

FIRST RETURN RECOVERY OF BAIRE CLASS ONE
FUNCTIONS ON ULTRAMETRIC SPACES

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To the two special women in my life – Janet and Daphne.

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Jonathan D. Duncan

First Return Recovery of Baire Class One Functions
on Ultrametric Spaces

In his 1992 paper *First return path derivatives*, R. J. O'Mally defines a *First Return Path System*, motivated by the Poincaré first return map of differential dynamics. While the original intent of the first return path system was to study questions of differentiability, the concept has found other applications. U. B. Darji and M. J. Evans applied first return paths to the question of recovering functions in *Recovering Baire one Functions*. The problem they posed was: for what types of functions $f : X \rightarrow Y$, where X and Y are separable metric spaces, can we recover the function based only on a countable dense subset of the domain and a simple recursive procedure which produces a first return path. The conclusion they reached was that for compact metric spaces, the functions recoverable in this way are the Baire class one functions.

The examination of this topic continued in *How can we recover Baire class one functions?* in which D. Lecomte showed that in certain ultrametric spaces, the class of recoverable functions are also exactly the class of Baire class one functions. Also in this paper, Lecomte gives an example of a Baire class one function on an ultrametric space which is not recoverable.

I continue the study of first return recoverability on ultrametric spaces by examining the conditions required on an ultrametric space in order to either find a non-recoverable Baire class one function, or to show that all Baire class one functions are first return recoverable. I provide criteria for ultrametric spaces both for the recoverability of Baire class one functions and for the existence of non-recoverable Baire class one functions in terms of the distances obtained at individual points. This considerably generalizes Lecomte's results and examples.

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CHAPTER 1

Introduction

1. Historical Background

In his 1992 paper [O], R. J. O'Mally defines a **First Return Path System**, motivated by the Poincaré first return map of differential dynamics. While the original intent of the first return path system was to study questions of differentiability, the concept has found other applications.

In [DE], U. B. Darji and M. J. Evans apply first return paths to the question of recovering functions. The problem posed is this: for what types of functions $f : X \rightarrow Y$, where X and Y are separable metric spaces, can we recover the value of the function at every point in X based only on the function's values on a countably dense subset of the domain. The conclusion reached in [DE] is that if X is a compact metric space, then the recoverable functions are exactly the Baire class one functions. It is important to note that in the proof that recoverable functions are Baire class one, Darji and Evans did not use compactness. Therefore, any function which can be recovered via first return methods must be Baire class one.

The examination of this topic continued in [Le]. In this paper, D. Lecomte did for σ -totally bounded spaces what Darji and Evans had done for compact spaces. Namely, he showed that if X is a countable union of totally bounded metric spaces, then again the recoverable functions are exactly the Baire class one functions. Later in this same paper, Lecomte proves that for a certain class of ultrametric space, all Baire class one functions on the ultrametric space are recoverable. He also provides, for the first time, an

example of a Baire class one function (on an ultrametric space) which is not first return recoverable.

I continue Lecomte's study of first return recoverability in ultrametric spaces by examining the conditions required on an ultrametric space in order to either find a Baire class one function which is not first return recoverable, or to show that all Baire class one functions are first return recoverable. I provide criteria for ultrametric spaces both for the recoverability of Baire class one functions and for the existence of non-recoverable Baire class one functions. These conditions deal with the distances attained at individual points in the ultrametric. This considerably generalizes Lecomte's results and examples.

2. Summary of Previous Results

Throughout the paper, we deal with separable complete metric spaces usually represented as (X, d) or (Y, δ) . A trajectory in X is a countable dense sequence $\{x_n\}_{n \in \omega}$ of points in X . If x is in X and r is in the range of d , we let $B(x, r)$ represent the open ball centered at x with radius r . The basic construction under discussion is that of a first return route. This was defined by Darji and Evans in [DE] and is restated below.

DEFINITION 1.1. (Darji and Evans) Let (X, d) be a separable metric space, $x \in X$, and $\{x_n\}$ a trajectory in X . Let $\xi(B(x, r))$ be the first element of the trajectory in $B(x, r)$. Then the **first return route to x based on $\{x_n\}$** , $\mathfrak{R}_x = \{s_k(x, \{x_n\})\}_{k=0}^\infty$, is defined recursively by:

$$s_0(x, \{x_n\}) = x_0$$

$$s_{k+1}(x, \{x_n\}) = \begin{cases} x & \text{if } s_k(x, \{x_n\}) = x \\ \xi(B(x, d(x, s_k(x, \{x_n\})))) & \text{otherwise} \end{cases}$$

According to this definition, the first return route to a point x based on a trajectory $\{x_n\}$ is a sequence starting with the first element of the trajectory, x_0 . Successive terms of the sequence are chosen in one of two ways. If the previous member of the return route sequence is x itself, then the next member is also x . If, however, the previous member of the sequence is not x , then the next member of the return route is the first element of the trajectory closer to x than the previous member of the return route.

In most cases, the particular trajectory being used will be clear from the context. In such instances, we will simplify notation by referring to the elements of the return route as $s_k(x)$. Based on the existence of an appropriate return route, the first return recoverability of a function is defined as follows.

DEFINITION 1.2. (Darji and Evans) Let (X, d) and (Y, δ) be separable metric spaces. Let $f : X \rightarrow Y$ be a function. We say that f is **first return recoverable with respect to the trajectory** $\{x_n\}$ if for every x in X , $f(s_k(x, \{x_n\})) \rightarrow f(x)$ as $k \rightarrow \infty$. A function f is called **first return recoverable** if there exists some trajectory in X with respect to which f is first return recoverable.

To conserve space we shall often refer to first return recoverable functions as **recoverable functions**. With these definitions taken care of, we cite the main result of Darji and Evans from [DE].

THEOREM 1.3. (Darji and Evans) *Let (X, d) be a compact metric space and (Y, δ) a separable metric space. A function $f : X \rightarrow Y$ is first return recoverable if, and only if it is Baire class one.*

Note that as mentioned previously, the proof that first return recoverable functions are Baire class one does not rely on compactness and holds for

all separable metric spaces. Therefore, the class of potentially recoverable functions is limited to Baire class one functions. However, the question remains if all Baire class one functions are recoverable in non-compact metric spaces.

Lecomte tackled this question in [Le] in two ways. He first found a broader class of metric spaces on which he also was able to show that all Baire class one functions are recoverable. This theorem is restated below.

THEOREM 1.4. *(Lecomte) Assume that X is a countable union of totally bounded spaces, and that f is Baire class one. Then f is first return recoverable.*

The question remained, however, under what conditions, if any, could a non-recoverable Baire class one function be constructed. To answer this, Lecomte turned to ultrametric spaces. Recall the definition of an ultrametric space.

DEFINITION 1.5. A metric space (X, d) is an **ultrametric** if for every $x, y, z \in X$, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Lecomte first noted that for a particular class of ultrametric spaces, all Baire class one functions are first return recoverable. He called these ultrametric spaces discrete. The definition of a discrete ultrametric is given below, followed by Lecomte's theorem.

DEFINITION 1.6. (Lecomte) We will say that an ultrametric space (X, d) is discrete if the following condition is satisfied.

$$\forall (d_n)_{n \in \omega} \subseteq d[X \times X] \left[(\forall n \in \omega \ d_{n+1} < d_n) \Rightarrow \left(\lim_{n \rightarrow \infty} d_n = 0 \right) \right]$$

THEOREM 1.7. (*Lecomte*) *If X is a discrete ultrametric space and f is Baire class one, then f is first return recoverable.*

However, Lecomte also found an ultrametric space Z together with a Baire class one function on Z which is not first return recoverable. This example of a non-recoverable function, together with the space on which it is defined, is given in the proposition below.

PROPOSITION 1.8. (*Lecomte*) *Define a space Z by*

$$Z = \{Q = (q_n)_{n \in \omega} \in \mathbb{Q}_+^\omega \mid \forall n \in \omega \ q_n < q_{n+1} \text{ and } \lim_{n \rightarrow \infty} q_n = \infty\}$$

and equip it with the metric

$$d(Q, Q') = \begin{cases} 2^{-\min\{q_{\min\{n \in \omega \mid q_n \neq q'_n\}}, q'_{\min\{n \in \omega \mid q_n \neq q'_n\}}\}} & \text{if } Q \neq Q' \\ 0 & \text{otherwise} \end{cases}$$

Then, the characteristic function of the closed set

$$F = \{Q \in Z \mid \forall n \in \omega \ n < q_n < n + 1\}$$

is a Baire class one function but is not first return recoverable.

This proposition shows that while every recoverable function is Baire class one, there do exist non-recoverable Baire class one functions. Thus, the inclusion proved by Darji and Evans can not be proper. We take the work done by Darji, Evans, and Lecomte and extend it to address the properties of ultrametrics which are needed in order to force all Baire class one functions to be recoverable. We also extend their work to determine under what conditions we can find a non-recoverable Baire class one function on an ultrametric space.

3. Summary of My Results

My results are divided into three chapters, each dealing with one aspect of the question of recoverability on ultrametric spaces. In chapter two, we start with the definition of an I -valued ultrametric space. We then define the morphisms between I - and J -valued ultrametric spaces. These morphisms are called **chain isometric embeddings**.

The main result of this chapter is a proof that if every Baire class one function on an ultrametric Y is first return recoverable, and if X can be chain isometrically embedded into Y , then every Baire class one function on X is also first return recoverable. The proof of this theorem does not rely on ultrametric properties, and will therefore work, after appropriate adjustments, in arbitrary metric spaces. It is in fact the contrapositive of this statement which is the most interesting form of this result. This is stated in the following corollary.

COROLLARY 2.7. If there exists a non-recoverable Baire class one function on a separable complete I -valued ultrametric X , and X can be chain isometrically embedded into a separable complete J -valued ultrametric Y , then there exists a non-recoverable Baire class one function on Y .

The third chapter contains my results on recoverability. We start with the definition of a **cutoff** for a point in an ultrametric.

DEFINITION 3.2. Let (X, d) be an I -valued ultrametric space and $I_x = \{i \in I \mid \exists y \in X \ d(x, y) = i\}$. Then $r \in I$ is a **cutoff** for x if for any sequence $\{d_n\}_{n \in \omega} \subseteq I_x$ with $d_{n+1} < d_n$ for all n and $d_0 \leq r$, we have $\lim_{n \rightarrow \infty} d_n = 0_I$.

This definition is then used to prove the main result of chapter three. That is, a condition on ultrametrics which forces any Baire class one function on the ultrametric to be first return recoverable. This condition is stated in

the following theorem.

THEOREM 3.3. *Let (X, d) be a separable complete I -valued ultrametric space covered by non-empty open sets U for which there exists $r \in I \setminus \{0_I\}$ so that r is a cutoff for each $x \in U$. Let (Y, δ) be a separable metric space. Then, $f : X \rightarrow Y$ is first return recoverable if, and only if it is Baire class one.*

In the fourth chapter, we examine the other side of this question. That is, under what conditions can we find a Baire class one function which is not first return recoverable. Using the concept of a cutoff once again, we give a criterium under which ultrametric spaces must have a Baire class one function which is not first return recoverable.

THEOREM 4.1. *Let (X, d) be a separable complete I -valued ultrametric space in which the set of points whose only cutoff is 0_I is dense. Then there exists a Baire class one function on X which is not first return recoverable.*

This hypothesis for this theorem, while not exactly the opposite of our condition for recoverability, makes clear the aspects of the ultrametric structure which play the deciding role in controlling the existence of non-recoverable Baire class one functions. That is, the existence of points at which complicated sets of distances are attained.

The final chapter relates some of our findings to the idea of a rich ultrametric space. Rich and n -rich ultrametric spaces were first introduced by F. Delon in [D]. These classifications of I -valued ultrametric spaces are particularly useful as they come with canonical examples which are unique prime models for a given I . In chapter five we show that there is a non-recoverable Baire class one function on the prime model for 2-rich I -valued ultrametrics for a certain set I . The proof of this is a specific application of Theorem 4.1.

CHAPTER 2

Morphisms and Recoverability

1. I -valued Ultrametric Spaces

We will find that the conditions required to either ensure that all Baire class one functions on an ultrametric space are recoverable, or that there exists a non-recoverable Baire class one function depend on the range of the ultrametric. In particular, the order type of that range will be of importance. To better study this idea, we examine a characterization of ultrametric spaces based on the set of distances attained by the metric.

In her 1984 paper [D], F. Delon introduced the general concept of ultrametric spaces in which the range of the ultrametric is a linearly ordered set I . This method of describing ultrametric spaces will be of particular use to us because it allows us to specify the order type of the range of the metric. Given a linearly ordered set I with least element 0_I , which we will call a chain, she makes the following definition.

DEFINITION 2.1. (Delon) An I -valued ultrametric space is a set X together with a surjective function $d : X \times X \rightarrow I$ such that for all $x, y, z \in X$:

- $d(x, y) = 0_I$ if, and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

The ultrametric spaces with which we work will always be separable. This automatically places a restriction on I . Namely, because of the separability of the ultrametric, I must be countable. The proof of this well-known fact is given in the following lemma.

LEMMA 2.2. *If (X, d) is a separable I -valued ultrametric space, then I is countable.*

PROOF. Let D be a countably dense subset of X . Pick $i \in I \setminus \{0_I\}$ and let x and y be elements of X with $d(x, y) = i$. Note that x and y are distinct by our choice of i . Consider the open balls $B(x, i)$ and $B(y, i)$.

As D is dense, there is an $x' \in B(x, i) \cap D$ and a $y' \in B(y, i) \cap D$. Now $d(x, x') < i$ and by the ultrametric inequality,

$$i = d(x, y) \leq \max\{d(x, x'), d(x', y)\}.$$

Hence $d(x', y) \geq i$. However, again by the ultrametric inequality,

$$d(x', y) \leq \max\{d(x', x), d(x, y)\} = i$$

showing that $d(x', y) = i$. A similar argument shows that $d(x', y') = i$.

Thus, for any $i \in I \setminus \{0_I\}$, we can find two points in D at distance i from each other. As D is countable, I can have only countably many values as well. \square

With the definition of an I -valued ultrametric space, we have a reasonable way to begin our investigation of the properties of ultrametrics which lead to recoverable or non-recoverable Baire class one functions. We will examine the effect of the order type of I on the recoverability of Baire class one functions on I -valued ultrametric spaces. We would like to ensure that discoveries we make in a given I -valued ultrametric space will be preserved in spaces into which the ultrametric embeds. Thus, we need to develop

morphisms for I -valued ultrametric spaces.

2. Chain Isometries

In this section we define the notion of a chain isometry, which we use as the morphism between I -valued ultrametric spaces. Let I and J be chains with least elements 0_I and 0_J respectively. Suppose that U and V are I - and J -valued ultrametric spaces respectively. A natural question to ask is, “what is the appropriate metric-preserving morphism between U and V ?” The morphism typically used between metric spaces is an isometric embedding. However, this only works in our situation if $I \subseteq J$. We therefore need a more general morphism.

As we shall see in the next chapter, our morphisms must especially preserve the metric of an I -valued ultrametric space around the least element 0_I . Thoughtful consideration reveals that we also need to preserve the relative order of values in I and J . That is, if $d_U(x, y) \leq d_U(w, z)$ for $x, y, w, z \in U$, we want to ensure that a morphism $\sigma : U \rightarrow V$ dictates that $d_V(\sigma(x), \sigma(y)) \leq d_V(\sigma(w), \sigma(z))$.

In the definition that follows, as well as in later chapters, we make use of limit notion in the context of a chain I with least element 0_I . Given a sequence $\{i_n\}_{n \in \omega} \subseteq I$, we understand the statement $\lim_{n \rightarrow \infty} i_n = 0_I$ to mean that for any $a \in I \setminus \{0_I\}$ there is an $N \in \omega$ such that for all $n \geq N$, $0_I \leq i_n < a$ in the chain I .

DEFINITION 2.3. Let U be an I -valued ultrametric space and V a J -valued ultrametric space. A surjective function $\sigma : U \rightarrow V$ is called a **chain isometry** if there is a map $\rho : I \rightarrow J$ such that:

- (1) $d_V(\sigma(x), \sigma(y)) = \rho(d_U(x, y))$.
- (2) Any sequence $\{i_n\}_{n \in \omega}$ in I has the property that $\lim_{n \rightarrow \infty} i_n = 0_I$ if, and only if $\lim_{n \rightarrow \infty} \rho(i_n) = 0_J$ in J .

(3) $i \leq j$ in I if, and only if $\rho(i) \leq \rho(j)$ in J .

If such a function exists, U and V are called **chain isometric**.

Note that condition (3) above implies that ρ is one-to-one. In the case that there is a non-surjective map $\sigma : U \rightarrow V$ which has these properties, we call σ a **chain isometric embedding** as U is chain isometric to $\sigma[U] \subseteq V$. There are several properties of chain isometries which will be important as we proceed. These properties are stated in the following lemmas.

LEMMA 2.4. *Let $\sigma : U \rightarrow V$ be a chain isometry. Then σ is one-to-one.*

PROOF. Let x and y be distinct points in U and let $i \in I \setminus \{0_I\}$ be such that $d_U(x, y) = i$. By condition (1) of the definition, $d_V(\sigma(x), \sigma(y)) = \rho(i) \in J$. Suppose by way of contradiction that $\rho(i) = 0_J$. It is a simple consequence of condition (2) that $\rho(0_I) = 0_J$. Therefore $\rho(i) \leq \rho(0_I)$ and $\rho(0_I) \leq \rho(i)$. Now by condition (3), $i \leq 0_I$ and $0_I \leq i$ making $i = 0_I$. But this contradicts our choice of distinct x and y . Hence, $\rho(i) \neq 0_J$ and therefore $\sigma(x) \neq \sigma(y)$ so that σ is in fact one-to-one. \square

LEMMA 2.5. *Let $\sigma : U \rightarrow V$ be a chain isometry. Then both σ and σ^{-1} are uniformly continuous.*

PROOF. First, note that by Lemma 2.4, σ^{-1} does exist. Let $\varepsilon \in J \setminus \{0_J\}$ be given. Pick $\delta \in I \setminus \{0_I\}$ so that $\rho(\delta) \leq \varepsilon$. Note that this can be done since for some $x, y \in U$, $d_V(\sigma(x), \sigma(y)) = \varepsilon > 0_J$. Hence $0_I < d_U(x, y)$ and $\rho(d_U(x, y)) = \varepsilon$.

Now for any z and w in U with $d_U(z, w) < \delta$, by properties (1) and (3) we have

$$d_V(\sigma(z), \sigma(w)) = \rho(d_U(z, w)) < \rho(\delta) \leq \varepsilon.$$

Therefore, σ is uniformly continuous as claimed. Similarly, with $\varepsilon \in I \setminus \{0_I\}$ by picking $\delta \in J \setminus \{0_J\}$ less than, or equal to $\rho(\varepsilon)$, we can show that σ^{-1} is

uniformly continuous. \square

3. Recoverability and Chain Isometric Embeddings

Using these properties of chain isometries, we can now examine how first return recoverability is preserved when an I -valued ultrametric space is chain isometrically embedded into a J -valued ultrametric space. This is done in the following theorem.

THEOREM 2.6. *Let (X, d) and (Y, δ) be separable complete I - and J -valued ultrametric spaces respectively. Let $\sigma : X \rightarrow Y$ be a chain isometric embedding. Then, if every Baire class one function $f : Y \rightarrow \mathbb{R}$ is first return recoverable, every Baire class one function $g : X \rightarrow \mathbb{R}$ is also first return recoverable.*

PROOF. Suppose that $\rho : I \rightarrow J$ witnesses the chain isometric embedding σ . Let $g : X \rightarrow \mathbb{R}$ be an arbitrary Baire class one function. By Lemma 2.4, σ is one-to-one. Thus we can define a function $f : Y \rightarrow \mathbb{R}$ as follows. A commutative diagram is also provided to illustrate this definition.

$$f(y) = \begin{cases} \frac{1}{\pi} (\arctan(g(\sigma^{-1}(y))) + \frac{\pi}{2}) & \text{for } y \in \sigma[X] \\ -1 & \text{otherwise} \end{cases} \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & Y \\ & \downarrow g & \swarrow f \\ & \mathbb{R} & \end{array}$$

We claim that f is a Baire class one function. To see this, first note that the range of f is $(0, 1) \cup \{-1\}$. Now, let U be an open subset of \mathbb{R} . If $-1 \notin U$, then $V = U \cap (0, 1)$ is also open and as the tangent function is continuous and g is Baire class one, $g^{-1}[\tan[\pi V - \frac{\pi}{2}]]$ is an F_σ set in X . By Lemma 2.5, σ is uniformly continuous, and therefore $W = \sigma[g^{-1}[\tan(\pi V - \frac{\pi}{2})]]$ is an F_σ set in $\sigma[X]$. But as X is complete, $\sigma[X]$ is closed in Y and therefore, W is an F_σ set in Y . Now $f^{-1}[U] = f^{-1}[V] = W$ so that $f^{-1}[U]$ is an F_σ set in Y . Finally, if U contains -1 , then $f^{-1}[U] = W \cup Y \setminus \sigma[X]$. Since

$Y \setminus \sigma[X]$ is open in Y , $f^{-1}[U]$ is an F_σ subset of Y for any open set U in \mathbb{R} . Therefore, f is a Baire class one function from Y to \mathbb{R} as desired.

Now by hypothesis, there is some trajectory $\{y_n\}_{n \in \omega}$ in Y with respect to which f is recoverable. Without loss of generality, assume that $y_0 \in \sigma[X]$ and define a sequence $\{x_n\}_{n \in \omega}$ in X recursively by setting:

$$x_0 = \sigma^{-1}(y_0)$$

$$x_{n+1} = \begin{cases} \sigma^{-1}(y_{n+1}) & \text{if } y_{n+1} \in \sigma[X] \\ x_n & \text{otherwise} \end{cases}$$

We claim that this sequence is a trajectory in X and that g is first return recoverable with respect to $\{x_n\}_{n \in \omega}$.

To show that $\{x_n\}_{n \in \omega}$ is a trajectory we need only show that it is dense in X . Let $x \in X$ and $\varepsilon \in I \setminus \{0_I\}$ be given. Let $y = \sigma(x)$. As f is recovered by the trajectory $\{y_n\}_{n \in \omega}$ in Y , the return route to y based on $\{y_n\}_{n \in \omega}$ approaches y . Hence, there is an N_0 such that for all $k \geq N_0$ we have $s_k(y, \{y_n\})$ in $B_Y(y, \rho(\varepsilon))$, the ball around y of radius $\rho(\varepsilon)$ in Y . Since $f(s_k(y, \{y_n\}))$ approaches $f(y)$, which is in $(0, 1)$, and $f(z) = -1$ for $z \in Y \setminus \sigma[X]$, there is an N_1 such that for $k \geq N_1$, $f(s_k(y, \{y_n\}))$ is in $(0, 1)$ so that $s_k(y, \{y_n\}) \in \sigma[X]$.

Let $N = \max\{N_0, N_1\}$. Then, for $k \geq N$, $s_k(y, \{y_n\}) \in \sigma[X] \cap B_Y(y, \rho(\varepsilon))$. Now pick l so that $y_l = s_N(y, \{y_n\})$. Then $y_l \in \sigma[X]$ and so by the definition of $\{x_n\}_{n \in \omega}$, $x_l = \sigma^{-1}(y_l)$. But as σ is a chain isometry,

$$\rho(d(x, x_l)) = \delta(\sigma(x), \sigma(x_l)) = \delta(y, y_l) < \rho(\varepsilon).$$

Therefore, $d(x, x_l) < \varepsilon$ so that $x_l \in B_X(x, \varepsilon)$ and $\{x_n\}_{n \in \omega}$ is dense in X as desired.

It remains to show that $\{x_n\}_{n \in \omega}$ recovers g . To prove this we first show that the return route to x based on $\{x_n\}_{n \in \omega}$ is eventually equal to inverse images under σ of the return route to $\sigma(x)$ based on $\{y_n\}_{n \in \omega}$. This is done by showing that there exists an M such that for $m > M$, we have x_m in $\{s_k(x, \{x_n\})\}_{k \in \omega}$ if, and only if $x_m = \sigma^{-1}(y_m)$ for y_m in $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$. We have previously shown that there exists an N_0 such that for $k \geq N_0$, $s_k(\sigma(x), \{y_n\}) \in \sigma[X]$. Let M be the index of $s_{N_0}(\sigma(x), \{y_n\})$ in $\{y_n\}_{n \in \omega}$.

Suppose, by way of contradiction, that there is an $l > M$ such that x_l is in the return route to x based on $\{x_n\}_{n \in \omega}$ but $x_l \neq \sigma^{-1}(y_l)$ for y_l in $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$. Then either x_l is not an inverse image under σ , or y_l is not in the return route to $\sigma(x)$ based on $\{y_n\}_{n \in \omega}$. But if x_l is not an inverse image under σ , then by definition, $x_l = x_{l-1}$ and x_{l-1} would have been chosen for the return route to x based on $\{x_n\}_{n \in \omega}$ instead of x_l . Thus, $x_l = \sigma^{-1}(y_l)$ but y_l is not in $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$.

Therefore, we can find p and q such that:

- (i) y_p and y_q are consecutive elements of the return route to $\sigma(x)$ based on $\{y_n\}_{n \in \omega}$.
- (ii) $M \leq p < l < q$.
- (iii) x_l is in the return route to x based on $\{x_n\}_{n \in \omega}$.

First, note that x_p is in $\{s_k(x, \{x_n\})\}_{k \in \omega}$. To see this, observe that as $p \geq M$ and y_p is in the return route, $y_p \in \sigma[X]$. Let m be the index in $\{x_n\}_{n \in \omega}$ of the last element before x_p which is in $\{s_k(x, \{x_n\})\}_{k \in \omega}$. Then,

$$\begin{aligned}
 x_p \in B(x, d(x, x_m)) &\Leftrightarrow d(x, x_p) < d(x, x_m) \\
 &\Leftrightarrow \delta(\sigma(x), \sigma(x_p)) < \delta(\sigma(x), \sigma(x_m)) \\
 &\Leftrightarrow \delta(\sigma(x), y_p) < \delta(\sigma(x), y_m)
 \end{aligned}$$

By (i) y_p is in $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$, so the last statement must be true. Therefore, $x_p \in B(x, d(x, x_m))$, and by our choice of m there is no x_h between x_m and x_p in this ball. Therefore, x_p is in $\{s_k(x, \{x_n\})\}_{k \in \omega}$ as claimed. In fact, this shows that in general, for $m > M$, if y_m is in the return route to $\sigma(x)$ based on $\{y_n\}_{n \in \omega}$, then $x_m = \sigma^{-1}(y_m)$ must be in the return route to x based on $\{x_n\}_{n \in \omega}$.

Now by the above claim, properties of chain isometries, and (iii) we have

$$\begin{aligned} d(x, x_l) < d(x, x_p) &\Rightarrow \delta(\sigma(x), \sigma(x_l)) < \delta(\sigma(x), \sigma(x_p)) \\ &\Rightarrow \delta(\sigma(x), y_l) < \delta(\sigma(x), y_p). \end{aligned}$$

But then y_l would have been chosen instead of y_q for the return route to $\sigma(x)$ based on $\{y_n\}_{n \in \omega}$. This is a contradiction, and hence no such l exists. Therefore, for $m > M$, x_m is in $\{s_k(x, \{x_n\})\}_{k \in \omega}$ if, and only if y_m is in $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$ so that $\{s_k(x, \{x_n\})\}_{k \in \omega}$ and $\{s_k(\sigma(x), \{y_n\})\}_{k \in \omega}$ eventually contain the same points.

Finally, by the above argument, the definition of f , the recoverability of f in Y , and the continuity of the tangent function,

$$\begin{aligned} \lim_{k \rightarrow \infty} g(s_k(x, \{x_n\})) &= \lim_{k \rightarrow \infty} g(\sigma^{-1}(s_k(\sigma(x), \{y_n\}))) \\ &= \lim_{k \rightarrow \infty} \tan\left(\pi f(s_k(\sigma(x), \{y_n\})) - \frac{\pi}{2}\right) \\ &= \tan\left(\pi f(\sigma(x)) - \frac{\pi}{2}\right) \\ &= g(\sigma^{-1}(\sigma(x))) \\ &= g(x). \end{aligned}$$

Therefore, g is recovered by $\{x_n\}_{n \in \omega}$ as desired. As g was an arbitrary Baire class one function from X to \mathbb{R} , every Baire class one function from X to \mathbb{R} is first return recoverable. \square

It is the contrapositive of this theorem which produces the most illuminating statement. This is given as a corollary below.

COROLLARY 2.7. *If there exists a non-recoverable Baire class one function on a separable complete I -valued ultrametric X , and X can be chain isometrically embedded into a separable complete J -valued ultrametric Y , then there exists a non-recoverable Baire class one function on Y .*

When I and J are subsets of \mathbb{R} , this corollary implies that if there is a non-recoverable Baire class one function on a subspace X of a separable metric space Y , then there is also a non-recoverable Baire class one function on Y .

It should be noted that the proof of Theorem 2.6 does not utilize any ultrametric properties of X and Y . Therefore, after appropriate adjustment, the theorem applies to general separable metric spaces. However, the proof does require that all Baire class one functions from Y to \mathbb{R} be recoverable. It is natural to ask if we can weaken this condition if we utilize the assumption that X is an ultrametric space. Slawomir Solecki showed in the following theorem that for a certain class of Baire class one functions, this is indeed the case.

THEOREM 2.8. *(Solecki) Let (Y, d) be a separable ultrametric space with $X \subseteq Y$ non-empty. Let $F \subseteq X$ be closed in X . Then, if the characteristic function χ_F is recoverable in (Y, d) , it is also recoverable in $(X, d|_X)$.*

PROOF. Let $\{y_n\}_{n \in \omega}$ be a trajectory in Y which recovers χ_F . Define a sequence $\{x_n\}_{n \in \omega} \subseteq X$ point-by-point as follows. For each $n \in \omega$:

- (a) if $y_n \in F$, let $x_n = y_n$.
- (b) otherwise, if there is a $y \in F$ such that $d(y_n, y) < d(y_n, X) + \frac{1}{n}$, let x_n be an element of F with $d(y_n, x_n) < d(y_n, X) + \frac{1}{n}$.

- (c) if there is no $y \in F$ as in (b), let x_n be an element of X with
- $$d(y_n, x_n) < d(y_n, X) + \frac{1}{n}.$$

We claim that $\{x_n\}_{n \in \omega}$ is dense in X and that it recovers χ_F in $(X, d|_X)$.

To show density, let $x \in X$ and $\varepsilon > 0$ be given and let n be such that $d(y_n, x) < \frac{\varepsilon}{2}$ and $\frac{1}{n} < \frac{\varepsilon}{2}$. Then, $d(y_n, X) < \frac{\varepsilon}{2}$, so that $d(y_n, x_n) < \frac{\varepsilon}{2} + \frac{1}{n} < \varepsilon$. Thus, by the ultrametric inequality, $d(x, x_n) \leq \max\{d(x, y_n), d(y_n, x_n)\} < \varepsilon$.

Next we show that $\{x_n\}_{n \in \omega}$ recovers χ_F in $(X, d|_X)$. Since F is closed, it is immediate that for any $x \in X \setminus F$ the return route to x based on $\{x_n\}_{n \in \omega}$ is from some point on included in $X \setminus F$, as it converges to x . So the value of χ_F on the return route is eventually 0 and $\chi_F(x) = 0$.

Now let $x \in F$. Assume towards a contradiction that the value of χ_F on the return route to x produced from $\{x_n\}_{n \in \omega}$ is not eventually $\chi_F(x) = 1$. Then infinitely many elements of this return route are in $X \setminus F$. Note also that points in the return route to x produced from $\{y_n\}_{n \in \omega}$ are eventually in F since χ_F is recovered by $\{y_n\}_{n \in \omega}$ in (Y, d) .

Thus, we can find y_m, y_p , and x_k such that:

- (i) y_m and y_p are consecutive elements of the return route to x produced from $\{y_n\}_{n \in \omega}$ in (Y, d) .
- (ii) y_m and y_p are in F .
- (iii) x_k is an element of the return route to x produced from $\{x_n\}_{n \in \omega}$ in $(X, d|_X)$.
- (iv) $x_k \in X \setminus F$.
- (v) $d(y_m, x) > d(x_k, x) \geq d(y_p, x)$.

Note first that $k < p$. To see this, observe that $y_p \in F$, so $y_p = x_p$. If $p < k$, then, since by (v) $d(x_p, x) \leq d(x_k, x)$, x_p would have been chosen instead of x_k in the return route to x produced from $\{x_n\}_{n \in \omega}$. Note that $p = k$ is ruled out since $x_p = y_p \in F$ and $x_k \notin F$.

Next, observe that $d(y_k, x) \geq d(y_m, x)$. To see this, note that if $d(y_k, x) < d(y_m, x)$, then by the above note, y_k would have been chosen instead of y_p as the element following y_m in the return route to x produced from $\{y_n\}_{n \in \omega}$ in (Y, d) . Now, from this observation and (v), we have

$$d(y_k, x) \geq d(y_m, x) > d(x_k, x).$$

Thus, since (Y, d) is an ultrametric,

$$d(y_k, x_k) \leq \max\{d(y_k, x), d(x, x_k)\} = d(y_k, x).$$

Since $x \in F$ and $x_k \notin F$ from (iv), this contradicts point (b) in the definition of x_k . That is,

$$d(y_k, x) + \frac{1}{k} > d(y_k, x_k) = d(y_k, x)$$

and $x \in F$ so that x_k must be chosen from F . Therefore, the value of χ_F on the return route to x produced from $\{x_n\}_{n \in \omega}$ is eventually $\chi_F(x) = 1$ and χ_F is recovered by $\{x_n\}_{n \in \omega}$ in $(X, d|_X)$ as claimed. \square

CHAPTER 3

Conditions for Recoverability

1. Ultrametrics on Which We Can Recover

We now turn our attention to the conditions under which we can recover all Baire class one functions on an ultrametric space. Lecomte showed in [Le] that there is a metric space on which we can define a Baire class one function which is not first return recoverable. This function was in fact a characteristic function of a closed set in an ultrametric space.

In that same paper, Lecomte gave a proof that on a class of ultrametric spaces which he called discrete, all Baire class one functions are recoverable. The definition of a discrete ultrametric is given below. Note that we have stated this definition in terms of I -valued ultrametrics where I is a chain with least element 0_I . However, Lecomte gave this definition under the assumption that $d : X \times X \rightarrow \mathbb{R}$.

DEFINITION 3.1. (Lecomte) We will say that an I -valued ultrametric space (X, d) is discrete if the following condition is satisfied.

$$\forall (d_n)_{n \in \omega} \subseteq I \left[(\forall n \in \omega \ d_{n+1} < d_n) \Rightarrow \left(\lim_{n \rightarrow \infty} d_n = 0_I \right) \right]$$

This definition deals with the distances attained on the ultrametric space. It is by focusing on the set of distances and the points at which they are attained that we improve on Lecomte's results. We have already seen that in separable ultrametrics, I will be a countable chain with least element 0_I . Under this assumption, we examine the set of distances attained at a single point. In particular, we are interested in the order type of this

set of distances. To this end, we make the following definition.

DEFINITION 3.2. Let (X, d) be an I -valued ultrametric space and $I_x = \{i \in I \mid \exists y \in X d(x, y) = i\}$. Then $r \in I$ is a **cutoff** for x if for any sequence $\{d_n\}_{n \in \omega} \subseteq I_x$ with $d_{n+1} < d_n$ for all n and $d_0 \leq r$, we have $\lim_{n \rightarrow \infty} d_n = 0_I$.

Note that 0_I is always a cutoff for any $x \in X$. Also, if $a > b$ in the chain I and a is a cutoff for some $x \in X$, then b is also a cutoff for x .

As we shall see, the location of cutoffs for points in an ultrametric space produces a partial criterium for determining if all Baire class one functions on the space are first return recoverable, or if there is an example of a Baire class one function on the space which is not first return recoverable. In this chapter, we give conditions for ultrametric spaces under which we can show that all Baire class one functions on the space are first return recoverable. This is done in the following theorem.

THEOREM 3.3. *Let (X, d) be a separable complete I -valued ultrametric space covered by non-empty open sets U for which there exists $r \in I \setminus \{0_I\}$ so that r is a cutoff for each $x \in U$. Let (Y, δ) be a separable metric space. Then, $f : X \rightarrow Y$ is first return recoverable if, and only if it is Baire class one.*

Before presenting the proof of this theorem we make a few general observations and state and prove several auxiliary lemmas. We also note that we need only prove that all Baire class one functions on such an ultrametric space are recoverable. Darji and Evans have already shown in [DE] that recoverable functions are necessarily Baire class one.

Our first observation is that any discrete ultrametric space, according to Lecomte's definition, will fulfill the assumptions of this theorem. Indeed, if an ultrametric space is discrete, then every value in the range of the

metric is a cutoff for every point in the space. Thus, any open covering of the ultrametric space will fulfill the conditions required in the theorem. It should be mentioned at this point that the conditions of Theorem 3.3 are substantially weaker than Lecomte's discreteness condition. For example, as exhibited in the second section of this chapter, there exists an I -valued ultrametric space such that each $r \in I$ is a cutoff for every point in the space, and yet the space is not discrete.

The next several lemmas, and the proof of Theorem 3.3, make use of the notion of a cutoff neighborhood. This auxiliary definition is stated below.

DEFINITION 3.4. Let (X, d) be an I -valued ultrametric space, $x \in X$. Let $A_x \subseteq I$ be the set of all $r > 0_I$ such that r is a cutoff for x , and if $y \in B(x, r)$ then r is also a cutoff for y . The **cutoff neighborhood** for x in X is the open set $U_x = \bigcup_{r \in A_x} B(x, r)$.

We first note that in general, not every point in an ultrametric space need have a non-empty cutoff neighborhood. The existence of a non-empty U_x around a point x is at least partially determined by the cutoffs of x . In particular, if the only cutoff for x is 0_I , then $A_x = \emptyset$ so that the cutoff neighborhood U_x is empty. If, on the other hand, x has cutoffs greater than 0_I , there is the potential for $A_x \neq \emptyset$ so that for any $r \in A_x$ we would have $x \in B(x, r) \subseteq U_x$.

LEMMA 3.5. *Let (X, d) be an I -valued ultrametric space, $x, y \in X$. Then the cutoff neighborhoods U_x and U_y are either equal or disjoint.*

PROOF. Suppose that $z \in U_x \cap U_y$. Let $w \in U_x$ be an arbitrary point. Now as cutoff neighborhoods are increasing unions of open balls, there exists $r_1 \in I$ such that $z, w \in B(x, r_1) \subseteq U_x$. Similarly, for some $r_2 \in I$, $z \in B(y, r_2) \subseteq U_y$. Now by ultrametric properties, since $z \in B(x, r_1) \cap B(y, r_2)$,

either $B(x, r_1) \subseteq B(y, r_2)$ or $B(y, r_2) \subseteq B(x, r_1)$. If $B(x, r_1) \subseteq B(y, r_2)$, then $w \in U_y$. On the other hand, if $B(y, r_2) \subseteq B(x, r_1)$ then $y \in B(x, r_1)$ so that $B(x, r_1) = B(y, r_1)$. But then, as $B(x, r_1) \subseteq U_x$, we must have $B(y, r_1) \subseteq U_y$ so that $w \in U_y$ again. Thus, as w was arbitrary, $U_x \subseteq U_y$.

Similarly, $U_y \subseteq U_x$. Therefore, the cutoff neighborhoods are either equal or disjoint as desired. \square

Before we begin our proof of Theorem 3.3, we must tie this notion of a cutoff neighborhood in with the theorem's hypotheses. In particular, we show that an open covering such as the one described in the theorem statement produces non-empty cutoff neighborhoods around each point in the ultrametric space. Our final lemma accomplishes this.

LEMMA 3.6. *Let (X, d) be an I -valued ultrametric space covered by non-empty open sets U for which there exists $r \in I \setminus \{0_I\}$ so that r is a cutoff for each $x \in U$. Then X can be partitioned into non-empty cutoff neighborhoods U_y for $y \in X$.*

PROOF. Let $y \in X$ be given. As X is covered by open sets U as described, $y \in U$ for some U . Now r is a cutoff for each point in U so that in particular, $B(y, r) \cap U$ is non-empty and thus, as U is open, $U_y \neq \emptyset$. But by Lemma 3.5 two cutoff neighborhoods are either equal or disjoint. Therefore, X can be partitioned into non-empty cutoff neighborhoods as claimed. \square

We now proceed to the proof of Theorem 3.3.

PROOF. Let $f : X \rightarrow Y$ be a Baire class one function. To complete this proof, we must construct a sequence $\{x_n\}_{n \in \omega}$ which is both dense and recovers f via first return methods. We build the trajectory in pieces and then show how to combine those pieces into a single sequence recovering f .

We first construct a family of closed sets in the domain which correspond to balls of decreasing diameter in the range. As the range is separable, we can find a countably dense subset $\{y_n\}_{n \in \omega}$ in Y and write $Y = \bigcup_{n \in \omega} B(y_n, \frac{1}{m})$ for any m . Now with this in mind, we construct a family of closed sets $\{F_s\}_{s \in \omega^{<\omega}}$ in the domain such that:

- (1) $F_\emptyset = X$.
- (2) the radius of $f[F_s]$ is less than $\frac{1}{|s|+1}$.
- (3) $F_s = \bigcup_{j \in \omega} F_{s \frown \langle j \rangle}$ for disjoint sets $\{F_{s \frown \langle j \rangle}\}_{j \in \omega}$.

This construction is performed recursively. As the basis, set $F_\emptyset = X$ satisfying (1). Now assume that F_s has been constructed and has the desired properties for $s \in \omega^{<\omega}$. As f is Baire class one, $f^{-1} \left[B \left(y_n, \frac{1}{|s|+1} \right) \right]$ is an F_σ set. Therefore, $F_s \cap f^{-1} \left[B \left(y_n, \frac{1}{|s|+1} \right) \right] = \bigcup_{i \in \omega} C_i^n$ where the C_i^n are closed. Using a bijective map $\varphi : \omega^2 \rightarrow \omega$ let $G_{\varphi(n,i)} = C_i^n$. Then for any $m \in \omega$, $G_m \setminus (\bigcup_{q < m} G_q)$ is an open subset of G_m and is therefore a countable disjoint union of clopen subsets of G_m , which are closed subsets of X . Let $F_{s \frown \langle j \rangle}$ range over the sets in these unions giving properties (2) and (3). This family of closed sets will be of use in selecting points for our trajectory in such a way as to ensure recoverability.

Let $B_\emptyset = X$ and x_\emptyset be an arbitrary point in X . By Lemma 3.6, $X \setminus \{x_\emptyset\}$ can be partitioned into non-empty cutoff neighborhoods U_x for $x \in X \setminus \{x_\emptyset\}$. Furthermore, as X is separable, there are at most countably many cutoff neighborhoods covering $X \setminus \{x_\emptyset\}$. For $i \in \omega$ let $B_{\langle i \rangle}$ range over these cutoff neighborhoods. Note that the sets $B_{\langle i \rangle}$ may not be balls themselves, but that they are increasing unions of open balls.

We now carry out a recursive construction in each of the open sets $B_{\langle i \rangle}$ to build a family of sub-balls B_s for $s \in \omega^{<\omega}$ with $|s| > 1$, and a set of points $\{x_s\}_{s \in \omega^{<\omega}}$ such that:

- (a) $x_s \in B_s \cap F_{\min\{t \in \omega^{|s|} \mid B_s \cap F_t \neq \emptyset\}}$ for $|s| > 1$.
- (b) $B_s = B(x_s, d(x_s, x_{s \setminus (|s|-1)}))$ for $|s| > 1$.
- (c) $B_s \setminus \{x_s\} = \bigcup_{i \in \omega} B_{s \setminus \langle i \rangle}$ for disjoint balls $\{B_{s \setminus \langle i \rangle}\}_{i \in \omega}$.

In part (a) the minimum is taken with respect to the lexicographical ordering of $\omega^{|s|}$. Starting with $B_{\langle i \rangle}$ as defined above, pick an arbitrary point in the set and call it $x_{\langle i \rangle}$. For any $x \in B_{\langle i \rangle}$, as $B_{\langle i \rangle}$ is an increasing union of open balls, there is an $r \in I \setminus \{0_I\}$ with $x, x_{\langle i \rangle} \in B(x, r)$. Therefore, $B(x, d(x, x_{\langle i \rangle})) \subseteq B_{\langle i \rangle}$. As X is an ultrametric, the balls $B(x, d(x, x_{\langle i \rangle}))$ and $B(y, d(y, x_{\langle i \rangle}))$ are either equal or disjoint for any $y \in B_{\langle i \rangle}$. Because X is separable, we can therefore divide $B_{\langle i \rangle} \setminus \{x_{\langle i \rangle}\}$ into countably many disjoint balls $B_{\langle i, j \rangle} = B(y_j, d(y_j, x_{\langle i \rangle}))$ for $j \in \omega$ satisfying (b) and (c). By our earlier construction of $\{F_s\}_{s \in \omega^{<\omega}}$, $B_{\langle i, j \rangle} = \bigcup_{t \in \omega^2} (B_{\langle i, j \rangle} \cap F_t)$, so that there is some minimum $t \in \omega^2$ such that $B_{\langle i, j \rangle} \cap F_t \neq \emptyset$. Choose $x_{\langle i, j \rangle} \in B_{\langle i, j \rangle} \cap F_t$ satisfying condition (a).

For any B_s we can repeat this same construction. First, we write $B_s \setminus \{x_s\} = \bigcup_{i \in \omega} B_{s \setminus \langle i \rangle}$, where $B_{s \setminus \langle i \rangle} = B(y_i, d(y_i, x_s))$ satisfying (b). As before, there are countably many disjoint balls of this form which cover $B_s \setminus \{x_s\}$ satisfying (c). Finally, we pick $x_{s \setminus \langle i \rangle}$ from the intersection of $B_{s \setminus \langle i \rangle}$ and the F_t with minimal $t \in \omega^{|s|+1}$ for which F_t intersects $B_{s \setminus \langle i \rangle}$ satisfying (a). This produces the family of balls $\{B_s\}_{s \in \omega^{<\omega}}$ and the set of points $\{x_s\}_{s \in \omega^{<\omega}}$ as desired.

Next we will show that the set $\{x_s\}_{s \in \omega^{<\omega}}$ is dense in X . Let $x \in X$ and $r > 0_I$ be given. If $x \in \{x_s\}_{s \in \omega^{<\omega}}$ then this set intersects $B(x, r)$ and we are done. Assume that $x \notin \{x_s\}_{s \in \omega^{<\omega}}$. Then by the above construction there is some $\alpha \in \omega^\omega$ such that $x \in \bigcap_n B_{\alpha|n}$. Now by (c), for any $n > 1$, $B_{\alpha|(n+1)} \subsetneq B_{\alpha|n}$. Furthermore, by (b), the radius of $B_{\alpha|n}$

is $d(x_{\alpha|n}, x_{\alpha|(n-1)})$. Thus, $\{d(x_{\alpha|n}, x_{\alpha|(n-1)})\}_{n \in \omega}$, and hence by ultrametric properties $\{d(x, x_{\alpha|(n-1)})\}_{n \in \omega}$, is a strictly decreasing sequence of distances from I_x attained in the cutoff neighborhood U_x . Therefore we have $\lim_{n \rightarrow \infty} d(x, x_{\alpha|(n-1)}) = 0_I$ so that $\{x_s\}_{s \in \omega^{<\omega}} \cap B(x, r) \neq \emptyset$ and $\{x_s\}_{s \in \omega^{<\omega}}$ is dense in X as desired.

It remains now to order the set $\{x_s\}_{s \in \omega^{<\omega}}$ into a trajectory $\{x_n\}_{n \in \omega}$ which will recover the function f . To do this, we first let $x_0 = x_1 = x_\emptyset$ and then for each $i \in \omega$, we let $x_{2(i+1)} = x_{\langle i \rangle}$. That is, we assigned the indices 0 and 1 to the initial point x_\emptyset and the remaining even indices to the initial representatives of the cutoff neighborhoods. Next, let $\{A_n\}_{n \in \omega}$ partition the odd indices greater than 1 so that:

- (i) $A_i \cap A_j = \emptyset$ for $i \neq j$.
- (ii) each A_i is infinite.
- (iii) $A_i \cap \{0, 1, \dots, 2(i+1)\} = \emptyset$.

Finally, set the x_k for $k \in A_i$ equal to the elements of the set $\{x_{\langle i \rangle \frown s}\}_{s \in \omega^{<\omega}}$ in such a way that if s is an initial segment of t , then $x_{\langle i \rangle \frown s}$ is given an index less than the one assigned to $x_{\langle i \rangle \frown t}$.

We now show that if $\{s_n(x)\}_{n \in \omega}$ is the recovery route to x based on $\{x_n\}_{n \in \omega}$, then $f(s_n(x)) \rightarrow f(x)$ and therefore f is first return recoverable. If $x \in \{x_n\}_{n \in \omega}$ then the result is trivial, so we assume $x \notin \{x_n\}_{n \in \omega}$. As before, this yields an $\alpha \in \omega^\omega$ such that $x \in \bigcap_{n \in \omega} B_{\alpha|n}$. This also yields a $\beta \in \omega^\omega$ such that $x \in \bigcap_{n \in \omega} F_{\beta|n}$.

First we claim that there exists a $p \in \omega$ such that for all $k \in \omega$, $s_{p+k}(x) = x_{\alpha|(k+1)}$. To see this, note that $x_{\alpha|1}$ was assigned index $2(\alpha(0) + 1)$ in $\{x_n\}_{n \in \omega}$. Also, by (iii), $x_{2(\alpha(0)+1)}$ is the first point from $\{x_n\}_{n \in \omega}$ in $B_{\alpha|1}$. Thus, $x_{2(\alpha(0)+1)}$ is in the return route to x based on $\{x_n\}_{n \in \omega}$. Because $s_0(x) = x_0 = x_\emptyset$, this yields some $p \geq 1$ such that $s_p(x) = x_{2(\alpha(0)+1)} = x_{\alpha|1}$. Now by our construction of $\{x_n\}_{n \in \omega}$, any point in this trajectory with index

less than $2(\alpha(0) + 1)$ comes from a different cutoff neighborhood than U_x . Therefore, such points can not be chosen in the return route to x so that $p = 1$ and $s_1(x) = x_{\alpha|1}$.

We now show that for all $k \in \omega$, $s_{k+1}(x) = x_{\alpha|(k+1)}$ by induction on k . For the base case $k = 0$ this has already been shown. Assume then that $s_{k+1}(x) = x_{\alpha|(k+1)}$. Let $s \in \omega^{<\omega}$ be such that $x_s = s_{(k+1)+1}(x)$. Note that then

$$x_s \in B(x, d(x, s_{k+1}(x))) = B(x, d(x, x_{\alpha|(k+1)})) = B_{\alpha|(k+2)}.$$

Thus, by (a) and (c) of our construction, $s = (\alpha|(k+2)) \frown t$ for some $t \in \omega^{<\omega}$. But $x_{\alpha|(k+2)} \in B_{\alpha|(k+2)}$ and in our ordering of $\{x_n\}_{n \in \omega}$, $x_{\alpha|(k+2)}$ comes before $x_{(\alpha|(k+2)) \frown t}$ for any $t \neq \emptyset$. Thus, $s = \alpha|(k+2)$ so that $s_{(k+1)+1}(x) = x_{\alpha|((k+1)+1)}$ as desired. Therefore, by induction, $s_{k+1}(x) = x_{\alpha|(k+1)}$ for all $k \in \omega$.

Next, we use the above result and the closed sets $\{F_{\beta|n}\}_{n \in \omega}$ constructed earlier to prove that $f(s_n(x)) \rightarrow f(x)$ for any $x \in X$. We do this by proving that from some point on, the return route to x is in $F_{\beta|l}$ for any $l \in \omega$. Fix $l \in \omega$. By (3) of our construction, $F_{\beta|l}$ is disjoint from the closed set $\bigcup_{q < l, p < \beta(q)} F_{(\beta|q) \frown \langle p \rangle}$. Thus, as $s_n(x)$ tends to x , there exists an $M_l > l$ such that $B(x, d(x, s_{M_l}(x)))$ and $\bigcup_{q < l, p < \beta(q)} F_{(\beta|q) \frown \langle p \rangle}$ are disjoint. Let $m \geq M_l$ be given. Note that then $m > l$ and $x \in B_{\alpha|m} \cap F_{\beta|l}$ so that by (3) there exists a $u \in \omega^{m-l}$ such that $x \in B_{\alpha|m} \cap F_{(\beta|l) \frown u}$. Now if $s_m(x) = x_{\alpha|m} \notin F_{\beta|l}$, then by (a) of our construction, there exists some $t \in \omega^m$ lexicographically preceding $(\beta|l) \frown u$ for any $u \in \omega^{m-l}$, such that $s_m(x) \in B_{\alpha|m} \cap F_t$. Hence, there is a $q < l$ and a $p < \beta(q)$ for which $(\beta|q) \frown \langle p \rangle$ is an initial segment of t . But then $s_m(x) \in F_{(\beta|q) \frown \langle p \rangle}$ contradicting the fact that $s_m(x) \in B(x, d(x, s_{M_l}(x)))$. Therefore, $s_m(x) \in F_{\beta|l}$ for all $m \geq M_l$ and the return route to x is eventually in $F_{\beta|l}$ as desired.

Finally, as $s_n(x)$ is eventually in $F_{\beta|l}$ and by (2) the radius of $f [F_{\beta|l}]$ is less than $\frac{1}{l+1}$, we have, for $m \geq M_l$, $\delta(f(s_m(x)), f(x)) < \frac{2}{l+1}$. Therefore, $f(s_n(x)) \rightarrow f(x)$ as $n \rightarrow \infty$ showing that f is recovered by $\{x_n\}_{n \in \omega}$ and completing the proof. \square

2. A Non-Discrete Ultrametric on Which We Can Recover

In this section we expand on our previous observation that the conditions used in Theorem 3.3 to ensure recoverability do not guarantee that the ultrametric space will be discrete. In fact, the space does not even need to be discrete for sequences of distances reaching below a fixed $\varepsilon \in I \setminus \{0_I\}$ (more formally, it does not fulfill the following condition for any $\varepsilon > 0_I$: $\forall (d_n)_{n \in \omega} \subseteq I [(\forall n \in \omega d_{n+1} < d_n < \varepsilon) \Rightarrow (\lim_{n \rightarrow \infty} d_n = 0_I)]$). Below we provide an example of a separable complete I -valued ultrametric space (X, d) which is not discrete, or even discrete below some $\varepsilon \in I \setminus \{0_I\}$. However, in this space all elements of I are cutoffs for every point. Hence, an open covering of X by itself fulfills the conditions of Theorem 3.3 so that every Baire class one function on X is recoverable. Therefore, Theorem 3.3 is in fact a proper strengthening of Lecomte's result from [Le] giving us a larger group of ultrametric spaces on which all Baire class one functions are recoverable.

EXAMPLE 3.7. We construct a separable complete I -valued ultrametric space (X, d) in which every open ball contains a sequence of points $\{x_n\}_{n \in \omega}$ with $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r > 0_I$. However, by Theorem 3.3, every Baire class one function on X is first return recoverable.

EXAMPLE CONSTRUCTION. Define I to be a set of distances of order type $(\omega^2 + 1)^*$ in which the limit points are designated by $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$, and $\{\lambda_i - n\}_{n \in \omega}$ is the strictly decreasing sequence of order type ω between

λ_i and λ_{i+1} with $\lambda_{i+1} = \inf_{n \in \omega} \{\lambda_i - n\}$. We build X based on this I .

To do this, let $R = \{s \in \omega^{<\omega} \mid s(i) \geq s(i-1) \text{ for } 0 < i < |s|\}$. We first construct a set of points $\{x_s\}_{s \in R}$ recursively based on $|s|$. We start with a single point x_\emptyset . Now, for $i \in \omega$, add points $x_{\langle i \rangle}$ to the set and let $d(x_{\langle i \rangle}, x_\emptyset) = \lambda_i$. If $j < i$ in ω , we must, by the ultrametric inequality, have

$$\begin{aligned} d(x_{\langle i \rangle}, x_{\langle j \rangle}) &\leq \max\{d(x_{\langle i \rangle}, x_\emptyset), d(x_\emptyset, x_{\langle j \rangle})\} \\ &= \max\{\lambda_i, \lambda_j\} \\ &= \lambda_j. \end{aligned}$$

As triangles in an ultrametric are isosceles with short base, and $\lambda_i < \lambda_j$, we must set $d(x_{\langle i \rangle}, x_{\langle j \rangle}) = \lambda_j$.

Assume now that points x_s have been constructed for $s \in R \cap \omega^{\leq k}$ and that all distances between these points have been defined consistently with the ultrametric inequality. For each such x_s with $|s| = k$, and for each $i \geq s(k-1)$, add a point $x_{s \frown \langle i \rangle}$ to the set and let $d(x_{s \frown \langle i \rangle}, x_s) = \lambda_i - k$. Now if $x_t \neq x_{s \frown \langle i \rangle}$ is a point already added to the set, then $|t| \leq k+1$. We consider three cases. First, if $t = s$, then $d(x_{s \frown \langle i \rangle}, x_t)$ is already defined. Second, if $t \neq s$ and $t \prec s \frown \langle i \rangle$, let $d(x_{s \frown \langle i \rangle}, x_t) = d(x_s, x_t)$. Note that as $d(x_{s \frown \langle i \rangle}, x_s) < d(x_s, x_t)$, this assignment satisfies the needed ultrametric inequalities. Finally, if $t \perp s \frown \langle i \rangle$, let u be the longest tuple in R with $u \prec s \frown \langle i \rangle$ and $u \prec t$. Define $d(x_{s \frown \langle i \rangle}, x_t) = \max\{d(x_{s \frown \langle i \rangle}, x_u), d(x_u, x_t)\}$. Note that both of these distances have been previously defined, and that this definition satisfies the required ultrametric inequalities.

This process results in a set $\{x_s\}_{s \in R}$, a portion of which is shown below.

After finishing this process, let X be the completion of $\{x_s\}_{s \in R}$ as an ultrametric. Note that X now has the following properties.

- X is complete. This is obvious as it is a completion.

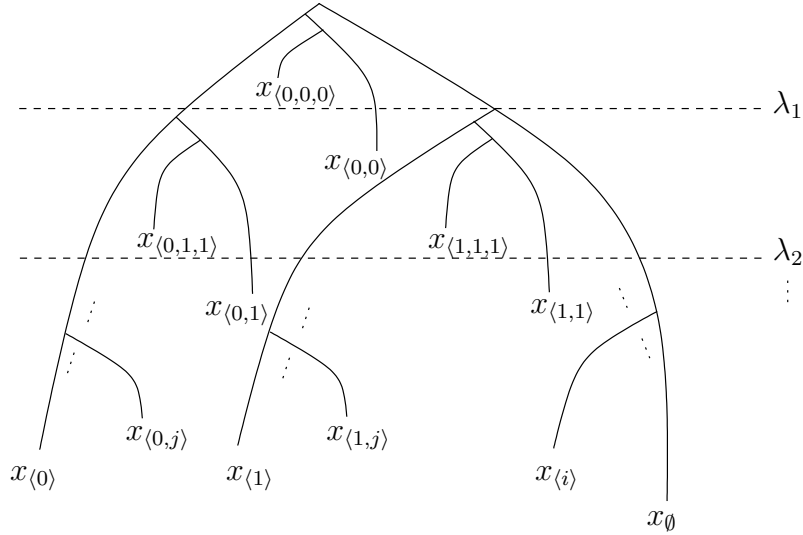


FIGURE 1. A Non-Discrete Ultrametric on which All Functions are Recoverable

- X is separable. As X is the completion of a countable set, this too is obvious.
- Each point in X realizes a discrete set of distances. That is, any decreasing sequence of distances attained at a given $x \in X$ must decrease to 0_I . Therefore, any $r \in I$ is a cutoff for x . To see this, we need only observe that x is either a point from $\{x_s\}_{s \in R}$ or a point added in the completion. If $x = x_s$ for some $s \in R$, then it realizes at most $|s|$ distances between λ_m and λ_{m+1} for any $m \in \omega$. If x was added in the completion then it is the limit of some sequence of points $\{x_k\}_{k \in \omega}$ from $\{x_s\}_{s \in R}$. Since x is the limit of this sequence, for each m there is a K_m such that $d(x, x_k) < \lambda_m$ for $k \geq K_m$. Finally, as X is an ultrametric, we see that since x_{K_m} has only finitely many points at distances between λ_j and λ_{j+1} for $j < m$, so must x .

- Finally, note that any open set U in X contains a sequence of points the distances between which decrease to something greater than 0_I . Namely, if $B(x_s, \lambda_i) \subseteq U$, then U contains a sequence of points starting with x_s with distances decreasing to λ_{i+1} . Therefore, X is not discrete, or even discrete below any $\varepsilon \in I \setminus \{0_I\}$.

□

CHAPTER 4

Conditions for Non-recoverability

In the last chapter we introduced the definition of a cutoff for a point in an ultrametric space. We then went on to give a criterium, involving cutoffs and their locations, for ultrametric spaces on which all Baire class one functions are recoverable. The other side of the question of recoverability is what conditions for an ultrametric space are required to guarantee the existence of a non-recoverable Baire class one function on that space.

D. Lecomte showed in [Le] that such spaces do in fact exist. However, he did this by exhibiting a specific function in a particular ultrametric space. In this chapter, we give a criterium for ultrametrics under which we are guaranteed the existence of a non-recoverable Baire class one function. We show how to construct that function in a general ultrametric space with this condition, and we then prove that the constructed function is indeed non-recoverable.

The construction of a non-recoverable Baire class one function depends on the existence of points whose only cutoff is 0_I . In order to build our function, we need to ensure that there are enough of these problematic points. A reasonable starting assumption is that the set of points whose only cutoff is 0_I is dense in the ultrametric. This turns out to be a sufficient condition to prove the following theorem.

THEOREM 4.1. *Let (X, d) be a separable complete I -valued ultrametric space in which the set of points whose only cutoff is 0_I is dense. Then there exists a Baire class one function on X which is not first return recoverable.*

Before proceeding with the proof of this theorem, we give an auxiliary definition which highlights the properties of points which will be important in constructing our non-recoverable functions. While this definition does not directly involve cutoffs, it is related to the notion of cutoffs as shown in Lemma 4.3. We again assume that I is a countable chain with least element 0_I and make use of the limit notation mentioned in a previous chapter.

DEFINITION 4.2. Let (X, d) be an I -valued ultrametric space, $x \in X$. Call x **bad** if $I_x = \{i \in I \mid \exists y \in X \ d(x, y) = i\}$ has the following properties:

- (1) there is a strictly decreasing sequence $\{\lambda_{x,i}\}_{i \in \omega}$ in I_x with

$$\lim_{i \rightarrow \infty} \lambda_{x,i} = 0_I.$$
- (2) for any $i \in \omega$ there exists a strictly decreasing sequence $\{d_j\}_{j \in \omega}$ in I_x between $\lambda_{x,i}$ and $\lambda_{x,i+1}$.

As the name suggests, bad points are points which can cause difficulties when trying to recover Baire class one functions. It therefore makes sense that they should be related to the points mentioned in the statement of Theorem 4.1 whose only cutoffs are 0_I . In fact, as shown in the following lemma, these points are the same.

LEMMA 4.3. *Let (X, d) be an I -valued ultrametric space with $\{j_n\}_{n \in \omega} \subseteq I$ such that $\lim_{n \rightarrow \infty} j_n = 0_I$. Then, a point $x \in X$ is bad if, and only if 0_I is the only cutoff for x .*

PROOF. Suppose that x is a bad point in X . Then, by property (1) of the definition, for any $r \in I \setminus \{0_I\}$ there is an N such that for any $i \geq N$ we have $\lambda_{x,i} < r$. Furthermore, by property (2) there is a strictly decreasing sequence of distances in I_x less than $\lambda_{x,i}$, the limit of which is either above 0_I or does not exist. Therefore, r can not be a cutoff for x . As r was arbitrary in $I \setminus \{0_I\}$, the only cutoff for x is 0_I .

Now suppose that the only cutoff for a given $x \in X$ is 0_I . Pick an arbitrary $\lambda_{x,0} \in I_x \setminus \{0_I\}$ less than j_0 . As $\lambda_{x,0}$ is not a cutoff for x , there is a strictly decreasing sequence of distances $\{d_n\}_{n \in \omega} \subseteq I_x$ with $d_0 < \lambda_{x,0}$ such that $\lim_{n \rightarrow \infty} d_n \neq 0_I$. Hence, we can find a $\lambda_{x,1}$ in $I_x \setminus \{0_I\}$ which is less than j_1 as well as d_n for each $n \in \omega$. Continuing this produces a strictly decreasing sequence $\{\lambda_{x,i}\}_{i \in \omega}$ approaching 0_I as in (1) with a strictly decreasing sequence in I_x between each $\lambda_{x,i}$ and $\lambda_{x,i+1}$ fulfilling (2). Thus, the point x is bad as desired. \square

With this definition and lemma, we are now ready to begin the proof of Theorem 4.1.

PROOF. We accomplish the proof of this theorem by constructing a closed set $F \subset X$. The characteristic function of this set, $\chi_F : X \rightarrow \mathbb{R}$, is Baire class one. However, we will show that χ_F is not first return recoverable with respect to any trajectory in X . Thus χ_F is the non-recoverable Baire class one function desired.

Before proceeding with this construction, note that we can assume that I contains a sequence $\{j_n\}_{n \in \omega}$ with $\lim_{n \rightarrow \infty} j_n = 0_I$, where the meaning of this limit expression is as defined in chapter 2. If this were not true, then there would exist some smallest $i \in I \setminus \{0_I\}$. This i would then be a cutoff for every point in X . But this contradicts our assumption that the set of points whose only cutoff is 0_I is dense in X . Therefore, the sequence mentioned above exists and by Lemma 4.3 the dense set of points whose only cutoff is 0_I is also a dense set of bad points in X .

We now turn to the construction of the set F . We start by assigning to each $u \in \omega^{<\omega}$ a bad point y_u , beginning with an arbitrary bad point y_\emptyset from X . This is done in such a way that for all $u \in \omega^{<\omega}$

$$(a) \text{ for any } i > j \in \omega, d(y_u, y_{u \smallfrown \langle i \rangle}) < d(y_u, y_{u \smallfrown \langle j \rangle}).$$

- (b) for any $i \in \omega$, $d(y_u, y_{u \smallfrown \langle i \rangle}) < \lambda_{y_\emptyset, |u|}$.
- (c) for any $i \in \omega$, $d(y_u, y_{u \smallfrown \langle i \rangle}) < d(y_u, y_{u \smallfrown \langle |u| - 1 \rangle})$ when $|u| \geq 1$.
- (d) there is a constant λ_u such that for all $v \succ u$, $d(y_u, y_v) > \lambda_u$.

At the first stage of this recursive construction, let $\{d_j\}_{j \in \omega}$ be the strictly decreasing sequence of distances realized at the bad point y_\emptyset between $\lambda_{y_\emptyset, 0}$ and $\lambda_{y_\emptyset, 1}$ as guaranteed by (2). For each $j \in \omega$ let $y_{\langle j \rangle}$ be a bad point with $d(y_{\langle j \rangle}, y_\emptyset) = d_j$. Note that this can be done as $d_j \in I_{y_\emptyset}$ so that for some $z \in X$ we have $d(y_\emptyset, z) = d_j$. By density, we can find a bad point $y_{\langle j \rangle} \in B(z, d_j)$ so that as $y_\emptyset \notin B(z, d_j)$

$$d(y_{\langle j \rangle}, y_\emptyset) = d(z, y_\emptyset) = d_j.$$

Note that the points $y_{\langle j \rangle}$ were all added at distinct distance less than $\lambda_{y_\emptyset, 0}$ from y_\emptyset satisfying (a) and (b). Also, for any $j \in \omega$,

$$d(y_{\langle j \rangle}, y_\emptyset) > \lambda_{y_\emptyset, 1} = \lambda_\emptyset$$

satisfying (d).

At the k th stage of the construction, assume that bad points y_w have been selected for all $|w| < k$ satisfying the conditions above. For each $u \in \omega^{k-1}$, let $\lambda_{y_u, i}$ be the first distance from (1) of the definition of the bad point y_u which is less than both $d(y_u, y_{u \smallfrown \langle |u| - 1 \rangle})$ and $\lambda_{y_\emptyset, |u|}$. By (2) there is a strictly decreasing sequence $\{d_j\}_{j \in \omega}$ in I_{y_u} between $\lambda_{y_u, i}$ and $\lambda_{y_u, i+1}$. For each d_j in this sequence we can, as before, find a bad point $y_{u \smallfrown \langle j \rangle}$ with $d(y_u, y_{u \smallfrown \langle j \rangle}) = d_j$. Note that the points $y_{u \smallfrown \langle j \rangle}$ were all added with distinct distances $d(y_u, y_{u \smallfrown \langle j \rangle}) < \lambda_{y_\emptyset, |u|}$ satisfying (a) and (b). As

$$d(y_u, y_{u \smallfrown \langle j \rangle}) = d_j < \lambda_{y_u, i} < d(y_u, y_{u \smallfrown \langle |u| - 1 \rangle})$$

(c) is also satisfied. Finally, note that

$$d(y_u, y_{u \smallfrown \langle j \rangle}) > \lambda_{y_u, i+1} = \lambda_u$$

and if $v \prec u$, then the triangle $y_v, y_u, y_{u \smallfrown \langle j \rangle}$ is isosceles with short base, and by (c) $d(y_{u \smallfrown \langle j \rangle}, y_u) < d(y_u, y_v)$. Thus, since (d) holds for $u \succ v$

$$d(y_{u \smallfrown \langle j \rangle}, y_v) = d(y_u, y_v) > \lambda_v$$

satisfying (d).

The first few steps of this process are depicted in the tree diagram below.

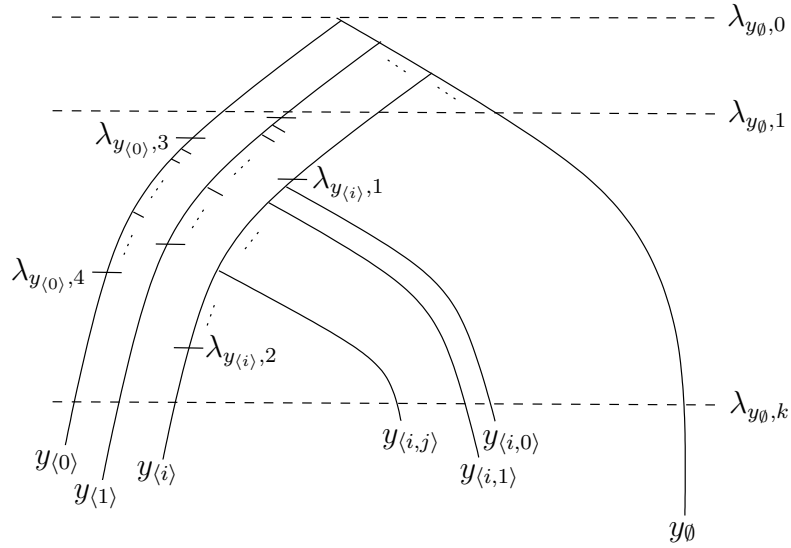


FIGURE 2. Construction of a Non-Recoverable Baire Class One Function

Note that for each $\beta \in \omega^\omega$, the sequence $\{y_{\beta|n}\}_{n \in \omega}$ from the family of points defined above is Cauchy. To see this, let $\varepsilon \in I \setminus \{0_I\}$ be given. By (2) there is an $n \in \omega$ such that $\lambda_{y_{\emptyset}, n} < \varepsilon$. Using ultrametric properties and (b) and (c) above, for any $m > n$, we see that

$$d(y_{\beta|n}, y_{\beta|m}) = d(y_{\beta|n}, y_{\beta|(n+1)}) < \lambda_{y_{\emptyset}, n} < \varepsilon$$

making the sequence Cauchy as desired. As X is complete, there must be a y_β in X to which the sequence $\{y_{\beta|n}\}_{n \in \omega}$ converges. Let $F = \{y_\beta \mid \beta \in \omega^\omega\}$. We claim that the following statements regarding F and points in F are true.

- (A) If y_β is in F , then for any $n \in \omega$, $d(y_\beta, y_{\beta|n}) = d(y_{\beta|n}, y_{\beta|(n+1)})$.
- (B) If y_β and $y_{\beta'}$ are distinct and $u \in \omega^{<\omega}$ is the longest initial segment common to β and β' , then $d(y_\beta, y_{\beta'}) = \max\{d(y_\beta, y_u), d(y_u, y_{\beta'})\}$.
- (C) F is closed.
- (D) F has no isolated points.
- (E) For each $u \in \omega^{<\omega}$, $B(y_u, \lambda_u) \cap F = \emptyset$, where λ_u is as in (d).

Let $n \in \omega$ be given. As y_β is the limit of the sequence $\{y_{\beta|m}\}_{m \in \omega}$ we can find an m such that $d(y_\beta, y_{\beta|m}) < d(y_{\beta|n}, y_{\beta|(n+1)})$. Now by repeated applications of (c), $d(y_{\beta|m}, y_{\beta|(n+1)}) < d(y_{\beta|n}, y_{\beta|(n+1)})$ so that using isosceles triangles, $d(y_{\beta|m}, y_{\beta|n}) = d(y_{\beta|n}, y_{\beta|(n+1)})$. Again using isosceles triangles, this gives

$$d(y_\beta, y_{\beta|n}) = d(y_{\beta|m}, y_{\beta|n}) = d(y_{\beta|n}, y_{\beta|(n+1)}).$$

Therefore, (A) holds.

To show that (B) is true, note that as in the proof of (A) we can find an $n > |u|$ such that both $d(y_\beta, y_{\beta|n}) < d(y_\beta, y_u)$ and $d(y_{\beta'}, y_{\beta'|n}) < d(y_{\beta'}, y_u)$. Thus, using isosceles triangles, $d(y_{\beta|n}, y_u) = d(y_\beta, y_u)$ and $d(y_{\beta'|n}, y_u) = d(y_{\beta'}, y_u)$. But u is the longest initial segment common to β and β' . Hence, by (a) and (c), $d(y_{\beta|n}, y_u) \neq d(y_{\beta'|n}, y_u)$. Therefore examining the triangle with vertices y_β , $y_{\beta'}$ and y_u we see that $d(y_\beta, y_u) \neq d(y_{\beta'}, y_u)$ so that $d(y_\beta, y_{\beta'}) = \max\{d(y_\beta, y_u), d(y_u, y_{\beta'})\}$ as desired.

Now to see (C), let $\{y_{\beta_m}\}_{m \in \omega}$ be a Cauchy sequence in F . We claim that for each $i \in \omega$, there is an $M_i \in \omega$ such that the sequence $\{\beta_m(i)\}_{m \geq M_i}$ is constant. Suppose not. Then there is some smallest i_0 for which there

are infinitely many $m \in \omega$ with $\beta_m(i_0) \neq \beta_{m+1}(i_0)$. Thus, $u = \beta_m|_{i_0}$ is the longest sequence in $\omega^{<\omega}$ for which there exists an M such that for all $m \geq M$, u is an initial segment of β_m . Now by (B), for such m $d(y_{\beta_m}, y_{\beta_{m+1}}) = \max\{d(y_{\beta_m}, y_u), d(y_u, y_{\beta_{m+1}})\}$. But by (d) and (A), both $d(y_{\beta_m}, y_u)$ and $d(y_{\beta_{m+1}}, y_u)$ are greater than $\lambda_u > 0_I$. Therefore, as this holds for infinitely many m , the sequence $\{y_{\beta_m}\}_{m \in \omega}$ can not be Cauchy, which is a contradiction.

Hence, for each $i \in \omega$, there is an M_i such that the sequence $\{\beta_m(i)\}_{m \geq M_i}$ is constant. Define $\beta \in \omega^\omega$ by setting $\beta(i) = \beta_{M_i}(i)$. We claim that y_β is the limit of $\{y_{\beta_m}\}_{m \in \omega}$. Let $\varepsilon \in I \setminus \{0_I\}$ be given. Then by (1) there exists an n such that $\lambda_{y_\beta, n} < \varepsilon$. Furthermore, we have shown that for $m \geq \max\{M_0, M_1, \dots, M_{n-1}\}$ we have $\beta_m|_n = \beta|_n$. Then by the ultrametric inequality, (A), and (b)

$$\begin{aligned} d(y_\beta, y_{\beta_m}) &\leq \max\{d(y_\beta, y_{\beta|_n}), d(y_{\beta|_n}, y_{\beta_m})\} \\ &= \max\{d(y_{\beta|_n}, y_{\beta|(n+1)}), d(y_{\beta_m|_n}, y_{\beta_m|(n+1)})\} \\ &< \varepsilon. \end{aligned}$$

Thus $y_\beta \in F$ is the limit of the sequence $\{y_{\beta_m}\}_{m \in \omega}$ as desired, making F closed.

We now show that (D) holds. Let y_β be a point in F . Let $\varepsilon \in I \setminus \{0_I\}$ be given. As y_β is the limit of the sequence $\{y_{\beta|_m}\}_{m \in \omega}$, there is an $n \in \omega$ such that $d(y_\beta, y_{\beta|_n}) < \varepsilon$. Let $\beta' \in \omega^\omega$ be such that $\beta'|_n = \beta|_n$ and $\beta'(n) = \beta(n) + 1$. Then using (B), (A) and (a)

$$\begin{aligned} d(y_\beta, y_{\beta'}) &= \max\{d(y_\beta, y_{\beta|_n}), d(y_{\beta|_n}, y_{\beta'})\} \\ &= d(y_\beta, y_{\beta|_n}) \\ &< \varepsilon. \end{aligned}$$

Hence, $y_{\beta'} \in B(y_\beta, \varepsilon)$ and y_β is not isolated in F .

Finally, we show (E). Let $u \in \omega^{<\omega}$ and let $y_\beta \in F$. If $u \prec \beta$ then by (A) and (d)

$$d(y_\beta, y_u) = d(y_{u \frown \langle \beta(|u|) \rangle}, y_u) > \lambda_u.$$

If u is not an initial segment of β , then let v be the longest initial segment common to both u and β . Then by (A) and (a),

$$d(y_\beta, y_v) = d(y_v, y_{v \frown \langle \beta(|v|) \rangle}) \neq d(y_v, y_u).$$

Hence, using isosceles triangles,

$$d(y_\beta, y_u) = \max\{d(y_\beta, y_v), d(y_v, y_u)\} \geq d(y_v, y_u).$$

But by (c), for any $i \in \omega$, $d(y_u, y_v) > d(y_u, y_{u \frown \langle i \rangle}) > \lambda_u$. Therefore, $d(y_\beta, y_u) > \lambda_u$ and $y_\beta \notin B(y_u, \lambda_u)$ as desired.

Now suppose by way of contradiction that there exists some countably dense set $\{x_n\}_{n \in \omega} \subseteq X$ with respect to which χ_F is first return recoverable. We use this trajectory to define a family of balls $\{B_s\}_{s \in \omega^{<\omega}}$ together with a set of points $\{x_s\}_{s \in \omega^{<\omega}}$ taken from $\{x_n\}_{n \in \omega}$ such that:

- (i) $B_\emptyset = X$.
- (ii) x_s is the first element from $\{x_n\}_{n \in \omega}$ in B_s .
- (iii) $B_s \setminus \{x_s\} = \bigcup_{i \in \omega} B_{s \frown \langle i \rangle}$ for disjoint balls $\{B_{s \frown \langle i \rangle}\}_{i \in \omega}$.

To accomplish this, set $B_\emptyset = X$ satisfying (i). Now let $x_\emptyset = x_0$ from $\{x_n\}_{n \in \omega}$. By ultrametric properties, $X \setminus \{x_\emptyset\}$ can be divided into countably many disjoint balls of the form $B(y_i, d(y_i, x_\emptyset))$. Let $B_{\langle i \rangle}$ range over these balls. Let $x_{\langle i \rangle}$ be the first element of $\{x_n\}_{n \in \omega}$ in $B_{\langle i \rangle}$ and note that by ultrametric properties, $B(y_i, d(y_i, x_\emptyset)) = B(x_{\langle i \rangle}, d(x_{\langle i \rangle}, x_\emptyset))$. Assume now that B_s has been constructed fulfilling the conditions above. Let x_s be the first element of $\{x_n\}_{n \in \omega}$ in B_s satisfying (ii). Finally, divide $B_s \setminus \{x_s\}$ into

disjoint balls of the form $B(y_i, d(y_i, x_s))$ and let $B_{s \smallfrown (i)}$ range over these balls satisfying (iii). Observe that for any $x \in X$ not in $\{x_n\}_{n \in \omega}$ there is a unique $\alpha \in \omega^\omega$ for which $x \in \bigcap_{n \in \omega} B_{\alpha|n}$. Furthermore, the return route to such an x based on $\{x_n\}_{n \in \omega}$ is the sequence $\{x_{\alpha|n}\}_{n \in \omega}$.

With this framework in place, we consider two complementary cases.

Case I: $\forall s \in \omega^{<\omega} [B_s \cap F = \emptyset \vee \exists t \succ_s (B_t \cap F \neq \emptyset \wedge x_t \notin F)]$

Restating this, each of the balls B_s constructed above is either disjoint from F or has a sub-ball which intersects F , but in which the first element from $\{x_n\}_{n \in \omega}$ is not in F . As $B_\emptyset \cap F$ is clearly non-empty, if this case holds we can find an $\alpha \in \omega^\omega$ for which there exist infinitely many n with $B_{\alpha|n} \cap F \neq \emptyset$ and $x_{\alpha|n} \notin F$. But then by (iii) of the construction above $B_{\alpha|n} \cap F \neq \emptyset$ for all $n \in \omega$. Using this α , we show that there must exist some $x \in \bigcap_n B_{\alpha|n}$. If this x is in F , then this contradicts our assumption that χ_F is recoverable as the return route to x is $\{x_{\alpha|n}\}_{n \in \omega}$, which is in $X \setminus F$ for infinitely many $n \in \omega$. If this x is not in F , we show that the return route $\{x_{\alpha|n}\}_{n \in \omega}$ does not approach x , contradicting the density of $\{x_n\}_{n \in \omega}$.

Now as $B_{\alpha|n}$ intersects F for each $n \in \omega$, we can find a point y_{β_n} in $B_{\alpha|n} \cap F$. For each $N \in \omega$ define s_N to be the longest sequence in $\omega^{<\omega}$ which is an initial segment of β_m for all $m \geq N$. Note that such an s_N exists as $\emptyset \prec \beta_m$ for all $m \in \omega$. Based on the sequence $\{s_N\}_{N \in \omega}$ we again consider two cases.

Suppose as our first sub-case that for every $l \in \omega$ there is an M_l such that for $N \geq M_l$ the length of $s_N > l$ and $s_N(l) = m_l$ for some constant $m_l \in \omega$. Now as y_\emptyset is a bad point, given any $\varepsilon \in I \setminus \{0_I\}$, there exists an i such that $\lambda_{y_\emptyset, i} < \varepsilon$. By our supposition for all $m \geq M = \max\{M_0, M_1, \dots, M_i\}$, we have $s_M|i \prec \beta_m$. But then by (A) and (b)

$$d(y_{\beta_m}, y_{s_M|i}) < \lambda_{y_\emptyset, i} < \varepsilon.$$

Let $m, k \geq M$. Then by the ultrametric inequality

$$d(y_{\beta_m}, y_{\beta_k}) \leq \max\{d(y_{\beta_m}, y_{s_M|i}), d(y_{s_M|i}, y_{\beta_k})\} < \varepsilon.$$

Therefore, the sequence $\{y_{\beta_m}\}_{m \in \omega}$ is Cauchy. Now as the metric space is complete and F is closed, there is a point $y \in F$ with $y_{\beta_m} \rightarrow y$ as $m \rightarrow \infty$. For any $n \in \omega$, we claim $y \in B_{\alpha|n}$. Indeed there exists an $m > n$ such that $d(y, y_{\beta_m}) < d(x_{\alpha|n}, x_{\alpha|(n-1)})$. Therefore, using the ultrametric inequality and the fact that $y_{\beta_m} \in B_{\alpha|n}$, we get

$$\begin{aligned} d(y, x_{\alpha|n}) &\leq \max\{d(y, y_{\beta_m}), d(y_{\beta_m}, x_{\alpha|n})\} \\ &< d(x_{\alpha|n}, x_{\alpha|(n-1)}). \end{aligned}$$

Hence $y \in B_{\alpha|n}$ for all $n \in \omega$. But then y is in F and $\bigcap_{n \in \omega} B_{\alpha|n}$. Thus the return route to y is $\{x_{\alpha|n}\}_{n \in \omega}$ which is in $X \setminus F$ for infinitely many n . Therefore χ_F is not recovered at y , a contradiction.

On the other hand, suppose that there is some smallest $l_0 \in \omega$ such that there is no m_{l_0} for which the sequence $\{s_N(l_0)\}_{N \in \omega}$ is eventually defined and equal to m_{l_0} . As l_0 is the smallest such index, there is an M such that for all $N \geq M$, $s_N|l_0 = s_M|l_0$. We claim that for any $n \in \omega$, the point $y_{s_M|l_0}$ is in $B_{\alpha|n}$. To see this note that we can find $m, k > \max\{M, n\}$ such that $y_{\beta_m}, y_{\beta_k} \in B_{\alpha|n}$ are distinct and $s_M|l_0$ is the longest common initial segment of β_m and β_k . Thus by (B)

$$d(y_{\beta_m}, y_{\beta_k}) = \max\{d(y_{\beta_m}, y_{s_M|l_0}), d(y_{s_M|l_0}, y_{\beta_k})\}.$$

But as $y_{\beta_m}, y_{\beta_k} \in B_{\alpha|n}$, we know that $d(y_{\beta_m}, y_{\beta_k}) < d(x_{\alpha|n}, x_{\alpha|(n-1)})$. Hence $d(y_{\beta_m}, y_{s_M|l_0}) < d(x_{\alpha|n}, x_{\alpha|(n-1)})$ so that

$$y_{s_M|l_0} \in B(y_{\beta_m}, d(x_{\alpha|n}, x_{\alpha|(n-1)})) = B(x_{\alpha|n}, d(x_{\alpha|n}, x_{\alpha|(n-1)})) = B_{\alpha|n}.$$

As this is true for any $n \in \omega$, we have $y_{s_M|l_0} \in \bigcap_{n \in \omega} B_{\alpha|n}$. Therefore, $\{x_{\alpha|n}\}_{n \in \omega}$ is the return route to $y_{s_M|l_0}$. However, each $B_{\alpha|n}$ intersects F and by (E), $B(y_{s_M|l_0}, \lambda_{s_M|l_0}) \cap F = \emptyset$. Therefore the ball $B_{\alpha|n}$ must have radius larger than $\lambda_{s_M|l_0}$ so that for all $n \in \omega$

$$d(y_{s_M}, x_{\alpha|(n-1)}) = d(x_{\alpha|n}, x_{\alpha|(n-1)}) > \lambda_{s_M|l_0} > 0I.$$

But then the return route to y_{s_M} does not approach y_{s_M} , contradicting the density of $\{x_n\}_{n \in \omega}$.

Therefore, in either of the possible sub-cases, we are able to reach a contradiction so that the recoverability of χ_F based on $\{x_n\}_{n \in \omega}$ is not possible.

Case II: $\exists s \in \omega^{<\omega} [B_s \cap F \neq \emptyset \wedge \forall t \succ s (B_t \cap F \neq \emptyset \Rightarrow x_t \in F)]$

In this case we can find an $s \in \omega^{<\omega}$ for which the ball B_s intersects F and any sub-ball B_t of B_s which intersects F has the point x_t indicated in (ii) chosen from F . Let s be the tuple above, or possibly an extension thereof, so that $x_s \in F$.

Let $t \succ s$ be any extension with $B_t \cap F \neq \emptyset$. Note that such a t exists by (iii) and (D). Now both x_s and x_t are in F . Thus, they have the form y_{β_s} and y_{β_t} for some $\beta_s, \beta_t \in \omega^\omega$. Let u be the longest sequence in $\omega^{<\omega}$ which is a common initial segment of β_s and β_t . Note that by (B) $d(y_{\beta_s}, y_{\beta_t}) = \max\{d(y_{\beta_s}, y_u), d(y_u, y_{\beta_t})\}$. As y_u is a bad point, there is an $i \in \omega$ such that $d(y_{\beta_s}, y_{\beta_t})$ is less than $\lambda_{y_u, i}$ but greater than $\lambda_{y_u, i+1}$. Pick d_{j_1} to be a distance from (2) of the definition of a bad point with $d(y_u, y_{\beta_s}) > d_{j_1} > \lambda_{y_u, i+1} = \lambda_u$. Note that by (a), $d(y_u, y_{u \smallfrown \langle j_1 \rangle}) < d(y_u, y_{\beta_s})$. Therefore, using isosceles triangles and recalling that $y_{\beta_s}, y_{\beta_t} \in B_s$,

$$d(y_{u \smallfrown \langle j_1 \rangle}, y_{\beta_s}) = d(y_u, y_{\beta_s}) \leq d(y_{\beta_s}, y_{\beta_t}) < d(x_s, x_{s \smallfrown \langle |s| - 1 \rangle}).$$

Thus $y_{u \smallfrown \langle j_1 \rangle} \in B_s \setminus \{x_s\}$ so that by (iii) $y_{u \smallfrown \langle j_1 \rangle} \in B_{s_1}$ with $s_1 = s \smallfrown \langle i \rangle$ for some $i \in \omega$. Note that

$$B_{s_1} = B(y_{u \smallfrown \langle j_1 \rangle}, d(y_{u \smallfrown \langle j_1 \rangle}, x_s)) = B(x_{s_1}, d(x_{s_1}, x_s)).$$

Furthermore, we claim that $B_{s_1} \cap F \neq \emptyset$. To see this, let $\beta \in \omega^\omega$ be such that $u \smallfrown \langle j_1 \rangle \prec \beta$. Then by (A), (c), and our choice of j_1 ,

$$d(y_\beta, y_{u \smallfrown \langle j_1 \rangle}) < d(y_u, y_{u \smallfrown \langle j_1 \rangle}) < d(y_u, x_s).$$

Now $d(y_{u \smallfrown \langle j_1 \rangle}, x_s) = d(y_u, x_s)$ by isosceles triangles. Therefore we have $d(y_\beta, y_{u \smallfrown \langle j_1 \rangle}) < d(y_{u \smallfrown \langle j_1 \rangle}, x_s)$ and hence $y_\beta \in B_{s_1}$ as claimed. Thus, $x_{s_1} \in F$ by the hypothesis of case II. Finally, as $d(y_u, y_{u \smallfrown \langle j_1 \rangle}) < d(y_{u \smallfrown \langle j_1 \rangle}, x_s)$, we also have $y_u \in B_{s_1}$.

Let β_{s_1} be such that $y_{\beta_{s_1}} = x_{s_1}$. We claim that $u \prec \beta_{s_1}$. Indeed, if $u \perp \beta_{s_1}$ then by (c) and (B), $d(y_{\beta_{s_1}}, x_s) > d(x_s, x_t)$ so that $y_{\beta_{s_1}} = x_{s_1} \notin B_s$, a contradiction. Therefore, by (A) and (d), $d(y_{\beta_{s_1}}, y_u) > \lambda_u$ and we can pick d_{j_2} to be a distance attained at y_u with $d(y_u, y_{\beta_{s_1}}) > d_{j_2} > \lambda_u$. Consider the point $y_{u \smallfrown \langle j_2 \rangle}$. By (a) $d(y_u, y_{u \smallfrown \langle j_2 \rangle}) < d(y_u, y_{\beta_{s_1}})$. Therefore, using isosceles triangles,

$$d(y_{u \smallfrown \langle j_2 \rangle}, y_{\beta_{s_1}}) = d(y_u, y_{\beta_{s_1}}) < d(y_{\beta_{s_1}}, x_s) = d(x_{s_1}, x_s).$$

So then $y_{u \smallfrown \langle j_2 \rangle} \in B_{s_1} \setminus \{x_{s_1}\}$ since $y_{u \smallfrown \langle j_2 \rangle} \notin F$ and hence not equal to $x_{s_1} \in F$. Now by (iii) $y_{u \smallfrown \langle j_2 \rangle} \in B_{s_2}$ with $s_2 = s_1 \smallfrown \langle i \rangle$ for some $i \in \omega$. Using the same argument as before, B_{s_2} must intersect F , and we proceed as above.

Continuing this process produces a sequence of balls

$$B_{s_1} \supset B_{s_2} \supset B_{s_3} \supset \cdots \supset B_{s_l} \supset \cdots$$

such that $y_u \in B_{s_l}$ for each $l \in \omega$. Hence, $y_u \in \bigcap_{l \in \omega} B_{s_l}$ and by (ii), the return route to y_u based on the trajectory $\{x_n\}_{n \in \omega}$ is, from some point on, equal to the sequence $\{x_{s_l}\}_{l \in \omega}$. But as these x_{s_l} are in F , by (E)

$$d(y_u, x_{s_l}) > \lambda_u > 0_I$$

for each $l \in \omega$. Therefore the recovery route does not converge to y_u . This contradicts the density of $\{x_n\}_{n \in \omega}$.

Exactly one of case I or case II must be true. However, both lead to contradictions. Therefore, we conclude that there is no trajectory with respect to which χ_F is first return recoverable, completing the proof. \square

The above theorem gives a general criterium for an ultrametric space which is sufficient to ensure the existence of a non-recoverable Baire class one function on that ultrametric. In the next chapter we examine a specific ultrametric space proposed by L. Nguyen Van The. We show that this space has the property required by Theorem 4.1. We also relate this space, and the question of recoverability, to the idea of richness introduced by F. Delon and B. Poizat.

CHAPTER 5

Connections to Other Topics

1. Rich and n -Rich I -valued Ultrametric Spaces

In order to investigate how the complexity of an I -valued ultrametric space affects recoverability, it is helpful to have a way to classify the complexity of the space. It would also be desirable to have canonical examples of I -valued ultrametrics at each level of complexity. One approach to such a classification was defined by F. Delon in [D] and reproduced in chapter 6 of B. Poizat's book [P], which acts as the primary source for the information in this chapter.

Poizat presented the following axioms for I -valued ultrametric spaces X with $i \in I$ and x, y , and x_j points in X .

$$A_0 : (\exists x)x = x$$

$$A_1 : (\forall i)(\forall x)(\exists y) d(x, y) = i$$

$$\vdots$$

$$A_n : (\forall i)(\forall x_1) \dots (\forall x_n)(\exists y) \bigwedge_{1 \leq \alpha < \beta < n} d(x_\alpha, x_\beta) = i \rightarrow \bigwedge_{1 \leq \alpha \leq n} d(x_\alpha, y) = i$$

The underlying idea of the axiom A_n is that every isosceles n -gon in X can be extended to an isosceles $(n + 1)$ -gon. With these axioms, Poizat made the following definition.

DEFINITION 5.1. (Poizat) An I -valued ultrametric is called rich if it satisfies axioms $A_0, A_1, \dots, A_n, \dots$. An I -valued ultrametric space is called n -rich, for $n \geq 2$, if it satisfies A_0, A_1, \dots, A_{n-1} and $\neg A_n$.

Rich I -valued ultrametric spaces can be thought of as having complicated sets of distances. This characterization is supported by the following Lemma, number 6.23 in [P].

LEMMA 5.2. (Poizat) *Every I -valued ultrametric space embeds into a rich I -valued space, and every I -valued ultrametric space with no $(n + 1)$ -gons embeds into an n -rich I -valued space.*

For a given I , Delon and Poizat defined the rich ultrametric space $E(I)$ by letting $E(I) = \{x \in \omega^I \mid x(t) = 0 \text{ at all but finitely many } t\}$. The metric on $E(I)$ is given by

$$d(x, y) = \begin{cases} \max\{i \in I \mid x(i) \neq y(i)\} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Delon showed that in cases such as ours, where I is countable, $E(I)$ is the unique prime rich I -valued ultrametric space. That is, $E(I)$ will embed into any model of a rich I -valued ultrametric space. Similarly, $E_n(I) = n^I$ together with the same metric produces the unique prime n -rich I -valued ultrametric space.

It is easy to see that in the case where I is countable $E(I)$ and $E_n(I)$ will also be countable. Since we have previously shown that all separable I -valued ultrametric spaces must have countable I 's, the $E(I)$ and $E_n(I)$ with which we might concern ourselves will necessarily be countable. Now recoverability in a countable space is trivial. We are therefore interested in the completion of these spaces. It should be noted that since every rich I -valued ultrametric contains a copy of $E(I)$, its completion will contain a copy of the completion of $E(I)$, which we call $\widehat{E(I)}$. The same holds true for $E_n(I)$ and $\widehat{E_n(I)}$.

It is clear that for a fixed n , the space $\widehat{E}_n(I)$ depends only on I . This gives us two obvious ways to vary the complexity of $\widehat{E}_n(I)$: by changing I or by changing n . The following table shows these two dimensions of complexity.

	I complexity \rightarrow			
richness	$\widehat{E}_2(I)$...	$\widehat{E}_2(J)$...
\downarrow	$\widehat{E}_3(I)$...	$\widehat{E}_3(J)$...
	\vdots		\vdots	
	$\widehat{E}(I)$...	$\widehat{E}(J)$...

In Theorem 3.3 and Theorem 4.1 we worked with both the complexity of I and the structure of the ultrametric space in order to develop criteria for recoverability and non-recoverability. While both types of complexity are involved in these theorems, we can still gain some understanding of the complexity of I required to produce a non-recoverable Baire class one function on an ultrametric space.

In the statement of Theorem 4.1, we require that the set of points whose only cutoff is 0_I be dense in the ultrametric. But if an ultrametric space has even one such point, then there is a subset of I of order type $(\omega^2 + 1)^*$. Indeed, if I did not have order type at least as complex as $(\omega^2 + 1)^*$, then we could find a single distance $r \in I \setminus \{0_I\}$ which is a cutoff for every point in any I -valued ultrametric space. Therefore, by Theorem 3.3, every Baire class one function on the ultrametric would be recoverable. However, this is not enough. In Example 3.7 we exhibit an ultrametric space in which I has order type $(\omega^2 + 1)^*$ but on which every Baire class one function is recoverable. Thus, recoverability is not simply a question of the complexity of I .

One of the goals of this chapter is to examine the role richness plays in determining the existence of a non-recoverable Baire class one function on an

ultrametric space. The idea of richness seems very attractive in this context as the definition of cutoffs deals with distances realized at given points. Furthermore, our work with chain isometries can be useful in determining if results in k -rich ultrametric spaces can be transferred to m -rich ultrametric spaces.

If we remove the requirement that n -rich I -valued ultrametric spaces not have $(n + 1)$ -gons, then $\widehat{E}_m(I)$ will be k -rich for all $k \leq m$. It is clear that under this modified definition, the n -rich I -valued ultrametric space $\widehat{E}_n(I)$ can be chain isometrically embedded into the $(n + 1)$ -rich I -valued ultrametric $\widehat{E}_{n+1}(I)$. Hence, applying Theorem 2.6, if we can show that there exists a non-recoverable Baire class one function on the k -rich ultrametric $\widehat{E}_k(I)$, then every $\widehat{E}_m(I)$ for $m \geq k$ must also admit a non-recoverable Baire class one function.

2. Richness and Non-recoverability

Lecomte gives an example of a non-recoverable Baire class one function on an ultrametric space in [Le]. However, the ultrametric space which he uses is more complicated than is required. In this section, we utilize Theorem 4.1 to prove that there exists a non-recoverable Baire class one function on the relatively simple rich ultrametric space $\widehat{E}(I_0)$ for a certain I_0 . We then show that in fact, utilizing this same I_0 , we can find a non-recoverable Baire class one function on the even simpler 2-rich ultrametric $\widehat{E}_2(I_0)$.

Much of the work of describing $\widehat{E}(I)$ was done by L. Nguyen Van The in [Li1] and [Li2]. Given a set of distances I , Nguyen Van The describes the rich I -valued ultrametric $E(I)$ as the set \mathbf{Q}_I of finitely supported elements of ω^I equipped with a distance d defined for $x, y \in \mathbf{Q}_I$ by

$$d(x, y) = \max\{s \in I \mid x(s) \neq y(s)\}.$$

The definition almost exactly matches that of $E(I)$ given by Delon and Poizat. To verify the richness of \mathbf{Q}_I , observe that given an $s_0 \in I$ we can find n distinct elements, x_1, \dots, x_n , of \mathbf{Q}_I which have distance s from each other by defining

$$x_i(s) = \begin{cases} 0 & \text{if } s \neq s_0, \\ i & \text{if } s = s_0. \end{cases}$$

The rational for using Nguyen Van The's definition of \mathbf{Q}_I instead of Delon and Poizat's space $E(I)$ is that the completion of the former space is more easily characterized. In [Li1], Nguyen Van The gives a characterization of this completion, but for a specific set I . We repeat the proposition and proof below in complete I -generality.

PROPOSITION 5.3. *The completion of \mathbf{Q}_I is the ultrametric space $\widehat{\mathbf{Q}}_I$ in which each point $x \in \omega^I$ has an underlying sequence $\{s_i\}_{i \in \omega} \subseteq I$ strictly decreasing to 0_I , such that:*

- (1) $x(s) = 0$ for $s > s_0$ in I .
- (2) x is constant on $I \cap (s_{i+1}, s_i]$.

The distance in $\widehat{\mathbf{Q}}_I$ is given by $d(x, y) = \inf\{s \in I \mid \forall t \in I (s < t \rightarrow x(t) = y(t))\}$.

PROOF. We will first show that \mathbf{Q}_I is dense in $\widehat{\mathbf{Q}}_I$. Let $x \in \widehat{\mathbf{Q}}_I$ have the associated sequence $\{s_i\}_{i \in \omega}$. For $n \in \omega$, let $x_n \in \mathbf{Q}_I$ be defined by $x_n(s) = x(s)$ whenever $s \geq s_n$ and $x_n(s) = 0$ otherwise. Then, in $\widehat{\mathbf{Q}}_I$, $d(x_n, x) \leq s_{n+1}$. But as the sequence $\{s_i\}_{i \in \omega}$ decreases to 0_I , the sequence $\{x_n\}_{n \in \omega} \subseteq \mathbf{Q}_I$ approaches x as $n \rightarrow \infty$. Therefore, \mathbf{Q}_I is dense in $\widehat{\mathbf{Q}}_I$.

The next task is to show that $\widehat{\mathbf{Q}}_I$ is complete. Let $\{x_n\}_{n \in \omega}$ be a Cauchy sequence in $\widehat{\mathbf{Q}}_I$. Note that given any $s \in I$, the sequence $\{x_n(s)\}_{n \in \omega}$ must be eventually constant as there exists an $N \in \omega$ for which $d(x_n, x_{n+1}) < s$ for all $n \geq N$. Define x by setting $x(s)$ equal to this constant value for each

$s \in I$. We claim that this x is in $\widehat{\mathbf{Q}}_I$. To verify this, we will check that x has an underlying sequence $\{s_i\}_{i \in \omega}$ which meets the criteria outlined in the proposition statement.

- (1) Let N be such that $d(x, x_n) < s_0$ for all $n \geq N$. Let $\{t_i\}_{i \in \omega}$ be the underlying sequence for x_N . Then, by condition (1), it must be the case that $x_N(s) = 0$ for all $s > t_0$ in I . But then, for any $s > s_0$ in I , $x(s)$ is constant and equal to $x_N(s)$. Thus, x fulfills condition (1) from the proposition statement.
- (2) To verify the second condition, it is enough to show that for any $s \in I$, there are $t < s < r$ all in I such that x is constant on $[t, s) \subset I$ and $(s, r] \subset I$. To do this, fix $t' < s$ in I and let $N \in \omega$ be such that for all $q, p \geq N$, $d(x_q, x_p) < t'$. Since x_N is in $\widehat{\mathbf{Q}}_I$, it fulfills condition (2). Thus, there are t and r in I such that $t' < t < s < r$ and x_N is constant on $(t, s] \subset I$ and $(s, r] \subset I$. But $x(s) = x_N(s)$ for all $s \geq t'$ in I , and thus is constant on the desired intervals.

Finally, note that for any $\varepsilon > 0_I$ and s with $0_I < s < \varepsilon$ in I , there is an N such that for all $q, p \geq N$, we have $d(x_q, x_p) < \varepsilon$. Therefore, for all $n \geq N$, $x_n(s) = x(s)$ for $s \geq \varepsilon$ in I . Hence, $d(x_n, x) < \varepsilon$ so that $x_n \rightarrow x$ as desired. Therefore, $\widehat{\mathbf{Q}}_I$ is complete. \square

Because of our requirement that the metric space be separable, the I we use in $\widehat{\mathbf{Q}}_I$ must be countable. As previously mentioned, this I must also contain a subset of order type $(\omega^2 + 1)^*$ if we wish to find a non-recoverable Baire class one function. To keep I as simple as possible while still allowing for points whose only cutoff is 0_I , we define I_0 to be a set of distances of exactly order type $(\omega^2 + 1)^*$. To show that there is a non-recoverable function on this particular $\widehat{\mathbf{Q}}_{I_0}$, we apply Theorem 4.1 as shown in the

following proposition.

PROPOSITION 5.4. *The space $\widehat{\mathbf{Q}}_{I_0}$ described above admits a non-recoverable Baire class one function.*

PROOF. We show that the set of bad points in $\widehat{\mathbf{Q}}_{I_0}$ is dense. Indeed, we claim that every point in $\widehat{\mathbf{Q}}_{I_0}$ is a bad point. Let $x \in \widehat{\mathbf{Q}}_{I_0}$ be given with underlying sequence $\{s_i\}_{i \in \omega}$. Note that for any $r \in I_0$, there is a $y \in \widehat{\mathbf{Q}}_{I_0}$ such that $d(x, y) = r$. To see this, observe that there is a minimum N such that $s_N > r$ in I_0 . Define y to be the point in $\widehat{\mathbf{Q}}_{I_0}$ such that

$$y(t) = \begin{cases} x(t) & \text{for } t > r \text{ in } I_0, \\ x(t) + 1 & \text{for } t \leq r \text{ in } I_0. \end{cases}$$

Note that y fulfills property (1) as $y(s) = 0$ for all $s > \max\{s_0, r\}$ in I_0 . Also, y has property (2) since $y(s)$ is either equal to $x(s)$ or $x(s) + 1$ and x has property (2). Therefore, y is in fact an element of $\widehat{\mathbf{Q}}_{I_0}$ as claimed. Furthermore,

$$d(x, y) = \min\{s \in I_0 \mid \forall t \in I_0 (s < t \rightarrow x(t) = y(t))\} = r.$$

Since this can be done for any $r \in I_0$, the set of distance realized at x is in fact all of I_0 which has order type $(\omega^2 + 1)^*$. Therefore, x is a bad point as desired.

But as every point in $\widehat{\mathbf{Q}}_{I_0}$ is a bad point, by Lemma 4.3 every point in $\widehat{\mathbf{Q}}_{I_0}$ has 0_{I_0} as its only cutoff. Therefore, as $\widehat{\mathbf{Q}}_{I_0}$ is a separable complete I_0 -valued ultrametric space, by Theorem 4.1 there exists a non-recoverable Baire class one function on $\widehat{\mathbf{Q}}_{I_0}$. \square

With this characterization of $E(I_0)$ and its completion $\widehat{E(I_0)}$, we are able to show that $\widehat{E(I_0)}$ admits a non-recoverable Baire class one function when I_0 has order type $(\omega^2 + 1)^*$. Thus, as $\widehat{E(I_0)}$ embeds into any separable

complete rich I_0 -valued ultrametric space in which $I_0 \subseteq I$, by Theorem 2.6, any separable complete rich I -valued ultrametric space in which I contains a subset of order type $(\omega^2 + 1)^*$ will admit a non-recoverable Baire class one function. But the question remains, is richness really required, and if not, is some level of n -richness enough? To answer this question, we modify Nguyen Van The's example to make the resulting space isometric to $\widehat{E_2(I_0)}$.

Define \mathbf{Q}_I^2 to be the set of finitely supported elements of 2^I equipped with the same distance used for \mathbf{Q}_I . Namely,

$$d(x, y) = \max\{s \in I \mid x(s) \neq y(s)\}.$$

Note that this definition matches that of $E_2(I)$. Also note that Nguyen Van The's characterization of the completion of \mathbf{Q}_I applies equally well to the completion of \mathbf{Q}_I^2 . That is, the completion of \mathbf{Q}_I^2 is the ultrametric space $\widehat{\mathbf{Q}_I^2}$ in which each point $x \in 2^I$ has an underlying sequence $\{s_i\}_{i \in \omega} \subseteq I$ strictly decreasing to 0_I , such that:

- (1) $x(s) = 0$ for $s > s_0$ in I .
- (2) x is constant on $I \cap (s_{i+1}, s_i]$

We also utilize the same distance for $\widehat{\mathbf{Q}_I^2}$. Namely, $d(x, y) = \inf\{s \in I \mid \forall t \in I (s < t \rightarrow x(t) = y(t))\}$.

While the same characterization of the completion $\widehat{\mathbf{Q}_I^2}$ works as did for the completion $\widehat{\mathbf{Q}_I}$, we do need to modify the proof that $\widehat{\mathbf{Q}_{I_0}^2}$ admits a non-recoverable Baire class one function slightly. If we interpret $x(t) + 1$ to mean boolean addition, so that $0 + 1 = 1$ and $1 + 1 = 0$, then this proof holds for $\widehat{\mathbf{Q}_{I_0}^2}$ as well. Thus, using the same argument given for rich I_0 -valued ultrametric spaces, we can find a non-recoverable Baire class one function in any 2-rich I -valued ultrametric in which I contains a subset of order type $(\omega^2 + 1)^*$.

Therefore, while the ideas behind richness appear related to the existence of a non-recoverable function, richness is not a determining factor. Even combining the least complex level of richness, together with the least complex order type for I allowing for the existence of a non-recoverable function yields an ultrametric space on which we can produce a non-recoverable Baire class one function.

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