is

$$C_{41}^{-1}(\sigma, \sigma') = \frac{\pi}{\lambda \cdot P} \Theta(\sigma, \sigma') = \frac{\pi}{\lambda \cdot P} \left(\Sigma_{+}(\sigma, \sigma') + \frac{\sigma}{\pi}\right). \tag{4.53}$$

Here the generalized step function $\Theta(\sigma, \sigma')$ obeys

$$\begin{aligned} &\frac{\partial}{\partial\sigma} \Theta(\sigma, \sigma') = \Delta(\sigma, \sigma') \\ &\left.\begin{aligned} \Theta(\sigma, \sigma') = 1 \quad , \quad \sigma > \sigma' \\ \Theta(\sigma, \sigma') = 0 \quad , \quad \sigma < \sigma' \end{aligned}\right\} \; 0 \leq \sigma \leq \pi, \quad 0 \leq \sigma' \leq \pi, \end{aligned} \tag{4.54}$$

while the periodic sawtooth functions

$$\begin{aligned} \Sigma_{\pm}(\sigma, \sigma') &= \frac{1}{2\pi i} \sum_{n \neq 0} \frac{1}{n} \left(e^{in(\sigma-\sigma')} \pm e^{in(\sigma+\sigma')}\right) \\ \Sigma_{+}(\sigma, \sigma') &= \Theta(\sigma, \sigma') - \frac{\sigma}{\pi} \end{aligned} \tag{4.55}$$

have the properties

$$\begin{aligned} &\Sigma_{+}(\sigma, \sigma') = -\Sigma_{-}(\sigma', \sigma) \\ &\Sigma_{\pm}(\sigma + 2\pi, \sigma') = \Sigma_{\pm}(\sigma, \sigma') \\ &\frac{\partial}{\partial\sigma} \Sigma_{+}(\sigma, \sigma') = \Delta_{+}(\sigma, \sigma') - \frac{1}{\pi} \\ &\frac{\partial}{\partial\sigma} \Sigma_{-}(\sigma, \sigma') = \Delta_{-}(\sigma, \sigma') = -\frac{\partial}{\partial\sigma} \Sigma_{+}(\sigma', \sigma). \end{aligned} \tag{4.56}$$

But now Eq. (4.53) and the requirement that

$$C_{41}^{-1}(\sigma, \sigma') = -C_{14}^{-1}(\sigma', \sigma) = \frac{\pi}{\lambda \cdot P} \left(\Sigma_{-}(\sigma, \sigma') - \frac{\sigma'}{\pi}\right)$$

Tell us immediately that

$$I_{11}(\sigma, \sigma') = \int d\sigma'' \frac{\partial}{\partial\sigma} \Delta_{+}(\sigma, \sigma') \left(\Sigma_{-}(\sigma'', \sigma') - \frac{\sigma'}{\pi}\right) = \Delta_{-}(\sigma, \sigma').$$

The other elements of the inverse matrix are

$$\begin{aligned} C_{32}^{-1}(\sigma, \sigma') &= -C_{23}^{-1}(\sigma, \sigma') = \frac{\pi}{2\lambda \cdot P} \Delta(\sigma, \sigma') \\ C_{13}^{-1}(\sigma, \sigma') &= \frac{\pi}{2\lambda \cdot P} C_{14}^{-1}(\sigma, \sigma') - \frac{1}{2\lambda \cdot P} \int_0^\pi d\sigma'' \, C_{14}^{-1}(\sigma, \sigma'') \\ &= \frac{\pi^2}{2(\lambda \cdot P)^2} \Sigma_{+}(\sigma, \sigma') \\ C_{31}^{-1}(\sigma, \sigma') &= -C_{13}^{-1}(\sigma', \sigma) = \frac{\pi^2}{2(\lambda \cdot P)^2} \Sigma_{-}(\sigma, \sigma'). \end{aligned} \tag{4.58}$$

The final result for C_{ij}^{-1} is therefore

$$C_{ij}^{-1}(\sigma,\sigma') = \frac{\pi}{\lambda\cdot P}\,\cdot \tag{4.59}$$

$$\times \begin{vmatrix} 0 & 0 & -\frac{\pi\lambda^2}{2\lambda\cdot P}\Sigma_+(\sigma,\sigma') & \Sigma_+(\sigma,\sigma')+\frac{\sigma}{\pi} \\ 0 & 0 & -\frac{1}{2}\Delta(\sigma,\sigma') & 0 \\ -\frac{\pi\lambda^2}{2\lambda\cdot P}\Sigma_-(\sigma,\sigma') & \frac{1}{2}\Delta(\sigma,\sigma') & 0 & 0 \\ \Sigma_-(\sigma,\sigma')-\frac{\sigma'}{\pi} & 0 & 0 & 0 \end{vmatrix}.$$

Equation (1.58) can now be used to compute the Dirac brackets compatible with the constraints (4.49). The results are

$$\begin{aligned}\{x^\mu(\sigma), x^\nu(\sigma')\}^* &= \frac{\pi}{\lambda\cdot P}\,u^\mu(\sigma)\left[\lambda^\nu - \frac{\pi\lambda^2}{\lambda\cdot P}\mathscr{P}^\nu(\sigma')\right]\Sigma_+(\sigma,\sigma') \\ &+ \frac{\pi}{\lambda\cdot P}\left[\lambda^\mu - \frac{\pi\lambda^2}{\lambda\cdot P}\mathscr{P}^\mu(\sigma)\right]u^\nu(\sigma')\,\Sigma_-(\sigma,\sigma') \\ &+ \frac{\tau}{N\lambda\cdot P}\left(\lambda^\mu\mathscr{P}^\nu(\sigma') - \lambda^\nu\mathscr{P}^\mu(\sigma)\right),\end{aligned} \tag{4.60 a}$$

$$\begin{aligned}\{\mathscr{P}^\mu(\sigma), \mathscr{P}^\nu(\sigma')\}^* &= \frac{\pi N^2}{\lambda\cdot P}\,\frac{\partial^2}{\partial\sigma\,\partial\sigma'}\Big[-\frac{\pi\lambda^2}{\lambda\cdot P}\mathscr{P}^\mu(\sigma)\,u^\nu(\sigma')\,\Sigma_+(\sigma,\sigma') \\ &- \frac{\pi\lambda^2}{\lambda\cdot P}\,u^\mu(\sigma)\,\mathscr{P}^\nu(\sigma')\,\Sigma_-(\sigma,\sigma') + \lambda^\mu u^\nu(\sigma')\left(\Sigma_+(\sigma,\sigma')+\frac{\sigma}{\pi}\right) + \\ &+ \lambda^\nu u^\mu(\sigma)\left(\Sigma_-(\sigma,\sigma') - \frac{\sigma'}{\pi}\right)\Big]\end{aligned} \tag{4.60 b}$$

$$\begin{aligned}\{\mathscr{P}^\mu(\sigma), x^\nu(\sigma')\}^* &= \left(-g^{\mu\nu} + \frac{\pi\lambda^\mu}{\lambda\cdot P}\mathscr{P}^\nu(\sigma')\right)\Delta(\sigma,\sigma') \\ &+ \frac{\partial}{\partial\sigma}\Big[\frac{\pi^2}{(\lambda\cdot P)^2}\mathscr{P}^\mu(\sigma)\,\Sigma_+(\sigma,\sigma')\left(-\lambda^2\mathscr{P}^\nu(\sigma') + \lambda^\nu\frac{\lambda\cdot P}{\pi}\right) \\ &- \frac{\lambda^2\pi^2N^2}{(\lambda\cdot P)^2}\,u^\mu(\sigma)\,u^\nu(\sigma')\,\Sigma_-(\sigma,\sigma') - \frac{\tau N}{\lambda\cdot P}\,u^\mu(\sigma)\,\lambda^\nu\Big].\end{aligned} \tag{4.60 c}$$

The constraints (4.49) are strongly valid in the Dirac bracket system. We may therefore solve the equations of motion (4.45 *a*),

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)x^\mu(\tau,\sigma) = 4\,\frac{\partial^2 x^\mu(\tau,\sigma)}{\partial(\tau+\sigma)\,\partial(\tau-\sigma)} = 0 \tag{4.61}$$

in the orthonormal system subject to the boundary conditions (4.45 *b*). The most general solution consistent with the reflection principle (4.22) takes the form

$$x^\mu(\tau,\sigma) = \frac{1}{2}\left(Q^\mu(\tau+\sigma) + Q^\mu(\tau-\sigma)\right) \tag{4.62}$$

where we may identify

$$Q^{\mu}(\theta) = x^{\mu}(\theta, 0) \tag{4.63}$$

as the coordinate of the end of the string. The momenta are

$$\mathscr{P}^{\mu}(\tau, \sigma) = \mathrm{N}\partial x^{\mu}/\partial\tau = \frac{\mathrm{N}}{2}\left(Q'^{\mu}(\tau + \sigma) + Q'^{\mu}(\tau - \sigma)\right) \tag{4.64}$$

where

$$Q'^{\mu}(\theta) \equiv \frac{\partial Q^{\mu}(\theta)}{\partial\theta} = \frac{1}{\mathrm{N}}\mathscr{P}^{\mu}(\theta, 0). \tag{4.65}$$

(Q'^{μ} is often written " P^{μ} " in the literature; we reserve this symbol for the translation generator (4.29).) By examining

$$u^{\mu}(\tau, \sigma) = \partial x^{\mu}/\partial\sigma = \frac{1}{2}\left(Q'^{\mu}(\tau + \sigma) - Q'^{\mu}(\tau - \sigma)\right) \tag{4.66}$$

we find that the boundary condition

$$u^{\mu}(\tau, 0) = u^{\mu}(\tau, \pi) = 0 \tag{4.67}$$

requires

$$Q'^{\mu}(\theta + 2\pi) - Q'^{\mu}(\theta) = 0\,. \tag{4.68}$$

Next, we notice that the four constraints (4.39) on x^{μ} and $\mathscr{P}^{\mu}$ can be expressed as two constraints on Q^{μ},

$$\begin{aligned} \chi_1 &= Q'^2(\theta) = 0 \\ \chi_2 &= \lambda \cdot Q(\theta) - \theta\,\frac{\lambda \cdot P}{\mathrm{N}\pi} = 0\,. \end{aligned} \tag{4.69}$$

Note also that

$$\lambda \cdot Q'(\theta) = \frac{\lambda \cdot P}{\mathrm{N}\pi}\,. \tag{4.70}$$

To find the brackets of $Q^{\mu}(\theta)$ compatible with the strong constraints (4.69), we insert Eq. (4.62) for $x^{\mu}(\tau, \sigma)$ into both sides of the Dirac bracket (4.60 *a*). After identifying terms of the same functional form on both sides of the equation, we find that

$$\begin{aligned} \{Q^{\mu}(\theta), Q^{\nu}(\theta')\} &= -\frac{1}{\mathrm{N}}g^{\mu\nu}\,\varepsilon(\theta - \theta') \\ &+ (\lambda \cdot P)^{-1}\left[\theta\lambda^{\mu}Q'^{\nu}(\theta') - \theta'\lambda^{\nu}Q'^{\mu}(\theta)\right] \\ &+ \frac{\rho(\theta - \theta')}{\mathrm{N}(\lambda \cdot P)^2}\left(\lambda^{\nu}Q'^{\mu}(\theta)(\lambda \cdot P) + \lambda^{\mu}Q'^{\nu}(\theta')(\lambda \cdot P) - \lambda^2 Q'^{\mu}(\theta)\,Q'^{\nu}(\theta')\right). \end{aligned} \tag{4.71}$$

Here we have defined the functions

$$\rho(\theta) = \frac{1}{i\pi} \sum_{n \neq 0} \frac{1}{n} e^{in\theta}$$

$$(4.72\,a) \qquad \varepsilon(\theta) = \rho(\theta) + \frac{\theta}{\pi}$$

$$\delta(\theta) = \frac{1}{2} \frac{d\varepsilon(\theta)}{d\theta} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} = \sum_{n} \delta_{\text{Dirac}}(\theta + 2n\pi)$$

with the periodic properties

$$\rho(\theta + 2\pi) = \rho(\theta)$$

$$(4.72\,b) \qquad \varepsilon(\theta + 2\pi) = \varepsilon(\theta) + 2$$

$$\delta(\theta + 2\pi) = \delta(\theta)\,.$$

One may verify that all of the brackets (4.60) follow from (4.71), (4.62) and (4.64). The brackets (4.71) are also compatible with the constraints (4.69). In fact, it is amusing to note that if the " canonical " bracket of $Q^{\mu}(\theta)$ is taken to be

$$(4.73) \qquad \{ Q^{\mu}(\theta)\,,\, Q^{\nu}(\theta')\} = -\frac{1}{N} g^{\mu\nu}\, \varepsilon(\theta - \theta')\,,$$

Eq. (4.71) is exactly the result one gets by using the Dirac procedure with the constraints (4.69).

Examining Eqs. (4.60) or (4.71), one sees that the Dirac brackets look like a plausible basis for a quantum theory *provided*

$$\lambda^2 = 0$$

(4.74) $\quad \lambda^{\mu} = 0 \quad$ for the D-2 independent degrees of freedom.

We are free to choose the gauge in such a way that the theory is as simple as possible. Therefore we take

$$\lambda_0 = \lambda_{D-1} = 1/\sqrt{2}$$

$$(4.75\,a) \qquad \lambda_i = 0\,, \qquad i = 1\,,\cdots, D-2$$

and define also a complementary vector $\overset{*}{\lambda}_{\mu}$ with

$$\overset{*}{\lambda}_0 = -\overset{*}{\lambda}_{D-1} = -1/\sqrt{2}$$

$$(4.75\,b) \qquad \overset{*}{\lambda}_i = 0$$

so that

$$\lambda^2 = \lambda^{*2} = 0$$

$$(4.75\,c) \qquad \lambda \cdot \lambda^* = 1\,.$$

Hereafter, any vector V^{μ} will be split into a transverse part V^{i}, $i = 1, \cdots,$ D-2 and

$$V^{+} = \lambda_{\mu} V^{\mu} = (V^{D-1} + V^{0})/\sqrt{2} = V_{-} \tag{4.76}$$
$$V^{-} = \lambda^{*}_{\mu} V^{\mu} = (V^{D-1} - V^{0})/\sqrt{2} = V^{+}$$

in accordance with the null-plane conventions listed in Appendix A. With the gauge choice (4.75), the nonzero brackets in Eq. (4.60) become

$$\{\mathscr{P}^{i}(\sigma), x^{j}(\sigma')\}^{*} = -\delta^{ij}\Delta(\sigma, \sigma') \tag{4.77}$$

$$\{x^{+}(\sigma), x^{-}(\sigma')\}^{*} = -\frac{\tau}{N\pi}$$

$$\{x^{-}(\sigma), x^{-}(\sigma')\}^{*} = \frac{P^{+}}{\pi}\Big[u^{-}(\sigma)\Sigma_{+}(\sigma, \sigma') + u^{-}(\sigma')\Sigma_{-}(\sigma, \sigma') + \\ + \frac{\tau}{N\pi}(\mathscr{P}^{-}(\sigma') - \mathscr{P}^{-}(\sigma))\Big]$$

$$\{x^{-}(\sigma), x^{i}(\sigma')\}^{*} = \frac{\pi}{P^{+}}\Big(u^{i}(\sigma')\Sigma_{-}(\sigma, \sigma') + \frac{\tau}{N\pi}\mathscr{P}^{i}(\sigma')\Big)$$

$$\{\mathscr{P}^{-}(\sigma), \mathscr{P}^{i}(\sigma')\}^{*} = \frac{N^{2}\pi}{P^{+}} u^{i}(\sigma)\frac{\partial\Delta(\sigma, \sigma')}{\partial\sigma}$$

$$\{\mathscr{P}^{+}(\sigma), x^{-}(\sigma')\}^{*} = -\frac{1}{\pi}$$

$$\{\mathscr{P}^{-}(\sigma), x^{-}(\sigma')\}^{*} = \frac{\pi}{P^{+}}\Big[\mathscr{P}^{-}(\sigma')\Delta(\sigma, \sigma') + \\ + \frac{\partial}{\partial\sigma}\Big(\mathscr{P}^{-}(\sigma)\Sigma_{+}(\sigma, \sigma') - \frac{N\tau}{\pi}u^{-}(\sigma)\Big)\Big]$$

$$\{\mathscr{P}^{-}(\sigma), x^{i}(\sigma')\}^{*} = \frac{\pi}{P^{+}}\mathscr{P}^{i}(\sigma')\Delta(\sigma, \sigma')$$

$$\{\mathscr{P}^{i}(\sigma), x^{-}(\sigma')\}^{*} = \frac{\pi}{P^{+}}\frac{\partial}{\partial\sigma}\Big(\mathscr{P}^{i}(\sigma)\Sigma_{+}(\sigma, \sigma') - \frac{N\tau}{\pi}u^{i}(\sigma)\Big),$$

For reference, we give also the nonvanishing brackets (4.71) with the gauge choice (4.75):

$$\{Q^{i}(\theta), Q^{j}(\theta')\}^{*} = -\frac{1}{N}\delta^{ij}\varepsilon(\theta - \theta') \tag{4.78}$$

$$\{Q^{+}(\theta), Q^{-}(\theta')\}^{*} = -\frac{\theta}{N\pi}$$

$$\{Q^{-}(\theta), Q^{-}(\theta')\}^{*} = \frac{\pi}{P^{+}}\Big[Q^{\overset{i}{-}}(\theta')\Big(\rho(\theta - \theta') + \frac{\theta}{\pi}\Big) + \\ + Q^{\overset{i}{-}}(\theta)\Big(\rho(\theta - \theta') - \frac{\theta'}{\pi}\Big)\Big]$$

$$\{Q^{-}(\theta), Q^{i}(\theta')\}^{*} = \frac{\pi}{P^{+}}Q'^{i}(\theta')\Big(\rho(\theta - \theta') + \frac{\theta}{\pi}\Big).$$

The action of the translation group generators (4.29) is now

$$
(4.79)\qquad
\begin{aligned}
\{ P^+, x^-(\sigma)\}^* &= -1 \\
\{ P^-, x^-(\sigma)\}^* &= \frac{\pi}{P^+}\, \mathscr{P}^-(\sigma) \\
\{ P^-, x^i(\sigma)\}^* &= \frac{\pi}{P^+}\, \mathscr{P}^i(\sigma) \\
\{ P^i, x^j(\sigma)\}^* &= -\delta^{ij}
\end{aligned}
$$

with all other brackets vanishing. Since

$$\tau = \frac{N\pi}{P^+}\, x^+$$

and P^- is the canonical generator of x^+ displacements, we may solve the constraint Eq. (4.13 *a*) for $\mathscr{P}^-$ to give the generator of τ-displacements,

$$
(4.80)\qquad H = -\frac{P^+ P^-}{N\pi} = \frac{1}{2N} \int_0^\pi d\sigma\, (\mathscr{P}\cdot\mathscr{P} + N^2\, \boldsymbol{u}\cdot\boldsymbol{u})\,.
$$

We verify that Eq. (4.80) gives Hamilton's equations of motion in τ for the independent variables x^i and $\mathscr{P}^i$:

$$
(4.81)\qquad
\begin{aligned}
\{ x^i(\tau, \sigma), H\}^* &= \frac{1}{N}\, \mathscr{P}^i(\tau, \sigma) = \frac{\partial x^i(\tau, \sigma)}{\partial \tau} \\
\{ \mathscr{P}^i(\tau, \sigma), H\}^* &= N\, \frac{\partial^2 x^i(\tau, \sigma)}{\partial \sigma^2} = \frac{\partial \mathscr{P}^i(\tau, \sigma)}{\partial \tau}\,.
\end{aligned}
$$

We next examine the Lorentz group generators (4.32). The rotation group generators M^{ij} transform x^i as a three-vector.

$$
(4.82)\qquad \{M^{ij}, x^k\}^* = \delta^{ik} x^j - \delta^{jk} x^i
$$

The boost generator M^{0i}, on the other hand, adds gauge transformations to the variables so as to preserve the constraint Eqs. (4.36) in the new Lorentz frame. We find

$$
(4.83)\qquad
\begin{aligned}
\{M^{-i}, x^j(\sigma)\}^* = &-\delta^{ij}\, x^-(\sigma) + \frac{\pi N}{P^+}\, \frac{\partial x^j(\sigma)}{\partial \tau} \left(-x^i(\sigma) + \frac{\tau P^i}{N\pi} \right) \\
&+ \frac{N\pi}{P^+}\, \frac{\partial x^j(\sigma)}{\partial \sigma} \int_0^\pi d\alpha\, \frac{\partial x^i(\tau, \alpha)}{\partial \tau}\, \Sigma_-(\alpha, \sigma)\,.
\end{aligned}
$$

The first term is expected from Lorentz covariance, while the remaining terms arise from the fact that a gauge transformation (a (τ, σ)-reparametrization) must be made to preserve the gauge in the new Lorentz frame. If the infinitesimal Lorentz transformation is written

$$
(4.84)\qquad P'^{\mu} = P^{\mu} + \frac{1}{2}\, \omega_{\alpha\beta}\, (g^{\mu\alpha} P^{\beta} - g^{\mu\beta} P^{\alpha}) = P^{\mu} + \omega^{\mu\nu} P_{\nu}
$$

then the change in $x^\mu(\tau, \sigma)$ under a Lorentz transformation will involve also an infinitesimal reparametrization $\tau, \sigma \to \tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma)$:

$$\begin{aligned} x^\mu(\tau, \sigma) \to x'^\mu(\tau, \sigma) &= x^\mu(\tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma)) + \omega^{\mu\nu} x_\nu(\tau, \sigma) \\ &= x^\mu(\tau, \sigma) + \omega^{\mu\nu} x_\nu(\tau, \sigma) + \Delta\tau\, v^\mu(\tau, \sigma) + \Delta\sigma\, u^\mu(\tau, \sigma) . \end{aligned} \tag{4.85}$$

The value of $\Delta\tau$ is fixed by requiring

$$x'^+ = \frac{P'^+ \tau}{N\pi} = \frac{\tau}{N\pi}(P^+ + \omega^{+\nu} P_\nu) . \tag{4.86}$$

Equation (5.85) then immediately gives

$$\Delta\tau = \frac{N\pi\omega^{+\nu}}{P+}\left(\frac{\tau P_\nu}{P+} - x_\nu\right) . \tag{4.87}$$

Once $\Delta\tau$ is fixed, $\Delta\sigma$ is completely determined by the requirement (4.46) that the orthonormality of the coordinate system, Eqs. (4.42), be preserved by the reparametrization,

$$\begin{aligned} \frac{\partial\Delta\tau}{\partial\tau} &= \frac{\partial\Delta\sigma}{\partial\sigma} = \frac{N\pi\omega^{+\nu}}{P+}\left(\frac{P_\nu}{N\pi} - v_\nu\right) \\ \frac{\partial\Delta\tau}{\partial\sigma} &= \frac{\partial\Delta\sigma}{\partial\tau} = -\frac{1}{P+} N\pi\omega^{+\nu} u_\nu . \end{aligned} \tag{4.88}$$

The solution of Eqs. (4.88) is

$$\begin{aligned} \Delta\sigma &= \frac{N\pi\omega^{+\nu}}{P+}\left(\frac{\sigma P_\nu}{N\pi} - \int_0^\sigma d\sigma'\, v_\nu(\tau, \sigma')\right) \\ &= \frac{N\pi\omega^{+\nu}}{P+}\int_0^\pi d\sigma'\, v_\nu(\tau, \sigma')\, \Sigma_-(\sigma', \sigma) \\ &= -\frac{N\pi\omega^{+\nu}}{P+}\int_0^\pi d\sigma'\, v_\nu(\tau, \sigma')\, \Sigma_+(\sigma, \sigma') . \end{aligned} \tag{4.89}$$

Thus when a Lorentz transformation is made, the required change of $x^\mu(\tau, \sigma)$ which preserves (4.86) and (4.42) is

$$\begin{aligned} x'^\mu(\tau, \sigma) = x^\mu(\tau, \sigma) &+ \omega^{\mu\nu} x_\nu(\tau, \sigma) + v^\mu(\tau, \sigma)\frac{N\pi\omega^{+\nu}}{P+}\left(\frac{\tau P_\nu}{N\pi} - x_\nu\right) \\ &- u^\mu(\tau, \sigma)\frac{N\pi\omega^{+\nu}}{P+}\int_0^\pi d\sigma'\, v_\nu(\tau, \sigma')\, \Sigma_+(\sigma, \sigma') , \end{aligned} \tag{4.89}$$

which agrees with the result (4.83) given *automatically* by the Dirac brackets.

D. Fourier Components of Variables

We begin the Fourier analysis by observing that the σ-averaged coordinates and momenta of the string are

$$q^\mu = \frac{1}{\pi}\int_0^\pi d\sigma\, x^\mu(0,\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \underset{\sim}{Q}^\mu(\theta) \tag{4.91}$$

$$P^\mu = \int_0^\pi d\sigma\, \mathscr{P}^\mu(0,\sigma) = N\int_0^\pi d\sigma\, v^\mu(0,\sigma)$$

$$= N\left(\underset{\sim}{Q}^\mu(\pi) - \underset{\sim}{Q}^\mu(-\pi)\right).$$

Then we may write

$$\begin{aligned} \underset{\sim}{Q}^\mu(\theta) &= q^\mu + \frac{\theta P^\mu}{N\pi} + f^\mu(\theta) \\ \underset{\sim}{Q}'^\mu(\theta) &= P^\mu/N\pi + f'^\mu(\theta), \end{aligned} \tag{4.92}$$

where one may deduce from (4.68) that f^μ is periodic,

$$f^\mu(\theta + 2\pi) = f^\mu(\theta), \tag{4.93}$$

Then, from (4.62), the canonical coordinates x^μ can be written as

$$x^\mu(\tau,\sigma) = q^\mu + \frac{\tau P^\mu}{N\pi} + \frac{1}{2}\left(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\right). \tag{4.94}$$

Next, we expand f^μ in harmonic oscillators using the DDF Fourier components (*) (Del Giudice, Di Vecchia and Fubini, 1972)

$$a_m^\mu = \frac{N}{2}\int_{-\pi}^{\pi} d\theta\, \underset{\sim}{Q}'^\mu(\theta)\exp\left[im\,\lambda\cdot \underset{\sim}{Q}(\theta)\, N\pi/\lambda\cdot P\right] \approx \frac{N}{2}\int_{-\pi}^{\pi} d\theta\, \underset{\sim}{Q}'^\mu(\theta)\, e^{im\theta} \tag{4.95}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} d\sigma\left(\mathscr{P}^\mu(\tau,\sigma) + N u^\mu(\tau,\sigma)\right) e^{im(\sigma+\tau)}.$$

From (4.92), we see that a_m^μ may be written

$$a_m^\mu = P^\mu\,\delta_{m,0} - im\,N\,\frac{1}{2}\int_{-\pi}^{\pi} d\theta'\, f^\mu(\theta')\, e^{im\theta'}. \tag{4.96}$$

(*) A modified form of these oscillators can be used to quantize the string even when the gauge is time-like (Goddard, Hanson and Ponzano, 1974).

Multiplying by $e^{-im\theta}/m$ and summing over $m \neq 0$, we may employ Eq. (4.72) to find an explicit expression for f^μ:

$$(4.97) \qquad f^\mu(\theta) = \frac{i}{N\pi} \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\theta}.$$

Thus the complete Fourier expansion of x^μ is

$$(4.98) \qquad x^\mu(\tau, \sigma) = q^\mu + \frac{\tau P^\mu}{N\pi} + \frac{i}{N\pi} \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu \cos m\sigma \, e^{-im\tau}$$

and $\mathscr{P}^\mu = N \, \partial x^\mu/\partial\tau$, $u^\mu = \partial x^\mu/\partial\sigma$.

Not all of the Fourier components q^μ, P^μ and α_μ^m are independent. Using, for example, the expressions of the Fourier components in terms of Q^μ and the Dirac brackets (4.71), we find the following nonvanishing Dirac brackets,

$$(4.99) \qquad \begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\}^* &= -i N\pi m g^{\mu\nu} \delta_{m,-n} \\ &\quad + \frac{iN\pi}{P^+}(m\lambda^\nu \alpha_{n+m}^\mu - n\lambda^\mu \alpha_{n+m}^\nu) \\ \{P^+, q^-\}^* &= -1 \\ \{P^i, q^j\}^* &= -\delta^{ij} \\ \{P^-, q^-\}^* &= \frac{P^-}{P^+} \\ \{P^-, q^i\}^* &= \frac{P^i}{P^+}, \end{aligned}$$

where λ^μ is the gauge parameter (4.75) and we bear in mind that

$$(4.100) \qquad \alpha_0^\mu = P^\mu.$$

These brackets are compatible with the strong constraints (4.49) or (4.69), which translate into the following constraints on the Fourier components:

$$(4.101) \qquad \begin{aligned} q^+ &= 0 \\ \alpha_m^+ &= \delta_{m,0} P^+ \\ \alpha_m^- &= -\frac{1}{2P^+} \sum_{n=-\infty}^{\infty} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{m-n} \\ P^- \equiv \alpha_0^- &= -\frac{1}{2P^+}\Big(\mathbf{P}\cdot\mathbf{P} + \sum_{n\neq 0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{-n}\Big). \end{aligned}$$

The expression for α_m^- in terms of the other variables can be derived, for example, by solving the constraint Eq. (4.13 a) for $\mathscr{P}^-$,

$$(4.102) \qquad \mathscr{P}^-(\tau, \sigma) = -\frac{\pi}{2P^+}(\boldsymbol{\mathscr{P}}\cdot\boldsymbol{\mathscr{P}} + N^2 \boldsymbol{u}\cdot\boldsymbol{u}),$$

expanding $\mathscr{P}^\mu$ and u^μ in the Fourier components (4.98), and identifying the coefficients of $\cos n\sigma\, e^{-in\tau}$.

Now we may take as our independent variables

$$(4.103)\qquad \begin{aligned} &a_0^+ = P^+,\ q^- \\ &q^i && i = 1, \cdots, D-2 \\ &a_n^i && \left\{ \begin{aligned} &i = 1, \cdots, D-2 \\ &n = -\infty, \cdots, \infty \end{aligned} \right. \\ &a_0^i = P^i \end{aligned}$$

with the nonvanishing brackets

$$(4.104)\qquad \begin{aligned} &\{a_m^j, a_n^k\}^* = -i N\pi\, m \delta^{jk}\, \delta_{m,-n} \\ &\{P^+, q^-\}^* = -1 \\ &\{P^i, q^j\}^* = -\delta^{ij}. \end{aligned}$$

Using (4.101), we easily confirm that the brackets (4.99) follow directly. Note that

$$(4.105)\qquad \begin{aligned} &\{a_m^-, a_n^k\}^* = -i N\pi\, (P^+)^{-1}\, n a_{n+m}^k \\ &\{a_m^-, a_n^-\}^* = i N\pi\, (P^+)^{-1}\, (m-n)\, a_{n+m}^- . \end{aligned}$$

In the literature, one often sees the notation

$$(4.106)\qquad \mathscr{L}_m = -\frac{P^+}{N\pi}\, a_m^- = \frac{1}{2N\pi} \sum_{n=-\infty}^{\infty} \boldsymbol{a}_n \cdot \boldsymbol{a}_{m-n}$$

where $\mathscr{L}_m$ obeys an algebra *isomorphic* to the Virasoro algebra (4.26) (Brower, 1972):

$$(4.107)\qquad \{\mathscr{L}_m, \mathscr{L}_n\}^* = -i\,(m-n)\, \mathscr{L}_{n+m} .$$

The important difference, however, is that $\mathscr{L}_m$ is *nonvanishing*, while the Virasoro operators vanish in the Dirac bracket algebra.

We may now use the independent variables (4.103) to write explicit expressions for the canonical variables and Poincaré group generators of the fully constrained system. The canonical coordinates are

$$(4.108)\qquad \begin{aligned} x^+ &= \frac{\tau P^+}{N\pi} \\ x^- &= q^- + \frac{\tau P^-}{N\pi} + \frac{i}{N\pi} \sum_{n \neq 0} \frac{1}{n}\, a_n^- \cos n\sigma\, e^{-in\tau} \\ &= q^- - \frac{\tau}{2N\pi P^+} \sum_m \boldsymbol{a}_m \cdot \boldsymbol{a}_{-m} \\ &\quad - \frac{i}{2N\pi P^+} \sum_{n \neq 0} \sum_{\text{all } m} \frac{1}{n}\, \boldsymbol{a}_m \cdot \boldsymbol{a}_{n-m} \cos n\sigma\, e^{-in\tau} \end{aligned}$$

$$x^k = q^k + \frac{\tau P^k}{N\pi} + \frac{i}{N\pi} \sum_{n \neq 0} \frac{1}{n}\, \alpha_n^k \cos n\sigma\, e^{-in\tau}$$

$$\mathscr{P}^+ = N \frac{\partial x^+}{\partial \tau} = \frac{P^+}{\pi}$$

$$\mathscr{P}^- = N \frac{\partial x^-}{\partial \tau} = -\frac{1}{2\pi P^+} \sum_m \left[\boldsymbol{a}_m \cdot \boldsymbol{a}_{-m} + \sum_{n \neq 0} \boldsymbol{a}_m \cdot \boldsymbol{a}_{n-m} \cos n\sigma\, e^{-in\tau} \right]$$

$$\mathscr{P}^k = N \frac{\partial x^k}{\partial \tau} = \frac{P^k}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \alpha_n^k \cos n\sigma\, e^{-in\tau}:$$

With some labor, one may check that these expressions for x^μ and $\mathscr{P}^\mu$ reproduce the Dirac bracket algebra (4.60) with the choice (4.75) for λ^μ.

The Fourier decomposition of the Hamiltonian (4.80) is

$$\text{(4.109)} \qquad H = -\frac{1}{N\pi} \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m \cdot \boldsymbol{a}_{-m} = -\frac{1}{N\pi} \left(\mathbf{P}^2 + 2 \sum_{m=1}^{\infty} \boldsymbol{a}_m \cdot \boldsymbol{a}_{-m} \right)$$

and continues to generate the equations of motion (4.81). The Lorentz-invariant mass-squared can be expressed in the form

$$\text{(4.110)} \qquad M^2 = -2\, P^+ P^- - \mathbf{P}^2 = 2 \sum_{m=1}^{\infty} \boldsymbol{a}_m \cdot \boldsymbol{a}_{-m}\, .$$

The translation generators are P^+, P^i, and the dependent variable P^- given in (4.101). Using the definition (4.32) of the Lorentz transformation generator $M^{\mu\nu}$ together with (4.108), we find

$$\text{(4.111)} \qquad M^{jk} = q^j P^k - q^k P^j + \frac{i}{2N\pi} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^j \alpha_{-n}^k - \alpha_n^k \alpha_{-n}^j)$$

$$M^{+k} = -q^k P^+$$

$$M^{+-} = -q^- P^+$$

$$M^{-k} = q^- P^k - q^k P^- + \frac{i}{2\pi N P^+} \sum_{n \neq 0} \sum_{\text{all } m} \frac{1}{n}\, \alpha_{-n}^k (\boldsymbol{a}_m \cdot \boldsymbol{a}_{n-m})\, .$$

In the classical system, the Poincaré group algebra (4.34) continues to hold. This question is more subtle in the quantum mechanical system, which is treated in the next section.

E. Quantum Mechanics

The quantum mechanical system corresponding to the classical string model described in the previous section has been treated extensively in the literature (Goddard *et al.*, 1973; Rebbi, 1974). We will give here only a brief summary of the results.

The following variables can be taken as operators with commutators corresponding to their Dirac brackets, using the convention $\{A, B\}^* \to \frac{1}{i}[A, B]$:

$$
\begin{aligned}
&[a_m^j, a_n^k] = N\pi\, m\delta^{jk}\,\delta_{n,-m} \\
(4.112)\qquad & i\,[P^+, q^-] = 1 \\
& i\,[P^i, q^j] = \delta^{ij}.
\end{aligned}
$$

All other commutators vanish, with the proviso that $a_0^i = P^i$.

The dependent variables are

$$
\begin{aligned}
(4.113)\qquad & q^+ = 0 \\
& a_n^+ = \delta_{n,0}\,P^+ \\
P^- = a_0^- + \frac{N\pi\alpha_0}{P^+} &= -\frac{1}{2\,P^+}\left(:\sum_{m=-\infty}^{\infty} \boldsymbol{a}_m\cdot\boldsymbol{a}_{-m}: -\, 2\,N\pi\alpha_0\right) \\
&= -\frac{N\pi}{P^+}\,(:\mathscr{L}_0: -\,\alpha_0) \\
a_n^- &= -\frac{1}{2\,P^+}\sum_{m=-\infty}^{\infty} \boldsymbol{a}_m\cdot\boldsymbol{a}_{n-m} \\
&= -\frac{N\pi}{P^+}\,\mathscr{L}_n\,, \quad n \neq 0.
\end{aligned}
$$

Here the colons $:(\ ):$ in P^- mean that the expression is to be normal-ordered, with the destruction operators a_m^i with $m > 0$ always on the right. Since P^- contains noncommuting operators, we may need to add for consistency a zero-point energy which we have written as $N\pi\alpha_0/P^+$. The Lorentz-invariant mass operator is thus

$$
(4.114)\qquad M^2 = 2\,N\pi\,(:\mathscr{L}_0: -\,\alpha_0) - \mathbf{P}^2.
$$

The requirement that a_0^- be expressed in normal ordered form causes the commutator of canonically conjugate pairs $[a_m^-, a_{-m}^-]$ to pick up a Schwinger term which was not present in the classical theory.

$$
\begin{aligned}
(4.115)\qquad [a_m^-, a_n^-] = &-\frac{N\pi}{P^+}\,(m-n)\,a_{n+m}^- \\
&+ \left(\frac{N\pi}{P^+}\right)^2 \frac{m^3-m}{12}\,(D-2)\,\delta_{n,-m}\,.
\end{aligned}
$$

Other commutators are unaffected:

$$
(4.116)\qquad [a_m^-, a_n^k] = \frac{N\pi}{P^+}\,n a_{n+m}^k\,.
$$

The Hamiltonian is now

$$\text{(4.117)} \qquad H = -\frac{P^+ P^-}{N\pi} = \frac{1}{2N\pi} (: \mathscr{L}_0 : - \alpha_0)$$

$$= \frac{1}{2N\pi} \left(\mathbf{P}^2 + 2 \sum_{m=1}^{\infty} a^i_{-m} a^i_m - \alpha_0 \right)$$

and generates Hamilton's equations of motion in τ for the dynamical variables. We observe that τ is dimensionless and

$$\text{(4.118)} \qquad \frac{\partial A(\tau, \sigma)}{\partial \tau} = \frac{\partial A}{\partial \left(\frac{N\pi x^+}{P^+} \right)}$$

due to the constraint on x^+. We find that the Fourier components have equations of motion

$$\text{(4.119)} \qquad \begin{aligned} i\,[H, q^k] &= P^k / N\pi \\ i\,[H, a^k_n] &= -in\, a^k_n \end{aligned}$$

so that the fields $x^k(\tau, \sigma)$ and momenta $\mathscr{P}^k(\tau, \sigma)$ obey

$$\text{(4.120)} \qquad \begin{aligned} i\,[H, x^k] &= \mathscr{P}^k / N = \dot{x}^k \\ i\,[H, \mathscr{P}^k] &= N \frac{\partial^2}{\partial \sigma^2} x^k(\tau, \sigma) = \dot{\mathscr{P}}^k . \end{aligned}$$

The most unusual feature of the quantized string is of course the fact that the Lorentz group algebra does not close unless certain conditions are met. We will not go through the calculation here, but merely state the result. One finds the because of hermiticity and normal ordering requirements, the commutator of the quantum operators M^{i-} and M^{j-} takes the form

$$\text{(4.121)} \qquad i\,[M^{i-}, M^{j-}] = -2\,(P^+)^{-2} \sum_{m=1}^{\infty} \left[m \left(1 - \frac{1}{24}(D-2) \right) + \right.$$

$$\left. + \frac{1}{m} \left(\frac{1}{24}(D-2) - \alpha_0 \right) \right] (a^i_{-m} a^j_m - a^j_{-m} a^i_m) .$$

This commutator must vanish if the Lorentz group algebra is to be satisfied, so we conclude that

$$\text{(4.122)} \qquad \begin{aligned} D &= 26 \\ \alpha_0 &= 1 . \end{aligned}$$

There are various methods of relaxing these restrictions, but the norm of some state always becomes negative for $D > 26$. No clear intuitive reason for such a phenomenon is understood at this time.

5. MAXWELL ELECTROMAGNETIC FIELD

The classic example of a theory with an invariance under an Abelian gauge group is Maxwell's theory of electromagnetism.

A. Electromagnetic Hamiltonian without Gauge Constraints

The application of the Dirac method to electromagnetism has been examined by Dirac (1951, 1964). (See also Anderson and Bergmann, 1951, and Kundt, 1966). We shall begin by considering a field $A^\mu(t, \boldsymbol{x})$ transforming as a four-vector on the index μ and possessing continuously infinite degrees of freedom labeled by the spatial coordinate $\boldsymbol{x}$, or some suitable combination of the x^μ depending on the type of dynamics chosen. The most general form for the action is assumed to be

$$\text{(5.1)} \qquad S = \int d^4 x \mathscr{L}(A^\mu(x), \partial A^\mu(x)/\partial x^\nu) .$$

Now we wish to make A^μ describe a massless vector field, which has only two independent degrees of freedom, not four. The time-tested method for doing this is to allow $\mathscr{L}$ (up to total divergences) to be a function only of

$$\text{(5.2)} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \equiv \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \equiv A^{\nu,\mu} - A^{\mu,\nu} .$$

$F^{\mu\nu}$ has the virtue that it is unchanged by the spacetime dependent gauge transformation

$$\text{(5.3)} \qquad A^\mu \rightarrow A'^\mu = A^\mu + \frac{\partial \Lambda(x)}{\partial x_\mu} .$$

The freedom to choose $\Lambda(x)$ may be used to eliminate the unphysical components of $A^\mu(x)$. Note that with the convention (5.2), the usual **E** and **B** fields are

$$\text{(5.4)} \qquad \begin{aligned} E^i &= F^{0i} = -\frac{\partial A^0}{\partial x^i} - \frac{\partial A^i}{\partial t} = -(\nabla A^0 + \dot{\mathbf{A}})^i \\ B^i &= \frac{1}{2}\varepsilon^{ijk} F^{jk} = \varepsilon^{ijk}\frac{\partial A^k}{\partial x^j} = (\nabla \times \mathbf{A})^i . \end{aligned}$$

We still have some latitude available in choosing the Lagrangian; one unconventional form would be the Born-Infeld electrodynamics

$$\text{(5.5)} \qquad \mathscr{L} = (-\mathrm{Det}\,(g_{\mu\nu} + F_{\mu\nu}))^{\frac{1}{2}}$$

which agrees with the Maxwell theory only in the weak-field limit (see Dirac, 1960). In what follows, however, we shall be content to investigate the usual Maxwell Lagrangian,

$$\text{(5.6)} \qquad \mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

in Lorentz-Heaviside units.

We first make the traditional choice of specifying the dynamics on constant-time surfaces. Then the canonical momenta are

$$\pi^{\mu}(x) = \frac{\partial \mathscr{L}}{\partial \dot{A}_{\mu}} = F^{\mu 0} = A^{0,\mu} - A^{\mu,0} . \tag{5.7}$$

The spatial components of π^{μ} are

$$\boldsymbol{\pi}(x) = -\mathbf{E}(x) = \dot{\mathbf{A}} + \nabla A^{0}, \tag{5.8}$$

while the time component vanishes weakly, giving the primary constraint

$$\pi^{0}(x) \approx 0 . \tag{5.9}$$

The canonical Poisson brackets of π^{μ} and A^{ν} are given by

$$\{\pi^{\mu}(t, \boldsymbol{x}), A_{\nu}(t, \boldsymbol{y})\} = -\delta^{\mu}_{\nu}\, \delta^{3}(\boldsymbol{x} - \boldsymbol{y}) \tag{5.10}$$

and are obviously incompatible with Eq. (5.9).

The variation of the fields $A^{\mu}(x)$ gives the usual Euler equations, which may be only weakly valid:

$$\partial_{\nu} \frac{\partial \mathscr{L}}{\partial(\partial_{\nu} A_{\mu})} = F^{\mu\nu}{}_{,\nu} = A^{\nu,\mu}{}_{,\nu} - A^{\mu,\nu}{}_{,\nu} \approx 0 . \tag{5.11}$$

The zero component can be written

$$\nabla \cdot (\nabla A^{0} + \dot{\mathbf{A}}) \approx 0 . \tag{5.12a}$$

Using Eq. (5.7), we rewrite this in the form

$$\nabla \cdot \boldsymbol{\pi} = -\nabla \cdot \mathbf{E} \approx 0 , \tag{5.12b}$$

which is indeed a weak equation because it is incompatible with the Poisson brackets (5.10).

Let us now begin to apply Dirac's approach to our system. First we recall that the Maxwell Lagrangian (5.6) is invariant under the Poincaré group of spacetime transformations [in fact, it is invariant under the full conformal group of which the Poincaré group is a subgroup]. We list below the effect of each Poincaré group transformation on x^{μ} and the field $A^{\mu}(x)$:

(5.13*a*) P^{α} = translation: $\delta x^{\mu} = \varepsilon_{\alpha}(g^{\mu\alpha})$

$$\delta A^{\mu} = A'^{\mu}(x') - A^{\mu}(x) = \varepsilon_{\alpha}(A^{\mu,\alpha})$$

(5.13*b*) $M^{\alpha\beta}$ = Lorentz transform: $\delta x^{\mu} = \omega_{\alpha\beta}(x^{\alpha} g^{\mu\beta} - x^{\beta} g^{\mu\alpha})$

$$\delta A^{\mu} = \omega_{\alpha\beta}\,[(x^{\alpha} g^{\lambda\beta} - x^{\beta} g^{\lambda\alpha})\, \partial_{\lambda} A^{\mu} + g^{\mu\alpha} A^{\beta} - g^{\mu\beta} A^{\alpha}] =$$

$$= \omega_{\alpha\beta}\,[x^{\alpha} A^{\mu,\beta} - x^{\beta} A^{\mu,\alpha} + \Sigma^{\alpha\beta}_{\lambda\nu}\, g^{\lambda\mu} A^{\nu}]$$

where

$$\Sigma^{\alpha\beta}_{\lambda\nu} = g^{\alpha}_{\lambda}\, g^{\beta}_{\nu} - g^{\alpha}_{\nu}\, g^{\beta}_{\lambda} .$$

Note that the tensor indices of A^μ transform oppositely to those of x^μ since it is the *inverse* transformation which relates the old tensor indices to the new coordinate system.

A canonical conserved Noether current is associated with each transformation. The canonical currents and the corresponding conserved Poincaré group generators are given in the equal-time convention by

$$\begin{aligned} &\theta_c^{\mu\nu} = -\frac{\partial\mathscr{L}}{\partial A_{\lambda,\mu}} A_{\lambda,}{}^{\nu} + g^{\mu\nu}\mathscr{L} = F^{\mu\lambda} A_{\lambda,}{}^{\nu} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \\ (5.14)\qquad &\partial_\mu \theta_c^{\mu\nu} = 0 \\ &P^\alpha = \int d^3 x \theta_c^{0\alpha} \end{aligned}$$

and

$$\begin{aligned} &\mathscr{M}^{\alpha\mu\nu} = x^\mu \theta_c^{\alpha\nu} - x^\nu \theta_c^{\alpha\mu} - \frac{\partial\mathscr{L}}{\partial A_{\lambda,\alpha}} \Sigma_{\lambda\sigma}^{\mu\nu} A^\sigma = \\ &\qquad = x^\mu \theta_c^{\alpha\nu} - x^\nu \theta_c^{\alpha\mu} + F^{\alpha\lambda} \Sigma_{\lambda\sigma}^{\mu\nu} A^\sigma \\ (5.14\text{ b})\qquad &\partial_\alpha \mathscr{M}^{\alpha\mu\nu} = 0 \\ &M^{\mu\nu} = \int d^3 x \mathscr{M}^{0\mu\nu}. \end{aligned}$$

Now we take the canonical Hamiltonian

$$(5.15)\qquad H_c = P^0 = \int d^3x\, \theta_c^{00} = \int d^3x \left(\pi^\mu \dot{A}_\mu + \frac{1}{2}\mathbf{B}^2 - \frac{1}{2}\boldsymbol{\pi}^2\right)$$

and add, according to Dirac's prescription, a multiple of the sole primary constraint $\pi^0 \approx 0$. We get

$$\begin{aligned} (5.16)\qquad H_0 &= H_c + \int d^3x\, v_1(x)\, \pi^0(x) = \\ &= \int d^3 x \left(\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}\mathbf{B}^2 - \boldsymbol{\pi}\cdot\nabla A^0 + v_1(x)\,\pi^0(x)\right). \end{aligned}$$

where we have used the identity

$$(5.17)\qquad \dot{\mathbf{A}} = -\nabla A^0 + \boldsymbol{\pi}.$$

We look for *secondary constraints* by computing the bracket of H_0 with the primary constraint $\pi^0 \approx 0$ and requiring the result to be ≈ 0 for consistency:

$$(5.18)\qquad \dot{\pi}^0 = \{\pi^0(x), H_0\} = \nabla\cdot\boldsymbol{\pi} \approx 0.$$

Our two (obviously first class) constraints are thus

$$(5.19)\qquad \begin{aligned} \psi_1 &= \pi^0 \approx 0 \\ \psi_2 &= \nabla\cdot\boldsymbol{\pi} \approx 0. \end{aligned}$$

We now add both ψ_1 and ψ_2 to the Hamiltonian with arbitrary coefficients. Using (5.17) and doing an integration by parts, we find

$$\text{(5.20)} \qquad \mathrm{H} = \mathrm{H}_c + \int d^3x\,(v_1\pi^0 + v_2\nabla\cdot\boldsymbol{\pi}) =$$

$$= \int d^3x\left(\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}\mathbf{B}^2 + v_1\pi^0 + (v_2 + \mathrm{A}^0)\nabla\cdot\boldsymbol{\pi}\right) -$$

$$-\int_{s_\infty} d\mathbf{s}\cdot\boldsymbol{\pi}\,\mathrm{A}^0\,.$$

Hereafter, we will neglect surface integrals such as the last term in (5.20).

By evaluating the brackets

$$\text{(5.21)} \qquad \begin{aligned} \dot{\mathrm{A}}^0 &\equiv \{\mathrm{A}^0, \mathrm{H}\} = -v_1 \\ \dot{\mathrm{A}}^i &\equiv \{\mathrm{A}^i, \mathrm{H}\} = \pi^i - \mathrm{A}^{0,i} - \partial v_2/\partial x^i \\ \dot{\pi}^i &\equiv \{\pi^i, \mathrm{H}\} = -\mathrm{F}^{ij,}{}_j = -(\nabla\times\mathbf{B})^i = -\dot{\mathrm{E}}^i\,, \end{aligned}$$

we find that we can set $v_1 = -\dot{\mathrm{A}}^0 = \mathrm{A}^{0,0}$ and $v_2 = 0$ to give the final result

$$\text{(5.22)} \qquad \mathrm{H} = \int d^3x\left(\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}\mathbf{B}^2 - \dot{\mathrm{A}}^0\pi^0 + \mathrm{A}^0\nabla\cdot\boldsymbol{\pi}\right).$$

Note that A^0 is a basically arbitrary function in the Hamiltonian which is eliminated from the equations of motion only after we choose a gauge. The last two terms in (5.22) are in fact the generating functional for infinitesimal gauge transformations with $\Lambda(x) = \mathrm{A}^0(x)$.

The functions v_1 and v_2 are not necessarily the same for all components of the energy-momentum tensor. We may in fact write

$$\text{(5.23)} \qquad \theta^{\mu\nu} = \theta_c^{\mu\nu} + v_1^{\mu\nu}\pi^0 + v_2^{\mu\nu}\nabla\cdot\boldsymbol{\pi}\,.$$

Arguments parallel to those which led to Eq. (5.22) give

$$\text{(5.24)} \qquad \mathbf{P} = \int d^3x\,(-\boldsymbol{\pi}\times\mathbf{B} + \pi^0\nabla\mathrm{A}^0 + \mathbf{A}(\nabla\cdot\boldsymbol{\pi}))\,,$$

so

$$v_1^{0\mu} = \mathrm{A}^{0,\mu}\,,\; v_2^{0\mu} = 0\,.$$

We see that Eqs. (5.22) and (5.24) are the appropriate modifications of Eq. (5.14a) to generate the spacetime translations of the theory without gauge constraints:

$$\text{(5.25)} \qquad \{\mathrm{P}^\mu, \mathrm{A}^\nu(x)\} = \partial\mathrm{A}^\nu/\partial x_\mu\,.$$

The other generators (5.14) of spacetime symmetries may be treated in a similar fashion.

B. Radiation Gauge Constraints

We now show that the Dirac brackets in the radiation gauge lead directly to the traditional quantum-mechanical structure (e.g. Bjorken and Drell, 1964).

Our objective is to use the gauge freedom (5.3) in our system to fix two components of A^μ so that the first class constraints (5.19) become second class. Since $\pi^0 \approx 0$, one logical choice is to set $A^0 \approx 0$. This is accomplished by the gauge transformation

$$A^\mu \rightarrow A^{\mu\prime} = A^\mu + \partial^\mu \Lambda = A^\mu(x) + \frac{\partial}{\partial x_\mu} \int_0^{x^0} dt\, A^0(t, \boldsymbol{x}), \tag{5.26}$$

for which the infinitesimal generator takes the form of the first class, weakly vanishing addition to Eq. (5.22):

$$L[\Lambda] = \int d^3x\,(\dot{\Lambda}(x)\,\pi^0(x) - \Lambda(x)\,\nabla \cdot \boldsymbol{\pi}(x)) = \int d^3x\,\pi^\mu\,\partial_\mu\,\Lambda\,.$$

Now the Euler equation (5.12) becomes

$$\nabla \cdot \dot{\mathbf{A}}' \approx 0\,, \tag{5.27}$$

so that a second time-independent gauge transformation can be made to fix $\mathbf{A}'$. The radiation gauge, $\nabla \cdot \mathbf{A}'' \approx 0$, is a convenient choice compatible with (5.27), and is achieved by making the gauge transformation

$$\begin{aligned} \mathbf{A}' &\rightarrow \mathbf{A}''(x) = \mathbf{A}'(x) + \nabla_x \int d^3y\, \frac{1}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}\,\nabla_y \cdot \mathbf{A}'(x^0, \boldsymbol{y}) \\ A^{0\prime} &= 0 \rightarrow A^{0\prime\prime}(x) = 0 + \frac{\partial}{\partial x_0} \int d^3y\, \frac{1}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}\,\nabla_y \cdot \mathbf{A}'(x^0, \boldsymbol{y}) = 0\,. \end{aligned} \tag{5.28}$$

Here $\mathbf{A}'$ is given by (5.26) and $\nabla^2(1/4\pi\,|\boldsymbol{x}-\boldsymbol{y}|) = -\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$, while the vanishing of $A^{0\prime\prime}$ follows directly from (5.27). Dropping all primes, we find the set of constraints

$$\begin{aligned} \varphi_1 &= \pi^0 \approx 0 \\ \varphi_2 &= \nabla \cdot \boldsymbol{\pi} \approx 0 \\ \varphi_3 &= A^0 \approx 0 \\ \varphi_4 &= \nabla \cdot \mathbf{A} \approx 0 \end{aligned} \tag{5.29}$$

in the new gauge.

The matrix of Poisson brackets of the constraints (5.29) is

$$C_{ij}(x, y) = \{\varphi_i(t, \boldsymbol{x}), \varphi_j(t, \boldsymbol{y})\} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nabla_x^2 \\ -1 & 0 & 0 & 0 \\ 0 & -\nabla_x^2 & 0 & 0 \end{bmatrix} \delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{5.30}$$

Imposing the boundary condition that the fields vanish at ∞, we find that the inverse of (5.30) exists and takes the form

$$C_{ij}^{-1}(x, y) = \begin{bmatrix} 0 & 0 & -\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 \\ 0 & 0 & 0 & \dfrac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\ \delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 & 0 & 0 \\ 0 & -\dfrac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} & 0 & 0 \end{bmatrix}. \tag{5.31}$$

Thus the Dirac brackets may be computed from Eq. (1.58) to be

$$\begin{aligned} \{\pi^\mu(t, \boldsymbol{x}), A^\nu(t, \boldsymbol{y})\}^* &= (-g^{\mu\nu} - g^{\mu 0} g^{\nu 0})\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) + \\ &\quad + \frac{\partial^2}{\partial x_\mu \partial y_\nu} \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\ \{\pi^\mu, \pi^\nu\}^* &= \{A^\mu, A^\nu\}^* = 0\,. \end{aligned} \tag{5.32}$$

Equations (5.32) are compatible with setting the constraints (5.29) strongly to zero. Therefore, only two of the A^i and two of the π^i are independent variables. Although we may solve $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \boldsymbol{\pi} = 0$ for the remaining two dependent variables if required, this is unnecessary; in the process of inverting the matrix C_{ij} to find the Dirac brackets (5.32), we have automatically given the dependent variables the same brackets they would have when expressed in terms of the independent variables.

We now verify Hamilton's equations of motion in the fully constrained system. The Hamiltonian is just

$$H = \int d^3x\theta^{00} = \int d^3x \left(\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}\mathbf{B}^2\right). \tag{5.33}$$

Thus the equations of motion are

$$\begin{aligned} \{\mathbf{A}(x), H\}^* &= \boldsymbol{\pi}(x) = \dot{\mathbf{A}} \\ \{\boldsymbol{\pi}(x), H\}^* &= -\nabla\times\mathbf{B} = -\dot{\mathbf{E}}\,. \end{aligned} \tag{5.34}$$

In deriving Eq. (5.34), we have assumed that $\mathbf{E}$ vanishes sufficiently rapidly at ∞ to drop a surface integral and have made use of Eq. (5.17).

Using Eq. (5.14), we find the remaining Poincaré group generators to be

$$\begin{aligned} \mathbf{P} &= \int d^3x(-\pi^i \nabla A^i) \\ M^{ij} &= \int d^3x(-\pi^k(x^i\partial^j - x^j\partial^i)A^k - \pi^i A^j + \pi^j A^i) \\ M^{0i} &= \int d^3x\left(-x^0\pi^j\partial^i A^j - \frac{1}{2}x^i(\boldsymbol{\pi}^2 + \mathbf{B}^2)\right). \end{aligned} \tag{5.35}$$

In the star brackets, P^μ and $M^{\mu\nu}$ continue to obey the Poincaré algebra (2.25). Under the action of translations and rotations, the A^i transform as

$$(5.36)\qquad \begin{aligned} \{P^i, A^j(x)\}^* &= \frac{\partial A^j(x)}{\partial x_i} = \partial^i A^j \\ \{M^{ij}, A^k(x)\}^* &= (x^i \partial^j - x^j \partial^i) A^k(x) + \delta^{ik} A^j(x) - \delta^{jk} A^i(x) . \end{aligned}$$

As usual, surface integrals have been dropped in Eq. (5.36). Under boosts, however, extra terms occur which cannot be neglected. We find

$$(5.37)\qquad \{M^{0i}, A^j(x)\}^* = (x^0 \partial^i - x^i \partial^0) A^j(x) - \frac{\partial}{\partial x_j} \int d^3 y \, \frac{\partial A^i(x^0, \boldsymbol{y})/\partial x_0}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} .$$

The extra term is required in order for M^{0i} to have vanishing star brackets with $\nabla \cdot \mathbf{A}$. It means that under an infinitesimal boost parametrized by ω_{0i}, $A^\mu(x)$ undergoes a gauge transformation of order ω_{0i} in order to keep $A^0 = 0$ and $\nabla \cdot \mathbf{A} = 0$ in the new Lorentz frame:

$$(5.38)\qquad \begin{aligned} A'^\mu(x) &= A^\mu(x) + \omega_{0i}(x^0 \partial^i - x^i \partial^0) A^\mu(x) + \\ &+ \omega_{0i}(g^{0\mu} A^i(x) - g^{i\mu} A^0(x)) - \omega_{0i} \frac{\partial}{\partial x_\mu} \int d^3 y \, \frac{\partial A^i(x^0, \boldsymbol{y})/\partial x_0}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} . \end{aligned}$$

To show that $A'^0 = 0$, we have used the fact that with our gauge choice (5.29), the Euler equations (5.11) reduce to

$$(5.39)\qquad \partial_\mu \partial^\mu A^i = 0$$

and integrated by parts.

C. Axial Gauge

Another amusing gauge which can be treated in the canonical formalism is the axial gauge (Kummer, 1961; Arnowitt and Fickler, 1962). We begin by using the gauge invariance of the theory to set

$$(5.40)\qquad A^3 \approx 0 .$$

We see that if we choose $v_2 = 0$ in Eq. (5.21), then Eq. (5.40) will hold for all times only if

$$(5.41)\qquad \dot{A}^3 = \pi^3 - A^{0,3} \approx 0.$$

Equation (5.41) is similar to a secondary contraint following from the gauge constraint (5.40); it replaces the condition $\nabla \cdot \mathbf{A} \approx 0$ used in the preceding section. The full set of constraints is then

$$
\begin{aligned}
\varphi_1 &= \pi^0 \approx 0 \\
\varphi_2 &= \nabla \cdot \boldsymbol{\pi} \approx 0 \\
\varphi_3 &= A^3 \approx 0 \\
\varphi_4 &= \pi^3 - A^{0,3} \approx 0 .
\end{aligned} \tag{5.42}
$$

The Poisson bracket matrix is

$$
C_{ij}(x, y) = \{\varphi_i(x), \varphi_j(y)\} = \tag{5.43}
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & (\partial/\partial x^3)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
0 & 0 & -(\partial/\partial x^3)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 \\
0 & -(\partial/\partial x^3)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 & +\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
(\partial/\partial x^3)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 & -\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0
\end{bmatrix} .
$$

We choose boundary conditions so that the inverse of (5.43) is

$$
C_{ij}^{-1}(x, y) = \begin{bmatrix}
0 & -g(x, y) & 0 & f(x, y) \\
g(x, y) & 0 & -f(x, y) & 0 \\
0 & -f(x, y) & 0 & 0 \\
f(x, y) & 0 & 0 & 0
\end{bmatrix} \tag{5.44}
$$

where

$$
\begin{aligned}
\frac{\partial g(x, y)}{\partial x^3} &= f(x, y) \\
\frac{\partial f(x, y)}{\partial x^3} &= \frac{\partial^2 g(x, y)}{(\partial x^3)^2} = \delta^3(\boldsymbol{x}-\boldsymbol{y}),
\end{aligned} \tag{5.45}
$$

and

$$
\begin{aligned}
g(x, y) &= \tfrac{1}{2}\,\delta(x^1 - y^1)\,\delta(x^2 - y^2)\,|x^3 - y^3| \\
f(x, y) &= \tfrac{1}{2}\,\delta(x^1 - y^1)\,\delta(x^2 - y^2)\,\varepsilon(x^3 - y^3) .
\end{aligned} \tag{5.46}
$$

Here $\varepsilon(x - y) =$ algebraic sign of $(x - y)$.

When we insert Eq. (5.44) into the Dirac bracket formula (1.58), we find

$$
\{\pi^\mu(t, \boldsymbol{x}), \pi^\nu(t, \boldsymbol{y})\}^* = 0 \tag{5.47}
$$

$$
\begin{aligned}
\{A^\mu(t, \boldsymbol{x}), A^\nu(t, \boldsymbol{y})\}^* &= (g^{\mu 0} g^{3\nu} + g^{\nu 0} g^{3\mu})\, f(x, y) \\
&\quad - \left(g^{\mu 0} \frac{\partial}{\partial x_\nu} + g^{\nu 0} \frac{\partial}{\partial x_\mu}\right) g(x, y)
\end{aligned} \tag{5.48}
$$

$$
\begin{aligned}
\{\pi^\mu(t, \boldsymbol{x}), A^\nu(t, \boldsymbol{y})\}^* &= (-g^{\mu\nu} - g^{\mu 0} g^{\nu 0})\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
&\quad + g^{\mu 3} \frac{\partial}{\partial x_\nu} f(x, y) .
\end{aligned} \tag{5.49}
$$

The nonvanishing brackets are

$$\{A^0, A^1\}^* = \frac{\partial g(x, y)}{\partial x^1} \tag{5.50}$$

$$\{A^0, A^2\}^* = \frac{\partial g(x, y)}{\partial x^2}$$

$$\{\pi^1, A^1\}^* = \{\pi^2, A^2\}^* = -\delta^3(\boldsymbol{x} - \boldsymbol{y})$$

$$\{\pi^3, A^1\}^* = +\frac{\partial f(x, y)}{\partial x^1}$$

$$\{\pi^3, A^2\}^* = +\frac{\partial f(x, y)}{\partial x^2},$$

so π^1, π^2, A^1 and A^2 still obey the *canonical* brackets (5.10). These brackets are manifestly compatible with all the constraints (5.42) and can be used as the basis of the axial gauge quantization scheme.

The Hamiltonian in the axial gauge is

$$H = \int d^3x \left[\tfrac{1}{2}(\pi^1)^2 + \tfrac{1}{2}(\pi^2)^2 - \pi^1 \partial_1 A^0 - \pi^2 \partial_2 A^0 - \tfrac{1}{2}(\partial_3 A^0)^2 + \tfrac{1}{2}\mathbf{B}^2\right]$$

where we have set the constraints (5.42) equal to zero. A^0 is a dependent variable found by solving the Euler equation

$$0 = \partial_i \pi^i = \partial_1 \pi^1 + \partial_2 \pi^2 + \partial_3 \partial_3 A^0$$

$$= \nabla^2 A^0 - \partial_1 \dot{A}^1 - \partial_2 \dot{A}^2$$

to give

$$A^0(x) = -\tfrac{1}{2}\int d\xi\, |x^3 - \xi|\, (\partial_1 \pi^1(tx^1 x^2 \xi) + \partial_2 \pi^2(tx^1 x^2 \xi)).$$

$$= -\int d^3y \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}(\partial_1 \dot{A}^1 + \partial_2 \dot{A}^2).$$

Using just π^1, π^2, A^1, A^2 as the independent variables, we easily confirm all of the brackets (5.50). Assuming appropriate boundary conditions, we find the Hamiltonian equation of motion

$$\{\pi^i, H\}^* = -(\nabla \times \mathbf{B})^i = -\dot{E}^i, \qquad i = 1, 2$$

$$\{A^i, H\}^* = \dot{A}^i, \qquad 1 = 1, 2,$$

for the independent canonical variables. The properties of the rest of the Poincaré group generators are treated as usual.

D. Null-Plane Brackets

As an illustration of how one can define a Hamiltonian field theory on a surface other than the constant-time surface, we examine the null-plane treatment of Kogut and Soper (1970). One interesting feature of null-plane dynamics is that the number of independent canonical variables is reduced to half the number present in equal-time dynamics.

We begin by establishing our notation (see also Appendix A). We define

$$(5.51)\qquad \begin{aligned} x^+ &= \frac{1}{\sqrt{2}}(x^3 + x^0) = x_- = \frac{1}{\sqrt{2}}(x_3 - x_0) \\ x^- &= \frac{1}{\sqrt{2}}(x^3 - x^0) = x_+ = \frac{1}{\sqrt{2}}(x_3 + x_0) \\ \partial_+ &= \frac{\partial}{\partial x^+} = \frac{1}{\sqrt{2}}(\partial_3 + \partial_0) = \partial^- = \frac{1}{\sqrt{2}}(\partial^3 - \partial^0) \\ \partial_- &= \frac{\partial}{\partial x^-} = \frac{1}{\sqrt{2}}(\partial_3 - \partial_0) = \partial^+ = \frac{1}{\sqrt{2}}(\partial^3 + \partial^0) \\ x^i &= x_i \qquad i = 1\,,\,2, \end{aligned}$$

so that the metric is

$$(5.52)\qquad g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\mu = (+\,,-\,,1\,,2)$. Latin indices of vector quantities (e.g. $\boldsymbol{x}$) will take on only the values $i = (1\,,2)$. The variable x^+ will now be used in place of x^0 to define the canonical momenta and the evolution of the system. The Poisson brackets and symmetry generators of the theory can then be written as indicated in Section 1. D.

The canonical momenta π^μ are conjugate to the x^+ derivatives of A_μ, and are therefore defined as

$$(5.53)\qquad \pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_+ A_\mu)} = F^{\mu+} = A^{+,\mu} - A^{\mu,+},$$

where $\mathscr{L}$ is still given by (5.6) because the volume element is unchanged. The canonical Poisson brackets are given on the surface $x^+ =$ constant, so

$$(5.54)\qquad \begin{aligned} &\{\pi^\mu(x^+, x^-, x^1, x^2)\,, A_\nu(x^+, y^-, y^1, y^2)\} \\ &\quad = -\,\delta^\mu_\nu\,\delta(x^- - y^-)\,\delta^2(\boldsymbol{x} - \boldsymbol{y}) \\ &\quad \equiv -\,\delta^\mu_\nu\,\delta^3(x - y). \end{aligned}$$

Since the metric is given by (5.52), A^- is the canonical conjugate to π^+, rather than A^+.

The first thing we notice is that the definition (5.53) of the canonical momenta gives rise not to one constraint but to *three primary constraints* which are incompatible with the canonical brackets (5.54):

$$\pi^{+} \approx 0 \tag{5.55}$$

$$\pi^{1} - \partial_1 A^{+} + \partial_- A^{1} \approx 0 \tag{5.56}$$

$$\pi^{2} - \partial_2 A^{+} + \partial_- A^{2} \approx 0. \tag{5.57}$$

The definition of the fourth momentum

$$\pi^{-} = \partial_+ A^{+} - \partial_- A^{-} \tag{5.58}$$

is not a constraint but a dynamical relation between the velocity $\partial_+ A^+$ and the canonical variables π^- and A^-. We observe that

$$\{\pi^{i} - \partial_i A^{+} + \partial_- A^{i}, \pi^{j} - \partial_j A^{+} + \partial_- A^{j}\} = 2\,\delta^{ij}\,\frac{\partial\delta^3(x-y)}{\partial x^-}\,. \tag{5.59}$$

so the constraints (5.56) and (5.57) are second class.

With null-plane dynamics, the formal expression for the canonical energy-momentum tensor is

$$\theta_c^{\mu\nu} = F^{\mu\lambda} A_{\lambda,}{}^{\nu} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

where the metric is given by (5.52). The canonical conserved translation generators are deduced from Section 1. D to be

$$P^{\mu} = \int dx^- \, d^2x\, \theta_c^{+\mu}(x) \tag{5.60}$$

and formally generate the transformations

$$\{P^{\mu}, A^{\nu}\} = \partial^{\mu} A^{\nu} \tag{5.61}$$

Since $P_+ = P^-$ generates x^+ displacements, the natural object to use as the Hamiltonian is

$$H_c = -P^- = -\int dx^- \, d^2x\, \theta_c^{+-}(x)\,. \tag{5.62}$$

where

$$\theta_c^{+-} = -\tfrac{1}{2}(\pi^-)^2 - \tfrac{1}{2} F^{12} F^{12} - (\pi^- \partial_- + \pi^i \partial_i) A^-. \tag{5.63}$$

The minus sign in (5.62) is dictated by the convention that the Hamiltonian be positive definite and the fact that our null plane metric (5.52) has no minus signs as did the ordinary metric (1.46) with $g^{00} = -1$.

Now we search for secondary constraints. We form a new Hamiltonian $\tilde{H}$ by adding multiples of the constraints (5.55) — (5.57) to the canonical Hamiltonian,

$$\tilde{H} = H_c + \int dx^- \, d^2x\,(v\pi^+(x) + u_i(\pi^i - \partial^i A^+ + \partial_- A^i)). \tag{5.64}$$

Then we require that $\tilde{\mathrm{H}}$ have vanishing canonical brackets with all of the constraints (5.55)—(5.57):

$$(5.65)\qquad \{\pi^+, \tilde{\mathrm{H}}\} = \partial_- \pi^- + \partial_i \pi^i \approx 0$$

$$(5.66)\qquad \{\pi^i - \partial^i \mathrm{A}^+ + \partial_- \mathrm{A}^i, \tilde{\mathrm{H}}\} = - \partial^i \pi^- + (\delta^{i2} \partial^1 - \delta^{i1} \partial^2)\, \mathrm{F}^{12} + 2\, \partial_- u^i \approx 0.$$

Equation (5.65) is a genuine secondary constraint, while Eq. (5.66) imposes conditions on the u^i. Of our final set of constraints (5.55)—(5.57) and (5.65), the only first class ones are

$$(5.67)\qquad \begin{aligned} \psi_1 &= \pi^+ \approx 0 \\ \psi_2 &= \partial_- \pi^- + \partial_i \pi^i \approx 0\,. \end{aligned}$$

The remaining two,

$$(5.68)\qquad \begin{aligned} \chi_1 &= \pi^1 - \partial^1 \mathrm{A}^+ + \partial_- \mathrm{A}^1 \approx 0 \\ \chi_2 &= \pi^2 - \partial^2 \mathrm{A}^+ + \partial_- \mathrm{A}^2 \approx 0 \end{aligned}$$

are second class due to Eq. (5.59).

Now we invert the bracket matrix $\{\chi_i, \chi_j\}$ given in Eq. (5.59) for the second class constraints with the result

$$(5.69)\qquad \mathrm{C}_{ij}^{-1}(x\,, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{4}\, \varepsilon(x^- - y^-)\, \delta^2(\boldsymbol{x} - \boldsymbol{y})\,,$$

where $\varepsilon(x^- - y^-) = \operatorname{sign}(x^- - y^-)$. Next we take $\tilde{\mathrm{H}}$ to be the *first-class* Hamiltonian analogous to (1.24),

$$(5.70)\qquad \tilde{\mathrm{H}} = \mathrm{H}' = \mathrm{H}_c - \int d^3x\, d^3y\, \{\mathrm{H}_c\,, \chi_i(x)\}\, \mathrm{C}_{ij}^{-1}(x\,, y)\, \chi_j(y)\,.$$

It is easy to verify that (5.70) contains the explicit solution of Eq. (5.66) for the u^i, as discussed in Section 1.A. Finally, we use Eq. (5.69) to compute the preliminary Dirac brackets, which we denote with a prime:

$$(5.71)\qquad \begin{aligned} \{\mathrm{A}^\mu(x)\,, \mathrm{A}^\nu(y)\}' &= (g^{\mu1} g^{1\nu} + g^{\mu2} g^{2\nu})\, \frac{1}{4}\, \varepsilon(x^- - y^-)\, \delta^2(\boldsymbol{x} - \boldsymbol{y}) \\ \{\pi^\mu(x)\,, \pi^\nu(y)\}' &= - g^{\mu+} g^{\nu+}\, \nabla_x^2\, \frac{1}{4}\, \varepsilon(x^- - y^-)\, \delta^2(\mathrm{x} - \mathrm{y}) + \\ &\quad + \frac{1}{2}\, (g^{\mu+} g^{\nu i}\, \partial_i^x + g^{\nu+} g^{\mu i}\, \partial_i^x - (g^{\mu1} g^{1\nu} + g^{\mu2} g^{2\nu})\, \partial_i^x)\, \delta^3(x - y) \\ \{\pi^\mu(x)\,, \mathrm{A}^\nu(y)\}' &= (- g_{\mu\nu} + \frac{1}{2}\, g^{\mu1} g^{1\nu} + \frac{1}{2}\, g^{\mu2} g^{2\nu})\, \delta^3(x - y) + \\ &\quad - g^{\mu+} g^{\nu i}\, \partial_i^x\, \frac{1}{4}\, \varepsilon(x^- - y^-)\, \delta^2(\boldsymbol{x} - \boldsymbol{y})\,. \end{aligned}$$

All equations of motion are now computed using the preliminary brackets (5.71) compatible with setting the second class constraints (5.68) *strongly* to zero.

Before choosing any gauge constraints, we may take our total Hamiltonian to be (5.70) plus arbitrary multiples of the first class constraints (5.67);

$$\text{(5.72)} \qquad H = \int dx^- \, d^2x \, [\tfrac{1}{2}(\pi^-)^2 + \tfrac{1}{2}(F^{12})^2 + v_1 \pi^+ + (v_2 - A^-)(\partial_- \pi^- + \partial_i \pi^i)]$$

where the multiples of χ_i in (5.70) can be dropped if we use the primed brackets. Hamilton's equations become

$$\text{(5.73 }a\text{)} \qquad \begin{aligned} \{A^+, H\}' &= \partial_+ A^+ - \partial_- v_2 \\ \{A^-, H\}' &= v_1 \\ \{A^i, H\}' &= \partial_+ A^i - \tfrac{1}{2} \partial^i v_2 \\ \{\pi^+, H\}' &= \partial_- \pi^- + \partial_i \pi^i \approx 0 \\ \{\pi^-, H\}' &= \partial_+ \pi^- \\ \{\pi^i, H\}' &= \partial_+ \pi^i . \end{aligned}$$

We may therefore choose to express v_1 and v_2 as

$$\text{(5.73 }b\text{)} \qquad \begin{aligned} v_1 &= \partial_+ A^- \\ v_2 &= 0 . \end{aligned}$$

The generating functional of the gauge transformation is thus

$$\text{(5.74)} \qquad L[A^-] = \int dx^- \, d^2x \, [\pi^+ \partial_+ A^- - A^-(\partial_- \pi^- + \partial_i \pi^i)] .$$

E. Null-Plane Radiation Gauge

Now let us eliminate the arbitrary functions from our theory by using gauge invariance to fix the remaining two degrees of freedom corresponding to the first class constraints (5.67). First it is convenient to recall that a straightforward derivation of the Euler equations gives the following results:

$$\text{(5.75)} \qquad \begin{aligned} F^{+\mu}{}_{,\mu} &= - \partial_- \pi^- - \partial_i \pi^i = - (\partial_+ \partial_- + \partial_i \partial_i) A^+ + \\ &\qquad + \partial_-(\partial_- A^- + \partial_i A^i) \approx 0 \\ F^{-\mu}{}_{,\mu} &= \partial_+ (\partial_\mu A^\mu) - \partial_\mu \partial^\mu A^- \approx 0 \\ F^{i\mu}{}_{,\mu} &= - \partial_\mu \partial^\mu A^i + \partial^i (\partial_\mu A^\mu) \approx 0 \end{aligned}$$

Now let us choose the null-plane analog of the Coulomb gauge by setting to zero the field $A_+ = A^-$ conjugate to $\pi^+ \approx 0$. A gauge transformation which accomplishes this is

$$\text{(5.76)} \qquad A^\mu \to A'^\mu = A^\mu(x) - \partial^\mu \int_0^{x^+} d\tau A^-(\tau, x^-, \boldsymbol{x})$$

where

$$A'^- \approx 0 .$$

From Eqs. (5.75) and (5.58), we find that

$$\partial_{+}(\partial_{+} A'^{+} + \partial_{i} A'^{i}) \approx \partial_{+}(\pi^{-} + \partial_{i} A'^{i}) \approx 0 . \tag{5.77}$$

Exactly as in the equal-time case, we may find a gauge transformation which sets $\pi^{-} + \partial_i A'''^{i}$ to zero while maintaining $A''^{-} \approx 0$ due to Eq. (5.77). The solution is

$$\mathbf{A}' \to \mathbf{A}'' = \mathbf{A}'(x) - \nabla_x \int d^2 y \, \frac{ln(\mathbf{x}-\mathbf{y})^2}{4\pi} \left(\nabla_y \cdot \mathbf{A}'(y) + \pi^{-}(y)\right)$$

$$\begin{aligned} A'^{+} \to A''^{+} &= A'^{+}(x) - \int d^2 y \, \frac{ln(\mathbf{x}-\mathbf{y})^2}{4\pi} \, \partial^x_{-} \left(\pi^{-} + \partial^y_i A'^{i}(x^{-}, \mathbf{y})\right) \\ A'^{-} \to A''^{-} &= 0 - \int d^2 y \, \frac{ln(\mathbf{x}-\mathbf{y})^2}{4\pi} \, \partial_{+} \left(\pi^{-} + \partial^y_i A'^{i}(x^{+}, x^{-}, \mathbf{y})\right) \\ &\approx 0 \cdot \end{aligned} \tag{5.78}$$

Dropping the primes, we find our final set of constraints

$$\begin{aligned} \varphi_1 &= \pi^{+} \approx 0 \\ \varphi_2 &= \partial_{-} \pi^{-} + \partial_i \pi^{i} \approx 0 \\ \varphi_3 &= A_{+} = A^{-} \approx 0 \\ \varphi_4 &= \pi^{-} + \partial_i A^{i} \approx 0 . \end{aligned} \tag{5.79}$$

Using the preliminary brackets (5.71), we compute the matrix of brackets of the constraints (5.79) to be

$$\{\varphi_i(x), \varphi_j(y)\}' = C_{ij}(x, y) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \nabla^2_x \\ 1 & 0 & 0 & 0 \\ 0 & -\nabla^2_x & 0 & 0 \end{bmatrix} \delta(x^{-} - y^{-}) \, \delta^2(\mathbf{x} - \mathbf{y}). \tag{5.80}$$

To specify the boundary conditions, we observe that when the (strong) constraints (5.68) are combined with φ_4 and inserted into the Euler equation φ_2, we find

$$\nabla^2 A^{+} - 2 \partial_{-}(\partial_i A^{i}) \approx 0 . \tag{5.81}$$

We then choose our boundary conditions (and exhaust all remaining gauge freedom) so that the solution of (5.81) is

$$A^{+}(x) = 2 \frac{\partial}{\partial x^{-}} \int d^2 z \, \frac{ln(\mathbf{z}-\mathbf{x})^2}{4\pi} \, \frac{\partial A^{i}(x^{+}, x^{-}, \mathbf{z})}{\partial z^{i}} . \tag{5.82}$$

Then the inverse of (5.8) exists and may be written

$$C^{-1}_{ij}(x, y) = \begin{bmatrix} 0 & 0 & \delta^3(x-y) & 0 \\ 0 & 0 & 0 & -G(x, y) \\ -\delta^3(x-y) & 0 & 0 & 0 \\ 0 & G(x, y) & 0 & 0 \end{bmatrix} \tag{5.83}$$

where

$$(5.84\,a)\qquad \begin{aligned} G(x, y) &= \frac{1}{4\pi}\, ln\,(\boldsymbol{x} - \boldsymbol{y})^2\, \delta(x^- - y^-) \\ \nabla_x^2\, G(x, y) &= \delta^2(\boldsymbol{x} - \boldsymbol{y})\, \delta(x^- - y^-) = \delta^3(x - y)\,. \end{aligned}$$

For use in the Dirac brackets we also define

$$(5.84\,b)\qquad \begin{aligned} F(x, y) &= \frac{1}{2}\, \varepsilon(x^- - y^-)\, \delta^2(\boldsymbol{x} - \boldsymbol{y}) \\ \frac{\partial}{\partial x^-}\, F(x, y) &= \delta^3(x - y) \end{aligned}$$

Plugging Eq. (5.83) into Eq. (1.58) with the primed brackets (5.71), we find the Dirac brackets

$$(5.85)\qquad \begin{aligned} \{\pi^\mu(x), \pi^\nu(y)\}^* &= -\tfrac{1}{2} g^{\mu+} g^{\nu+} \nabla_x^2\, F(x, y) + \\ &\quad + \tfrac{1}{2}\,[-(g^{\mu 1} g^{\nu 1} + g^{\mu 2} g^{\nu 2})\, \partial_-^x + g^{\mu+} g^{\nu i}\, \partial_i^x + \\ &\quad + g^{\mu i} g^{\nu+}\, \partial_i^x]\, \delta^3(x - y) \\ \{A^\mu(x), A^\nu(y)\}^* &= \tfrac{1}{2}\,(g^{\mu 1} g^{\nu 1} + g^{\mu 2} g^{\nu 2})\, F(x, y) + 2 g^{\mu-} g^{\nu-}\, \partial^x\, G(x, y) + \\ &\quad + (g^{\mu i} g^{\nu-} + g^{\mu-} g^{\nu i})\, \partial_i^x\, G(x, y) \\ \{\pi^\mu(x), A^\nu(y)\}^* &= (-g^{\mu\nu} + g^{\mu-} g^{\nu+} + \tfrac{1}{2} g^{\mu 1} g^{\nu 1} + \tfrac{1}{2} g^{\mu 2} g^{\nu 2})\, \delta^3(x - y) + \\ &\quad + g^{\mu i} g^{\nu-}\, \partial_i^x \partial_-^x\, G(x, y) + g^{\mu i} g^{\nu j}\, \partial_i^x \partial_j^x\, G(x, y) - \\ &\quad - \tfrac{1}{2} g^{\mu+} g^{\nu i}\, \partial_i^x\, F(x, y) \end{aligned}$$

which are manifestly compatible with all the constraints (5.68) and (5.79).

Using Eq. (5.82) combined with the constraints (5.68) and (5.79) it is easy to verify that all of the Dirac brackets (5.85) follow directly from the *nonlocal* Dirac brackets

$$(5.86)\qquad \{A^i(x^+ x^- \boldsymbol{x}), A^j(x^+ y^- \boldsymbol{y})\}^* = +\frac{1}{4}\, \delta^{ij}\, \varepsilon(x^- - y^-)\, \delta^2(\boldsymbol{x} - \boldsymbol{y})$$

of the two independent canonical variables A^1 and A^2.

Equation (5.86) serves as the starting point for the quantum theory.

Applying all of our constraints, we find that θ^{+-} is simply

$$(5.87)\qquad \theta^{+-} = -\tfrac{1}{2}\,((\partial_1 A^2 - \partial_2 A^1)^2 + (\partial_1 A^1 + \partial_2 A^2)^2)$$

and the Hamiltonian is

$$(5.88)\qquad H = -\int dx^-\, d^2 x\, \theta^{+-}(x)$$

The Hamiltonian equations of motion then become

$$(5.89)\qquad \partial_+ A^i \equiv \{A^i(x), H\}^* = -\frac{1}{4} \int d\xi\, \varepsilon(x^- - \xi)\, \partial_j \partial_j\, A^i(x^+ \xi \boldsymbol{x})\,.$$

Differentiating by x^-, we find

$$\partial_- \partial_+ A^i = \{\partial_- A^i, H\}^* = -\tfrac{1}{2}\partial_j \partial_j A^i$$

so that Eq. (5.89) can be understood very simply as the x^- integral of the Klein-Gordon equation

$$\partial_\mu \partial^\mu A^i = (2\,\partial_+ \partial_- + \partial_j \partial_j)\, A^i = 0\,. \tag{5.90}$$

We leave as an exercise for the reader the development of null-plane electrodymanics in the gauge $A^+ \approx 0$ chosen by Kogut and Soper (1970). We remark that the choice $A^+ \approx 0$ a produces a theory very similar to the equal-time electrodynamics in the axial gauge $A^3 \approx 0$.

6. YANG-MILLS GAUGE FIELD

We now turn our attention to theories with a set of vector fields whose Lagrangian is invariant under a non-Abelian gauge transformation, as proposed by Yang and Mills (1954) [see, e.g., Weinberg, 1973; Abers and Lee, 1973]. Since the fields in the multiplet mix among themselves under the gauge transformation, this type of theory is similar in some ways to the theory of gravitation given in the next chapter. The Yang-Mills fields are in fact self-interacting fields, again like gravitation. Since we have until now restricted ourselves to free systems, this will be our first exposure to the difficulties of interactions.

A. Lie Groups

We begin with a brief digression on Lie groups (Racah, 1965). Consider an n-parameter Lie group $\mathscr{J}$ and the associated Lie algebra $\mathscr{L}(\mathscr{J})$. Let T_a, $a = 1, \cdots, n$, be a complete set of linearly-independent elements of $\mathscr{L}(\mathscr{J})$. The composition rule in $\mathscr{L}(\mathscr{J})$ for these elements is

$$i\,[T_a, T_b] = -\, C_{ab}{}^c\, T_c\,. \tag{6.1}$$

The antisymmetry of (6.1) and the Jacobi identity imply that $C_{ab}{}^c$ satisfies

$$C_{ab}{}^c = -\, C_{ba}{}^c \tag{6.2}$$

$$C_{ab}{}^e\, C_{ec}{}^d + C_{bc}{}^e\, C_{ea}{}^d + C_{ca}{}^e\, C_{eb}{}^d = 0\,. \tag{6.3}$$

We can now define quadratic polynomials on the enveloping Lie algebra using the symmetric second rank " metric tensor "

$$g_{ab} = C_{ad}{}^e\, C_{be}{}^d \tag{6.4}$$

Det $|g_{ab}|$ is nonvanishing provided the group is semi-simple (has no Abelian invariant subgroups). We now define

$$C_{abc} = C_{ab}{}^d\, g_{dc}\,. \tag{6.5}$$

Using Eqs. (6.2) and (6.3) we can show that C_{abc} is totally antisymmetric in its indices.

Next, we note that any given Lie group will have a specific number of Casimir invariants G_i which are polynomials in the elements T_a and which are in the center of the enveloping algebra:

(6.6) $$[T_a, G_i] = 0, \quad \text{for all } a \text{ and all } i.$$

The most familiar Casimir invariant is the quadratic one, which can be written

(6.7) $$G = g^{ab} T_a T_b$$

where $g^{ab} g_{bc} = \delta^a_c$ and the validity of Eq. (6.6) follows from Eq. (6.5).

Finally, we note that the $n \times n$ matrices

(6.8) $$[iM_a]^c_b = C_{ab}{}^c$$

form the *adjoint representation* of the Lie algebra. M_a satisfies Eq. (6.1) trivially due to the Jacobi identity (6.3).

B. System without Gauge Constraints

For any semisimple Lie group $\mathscr{J}$ with structure constants $C_a{}^{bc}$, we consider an action functional

(6.9) $$S = \int d^4x \, \mathscr{L}(A^\mu_a(x), A^{\mu,\nu}_a(x))$$

where the n vector fields $A^\mu_a(x)$, $a = 1, \cdots, n$, are taken to transform as follows under an infinitesimal gauge transformation:

(6.10) $$\delta A^\mu_a = A'^\mu_a - A^\mu_a = C_a{}^{bc} \Lambda_b(x) A^\mu_c(x) + \frac{1}{g} \frac{\partial \Lambda_a(x)}{\partial x_\mu}$$

It is a standard result (Utiyama, 1956) that although A^μ_a does not transform as the adjoint representation of $\mathscr{J}$ on its a-index under the gauge transformation, the quantity $F^{\mu\nu}_a$ does:

(6.11) $$\begin{aligned} F^{\mu\nu}_a &= \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g C_a{}^{bc} A^\mu_b A^\nu_c \\ \delta F^{\mu\nu}_a &= C_a{}^{bc} \Lambda_b F^{\mu\nu}_c . \end{aligned}$$

For convenience, we define

(6.12) $$\begin{aligned} E^i_a &= F^{oi}_a \\ B^i_a &= \tfrac{1}{2} \varepsilon^{ijk} F^{jk}_a . \end{aligned}$$

Our objective is now to find an action (6.9) which is invariant under the gauge transformation (6.10), so that two components of each A^μ_a are unphysical, and the A^μ_a's describe massless spin one particles. In principle, any Lorentz scalar function of the $F_a^{\mu\nu}$ with the a-indices combined in the form of the *Casimir invariants* (6.6) will do. The traditional form is

$$\mathscr{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \tag{6.13}$$

and includes only the term analogous to the quadratic Casimir invariant (6.7).

In the rest of this chapter we will restrict ourselves for convenience to the Lagrangian (6.13) and to the group SU (2). Since SU (2) has only three generators and

$$\begin{aligned} C_{ab}{}^{c} &= \varepsilon_{abc} \\ g_{ab} &= 2\,\delta_{ab} \end{aligned} \tag{6.14}$$

where ε_{abc} is the totally antisymmetric tensor in three dimensions, we can now redefine everything in such a way that we can forget about raising and lowering group indices. All of what follows can easily be generalized to any Lie group by the interested reader.

If we now specify the dynamics on constant-time surfaces, the canonical momenta are

$$\pi_a^{\mu} = \frac{\partial \mathscr{L}}{\partial \dot{A}_{\mu}^{a}} = -F_a^{0\mu}. \tag{6.15}$$

We see that

$$\boldsymbol{\pi}_a = -\mathbf{E}_a = +\dot{\mathbf{A}}_a + \nabla A_a^0 + g\varepsilon_{abc} A_b^0 \mathbf{A}_c\,. \tag{6.16}$$

This is not a constraint equation but serves to define the dynamical properties of $\dot{\mathbf{A}}_a$. On the other hand, for $\mu = 0$, we find the expected n primary constraints:

$$\pi_a^0 \approx 0\,. \tag{6.17}$$

The canonical Poisson brackets are

$$\{\pi_a^{\mu}(t\,,\boldsymbol{x})\,,\,A_b^{\nu}(t\,,\boldsymbol{y})\} = -g^{\mu\nu}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{6.18}$$

These are incompatible with (6.17).

Now since

$$\frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}^{a}} = F_a^{\mu\nu} \quad , \quad \frac{\partial \mathscr{L}}{\partial A_{\mu}^{a}} = -g\varepsilon_{abc} F_b^{\mu\nu} A_\nu^c\,, \tag{6.19}$$

we find the Euler equations

$$D_\nu^{ab} F_b^{\mu\nu} \equiv F_{a,\nu}^{\mu\nu} + g\varepsilon_{abc} F_b^{\mu\nu} A_\nu^c \approx 0\,. \tag{6.20}$$

The zero component can be written

$$\nabla\cdot\boldsymbol{\pi}_a + g\varepsilon_{abc}\,\boldsymbol{\pi}_b\cdot\mathbf{A}_c \approx 0\,. \tag{6.21a}$$

This, as usual, is also incompatible with the Poisson brackets. The space components of (6.20) are

$$\frac{\partial \boldsymbol{\pi}_a}{\partial t} + \nabla \times \mathbf{B}_a - g\varepsilon_{abc}\, \boldsymbol{\pi}_b\, A_c^0 - g\varepsilon_{abc}\, \mathbf{B}_b \times \mathbf{A}_c = 0\,. \tag{6.21 b}$$

Due to the conformal space-time symmetry of the system, conserved Noether currents exist which generate the full algebra of the conformal group. Here we shall concern ourselves only with the canonical energy-momentum tensor, which takes the form

$$\begin{aligned} \theta_c^{\mu\nu} &= -\frac{\partial \mathscr{L}}{\partial A^a_{\lambda,\mu}}\, A_{a\lambda,}{}^{\nu} + g^{\mu\nu}\, \mathscr{L} = \\ &= F_a^{\mu\lambda}\, A_{a\lambda,}{}^{\nu} - \frac{1}{4}\, g^{\mu\nu}\, F_a^{\alpha\beta}\, F^a_{\alpha\beta}\,. \end{aligned} \tag{6.22}$$

The canonical Hamiltonian is then

$$H_c = P^0 = \int d^3x\, \theta_c^{00} = \int d^3x \left(\boldsymbol{\pi}_a \cdot \dot{\mathbf{A}}_a + \frac{1}{2}\, \mathbf{B}_a \cdot \mathbf{B}_a - \frac{1}{2}\, \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_a\right). \tag{6.23}$$

We now add arbitrary multiples of the primary constraints (6.17) and use (6.16) to get the preliminary Hamiltonian

$$\begin{aligned} H' = \int d^3x \Big(\frac{1}{2}\, \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_a &+ \frac{1}{2}\, \mathbf{B}_a \cdot \mathbf{B}_a - \boldsymbol{\pi}_a \cdot \nabla A_a^0 + \\ &+ g\varepsilon_{abc}\, A_a^0\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c + v_a\, \pi_a^0\Big)\,. \end{aligned} \tag{6.24}$$

As usual, the secondary constraints are found by computing

$$\dot{\pi}_a^0 = \{\pi_a^0(x)\,,\, H'\} = \nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c \approx 0 \tag{6.25}$$

which agrees with (6.21 *a*). Our $2n$ first class constraints are now

$$\begin{aligned} \psi_1^a &= \boldsymbol{\pi}_a^0 \approx 0 \\ \psi_2^a &= \nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c \approx 0\,. \end{aligned} \tag{6.26}$$

We add them to H' with arbitrary coefficients and integrate by parts to get the Hamiltonian

$$\begin{aligned} H = H_c + \int d^3x\, (v_1^a\, \psi_1^a + v_2^a\, \psi_2^a) &= \\ = \int d^3x \Big(\frac{1}{2}\, \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_a + \frac{1}{2}\, \mathbf{B}_a \cdot \mathbf{B}_a &+ v_1^a\, \pi_a^0 \\ &+ (v_2^a + A_a^0)\, (\nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c)\Big) - \\ &- \int_{s_\infty} d\boldsymbol{s} \cdot \boldsymbol{\pi}_a\, A_a^0\,. \end{aligned} \tag{6.27 a}$$

As before, the surface term will be dropped. Examining $\{A^{\mu}, H\}$, we find that the final form of the Hamiltonian can be taken to be

$$H = P^0 = \int d^3x \left(\frac{1}{2}\,\boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_a + \frac{1}{2}\,\mathbf{B}_a \cdot \mathbf{B}_a - \right.$$
$$\left. - \dot{A}_a^0 \pi_a^0 + A_a^0 (\nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c)\right). \tag{6.27 b}$$

Parallel arguments give for the translation generator

$$\mathbf{P} = \int d^3x (- \boldsymbol{\pi}_a \times \mathbf{B}_a + \pi_a^0\, \nabla A_a^0 +$$
$$+ \mathbf{A}_a (\nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c)). \tag{6.28}$$

Unlike the Maxwell field, the present theory has additional Noether currents due to the local invariance of the Lagrangian under the non-Abelian Lie group. Suppose we make a space-time independent transformation with n infinitesimal parameters λ_a, so that (6.10) becomes

$$\delta_\lambda A_a^\mu = \varepsilon_{abc}\, \lambda_b\, A_c^\mu .$$

Then the standard arguments tell us that the conserved current is

$$\begin{aligned} J_a^\mu &= \varepsilon_{abc}\, F_b^{\mu\lambda}\, A_\lambda^c \\ \partial_\mu J_a^\mu &= 0 . \end{aligned} \tag{6.29}$$

With constant-time Poisson brackets, the time-independent generators of the symmetry transformations are

$$Q_a = \int d^3x\, [\varepsilon_{abc}\, \mathbf{A}_b \cdot \boldsymbol{\pi}_c + \varepsilon_{abc}\, \pi_b^0\, A_c^0] = \int d^3x\, \varepsilon_{abc}\, A_b^\mu\, \pi_\mu^c . \tag{6.30}$$

The Q_a obey an algebra in the Poisson brackets isomorphic to the Lie algebra (6.1),

$$\{Q_a , Q_b\} = \varepsilon_{abc}\, Q_c . \tag{6.31}$$

We see also that the fields A_a^μ transform under Q_a as the adjoint representation of the group:

$$\{Q_a , A_b^\mu\} = \varepsilon_{abc}\, A_c^\mu . \tag{6.32}$$

We may easily verify that the infinitesimal generator of the full gauge transformation (6.10) is

$$L[\Lambda_a] = g^{-1} \int d^3x\, \pi_a^\mu (\partial_\mu \Lambda_a + g\varepsilon_{abc}\, \Lambda_b\, A_\mu^c) \tag{6.33 a}$$

and obeys the algebra

(6.33*b*) $$\{L[\Lambda_a], L[\tilde{\Lambda}_a]\} = L[\varepsilon_{abc}\Lambda_b\tilde{\Lambda}_c].$$

However, the last two terms of the Hamiltonian (6.27) consist of another gauge-like transformation

(6.34) $$K[\Lambda_a] = g^{-1}\int d^3x\,[\pi_a^0\,\partial_0\Lambda_a - (\nabla\cdot\boldsymbol{\pi}_a + g\varepsilon_{abc}\boldsymbol{\pi}_b\cdot\mathbf{A}_c)\Lambda_a] \approx 0$$

with $\Lambda_a = -gA_a^0$. $L[\Lambda_a]$ and $K[\Lambda_a]$ differ by a functional of the form

$$Q[\Lambda_a] = L[\Lambda_a] - K[\Lambda_a] = g^{-1}\int d^3x\,(\nabla\cdot(\boldsymbol{\pi}_a\Lambda_a) + g\,\varepsilon_{abc}\Lambda_a\pi_b^0 A_c^0).$$

If A_a^0 vanishes like $1/r$ at ∞, $Q[-gA_a^0] = 0$ and $L[-gA_a^0] = K[-gA_a^0] \approx 0$. If Λ_a does not decrease rapidly enough, $L[\Lambda_a]$ and $Q[\Lambda_a]$ will *not* vanish weakly. The nonvanishing charge (6.30) generating the symmetry group algebra is in fact seen to be a particular case of $Q[\Lambda_a]$, namely

$$Q_a = Q[\Lambda_b = \delta_{ab}] = g^{-1}\int d^3x\,(\nabla\cdot\boldsymbol{\pi}_a + g\varepsilon_{abc}\pi_b^0 A_c^0).$$

C. Radiation Gauge Constraints

The problem of finding a suitable canonical quantization scheme for the Yang-Mills field in the radiation gauge is well-known to be quite complex (Schwinger, 1962*a*, 1962*b*; Mohapatra, 1971*a*). We now proceed to attack this problem by imposing the gauge constraints in the context of the Dirac method.

We begin by choosing the traditional radiation gauge condition

(6.35) $$\nabla\cdot\mathbf{A}_a \approx 0.$$

We will find it useful to define a Green's function $G_{ab}(x, y, A)$ solving

(6.36) $$(\delta_{ab}\nabla^2 + g\varepsilon_{abd}\mathbf{A}_d(x)\cdot\nabla_x)\,G_{bc}(x, y, A) = \delta_{ac}\,\delta^3(\mathbf{x}-\mathbf{y}),$$

with the boundary condition that G_{ab} fall like $1/r$ at infinity. While no exact solution is known for G_{ab}, it can be computed as a power series in g:

(6.37) $$G_{ab}(x, y, A) = -\frac{\delta_{ab}}{4\pi|\mathbf{x}-\mathbf{y}|} - g\int d^3z\,\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\varepsilon_{abc}A_c^i(z)\,\frac{\partial}{\partial z^i}\,\frac{1}{4\pi|\mathbf{z}-\mathbf{y}|} + \cdots$$

We now take as our preliminary constraints the second class pair

(6.38) $$\varphi_1^a = \nabla\cdot\boldsymbol{\pi}_a + g\varepsilon_{abc}\boldsymbol{\pi}_b\cdot\mathbf{A}_c \approx 0$$

(6.39) $$\varphi_2^a = \nabla\cdot\mathbf{A}_a \approx 0.$$

The Poisson brackets of the constraints (6.38) and (6.39) are

$$(6.40)\quad \begin{aligned}
\{\varphi_1^a(x), \varphi_1^b(y)\} &= -g\varepsilon_{abc}(\nabla\cdot\boldsymbol{\pi}_c(x) + g\varepsilon_{cde}\,\boldsymbol{\pi}_d(x)\cdot\mathbf{A}_e(x))\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \approx 0\\
\{\varphi_1^a(x), \varphi_2^b(y)\} &= (\delta_{ab}\nabla_x^2 + g\varepsilon_{abc}\,\mathbf{A}_c(x)\cdot\nabla_x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\\
\{\varphi_2^a(x), \varphi_1^b(y)\} &= -(\delta_{ab}\nabla_x^2 + g\varepsilon_{abc}\,\mathbf{A}_c(x)\cdot\nabla_x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\\
\{\varphi_2^a(x), \varphi_2^b(y)\} &= 0\,.
\end{aligned}$$

The inverse of the matrix $C_{ij}^{ab}(x, y) = \{\varphi_i^a(x), \varphi_j^b(y)\}$ can thus be chosen to be

$$(6.41)\qquad [C^{-1}]_{ij}^{ab}(x, y) = \begin{bmatrix} 0 & -G_{ab}(x, y, A)\\ G_{ab}(x, y, A) & 0\end{bmatrix}$$

where $G_{ab}(x, y, A)$ is the solution to Eq. (6.36) introduced at the beginning of this section.

Equation (6.41) is then used to compute the preliminary Dirac brackets compatible with the strong constraints (6.38) and (6.39):

$$\begin{aligned}
\{\pi_a^i(t, \boldsymbol{x}), A_b^j(t, \boldsymbol{y})\}' &= -\delta^{ij}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\\
&\quad - \left[\frac{\partial}{\partial y^j}\,\delta_{bc} + g\varepsilon_{bcd}\,A_d^j(y)\right]\frac{\partial G_{ac}(x, y)}{\partial x^i}\\
\{A_a^i(t, \boldsymbol{x}), \pi_b^j(t, \boldsymbol{y})\}' &= \delta^{ij}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) +\\
&\quad + \left[\delta_{ac}\,\frac{\partial}{\partial x^i} + g\varepsilon_{acd}\,A_d^i(x)\right]\frac{\partial G_{cb}(x, y)}{\partial y^j}\\
\{\pi_a^i(t, \boldsymbol{x}), \pi_b^j(t, \boldsymbol{y})\}' &= -g\varepsilon_{acd}\,\pi_c^i(x)\,\frac{\partial G_{db}(x, y)}{\partial y^j}\\
&\quad - g\,\frac{\partial G_{ad}(x, y)}{\partial x^i}\,\varepsilon_{dcb}\,\pi_c^j(y)
\end{aligned}$$

$$(6.42)\quad \begin{aligned}
\{A_a^\mu(t, \boldsymbol{x}), A_b^\nu(t, \boldsymbol{y})\}' &= 0\\
\{\pi_a^0(t, \boldsymbol{x}), \pi_b^\nu(t, \boldsymbol{y})\}' &= 0\\
\{\pi_a^0(t, \boldsymbol{x}), A_b^\nu(t, \boldsymbol{y})\}' &= -g^{0\nu}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\\
\{\pi_a^\mu(t, \boldsymbol{x}), A_b^0(t, \boldsymbol{y})\}' &= -g^{0\mu}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})
\end{aligned}$$

Now we must see what becomes of A_a^0 when we impose the strong constraints (6.38) and (6.39). Taking the divergence of Eq. (6.16), we find

$$(6.43)\qquad \begin{aligned}
\nabla\cdot\boldsymbol{\pi}_a &= +\nabla^2 A_a^0 + g\varepsilon_{abc}(\nabla A_b^0)\cdot\mathbf{A}_c\\
&= -g\varepsilon_{abc}\,\boldsymbol{\pi}_b\cdot\mathbf{A}_c\,.
\end{aligned}$$

Using Eq. (6.36), we solve Eq. (6.43) for A_a^0:

$$(6.44)\qquad A_a^0(x) \approx -\int d^3y G_{ab}(x, y, A)\, g\varepsilon_{bcd}\, \boldsymbol{\pi}_c(y)\cdot \mathbf{A}_d(y)$$

$$\approx +\int d^3y G_{ab}(x, y, A)\, \nabla_y \cdot \boldsymbol{\pi}_b(y) .$$

Equation (6.44) is seen to be a *weak* equation because the right-hand side has vanishing brackets (6.42) with π_b^0, while A_a^0 does not. It is clear that our final set of constraints is

$$(6.45)\qquad \chi_1^a = \pi_a^0(x) \approx 0$$

$$\chi_2^a = A_a^0(x) + g\int d^3y G_{ab}(x, y, A)\, \varepsilon_{bcd}\, \boldsymbol{\pi}_c(y)\cdot \mathbf{A}_d(y) \approx 0$$

$$= A_a^0(x) - \int d^3y G_{ab}(x, y, A)\, \nabla_y \cdot \boldsymbol{\pi}_b(y) \approx 0 .$$

The primed brackets (6.42) of these constraints form the matrix

$$C_{11}^{ab} = \{\chi_1^a(x), \chi_1^b(y)\}' = 0$$

$$(6.46)\qquad C_{12}^{ab} = \{\chi_1^a(x), \chi_2^b(y)\}' = -\{\chi_2^a(x), \chi_1^b(y)\}' = +\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$C_{22}^{ab} = \{\chi_2^a(x), \chi_2^b(y)\}' = M_{ab}(x, y)$$

where $M_{ab}(x, y)$ can be explicitly computed as a power series in g if desired. We need not compute $M_{ab}(x, y)$ here because the inverse of $C_{ij}^{ab}(x, y)$ is seen by inspection to be

$$(6.47)\qquad [C^{-1}]_{ij}^{ab}(x, y) = \begin{bmatrix} M_{ab}(x, y) & -\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ +\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 \end{bmatrix} .$$

We thus find that the final Dirac bracket system is

$$(6.48)\qquad \{A_a^0(t, \boldsymbol{x}), A_b^0(t, \boldsymbol{y})\}^* = M_{ab}(x, y)$$

$$\{A_a^i(t, \boldsymbol{x}), A_b^j(t, \boldsymbol{y})\}^* = 0$$

$$\{A_a^0(t, \boldsymbol{x}), A_b^j(t, \boldsymbol{y})\}^* = -gG_{ac}(x, y, A)\,\varepsilon_{cdb}\,A_d^j(y)$$

$$-g\int d^3z G_{ac}(x, z, A)\,\varepsilon_{ced}\,\mathbf{A}_e(z)\cdot\nabla_z\cdot$$

$$\left[\frac{\partial}{\partial y^j}\delta_{fb} + g\varepsilon_{fbg}A_g^j(y)\right] G_{df}(z, y, A)$$

$$\{\pi_a^0(t, \boldsymbol{x}), \pi_b^\mu(t, \boldsymbol{y})\}^* = \{\pi_a^0(t, \boldsymbol{x}), A_b^\mu(t, \boldsymbol{y})\}^* = 0$$

$$\{\pi_a^i(t, \boldsymbol{x}), \pi_b^j(t, \boldsymbol{y})\}^* = \{\pi_a^i(t, \boldsymbol{x}), \pi_b^j(t, \boldsymbol{y})\}'$$

$$\{\pi_a^i(t, \boldsymbol{x}), A_b^j(t, \boldsymbol{y})\}^* = \{\pi_a^i(t, \boldsymbol{x}), A_b^j(t, \boldsymbol{y})\}' .$$

We may now take the Hamiltonian to be

$$\mathrm{H} = \int d^3x \left(\tfrac{1}{2}\, \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_a + \tfrac{1}{2}\, \mathbf{B}_a \cdot \mathbf{B}_a \right). \tag{6.49}$$

In the Dirac bracket system (6.48), H indeed generates the correct equations of motion:

$$\begin{aligned} \{ \mathbf{A}_a(x) \,,\, \mathrm{H} \}^* &= \dot{\mathbf{A}}_a(x) \\ \{ \boldsymbol{\pi}_a(x) \,,\, \mathrm{H} \}^* &= \dot{\boldsymbol{\pi}}_a(x) = -\dot{\mathbf{E}}_a(x) \\ &= -\nabla \times \mathbf{B}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b\, A_c^0 + g\varepsilon_{abc}\, \mathbf{B}_b \times \mathbf{A}_c \,. \end{aligned} \tag{6.50}$$

Schwinger (1962*b*) has argued that the brackets (6.48) produce a Lorentz-covariant theory. We leave this as an exercise.

D. Alternate Radiation Gauge Techniques

All four sets of constraints (6.39) and (6.45) hold strongly in our Dirac bracket system (6.48). However, the field-dependent terms on the right-hand sides of the Dirac brackets make it extremely difficult to use these brackets as the basis for a canonical quantization scheme. Several procedures are available for circumventing these problems. One example is the direct path-integral method (Faddeev and Popov, 1967*a*, 1967*b*) for developing a Feynman diagram expansion consistent with the constraints. The "ghost" particles which occur in this procedure correspond precisely to the integrals which appear in G_{ab} on the right hand sides of our Dirac brackets.

Another popular procedure (Schwinger, 1962*a*; Mohapatra, 1971*a*) which we will sketch for completeness gives up dealing with the strictly canonical momenta π_a^0. One ignores π_a^0 completely and transforms the fields to the radiation gauge

$$\nabla \cdot \mathbf{A}_a \approx 0 \,. \tag{6.51}$$

Then $\boldsymbol{\pi}_a$ is split into a *transverse* part $\boldsymbol{p}_a$ and a *longitudinal* part equal to the gradient of a scalar field Φ_a,

$$\boldsymbol{\pi}_a = \boldsymbol{p}_a + \nabla \Phi_a \,, \tag{6.52}$$

where $\boldsymbol{p}_a$ and Φ_a have their canonical properties *defined* by

$$\boldsymbol{p}_a = \dot{\mathbf{A}}_a \tag{6.53}$$

$$\nabla \Phi_a = \nabla A_a^0 + g\varepsilon_{abc}\, A_b^0\, \mathbf{A}_c \,. \tag{6.54}$$

Thus the Euler equation (6.25) and the gauge choice (6.51) imply

(6.55) $$\nabla \cdot \boldsymbol{p}_a \approx 0 .$$

Then one computes the Dirac brackets consistent with (6.51) and (6.55) finding that $\boldsymbol{p}_a$ and $\mathbf{A}_a$ obey the familiar quantizable Maxwell field Dirac brackets (5.32). At this point, one can use Eqs. (6.16) and (6.21*a*) to find the following expressions:

(6.56) $$(\nabla^2 \delta_{ab} + g\varepsilon_{abc}\, \mathbf{A}_c \cdot \nabla)\, \Phi_b = -g\varepsilon_{abc}\, \boldsymbol{p}_b \cdot \mathbf{A}_c$$

(6.57) $$(\nabla^2 \delta_{ab} + g\varepsilon_{abc}\, \mathbf{A}_c \cdot \nabla)\, \mathrm{A}^0_b = -g\varepsilon_{abc}(\boldsymbol{p}_b + \nabla\Phi_b) \cdot \mathbf{A}_c = \nabla^2 \Phi_a$$

Using Eq. (6.36), one then solves for Φ_a,

(6.58) $$\Phi_a(x) = -g\varepsilon_{bcd} \int \mathrm{d}^3y\, \mathrm{G}_{ab}(x\,, y\,, \mathrm{A})\, \boldsymbol{p}_c \cdot \mathbf{A}_d\,,$$

and expresses A^0_a in terms of Φ_a,

(6.59) $$\mathrm{A}^0_a(x) = \int \mathrm{d}^3y\, \mathrm{G}_{ab}(x\,, y\,, \mathrm{A})\, \nabla^2_y\, \Phi_b(y)\,.$$

Since π^0_a has been ignored, Eqs. (6.58) and (6.59) are taken to define the properties of Φ_a and A^0_a. The Hamiltonian can then be written

(6.60) $$\mathrm{H} = \tfrac{1}{2} \int \mathrm{d}^3x\, (\boldsymbol{p}_a \cdot \boldsymbol{p}_a + \mathbf{B}_a \cdot \mathbf{B}_a + \nabla\Phi_a \cdot \nabla\Phi_a)\,,$$

where the $\nabla\Phi_a \cdot \nabla\Phi_a$ term is compared to the instantaneous Coulomb energy which appears in interacting Maxwell electrodynamics in the radiation gauge.

We prefer the techniques of the previous section, where the variables are all treated in a strictly canonical fashion. Moreover, there is a gauge in which the canonical procedure can be used directly to define the quantum theory: this is the subject of the next section.

E. Axial Gauge

We now explore the properties of the Yang-Mills field in the axial gauge

(6.61) $$\mathrm{A}^3_a \approx 0\,.$$

introduced by Arnowitt and Fickler (1962) (see also Mohapatra, 1971 *b*; Konetschny and Kummer, 1975). Using the Hamiltonian (6.27 *b*) we find that

(6.62) $$\pi^3_a = \dot{\mathrm{A}}^3_a + \mathrm{A}^{0,3}_a + g\varepsilon_{abc}\, \mathrm{A}^0_b\, \mathrm{A}^3_c.$$

Therefore Eq. (6.61) will not hold for all time unless

$$\dot{A}_a^3 = \pi_a^3 - A_a^{0,3} \approx 0 . \tag{6.63}$$

We now adjoin (6.61) and (6.63) to the original constraints (6.26) to form the set

$$\begin{aligned} \varphi_1^a &= \pi_a^0 \approx 0 \\ \varphi_2^a &= \nabla \cdot \boldsymbol{\pi}_a + g\varepsilon_{abc}\, \boldsymbol{\pi}_b \cdot \mathbf{A}_c \approx 0 \\ \varphi_3^a &= A_a^3 \approx 0 \\ \varphi_4^a &= \pi_a^3 - A_a^{0,3} \approx 0 . \end{aligned} \tag{6.64}$$

The matrix of Poisson brackets is found to be

$$C_{ij}^{ab}(x, y) = \{ \varphi_i^a(x) , \varphi_i^b(y)\} = \tag{6.65}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \delta_{ab}\,\partial_3^x \\ 0 & 0 & -\delta_{ab}\,\partial_3^x & -g\varepsilon_{abc}\,\pi_c^3(x) \\ 0 & -\delta_{ab}\,\partial_3^x & 0 & \delta_{ab} \\ \delta_{ab}\,\partial_3^x & -g\varepsilon_{abc}\,\pi_c^3(x) & -\delta_{ab} & 0 \end{bmatrix} \delta^3(\boldsymbol{x} - \boldsymbol{y}) .$$

With appropriate boundary conditions giving the explicit solutions for the dependent variables to be written later, we find the inverse of (6.65):

$$[C^{-1}]_{ij}^{ab}(x, y) = \tag{6.66}$$

$$= \begin{bmatrix} I_{ab}(x, y) & -\delta_{ab}G(x, y) & H_{ab}(x, y) & \delta_{ab}F(x, y) \\ \delta_{ab}G(x, y) & 0 & -\delta_{ab}F(x, y) & 0 \\ H_{ab}(x, y) & -\delta_{ab}F(x, y) & 0 & 0 \\ \delta_{ab}F(x, y) & 0 & 0 & 0 \end{bmatrix} .$$

The functions F, G, H_{ab} and I_{ab} obey

$$\begin{aligned} & F(x, y) = \partial_3^x G(x, y) \\ & \partial_3^x F(x, y) = \partial_3^x \partial_3^x G(x, y) = \delta^3(\boldsymbol{x} - \boldsymbol{y}) \\ & \partial_3^x H_{ab}(x, y) + g\varepsilon_{abc}\,\pi_c^3(x)\, F(x, y) = 0 \\ & \partial_3^x I_{ab}(x, y) - g\varepsilon_{abc}\,\pi_c^3(x)\, G(x, y) - H_{ab}(x, y) = 0 \end{aligned} \tag{6.67}$$

and are chosen to have the explicit forms

(6.68) $$G(x,y) = \tfrac{1}{2}\,|x^3 - y^3|\,\delta^2(x-y)$$
$$F(x,y) = \tfrac{1}{2}\,\varepsilon(x^3 - y^3)\,\delta^2(x-y)$$
$$H_{ab}(x,y) = \partial_3^x I_{ab}(x,y) - g\varepsilon_{abc}\,\pi_c^3(x)\,G(x,y)$$
$$I_{ab}(x,y) = \tfrac{1}{4}\,g\varepsilon_{abc}\,\delta^2(x-y)\int d\xi\,|x^3-\xi|\,|\xi-y^3|\,\frac{\partial\pi_c^3(x^0\,x^1\,x^2\,\xi)}{\partial\xi}\,.$$

where $\varepsilon(x-y)$ is the algebraic sign of $(x-y)$. All remaining gauge freedom is exhausted by the choice of boundary conditions implied by (6.68). The Dirac brackets corresponding to Eq. (6.66) are

(6.69) $$\{A_a^\mu(x), A_b^\nu(y)\}^* = \delta_{ab}(g^{\mu 0}g^{\nu 3} + g^{\mu 3}g^{\nu 0})\,F(x,y)$$
$$- g\varepsilon_{abc}\,[g^{\mu 0}g^{\nu i}A_c^i(y) + g^{\mu i}g^{\nu 0}A_c^i(x)]\,G(x,y)$$
$$- \delta_{ab}(g^{\mu 0}g^{\nu i} + g^{\mu i}g^{\nu 0})\,\partial_i^x G(x,y) + g^{\mu 0}g^{\nu 0}\,I_{ab}(x,y)$$

(6.70) $$\{\pi_a^\mu(x), \pi_b^\nu(y)\}^* = g\varepsilon_{abc}(-g^{\mu i}g^{3\nu}\pi_c^i(x) + g^{\mu 3}g^{\nu i}\pi_c^i(y))\,F(x,y)$$

(6.71) $$\{\pi_a^\mu(x), A_b^\nu(y)\}^* = (-g^{\mu\nu} - g^{\mu 0}g^{\nu 0})\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$$
$$+ g^{\mu 3}g^{\nu i}\,\delta_{ab}\,\partial_i^x F(x,y) - g^{\mu i}g^{\nu 0}g\varepsilon_{abc}\,\pi_c^i(x)\,G(x,y)$$
$$+ g^{\mu 3}g^{\nu i}g\varepsilon_{abc}\,A_c^i(y)\,F(x,y) - g^{\mu 3}g^{\nu 0}\,H_{ab}(x,y)\,.$$

We now take as our four independent variables π_a^i, A_a^i, $i = 1, 2$, which have the following Dirac brackets among themselves:

(6.72) $$\{\pi_a^i(x), \pi_b^j(y)\}^* = 0$$
$$\{A_a^i(x), A_b^j(y)\}^* = 0$$
$$\{\pi_a^i(x), A_b^j(y)\}^* = -\,\delta^{ij}\,\delta_{ab}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,.$$

These are *canonical* brackets, with no fields on the right-hand side. The strong constraints φ_2^a, φ_3^a and φ_4^a can now be used to show

(6.73) $$\partial_3\,\partial_3\,A_a^0 = \partial_1\pi_a^1 + \partial_2\pi_a^2 + g\varepsilon_{abc}(\pi_b^1 A_c^1 + \pi_b^2 A_c^2)$$
$$= \partial_i\,\pi_a^i + g\varepsilon_{abc}\,\pi_b^i A_c^i$$

so that A_a^0 is a dependent variable:

(6.74) $$A_a^0(x) = \tfrac{1}{2}\int d\xi\,|x^3-\xi|\,\cdot$$
$$\cdot\,[\partial_i\,\pi_a^i(x^0, x^1, x^2, \xi) + g\varepsilon_{abc}\,\pi_b^i(x^0, x^1, x^2, \xi)\,A_c^i(x^0, x^1, x^2, \xi)]\,.$$

Then $\varphi_4^a = 0$ also implies

(6.75) $$\pi_a^3 = \tfrac{1}{2}\int d\xi\,\varepsilon(x^3-\xi)\,[\partial_i\,\pi_a^i(\xi) + g\varepsilon_{abc}\,\pi_b^i(\xi)\,A_c^i(\xi)]\,.$$