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ASYMPTOTIC ANALYSIS OF THE NAVIER-STOKES EQUATIONS IN A CURVED DOMAIN WITH A NON-CHARACTERISTIC BOUNDARY

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Abstract. We consider the Navier-Stokes equations of an incompressible fluid in a three dimensional curved domain with permeable walls in the limit of small viscosity. Using a curvilinear coordinate system, adapted to the boundary, we construct a corrector function at order ε^j, $j = 0, 1$, where ε is the (small) viscosity parameter. This allows us to obtain an asymptotic expansion of the Navier-Stokes solution at order ε^j, $j = 0, 1$, for ε small . Using the asymptotic expansion, we prove that the Navier-Stokes solutions converge, as the viscosity parameter tends to zero, to the corresponding Euler solution in the natural energy norm. This work generalizes earlier results in [14] or [26], which discussed the case of a channel domain, while here the domain is curved.

1. **Introduction.** We aim to study the flow governed by the Navier-Stokes equations (NSE), in a general bounded (curved) domain in $\mathbb{R}^3$, when the boundary is not characteristic and the viscosity is small. Such flows occur, e.g., in certain devices proposed to reduce the drag of an airplane, by blowing and suction of air through the airfoils, minimizing thus the effects of turbulence.

From the mathematical point of view, we consider the following equations:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = f, \text{ in } \Omega \times (0,T), \\ \text{div } u^\varepsilon = 0, \text{ in } \Omega \times (0,T), \\ u^\varepsilon \big|_{t=0} = u_0, \text{ in } \Omega, \end{cases} \tag{1.1}$$

where ε is a small but strictly positive viscosity parameter, $T > 0$ is a fixed time, and f and u_0 are given smooth functions; see (1.8).

In our study, we consider a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ. Although we treat the more complicated case of space dimension 3, our results are

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also valid in space dimension 2 as all a priori estimates for the nonlinear terms in space dimension 3 are valid in space dimension 2 as well.

The boundary Γ of Ω consists of two components Γ_i, $i = O, I$, such that

$$\Gamma = \Gamma_O \cup \Gamma_I, \tag{1.2}$$

where each Γ_i is a simply connected, compact and smooth 2-manifold in $\mathbb{R}^3$. Here Γ_O and Γ_I respectively denote "outer" and "inner" boundaries of Ω.

On each Γ_i, $i = O, I$, we assume that the velocity u^ε is known, $u^\varepsilon|_{\Gamma_i} = U^i \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit normal vector to Γ, pointing outward of Ω, with $|U^i| > 0$, $i = O, I$. For the sake of simplicity, we take U^i constant on Γ_i, ($U^i > 0$ or $U^i < 0$), $i = O, I$. Then, the conservation of mass equation (1.1)$_2$ gives, by integrating over Ω and using the Stokes formula, that

$$\int_\Omega \text{div } u^\varepsilon d\Omega = \int_\Gamma u^\varepsilon \cdot \boldsymbol{n} d\Gamma = U^O |\Gamma_O| - U^I |\Gamma_I| = 0, \tag{1.3}$$

where $|\Gamma_i|$ is the measure of Γ_i, $i = O, I$.

Using (1.3), we supplement the Navier-Stokes equations, (1.1) with the following permeable boundary condition,

$$u^\varepsilon = \begin{cases} U^O \boldsymbol{n}, & \text{on} \quad \Gamma_O, \\ -U^I \boldsymbol{n}, & \text{on} \quad \Gamma_I, \end{cases} \tag{1.4}$$

where

$$U^O |\Gamma_O| - U^I |\Gamma_I| = 0, \ U^O, U^I > 0. \tag{1.5}$$

Hence Γ_O and Γ_I respectively correspond to the outgoing and entering parts of the flow; see Figure 1 which is drawn in space dimension two for clarity.

To make more physical sense of the problem, we assume that each of Γ_O and Γ_I is itself diffeomorphic to a torus, having no umbilical points, so that the domain Ω is diffeomorphic to a 3D channel. However, since the boundary layer is a highly local phenomenon, all the analysis in this article can easily extend to treat more general domains in $\mathbb{R}^3$, enclosed by finitely many compact and simply connected boundaries, which do not intersect with each other; see Remark 3.1. In addition, it is noteworthy that our analysis in this article remains valid for more general boundary conditions than (1.4) and (1.5) by taking U^O and U^I, satisfying (1.3), as any positive and smooth functions on Γ_O and Γ_I respectively.

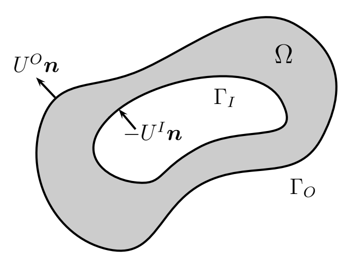

FIGURE 1. An annulus like domain $\Omega \subset \mathbb{R}^2$, with its smooth boundary $\Gamma = \Gamma_O \cup \Gamma_I$. The flow comes in across the "inner" boundary Γ_I with a constant velocity $U^I > 0$, and goes out across the "outer" boundary Γ_O with a constant velocity $U^O > 0$.

We impose a consistency condition on the initial data:

$$u_0 = U^O \boldsymbol{n}, \text{ on } \Gamma_O, \text{ and } -U^I \boldsymbol{n}, \text{ on } \Gamma_I; \tag{1.6}$$

concerning the compatibility issue of the initial data, see, e.g., [3, 4, 24].

By formally setting $\varepsilon = 0$ in the Navier-Stokes equations (1.1) with the non-characteristic boundary condition (1.4), and by using the so-called "upwind" boundary condition on Γ_I (where the flow enters in) as proposed in, e.g., [14] or [26], we obtain the Euler system:

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f, \text{ in } \Omega \times (0,T), \\ \text{div } u^0 = 0, \text{ in } \Omega \times (0,T), \\ u^0 \cdot \boldsymbol{n} = U^O \boldsymbol{n}, \text{ on } \Gamma_O \times (0,T), \\ u^0 = -U^I \boldsymbol{n}, \text{ on } \Gamma_I \times (0,T), \\ u^0\big|_{t=0} = u_0, \text{ in } \Omega. \end{cases} \tag{1.7}$$

For the sake of simplicity, we assume that the data are as regular as needed,

$$\Gamma \text{ is of class } C^\infty, \ u_0 \in C^\infty(\overline{\Omega}), \ f \in C^\infty([0,T] \times \Omega). \tag{1.8}$$

In particular, in view of the classical results on the three dimensional Navier-Stokes equations, (1.1) possesses a unique smooth solution on some interval of time $[0, T_*]$, for $T_* > 0$ sufficiently small.

Similarly for the Euler equations (1.7), in our case of a curved domain, we assume that the domain Ω is diffeomorphic to a 3D channel. For a 3D channel, the local existence of a unique regular solution of (1.7) is well-studied in [22], in relation with the methods of [1] and [27]. Hence we expect that the same results hold for (1.7) if the data are smooth enough as in (1.8), and that they can be proved by the same methods. Deferring this point to a future work, we assume here that if (1.8) holds, there exists a unique solution (u^0, p^0) of (1.7), on some interval of time $[0, T_*]$, satisfying

$$u^0 \in C^m([0,T_*] \times \Omega), \ \nabla p^0 \in C^{m-1}([0,T_*] \times \Omega), \tag{1.9}$$

for arbitrary m, as needed; reducing possibly T_*, this time will be the same for the Navier-Stokes and Euler equations.

Remark 1.1. The assumption (1.9) requires some compatibility conditions between the data, u_0, f, U^O and U^I as described in [24]. As shown in [24] for the Navier-Stokes equations, we expect these compatibility conditions to be nonlocal and hence non explicit.

The main goal of this article is to obtain an asymptotic expansion of u^ε, solution of (1.1) and (1.4) at order ε^j, $j = 0, 1$, with respect to the viscosity parameter ε, and to prove the convergence of u^ε to u^0, solution of (1.7) as ε goes to zero on some interval of time $[0, T]$, $0 < T \leq T_*$.

For the general theory of boundary layer analysis, see, for example, [8, 9, 11, 13, 16, 19, 21, 23, 32]. Concerning the boundary layer analysis related to the Navier-Stokes equations, we refer the readers to, e.g., [6, 7, 10, 12, 14, 15, 17, 20, 26, 29, 30, 31].

To study the asymptotic behavior of the Navier-Stokes solutions of (1.1) with the permeable boundary condition (1.4), we need to homogenize the problem (1.1), and, to do so, it is necessary to find a sufficiently smooth vector field U in Ω, which is divergence free, and satisfies the boundary condition (1.4). By finding such a smooth vector field U, in Section 2, we derive the homogenized Navier-Stokes and Euler systems, and state the main result of this article. To prove the main result, in Section 3, we first introduce a curvilinear coordinate system, adapted to the boundary Γ_O, where the boundary layers will occur. More precisely, following the approach in [12], we start from *the principal curvature coordinate system* on the boundary Γ_O and extend it along the normal direction inside of the domain to have a triply orthogonal system. Then, in Section 4, we discuss an asymptotic expansion, at order ε^0, of the Navier-Stokes solutions, and the next order expansion, at order ε^1, is considered in Section 5. Using the asymptotic expansion of the Navier-Stokes solutions, as stated in Theorem 2.1, we obtain the convergence result of the Navier-Stokes solutions to the Euler solution as the viscosity parameter tends to zero with explicit (optimal) convergence rates with respect to the viscosity.

2. **Homogenization of the problem and main result.** To homogenize the boundary conditions for the Navier-Stokes equations (1.1) with non-characteristic boundary condition (1.4), we need to find a smooth divergence free vector field that satisfies the boundary condition (1.4). In this direction, we consider the following problem:

Given smooth function $\mathcal{G} \in H^{m+3/2}(\Gamma)$, $m \geq -1$, find a smooth vector field U on Ω such that

$$\begin{cases} -\Delta U + \nabla \pi = 0, \text{ in } \Omega, \\ \operatorname{div} U = 0, \text{ in } \Omega, \\ U = \mathcal{G}\, \boldsymbol{n}, \text{ on } \Gamma, \end{cases} \tag{2.1}$$

where

$$\int_\Gamma \mathcal{G}\, dS = 0. \tag{2.2}$$

Under the assumption (1.8), thanks to Proposition 2.3 in [28] (see also [2]), there exists a unique solution (U, π) (π is unique up to an additive constant) of (2.1) such that

$$\|U\|_{H^{m+2}(\Omega)} + \|\pi\|_{H^{m+1}(\Omega)/\mathbb{R}} \leq c_0 \|\mathcal{G}\|_{H^{m+\frac{3}{2}}(\Gamma)}, \tag{2.3}$$

for a constant $c_0 = c_0(m, \Omega)$, independent of ε.

As an application, we choose

$$\mathcal{G} = U^O, \text{ on } \Gamma_O, \text{ and } -U^I, \text{ on } \Gamma_I. \tag{2.4}$$

Using (1.5), we see that $\mathcal{G}$ in (2.4), which is of class C^∞, satisfies (2.2). Hence, in particular, the solution U of (2.1) with (2.4) satisfies

$$\begin{cases} \operatorname{div} U = 0, \text{ in } \Omega, \\ U = U^O \boldsymbol{n}, \text{ on } \Gamma_O, \text{ and } -U^I \boldsymbol{n}, \text{ on } \Gamma_I. \end{cases} \tag{2.5}$$

Moreover, since U enjoys the estimate (2.3) for any $m \geq -1$, U is of class C^∞ in Ω.

Now, we set

$$v^\varepsilon := u^\varepsilon - U, \tag{2.6}$$

and homogenize the problem (1.1), (1.4):

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} - \varepsilon\Delta v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + (U\cdot\nabla)v^\varepsilon + (v^\varepsilon\cdot\nabla)U + \nabla p^\varepsilon \\ \qquad\qquad\qquad = f + \varepsilon\Delta U - (U\cdot\nabla)U, \text{ in } \Omega\times(0,T), \\ \operatorname{div} v^\varepsilon = 0, \text{ in } \Omega\times(0,T), \\ v^\varepsilon = 0, \text{ on } \Gamma\times(0,T), \\ v^\varepsilon\big|_{t=0} = v_0, \text{ in } \Omega. \end{cases} \tag{2.7}$$

where $v_0 := u_0 - U$ satisfies the compatibility condition (see (1.6)):

$$v_0 = 0, \text{ on } \partial\Omega. \tag{2.8}$$

Setting $v^0 := u^0 - U$ in (1.7), we find the corresponding limit problem of (2.7):

$$\begin{cases} \dfrac{\partial v^0}{\partial t} + (v^0\cdot\nabla)v^0 + (U\cdot\nabla)v^0 + (v^0\cdot\nabla)U + \nabla p^0 = f - (U\cdot\nabla)U, \text{ in } \Omega\times(0,T), \\ \operatorname{div} v^0 = 0, \text{ in } \Omega\times(0,T), \\ v^0\cdot\boldsymbol{n} = 0, \text{ on } \Gamma_O\times(0,T), \text{ (where the flow goes out)}, \\ v^0 = 0, \text{ on } \Gamma_I\times(0,T), \text{ (where the flow comes in)}, \\ v^0\big|_{t=0} = v_0, \text{ in } \Omega. \end{cases} \tag{2.9}$$

In what follows, we study the asymptotic behavior of v^ε, solution of (2.7) associated with its formal limit v^0, solution of (2.9), which is equivalent to the asymptotic behavior of u^ε associated with u^0, because

$$v^\varepsilon - v^0 = u^\varepsilon - u^0.$$

Now, using Θ^0, θ^j, $j = 0, 1$, and v^1, defined below in (4.12), (4.13), (5.4), (5.30) and (5.31), we state our main result:

Theorem 2.1. *Assuming that the data Γ, u_0 and f satisfy the regularity assumptions (1.6) and (1.8), there exists a time $0 < T^0 \,(\leq T)$, defined in (4.33), such that*

$$\begin{cases} \|v^\varepsilon - (v^0 + \theta^0)\|_{L^\infty(0,T^0;L^2(\Omega))} \leq \kappa\varepsilon, \\ \|v^\varepsilon - (v^0 + \theta^0)\|_{L^2(0,T^0;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}, \end{cases} \tag{2.10}$$

and

$$\begin{cases} \|v^\varepsilon - (v^0 + \Theta^0) - \varepsilon(v^1 + \theta^1)\|_{L^\infty(0,T^0;L^2(\Omega))} \leq \kappa\varepsilon^2, \\ \|v^\varepsilon - (v^0 + \Theta^0) - \varepsilon(v^1 + \theta^1)\|_{L^2(0,T^0;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{3}{2}}, \end{cases} \tag{2.11}$$

for a constant $\kappa = \kappa(\Omega, v_0, f, T^0) > 0$, independent of ε. Moreover, as the viscosity parameter ε tends to zero, the Navier-Stokes solutions v^ε of (2.7) converge to v^0, solution of (2.9) in the sense that

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T^0;L^2(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}. \tag{2.12}$$

3. **Curvilinear coordinate system and differential operators.** We let $\boldsymbol{x} = (x_1, x_2, x_3)$ denote the Cartesian coordinates of a point in $\mathbb{R}^3$.

As appearing in (1.2), the multi connected domain Ω in $\mathbb{R}^3$ is enclosed by $\Gamma = \Gamma_O \cup \Gamma_I$ which satisfies (1.8). Each of Γ_O and Γ_I is assumed to be diffeomorphic to a torus in $\mathbb{R}^3$ without any umbilical point. That is, at any point on Γ_O or Γ_I, the two principal curvatures are different; see Remark 3.1.

As we will see below, due to the choice of the boundary condition (1.4), boundary layers at small viscosity, associated with the problems (1.1) and (1.7), will occur only near Γ_O. Hence, from this point, we mainly focus on constructing a curvilinear coordinate system adapted to Γ_O. However, a curvilinear system near Γ_I can be constructed in the same manner.

Since Γ_O is a topological torus with no umbilical points, thanks to Lemma 3.6.6 of [18], we can construct a curvilinear system globally on Γ_O in which the metric tensor is diagonal and the coordinate lines at each point are parallel to the principal directions. Such a coordinate system is called *the principal curvature coordinate system*. More precisely, using (1.8), we define a C^{m+2} map $\widetilde{\psi}$, $m \geq 0$, $\widetilde{\psi} : \Gamma_{O,\xi'} \to \Gamma_O$, such that *the principal curvature coordinates* $\xi' = (\xi_1, \xi_2)$ in $\Gamma_{O,\xi'}$, for $\widetilde{\boldsymbol{x}} \in \Gamma_O$, satisfies

$$\widetilde{\boldsymbol{x}} = \widetilde{\psi}(\xi'), \quad \xi' = (\xi_1, \xi_2) \in \Gamma_{O,\xi'}. \tag{3.1}$$

Here $\Gamma_{O,\xi'}$, which is a bounded set in $\mathbb{R}^2_{\xi'}$ with respect to the ξ' variables, denotes the preimage of $\Gamma_O \subset \mathbb{R}^3$ via $\widetilde{\psi}^{-1}$.

Remark 3.1. The assumption on Γ_O (or Γ_I), being diffeomorphic to a torus, is due to the physical relevance of the problem (1.1) with (1.4). Then, by assuming Γ_O has no umbilical points as well, *the principal curvature coordinate system* is globally defined on Γ_O. However, from the mathematical point of view, these assumptions can be omitted. In general, on any compact and simply connected surface Γ in $\mathbb{R}^3$, one can construct an atlas of finitely many charts. Then, using such an atlas, all the analysis in this article can be performed as well.

Differentiating (3.1) with respect to the ξ_i variables, $i = 1, 2$, we obtain the covariant basis on $\Gamma_{O,\xi'}$ and the metric tensor:

$$\widetilde{\boldsymbol{g}}_i(\xi') := \frac{\partial \widetilde{\boldsymbol{x}}}{\partial \xi_i}, \quad i = 1, 2, \tag{3.2}$$

and

$$\big(\widetilde{g}_{ij}(\xi')\big)_{1\leq i,j\leq 2} := \big(\widetilde{\boldsymbol{g}}_i \cdot \widetilde{\boldsymbol{g}}_j\big)_{1\leq i,j\leq 2} = \operatorname{diag}\big(\widetilde{\boldsymbol{g}}_1 \cdot \widetilde{\boldsymbol{g}}_1,\ \widetilde{\boldsymbol{g}}_2 \cdot \widetilde{\boldsymbol{g}}_2\big). \tag{3.3}$$

The determinant of the metric tensor is strictly positive;

$$\widetilde{g}(\xi') := \det(\widetilde{g}_{ij}) > 0, \text{ for all } \xi' \text{ in the closure of } \Gamma_{O,\xi'}. \tag{3.4}$$

Denoting by $\Omega^O_{3\delta}$ the interior tubular neighborhood, near Γ_O, of Ω with width 3δ for a (fixed) sufficiently small $\delta > 0$, we globally define the coordinate ξ_3 on $\Omega^O_{3\delta}$ to be distance from the boundary Γ_O, with positive distances directed inward.

We choose the orientation of the ξ' variables on Γ_O so that

$$\boldsymbol{n}(\xi') := -\frac{\widetilde{\boldsymbol{g}}_1 \times \widetilde{\boldsymbol{g}}_2}{|\widetilde{\boldsymbol{g}}_1 \times \widetilde{\boldsymbol{g}}_2|}(\widetilde{\psi}(\xi')), \tag{3.5}$$

where $\boldsymbol{n}(\xi')$ is the unit outer normal vector on Γ_O. Then, setting $\Omega^O_{3\delta,\boldsymbol{\xi}} = \Gamma_{O,\xi'} \times (0, 3\delta)$, we define $\psi\colon \Omega^O_{3\delta,\boldsymbol{\xi}} \to \Omega^O_{3\delta}$ such that, for $\boldsymbol{x} = (x_1, x_2, x_3) \in \Omega^O_{3\delta}$,

$$\boldsymbol{x} = \psi(\boldsymbol{\xi}) = \widetilde{\psi}(\xi') - \xi_3 \boldsymbol{n}(\xi'), \quad \boldsymbol{\xi} \in \Omega^O_{3\delta,\boldsymbol{\xi}}. \tag{3.6}$$

By differentiating ψ in the $\boldsymbol{\xi}$ variables and using (3.2), we find the covariant basis of the curvilinear system $\boldsymbol{\xi}$:

$$\boldsymbol{g}_i(\boldsymbol{\xi}) = \widetilde{\boldsymbol{g}}_i(\xi') - \xi_3 \frac{\partial \boldsymbol{n}}{\partial \xi_i}(\xi'), \; i = 1, 2, \quad \boldsymbol{g}_3(\boldsymbol{\xi}) = -\boldsymbol{n}(\xi'). \tag{3.7}$$

Hence, from (3.5) and (3.7), we see that the covariant basis has a positive orientation.

For *the principal curvature coordinate system*, we have

$$\frac{\partial \boldsymbol{n}}{\partial \xi_i} = \kappa_i(\xi')\widetilde{\boldsymbol{g}}_i, \quad i = 1, 2, \tag{3.8}$$

where $\kappa_i(\xi')$ is the principal curvature in the principal direction $\widetilde{\boldsymbol{g}}_i$, $i = 1, 2$. Thus, using (3.3), (3.7) and (3.8), we write the metric tensor of $\boldsymbol{\xi}$:

$$\begin{aligned}(g_{ij})_{1\le i,j\le 3} &:= (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)_{1\le i,j\le 3} \\ &= \begin{pmatrix} \big(1-\kappa_1(\xi')\xi_3\big)^2\,\widetilde{g}_{11}(\xi') & 0 & 0 \\ 0 & \big(1-\kappa_2(\xi')\xi_3\big)^2\,\widetilde{g}_{22}(\xi') & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned} \tag{3.9}$$

Thanks to (3.4), by choosing the thickness $3\delta > 0$ of the tubular neighborhood $\Omega^O_{3\delta}$ small enough, we obtain that

$$g(\boldsymbol{\xi}) := \det(g_{ij})_{1\le i,j\le 3} > 0 \text{ for all } \boldsymbol{\xi} \text{ in the closure of } \Omega^O_{3\delta,\boldsymbol{\xi}} = \Gamma_{O,\xi'} \times (0, 3\delta). \tag{3.10}$$

The matrix of the contravariant metric components are defined in the closure of $\Omega^O_{3\delta,\boldsymbol{\xi}}$ as well:

$$(g^{ij})_{1\le i,j\le 3} = (g_{ij})^{-1}_{1\le i,j\le 3} = \begin{pmatrix} 1/g_{11} & 0 & 0 \\ 0 & 1/g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

We introduce the normalized covariant vectors:

$$\boldsymbol{e}_i = \frac{\boldsymbol{g}_i}{|\boldsymbol{g}_i|}, \; 1 \le i \le 3, \tag{3.12}$$

and set

$$h_i(\boldsymbol{\xi}) = \sqrt{g_{ii}}, \; i = 1, 2, \qquad h(\boldsymbol{\xi}) = \sqrt{g}; \tag{3.13}$$

the function $h(\boldsymbol{\xi}) > 0$ is the magnitude of the Jacobian determinant of ψ.

For a scalar function ϕ, defined in $\Omega^O_{3\delta,\boldsymbol{\xi}}$, we write the gradient of ϕ in the $\boldsymbol{\xi}$ variables:

$$\nabla\phi = \sum_{i=1}^{2} \frac{1}{h_i}\frac{\partial \phi}{\partial \xi_i}\boldsymbol{e}_i + \frac{\partial \phi}{\partial \xi_3}\boldsymbol{e}_3. \tag{3.14}$$

For a vector valued function F, defined in $\Omega^O_{3\delta,\boldsymbol{\xi}}$, in the form

$$F = \sum_{i=1}^{3} F^i(\boldsymbol{\xi})\boldsymbol{e}_i,$$

one can classically express the divergence operator acting on F in the $\boldsymbol{\xi}$ variables (see [5] or [18]) as

$$\text{div } F = \frac{1}{h}\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i}\Big(\frac{h}{h_i}F^i\Big) + \frac{1}{h}\frac{\partial (hF^3)}{\partial \xi_3}. \tag{3.15}$$

The Laplacian of F is given in the form,

$$\Delta F = \sum_{i=1}^{3} \Big(\mathcal{S}^i F + \mathcal{L}^i F^i + \frac{\partial^2 F^i}{\partial \xi_3^2} \Big) \boldsymbol{e}_i, \tag{3.16}$$

where

$$\begin{cases} \mathcal{S}^i F = \Big(\begin{array}{l} \text{linear combination of tangential derivatives} \\ \text{of } F^j,\ 1 \le j \le 3, \text{ in } \xi', \text{ up to order 2} \end{array} \Big), \\ \mathcal{L}^i F^i = \Big(\text{proportional to } \dfrac{\partial F^i}{\partial \xi_3} \Big). \end{cases} \tag{3.17}$$

Remark 3.2. Note that the coefficients of $\mathcal{S}^i$ and $\mathcal{L}^i$, $1 \le i \le 3$ in (3.17), are multiples of h , $1/h$, h_i , $1/h_i$, $i = 1, 2$ and their derivatives. Thanks to (3.10), all these quantities are well-defined because of the regularity assumption (1.8) on Γ. The explicit expression of (3.16) appears in, e.g., [10].

For smooth vector fields F, $G : \Omega^{\mathcal{O}}_{3\delta, \boldsymbol{\xi}} \to \mathbb{R}^3$, we consider $\nabla_F G$, the covariant derivative of G in the direction F, which gives $F \cdot \nabla G$ in the Cartesian coordinate system. More precisely, let us consider the smooth functions F and G in the form

$$F = \sum_{i=1}^{3} F^i(\boldsymbol{\xi}) \boldsymbol{e}_i, \qquad G = \sum_{i=1}^{3} G^i(\boldsymbol{\xi}) \boldsymbol{e}_i.$$

Then, one can write $\nabla_F G$ in the $\boldsymbol{\xi}$ variables as follows:

$$\nabla_F G = \sum_{i=1}^{3} \Big\{ \mathcal{P}^i(F, G) + F^3 \frac{\partial G^i}{\partial \xi_3} + \mathcal{Q}^i(F, G) + \mathcal{R}^i(F, G) \Big\} \boldsymbol{e}_i, \tag{3.18}$$

where

$$\mathcal{P}^i(F, G) = \sum_{j=1}^{2} \frac{1}{h_j} F^j \frac{\partial G^i}{\partial \xi_j},\ 1 \le i \le 3, \tag{3.19}$$

$$\mathcal{Q}^i(F, G) = \begin{cases} \dfrac{1}{h_1 h_2} \Big(\dfrac{\partial h_i}{\partial \xi_{3-i}} F^i - \dfrac{\partial h_{3-i}}{\partial \xi_i} F^{3-i} \Big) G^{3-i},\ i = 1, 2, \\ -\displaystyle\sum_{j=1}^{2} \frac{1}{h_j} \frac{\partial h_j}{\partial \xi_3} F^j G^j,\ i = 3, \end{cases} \tag{3.20}$$

$$\mathcal{R}^i(F, G) = \frac{1}{h_i} \frac{\partial h_i}{\partial \xi_3} F^i G^3,\ i = 1, 2, \qquad \mathcal{R}^3(F, G) = 0. \tag{3.21}$$

Remark 3.3. $\mathcal{Q}^i(F, G)$ and $\mathcal{R}^i(F, G)$, $1 \le i \le 3$, are related to the Christoffel symbols of the second kind, which reflect the twisting effects of the curvilinear system $\boldsymbol{\xi}$.

4. First order asymptotic expansion at order ε^0.

To study the asymptotic behavior of v^ε, solution of (2.7), as ε tends to zero, we propose the following expansion of v^ε:

$$v^\varepsilon \simeq v^0 + \Theta^0, \tag{4.1}$$

where v^0 is the solution of the limit problem (2.9), and Θ^0 is a divergence free corrector which corrects the difference, $v^\varepsilon - v^0$, on the boundary Γ_O; it will be constructed below in the form:

$$\Theta^0 := \sum_{i=1}^{3} \Theta^{0,\,i} \boldsymbol{e}_i. \tag{4.2}$$

We insert $\Theta^0 \simeq v^\varepsilon - v^0$ in the difference of equations (2.7) and (2.9). Then, by omitting the pressures and lower order terms with respect to ε, we find

$$\frac{\partial \Theta^0}{\partial t} - \varepsilon \Delta \Theta^0 + (U\cdot\nabla)\Theta^0 + (\Theta^0\cdot\nabla)U + (v^0\cdot\nabla)\Theta^0 + (\Theta^0\cdot\nabla)v^0 + (\Theta^0\cdot\nabla)\Theta^0 \simeq 0. \tag{4.3}$$

Considering the boundary layer that arises near Γ_O, and using the Prandtl theory, we find that the stretched variable associated with (4.3) should be $\overline{\xi}_3 = \varepsilon^{-1}\xi_3$ in $\Omega^O_{3\delta}$, that is, the thickness of the boundary layer is of order ε. Hence, by using the expressions of the Laplacian and convective terms in (3.16) and (3.18), we collect all the leading order terms with respect to ε in (4.3), and obtain the following equation of the corrector Θ^0, defined in (4.2):

$$-\varepsilon \frac{\partial^2 \Theta^{0,\,i}}{\partial \xi_3^2} - U^O \frac{\partial \Theta^{0,\,i}}{\partial \xi_3} = 0, \ \ 1 \le i \le 3. \tag{4.4}$$

Here, thanks to the Taylor expansion in ξ_3, at $\xi_3 = 0$, of $U = \sum_{i=1}^3 U^i \boldsymbol{e}_i$ in $\Omega^O_{3\delta,\,\boldsymbol{\xi}}$, and the fact that $\boldsymbol{n} = -\boldsymbol{e}_3$ on Γ_O, we used the following expression of U^3:

$$U^3(\boldsymbol{\xi}) \simeq -U^O + \frac{\partial U^3}{\partial \xi_3}(\xi', 0)\, \xi_3, \ \text{ in } \Omega^O_{3\delta,\,\boldsymbol{\xi}}. \tag{4.5}$$

Hence (4.4) is the proposed equation of the corrector Θ^0.

On the other hand, as the main requirement of the corrector Θ^0 is to balance the difference $v^\varepsilon - v^0$ on the outer boundary Γ_O, we use (2.7) and (2.9), and find the boundary condition of Θ^0:

$$\Theta^0\big|_{\xi_3=0} = -\sum_{i=1}^{2} \left(v^0 \cdot \boldsymbol{e}_i\right)|_{\xi_3=0}\, \boldsymbol{e}_i|_{\xi_3=0}. \tag{4.6}$$

We define a cut-off function $\sigma = \sigma(\xi_3)$, which belongs to $\mathcal{C}^\infty(\mathbb{R}_+)$, such that

$$\sigma(\xi_3) = \begin{cases} 1, & 0 \le \xi_3 \le \delta, \\ 0, & \xi_3 \ge 2\delta. \end{cases} \tag{4.7}$$

Now, to define Θ^0, we first notice that the exponentially decaying function $e^{-U^O \xi_3/\varepsilon}$ satisfies the equation (4.4). Then, using the cut-off function $\sigma(\xi_3)$ in (4.7), and letting

$$\widetilde{v}^{0,\,i}(\xi'; t) := \frac{h}{h_i}\Big|_{\xi_3=0} \left(v^0 \cdot \boldsymbol{e}_i\right)|_{\xi_3=0}, \tag{4.8}$$

we define the tangential components $\Theta^{0,\,i}$ of Θ^0:

$$\Theta^{0,\,i}(\boldsymbol{\xi}; t) := -\varepsilon \frac{1}{U^O}\, \widetilde{v}^{0,\,i}(\xi'; t)\, \frac{h_i}{h}(\boldsymbol{\xi}) \frac{\partial}{\partial \xi_3}\Big(\sigma(\xi_3)\big(1 - e^{-U^O \xi_3/\varepsilon}\big)\Big), \ i = 1, 2. \tag{4.9}$$

By enforcing the divergence free condition on Θ^0 and using (3.15), we define the normal component $\Theta^{0,\,3}$ of Θ^0 in the form:

$$\Theta^{0,\,3}(\boldsymbol{\xi}; t) := \varepsilon \frac{1}{U^O}\Big(\sum_{i=1}^{2} \frac{\partial \widetilde{v}^{0,\,i}}{\partial \xi_i}(\xi'; t)\Big) \frac{1}{h}(\boldsymbol{\xi}) \sigma(\xi_3)\big(1 - e^{-U^O \xi_3/\varepsilon}\big). \tag{4.10}$$

The expression of Θ^0 (or Θ^1 in (4.9) and (5.27)) is a modification of the corrector function in [12], and originated from that of [26].

It is easy to see that the divergence free corrector Θ^0, defined by (4.2), (4.9) and (4.10), satisfies the desired boundary condition (4.6). Moreover, due to the presence of σ in (4.9) and (4.10), we notice that

$$\frac{\partial^{k+l}\Theta^0}{\partial t^k \partial \xi_i^l}\Big|_{\xi_3 \geq 2\delta} = 0, \ 1 \leq i \leq 3, \ k, l \geq 0. \tag{4.11}$$

We observe, since σ' vanishes for $0 \leq \xi_3 \leq \delta$, that $\sigma'(\xi_3)e^{-U^O \xi_3/\varepsilon}$ is an *e.s.t.* in all of Ω, and 0 on Γ_O, where *e.s.t.* stands for a function (or a constant) whose norm in all Sobolev spaces H^s (and thus spaces $\mathcal{C}^s$) is exponentially small with a bound of the form $c_1 \exp(-c_2/\varepsilon^\gamma)$, $c_1, c_2, \gamma > 0$, for each s. We then write each component $\Theta^{0,\,i}$, $1 \leq i \leq 3$ in (4.9) and (4.10), in the form:

$$\Theta^{0,\,i}(\boldsymbol{\xi}; t) = \theta^{0,\,i} + \varepsilon \varphi^{0,\,i} + e.s.t., \tag{4.12}$$

where

$$\theta^{0,\,i}(\boldsymbol{\xi}; t) = \begin{cases} -\widetilde{v}^{0,\,i}(\xi'; t)\dfrac{h_i}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-U^O \xi_3/\varepsilon}, \ i = 1, 2, \\ -\varepsilon \dfrac{1}{U^O}\Big(\displaystyle\sum_{i=1}^{2} \frac{\partial \widetilde{v}^{0,\,i}}{\partial \xi_i}(\xi'; t)\Big)\dfrac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-U^O \xi_3/\varepsilon}, \ i = 3, \end{cases} \tag{4.13}$$

and

$$\varphi^{0,\,i}(\boldsymbol{\xi}; t) = \begin{cases} -\dfrac{1}{U^O}\,\widetilde{v}^{0,\,i}(\xi'; t)\dfrac{h_i}{h}(\boldsymbol{\xi})\sigma'(\xi_3), \ i = 1, 2, \\ \dfrac{1}{U^O}\Big(\displaystyle\sum_{i=1}^{2} \frac{\partial \widetilde{v}^{0,\,i}}{\partial \xi_i}(\xi'; t)\Big)\dfrac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3), \ i = 3. \end{cases} \tag{4.14}$$

Using again the fact that σ' vanishes for $0 \leq \xi_3 \leq \delta$, one can easily verify the following estimates on $\theta^0 = \sum_{i=1}^3 \theta^{0,\,i}\boldsymbol{e}_i$ and $\varphi^0 = \sum_{i=1}^3 \varphi^{0,\,i}\boldsymbol{e}_i$: For $r, k, l, m \geq 0$, and $i = 1, 2$,

$$\begin{cases} \Big\| \Big(\dfrac{\xi_3}{\varepsilon}\Big)^r \dfrac{\partial^{k+l+m}\theta^{0,\,i}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \Big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon^{\frac{1}{2}-m}, \\ \Big\| \Big(\dfrac{\xi_3}{\varepsilon}\Big)^r \dfrac{\partial^{k+l+m}\theta^{0,\,3}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \Big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon^{\frac{3}{2}-m}, \end{cases} \tag{4.15}$$

$$\begin{cases} \Big\| \Big(\dfrac{\xi_3}{\varepsilon}\Big)^r \dfrac{\partial^{k+l+m}\theta^{0,\,i}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \Big\|_{L^\infty([0,T]\times\Omega)} \leq \kappa \varepsilon^{-m}, \\ \Big\| \Big(\dfrac{\xi_3}{\varepsilon}\Big)^r \dfrac{\partial^{k+l+m}\theta^{0,\,3}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \Big\|_{L^\infty([0,T]\times\Omega)} \leq \kappa \varepsilon^{1-m}, \end{cases} \tag{4.16}$$

and

$$\Big\| \frac{\partial^{k+l+m}\varphi^{0,\,j}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \Big\|_{L^\infty([0,T]\times\Omega)} \leq \kappa, \quad 1 \leq j \leq 3, \tag{4.17}$$

where $\partial^k/\partial \boldsymbol{\tau}^k$ denotes any tangential derivative in ξ' of order $k \geq 0$. Here and throughout this article, we call κ a positive constant depending on the data, Γ, v_0, f and T, but independent of ε. Note that κ may be different at different places.

Performing direct computations, one can verify that $\theta^{0,\,i}$, $1 \le i \le 3$, satisfies the proposed equation (4.4) up to a small error:

$$\begin{cases} -\varepsilon \dfrac{\partial^2 \theta^{0,\,i}}{\partial \xi_3^2} - U^O \dfrac{\partial \theta^{0,\,i}}{\partial \xi_3} = E^{0,\,i}(\boldsymbol{\xi}; t)\, \sigma(\xi_3)\, e^{-U^O \xi_3/\varepsilon} + e.s.t., \quad 1 \le i \le 2, \\ -\varepsilon \dfrac{\partial^2 \theta^{0,\,3}}{\partial \xi_3^2} - U^O \dfrac{\partial \theta^{0,\,3}}{\partial \xi_3} = \varepsilon E^{0,\,3}(\boldsymbol{\xi}; t)\, \sigma(\xi_3)\, e^{-U^O \xi_3/\varepsilon} + e.s.t., \end{cases} \tag{4.18}$$

where

$$\begin{cases} E^{0,\,i}(\boldsymbol{\xi}; t) := -U^O\, \widetilde{v}^{0,\,i} \dfrac{\partial}{\partial \xi_3}\Big(\dfrac{h_i}{h}\Big) + \varepsilon\, \widetilde{v}^{0,\,i} \dfrac{\partial^2}{\partial \xi_3^2}\Big(\dfrac{h_i}{h}\Big), \ i = 1, 2, \\ E^{0,\,3}(\boldsymbol{\xi}; t) := -\Big(\displaystyle\sum_{i=1}^{2} \dfrac{\partial \widetilde{v}^{0,\,i}}{\partial \xi_i}\Big) \dfrac{\partial}{\partial \xi_3}\Big(\dfrac{1}{h}\Big) + \varepsilon \dfrac{1}{U^O}\Big(\displaystyle\sum_{i=1}^{2} \dfrac{\partial \widetilde{v}^{0,\,i}}{\partial \xi_i}\Big) \dfrac{\partial^2}{\partial \xi_3^2}\Big(\dfrac{1}{h}\Big). \end{cases} \tag{4.19}$$

4.1. **Proof of Theorem 2.1 at order ε^0.** We define the remainder w_ε^0,

$$w_\varepsilon^0 := v^\varepsilon - v^0 - \Theta^0. \tag{4.20}$$

Since Θ^0 is divergence free, using (2.8), (2.7), (2.9), (4.6), the equations satisfied by w_ε^0 read:

$$\begin{cases} \dfrac{\partial w_\varepsilon^0}{\partial t} - \varepsilon \Delta w_\varepsilon^0 + (U \cdot \nabla) w_\varepsilon^0 + (w_\varepsilon^0 \cdot \nabla) U + \nabla(p^\varepsilon - p^0) \\ \qquad\qquad = \varepsilon \Delta v^0 + \varepsilon \Delta U + R_\varepsilon^0(\Theta^0) - J_\varepsilon^0(v^\varepsilon, v^0), \ \text{in } \Omega \times (0,T), \\ \operatorname{div}\, w_\varepsilon^0 = 0, \ \text{in } \Omega \times (0,T), \\ w_\varepsilon^0 = 0, \ \text{on } \Gamma \times (0,T), \\ w_\varepsilon^0\big|_{t=0} = 0, \ \text{in } \Omega. \end{cases} \tag{4.21}$$

Here, for any (smooth) vector field $G : \Omega \times (0,T) \to \mathbb{R}^3$,

$$R_\varepsilon^0(G) := -\frac{\partial G}{\partial t} + \varepsilon \Delta G - (U \cdot \nabla) G - (G \cdot \nabla) U, \tag{4.22}$$

and

$$J_\varepsilon^0(v^\varepsilon, v^0) := (v^\varepsilon \cdot \nabla) v^\varepsilon - (v^0 \cdot \nabla) v^0. \tag{4.23}$$

Using the linearity of $R_\varepsilon^0(\cdot)$, (4.12) and (4.17), we have, pointwise:

$$|R_\varepsilon^0(\Theta^0)| \le |R_\varepsilon^0(\theta^0)| + \kappa\varepsilon.$$

Thus, by multiplying $(4.21)_1$ by w_ε^0, integrating over Ω, and then using the Schwarz and Young inequalities, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_\varepsilon^0\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla w_\varepsilon^0\|_{L^2(\Omega)}^2 \\ &\qquad \le \kappa\varepsilon^2 + \big\|R_\varepsilon^0(\theta^0) \cdot w_\varepsilon^0\big\|_{L^1(\Omega)} + \big\|J_\varepsilon^0(v^\varepsilon, v^0) \cdot w_\varepsilon^0\big\|_{L^1(\Omega)} + \kappa \|w_\varepsilon^0\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.24}$$

To estimate the second term on the right hand side of (4.24), we use (4.13) and the Hardy inequality with $(4.21)_3$, and we find

$$\begin{aligned} \big\|R_\varepsilon^0(\theta^0) \cdot w_\varepsilon^0\big\|_{L^1(\Omega)} &\le \|\xi_3 R_\varepsilon^0(\theta^0)\|_{L^2(\Omega_{2\delta}^O)} \|\xi_3^{-1} w_\varepsilon^0\|_{L^2(\Omega_{2\delta}^O)} \\ &\le \|\xi_3 R_\varepsilon^0(\theta^0)\|_{L^2(\Omega_{2\delta}^O)} \|\nabla w_\varepsilon^0\|_{L^2(\Omega)} \\ &\le \kappa\varepsilon \|\varepsilon^{-1} \xi_3 R_\varepsilon^0(\theta^0)\|_{L^2(\Omega_{2\delta}^O)}^2 + \frac{1}{10}\varepsilon \|\nabla w_\varepsilon^0\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.25}$$

Here, thanks to (4.13), we use the fact that θ^0 or any derivative of θ^0 vanishes outside of $\Omega^O_{2\delta}$. Using (3.16), (3.18) and (4.15), we find that

$$\|\varepsilon^{-1}\xi_3 R^0_\varepsilon(\theta^0)\|^2_{L^2(\Omega^O_{2\delta})} \le \kappa\varepsilon + \sum_{i=1}^{3} \Big\| \frac{\xi_3}{\varepsilon}\Big(\varepsilon \frac{\partial^2 \theta^{0,\,i}}{\partial \xi_3^2} - U^3(\boldsymbol{\xi})\frac{\partial \theta^{0,\,i}}{\partial \xi_3}\Big)\Big\|^2_{L^2(\Omega)},$$

and, using (4.5), (4.15), (4.18) and (4.19), we see that

$$\sum_{i=1}^{3} \Big\| \frac{\xi_3}{\varepsilon}\Big(\varepsilon \frac{\partial^2 \theta^{0,\,i}}{\partial \xi_3^2} - U^3(\boldsymbol{\xi})\frac{\partial \theta^{0,\,i}}{\partial \xi_3}\Big)\Big\|^2_{L^2(\Omega)} \le \kappa\varepsilon.$$

Hence, from these bounds, (4.25) yields

$$\big\|R^0_\varepsilon(\theta^0)\cdot w^0_\varepsilon\big\|_{L^1(\Omega)} \le \kappa\varepsilon^2 + \frac{1}{10}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}, \tag{4.26}$$

and, combining (4.24) and (4.26), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w^0_\varepsilon\|^2_{L^2(\Omega)} + \frac{9}{10}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} \le \kappa\varepsilon^2 + \big\|J^0_\varepsilon(v^\varepsilon, v^0)\cdot w^0_\varepsilon\big\|_{L^1(\Omega)} + \kappa\|w^0_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.27}$$

To estimate the second term in the right hand side of (4.27), using (4.20), we notice that

$$J^0_\varepsilon(v^\varepsilon, v^0) = (v^\varepsilon\cdot\nabla)w^0_\varepsilon + (w^0_\varepsilon\cdot\nabla)v^0 + (w^0_\varepsilon\cdot\nabla)\Theta^0 + (\Theta^0\cdot\nabla)v^0 + (v^0\cdot\nabla)\Theta^0 + (\Theta^0\cdot\nabla)\Theta^0,$$

and find that

$$\big\|J^0_\varepsilon(v^\varepsilon, v^0)\cdot w^0_\varepsilon\big\|_{L^1(\Omega)} \le \sum_{j=1}^{5} \mathcal{J}^{0,\,j} \tag{4.28}$$

where

$$\begin{cases} \mathcal{J}^{0,\,1} := \|(w^0_\varepsilon\cdot\nabla)v^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}, \\ \mathcal{J}^{0,\,2} := \|(w^0_\varepsilon\cdot\nabla)\Theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}, \\ \mathcal{J}^{0,\,3} := \|(\Theta^0\cdot\nabla)v^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}, \\ \mathcal{J}^{0,\,4} := \|(v^0\cdot\nabla)\Theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}, \\ \mathcal{J}^{0,\,5} := \|(\Theta^0\cdot\nabla)\Theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}. \end{cases} \tag{4.29}$$

The term $\mathcal{J}^{0,\,1}$ is easy to estimate:

$$\mathcal{J}^{0,\,1} \le \kappa\|w^0_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.30}$$

To estimate $\mathcal{J}^{0,\,2}$, using (4.12) and (4.17), we write

$$\mathcal{J}^{0,\,2} \le \|(w^0_\varepsilon\cdot\nabla)\theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon\|w^0_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.31}$$

Then, using (3.18) with (3.19), (3.20) and (3.21), and using (4.16), we find

$$\begin{aligned}\|(w^0_\varepsilon \cdot \nabla)\theta^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} &\leq \kappa \|w^0_\varepsilon\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \Big\| \Big(w^{0,\,3}_\varepsilon \frac{\partial \theta^{0,\,i}}{\partial \xi_3} \Big) \cdot w^{0,\,i}_\varepsilon \Big\|_{L^1(\Omega)} \\ &\leq \kappa \|w^0_\varepsilon\|^2_{L^2(\Omega)} + 2\varepsilon^2 \sum_{i=1}^{2} \Big\| \Big(\frac{\xi_3}{\varepsilon}\Big)^2 \frac{\partial \theta^{0,\,i}}{\partial \xi_3} \Big\|_{L^\infty(\Omega)} \Big\| \frac{w^0_\varepsilon}{\xi_3} \Big\|^2_{L^2(\Omega)} \\ &\leq (\text{using the Hardy inequality for } w^0_\varepsilon) \\ &\leq \kappa \|w^0_\varepsilon\|^2_{L^2(\Omega)} + 2\varepsilon^2 \sum_{i=1}^{2} \Big\| \Big(\frac{\xi_3}{\varepsilon}\Big)^2 \frac{\partial \theta^{0,\,i}}{\partial \xi_3} \Big\|_{L^\infty(\Omega)} \|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.32}$$

On the other hand, since v^0 satisfies the compatibility condition (2.8), from (4.13), we infer that there exists $0 < T^0 \leq T$ such that, for $i = 1, 2$,

$$\big\| (\gamma_0 v^0)(t) \big\|_{L^\infty(\Gamma)} \leq \frac{1}{80} U^O e^2 \Big(\Big\| \frac{h_i}{h} \Big\|_{L^\infty(\Omega^O_{3\delta})} \Big\| \frac{h}{h_i} \Big\|_{L^\infty(\Gamma_O)} \Big)^{-1}, \ 0 \leq t \leq T^0, \tag{4.33}$$

where $\gamma_0 : H^{1/2}(\Omega) \to L^2(\Gamma_O)$ is the usual trace operator. Then we conclude that

$$\begin{aligned}&\Big\| \Big(\frac{\xi_3}{\varepsilon}\Big)^2 \frac{\partial \theta^{0,\,i}}{\partial \xi_3} \Big\|_{L^\infty(\Omega)} \\ &\leq \frac{U^O}{\varepsilon} \Big\| \frac{h_i}{h} \Big\|_{L^\infty(\Omega^O_{3\delta})} \Big\| \frac{h}{h_i} \Big\|_{L^\infty(\Gamma_O)} \Big\| \Big(\frac{\xi_3}{\varepsilon}\Big)^2 e^{-U^O \xi_3/\varepsilon} \Big\|_{L^\infty(0,\,3\delta)} \big\| (\gamma_0 v^0)(t) \big\|_{L^\infty(\Gamma)} \\ &\quad + l.o.t. \\ &\leq \frac{1}{40} \varepsilon^{-1}, \ 0 \leq t \leq T^0, \ i = 1, 2.\end{aligned} \tag{4.34}$$

Combining (4.31), (4.32) and(4.34), we find that

$$\mathcal{J}^{0,\,2} \leq \frac{1}{10} \varepsilon \|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} + \kappa \|w^0_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.35}$$

To estimate $\mathcal{J}^{0,\,3}$, from (4.12) and (4.17), we first infer that

$$\begin{aligned}\mathcal{J}^{0,\,3} &\leq \|(\theta^0 \cdot \nabla) v^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon \|w^0_\varepsilon\|_{L^2(\Omega)} \\ &\leq \|(\theta^0 \cdot \nabla) v^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon^2 + \|w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Then, using the Hardy inequality for w^0_ε and (4.15), we find

$$\begin{aligned}\|(\theta^0 \cdot \nabla) v^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} &\leq \kappa\varepsilon \Big\| \frac{\xi_3}{\varepsilon} \theta^0 \Big\|_{L^2(\Omega)} \Big\| \frac{w^0_\varepsilon}{\xi_3} \Big\|_{L^2(\Omega)} \\ &\leq \kappa \varepsilon^{\frac{3}{2}} \|\nabla w^0_\varepsilon\|_{L^2(\Omega)} \leq \kappa\varepsilon^2 + \frac{1}{10} \varepsilon \|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the bounds above, we obtain

$$\mathcal{J}^{0,\,3} \leq \kappa\varepsilon^2 + \frac{1}{10} \varepsilon \|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} + \|w^0_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.36}$$

From (4.12) and (4.17), we infer that

$$\begin{aligned}\mathcal{J}^{0,\,4} &\leq \|(v^0 \cdot \nabla) \theta^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon \|w^0_\varepsilon\|_{L^2(\Omega)} \\ &\leq \|(v^0 \cdot \nabla) \theta^0 \cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon^2 + \|w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.37}$$

Using the Hardy inequality for w^0_ε, we write

$$\begin{aligned}\|(v^0\cdot\nabla)\theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)} &\le \varepsilon\Big\|\frac{\xi_3}{\varepsilon}(v^0\cdot\nabla)\theta^0\Big\|_{L^2(\Omega)}\|\nabla w^0_\varepsilon\|_{L^2(\Omega)}\\ &\le \kappa\varepsilon\Big\|\frac{\xi_3}{\varepsilon}(v^0\cdot\nabla)\theta^0\Big\|^2_{L^2(\Omega)} + \frac{1}{10}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.38}$$

Moreover, using (3.18), we find

$$\begin{aligned}\Big\|\frac{\xi_3}{\varepsilon}(v^0\cdot\nabla)\theta^0\Big\|^2_{L^2(\Omega)} &\le \kappa\varepsilon + \sum_{i=1}^{2}\Big\|\frac{\xi_3}{\varepsilon}(v^0\cdot\boldsymbol{e}_3)\frac{\partial\theta^{0,\,i}}{\partial\xi_3}\Big\|^2_{L^2(\Omega)}\\ &\le \kappa\varepsilon + \varepsilon^2\Big\|\frac{v^0\cdot\boldsymbol{e}_3}{\xi_3}\Big\|^2_{L^\infty(\Omega)}\sum_{i=1}^{2}\Big\|\Big(\frac{\xi_3}{\varepsilon}\Big)^2\frac{\partial\theta^{0,\,i}}{\partial\xi_3}\Big\|^2_{L^2(\Omega)}\\ &\le \left(\begin{array}{l}\text{using (4.15) and the regularity of } v^0\\ \text{with } (v^0\cdot\boldsymbol{e}_3)|_{\xi_3=0}=0\end{array}\right)\\ &\le \kappa\varepsilon.\end{aligned}\tag{4.39}$$

Combining (4.37)-(4.39), we obtain

$$\mathcal{J}^{0,\,4} \le \kappa\varepsilon^2 + \frac{1}{10}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} + \|w^0_\varepsilon\|^2_{L^2(\Omega)}.\tag{4.40}$$

For the term $\mathcal{J}^{0,\,5}$, using (4.12), (4.15), (4.17) and the Hardy inequality for w^0_ε, we write

$$\begin{aligned}\mathcal{J}^{0,\,5} &\le \|(\theta^0\cdot\nabla)\theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\big(\varepsilon\|\theta^0\|_{L^2(\Omega)}+\varepsilon^2\big)\|w^0_\varepsilon\|_{L^2(\Omega)}\\ &\quad+\varepsilon^2\Big\|\frac{\xi_3}{\varepsilon}\nabla\theta^0\Big\|_{L^2(\Omega)}\|\nabla w^0_\varepsilon\|_{L^2(\Omega)}\\ &\le \|(\theta^0\cdot\nabla)\theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)} + \kappa\varepsilon^2 + \|w^0_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{20}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.41}$$

Using (3.18), (4.15) and the Hardy inequality for w^0_ε, we find

$$\begin{aligned}&\|(\theta^0\cdot\nabla)\theta^0\cdot w^0_\varepsilon\|_{L^1(\Omega)}\\ &\quad\le \varepsilon\Big\|\frac{\xi_3}{\varepsilon}(\theta^0\cdot\nabla)\theta^0\Big\|_{L^2(\Omega)}\|\nabla w^0_\varepsilon\|_{L^2(\Omega)} \le \kappa\varepsilon^{\frac{3}{2}}\|\nabla w^0_\varepsilon\|_{L^2(\Omega)}\\ &\quad\le \kappa\varepsilon^2 + \frac{1}{20}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence, from this bound, (4.41) yields that

$$\mathcal{J}^{0,\,5} \le \kappa\varepsilon^2 + \frac{1}{10}\varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} + \|w^0_\varepsilon\|^2_{L^2(\Omega)}.\tag{4.42}$$

Thanks to (4.28), (4.30), (4.35), (4.36), (4.40) and (4.42), we find from (4.24) that

$$\frac{d}{dt}\|w^0_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\nabla w^0_\varepsilon\|^2_{L^2(\Omega)} \le \kappa\varepsilon^2 + \kappa\|w^0_\varepsilon\|^2_{L^2(\Omega)},\ 0\le t\le T^0.\tag{4.43}$$

Applying the Gronwall inequality, we deduce that

$$\|w^0_\varepsilon\|_{L^\infty(0,T^0;L^2(\Omega))} \le \kappa\varepsilon, \qquad \|w^0_\varepsilon\|_{L^2(0,T^0;H^1(\Omega))} \le \kappa\varepsilon^{\frac{1}{2}}.\tag{4.44}$$

Using (4.12), (4.17), (4.20) and (4.44), we obtain (2.10). Then, (2.12) follows from $(2.10)_1$ with (4.15). This completes the proof of Theorem 2.1 at order ε^0.

Remark 4.1. In many boundary layer analysis problems related to the Navier-Stokes equations, it is crucial to handle the term on the left hand side of (4.32). For some related issues on this problematic term, see, e.g., [33].

5. **Asymptotic expansion at order ε^1.** In this section, aiming to obtain the next term for the asymptotic expansion of $(v^\varepsilon,\, p^\varepsilon)$ at order ε^1, we write

$$v^\varepsilon \simeq (v^0+\Theta^0)+\varepsilon(v^1+\Theta^1), \quad p^\varepsilon \simeq p^0+\varepsilon(p^1+q^1), \tag{5.1}$$

where v^1, p^1, Θ^1 and q^1 will be determined below.

It is worthy to stress that our methodology used in this section is applicable to obtain asymptotic expansions of v^ε at any order ε^j, $j \geq 1$, and it is valid for any domain with a flat boundary as well. In fact, for the case of a channel domain, one can use the Cartesian coordinates to define a corrector where the metric tensor in (3.9) is an identity matrix. For higher order expansions in a channel domain, see, e.g., [14] and [25].

5.1. **Outer expansion at order ε^1.** To complete the outer expansion, outside of the boundary layer, as $v^\varepsilon \simeq v^0+\varepsilon v^1$ and $p^\varepsilon \simeq p^0+\varepsilon p^1$, we insert (5.1) into the difference of equations (2.7) and (2.9). Then, using (4.12) as well, we find

$$\begin{aligned}
&\frac{\partial(\varepsilon v^1+\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)}{\partial t}-\varepsilon\Delta(\varepsilon v^1+\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)+v^\varepsilon\cdot\nabla v^\varepsilon-v^0\cdot\nabla v^0\\
&+U\cdot\nabla(\varepsilon v^1+\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)+(\varepsilon v^1+\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)\cdot\nabla U+\varepsilon\nabla p^1\\
&\quad\simeq \varepsilon\Delta(v^0+U),
\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}
v^\varepsilon\cdot\nabla v^\varepsilon-v^0\cdot\nabla v^0 \simeq\; &(\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)\cdot\nabla(\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)\\
&+(\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)\cdot\nabla(v^0+\varepsilon v^1)\\
&+(v^0+\varepsilon v^1)\cdot\nabla(\theta^0+\varepsilon\varphi^0+\varepsilon\Theta^1)\\
&+\varepsilon(v^0\cdot\nabla v^1)+\varepsilon(v^1\cdot\nabla v^0)+\varepsilon^2 v^1\cdot\nabla v^1.
\end{aligned}\tag{5.3}$$

By noticing from (4.17) that φ^0 is of order ε^0 in any Sobolev space, we collect all terms of order ε^1 in (5.2), which involve in v^0, v^1, U and φ^0 only. As a result, we obtain the equations for v^1 as the linear system below:

$$\begin{cases}
\dfrac{\partial v^1}{\partial t}+(U+v^0)\cdot\nabla v^1+v^1\cdot\nabla(U+v^0)+\nabla p^1\\
\qquad=\Delta(U+v^0)-\Big(\dfrac{\partial\varphi^0}{\partial t}+(U+v^0)\cdot\nabla\varphi^0+\varphi^0\cdot\nabla(U+v^0)\Big),\ \text{in } \Omega\times(0,T),\\
\operatorname{div} v^1=0,\ \text{in } \Omega\times(0,T),\\
v^1\cdot\boldsymbol{n}=0,\ \text{on } \Gamma_O\times(0,T),\\
v^1=0,\ \text{on } \Gamma_I\times(0,T),\\
v^1\big|_{t=0}=0,\ \text{in } \Omega,
\end{cases}\tag{5.4}$$

As the right hand side of $(5.4)_1$ is a smooth function, of class $C^\infty([0,T]\times\Omega)$, whose norm in any Sobolev space is independent of ε, we expect as for the case where $v^1 \cdot \boldsymbol{n} = 0$ (see [27]) that the system (5.4) has a unique solution v^1 such that

$$v^1 \in C^\infty([0,T] \times \Omega), \tag{5.5}$$

and we assume so. Concerning the case of a 3D channel domain, the proof of (5.5) appears in [14].

5.2. **Inner expansion at order ε^1.** To complete the proposed expansion (5.1), by omitting the terms appearing in (5.4) and the lower order terms with respect to ε, we rewrite (5.2) in the form,

$$-\varepsilon^2 \Delta\Theta^1 + \varepsilon U \cdot \nabla\Theta^1 \simeq -\Big(\frac{\partial \theta^0}{\partial t} - \varepsilon\Delta\theta^0 + (U + v^0 + \varepsilon\varphi^0)\cdot\nabla\theta^0 + \theta^0 \cdot \nabla(U + v^0) + \theta^0 \cdot \nabla\theta^0\Big). \tag{5.6}$$

Using (3.16), (3.18) and (4.5), we collect all terms of order ε^0 in the left hand side of (5.6), inside of the boundary layer, and find

$$\sum_{i=1}^{3} \Big\{ -\varepsilon^2 \frac{\partial^2 \Theta^{1,\,i}}{\partial \xi_3^2} - \varepsilon U^O \frac{\partial \Theta^{1,\,i}}{\partial \xi_3} \Big\} \boldsymbol{e}_i. \tag{5.7}$$

Using Taylor expansions of h_i/h and $1/h$ in ξ_3 at $\xi_3 = 0$, we write θ^0, defined in (4.13), in the form:

$$\theta^0 \simeq \sum_{i=1}^{2} \Big(\sum_{j=0}^{\infty} \varepsilon^j \Big(\frac{\xi_3}{\varepsilon}\Big)^j \theta_j^{0,\,i} \Big) \boldsymbol{e}_i + \sum_{j=0}^{\infty} \varepsilon^{j+1} \Big(\frac{\xi_3}{\varepsilon}\Big)^j \theta_j^{0,\,3} \boldsymbol{e}_3, \tag{5.8}$$

where $\theta_j^{0,\,i}$, $1 \le i \le 3$, is given by the right hand side of (4.13) with h_i/h or $1/h$ replaced by its j-th order derivative in ξ_3 evaluated at $\xi_3 = 0$. Moreover, for our convenience, we write the ε^0-order part of θ^0 as

$$\theta_0^0 = \sum_{i=1}^{2} \theta_0^{0,\,i} \boldsymbol{e}_i. \tag{5.9}$$

We also write, in $\Omega_{3\delta}^O$ where the curvilinear system $\boldsymbol{\xi}$ is well-defined,

$$\begin{cases} U \simeq \displaystyle\sum_{i=1}^{3} U^i(\boldsymbol{\xi})\boldsymbol{e}_i \simeq \sum_{i=1}^{3}\sum_{j=0}^{\infty} \varepsilon^j \Big(\frac{\xi_3}{\varepsilon}^j\Big) U_j^i(\xi')\boldsymbol{e}_i, \\ v^0 \simeq \displaystyle\sum_{i=1}^{3} v^{0,\,i}(\boldsymbol{\xi})\boldsymbol{e}_i \simeq \sum_{i=1}^{3}\sum_{j=0}^{\infty} \varepsilon^j \Big(\frac{\xi_3}{\varepsilon}^j\Big) v_j^{0,\,i}(\xi')\boldsymbol{e}_i, \\ \varphi^0 \simeq \displaystyle\sum_{i=1}^{3} \varphi^{0,\,i}(\boldsymbol{\xi})\boldsymbol{e}_i \simeq \sum_{i=1}^{3}\sum_{j=0}^{\infty} \varepsilon^j \Big(\frac{\xi_3}{\varepsilon}^j\Big) \varphi_j^{0,\,i}(\xi')\boldsymbol{e}_i, \end{cases} \tag{5.10}$$

where $U_j^i(\xi')$, ($v_j^{0,\,i}(\xi')$ or $\varphi_j^{0,\,i}(\xi')$), is the j-th order derivative in ξ_3 of the function evaluated at $\xi_3 = 0$. The ε^0-order part of U, v^0 and φ^0 are written in the form,

$$U_0 = \sum_{i=1}^{3} U_0^i \boldsymbol{e}_i, \qquad v_0^0 = \sum_{i=1}^{3} v_0^{0,\,i}\boldsymbol{e}_i, \qquad \varphi_0^0 = \sum_{i=1}^{3} \varphi_0^{0,\,i}\boldsymbol{e}_i. \tag{5.11}$$

Now, for the first two terms in the right hand side of (5.6), noticing that $\varepsilon\partial^2\theta^{0,\,i}/\partial\xi_3^2$ is already taken into account by the equation of $\theta^{0,\,i}$ in (4.18), we use (3.16) and (5.8), and collect the terms of order ε^0:

$$-\frac{\partial\theta^0}{\partial t}+\varepsilon\Delta\theta^0\simeq\sum_{i=1}^{2}\Big\{-\frac{\partial\theta_0^{0,\,i}}{\partial t}+\varepsilon\mathcal{L}_0^i\theta_0^{0,\,i}\Big\}\boldsymbol{e}_i, \tag{5.12}$$

where $\mathcal{L}^i\simeq\sum_{j=0}^{\infty}\varepsilon^j(\varepsilon^{-1}\xi_3)^j\mathcal{L}_j^i(\xi')$ and each $\mathcal{L}_j^i(\xi')$ is proportional to $\partial/\partial\xi_3$ with a coefficient independent of ξ_3 and ε.

To handle the nonlinear terms in (5.6), using Taylor expansions again, we rewrite $\mathcal{P}^i$, $\mathcal{Q}^i$ and $\mathcal{R}^i$, $1\le i\le 3$, in (3.18) in the form:

$$\nabla_F G=\sum_{i=1}^{3}\Big\{F^3\frac{\partial G^i}{\partial\xi_3}+\sum_{j=0}^{\infty}\varepsilon^j\Big(\frac{\xi_3}{\varepsilon}\Big)^j\big[\mathcal{P}_j^i(F,G)+\mathcal{Q}_j^i(F,G)+\mathcal{R}_j^i(F,G)\big]\Big\}\boldsymbol{e}_i, \tag{5.13}$$

where $\mathcal{P}_j^i(F,G)$, $\mathcal{Q}_j^i(F,G)$ and $\mathcal{R}_j^i(F,G)$ are given by (3.19), (3.20) and (3.21) with the smooth functions involved in h_i, $1/h_i$ and their derivatives replaced by their j-th order derivatives in ξ_3 at $\xi_3=0$.

Note that $\mathcal{P}_j^i(F,G)$, $\mathcal{Q}_j^i(F,G)$ or $\mathcal{R}_j^i(F,G)$ contains no differential operators in the ξ_3 variable, and that their dependency on ξ_3 or ε appears in the input functions F and G only.

Recalling that $U^O\partial\theta^{0,\,i}/\partial\xi_3$, $1\le i\le 3$, are already used in the equation of θ^0, and that $v^{0,\,3}=-v^0\cdot\boldsymbol{n}=0$ at $\xi_3=0$, using (5.8)-(5.11) and (5.13), we collect all the terms of order ε^0 from the bilinear terms in (5.6):

$$\begin{aligned}
&-(U+v^0+\varepsilon\varphi^0)\cdot\nabla\theta^0\\
&\quad\simeq\sum_{i=1}^{2}\Big\{-\varepsilon\varphi_0^{0,\,3}\frac{\partial\theta_0^{0,\,i}}{\partial\xi_3}-\varepsilon(U_1^3+v_1^{0,\,3})\frac{\xi_3}{\varepsilon}\frac{\partial\theta_0^{0,\,i}}{\partial\xi_3}+\varepsilon U^O\frac{\xi_3}{\varepsilon}\frac{\partial\theta_1^{0,\,i}}{\partial\xi_3}\Big\}\boldsymbol{e}_i\\
&\qquad+\sum_{i=1}^{2}\Big\{-\mathcal{P}_0^i(U_0+v_0^0,\theta_0^0)-\mathcal{Q}_0^i(U_0+v_0^0,\theta_0^0)\Big\}\boldsymbol{e}_i\\
&\qquad-\mathcal{Q}_0^3(U_0+v_0^0,\theta_0^0)\boldsymbol{e}_3,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&-\theta^0\cdot\nabla(U+v^0)\\
&\quad\simeq\sum_{i=1}^{2}\Big\{-\mathcal{P}_0^i(\theta_0^0,U_0+v_0^0)-\mathcal{Q}_0^i(\theta_0^0,U_0+v_0^0)-\mathcal{R}_0^i(\theta_0^0,U_0+v_0^0)\Big\}\boldsymbol{e}_i\\
&\qquad-\{\mathcal{P}_0^3(\theta_0^0,U_0+v_0^0)+\mathcal{Q}_0^3(\theta_0^0,U_0+v_0^0)\}\boldsymbol{e}_3,
\end{aligned} \tag{5.15}$$

$$-\theta^0\cdot\nabla\theta^0\simeq\sum_{i=1}^{2}\Big\{-\mathcal{P}_0^i(\theta_0^0,\theta_0^0)-\varepsilon\theta_0^{0,\,3}\frac{\partial\theta_0^{0,\,i}}{\partial\xi_3}-\mathcal{Q}_0^i(\theta_0^0,\theta_0^0)\Big\}\boldsymbol{e}_i-\mathcal{Q}_0^3(\theta_0^0,\theta_0^0)\boldsymbol{e}_3. \tag{5.16}$$

Finally, using (4.18) and (4.19), we write

$$\sum_{i=1}^{3}\Big(\varepsilon\frac{\partial^2\theta^{0,\,i}}{\partial\xi_3^2}+U^O\frac{\partial\theta^{0,\,i}}{\partial\xi_3}\Big)\boldsymbol{e}_i\simeq\sum_{i=1}^{2}U^O\,\tilde{v}^{0,\,i}\frac{\partial}{\partial\xi_3}\Big(\frac{h_i}{h}\Big)\Big|_{\xi_3=0}\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\boldsymbol{e}_i. \tag{5.17}$$

Inserting (5.7), (5.12) and (5.14)-(5.17) into the equation (5.6), we propose the following equation for the corrector $\Theta^1 = \sum_{i=1}^3 \Theta^{1,\,i}\boldsymbol{e}_i$:

$$\left\{\begin{aligned} & -\varepsilon\frac{\partial^2\Theta^{1,\,i}}{\partial\xi_3^2} - U^O\frac{\partial\Theta^{1,\,i}}{\partial\xi_3} = f^{1,\,i},\ 1\le i\le 3,\\ & \Theta^{1,\,i}\big|_{\xi_3=0} = -(v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0},\ i=1,2,\\ & \Theta^{1,\,3}\big|_{\xi_3=0} = 0,\end{aligned}\right. \tag{5.18}$$

where

$$f^{1,\,i} = \frac{1}{\varepsilon}(\text{sum of the right hand sides of (5.12) and (5.14)-(5.17)})\cdot\boldsymbol{e}_i. \tag{5.19}$$

Then, using (5.8), one can write $f^{1,\,i}$ in the form:

$$\left\{\begin{aligned} f^{1,\,i}(\boldsymbol{\xi},t) &= \frac{1}{\varepsilon}\Big\{a_0^i(\xi',t) + a_1^i(\xi',t)\frac{\xi_3}{\varepsilon}\Big\}\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\\ &\qquad + \frac{1}{\varepsilon}b_0^i(\xi',t)\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon},\ i=1,2,\\ f^{1,\,3}(\boldsymbol{\xi},t) &= \frac{1}{\varepsilon}a_0^3(\xi',t)\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon} + \frac{1}{\varepsilon}b_0^3(\xi',t)\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}.\end{aligned}\right. \tag{5.20}$$

Here, $a_k^i(\xi',t)$ and $b_k^i(\xi',t)$, for $1\le i\le 3$ and $k=1$ or 2, are smooth functions in ξ' and t, independent of ξ_3 or ε.

Aiming to find a solution of (5.18), we recall some elementary lemmas from [14]:

Lemma 5.1. *For any $j\ge 0$, we have*

$$\int_0^{\xi_3}\Big(\frac{\eta}{\varepsilon}\Big)^j e^{-U^O\eta/\varepsilon}\,d\eta = -\varepsilon\sum_{k=0}^{j}\Big(\frac{1}{U^O}\Big)^{j-k+1}\frac{j!}{k!}\Big(\frac{\xi_3}{\varepsilon}\Big)^k e^{-U^O\xi_3/\varepsilon}.$$

Lemma 5.2. *A particular solution of*

$$-\varepsilon\frac{\partial^2\Psi}{\partial\xi_3^2} - U^O\frac{\partial\Psi}{\partial\xi_3} = \frac{1}{\varepsilon}\Big(\frac{\xi_3}{\varepsilon}\Big)^j e^{-U^O\xi_3/\varepsilon},\ j\ge 0,$$

is given by

$$\Psi = \sum_{k=0}^{j}(U^O)^{k-j-1}\frac{j!}{(k+1)!}\Big(\frac{\xi_3}{\varepsilon}\Big)^{k+1}e^{-U^O\xi_3/\varepsilon}.$$

We define

$$\overline{f}^{1,\,i} := f^{1,\,i}/\sigma(\xi_3),\ 1\le i\le 3, \tag{5.21}$$

and notice that

$$\overline{f}^{1,\,i} - f^{1,\,i} = e.s.t.. \tag{5.22}$$

Thanks to Lemma 5.2, we see that equation (5.18) for $i=1,2$ with $f^{1,\,i}$ replaced by $\overline{f}^{1,\,i}$ has a solution,

$$\begin{aligned}\Theta^{1,\,i}_{\text{temp}} := &\Big(b_0^i\frac{1}{2(U^O)^2} - (v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)e^{-U^O\xi_3/\varepsilon} - b_0^i\frac{1}{2(U^O)^2}e^{-2U^O\xi_3/\varepsilon}\\ &+\Big\{\Big(a_0^i\frac{1}{U^O} + a_1^i\frac{1}{(U^O)^2}\Big)\frac{\xi_3}{\varepsilon} + a_1^i\frac{1}{2U^O}\Big(\frac{\xi_3}{\varepsilon}\Big)^2\Big\}e^{-U^O\xi_3/\varepsilon},\ i=1,2.\end{aligned} \tag{5.23}$$

Thanks to Lemma 5.1, we find an antiderivative, with respect to the ξ_3 variable, of the third term on the right hand side of (5.23),

$$
\begin{aligned}
W^i(\boldsymbol{\xi},t) \, = & -\varepsilon\left(a_0^i\frac{1}{U^O}+a_1^i\frac{1}{(U^O)^2}\right)\sum_{k=0}^{1}(U^O)^{k-2}\frac{1}{k!}\left(\frac{\xi_3}{\varepsilon}\right)^k e^{-U^O\xi_3/\varepsilon}\\
& -\varepsilon a_1^i\frac{1}{U^O}\sum_{k=0}^{2}(U^O)^{k-3}\frac{1}{k!}\left(\frac{\xi_3}{\varepsilon}\right)^k e^{-U^O\xi_3/\varepsilon}+\varepsilon\Big(a_0^i\frac{1}{(U^O)^3}+a_1^i\frac{2}{(U^O)^4}\Big),
\end{aligned}
\tag{5.24}
$$

that satisfies

$$
\begin{cases}
\dfrac{\partial W^i}{\partial \xi_3}=\Big\{\Big(a_0^i\dfrac{1}{U^O}+a_1^i\dfrac{1}{(U^O)^2}\Big)\dfrac{\xi_3}{\varepsilon}+a_1^i\dfrac{1}{2U^O}\Big(\dfrac{\xi_3}{\varepsilon}\Big)^2\Big\}e^{-U^O\xi_3/\varepsilon},\\
W^i|_{\xi_3=0}=0,\quad \dfrac{\partial W^i}{\partial \xi_k}\Big|_{\xi_3=0}=0,\ 1\le k\le 3.
\end{cases}
\tag{5.25}
$$

Now, to define the tangential components $\Theta^{1,\,i}$, $i=1,2$, of the corrector Θ^1, we use the same strategy that we used for $\Theta^{0,\,i}$ in (4.9). Then, using (5.24) as well, we define: For $i=1,2$,

$$
\begin{aligned}
\Theta^{1,\,i} \, = \, & \varepsilon\frac{1}{U^O}\Big(b_0^i\frac{1}{2(U^O)^2}-(v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\frac{\partial}{\partial\xi_3}\Big(\sigma(\xi_3)\big(1-e^{-U^O\xi_3/\varepsilon}\big)\Big)\\
& -\varepsilon\frac{1}{4(U^O)^3}b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\frac{\partial}{\partial\xi_3}\Big(\sigma(\xi_3)\big(1-e^{-2U^O\xi_3/\varepsilon}\big)\Big)\\
& +\frac{h_i}{h}(\boldsymbol{\xi})\frac{\partial(\sigma(\xi_3)W^i)}{\partial\xi_3}.
\end{aligned}
\tag{5.26}
$$

By enforcing the divergence free condition on Θ^1 and using (3.15), we define the normal component $\Theta^{1,\,3}$ of Θ^1 in the form:

$$
\begin{aligned}
& \Theta^{1,\,3}\\
& = -\varepsilon\frac{1}{U^O}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{\Big(b_0^i\frac{1}{2(U^O)^2}-(v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)\big(1-e^{-U^O\xi_3/\varepsilon}\big)\\
& \quad +\varepsilon\frac{1}{4(U^O)^3}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)\big(1-e^{-2U^O\xi_3/\varepsilon}\big)\\
& \quad -\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)\sum_{i=1}^{2}\frac{\partial W^i}{\partial\xi_i}.
\end{aligned}
\tag{5.27}
$$

Due to $(5.25)_2$, one can verify that the divergence free corrector $\Theta^1=\sum_{i=1}^3\Theta^{1,\,i}\boldsymbol{e}_i$, defined by (5.26) and (5.27), satisfies the desired boundary conditions $(5.18)_{2,3}$. Moreover, due to the presence of σ in (5.26) and (5.27), we notice that

$$
\frac{\partial^{k+l}\Theta^1}{\partial t^k\partial\xi_i^l}\Big|_{\xi_3\ge 2\delta}=0,\ 1\le i\le 3,\ k,l\ge 0.
\tag{5.28}
$$

Using the fact that $\sigma'(\xi_3)e^{-U^O\xi_3/\varepsilon}$ is an *e.s.t.* in Ω, and is equal to zero on Γ_O, and using (5.24) and (5.25), we write each component $\Theta^{1,\,i}$, $1\le i\le 3$ in (5.26) and

(5.27), in the form:

$$\Theta^{1,i}(\boldsymbol{\xi};t) = \theta^{1,i} + \varepsilon\varphi^{1,i} + e.s.t., \tag{5.29}$$

where

$$\begin{aligned}\theta^{1,i} = &\Big(b_0^i\frac{1}{2(U^O)^2} - (v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\\ &-\frac{1}{2(U^O)^2}b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}\\ &+\Big\{\Big(a_0^i\frac{1}{U^O}+a_1^i\frac{1}{(U^O)^2}\Big)\frac{\xi_3}{\varepsilon}+a_1^i\frac{1}{2U^O}\Big(\frac{\xi_3}{\varepsilon}\Big)^2\Big\}\frac{h_i}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon},\ i=1,2,\end{aligned} \tag{5.30}$$

$$\begin{aligned}\theta^{1,3} = &\,\varepsilon\frac{1}{U^O}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{\Big(b_0^i\frac{1}{2(U^O)^2} - (v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\\ &-\varepsilon\frac{1}{4(U^O)^3}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}\\ &+\varepsilon\sum_{i=1}^{2}\Big\{\frac{\partial}{\partial\xi_i}\Big(a_0^i\frac{1}{U^O}+a_1^i\frac{1}{(U^O)^2}\Big)\sum_{k=0}^{1}(U^O)^{k-2}\frac{1}{k!}\Big(\frac{\xi_3}{\varepsilon}\Big)^k\\ &+\frac{\partial a_1^i}{\partial\xi_i}\frac{1}{U^O}\sum_{k=0}^{2}(U^O)^{k-3}\frac{1}{k!}\Big(\frac{\xi_3}{\varepsilon}\Big)^k\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon},\end{aligned} \tag{5.31}$$

$$\begin{aligned}\varphi^{1,i} = &\frac{1}{U^O}\Big(b_0^i\frac{1}{2(U^O)^2} - (v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\sigma'(\xi_3)\\ &-\frac{1}{4(U^O)^3}b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\frac{h_i}{h}(\boldsymbol{\xi})\sigma'(\xi_3)\\ &+\Big(a_0^i\frac{1}{(U^O)^3}+a_1^i\frac{2}{(U^O)^4}\Big)\frac{h_i}{h}(\boldsymbol{\xi})\sigma'(\xi_3),\quad i=1,2,\end{aligned} \tag{5.32}$$

$$\begin{aligned}\varphi^{1,3} = &-\frac{1}{U^O}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{\Big(b_0^i\frac{1}{2(U^O)^2} - (v^1\cdot\boldsymbol{e}_i)|_{\xi_3=0}\Big)\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)\\ &+\frac{1}{4(U^O)^3}\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big\{b_0^i\frac{h}{h_i}\Big|_{\xi_3=0}\Big\}\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3)\\ &-\sum_{i=1}^{2}\frac{\partial}{\partial\xi_i}\Big(a_0^i\frac{1}{(U^O)^3}+a_1^i\frac{2}{(U^O)^4}\Big)\frac{1}{h}(\boldsymbol{\xi})\sigma(\xi_3).\end{aligned} \tag{5.33}$$

One can derive the estimates on $\theta^1 = \sum_{i=1}^3\theta^{1,i}\boldsymbol{e}_i$ and $\varphi^1 = \sum_{i=1}^3\varphi^{1,i}\boldsymbol{e}_i$: For $r,k,l,m\geq 0$, and $i=1,2$,

$$\begin{cases}\Big\|\Big(\frac{\xi_3}{\varepsilon}\Big)^r\frac{\partial^{k+l+m}\theta^{1,i}}{\partial t^k\partial\boldsymbol{\tau}^l\partial\xi_3^m}\Big\|_{L^\infty(0,T;L^2(\Omega))}\leq\kappa\varepsilon^{\frac12-m},\\ \Big\|\Big(\frac{\xi_3}{\varepsilon}\Big)^r\frac{\partial^{k+l+m}\theta^{1,3}}{\partial t^k\partial\boldsymbol{\tau}^l\partial\xi_3^m}\Big\|_{L^\infty(0,T;L^2(\Omega))}\leq\kappa\varepsilon^{\frac32-m},\end{cases} \tag{5.34}$$

$$\begin{cases} \left\| \left(\frac{\xi_3}{\varepsilon}\right)^r \frac{\partial^{k+l+m}\theta^{1,i}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \right\|_{L^\infty([0,T]\times\Omega)} \le \kappa\varepsilon^{-m}, \\ \left\| \left(\frac{\xi_3}{\varepsilon}\right)^r \frac{\partial^{k+l+m}\theta^{1,3}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \right\|_{L^\infty([0,T]\times\Omega)} \le \kappa\varepsilon^{1-m}, \end{cases} \tag{5.35}$$

and

$$\left\| \frac{\partial^{k+l+m}\varphi^{1,j}}{\partial t^k \partial \boldsymbol{\tau}^l \partial \xi_3^m} \right\|_{L^\infty([0,T]\times\Omega)} \le \kappa, \quad 1 \le j \le 3. \tag{5.36}$$

Performing direct computations, one can verify that $\theta^{1,i}$, $i = 1,2$, satisfies the proposed equation (5.18) up to a small error: For $i = 1,2$,

$$-\varepsilon\frac{\partial^2\theta^{1,i}}{\partial\xi_3^2} - U^O\frac{\partial\theta^{1,i}}{\partial\xi_3} = f^{1,i} + E^{1,i}(\boldsymbol{\xi};t)\,\sigma(\xi_3)\big(e^{-U^O\xi_3/\varepsilon} + e^{-2U^O\xi_3/\varepsilon}\big) + e.s.t., \tag{5.37}$$

where

$$\|E^{1,i}(\boldsymbol{\xi};t)\|_{H^k(\Omega)} \le \kappa,\ i = 1,2,\ k \ge 0. \tag{5.38}$$

Moreover, the normal component $\theta^{1,3}$ of θ^1 satisfies the equation (5.18) without the source term $f^{1,3}$ up to a small error:

$$-\varepsilon\frac{\partial^2\theta^{1,3}}{\partial\xi_3^2} - U^O\frac{\partial\theta^{1,3}}{\partial\xi_3} = \varepsilon E^{1,3}(\boldsymbol{\xi};t)\,\sigma(\xi_3)\big(e^{-U^O\xi_3/\varepsilon} + e^{-2U^O\xi_3/\varepsilon}\big) + e.s.t., \tag{5.39}$$

where

$$\|E^{1,3}(\boldsymbol{\xi};t)\|_{H^k(\Omega)} \le \kappa,\ k \ge 0. \tag{5.40}$$

5.3. **Corrector q^1 of the pressure p^ε.** Our remaining task is to evaluate the error $f^{1,3}$, appearing in (5.18), in the normal direction. In the process of defining the velocity corrector Θ^1, the normal component of $\Theta^{1,3}$ (or $\theta^{1,3}$) is deduced from the divergence free condition, and for this reason, we have no control of the error $f^{1,3}$ in the normal direction; see the equations (5.18) and (5.39). To overcome this difficulty, we propose to use an asymptotic expansion of the pressure p^ε as it appears in $(5.1)_2$ where the pressure corrector q^1 is defined by the equation,

$$\frac{\partial q^1}{\partial\xi_3} = f^{1,3} = (\text{using } (5.20)_2) = \frac{1}{\varepsilon}a_0^3(\xi',t)\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon} + \frac{1}{\varepsilon}b_0^3(\xi',t)\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}. \tag{5.41}$$

Using (5.41), we define

$$q^1 := -\frac{1}{U^O}a_0^3(\xi',t)\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon} - \frac{1}{2U^O}b_0^3(\xi',t)\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}. \tag{5.42}$$

Then, using (3.14), we see that, up to an $e.s.t.$,

$$\nabla q^1 = -\frac{1}{U^O}\sum_{i=1}^{2}\frac{1}{h_i}\frac{\partial a_0^3}{\partial\xi_i}\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\boldsymbol{e}_i - \frac{1}{2U^O}\sum_{i=1}^{2}\frac{1}{h_i}\frac{\partial b_0^3}{\partial\xi_i}\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}\boldsymbol{e}_i + f^{1,3}\boldsymbol{e}_3. \tag{5.43}$$

Hence, we expect that the normal component of $-\varepsilon\nabla q^1$, which will be introduced in the error analysis, balances the term $\varepsilon f^{1,3}\boldsymbol{e}_3$, and that, as appearing in (5.43), the tangential component of $-\varepsilon\nabla q^1$ produces a lower order error with respect to ε.

5.4. **Proof of Theorem 2.1 at order ε^1.** We define the remainder at order ε^1 as

$$w_\varepsilon^1 := v^\varepsilon - (v^0 + \Theta^0) - \varepsilon(v^1 + \Theta^1). \tag{5.44}$$

Then, since Θ^0 and Θ^1 are divergence free, using the equations (2.7), (2.9) and (5.4), and using (4.6), $(5.18)_{2,3}$ and (5.3), the equations for w_ε^1 read:

$$\begin{cases} \dfrac{\partial w_\varepsilon^1}{\partial t} - \varepsilon \Delta w_\varepsilon^1 + (U \cdot \nabla) w_\varepsilon^1 + (w_\varepsilon^1 \cdot \nabla) U + \nabla(p^\varepsilon - p^0 - \varepsilon p^1 - \varepsilon q^1) \\ \qquad = \varepsilon^2 \big(\Delta(\varphi^0 + v^1) + R_\varepsilon^0(\varphi^1)\big) + R_\varepsilon^0(\theta^0 + \varepsilon\theta^1) - J_\varepsilon^1(v^\varepsilon, v^0) - \varepsilon \nabla q^1 + e.s.t., \\ \qquad \text{in } \Omega \times (0,T), \\ \text{div } w_\varepsilon^1 = 0, \text{ in } \Omega \times (0,T), \\ w_\varepsilon^1 = 0, \text{ on } \Gamma \times (0,T), \\ w_\varepsilon^1\big|_{t=0} = 0, \text{ in } \Omega, \end{cases} \tag{5.45}$$

where $R_\varepsilon^0(\cdot)$ is defined in (4.22).

The term $J_\varepsilon^1(v^\varepsilon, v^0)$, appearing in $(5.45)_1$, is defined by

$$J_\varepsilon^1(v^\varepsilon, v^0) := \sum_{k=1}^{6} J_\varepsilon^{1,k}, \tag{5.46}$$

where

$$\begin{cases} J_\varepsilon^{1,1} = & v^\varepsilon \cdot \nabla w_\varepsilon^1 + w_\varepsilon^1 \cdot \nabla(v^0 + \varepsilon v^1 + \varepsilon\varphi^0 + \varepsilon^2\varphi^1) \\ & + \varepsilon^2 (v^0 \cdot \nabla\varphi^1 + \varphi^1 \cdot \nabla v^0 + (v^1 + \varphi^0 + \varepsilon\varphi^1) \cdot \nabla(v^1 + \varphi^0 + \varepsilon\varphi^1)), \\ J_\varepsilon^{1,2} = & w_\varepsilon^1 \cdot \nabla(\theta^0 + \varepsilon\theta^1), \\ J_\varepsilon^{1,3} = & \varepsilon\theta^0 \cdot \nabla(v^1 + \varphi^0 + \varepsilon\varphi^1) + \varepsilon\theta^1 \cdot \nabla(v^0 + \varepsilon v^1 + \varepsilon\varphi^0 + \varepsilon^2\varphi^1), \\ J_\varepsilon^{1,4} = & \varepsilon(v^1 + \varepsilon\varphi^1) \cdot \nabla\theta^0 + \varepsilon(v^0 + \varepsilon v^1 + \varepsilon\varphi^0 + \varepsilon^2\varphi^1) \cdot \nabla\theta^1, \\ J_\varepsilon^{1,5} = & \varepsilon\theta^0 \cdot \nabla\theta^1 + \varepsilon\theta^1 \cdot \nabla(\theta^0 + \varepsilon\theta^1). \\ J_\varepsilon^{1,6} = & \theta^0 \cdot \nabla v^0 + v^0 \cdot \nabla\theta^0 + \varepsilon\varphi^0 \cdot \nabla\theta^0 + \varepsilon\theta^0 \cdot \nabla\theta^0. \end{cases} \tag{5.47}$$

We multiply $(5.45)_1$ by w_ε^1 and integrate by parts. Then, using (5.36), we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_\varepsilon^1\|_{L^2(\Omega)}^2 + \varepsilon\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2 \\ &\qquad \le \kappa\varepsilon^4 + \|w_\varepsilon^1\|_{L^2(\Omega)}^2 + \sum_{k=1}^{5} \mathcal{J}^{1,k} + \big\|\big(R_\varepsilon^0(\theta^0 + \varepsilon\theta^1) - J_\varepsilon^{1,6} - \varepsilon\nabla q^1\big) \cdot w_\varepsilon^1\big\|_{L^1(\Omega)}, \end{aligned} \tag{5.48}$$

where

$$\mathcal{J}^{1,k} = \big\|J_\varepsilon^{1,k} \cdot w_\varepsilon^1\big\|_{L^1(\Omega)}, \ 1 \le k \le 5. \tag{5.49}$$

The term $\mathcal{J}^{1,1}$ is easy to estimate:

$$\mathcal{J}^{1,1} \le \kappa\varepsilon^4 + \kappa\|w_\varepsilon^1\|_{L^2(\Omega)}^2. \tag{5.50}$$

For $\mathcal{J}^{1,2}$, using the same estimate as in (4.32), we write

$$\begin{aligned}\mathcal{J}^{1,2} &\leq \|(w_\varepsilon^1\cdot\nabla)\theta^0\cdot w_\varepsilon^1\|_{L^1(\Omega)} + \varepsilon\|(w_\varepsilon^1\cdot\nabla)\theta^1\cdot w_\varepsilon^1\|_{L^1(\Omega)}\\ &\leq \kappa\|w_\varepsilon^1\|_{L^2(\Omega)}^2 + 2\varepsilon^2\sum_{i=1}^2\Big\|\Big(\frac{\xi_3}{\varepsilon}\Big)^2\frac{\partial\theta^{0,i}}{\partial\xi_3}\Big\|_{L^\infty(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2\\ &\quad+2\varepsilon^3\sum_{i=1}^2\Big\|\Big(\frac{\xi_3}{\varepsilon}\Big)^2\frac{\partial\theta^{1,i}}{\partial\xi_3}\Big\|_{L^\infty(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2.\end{aligned} \tag{5.51}$$

Then, using (4.33) for the second term, and using (5.35) with $\varepsilon \ll 1$ for the last term on the right hand side of (5.51), we find

$$\mathcal{J}^{1,2} \leq \frac{2}{10}\varepsilon\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2 + \kappa\|w_\varepsilon^1\|_{L^2(\Omega)}^2, \quad 0\leq t\leq T_0. \tag{5.52}$$

It is easy to see that $\varepsilon^{-1}\mathcal{J}^{1,3}$ and $\varepsilon^{-1}\mathcal{J}^{1,5}$ lead to the same estimates as $\mathcal{J}^{0,3}$ and $\mathcal{J}^{0,5}$, as they appear in (4.36) and (4.42). Hence, we find that

$$\mathcal{J}^{1,3} + \mathcal{J}^{1,5} \leq \kappa\varepsilon^4 + \frac{2}{10}\varepsilon\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2 + 2\|w_\varepsilon^1\|_{L^2(\Omega)}^2. \tag{5.53}$$

To estimate $\mathcal{J}^{1,4}$, using the Hardy inequality for w_ε^1, we write

$$\begin{aligned}\mathcal{J}^{1,4} &\leq \varepsilon\|v^1\cdot\nabla\theta^0\cdot w_\varepsilon^1\|_{L^1(\Omega)} + \varepsilon\|v^0\cdot\nabla\theta^1\cdot w_\varepsilon^1\|_{L^1(\Omega)}\\ &\quad+\varepsilon^2\|\varphi^1\cdot\nabla\theta^0\cdot w_\varepsilon^1\|_{L^1(\Omega)} + \varepsilon^2\|(v^1+\varphi^0+\varepsilon\varphi^1)\cdot\nabla\theta^1\cdot w_\varepsilon^1\|_{L^1(\Omega)}\\ &\leq \varepsilon^2\Big\|\frac{\xi_3}{\varepsilon}v^1\cdot\nabla\theta^0\Big\|_{L^2(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)} + \varepsilon^2\Big\|\frac{\xi_3}{\varepsilon}v^0\cdot\nabla\theta^1\Big\|_{L^2(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}\\ &\quad+\varepsilon^3\Big\|\frac{\xi_3}{\varepsilon}\varphi^1\cdot\nabla\theta^0\Big\|_{L^2(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}\\ &\quad+\varepsilon^3\Big\|\frac{\xi_3}{\varepsilon}(v^1+\varphi^0+\varepsilon\varphi^1)\cdot\nabla\theta^1\Big\|_{L^2(\Omega)}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}.\end{aligned} \tag{5.54}$$

Since $v^0\cdot\boldsymbol{n} = v^1\cdot\boldsymbol{n} = 0$ on Γ, for the first two terms on the right hand side of (5.54), we use the same estimate as in (4.39). Then, using (4.15) and (5.34) for the last two terms in (5.54) as well, we find

$$\mathcal{J}^{1,4} \leq \kappa\varepsilon^{\frac{5}{2}}\|\nabla w_\varepsilon^1\|_{L^2(\Omega)} \leq \kappa\varepsilon^4 + \frac{1}{10}\varepsilon\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2. \tag{5.55}$$

Using (5.50), (5.52), (5.53) and (5.55), (5.48) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_\varepsilon^1\|_{L^2(\Omega)}^2 + \frac{1}{2}\varepsilon\|\nabla w_\varepsilon^1\|_{L^2(\Omega)}^2\\ &\quad\leq \kappa\varepsilon^4 + \kappa\|w_\varepsilon^1\|_{L^2(\Omega)}^2 + \big\|\big(R_\varepsilon^0(\theta^0+\varepsilon\theta^1) - J_\varepsilon^{1,6} - \varepsilon\nabla q^1\big)\cdot w_\varepsilon^1\big\|_{L^1(\Omega)}.\end{aligned} \tag{5.56}$$

To estimate the last term on the right hand side of (5.56), using (4.18), (5.37), (5.39) and (5.43) with (5.19), we find, for $i = 1, 2$,

$$
\begin{aligned}
&\big(R^0_\varepsilon(\theta^0 + \varepsilon\theta^1) - J^{1,6}_\varepsilon - \varepsilon\nabla q^1\big)\boldsymbol{e}_i\\
&\quad= -\frac{\partial}{\partial t}(\theta^{0,i} - \theta^{0,i}_0) + \varepsilon\mathcal{S}^i\theta^0 + \varepsilon\big(\mathcal{L}^i\theta^{0,i} - \mathcal{L}^i_0\theta^{0,i}_0\big)\\
&\qquad -\Big\{(U\cdot\nabla\theta^0)\boldsymbol{e}_i - U^3_0\frac{\partial\theta^{0,i}}{\partial\xi_3} - \varepsilon U^3_1\frac{\xi_3}{\varepsilon}\frac{\partial\theta^{0,i}_0}{\partial\xi_3} - \varepsilon U^3_0\frac{\xi_3}{\varepsilon}\frac{\partial\theta^{0,i}_1}{\partial\xi_3}\Big\}\\
&\qquad -\Big\{-\mathcal{P}^i_0(U_0,\theta^0_0) - \mathcal{Q}^i_0(U_0,\theta^0_0)\Big\}\\
&\qquad -\Big\{(\theta^0\cdot\nabla U)\boldsymbol{e}_i - \mathcal{P}^i_0(\theta^0_0,U_0) - \mathcal{Q}^i_0(\theta^0_0,U_0) - \mathcal{R}^i_0(\theta^0_0,U_0)\Big\}\\
&\qquad -\Big\{(\theta^0\cdot\nabla v^0)\boldsymbol{e}_i - \mathcal{P}^i_0(\theta^0_0,v^0_0) - \mathcal{Q}^i_0(\theta^0_0,v^0_0) - \mathcal{R}^i_0(\theta^0_0,v^0_0)\Big\}\\
&\qquad -\Big\{\varepsilon(\varphi^0\cdot\nabla\theta^0)\boldsymbol{e}_i - \varepsilon\varphi^{0,3}_0\frac{\partial\theta^{0,i}_0}{\partial\xi_3}\Big\} \qquad\qquad (5.57)\\
&\qquad -\Big\{(v^0\cdot\nabla\theta^0)\boldsymbol{e}_i - \mathcal{P}^i_0(v^0_0,\theta^0_0) - \varepsilon v^{0,3}_1\frac{\xi_3}{\varepsilon}\frac{\partial\theta^{0,i}_0}{\partial\xi_3} - \mathcal{Q}^i_0(v^0_0,\theta^0_0)\Big\}\\
&\qquad -\Big\{(\theta^0\cdot\nabla\theta^0)\boldsymbol{e}_i - \mathcal{P}^i_0(\theta^0_0,\theta^0_0) - \varepsilon\theta^{0,3}_0\frac{\partial\theta^{0,i}_0}{\partial\xi_3} - \mathcal{Q}^i_0(\theta^0_0,\theta^0_0)\Big\}\\
&\qquad -\varepsilon\frac{\partial\theta^{1,i}}{\partial t} + \varepsilon^2\mathcal{S}^i\theta^1 + \varepsilon^2\mathcal{L}^i\theta^{1,i} - \varepsilon\Big\{(U\cdot\nabla\theta^1)\boldsymbol{e}_i - U^3_0\frac{\partial\theta^{1,i}}{\partial\xi_3}\Big\}\\
&\qquad -\varepsilon(\theta^1\cdot\nabla U + \nabla q^1)\cdot\boldsymbol{e}_i\\
&\qquad +\varepsilon\Big\{U^O\,\widetilde{v}^{0,i}\frac{\partial^2}{\partial\xi_3^2}\Big(\frac{h_i}{h}\Big)\Big|_{\xi_3=0}\frac{\xi_3}{\varepsilon} - \widetilde{v}^{0,i}\frac{\partial^2}{\partial\xi_3^2}\Big(\frac{h_i}{h}\Big) - E^{1,i}\Big\}\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon}\\
&\qquad -\varepsilon E^{1,i}\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(R^0_\varepsilon(\theta^0 + \varepsilon\theta^1) - J^{1,6}_\varepsilon - \varepsilon\nabla q^1\big)\boldsymbol{e}_3\\
&= -\frac{\partial\theta^{0,3}}{\partial t} + \varepsilon\mathcal{S}^3\theta^0 + \varepsilon\mathcal{L}^3\theta^{0,3} - \Big\{(U\cdot\nabla\theta^0)\boldsymbol{e}_3 - U^3_0\frac{\partial\theta^{0,3}}{\partial\xi_3} - \mathcal{Q}^3_0(U_0,\theta^0_0)\Big\}\\
&\quad -\Big\{(\theta^0\cdot\nabla U)\boldsymbol{e}_3 - \mathcal{P}^3_0(\theta^0_0,U_0) - \mathcal{Q}^3_0(\theta^0_0,U_0)\Big\}\\
&\quad -\Big\{(\theta^0\cdot\nabla v^0)\boldsymbol{e}_3 - \mathcal{P}^3_0(\theta^0_0,v^0_0) - \mathcal{Q}^3_0(\theta^0_0,v^0_0)\Big\} - \varepsilon(\varphi^0\cdot\nabla\theta^0)\boldsymbol{e}_3\\
&\quad -\Big\{(v^0\cdot\nabla\theta^0)\boldsymbol{e}_3 - \mathcal{Q}^3_0(v^0_0,\theta^0_0)\Big\} - \Big\{(\theta^0\cdot\nabla\theta^0)\boldsymbol{e}_3 - \mathcal{Q}^3_0(\theta^0_0,\theta^0_0)\Big\}\\
&\quad -\varepsilon\frac{\partial\theta^{1,3}}{\partial t} + \varepsilon^2\mathcal{S}^3\theta^1 + \varepsilon^2\mathcal{L}^3\theta^{1,3} - \varepsilon\Big\{(U\cdot\nabla\theta^1)\boldsymbol{e}_3 - U^3_0\frac{\partial\theta^{1,3}}{\partial\xi_3}\Big\} - \varepsilon\theta^1\cdot\nabla U\boldsymbol{e}_3\\
&\quad -\varepsilon(E^{0,3} + \varepsilon E^{1,3})\sigma(\xi_3)e^{-U^O\xi_3/\varepsilon} - \varepsilon^2E^{1,3}\sigma(\xi_3)e^{-2U^O\xi_3/\varepsilon}.
\end{aligned}
\tag{5.58}
$$

Thanks to (3.18), (4.13), (4.19), (5.8), (5.10), (5.13), (5.30), (5.31), (5.38), (5.40) and (5.43), one can verify that

$$\sum_{i=1}^{3} \big|\big(R^0_\varepsilon(\theta^0+\varepsilon\theta^1) - J^{1,6}_\varepsilon - \varepsilon\nabla q^1\big)\boldsymbol{e}_i\big| \le \varepsilon \boldsymbol{P}_2(\xi_3/\varepsilon)\sigma(\xi_3)\big(e^{-U^O\xi_3/\varepsilon} + e^{-2U^O\xi_3/\varepsilon}\big),$$

where $\boldsymbol{P}_2(\xi_3/\varepsilon)$ is a polynomial in ξ_3/ε of order 2 with positive constant coefficients, independent of $\boldsymbol{\xi}$ or ε. Then, using the Hardy inequality for w^1_ε, we find

$$\begin{aligned} &\big\|\big(R^0_\varepsilon(\theta^0+\varepsilon\theta^1) - J^{1,6}_\varepsilon - \varepsilon\nabla q^1\big)\cdot w^1_\varepsilon\big\|_{L^1(\Omega)} \\ &\le \varepsilon^2\Big\|\frac{\xi_3}{\varepsilon}\boldsymbol{P}_2(\xi_3/\varepsilon)\sigma(\xi_3)\big(e^{-U^O\xi_3/\varepsilon}+e^{-2U^O\xi_3/\varepsilon}\big)\Big\|_{L^2(\Omega)}\|\nabla w^1_\varepsilon\|_{L^2(\Omega)} \qquad (5.59)\\ &\le \kappa\varepsilon^{\frac{5}{2}}\|\nabla w^1_\varepsilon\|_{L^2(\Omega)} \le \kappa\varepsilon^4 + \frac{1}{4}\varepsilon\|\nabla w^1_\varepsilon\|^2_{L^2(\Omega)}. \end{aligned}$$

Combining (5.56) and (5.59), we see that

$$\frac{d}{dt}\|w^1_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\varepsilon\|\nabla w^1_\varepsilon\|^2_{L^2(\Omega)} \le \kappa\varepsilon^4 + \kappa\|w^1_\varepsilon\|^2_{L^2(\Omega)}. \qquad (5.60)$$

Applying the Gronwall inequality, we deduce that

$$\|w^1_\varepsilon\|_{L^\infty(0,T^0;L^2(\Omega))} \le \kappa\varepsilon^2, \qquad \|w^1_\varepsilon\|_{L^2(0,T^0;H^1(\Omega))} \le \kappa\varepsilon^{\frac{3}{2}}. \qquad (5.61)$$

Now, using (5.29), (5.36), (5.44) and (5.61), we finally obtain (2.11). This completes the proof of Theorem 2.1 at order ε^1.

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