of the intermediate cohomology of the Calabi-Yau and using this decomposition to formulate a generalization of the concept of complex multiplication of black hole attractor varieties. To achieve this we formulate complex multiplication in this more general context by analyzing in some detail the cohomology group $\mathrm{H}^3(X)$ of weighted Fermat hypersurfaces.

The paper is organized as follows. In order to make the presentation more self-contained we briefly review in Section 2 the physical setting of black hole attractors in type IIB theories, as well as Moore's solution of the $\mathrm{K3} \times \mathrm{E}$ solution of the attractor equations. In Section 3 we describe the necessary background of abelian varieties, and in Section 4 we show how abelian varieties can be derived from Calabi-Yau hypersurfaces by showing that the cohomology of such varieties can be constructed from the cohomology of curves embedded in these higher dimensional varieties. This leads us to abelian varieties defined by the Jacobians of curves. Such abelian varieties do not, in general, admit complex multiplication. What can be shown, however, is that Jacobians of projective Fermat curves split into abelian factors which *do* admit complex multiplication. We briefly describe this construction and generalize the discussion to curves of Brieskorn-Pham type. Combining these results shows that we can define the complex multiplication type of Calabi-Yau varieties with finite fundamental groups via the CM types of their underlying Jacobians. In Section 5 we indicate some of the arithmetic consequences for Calabi-Yau varieties that derive from the emergence of abelian varieties with complex multiplication in the context of black hole attractors.

In the process of our analysis we will recover the same fields which Moore uncovered by considering the fields generated by periods of higher dimensional varieties. Even though our approach is very different from Moore's, it is not completely unexpected that we should be able to recover the field of periods by considering the complex multiplication type. The reason for this is a conjecture of Deligne [12] which states that the field determined by the periods of a critical motive is determined by its L-function. Because Deligne's conjecture is important for our general view of the issue at hand, we briefly describe this conjecture in Section 6 in order to provide the appropriate perspective. Deligne's conjecture is in fact a theorem in the context of projective Fermat hypersurfaces [13], but has not been proven in the context of weighted hypersurfaces. Our results in essence can be viewed as support of this conjecture even in this more general context. In Section 7 we summarize our results and indicate possible generalizations.

2 Arithmetic of Elliptic Attractor Varieties

2.1 Attractor Varieties

In this paper we consider type IIB string theory compactified on Calabi-Yau threefold varieties. The field content of the string theory in 10D space X^{10} splits into two sectors according to the boundary conditions on the world sheet. The Neveu-Schwarz fields are given by the metric $g \in \Gamma(X^{10}, \mathrm{T}^*X^{10} \otimes \mathrm{T}^*X^{10})$, an antisymmetric tensor field $B \in \Gamma(X^{10}, \Omega^2)$ and the dilaton scalar $\phi \in C^\infty(X^{10}, \mathbb{R})$. The Ramond sector is spanned by even antisymmetric forms $A^p \in \Gamma(X^{10}, \Omega^p)$ of rank p zero, two, and four. Here $\Omega^p \longrightarrow X$ denotes the bundle of p-forms over the variety X.

In the context of the black hole solutions considered in [8] the pertinent sectors are given by the metric and the five-form field strength $\mathbf{F}$ of the Ramond-Ramond 4-form $\mathbf{A}^4$. The metric is assumed to be a static spherically symmetric spacetime which is asymptotically Minkowskian and describes an extremally charged black hole, leading to the ansatz

$$ds^2 = -e^{2U(r)} dt \otimes dt + e^{-2U(r)}(dr \otimes dr + r^2\sigma_2), \tag{1}$$

where r is the spatial three dimensional radius, σ_2 is the 2D angular element, and the asymptotic behavior is encoded via $e^{-U(r)} \to \infty$ for $r \to \infty$. The expansion of the five-form $\mathbf{F}$ leads to a number of different 4D fields, the most important in the present context being the field strengths F^L of four dimensional abelian fields, the number of which depends on the dimension of the cohomology group $\mathrm{H}^3(X)$ via

$$\mathbf{A}^4_{\mu mnp}(x, y) = \sum_L A^{4L}_\mu(x)\omega^L_{mnp}(y), \tag{2}$$

where $\{\omega_L\}_{L=1,\ldots,b_3}$ is a basis of $\mathrm{H}^3(X)$. This is usually written in terms of a symplectic basis $\{\alpha_a, \beta^a\}_{a=0,\ldots,h^{2,1}}$, for which $\int_X \alpha_a \wedge \beta^b = \delta^b_a$, leading to an expansion of the field strength of the form

$$\mathbf{F}(x, y) = \mathbf{F}^a(x) \wedge \alpha_a - \mathbf{G}_a(x) \wedge \beta^a. \tag{3}$$

Being a five-form in ten dimensions the field strength $\mathbf{F}$ admits (anti)self-duality constraints with respect to Hodge duality, $\mathbf{F} = \pm *_{10}\mathbf{F}$. The ten dimensional Hodge operator $*_{10}$ factorizes into a 4D and a 6D part $*_{10} = *_4 *_6$. A solution to the anti-selfduality constraint in 10D as well as the Bianchi identity $d\mathbf{F} = 0$ can be obtained by considering [5]

$$\mathbf{F} = \mathrm{Re}\left(\mathbf{E} \wedge (\omega^{2,1} + \omega^{0,3})\right), \tag{4}$$

where [14]

$$\mathbf{E} \equiv q \sin\theta d\theta \wedge d\phi - iq\frac{e^{2U(r)}}{r^2} dt \wedge dr \tag{5}$$

is a 2-form for which the four dimensional Hodge duality operator leads to $*_4\mathbf{E} = i\mathbf{E}$. The 6D Hodge dual on $\mathrm{H}^3(X)$ is defined by

$$*_6 = -i\Pi^{3,0} \oplus i\Pi^{2,1} \oplus -i\Pi^{1,2} \oplus i\Pi^{0,3}, \tag{6}$$

leading to a purely imaginary duality transformation of the internal part of $\mathbf{E}$.

Two standard maneuvers to derive the dynamics of a string background configuration are provided by the reduced IIB effective action with a sort of small superspace ansatz [11], and the supersymmetry variation constraints of the fermions in nontrivial backgrounds, in particular the gravitino and gaugino variations. We adopt the notation of [15]. Defining an inner product $<\cdot,\cdot>$ on $\mathrm{H}^3(X)$ via

$$< \omega, \eta >= \int_X \omega \wedge \eta, \tag{7}$$

the gravitino equation involves the integrated version of the 5-form field strength defined as [16] [17]

$$\mathbf{T}^- = e^{K/2} < \Omega, \mathbf{F}^- >= e^{K/2} \left(\mathcal{G}_a \mathbf{F}^{a-}(x) - z^a \mathbf{G}_a^-(x) \right), \tag{8}$$

with Kähler potential

$$e^{-K} = i < \Omega, \bar{\Omega} >= -i(z^a \bar{\mathcal{G}}_a - \bar{z}^a \mathcal{G}_a). \tag{9}$$

Here the second equation is written in terms of the periods

$$\begin{aligned} z^a &= < \Omega, \beta^a >= \int_{A^a} \Omega \\ \mathcal{G}_a &= < \Omega, \alpha_a >= \int_{B_a} \Omega, \end{aligned} \tag{10}$$

with respect to a symplectic homological basis $\{A^a, B_a\}_{a=0,\ldots,h^{2,1}} \subset \mathrm{H}_3(X)$ which is dual to the cohomological basis $\{\alpha_a, \beta^a\}_{a=0,\ldots,h^{2,1}} \subset \mathrm{H}^3(X)$. The holomorphic three-form thus can be expanded as

$$\Omega = z^a \alpha_a - \mathcal{G}_a \beta^a. \tag{11}$$

The supersymmetry transformation of the gravitino $\psi^A = \psi^A_\mu dx^\mu$ can then be written in terms of the components of $\mathbf{T}^-$ as

$$\delta\psi^A = D\varepsilon^A + dx^\mu \mathrm{T}^-_{\mu\nu}\gamma^\nu(\epsilon\varepsilon)^A, \tag{12}$$

where γ^μ denotes the covariant Dirac matrices and

$$D = dx^\mu D_\mu = dx^\mu \left(\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + iQ_\mu \right) \tag{13}$$

is a derivative covariant with respect to both the Lorentz and the Kähler transformations in terms of the spin connection ω_μ^{ab} and the Kähler connection Q_μ. The variation of the gaugino of the abelian multiplets takes the form

$$\delta\lambda^{iA} = i\gamma^\mu\partial_\mu z^i\epsilon^A + \frac{i}{2}G_{\mu\nu}^{i-}\gamma^{\mu\nu}(\epsilon\varepsilon)^A. \tag{14}$$

Plugging these ingredients into the supersymmetry transformation behavior of the gravitino and the gaugino fields, and demanding that the vacuum remains fermion free, leads to the following equations for the moduli z^i and the spacetime function $U(r)$

$$\begin{aligned} \frac{dU}{d\rho} &= -e^U|Z| \\ \frac{dz^i}{d\rho} &= -2e^U g^{i\bar{j}}\partial_{\bar{j}}|Z|, \end{aligned} \tag{15}$$

where $\rho = 1/r$, $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ is the metric derived from the Kähler potential K, and

$$Z(\Gamma) = e^{K/2}\int_\Gamma \Omega = e^{K/2}\int_X \eta_\Gamma \wedge \Omega \tag{16}$$

is the central charge of the cycle $\Gamma \in \mathrm{H}_3(X)$ with Poincare dual $\eta_\Gamma \in \mathrm{H}^3(X)$. To make the moduli and charge dependence of the central charge explicit one can alternatively view $Z(\Gamma)$ as the integral of the graviphoton form

$$Z(z^a, p^a, q_a) = \int_{S^2} \mathbf{T}^- = e^{K/2}\left(\mathcal{G}_a p^a - z^a q_a\right) \tag{17}$$

in terms of the charges

$$\begin{aligned} p^a &= \int_{S^2} \mathbf{F}^{a-} \\ q_a &= \int_{S^2} \mathbf{G}_a^-. \end{aligned} \tag{18}$$

The fixed point condition of the attractor equations can be written in a geometrical way as the Hodge condition

$$\mathrm{H}^3(X, \mathbb{Z}) \ni \omega = \omega^{3,0} + \omega^{0,3}. \tag{19}$$

Writing $\omega^{3,0} = -i\bar{C}\Omega$ this can be formulated as

$$\begin{aligned} ip^a &= \bar{C}z^a - C\bar{z}^a \\ iq_a &= \bar{C}\mathcal{G}_a - C\bar{\mathcal{G}}_a, \end{aligned} \tag{20}$$

where $C = e^{K/2} Z$.

The system (20) describes a set of $b_3(X)$ charges (p^a, q_a) determined by the physical 4-dimensional input, which in turn determines the complex periods of the Calabi-Yau variety. Hence the system should be solvable. The interesting structure of the fixed point which emerges is that the central charges are determined completely in terms of the charges of the four-dimensional theory. As a consequence the 4D geometry is such that the horizon is a moduli independent quantity. This is precisely as expected because the black hole entropy should not depend on adiabatic changes of the environment [18].

2.2 Arithmetic of Attractor Elliptic Curves

In reference [5] Moore noted that two types of solutions of the attractor equations have particularly interesting properties. The first of these is provided by the triple product of a torus, while the second is a product of a K3 surface and a torus. Both solutions are special in the sense that they involve elliptic curves. In the case of the product threefold $X = \mathrm{K3} \times \mathrm{E}$ the simplifying feature is that via Künneth's theorem one finds $\mathrm{H}^3(\mathrm{K3} \times \mathrm{E}) \cong \mathrm{H}^2(\mathrm{K3}) \otimes \mathrm{H}^1(\mathrm{E})$, and therefore the cohomology group of the threefold in the middle dimension is isomorphic to two copies of the cohomology group $\mathrm{H}^2(\mathrm{K3})$ because $\mathrm{H}^1(\mathrm{E})$ is two dimensional. The attractor equations for such threefolds have been considered in [19]. The resulting constraints determine the holomorphic form of both factors in terms of the charges (p, q) of the fields. The complex structure τ of the elliptic curve $\mathrm{E}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is solved as

$$\tau_{p,q} = \frac{p \cdot q + \sqrt{D_{p,q}}}{p^2}, \tag{21}$$

where $D_{p,q} = (p \cdot q)^2 - p^2 q^2$ is the discriminant of a BPS state labelled by

$$\omega = (p, q) \in \mathrm{H}^3(\mathrm{K3} \times \mathrm{E}, \mathbb{Z}). \tag{22}$$

The holomorphic two form on K3 is determined as $\Omega^{2,0} = \mathcal{C}(q - \bar{\tau} p)$, where $\mathcal{C}$ is a constant.

Moore makes the interesting observation that this result is known to imply that the elliptic curve determined by the attractor equation is distinguished by exhibiting a particularly symmetric structure. Elliptic curves are groups and therefore one can consider the endomorphism algebra End(E). This algebra can take one of three forms. Generally, End(E) is just the ring $\mathbb{Z}$ of rational integers. For special curves however there are two other possibilities for which End(E) is either an order of a quadratic imaginary field F, i.e. it is a subring $\mathcal{O}_F$ which generates F as a $\mathbb{Q}$-vector space and is finitely generated as a $\mathbb{Z}-$module, or it is a maximal order in a quaternion algebra. The latter possibility can occur only when the field K over which E is defined has positive characteristics. Elliptic curves are said to admit complex multiplication if the endomorphism algebra is strictly larger than the ring of rational integers[1].

The point here is that the property of complex multiplication appears if and only if the j-invariant $j(\tau)$ of an elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is an algebraic integer, i.e. it solves a polynomial equation with rational coefficients such that the coefficient of the leading term is unity. This happens if and only if the modulus τ is an imaginary quadratic number, i.e. it solves an equation $A\tau^2 + B\tau + C = 0$ for $A, B, C \in \mathbb{Z}$. The j-invariant of the elliptic curve E_τ can be defined in terms of the Eisenstein series

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ m,n \text{ coprime}}} \frac{1}{(m\tau + n)^k} \tag{23}$$

as

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \tag{24}$$

where $1728\Delta(\tau) = E_4(\tau)^3 - E_6(\tau)^2$. In general $j(\tau)$ does not take algebraic values, not to mention values in an imaginary quadratic field. Even if τ is an algebraic number will $j(\tau)$ be a transcendental number unless τ is imaginary quadratic. Thus we see that in the framework of toroidal compactification the solutions of the attractor equations can be characterized as varieties which are unusually symmetric and which admit complex multiplication by a quadratic imaginary field $F = \mathbb{Q}(i\sqrt{|D|})$.

Once this is recognized several classical results about elliptic curves with complex multiplication are available to illuminate the nature of the attractor variety. Exploring these consequences is of interest because it provides tools that allow a characterization of attractor varieties that involve elliptic factors. The nature of attractor varieties without elliptic factors

[1] A brief review of the arithmetic theory of elliptic curves can be found in [20]. A more extended source [21].

is at present not understood, and finding generalizations of the arithmetic results obtained in the elliptic context provides a framework in which Calabi-Yau varieties with finite fundamental groups (which may be trivial) can be explored.

One of the important number theoretic results associated to elliptic curves with complex multiplication is that the extension $F(j(\tau))$ obtained by adjoining the j-value to F is the maximal unramified extension of F with an abelian Galois group, i.e. the Hilbert class field. Geometrically there is a Weierstrass model, i.e. a projective embedding of the elliptic curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{25}$$

that is defined over this extension $F(j(\tau))$.

Even more interesting is that it is possible to construct from the geometry of the elliptic curve the maximal abelian extension $F_{\rm ab}$ of F by considering the torsion points $\mathrm{E}_{\rm tor}$ on the curve E, i.e. points of finite order with respect to the group law. The field $F(j(\tau), \mathrm{E}_{\rm tor})$, defined by the $j-$function and the torsion points, is in general not an abelian extension of F, but contains the maximal abelian extension $F_{\rm ab}$. It is possible to isolate $F_{\rm ab}$ by mapping the torsion points via the Weber function

$$\Phi_{\mathrm{E}} : \mathrm{E} \longrightarrow \mathbb{P}_1, \tag{26}$$

of the curve E, whose definition depends on the automorphism group of the curve. Assuming that the characteristic of the field F is different from 2 or 3, the elliptic curve can be embedded via the simplified Weierstrass form

$$y^2 = x^3 + Ax + B. \tag{27}$$

If the discriminant

$$\Delta = -16(4A^3 + 27B^2) \tag{28}$$

does not vanish the elliptic curve is smooth, and the automorphism group Aut(E) can be shown to take one of the following forms, depending on the value of the $j-$invariant

$$\mathrm{Aut}(\mathrm{E}) = \left\{ \begin{array}{ll} \{\pm 1\} & \text{if } j(\mathrm{E}) \neq 0 \text{ or } 1728, \text{ i.e. } AB \neq 0 \\ \{\pm 1, \pm i\} & \text{if } j(\mathrm{E}) = 1728, \text{ i.e. } B = 0 \\ \{\pm 1, \pm \xi_3, \pm \xi_3^2\} & \text{if } j(\mathrm{E}) = 0, \text{ i.e. } A = 0, \end{array} \right\} \tag{29}$$

where $\xi_3 = e^{2\pi i/3}$ is a primitive third root of unity.

The Weber function can then be defined as

$$\Phi_{\mathrm{E}}(p) = \left\{ \begin{array}{ll} \frac{AB}{\Delta} x(p) & \text{if } j(\mathrm{E}) \neq 0 \text{ or } 1728 \\ \frac{A^2}{\Delta} x^2(p) & \text{if } j(\mathrm{E}) = 1728 \\ \frac{B}{\Delta} x^3(p) & \text{if } j(\mathrm{E}) = 0. \end{array} \right\} \tag{30}$$

and the Hilbert class field $F(j(\tau))$ can be extended to the maximal abelian extension F_{ab} of F by adjoining the Weber values of the torsion points

$$F_{\mathrm{ab}} = F(j(\tau), \{\Phi_{\mathrm{E}}(t) \mid t \in \mathrm{E}_{\mathrm{tor}}\}). \tag{31}$$

We see from these results that the attractor equations pick out special elliptic curves with an enhanced symmetry group. This is an infinite discrete group which in turn leads to a rich arithmetic structure. It is this set of tools which we wish to generalize to the framework of Calabi-Yau varieties proper, i.e. those with finite fundamental group.

3 Abelian Varieties with Complex Multiplication

We first review some pertinent definitions of abelian varieties. An abelian variety over some number field K is a smooth, geometrically connected, projective variety which is also an algebraic group with a group law $A \times A \longrightarrow A$ defined over K. A concrete way to construct abelian varieties is via complex tori $\mathbb{C}^n/\Lambda$ with respect to some lattice Λ, that is not necessarily integral, and admits a Riemann form. The latter is defined as an $\mathbb{R}$-bilinear form $<,>$ on $\mathbb{C}^n$ such that $< x, y >$ takes integral values for all $x, y \in \Lambda$, and satisfies the relations $< x, y >= - < y, x >$. Furthermore, $< x, iy >$ is a positive symmetric form, not necessarily non-degenerate. The result then is that a complex torus $\mathbb{C}^n/\Lambda$ has the structure of an abelian variety if and only if there exists a non-degenerate Riemann form on $\mathbb{C}^n/\Lambda$.

A special class of abelian varieties are those of CM type, so-called complex multiplication type. The reason these varieties are special is because, as in the lower dimensional case of elliptic curves, certain number theoretic question can be addressed in a systematic fashion for this class. Consider a number field F over the rational numbers $\mathbb{Q}$ and denote by $[F : \mathbb{Q}]$ the degree of the field F over $\mathbb{Q}$, i.e. the dimension of F over the subfield $\mathbb{Q}$. An abelian variety A of dimension n is called a CM−variety if there exists an algebraic number field F of degree $[F : \mathbb{Q}] = 2n$ over the rationals $\mathbb{Q}$ which can be embedded into the endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q}$ of the variety. More precisely, a CM variety is a pair (A, θ) with $\theta : F \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ an embedding of F. It follows from this that the field F necessarily is a CM field, i.e. a totally imaginary quadratic extension of a totally real field. The important ingredient here is that restriction to $\theta(F) \subset \mathrm{End}(A) \otimes \mathbb{Q}$ is equivalent to the direct sum of n isomorphisms $\phi_1, ..., \phi_n \in \mathrm{Iso}(F, \mathbb{C})$ such that $\mathrm{Iso}(F, \mathbb{C}) = \{\phi_1, ..., \phi_n, \rho\phi_1, ..., \rho\phi_n\}$, where ρ denotes complex conjugation. These considerations lead to the definition of calling the pair $(F, \{\phi_i\})$ a CM type, in the present context, the CM type of a CM variety (A, θ).

The context in which these concepts will appear in this paper is provided by varieties which have complex multiplication by a cyclotomic field $F = \mathbb{Q}(\mu_n)$, where μ_n denotes the cyclic group generated by a primitive n'th root of unity ξ_n. The field $\mathbb{Q}(\mu_n)$ is the imaginary quadratic extension of the totally real field $\mathbb{Q}(\xi_n + \bar{\xi}_n) = \mathbb{Q}(\cos(2\pi/n))$ and therefore is a CM field. The degree of $\mathbb{Q}(\mu_n)$ is given by $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \varphi(n)$, where $\varphi(n) = \#\{m \in \mathbb{N} \mid m < n,\ \gcd(m, n) = 1\}$ is the Euler function. Hence the abelian varieties we will encounter will have complex dimension $\varphi(n)/2$. Standard references for abelian varieties with complex multiplication have been provided by Shimura [22, 23, 24].

In the following parts we first reduce the cohomology of the Brieskorn-Pham varieties to that generated by curves and then analyze the structure of the resulting weighted curve Jacobians.

4 Abelian Varieties from Brieskorn-Pham type hypersurfaces

4.1 Curves and the cohomology of threefolds

The difficulty of higher dimensional varieties is that there is no immediate way to recover abelian varieties, thus making it non-obvious how to generalize the concept of complex multiplication from one-dimensional Calabi-Yau varieties, which are abelian varieties, to K3 surfaces and higher dimensional spaces. As a first step we need to disentangle the Jacobian of the elliptic curve from the curve itself. This would lead us to the concept of the middle-dimensional cohomology, more precisely the intermediate (Griffiths) Jacobian which is the appropriate generalization of the Jacobian of complex curves. The problem with this intermediate Jacobian is that it is not, in general, an abelian variety.

We will show now that it is possible nevertheless to recover abelian varieties as the basic building blocks of the intermediate cohomology in the case of weighted projective hypersurfaces. The basic reason for this is that the cohomology $\mathrm{H}^3(X)$ for these varieties decomposes into the monomial part and the part coming from the resolution. The monomial part of the intermediate cohomology can be obtained from the cohomology of a projective hypersurface of the same degree by realizing the weighted projective space as a quotient variety with respect to a product of discrete groups determined by the weights of the coordinates. For projective varieties

$$X_d^n = \left\{(z_0, ..., z_{n+1}) \in \mathbb{P}_{n+1} \mid z_0^d + \cdots + z_{n+1}^d = 0\right\} \subset \mathbb{P}_{n+1} \quad (32)$$

it was shown in [26] that the intermediate cohomology can be determined by lower-dimensional varieties in combination with Tate twists by reconstructing the higher dimensional variety X_d^n of degree d and dimension n in terms of lower dimensional varieties X_d^r and X_d^s of the same degree with $n = r + s$. Briefly, this works as follows. The decomposition of X_d^n is given as

$$X_d^{r+s} \;\cong\; B_{Z_1,Z_2}\left(\left(\pi_Y^{-1}(X_d^r \times X_d^s)\right)/\mu_d\right), \quad (33)$$

which involves the following ingredients.

(1) $\pi_Y^{-1}(X_d^r \times X_d^s)$ denotes the blow-up of $X_d^r \times X_d^s$ along the subvariety

$$Y = X_d^{r-1} \times X_d^{s-1} \subset X_d^r \times X_d^s. \tag{34}$$

The variety Y is determined by the fact that the initial map which establishes the relation between the three varieties X_d^{r+s}, X_d^r, X_d^s is defined on the ambient spaces as

$$((x_0, ..., x_{r+1}), (y_0, ..., y_{s+1}) \ \mapsto \ (x_0 y_{s+1}, ..., x_r y_{s+1}, x_{r+1} y_0, ..., x_{r+1} y_s). \tag{35}$$

This map is not defined on the subvariety Y.

(2) $\pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d$ denotes the quotient of the blow-up $\pi_Y^{-1}(X_d^r \times X_d^s)$ with respect to the action of

$$\mu_d \ni \xi : ((x_0, ..., x_r, x_{r+1}), (y_0, ..., y_s, y_{s+1})) \ \mapsto \ ((x_0, ..., x_r, \xi x_{r+1}), (y_0, ..., y_s, \xi y_{s+1})).$$

(3) $B_{Z_1,Z_2}\left(\left(\pi_Y^{-1}(X_d^r \times X_d^s)\right)/\mu_d\right)$ denotes the blow-down in $\pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d$ of the two subvarieties

$$Z_1 = \mathbb{P}_r \times X_d^{s-1}, \qquad Z_2 = X_d^{r-1} \times \mathbb{P}_s.$$

This construction leads to an iterative decomposition of the cohomology which takes the following form. Denote the Tate twist by

$$\mathrm{H}^i(X)(j) := \mathrm{H}^i(X) \otimes W^{\otimes j} \tag{36}$$

with $W = \mathrm{H}^2(\mathbb{P}_1)$ and let X_d^{r+s} be a Fermat variety of degree d and dimension $r+s$. Then

$$\begin{aligned} \mathrm{H}^{r+s}(X_d^{r+s}) \quad &\oplus \quad \sum_{j=1}^{r} \mathrm{H}^{r+s-2j}(X_d^{r-1})(j) \oplus \sum_{k=1}^{s} \mathrm{H}^{r+s-2k}(X_d^{s-1})(k) \\ &\qquad \cong \mathrm{H}^{r+s}(X_d^r \times X_d^s)^{\mu_d} \oplus \mathrm{H}^{r+s-2}(X_d^{r-1} \times X_d^{s-1})(1). \end{aligned} \tag{37}$$

This allows us to trace the cohomology of higher dimensional varieties to that of curves.

Weighted projective hypersurfaces can be viewed as resolved quotients of hypersurfaces embedded in ordinary projective space. The resulting cohomology has two components, the invariant part coming from the projection of the quotient, and the resolution part. As described in [27], the only singular sets on arbitrary weighted hypersurface Calabi-Yau threefolds are either points or curves. The resolution of singular points contributes to the even cohomology group $\mathrm{H}^2(X)$ of the variety, but does not contribute to the middle-dimensional cohomology group $H^3(X)$. Hence we need to be concerned only with the resolution of curves (see e.g. [28]). This can be described for general CY hypersurface threefolds as follows. If a discrete symmetry group $\mathbb{Z}/n\mathbb{Z}$ of order n acting on the threefold leaves invariant a curve then the normal bundle has fibres $\mathbb{C}_2$ and the discrete group induces an action on these fibres which can be described by a matrix

$$\begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix}, \tag{38}$$

where α is an n'th root of unity and (q, n) have no common divisor. The quotient $\mathbb{C}_2/(\mathbb{Z}/n\mathbb{Z})$ by this action has an isolated singularity which can be described as the singular set of the surface in $\mathbb{C}_3$ given by the equation

$$S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 \mid z_3^n = z_1 z_2^{n-q}\}. \tag{39}$$

The resolution of such a singularity is completely determined by the type (n, q) of the action by computing the continued fraction of $\frac{n}{q}$

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}} \equiv [b_1, ..., b_s]. \tag{40}$$

The numbers b_i specify completely the plumbing process that replaces the singularity and in particular determine the additional generator to the cohomology $\mathrm{H}^*(X)$ because the number of $\mathbb{P}_1$s introduced in this process is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, ..., b_s]$. This can be traced to the fact that the singularity is resolved by a bundle which is constructed out of $s + 1$ patches with s transition functions that are specified by the numbers b_i. Each of these glueing steps introduces a sphere, which in turn supports a (1,1)-form. The intersection properties of these 2-spheres are described by Hirzebruch-Jung trees, which for a $\mathbb{Z}/n\mathbb{Z}$ action is just an $SU(n + 1)$ Dynkin diagram, while the numbers b_i describe the intersection numbers. We see from this that the resolution of a curve of genus g thus introduces s additional generators to the second cohomology group $\mathrm{H}^2(X)$, and $g \times s$ generators to the intermediate cohomology $\mathrm{H}^3(X)$.

Hence we have shown that the cohomology of weighted hypersurfaces is determined completely by the cohomology of curves. Because the Jacobian, which we will describe in the next subsection, is the only motivic invariant of a smooth projective curve this says that for weighted hypersurfaces the main motivic structure is carried by their embedded curves. We will come back to the motivic structure of Calabi-Yau varieties in Section 6.

4.2 Cohomology of weighted curves

For smooth algebraic curves C of genus g the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(C)$ decomposes (over the complex number field $\mathbb{C}$) as

$$\mathrm{H}^1_{\mathrm{dR}}(C) \;\cong\; \mathrm{H}^0(C,\Omega^1) \oplus \mathrm{H}^1(C,\mathcal{O}). \tag{41}$$

The Jacobian $J(C)$ of a curve C of genus g can be identified with

$$J(C) = \mathbb{C}^g/\Lambda, \tag{42}$$

where Λ is the period lattice

$$\Lambda := \left\{ \left(\ldots, \int_a \omega_i, \ldots\right)_{i=1,\ldots,g} \;\Big|\; a \in \mathrm{H}_1(C,\mathbb{Z}),\ \omega_i \in \mathrm{H}^0(C,\Omega^1) \right\}, \tag{43}$$

where the ω_i form a basis. Given a fixed point $p_0 \in C$ on the curve there is a canonical map from the curve to the Jacobian, called the Abel-Jacobi map

$$\Psi : C \longrightarrow J(C), \tag{44}$$

defined as

$$p \mapsto \left(\ldots, \int_{p_0}^{p} \omega_i, \ldots\right) \mathrm{mod}\ \Lambda. \tag{45}$$

We are interested in curves of Brieskorn-Pham type, i.e. curves of the form

$$C_d = \left\{x^d + y^a + z^b = 0\right\} \in \mathbb{P}_{(1,k,\ell)}[d], \tag{46}$$

such that $a = d/k$ and $b = d/\ell$ are positive rational integers. Without loss of generality we can assume that $(k,\ell) = 1$. The genus of these curves is given by

$$g(C_d) = \frac{1}{2}(2-\chi) = \frac{(d-k)(d-\ell) + (k\ell - d)}{2k\ell}. \tag{47}$$

For non-degenerate curves in the configurations $\mathbb{P}_{(1,k,\ell)}[d]$ the set of forms

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}_{(1,k,\ell)}[d]) = \left\{ \omega_{r,s,t} = y^{s-1} z^{t-d/\ell} dy \;\middle|\; r + ks + \ell t = 0 \bmod d, \quad \begin{pmatrix} 1 \le r \le d-1, \\ 1 \le s \le \frac{d}{k} - 1, \\ 1 \le t \le \frac{d}{\ell} - 1 \end{pmatrix} \right\} \tag{48}$$

defines a basis for the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(C_d)$ whose Hodge split is given by

$$\begin{aligned} \mathrm{H}^0\left(C_d, \Omega^1_{\mathbb{C}}\right) &= \{\omega_{r,s,t} \mid r + ks + \ell t = d\} \\ \mathrm{H}^1\left(C_d, \mathcal{O}_{\mathbb{C}}\right) &= \{\omega_{r,s,t} \mid r + ks + \ell t = 2d\}. \end{aligned} \tag{49}$$

In order to show this we view the weighted projective space as the quotient of projective space with respect to the actions $\mathbb{Z}_k : [0\ 1\ 0]$ and $\mathbb{Z}_\ell : [0\ 0\ 1]$, where we use the abbreviation $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ and for any group $\mathbb{Z}_r$ the notation $[a, b, c]$ indicates the action

$$[a, b, c] : \ (x, y, z) \mapsto (\gamma^a x, \gamma^b y, \gamma^c z), \tag{50}$$

where γ is a generator of the group. This allows us to view the weighted curve as the quotient of a projective Fermat type curve

$$\mathbb{P}_{(1,k,\ell)}[d] = \mathbb{P}_2[d]/\mathbb{Z}_k \times \mathbb{Z}_\ell : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

These weighted curves are smooth and hence their cohomology is determined by considering those forms on the projective curve $\mathbb{P}_2[d]$ which are invariant with respect to the group actions. A basis for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}_2[d])$ is given by the set of forms

$$\mathrm{H}^1(\mathbb{P}_2[d]) = \left\{ \omega_{r,s,t} = y^{s-1} z^{t-d} dy \mid 0 < r, s, t < d, \quad r + s + t = 0 \ (\mathrm{mod}\ d), \quad r, s, t \in \mathbb{N} \right\}. \tag{52}$$

Denote the generator of the $\mathbb{Z}_k$ action by α and consider the induced action on $\omega_{r,s,t}$

$$\mathbb{Z}_k : \quad \omega_{r,s,t} \mapsto \alpha^s \omega_{r,s,t}. \tag{53}$$

It follows that the only forms that descend to the quotient with respect to $\mathbb{Z}_k$ are those for which $s = 0 (\mathrm{mod}\ k)$. Similarly we denote by β the generator of the action $\mathbb{Z}_\ell$ and consider the induced action on the forms $\omega_{r,s,t}$

$$\mathbb{Z}_\ell : \quad \omega_{r,s,t} \mapsto \beta^{t-d} \omega_{r,s,t}. \tag{54}$$

Again we see that the only forms that descend to the quotient are those for which $t = 0 (\mathrm{mod}\ \ell)$.

4.3 Abelian varieties from weighted Jacobians

Jacobian varieties in general are not abelian varieties with complex multiplication. The question we can ask, however, is whether the Jacobians of the curves that determine the cohomology of the Calabi-Yau varieties can be decomposed such that the individual factors admit complex multiplication by an order of a number field. In this section we show that this is indeed the case and therefore we can define the complex multiplication type of a Calabi-Yau variety in terms of the CM types induced by the Jacobians of its curves.

It was shown by Faddeev [29][4] that the Jacobian variety $J(C_d)$ of Fermat curves $C_d \subset \mathbb{P}_2$ splits into a product of abelian factors $A_{\mathcal{O}_i}$

$$J(C_d) \cong \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \qquad (55)$$

where the set $\mathcal{I}$ provides a parametrization of the cohomology of C_d, and the sets $\mathcal{O}_i$ are orbits in $\mathcal{I}$ of the multiplicative subgroup $(\mathbb{Z}/d\mathbb{Z})^\times$ of the group $\mathbb{Z}/d\mathbb{Z}$. More precisely it was shown that there is an isogeny

$$i : J(C_d) \longrightarrow \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \qquad (56)$$

where an isogeny $i : A \to B$ between abelian varieties is defined to be a surjective homomorphism with finite kernel. In the parametrization used in the previous subsection $\mathcal{I}$ is the set of triplets (r,s,t) in (52) and the periods of the Fermat curve have been computed by Rohrlich [32] to be

$$\int_{A^j B^k \kappa} \omega_{r,s,t} = \frac{1}{d} B\left(\frac{s}{d}, \frac{t}{d}\right)(1-\xi^s)(1-\xi^t)\xi^{js+kt}, \qquad (57)$$

where ξ is a primitive $d-$th root of unity, and

$$B(u,v) = \int_0^1 t^{u-1}(1-v)^{v-1} dt \qquad (58)$$

is the classical beta function. A, B are the two automorphism generators

$$\begin{aligned} A(1,y,z) &= (1,\xi y, z) \\ B(1,y,z) &= (1,y,\xi z) \end{aligned} \qquad (59)$$

[4]More accessible are the references [30] [31], [32] on the subject.

and κ is the generator of $\mathrm{H}_1(C_d)$ as a cyclic module over $\mathbb{Z}[A,B]$. The period lattice of the Fermat curve therefore is the span of

$$\left(\ldots, \xi^{jr+ks}(1-\xi^r)(1-\xi^s)\frac{1}{d}B\left(\frac{r}{d},\frac{s}{d}\right),\ldots\right)_{\substack{1\leq r,s,t\leq d-1\\ r+s+t=d}}, \quad \forall 0\leq j,k\leq d-1. \tag{60}$$

The abelian factor $A_{[(r,s,t)]}$ associated to the orbit $\mathcal{O}_{r,s,t} = [(r,s,t)]$ can be obtained as the quotient

$$A_{[(r,s,t)]} = \mathbb{C}^{\varphi(d_0)/2}/\Lambda_{r,s,t}, \tag{61}$$

where $d_0 = d/\mathrm{gcd}(r,s,t)$ and the lattice $\Lambda_{r,s,t}$ is generated by elements of the form

$$\sigma_a(z)(1-\xi^{as})(1-\xi^{at})\frac{1}{d}B\left(\frac{<as>}{d},\frac{<at>}{d}\right), \tag{62}$$

where $z\in\mathbb{Z}[\mu_{d_0}]$, $\sigma_a\in\mathrm{Gal}(\mathbb{Q}(\mu_{d_0}/\mathbb{Q})$ runs through subgroups of the Galois group of the cyclotomic field $\mathbb{Q}(\mu_{d_0})$ and $<x>$ is the smallest integer $0\leq x<1$ congruent to x mod d.

We adapt this discussion to the weighted case. Denote the index set of triples (r,s,t) parametrizing the one-forms of the weighted curves $C_d\in\mathbb{P}_{(1,k,\ell)}[d]$ again by $\mathcal{I}$. The cyclic group $(\mathbb{Z}/d\mathbb{Z})^\times$ again acts on $\mathcal{I}$ and produces a set of orbits

$$\mathcal{O}_{r,s,t} = [(r,s,t)] \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times. \tag{63}$$

Each of these orbits leads to an abelian variety $A_{[(r,s,t)]}$ of dimension

$$\mathrm{dim}A_{[(r,s,t)]} = \frac{1}{2}\varphi\left(d_0\right), \tag{64}$$

and complex multiplication with respect to the field $F_{[(r,s,t)]} = \mathbb{Q}(\mu_{d_0})$, where $d_0 = d/\mathrm{gcd}(r,ks,\ell t)$. This leads to an isogeny

$$i: J(\mathbb{P}_{(1,k,\ell)}[d]) \longrightarrow \prod_{[(r,s,t)]\in\mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{[(r,s,t)]}. \tag{65}$$

The complex multiplication type of the abelian factors $A_{r,s,t}$ of the Jacobian $J(C)$ can be identified with the set

$$\mathrm{H}_{r,s,t} := \left\{a\in(\mathbb{Z}/d\mathbb{Z})^\times \mid\ <ar>+<aks>+<a\ell t>=d\right\} \tag{66}$$

via a homomorphism from $\mathrm{H}_{r,s,t}$ to the Galois group. More precisely, the CM type is determined by the subgroup $G_{r,s,t}$ of the Galois group of the cyclotomic field that is parametrized by $\mathrm{H}_{r,s,t}$

$$G_{r,s,t} = \{\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\mu_{d_0})/\mathbb{Q}) \mid a \in \mathrm{H}_{r,s,t}\} \tag{67}$$

by considering

$$(F, \{\phi_a\}) = (\mathbb{Q}(\mu_{d_0}), \{\sigma_a \mid \sigma_a \in G_{r,s,t}\}). \tag{68}$$

5 Arithmetic of Abelian Varieties

Abelian varieties with complex multiplication have special properties because of their particular symmetries. It turns out that even though the theory of CM fields associated to higher dimensional varieties is not as complete as the theory associated to elliptic curves with complex multiplication, a number of key results of the elliptic theory have been generalized to abelian varieties, mostly by Shimura.

Suppose that the abelian variety A of dimension n has complex multiplication by the ring of integers $\mathcal{O}_F$ of some CM field F (or by an order in F), and that the CM type of the variety is given by $(F, \{\phi_i\}_{i=1,\dots,n})$. The arithmetic structure induced by higher dimensional varieties is concerned not with F itself but the so-called reflex field $\hat{F}$, which depends not only on the field F, but also on the CM type of F. $\hat{F}$ is defined as the extension $\mathbb{Q}(\sum_{i=1}^n a^{\phi_i})$ of the field of rational numbers by adjoining certain traces of elements $a \in F$. The higher dimensional analog of the elliptic field of moduli then gives an unramified abelian extension of the field $\hat{F}$ [23]. Even though ramified class fields over $\hat{F}$ can be obtained as well, the theory leads to less complete results because it does not give all abelian extensions of $\hat{F}$. For this reason Hilbert's twelfth problem is still not solved for CM fields.

The basic question is whether there is some simple way to characterize the kind of subfield of the maximal abelian extension F_{ab} of a CM field F that can be obtained by adjoining to the reflex field the moduli fields of abelian varieties as well as the points of finite order. A nice result in this direction has been obtained by Wei [33]. In brief, her theorem states that given a CM field F with totally real subfield $F_{\mathbf{R}}$, the field $F_{\rm mod}$ generated by the moduli and torsion points of all polarized abelian varieties of CM type whose reflex field is contained in F, is the subfield of $F_{\rm ab}$ that is fixed under the subgroup H of the Galois group ${\rm Gal}(F_{\rm ab}/F)$ generated by the verlagerungs map

$$\mathrm{Ver}: \mathrm{Gal}(\bar{\mathbb{Q}}/F_{\mathbf{R}})_{\rm ab} \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/F). \tag{69}$$

More concisely,

$$F_{\rm mod} = (F_{\rm ab})^H. \tag{70}$$

The verlagerungs map Ver involved here is a general construction which assigns to a subgroup H of a group G a homomorphism between the abelianizations $G_{\rm ab} = G/(G,G)$ and $H_{\rm ab} = H/(H,H)$ of the pair of groups

$$\mathrm{Ver}: G_{\rm ab} \longrightarrow H_{\rm ab}. \tag{71}$$

Consider a system of C of representatives for the left cosets of H in G. For each $g \in G$ decompose the translate ga for any $a \in C$ as $ga = a'g_a$, with $g_a \in H$ and $a' \in C$. The verlagerungs map is then defined as

$$\mathrm{Ver}(g \bmod (G,G)) = \prod_{a\in C} g_a \bmod (H,H). \tag{72}$$

More details can be found in [34, 35].

We see from this that, even though the results are weaker, the generalization from the imaginary quadratic fields of one-dimensional abelian varieties to the CM fields of higher dimensions allows for a fairly nice characterization.

6 Deligne's Period Conjecture

Our focus in this paper is on the fields of complex multiplication that we derive from the abelian varieties which we construct from Calabi-Yau varieties. In Moore's analysis of higher dimensional manifolds the focus is on fields derived from the periods of the variety. In this section we describe how the period approach can be recovered from our higher-dimensional complex multiplication point of view via Deligne's conjecture formulated in [12]. Precursors to Deligne's formulation can be found in Shimura's work [36, 37, 38].

Deligne's conjecture in its motivic formulation is also useful in the present context because it allows us to provide a general perspective for our results which will furnish what we expect to be a useful general framework in which to explore further the arithmetic nature of attractor varieties. Motives are somewhat complicated objects whose status is reminiscent of string theory: different realizations are used to probe what is believed to be some yet unknown unifying universal cohomology theory of varieties which satisfies a number of expected functorial properties. More precisely, motives are characterized by a triplet of different cohomology theories together with a pair of compatibility homomorphisms. In terms of these ingredients a motive then can be described by the quintuplet of objects

$$(\mathrm{M_B}, \mathrm{M_{dR}}, \mathrm{M}_\ell, \mathrm{I}_{\mathrm{B},\sigma}, \mathrm{I}_{\ell,\bar{\sigma}}), \tag{73}$$

where the three first entries are cohomological objects constructed via Tate twists from the Betti, de Rham, and étale cohomology, respectively. Furthermore $\mathrm{I}_{\mathrm{B},\sigma}$ describes a map between the Betti and de Rham cohomology, while $\mathrm{I}_{\ell,\bar{\sigma}}$ is a map between Betti and étale cohomology[2]. The focus in the present paper is mostly on motives derived from the first (co)homology groups $\mathrm{H}^1(A)$ and $\mathrm{H}_1(A)$ of abelian varieties A, as well as the primitive cohomology of Fermat hypersurfaces.

The second ingredient in Deligne's conjecture is the concept of an $L-$function. This can be described in a number of equivalent ways. Conceptually, the perhaps simplest approach results when it can be derived via Artin's zeta function as the Hasse-Weil L-function induced by the underlying variety, i.e. by counting solutions of the variety over finite fields[3]. The complete L-function receives contributions from two fundamentally different factors, $\Lambda(\mathrm{M}, s) = L_\infty(\mathrm{M}, s)L(\mathrm{M}, s)$. The infinity term $L_\infty(\mathrm{M}, s)$ originates from those fields over which the underlying variety has bad reduction, i.e. it is singular, while the second term $L(\mathrm{M}, s)$ collects

[2]Detailed reviews of motives can be found in [25].

[3]Ref.[1] contains a brief description of this construction.

all the information obtained from the finite fields over which the variety is smooth. The complete L-function is in general expected to satisfy a functional equation, relating its values at s and $1-s$. A motive is called critical if neither of the infinity factors in the functional equation has a pole at $s=0$.

The final ingredient is the concept of the period of a motive, a generalization of ordinary periods of varieties. Viewing the motive M as a generalized cohomology theory, Deligne formulates the notion of a period $c^+(\mathrm{M}) \in \mathbb{C}^\times/\mathbb{Q}^\times$ by taking the determinant of the compatibility homomorphism

$$I_{\mathrm{B},\sigma} : \mathrm{M}_{\mathrm{B}} \longrightarrow \mathrm{M}_{\mathrm{dR}} \tag{74}$$

between the Betti and the deRham realizations of the motive M. Deligne's basic conjecture then relates the period and the L-function via $L(\mathrm{M},0)/c^+(\mathrm{M}) \in \mathbb{Q}$. Contact with the Hasse-Weil L-function is made by noting that for motives of the type $\mathrm{M} = \mathrm{H}(X)(m)$ with Tate twists one has $L(\mathrm{M},0) = L(X,m)$.

Important for us is a generalization of this conjecture which involves motives with coefficients. Such motives can best be described via algebraic Hecke characters, which are of particular interest for us because they come up in the L-function of projective Fermat varieties. Algebraic Hecke characters were first introduced by Weil and called Hecke characters of type A_0, which is what they are called in the older literature. The beauty of algebraic Hecke characters is that one is immediately led to a clear distinction between the defining field K and the field in which a character lives, i.e. the field F of values. In the context of motives constructed from these characters the field F becomes the field of complex multiplication. Deligne's conjecture emerges in the following way.

Deligne Period Conjecture:

$$\frac{L(\mathrm{M},0)}{c^+(\mathrm{M})} \in F. \tag{75}$$

This shows why the Deligne conjecture is of interest to us. The period and the L-function determine the same field, which is the CM field of the motive. Deligne's conjecture has been proven by Blasius for Fermat hypersurfaces [13].

7 Summary and Generalizations

We have shown that the concepts used to describe attractor varieties in the context of elliptic compactifications can be generalized to Calabi-Yau varieties with finite fundamental groups. We have mentioned above that the abelian property is neither carried by the variety itself nor the generalized intermediate Jacobian

$$J^n(X) = \mathrm{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C})/\mathrm{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{Z}(n)) + \mathrm{F}^n\mathrm{H}^{2n-1}(X_{\mathrm{an}}, \mathbb{C}), \tag{76}$$

but by the Jacobians of the curves that are the building blocks of the middle-dimensional cohomology $\mathrm{H}^{\dim_{\mathbb{C}} X}(X)$. These Jacobians themselves do not admit complex multiplication, unlike the situation in the elliptic case, but instead split into different factors which admit different types of complex multiplication, in general. Furthermore the ring class field can be generalized to be the field of moduli, and we can consider also points on the abelian variety that are of finite order, i.e. torsion points, and the field extensions they generate.

This allows us to answer a question posed in [5] which asked whether the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ could play a role in the context of $N = 2$ compactifications of type IIB strings. This is indeed the case. Suppose we have given an abelian variety A defined over a field K with complex multiplication by a field F. Then there is an action of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ of the closure $\bar{K}$ of K on the torsion points of A. This action is described by a Hecke character which is associated to the fields (K, F) [23].

We have mentioned already that in general the (Griffiths) intermediate Jacobian is only a torus, not an abelian variety. Even in those cases it is however possible to envision the existence of motives via abelian varieties associated to a variety X. Consider the Chow groups $\mathrm{CH}^p(X)$ of codimension p cycles modulo rational equivalence and denote by $\mathrm{CH}^p(X)_{\mathrm{hom}}$ the subgroup of cycles homologically equivalent to zero. Then there is a homomorphism, the Abel-Jacobi homomorphism, which embeds $\mathrm{CH}^p(X)_{\mathrm{hom}}$ into the intermediate Jacobian

$$\Psi : \mathrm{CH}^p(X)_{\mathrm{hom}} \longrightarrow J^p(X). \tag{77}$$

The image of Ψ on the subgroup $\mathcal{A}^p(X)$ defined by cycles algebraically equivalent to zero does in fact define an abelian variety, even if $J^p(X)$ is not an abelian variety but only a torus [39].

Hence we can ask whether attractor varieties are distinguished by Abel-Jacobi images which admit complex multiplication.

Even more general, we can formulate this question in the framework of motives because of Deligne's conjecture. Thinking of motives as universal cohomology theories, it is conceivable that attractor varieties lead to motives in the abelian category with (potential) complex multiplication. The standard cycle class map construction of $\mathrm{CH}^p(X)_{\mathrm{hom}}$ is replaced by the first term of a (conjectured) filtration in the resulting K-theory.

Combining the two threads of our analysis illustrates that the two separate discussions in [5] characterizing toroidal attractor varieties via complex multiplication on the one hand, and Calabi-Yau hypersurfaces via periods on the other, are two aspects of our way of looking at this problem. This is the case precisely because of Deligne's period conjecture which relates the field of the periods to the field of complex multiplication via the L-function of the variety (or motive). Thus a very pretty unified picture emerges.

Acknowledgement

We are grateful to G. Moore and P. Deligne for discussions. ML and RS thank the Kavli Institute for Theoretical Physics, Santa Barbara, for hospitality and support through KITP Scholarships during the course of part of this work. RS would also like to thank the organizers of the Duality Workshop at the ITP in 2001. This research was supported in part by the National Science Foundation under grant No. PHY99-07949. The work of M.L. and R.S. has been supported in part by NATO under grant CRG 9710045. The work of VP was supported by NSF under grant No. PHY98-02484.

References

[1] R. Schimmrigk, *On the Beilinson-Bloch conjecture and Rational Conformal Field Theory*, Lecture at Bonn University, 1995

[2] R. Schimmrigk, *Arithmetic Geometry of Calabi-Yau Manifolds and Rational Conformal Field Theory*, J. Geom. Phys. **44** (2003) 555, arXiv: hep-th/0111226;
Aspects of Conformal Field Theory from Calabi-Yau Arithmetic, to appear in the proceedings of the Workshop on Arithmetic, Geometry, and Physics around Calabi-Yau Varieties and Mirror Symmetry, eds. J. Lewis and N. Yui, 2003, arXiv: math.AG/0209168

[3] R. Schimmrigk and S. Underwood, *The Shimura-Taniyama Conjecture and Conformal Field Theory*, to appear in J. Geom. Phys., arXiv: hep-th/0211284

[4] S. Gukov and C. Vafa, *Rational Conformal Field Theory and Complex Multiplication*, arXiv: hep-th/0203213

[5] G. Moore, *Attractors and Arithmetic*, arXiv: hep-th/9807056;
Arithmetic and Attractors, arXiv: hep-th/9807087

[6] S. D. Miller and G. Moore, *Landau-Siegel Zeroes and Black Hole Entropy*, Asian J. Math. **4** (2000) 183 - 212, arXiv: hep-th/9903267

[7] P. Candelas, X. de la Ossa, F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields*, arXiv: hep-th/0012233

[8] S. Ferrara, R. Kallosh and A. Strominger, *N=2 Extremal Black Holes*, Phys.Rev. **D52** (1995) 5412-5416, arXiv: hep-th/9508072

[9] A. Strominger, *Macroscopic Entropy of N=2 Extremal Black Holes*, Phys.Lett. **B383** (1996) 39-43, arXiv: hep-th/9602111

[10] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54** (1996) 1514, arXiv: hep-th/9602136;
Universality of Supersymmetric Attractors, Phys. Rev. **D54** (1996) 1525, arXiv: hep-th/9603090

[11] S. Ferrara, Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500** (1997) 75, arXiv: hep-th/9702103

[12] P. Deligne, *Valeurs de fonctions L et periodes d'integrales*, Proc. Symp. Pure Math AMS **33**(1979)313-346

[13] D. Blasius, *On the critical values of Hecke L-series*, Ann. Math. **124**(1986)23-63

[14] P. Fre, *Supersymmetry and First Order Equations for Extremal States: Monopoles, Hyperinstantons, Black Holes and p-branes*, Nucl. Phys. Proc. Suppl. **57**(1997)52, arXiv: hep-th/9701054

[15] P. Candelas and X. de la Ossa, *Moduli Space of Calabi-Yau Manifolds*, Nucl. Phys. **B355** (1991) 455

[16] M. Billo, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre, T. Regge, P. Soriani and A. van Proeyen, *A Search for Non-Perturbative Dualities of Local N=2 Yang-Mills Theories from Calabi-Yau Threefolds*, Class. Quant. Grav. **13**(1996)831, arXiv: hep-th/9506075

[17] A. Ceresole, R. D'Auria and S. Ferrara, *The Symplectic Structure of N=2 Supergravity and its Central Extensions*, Nucl. Phys. Proc. Suppl. **46** (1996)67 , arXiv: hep-th/9509160

[18] F. Larsen and F. Wilczek, *Internal Structure of Black Holes*, Phys. Lett. **B375** (1996) 37, arXiv: hep-th/9511064

[19] L. Andrianopoli, R. D'Auria and S. Ferrara, *U-Duality and Central Charges in Various Dimensions Revisited*, Int. J. Mod. Phys. **A13** (1998) 431-490, arXiv: hep-th/9612105; *Flat Symplectic Bundles of N-Extended Supergravities, Central Charges and Black Hole Entropy*, in Proceedings of the 5th Winter School on Mathematical Physics held at the Asia Pacific Center for Theoretical Physics, February 1997, arXiv: hep-th/9707203

[20] J. H. Silverman, *A Survey of the Arithmetic Theory of Elliptic Curves*, in Modular Forms and Fermat's Last Theorem, Springer 1997

[21] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer 1994

[22] G. Shimura and Y. Taniyama, *Complex Multiplication of Abelian Varieties and its Application to Number Theory*, 1961

[23] G. Shimura, *Introduction to Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971

[24] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton University Press 1998

[25] U. Jannsen, S. Kleiman, J. P. Serre, *Motives*, AMS 1994

[26] T. Shioda and T. Katsura, *On Fermat varieties*, Tohoku Math. J. **31** (1979)97-115

[27] P. Candelas, M. Lynker, and R. Schimmrigk, *Calabi-Yau Manifolds in Weighted* $\mathbb{P}_4$, Nucl. Phys. **B341** (1990) 383, KEK archive 199001117

[28] R. Schimmrigk, *A New Construction of a Three-generation Calabi-Yau Manifold*, Phys. Lett. **B193** (1987) 175, KEK archive 198706386;
Heterotic RG Flow Fixed Points with Non-Diagonal Affine Invariants, Phys. Lett. **B229** (1989) 227, KEK archive 198905565

[29] D. K. Faddeev, *On the Divisor Groups of some Algebraic Curves*, Dokl. Tom **136** (1961) 296-298 [Sov. Math **2**(1961) 67-69]
Invariants of Divisor Classes for the Curves $y^\ell = x^k(1-x)$ *in an* ℓ*-adic Cyclotomic Field*, Trudy Mat. Inst. Steklova **64** (1961) 284 - 293 (In Russian)

[30] A. Weil, *Sur les Périodes des Intégrales Abeliennes*, Comm. Pure Appl. Math **29** (1976) 813

[31] B. H. Gross, *On the Periods of Abelian Integrals and a Formula of Chowla and Selberg*, Inv. math. **45** (1978) 193

[32] D. E. Rohrlich, *The Periods of the Fermat Curve*, Appendix to Gross 1978, Inv. math. **45** (1978) 208

[33] W. Wei, *Moduli Fields of CM Motives applied to Hilbert's 12TH Problem*, preprint

[34] J. Neukirch, *Algebraic Number Theory*, Springer 1999

[35] D. Ramakrishnan and R. Valenza, *Fourier Analysis on Number Fields*, Springer 1998

[36] G. Shimura, *On Some Arithmetic Properties of Modular Forms of One and Several Variables*, Ann. Math. **102** (1975) 491 - 515

[37] G. Shimura, *The Special Values of the Zeta Functions Associated With Cusp Forms*, Comm. Pure Appl. Math. **29** (1976) 783 - 804

[38] G. Shimura, *On the Periods of Modular Forms*, Math. Ann. **229** (1977) 211 - 221

[39] H. Saito, *Abelian Varieties attached to Cycles of Intermediate Jacobians*, Nagoya Math. J. **75** (1979) 95 - 119