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USING THE 11-POINT BIPLANE AND $L_2(11)$ TO UNDERSTAND J_1

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In this paper, we use the 11-point biplane and its automorphisms in $L_2(11)$ to label and study the Livingstone graph (Γ) and J_1, with an aim of using the simplest methods possible. We detail the action of J_1 on Γ, along with the adjacencies and coadjacencies (vertices at maximum distance) in Γ. In the last section, we use this apparatus to describe the generation of subgroups of the form $2^3 : 7 : 3$ and an elegant substructure of Γ fixed by a maximal subgroup of J_1 isomorphic to $19 : 6$.

1. Setting the stage

J_1 is a sporadic group of order 175560. Our task is to develop a notation and computational framework for understanding J_1 and its related objects. As sporadic groups go, J_1 is very small. On the other hand, it is one of six "pariah" sporadic groups that are not subquotients of the monster group M [9], and so its existence cannot be understood by the framework surrounding M. The path we will take involves several important objects, so it is perhaps best to start with an overview. Keep in mind that, throughout, we seek to use as little technical information about J_1 as possible.

We begin by using the residues modulo 11 to define the 11-point biplane, B, which is an incidence system consisting of 11 points ($E = \{0, 1, \dots, 9, \mathrm{X}\}$) and 11 lines. We study its various symmetries (that leave points and lines fixed as sets), obtaining the group $\mathrm{PSL}_2(11)$, which we will usually refer to as $L_2(11)$ or just L. Using its action on E, we get a notation for its elements.

However, L also has a transitive and faithful action on a set, T, of 12 letters. Although the elements of this set and E are distinct, it will be helpful to have an overlap in notation and call them $\{0, 1, \dots, 9, \mathrm{X}, \infty\}$. Understanding both of these actions allows us to name elements of L by either action (or both). As an example, the following shows a particular element of L, where the parenthetical action is on E, and the bracketed action is on T.

$$\gamma = [0\,\infty][1\,\mathrm{X}][2\,5][3\,7][4\,8][6\,9] \bullet (3\,4)(5\,9)(2\,\mathrm{X})(6\,7).$$

Computationally this is great, but notationally it is a bit verbose. To help in this respect, we also develop a sort of shorthand notation for most elements. For instance, we will be able to identify the above γ with the transposition $(3\,4)$, giving us the shorter name γ_{34}, from which we can extrapolate the full permutation notation if necessary.

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Why study $L_2(11)$? The group $L_2(11)$ is crucial in understanding J_1 as it is its maximal proper subgroup (to index 266). Furthermore, J_1's lowest degree faithful permutation representation can be obtained by its action on its $L_2(11)$ subgroups by conjugation. This action can be represented by an object known as the Livingstone graph, Γ. This graph, which we will construct, is a 266-vertex graph whose symmetry group is J_1 and whose vertex stabilizer is $L_2(11)$. As a result, each vertex of the graph corresponds to a unique conjugate of $L_2(11)$ in J_1. To have as few edges as possible, we define adjacency so that two vertices will be adjacent exactly when their copies of L intersect maximally, in an A_5. (A_5 is indeed the maximal proper subgroup of $L_2(11)$.) The normalizer of such an A_5 in J_1 is $2 \times A_5$, giving a correspondence between edges of Γ and involutions of J_1.

Much of our work will lie in using our notations for B and L to create a notation for the vertices of Γ. The key is that the action of an $L_2(11)$ on the neighbors of the vertex it stabilizes is the same as its action on E. This allows us to name the neighbors using these letters. Calling our starting vertex $\bullet$, its neighbors will be the vertices named $\{\bullet i : i \in E\}$. Similarly, the central involution that stabilizes the edge connecting $\bullet$ and $\bullet i$ will also be named i. This involution switches the two vertices, and so we can also think of the name $\bullet i$ as meaning "$\bullet$ acted upon by the involution i". In general, the vertex $\bullet a_1 \cdots a_n$ will be adjacent to exactly $\{\bullet i a_1 \cdots a_n : i \in E\}$. We show that no more than four such involutions are needed to name all vertices, but that these names are not unique. So further work is done, now incorporating the 12-element set, to simplify things, giving unique standard names to each vertex. Our final list of vertices will be

$$\{\bullet, \bullet i, \bullet ij, x \bullet i, x\bullet : i, j \in E, x \in T, i \neq j\}.$$

Finally, this gives us a useful notation for elements of J_1. Remembering that L is maximal in J_1, $\langle L, 0\rangle$ gives all of J_1. The orbit of 0 under the action of L is our set of transpositions $\{0, 1, \ldots, 9, \mathrm{X}\}$, and so we will be able to write every element of J_1 in the form $\sigma\iota$, where $\sigma \in L$, and ι is a product of no more than four of the these transpositions. For example, in our final section, we will demonstrate that the following is an element of order 19.

$$\pi_{386}374 = [0\mathrm{X}2365][19\infty748] \bullet (02)(15497\mathrm{X})(386)374.$$

Using this set of tools, one can make strong connections between substructures of the group J_1 and those of Γ, giving a sort of visual reference to these group theoretic objects. As an example, in our last section we find that maximal subgroups of the form $19:6$ correspond to certain hexagonal structures related to Γ.

With all of this context in mind, we are ready to begin.

2. The 11-point biplane and its automorphisms

The overall procedure we follow will be parallel to portions of my dissertation [5], with various improvements in exposition, clarity, and proof. We first study the 11-point biplane and its automorphisms, as this gives us the language to efficiently talk about the Livingstone graph.

2.1. Basic properties of the biplane. Throughout, we let $E = \{0, 1, 2, \ldots, 9, \mathrm{X}\}$. The elements of E are ostensibly just 11 letters, but we will often also think of them as residues modulo 11, with X standing for 10.

2	6	7	X	8
3	7	8	0	9
4	8	9	1	X
5	9	X	2	0
6	X	0	3	1
7	0	1	4	2
8	1	2	5	3
9	2	3	6	4
X	3	4	7	5
0	4	5	8	6
1	5	6	9	7

Figure 1. The lines of the biplane.

Definition 1. The 11*-point biplane* B is the set of translates $\{\{2, 6, 7, X, 8\} + i : i \in \mathbb{Z}_{11}\}$. Each element of E is called a *point*, and each set in B is called a *line*. Two or more points in E are *collinear* if there is some line containing them.

In what follows, it will be useful to have the lines of B at hand, so these are produced in Figure 1. The following proposition can be readily checked and is in fact what makes B into a biplane.

Proposition 2. *B has the following properties*:

(1) *B is comprised of* 11 *lines from a population of* 11 *points.*

(2) *Every line has exactly* 5 *points*, *and every point is on exactly* 5 *lines.*

(3) *Each pair of lines intersects in a unique pair of points*, *and every pair of points is contained in a unique pair of lines.*

Specifically, this makes B into an $(11, 5, 2)$-symmetric block design. A more thorough discussion of these designs, including biplanes, can be found in [1] or [3]. In [2], the author discusses this particular biplane and its connection with various mathematical objects, with similar notation, although their lines are the negatives of ours. Next, we prove a simple lemma.

Lemma 3. *If $a, b, c \in E$ are collinear*, *then there is a unique point $d \in E$ that is not collinear with either of the pairs $\{a, b\}$ or $\{a, c\}$.*

Proof. Let L_{abc} be the unique line containing a, b, and c. (Lines cannot intersect in three or more points.) Let L_{ab} be the unique line containing a and b but not c. Finally, let L_{ac} be the unique line containing a and c, but not b. Considering these three lines as sets, we compute the size of their union. Each individual line has five elements, and pairwise intersections have two, so the only piece left to consider is their (threeway) intersection. We know $L_{abc} \cap L_{ab} = \{a, b\}$, and that L_{ac} contains a but not b, so this three-way intersection has size 1. Thus

$$|L_{abc} \cup L_{ab} \cup L_{ac}| = 5+5+5-2-2-2+1 = 10$$

Therefore, there is a unique point not collinear with $\{a, b\}$ or $\{a, c\}$. □

2.2. Automorphisms of B. Our present aim is to study the symmetries of B. It is known that this group is isomorphic to $L_2(11)$, as discussed in [2]. However, we want rather more detail, and so we construct this group independently. In particular, we want to describe these elements as permutations on E (which of course determine an action on B as well).

As we proceed, we will want to write our automorphisms as permutations on E with a *right action.* Therefore, multiplication of cycles is done left-to-right, and for $\sigma, \tau \in \mathrm{Aut}(B)$, $\sigma^\tau = \tau^{-1}\sigma\tau$. We begin with a lemma on transitivity.

Lemma 4. $\mathrm{Aut}(B)$ *is* 2*-transitive on* E, *but not* 3*-transitive.*

Proof. The elements $\alpha = (0\,1\,2\,\cdots\,9\,\mathrm{X})$, that is, $a \mapsto a+1 \mod 11$, $\beta = (1\,3\,9\,5\,4)(2\,6\,7\,\mathrm{X}\,8)$, that is, $a \mapsto 3a \mod 11$, and $\tau = (0\,1)(2\,6)(3\,4)(7\,\mathrm{X})$ all send lines to lines, and so are elements of $\mathrm{Aut}(B)$. Under the action of $\langle\alpha, \beta\rangle$, the ordered pair $(0, 1)$ has orbit $(a, a+q)$, where $q \in \{1, 3, 4, 5, 9\}$ and $(1, 0)$ has orbit $(a, a+n)$, where $n \in \{2, 6, 7, \mathrm{X}, 8\}$. The action by τ joins these two orbits into a single orbit of size 110, as it switches $(0, 1)$ and $(1, 0)$. Therefore $\mathrm{Aut}(B)$ is 2-transitive. An element of $\mathrm{Aut}(B)$ cannot send a noncollinear triple to a collinear triple, so 3-transitivity is impossible. □

Proposition 5. *An automorphism of B is determined by its action on three collinear points.* $|\mathrm{Aut}(B)| \leq 660$.

Proof. We will demonstrate that labeling some three collinear points determines the roles of all other points of the biplane. We do not want to assume that collinear triples are indistinguishable, so we work with a general one, although following this argument with the specific case $\{0, 1, 2\}$ may be helpful.

Suppose $\{a, b, c\} \subset E$ is a collinear triple. Then we have a unique line, say $L_{abc} = \{a, b, c, d, e\}$. We also have lines L_{ab}, L_{ac}, L_{bc} containing the subscripted pair of points, but not the remaining point of the triple. Note that these three lines cannot contain d or e, and their threeway intersection is empty. As a result, we may label the distinct points

$$\{f\} = L_{ab} \cap L_{ac} \setminus \{a\}$$
$$\{g\} = L_{ab} \cap L_{bc} \setminus \{b\}$$
$$\{h\} = L_{bc} \cap L_{ac} \setminus \{c\}$$

Now see that L_{bc}, for example, contains $\{b, c, g, h\}$, which already lists two points each from L_{abc}, L_{ab}, and L_{ac}. Therefore the last point of L_{bc} must be the (unique by Lemma 3) point not in $L_{abc} \cup L_{ab} \cup L_{ac}$. Thus, we can make three more designations

$$\{i\} = E \setminus (L_{abc} \cup L_{ab} \cup L_{ac})$$
$$\{j\} = E \setminus (L_{abc} \cup L_{ab} \cup L_{bc})$$
$$\{k\} = E \setminus (L_{abc} \cup L_{ac} \cup L_{bc})$$

To be clear, these are distinct, as $i \in L_{bc}$, while $j, k \notin L_{bc}$, for example.

Let L_{ad} and L_{ae} be the lines other than L_{abc} containing those pairs of points. Since i must be on two of the five lines inhabited by a, it must be on each of these. Now consider the intersections of $L_{bc} = \{b, c, g, h, i\}$ with L_{ad} and L_{ae}. One intersection must be $\{g, i\}$, and the other must be $\{h, i\}$. We then label d as the point whose other line with a contains g, and e as the point whose other line with a contains h.

1	5	6	9	7	1	5	6	9	7	1	5	6	9	7
	0		4	5	6	0	8	4	5	6	0	8	4	5
	7		9	0	8	7	3	9	0	8	7	3	9	0
	5		4	7	3	5	X	4	7	3	5	X	4	7
	0		9	5		0		9	5	X	0	2	9	5
	7		4	0		7		4	0	2	7	1	4	0

Figure 2. Using the lines of B to extend a partial action on E, $(0\,7\,5)(4\,9)$, to an element of $\mathrm{Aut}(B)$.

Finally, as a given pair of points lie on a pair of lines, each with three additional (and distinct) points, the number of ordered collinear triples is $11 \times 10 \times 6$. Thus we have established the bound on $|\mathrm{Aut}(B)|$. □

Corollary 6. *An automorphism of B is determined by its action on any four points.*

Proof. We show that every set of four points of B contains at least one collinear triple. Suppose that $\{a, b, c, d\}$ were a counterexample. Forming the six pairs of points from this set would then have to produce 12 distinct lines. Of course, B only has 11 lines, so this is impossible. □

Example 7. In practice, the way we extend a partial known action of an automorphism to its full action is a little different from the argument in the proof of Proposition 5. Suppose we know that an element has $(0\,7\,5)$ as a 3-cycle and $(4\,9)$ as a 2-cycle, as we will have in a later section. First, we find a line that contains points whose actions we know. In this case, we can choose 15697. We now let our element act on this line, going down, filling out what we can. The result is the leftmost column of lines in Figure 2.

Three points on a line is enough to determine it completely, so we know that the second line is missing $\{6, 8\}$, the third line is missing $\{3, 8\}$, and the fourth line is missing $\{3, \mathrm{X}\}$. The 6 on the first line cannot go to a 6 on the second line without repeating further (which it does not), so we must have $6 \to 8$, similarly that 8 must go to a 3 on the next line, and a X on the following, giving us the middle set of lines in the figure. Continuing with this, we get the finished lines on the right. This tells us our element must be $(0\,7\,5)(4\,9)(1\,6\,8\,3\,\mathrm{X}\,2)$.

2.3. Elements of order 2. Suppose $\gamma \in \mathrm{Aut}(B)$ with order two. Without loss of generality assume γ switches the points 3 and 4. Then γ must act on the triple of points not in line with $\{3, 4\}$, that being $\{0, 1, 8\}$. At least one of these must be fixed by γ, so let us choose 0. Then either γ fixes the other two as well, or permutes them. Either way, this defines γ totally:

$$\gamma = (3\,4)(5\,9)(2\,\mathrm{X})(6\,7)(0)(1)(8). \quad \gamma' = (3\,4)(1\,8)(2\,9)(5\,\mathrm{X})(0)(6)(7).$$

The first of these is special with respect to the pair $\{3, 4\}$, as it switches the two points and the two lines containing them (but leaves the three points of these two lines fixed pointwise). You can check that γ' does this very thing for the pair $\{1, 8\}$. Had we chosen γ' to fix 1 instead of 0, the resulting involution would switch the two lines containing $\{0, 8\}$.

The above shows that each element of order two has a distinguished transposition such that the element not only switches those points, but also the two lines containing them, fixing the three points off those lines. We will use the following notation:

$$\gamma_{34} = (\underline{3\,4})(5\,9)(2\,\mathrm{X})(6\,7)(0)(1)(8).$$

Definition 8. The element γ_{ab} is the unique involution of $\mathrm{Aut}(B)$ that permutes a and b, as well as the two lines containing $\{a, b\}$, while fixing pointwise the three points out of line with $\{a, b\}$.

2.4. Elements of order 3. Suppose $\epsilon \in \mathrm{Aut}(B)$ has order three. A nonidentity element cannot fix four or more points, so ϵ must have three 3-cycles and two fixed points. Reminding again that pairs of points are indistinguishable, we may assume ϵ fixes 4 and 9. So ϵ must also fix as sets each of the two lines containing $\{4, 9\}$: 4891X and 92364, and the set three points not on these lines, $\{0, 5, 7\}$. Since ϵ has order three and fixes only 4 and 9, we see that it must permute the elements within $\{2, 3, 6\}$, $\{1, 8, \mathrm{X}\}$, and $\{0, 5, 7\}$.

By Proposition 5, choosing the image of any additional element in line with $\{4, 9\}$ will determine ϵ totally. Choosing 2, we know that sending 2 to itself must give the identity. However sending $2 \to 3$ or $2 \to 6$ each yields an order-3 automorphism, forming a pair of inverses. Selecting the latter, we get

$$\epsilon_{057} = (\underline{0\,5\,7})(2\,6\,3)(1\,8\,\mathrm{X})(4)(9).$$

The distinguished 3-cycle is special for two related reasons: it is the set of three points off of the pair of fixed points of ϵ, and it is the only 3-cycle comprised of a noncollinear triple.

This shows that any element of order three has exactly two fixed points $\{a, b\}$ and a distinguished 3-cycle $(\underline{c\,d\,e})$ such that $\{c, d, e\}$ are the three points out of line with the pair $\{a, b\}$.

Definition 9. Let $a, b, n \in E$ be noncollinear. The element ϵ_{abn} is the unique order-3 element of $\mathrm{Aut}(B)$ that contains the three cycle $(a\,b\,n)$.

2.5. Elements of order 5. Suppose $\beta \in \mathrm{Aut}(B)$ has order five. Acting on E, it must have two 5-cycles and a unique fixed point. However, this is also true when acting upon the lines. (Fixing any pair of lines would fix as a set the two points common to those lines. Being odd order, it would have to fix them pointwise, which contradicts our action on points.) Therefore β fixes some unique line, whose points comprise one of its 5-cycles.

Assume β fixes the line $L_\beta = \{2, 6, 7, \mathrm{X}, 8\}$. Now there are six points not on this line, and these form a single orbit under $\mathrm{Stab}(L_\beta)$: $(2\,6\,7\,\mathrm{X}\,8)(1\,3\,9\,5\,4)$ contains the cycle $(1\,3\,9\,5\,4)$ and γ_{01} (seen in the proof of Lemma 4) contains the cycle $(0\,1)$. If β fixes one of these points, then so would all four nonidentity powers of β. So there must be at least 24 elements of order five that fix this line, if there are any. On the other hand, there are only 24 distinct possible 5-cycles one can make from $\{2, 6, 7, \mathrm{X}, 8\}$, and each one would determine the rest of the action. Given that we already know $(2\,6\,7\,\mathrm{X}\,8)(1\,3\,9\,5\,4)(0)$ is an automorphism of B, we conclude that all 24 possible 5-cycles must extend.

$$\beta_{267\mathrm{X}8} = (\underline{2\,6\,7\,\mathrm{X}\,8})(1\,3\,9\,5\,4)(0).$$

The first 5-cycle, as written, is distinguished as it forms the unique line fixed by β.

This shows that any element of order five can be formed by extending the appropriate 5-cycle fixing a line.

Definition 10. Let $a, b, c, d, e \in E$ comprise a line. Then β_{abcde} is the unique element of $\mathrm{Aut}(B)$ containing the 5-cycle $(a\,b\,c\,d\,e)$.

2.6. Elements of order 6. Suppose $\pi \in \mathrm{Aut}(B)$ has order six. Its square must be of the form ϵ_{cde}, so without loss of generality, let us say

$$\pi^2 = \epsilon_{057} = (\underline{0\,5\,7})(2\,6\,3)(1\,8\,\mathrm{X})(4)(9).$$

If π did not contain $\{0, 5, 7\}$ in a 3-cycle, it would have to be interwoven with either $\{2, 6, 3\}$ or $\{1, 8, \mathrm{X}\}$ into a 6-cycle. However, this would imply that π would send a noncollinear triple, $\{0, 5, 7\}$, to a collinear one (either of the others), which is impossible. Therefore π contains the 3-cycle $(0\,7\,5)$. If it fixed either 4 or 9, it would match $\epsilon_{057}^{-1} = \epsilon_{075}$ in its image of four elements, contradicting Corollary 6, and so π must contain the transposition $(4\,9)$. This fully determines the element, via our work in Example 7:

$$\pi_{075} = (\underline{0\,7\,5})(4\,9)(1\,6\,8\,3\,\mathrm{X}\,2).$$

Definition 11. Let $a, b, n \in E$ be a noncollinear triple. Then π_{abn} is the unique order-6 element containing $(a\,b\,n)$ as a 3-cycle.

2.7. Elements of order 11. Suppose $\alpha \in \mathrm{Aut}(B)$ has order 11. It must therefore fix no lines and no points. Pairs of points are indistinguishable, so assume $\alpha : 0 \to 1 \to 2$. Since $\{0, 1, 2\}$ is collinear, the image of 2 will determine α completely and will have to be in line with $\{1, 2\}$. Therefore α sends 2 to an element of $\{0, 3, 4, 5, 7, 8\}$. If it is either 0, 4, or 7, we end up fixing the line 70142, which cannot happen. If it is 8, then the unique point not in line with $\{0, 1\}$ and $\{1, 2\}$ would be sent to the unique point not in line with $\{1, 2\}$ and $\{2, 8\}$. Unfortunately, both of these points are 9, which we know cannot be fixed. Therefore it must be 3 or 5. Both of these can extend successfully:

$$\alpha = (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,\mathrm{X}) \qquad \alpha' = (0\,1\,2\,5\,9\,6\,4\,8\,\mathrm{X}\,7\,3)$$

While α and α' are not conjugate in $\mathrm{Aut}(B)$, we do have $\alpha^{\beta_{34\mathrm{X}75}} = (\alpha')^2$, so the cyclic subgroups they generate are conjugate.

2.8. Identifying Aut(B). For the remainder of the paper, we reserve the following notation:

$$\alpha = (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,\mathrm{X}).$$
$$\beta = (\underline{2\,6\,7\,\mathrm{X}\,8})(1\,3\,9\,5\,4).$$
$$\gamma = (\underline{3\,4})(1\,8)(2\,9)(5\,\mathrm{X}).$$

Proposition 12. $\mathrm{Aut}(B) = \langle \alpha, \beta, \gamma \rangle$, *a group of order* 660.

Proof. Let $\sigma \in \mathrm{Aut}(B)$. By 2-transitivity, assume σ fixes 0 and 1. As 2 is in line with $\{0, 1\}$, its image must be one of $\{2, 3, 4, 6, 7, \mathrm{X}\}$ and this choice determines σ totally. We can easily check that each of these extend to an automorphism of B. In particular, $\mathrm{Stab}(0) \cap \mathrm{Stab}(1) = \langle \gamma, \epsilon_{598} \rangle \simeq D_6$. Therefore $|\mathrm{Aut}(B)| = 11 \times 10 \times 6 = 660$. □

We identify the following involutions:

$$a = \gamma^{\alpha^3} = (\underline{6\,7})(1\,8)(2\,5)(9\,\mathrm{X}).$$
$$b = \gamma^{\beta} = (\underline{1\,9})(4\,5)(6\,8)(7\,\mathrm{X}).$$
$$c = \gamma = (\underline{3\,4})(5\,9)(2\,\mathrm{X})(6\,7).$$
$$d = \gamma^{\beta\alpha^2} = (\underline{0\,3})(6\,7)(8\,\mathrm{X})(1\,9).$$

It is obvious that $\langle a, b, c, d\rangle \leq \langle \alpha, \beta, \gamma\rangle$. However the opposite is also true. For example, one can check that $\gamma = c$, $\beta = (bc)^2$, and $\alpha = ad^{\beta}b$. Furthermore, if we let $s = \gamma_{67}$ and $t = \gamma_{09}\gamma_{19}$, then we have relations $s^2 = t^3 = (st)^{11} = [s, tstst]^2 = \mathrm{id}$. This is a standard presentation for $L_2(11)$[10], whose order is 660, showing that $\langle s, t\rangle$ generates the entire group.

$$\mathrm{Aut}(B) = \langle s, t \mid s^2 = t^3 = (st)^{11} = [s, tstst]^2 = \mathrm{id}\rangle = L_2(11)$$

In general, $L_2(p)$ is thought of as acting transitively on a set of $p+1$ letters, but for $p \leq 11$, $L_2(p)$ can act on p letters faithfully. It is this unusual action that we have studied so far. We now want to identify a 12-letter population, T, and name its elements in an advantageous way.

We begin with naming the elements of $T = \{0, 1, 2, \ldots, 9, \mathrm{X}, \infty\}$. Strictly speaking, these elements are not the same as those in E. On the other hand, there is an obvious meaning to statements like $t = a$, where $t \in T$, $a \in E$. We will denote an element's action on E as we have previously, and its action on T using square brackets to delimit its cycles. When both are desired, we will separate the actions with $\bullet$:

$$[\text{action on } T] \bullet (\text{action on } E).$$

Proposition 13. *T may be labeled so that we have the following actions*:

$$\alpha = [0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,X][\infty].$$
$$\beta = [1\,3\,9\,5\,4][2\,6\,7\,X\,8][0][\infty].$$
$$\gamma = [0\,\infty][1\,X][2\,5][3\,7][4\,8][6\,9].$$

Proof. The element α, being order 11, must fix some element of T and cyclically permute the rest. As we have not identified any of these elements yet, we can choose that α's action on T is $0 \to 1 \to 2 \to \cdots \to 9 \to \mathrm{X} \to 0$, fixing ∞. This gives us the desired α.

Since $\alpha^{\beta} = \alpha^3$, which also fixes ∞, β must fix ∞. We claim β cannot have seven fixed points in T. If it did, it would have to fix at least two consecutive points in the 11-cycle of α (acting on T), but then $\alpha^{\beta} \neq \alpha^3$. Thus β fixes exactly two points in T. The unique fixed point of β not fixed by α we call 0. (Note that the other elements of T had not yet been distinguished.) Once again using $\alpha^{\beta} = \alpha^3$ tells us that $\beta : 1 \to 3 \to 9 \to 5 \to 4$ and $2 \to 6 \to 7 \to \mathrm{X} \to 8$. Therefore we have determined β as desired.

We now have that i is the unique fixed point of β^{α^i} not fixed by α, which means that we have fully named all the letters of T. So this lettering must imply the action of γ. Knowing that $\beta^{\gamma} = \beta^{-1}$, we have that γ either fixes 0 and ∞ or switches them. If you choose it to fix ∞, then our action on T just mirrors that on E, with ∞ fixed by everything. As we want the transitive action of $L_2(11)$ on T, we need γ to switch the two. Now we use the fact that $\alpha\gamma$ is order 3.

$$\infty \xrightarrow{\alpha\gamma} 0 \xrightarrow{\alpha\gamma} ? \xrightarrow{\alpha\gamma} \infty$$

The missing value, then is X, as $\mathrm{X} \xrightarrow{\alpha} 0 \xrightarrow{\gamma} \infty$. This, along with $\beta^\gamma = \beta^{-1}$, now completely determines γ as desired. □

3. The 266-point graph

3.1. The 66 augmented lines. Now we make a jump from $L_2(11)$ to the first Janko group, J_1. A great deal of technical information about J_1 is available, both from the Atlas [4] and even in the original articles announcing and detailing J_1 [6], [7]. However, we seek to use as little information about J_1 as possible.

We create a graph Γ whose vertices correspond to conjugates of $L_2(11)$ in J_1. Two vertices are adjacent exactly if the corresponding conjugates intersect as large as properly possible, that is, in an A_5. As a result:

- For $v \in V(\Gamma)$, $\mathrm{Stab}(v) \simeq L_2(11)$ (largest proper subgroup of J_1).
- For $e \in E(\Gamma)$, $\mathrm{Stab}(e) \simeq 2 \times A_5$. (The central involution normalizing the A_5 flips the edge.)
- Γ is an 11-regular graph on 266 vertices.
- $\mathrm{Aut}(\Gamma) \simeq J_1$ which acts transitively.

For the last point, note any action on Γ gives an action on the involutions of J_1 via its edge action. J_1 is generated by its involutions and has no outer automorphisms.

The result is known as the Livingstone graph [8]. Our main object in what follows is not necessarily to show that such an object exists. Rather we endeavor to demonstrate how such an object (and J_1) can be understood and explored using $L_2(11)$ and the biplane.

Before we begin, we must make a subtle connection between the populations E and T as we have chosen to label them. We do this by exploring a way of distinguishing between the two elements of T fixed by an element of order five.

As a working example, we choose points 0 and ∞ in T. The elements of order five that fix these two points fix the unique line 267X8. Now look at all elements of order three that have 3-cycle $[0\,?\,\infty]$ on T. To find one such element, first take $\gamma\gamma^{\alpha^2} = \pi_{143}$, then square to give ϵ_{134}. Now act by α^4 to produce ϵ_{578}. This will yield the following element (in the interest of saving space, whitespace will be omitted between letters in cycles hereafter):

$$[0\mathrm{X}\infty][742][591][863] \bullet (578)(1\mathrm{X}0)(492).$$

Acting by β gives you four more. Acting by γ on these switches $[0?\infty]$ to $[\infty?0]$, so then taking inverses, we get the other five. The result is Figure 3.

Looking at the action on E, when elements of the line 267X8 are sent to each other, it is done according to the cycle

$$2 \to 7 \to 8 \to 6 \to \mathrm{X} \to 2.$$

We denote this behavior by writing

$$\mathbf{0}(2786\mathrm{X})\infty.$$

By this construction, we have equivalences

$$\boldsymbol{x}(abcde)\boldsymbol{y} \iff \boldsymbol{x}(bcdea)\boldsymbol{y} \iff \boldsymbol{y}(edcba)\boldsymbol{x}.$$

```
[02∞][837][149][6X5]•(186)(920)(347)(X)(5)
[06∞][29X][315][784]•(327)(560)(91X)(8)(4)
[07∞][658][934][X21]•(96X)(470)(538)(2)(1)
[0X∞][742][591][863]•(578)(1X0)(492)(6)(3)
[08∞][X16][453][279]•(4X2)(380)(156)(7)(9)

[09∞][516][72X][384]•(521)(907)(46X)(8)(3)
[05∞][437][X68][921]•(463)(50X)(178)(2)(9)
[04∞][19X][872][563]•(179)(408)(3X2)(6)(5)
[01∞][358][2X6][479]•(3X5)(102)(986)(7)(4)
[03∞][942][687][1X5]•(984)(306)(527)(X)(1)
```

Figure 3. The 10 order-3 elements with [0?∞].

0 (2786X) ∞	**1** (26X78) **2**	**3** (678X2) **6**	**9** (7X286) **7**	**5** (X8627) **X**	**4** (8276X) **8**
1 (38970) ∞	**2** (37089) **3**	**4** (78903) **7**	**X** (80397) **8**	**6** (09738) **0**	**5** (93870) **9**
2 (49X81) ∞	**3** (4819X) **4**	**5** (89X14) **8**	**0** (914X8) **9**	**7** (1X849) **1**	**6** (X4981) **X**
3 (5X092) ∞	**4** (592X0) **5**	**6** (9X025) **9**	**1** (X2509) **X**	**8** (2095X) **2**	**7** (05X92) **0**
4 (601X3) ∞	**5** (6X301) **6**	**7** (X0136) **X**	**2** (0361X) **0**	**9** (31X60) **3**	**8** (160X3) **1**
5 (71204) ∞	**6** (70412) **7**	**8** (01247) **0**	**3** (14720) **1**	**X** (42071) **4**	**9** (27104) **2**
6 (82315) ∞	**7** (81523) **8**	**9** (12358) **1**	**4** (25831) **2**	**0** (53182) **5**	**X** (38215) **3**
7 (93426) ∞	**8** (92634) **9**	**X** (23469) **2**	**5** (36942) **3**	**1** (64293) **6**	**0** (49326) **4**
8 (X4537) ∞	**9** (X3745) **X**	**0** (3457X) **3**	**6** (47X53) **4**	**2** (753X4) **7**	**1** (5X437) **5**
9 (05648) ∞	**X** (04856) **0**	**1** (45680) **4**	**7** (58064) **5**	**3** (86405) **8**	**2** (60548) **6**
X (16759) ∞	**0** (15967) **1**	**2** (56791) **5**	**8** (69175) **6**	**4** (97516) **9**	**3** (71659) **7**

Figure 4. The 66 augmented lines.

That is, shifting the 5-cycle changes nothing, and switching the outer symbols reverses the cycle. We call this symbol an *augmented line*. Letting $\langle \alpha, \beta, \gamma \rangle$ act on our original augmented line gives us the table in Figure 4, which is indispensable for what follows.

Given $v \in V(\Gamma)$ we would like to know the sizes of the orbits as $\mathrm{Stab}(v)$ acts on $V(\Gamma)$.

Lemma 14. *Let $v \in V(\Gamma)$. Acting on $V(\Gamma)$, $L = \mathrm{Stab}(v)$ produces a unique orbit of size 1 and at least one orbit each of sizes 11 and 12.*

Proof. We claim $\{v\}$ is the only orbit of size 1. If L were to fix both v and w, say, then we could select $\sigma \in \mathrm{Aut}(\Gamma)$ with $\sigma(v) = w$. But then $L^\sigma = L$, with $\sigma \notin L$. This cannot be, as L is a maximal subgroup of a simple group.

All other orbits must have order equal to an index of a proper subgroup of $L_2(11)$ less than 266. This gives possibilities of

$$\{11, 12, 55, 60, 66, 110, 132, 165, 220\}.$$

The 11 neighbors of v must comprise a union of orbits, so at least one orbit has size 11.

If we let $n_k \geq 0$ be the number of orbits of size k, then we need $\sum k n_k = 266$, where the sum may be restricted to values of k in our above list (also including the term corresponding to $k = 1$). Reducing modulo 11, all but three terms drop out, leaving us with

$$1 + 12n_{12} + 60n_{60} \equiv 266 \bmod 11 \Longrightarrow n_{12} + 5n_{60} \equiv 2 \bmod 11.$$

If $n_{12} = 0$, we would need that $n_{60} \equiv 7 \bmod 11$. However this gives us far too many vertices. Therefore $n_{12} \geq 1$. □

Now we describe adjacency in Γ. We choose a vertex and name it $\bullet$. We know $L = \mathrm{Stab}(\bullet) = \langle \alpha, \beta, \gamma \rangle \simeq L_2(11)$ and that L will act on the 11 neighbors of $\bullet$ as it does on E. Each edge emanating from $\bullet$ is stabilized by a $2 \times A_5$ containing a unique central involution (which we can identify with the edge). Name one such involution 0, with the vertex on the other end now named $\bullet 0$. Now let α act, thereby naming the involutions $I = \{0, 1, \ldots, \mathrm{X}\}$ and neighbors $\{\bullet 0, \bullet 1, \ldots, \bullet \mathrm{X}\}$.

We continue to name vertices in this manner, from $\bullet$. That is, for $a_1, \ldots, a_k \in I$, $\bullet a_1 \cdots a_k$ is the vertex obtained by acting on $\bullet$ by a_1, then a_2, and so on. Now, since the neighbors of $\bullet$ are the set $\{\bullet i : i \in I\}$. We have that in general the neighbors of $\bullet a_1 \cdots a_k$ are the set $\{\bullet i a_1 \cdots a_k : i \in I\}$. If $v, w \in V(\Gamma)$ are neighbors, we say v and w are *joined* and write $v - w$. Our next focus, then must be to shorten these names, and to tell when two reduced names are equivalent.

3.2. Length-2 products: the pentagon relation. We know that the points $\bullet i$ are distinct, as there must be 11 of them, but what about products $\bullet i j$? Clearly, if $i = j$, we get $\bullet$, so we need only to consider the case where $i \neq j$. The following arguments not only show that these will all be distinct elements, but describe the smallest nontrivial cycles in Γ.

Lemma 15. *Let $H \leq J_1$ be isomorphic to D_6. Then H is contained in five conjugates of L.*

Proof. All copies of D_6 in J_1 are conjugate, and the same is true of those in $L_2(11)$. We have $N_{J_1}(D_6) \simeq D_6 \times D_{10}$, which has index 2926. $N_{L_2(11)}(D_6) \simeq D_{12}$, which has index 55. As there are 266 copies of L in J_1, the number of such copies containing H must be $55 \times 266/2926 = 5$. □

Lemma 16. *Let i, j be distinct elements of I. Then $\bullet i$ is not adjacent to $\bullet j$.*

Proof. Suppose $\bullet i - \bullet j$. By 2-transitivity of L, this would be true for any pair of distinct neighbors to $\bullet$. Then $\bullet$ and its neighbors would form a K_{12} with no further adjacencies. By transitivity, Γ would decompose into a union of K_{12} graphs, impossible for a graph on 266 vertices. □

Lemma 17. *Let i, j be distinct elements of I. Then Γ contains a 5-cycle of the form $\bullet - \bullet i - \bullet j i - \bullet i j - \bullet j (-\bullet)$.*

Proof. Without loss of generality choose $i = 0$, $j = 1$. Let $H = \mathrm{Stab}(\bullet) \cap \mathrm{Stab}(\bullet 0) \cap \mathrm{Stab}(\bullet 1)$, $P = \{\bullet, \bullet 0, \bullet 1, \bullet 01, \bullet 10\}$, $F = \{v \in V(\Gamma) : H \subset \mathrm{Stab}(v)\}$. H is the subgroup of L that stabilizes points 0 and 1, which is isomorphic to D_6. F is the set of vertices fixed by H. Clearly $P \subseteq F$. We want to show that $P = F$. By Lemma 15, $|F| = 5$, but we need to be sure the elements in P are distinct.

We know $\bullet$ and its neighbors are distinct, so we need only check a few cases. If $\bullet = \bullet 01$, then acting by 1 gives $\bullet 1 = \bullet 0$, which is impossible. If $\bullet 0 = \bullet 10$, then $\bullet = \bullet 1$, which is no good. On the other hand, if $\bullet 0 = \bullet 01$, then $\bullet 0$ would be adjacent to $\bullet 1$, which is disproved by Lemma 16. So finally, we consider $\bullet 01 = \bullet 10$. In this case, Γ would contain the 4-cycle $\bullet - \bullet 0 - \bullet 10 = \bullet 01 - \bullet 1 (-\bullet)$. We show this is false.

H is normalized by a $D_6 \times D_{10}$ in J_1, but only a D_{12} in L. Let σ be an order 5 element in $N_{J_1}(H)$. Clearly then $\sigma \notin L$, and so it does not fix $\bullet$. On other hand, it must fix F as a set. Thus σ cyclically permutes the five points in F.

The only neighbors of $\bullet$ contained in F are $\bullet 0$ and $\bullet 1$, as no other $\bullet i$ would be fixed by, say $\epsilon_{589} = (589)(247)(3\mathrm{X}6)$. Since $\bullet$ is adjacent to exactly two points of F, so must be $\sigma^i(\bullet)$, proving that the elements of F form a 5-cycle in Γ with no other adjacencies among them. This is impossible if there is a 4-cycle among them. Therefore $\bullet 01 \neq \bullet 10$, $P = F$, and $\bullet - \bullet 0 - \bullet 10 - \bullet 01 - \bullet 1(-\bullet)$ forms a 5-cycle in Γ. □

Theorem 18 (The Pentagon Relation). *Let i, j be distinct elements of I. Then $\bullet iji = \bullet ij$, that is, $\bullet ij$ and $\bullet ji$ are adjacent, with $ijiji \in L$.*

Proof. Let σ be the element in the preceding proof. By perhaps replacing σ with one of its powers, we may assume σ sends each vertex to one of its neighbors along the cycle, so suppose $\bullet \to \bullet 0 \to \bullet 10 \to \bullet 01 \to \bullet 1 \to \bullet$. By $\bullet \to \bullet 0$, we know $\sigma = \tau 0$, where $\tau \in L$. Now given $\bullet 0 \to \bullet 10$, that is $\bullet 0\tau 0 = \bullet 10$, we see that $\tau : 0 \to 1$. Using $\bullet 1 \to \bullet$, we have $\bullet 1\tau 0 = \bullet \Longrightarrow \bullet 1\tau = \bullet 0$, so $\tau : 1 \to 0$. Finally $\bullet 10\tau 0 = \bullet 01 \Longrightarrow \bullet 010 = \bullet 01$. Acting by L, we get the relation. Our adjacency condition implies that $\bullet ij$ and $\bullet ji$ are adjacent, and acting by ji on both sides gives $\bullet ijiji = \bullet$, which tells us $ijiji \in \mathrm{Stab}(\bullet) = L$. □

What is the size of the orbit of $\bullet ij$ under L? $\mathrm{Stab}_L(\bullet ij)$ definitely contains the D_6 that stabilizes i and j in I, so the orbit size must be a divisor of 110. Using our previous list of possible orders, this narrows the possibilities down to 11, 55, and 110. The following shows that 55 is impossible.

Lemma 19. $\mathrm{Stab}_L(\bullet 01) \not\simeq D_{12}$.

Proof. Deny and let $\rho \in \mathrm{Stab}_L(\bullet 01)$ of order 6. Such elements of L fix no element of E. However, ρ^2 is in the D_6 that stabilizes exactly the points 0 and 1. Therefore ρ must switch 0 and 1. This would mean that $\bullet 01 = \bullet 01\rho = \bullet 10$, which contradicts the Pentagon Relation. □

For the size of the orbit to be 11, $\mathrm{Stab}_L(\bullet 01)$ would have to be an A_5, and therefore either stabilize a line or a point of B. We now show this cannot be.

Lemma 20. $\mathrm{Stab}_L(\bullet 01)$ *is not the stabilizer of a line or a point of B.*

Proof. Let $S = \mathrm{Stab}_L(\bullet 01)$. No line is fixed by both γ and $\epsilon_{589} = (589)(247)(3\mathrm{X}6)$, each of which are in S. Therefore S stabilizes no line. The only points of B that are fixed by these two elements are of course 0 and 1. Suppose $S = \mathrm{Stab}(1)$. Since this acts transitively on the other points of B, we would have

$$\bullet 01 = \bullet 21 = \bullet 31 = \cdots = \bullet \mathrm{X}1$$

and therefore

$$\bullet 0 = \bullet 2 = \bullet 3 = \cdots = \bullet \mathrm{X}$$

which is nonsense. On the other hand, if $S = \mathrm{Stab}(0)$, then $\bullet 01 = \bullet 02 = \cdots = \bullet 0\mathrm{X}$. Acting by α, we would get $\bullet ij = \bullet ik$, that is $\bullet ijk = \bullet i$ for all i, j, k distinct. As a result, every length-3 product (excluding the degenerate cases $\bullet iii = \bullet i$, $\bullet iij = \bullet j$, and $\bullet ijj = \bullet i$) would be $\bullet iji = \bullet ij$ or $\bullet ijk = \bullet i$, and our total number of vertices would be no more than $1 + 11 + 110 = 122$, a contradiction. □

As a result of the preceding arguments, we can conclude the following.

Proposition 21. *Let i, j be distinct elements of I.* $\mathrm{Stab}_L(\bullet ij) \simeq D_6$. *The orbit of $\bullet ij$ under L has size* 110, *and the* 110 *names $\bullet ab$ ($a, b \in I$, $a \neq b$) correspond to distinct points.*

3.3. Length-3 products: The Pentagram Relation. What can we say about the neighbors of $\bullet ab$, where $a \neq b$? These are the points of the form $\bullet iab$, $i \in I$. Under the action of L, we can distinguish four cases: $i = a$, $i = b$, i is a point not contained in but collinear with $\{a, b\}$, i is a point not in line with $\{a, b\}$. The first two cases yield $\bullet aab = \bullet b$ and $\bullet bab = \bullet ba$, so only the last two cases are new.

Theorem 22 (The Pentagram Relation). *Let $\{a, b, n\}$ be a noncollinear triple. Then $\bullet abn = \bullet ba$, that is, $abnab \in L$.*

Proof. All involutions are conjugate in J_1, and each is normalized by a $2\times A_5$, so J_1 contains $|J_1|/|2\times A_5| = 1463$ such elements. We know that each involution of L is of the form γ_{ab}, of which there are 55. There are 266 conjugates of L, so each involution in J_1 is contained in $266 \times 55/1463 = 10$ conjugates. These conjugates each fix a unique point in Γ, so each involution will fix exactly 10 points.

L acts transitively on pairs of points, and given a pair (a, b), there is an appropriate ϵ_{ijk} that fixes them while cycling through the three points not in line with them. As a result, L acts transitively on noncollinear triples, so we will specifically examine $\bullet 018$. The involution γ fixes $0, 1, 8 \in E$, so it must fix the following ten points, all of which we know to be distinct

$$\{\bullet, \bullet 0, \bullet 1, \bullet 8, \bullet 01, \bullet 10, \bullet 08, \bullet 80, \bullet 18, \bullet 81\}.$$

At the same time, γ must fix $\bullet 018$. If $\bullet 018$ were equal to $\bullet$, $\bullet i$, or $\bullet i8$, we would have some nontrivial equality between products of length 2 or less, which we know is impossible. Furthermore, $\bullet 018 = \bullet 8i \Rightarrow \bullet 01 = \bullet 8i8 = \bullet 8i$, which gives the same result. Thus $\bullet 018 \in \{\bullet 01, \bullet 10\}$.

Assume $\bullet 018 = \bullet 01$. Then $\bullet 01810 = \bullet$ and so $01810 \in L$. In turn, then, we would have $banab \in L$ for all such noncollinear triples $\{a, b, n\}$. Now apply 10801 to the adjacent points $\bullet 0$ and $\bullet 80$. Note in the second case that 80108 is of the form $banab$ so is in L:

$$\bullet 010801 = \bullet 01801 = \bullet 0101 = \bullet 011 = \bullet 0 \qquad \bullet 8010801 = \bullet(80108)01 = \bullet 01$$

The two resulting points are not adjacent, and so we have a contradiction. Therefore $\bullet 018 = \bullet 10$, and the relation $\bullet abn = \bullet ba$ is established. □

Now that we have The Pentagon and Pentagram Relations, we want to identify the elements $ababa$ and $abnab$ within L.

Proposition 23. *Let $\{a, b, n\}$ be a noncollinear triple. Then $ababa = babab = \gamma_{ab}$ and $abnab = \pi_{ban}$.*

Proof. $ababa$ is clearly order-2, and since $\bullet n(ababa) = (\bullet nab)aba = \bullet anaba = \bullet anba = \bullet naa = \bullet n$, $ababa$ is an involution of L that fixes each of the three points not in line with $\{a, b\}$. This is exactly γ_{ab}. Of course, $\gamma_{ab} = \gamma_{ba}$, so $ababa = babab$.

Now look at $abnab$. Acting on $\bullet a$, $\bullet b$, and $\bullet n$, we see that they are cyclically permuted:

$$\bullet aabnab = \bullet bnab = \bullet nbb = \bullet n.$$

$$\bullet babnab = \bullet banab = \bullet abab = \bullet abb = \bullet a.$$

$$\bullet nabnab = \bullet annab = \bullet b.$$

So $abnab$ contains the 3-cycle $(b\,a\,n)$. The only such elements in L are ϵ_{ban} and π_{ban}. Under the action of ϵ_{ban}, there are two fixed points, x and y, such that $\{a, b, n\}$ is the triple of points not in line with $\{x, y\}$. So, if $abnab = \epsilon_{ban}$, we would have $\bullet xy(abnab) = \bullet xy$, but now knowing that each of a, b, and

n are noncollinear with $\{x, y\}$, we can reduce the left-hand side with five uses of The Pentagram Relation, yielding $\bullet yx = \bullet xy$. Therefore $abnab = \pi_{ban}$. □

The only length-3 products left unexamined are those of the form $\bullet abc$, where $\{a, b, c\}$ is a collinear triple. In examining these, we actually get a result about certain length-4 products. In the following discussion, we will make heavy use of the pentagram relation.

Proposition 24. *The point* $\bullet 3890$ *is contained in an orbit of size* 12.

Proof. We have

$$\bullet 1X3 = \bullet 1\pi_{3X9}3X9 = \bullet 83X9 = \bullet\pi_{38X}389 = \bullet 389 = (\bullet 1X3)\beta.$$

In other words, $\bullet 1X3$ is fixed by β, and so the five names $\bullet 1X3$, $\bullet 389$, $\bullet 925$, $\bullet 564$, and $\bullet 471$ correspond to the same point. But we can do even better.

$$\bullet 3890 = \bullet 38\pi_{091}091 = \bullet 84091 = \bullet 8\pi_{049}041 = \bullet 2041 = (\bullet 1X30)\alpha = (\bullet 3890)\alpha.$$

So the point $\bullet 3890$ is fixed by α. But since β fixes 0 and $\bullet 389$ it must also fix $\bullet 3890$. Since $\langle\alpha, \beta\rangle \simeq 11 : 5$, the orbit of $\bullet 3890$ must have order dividing 12. It cannot be the 1-orbit, as then $\bullet 38 = \bullet 09$, so considering our previous list of possible orbit orders, we see it must be 12. □

We have that $\mathrm{Stab}_L(\bullet 3890) = \langle\alpha, \beta\rangle = \mathrm{Stab}_L(\infty)$. Therefore we attach to $\bullet 3890$ (and its 54 other representations as length-4 products) the unique name $\infty\bullet$. Letting L act on these points and this new name gives unique names to each of the points in the 12-orbit. Namely, for $x \in T$, the point $x\bullet$ is the unique point stabilized by $\mathrm{Stab}_L(x)$.

Now, finally, we can close the book on length-3 products, which will actually finish the graph.

Proposition 25. *The points* $\{\bullet abc\}$, *where* $\{a, b, c\}$ *is a collinear triple, form an orbit of size* 132 *under* L.

Proof. We count edges in Γ, using $\bullet 890$ as an example point. (In the proof of Proposition 12, we showed that L is transitive on collinear triples.) Let n equal the size of the orbit of $\bullet 890$ under L. We know $\bullet 90 - \bullet 890 - \bullet 3890$, so points in the $\bullet abc$ orbit share edges with points in the 110-orbit and the 12-orbit. Each point $\bullet bc$ must join exactly six different $\bullet abc$ points. (Note $\bullet abc = \bullet a'bc$ is impossible, so the six points really are distinct.) Therefore there are 660 edges between the $\{\bullet ab\}$ and the $\{\bullet abc\}$ orbits. On the other hand we know $\bullet 890$ is fixed by an element of order five, so n must divide 132. Each $\bullet abc$ has only 11 edges emanating from it, and one of those goes to the 12-orbit, therefore at most 10 go to the 110-orbit. As a result, n must be 66 or 132. If there were only 66 elements in the $\bullet abc$ orbit, then they would each have 10 edges into the 110-orbit and a single edge each going into the 12-orbit. However this gives the number of edges into the 12-orbit as 66, which is impossible, as it needs to be a multiple of 12. □

We have now completed our graph, as $1 + 11 + 110 + 132 + 12 = 266 = |V(\Gamma)|$. At the same, we have established that Γ has diameter 4, so we now introduce the concept of coadjacency.

Definition 26. Let $v, w \in V(\Gamma)$. If the shortest path between v and w is length 4, then we say that v and w are *cojoined*, and denote this with $v \leftrightsquigarrow w$.

Before we start summarizing properties of Γ, we need to simplify the names of the points $\bullet abc$. As it stands, they have five different names as length-3 products. We can remedy this using our names for the points in the 12-orbit as a guide.

Definition 27. For $x \in T$, $a \in E$, the symbol $x \bullet a$ means $(x\bullet)a$, that is, the point $x\bullet$ acted upon by the involution a.

Proposition 28. *The* 132*-orbit is comprised exactly of the points represented by the* (132 *distinct*) *names* $x \bullet a$.

Proof. Let $x \in T$, $a \in E$. The point $x\bullet$ is stabilized by an element of order 11 which acts transitively on E. Therefore, we can write $x\bullet = \bullet a_1 a_2 a_3 a$, where $\{a_1, a_2, a_3\}$ are a collinear triple. (Were they noncollinear, The Pentagram Relation would shorten the name of $x\bullet$.) As a result, $x \bullet a = \bullet a_1 a_2 a_3$, and so these points comprise the 132-orbit. □

3.4. Summary of Γ. The vertices of Γ are exactly those with names $\bullet$, $\bullet a$, $\bullet ab$, $x \bullet a$, $x\bullet$, where $a, b \in E$, $x \in T$. In this section we want to make exact the action of I on these points and the adjacencies between them.

The action of I on the points of form $\bullet$, $\bullet a$, and $x\bullet$ is obvious. We need only study the action on the points $\bullet ab$ and $x \bullet a$.

Theorem 29. *If* $\mathbf{x}(abcde)\mathbf{y}$ *is an augmented line, and* n *is noncollinear with* $\{a, b\}$, *then*:

$$\bullet aba = \bullet ab.$$
$$\bullet abn = \bullet ba.$$
$$\bullet abc = y \bullet e.$$
$$x \bullet ab = y \bullet c.$$
$$x \bullet ac = \bullet ed.$$

Proof. The first two lines are The Pentagon and Pentagram Relations, respectively. The equality $\bullet 389 = \infty \bullet 0$ has already been established and corresponds to augmented line $\mathbf{1}(38970)\infty$. Since L acts on the set of augmented lines, and is transitive on names $\bullet abc$, we get $\bullet abc = y \bullet e$. We also know that $\infty \bullet 0 = \bullet 471$, so $\infty \bullet 01 = \bullet 47$. This corresponds to augmented line $\mathbf{5}(71204)\infty = \infty(02174)\mathbf{5}$. Again, under the action of L this establishes $\infty \bullet ac = \bullet ed$.

Simplifying $x \bullet ab$ takes a bit more work. We know $\infty\bullet = \bullet 1X30$, but being fixed by α^2, it is also $\bullet 3152$. This gives

$$\infty \bullet 02 = \bullet 315202 = \bullet 315\gamma_{02}20 = \bullet 39720.$$

We can now reduce this using the $\bullet abc$ and then $x \bullet ac$ rules

$$\bullet 39720 = 8 \bullet 020 = \bullet 740 = 5 \bullet 1.$$

The resulting equation $\infty \bullet 02 = 5 \bullet 1$ corresponds to $\infty(02174)\mathbf{5}$, so allowing L to act yields $x \bullet ab = y \bullet c$. □

We now tackle the issue of coadjacency, which actually helps us solve a particularly tricky case of adjacency later.

Proposition 30. *If* $\boldsymbol{x}(abcde)\boldsymbol{y}$ *is an augmented line, then:*

$$\bullet \leftrightsquigarrow x\bullet.$$

$$\bullet a \leftrightsquigarrow x\bullet a.$$

$$\bullet ab \leftrightsquigarrow \bullet de,\ y\bullet c.$$

$$x\bullet a \leftrightsquigarrow \bullet a,\ \bullet cb,\ y\bullet b,\ x\bullet.$$

$$x\bullet \leftrightsquigarrow \bullet,\ x\bullet a.$$

Proof. We already know the relations $\bullet \leftrightsquigarrow x\bullet$ and $\bullet a \leftrightsquigarrow x\bullet a$, so we begin our study looking at the 110-orbit. Applying b to both sides of the latter preceding relation, we get $\bullet ab \leftrightsquigarrow x\bullet ab = y\bullet c$. This gives six cojoins. From our actions of I, we know that $y\bullet c = \bullet dea$, so $\bullet ab \leftrightsquigarrow \bullet dea$. Acting by a yields $\bullet ab \leftrightsquigarrow \bullet de$, bringing the total to 12.

With regards to the cojoins of the 132-orbit, we have already established $x\bullet a \leftrightsquigarrow \bullet a$, giving a single cojoin. Given the relation $\bullet ab \leftrightsquigarrow y\bullet c$, we can reverse and shift our line

$$\boldsymbol{x}(abcde)\boldsymbol{y} \Longrightarrow \boldsymbol{y}(cbaed)\boldsymbol{x}$$

to see this is equivalent to $x\bullet a \leftrightsquigarrow \bullet cb$. This gives us five more. Furthermore,

$$\bullet \leftrightsquigarrow x\bullet \Longrightarrow \bullet abc \leftrightsquigarrow x\bullet abc \Longrightarrow y\bullet e \leftrightsquigarrow y\bullet$$

Acting by L gives $x\bullet a \leftrightsquigarrow x\bullet$. We now have seven cojoins. Finally, $\bullet abced \leftrightsquigarrow x\bullet abced \Longrightarrow y\bullet d \leftrightsquigarrow x\bullet c$. Cycling the augmented line gives $x\bullet a \leftrightsquigarrow y\bullet b$. We have now accounted for all 12 cojoins.

The two rules $\bullet \leftrightsquigarrow x\bullet$ and $x\bullet a \leftrightsquigarrow x\bullet$ exhaust the cojoins for the 12-orbit, so we are done. □

Finally, we are ready to fully describe adjacency in Γ. We deal with one case separately. From the proof in Proposition 25, we know that a point $x\bullet a$ joins exactly one point of the 12-orbit, but which one? We know this point must be fixed by the order-5 element fixing $x\bullet a$. Acting on T, this element fixes x and one other element, call it z. Our two candidates then are $x\bullet$ and $z\bullet$, but we have already found that $x\bullet$ is in fact cojoined to $x\bullet a$, so the final join must be $z\bullet$.

We need to describe the relationship between x, a, and z in a more convenient way. According to the above discussion, $\infty\bullet 0 - 0\bullet$, since β fixes $\infty\bullet 0$, and the other element of T fixed by β is 0. Acting by β^a we get $\infty\bullet a - a\bullet$ and thus $\infty\bullet -a\bullet a$. Therefore, $x=\infty \Longleftrightarrow a=z$, and $z=\infty \Longleftrightarrow x=a$.

Applying γ to $\infty\bullet a - a\bullet$ gives:

$0\bullet 1 - \mathrm{X}\bullet$	$0\bullet 2 - 1\bullet$
$0\bullet 3 - 8\bullet$	$0\bullet 6 - 3\bullet$
$0\bullet 9 - 2\bullet$	$0\bullet 7 - 9\bullet$
$0\bullet 5 - 6\bullet$	$0\bullet \mathrm{X} - 5\bullet$
$0\bullet 4 - 7\bullet$	$0\bullet 8 - 4\bullet$

(The case of $0\bullet 0$ is dealt with under the case $x=a$.) This table allows us to prove the following.

Lemma 31. *Let* $x \neq \infty$. *Consider* x *and* a *as integers* $\bmod 11$. *If* $x \neq a$, *then* $x \bullet a - z\bullet$, *where either* z, x, a *or* x, z, a *is an arithmetic progression, modulo* 11, *whose common difference is a quadratic residue.*

Proof. Applying α^a to $0 \bullet 0 - \infty\bullet$ gives the first result. Now suppose $x \neq a$. Then either $a - x$ is a quadratic residue or twice a quadratic residue, since multiplication by 2 modulo 11 switches nonzero quadratic residues and quadratic nonresidues. In the first case, it reduces, via α, to an example in the first column. In the second case, it reduces to one in the second column. Since the result is true in both columns, and α preserves differences, the lemma is proven. ☐

It is useful to have a notation for this relationship.

Definition 32. Let $x \in T$, $a \in E$. Then $x \wedge a = z$, where $x \bullet a - z\bullet$, that is,

$$x \wedge a = \begin{cases} \infty & x = a, \\ a & x = \infty, \\ x - q & a - x = q \in Q, \\ x + q/2 & a - x = q \notin Q, \end{cases}$$

where Q is the set of nonzero quadratic residues in $\mathbb{Z}_{11}$. If $x \wedge a = z$, we say $z \vee a = x$.

Note that the last two cases in the definition of $x \wedge a$ assure that x, z, a or z, x, a form an arithmetic progression, modulo 11, whose common difference is a nonzero quadratic residue. Finally, we can describe the adjacencies among the simplified names of $V(\Gamma)$ in a concise manner.

Proposition 33. *If* $\boldsymbol{x}(abcde)\boldsymbol{y}$ *is an augmented line, and* n *is a point noncollinear with* $\{a, b\}$, *then*

$$\bullet - \bullet a.$$

$$\bullet a - \bullet, \bullet ba.$$

$$\bullet ab - \bullet b, \bullet ba, \bullet an, y \bullet d.$$

$$x \bullet a - \bullet dc, y \bullet d, z \bullet.$$

$$x\bullet - w \bullet a.$$

Where $z = x \wedge a$ *and* $w = x \vee a$.

Proof. Once again, the nontrivial cases begin with the 110-orbit. The point $\bullet ab$ will be joined to $\bullet b$ by definition, but also $\bullet ba$ by the Pentagon Relation and to each $\bullet an$, where n is noncollinear with $\{a, n\}$, by the Pentagram Relation. Each of the six points in line with $\{a, b\}$ serves as the end of some augmented line $\boldsymbol{x}(abcde)\boldsymbol{y} = \boldsymbol{x}(eabcd)\boldsymbol{y}$, so that $\bullet eab = y \bullet d$, giving the last six neighbors to $\bullet ab$.

Now we examine a point $x \bullet a$. The rule $\bullet ab - y \bullet d$ is equivalent to $x \bullet a - \bullet dc$. This gives five adjacencies. Furthermore

$$\bullet ab - y \bullet d \Longrightarrow \bullet ea - y \bullet c \Longrightarrow y \bullet db - x \bullet ab \Longrightarrow y \bullet d - x \bullet a.$$

yields another five. The previous lemma gives the $x \bullet a - z\bullet$ and (the equivalent) $x \bullet - w \bullet a$ adjacencies. ☐

4. Applications

We end by exploring a couple of facets of J_1 and Γ, using the structure we have built. In the first example, we begin with the stabilizer of an edge and proceed to find a way to produce elements of order seven and subgroups of the form $2^3 : 7 : 3$. In the second example, we study special hexagons made of cojoins and how they can be put together to produce an object displaying the $19 : 6$ subgroup.

4.1. An element of order 7. Consider $\mathrm{Stab}(\{\bullet, \bullet 0\})$. We know this is isomorphic to $2 \times A_5$. In particular, the central involution is the one we call 0. Its A_5 intersection with $L = \mathrm{Stab}(\bullet)$ is $\mathrm{Stab}(0)$. A copy of A_5 contains five subgroups isomorphic to 2^2, which pairwise intersect in the identity. One such subgroup of $\mathrm{Stab}(0)$ is $\{\mathrm{id}, \gamma_{34}, \gamma_{67}, \gamma_{18}\}$. Even within $\mathrm{Stab}(0)$, this subgroup can be extended by an element of order three. Without loss of generality assume it fixes 0, sends 1 to 4, and 8 to 3. The element 4 can be then sent to either 6 or 7, and both options work. In fact, the resulting elements are conjugate via γ_{18}, so they give rise to the same extension. We choose $\epsilon_{146} = (\underline{146})(25\mathrm{X})(378)$ for our extending element.

Combined with 0, we get $S = \langle \gamma_{34}, \gamma_{67}, 0, \epsilon_{146} \rangle \simeq 2^3 : 3$. There are seven involutions in S, and each of these corresponds to the central involution of an edge stabilizer, and thus to an edge. We know 0 corresponds to the edge $\{\bullet, \bullet 0\}$, but what are the other correspondences? If $a \neq 0$, then we have $\gamma_{ab} = ababa = a^{ba} = 0^{\gamma_{0a} ba}$, so $\gamma_{ab} \sim \{\bullet \gamma_{0a} ba, \bullet 0 \gamma_{0a} ba\} = \{\bullet ba, \bullet ab\}$. For the involutions $\gamma_{ab}0$, we first need a lemma.

Lemma 34. *If $\{a, b, n\}$ form a noncollinear triple, then $anbnan = \gamma_{ab}$.*

Proof. $anbnan = (anban)(nanan) = \pi_{nab}\gamma_{an}$. Since L is transitive on noncollinear triples, we check

$$\pi_{075}\gamma_{70} \stackrel{?}{=} \gamma_{75},$$
$$(075)(1683\mathrm{X}2)(49)(07)(18)(23)(49) \stackrel{?}{=} (16)(28)(3\mathrm{X})(57),$$
$$(16)(28)(3\mathrm{X})(57) \stackrel{\checkmark}{=} (16)(28)(3\mathrm{X})(57). \qquad \square$$

Now we can say, if $\{a, b, 0\}$ is a noncollinear triple,

$$\gamma_{ab}0 = a0b0a00 = a0b0a = b^{0a} = 0^{\gamma_{0b}0a}.$$

Therefore $\gamma_{ab}0 \sim \{\bullet \gamma_{0b} 0a, \bullet 0 \gamma_{0b} 0a\} = \{\bullet 0a, \bullet 0b\}$. In our copy of 2^3, then we have the following (in addition to the original correspondence $0 \sim \{\bullet, \bullet 0\}$).:

$$\gamma_{34} \sim \{\bullet 34, \bullet 43\} \quad \gamma_{18} \sim \{\bullet 18, \bullet 81\} \quad \gamma_{67} \sim \{\bullet 67, \bullet 76\}$$
$$\gamma_{34}0 \sim \{\bullet 03, \bullet 04\} \quad \gamma_{18}0 \sim \{\bullet 01, \bullet 08\} \quad \gamma_{67}0 \sim \{\bullet 06, \bullet 07\}$$

One maximal subgroup of J_1 is $2^3 : 7 : 3$, but how do we produce such an element of order seven? We seek an element σ that cyclically permutes the seven involutions in our 2^3 (and thus these edges) and is acted upon by ϵ_{146} appropriately, that is, $\sigma^{\epsilon_{146}} = \sigma^2$.

By taking appropriate powers and conjugating by γ_{18}, we may assume σ has actions $\bullet \to \bullet 81$ and $\bullet 0 \to \bullet 18$. Therefore $\sigma = \tau 81$, for some $\tau \in L$. At the same time

$$\bullet 0 \tau 81 = \bullet 18,$$
$$\bullet 0 \tau = \bullet 1818 = \bullet 188 = \bullet 1$$

so τ sends $0 \to 1$ on E. Now use the action by ϵ_{146}:

$$\bullet\sigma^2 = \bullet\sigma^{\epsilon_{146}} = \bullet\sigma\epsilon_{146} = \bullet\tau 81\epsilon_{146} = \bullet 34.$$

$$\bullet\sigma^4 = \bullet\sigma^{\epsilon_{146}^2} = \bullet\sigma\epsilon_{146}^2 = \bullet\tau 81\epsilon_{146}^2 = \bullet 76.$$

Thus

$$\bullet\tau 81\tau 81 = \bullet 34 \Longrightarrow \bullet 81\tau = \bullet 3418 = \bullet 34.$$

so τ sends $8 \to 3$, $1 \to 4$ on E, and

$$\bullet 34\tau 81\tau 81 = \bullet 76, \quad \bullet 34\tau^2 3481 = \bullet 76 \quad \text{and} \quad \bullet\, 34\tau^2 = \bullet 761843 = \mathrm{X} \bullet 5843 = \bullet 1243 = 0 \bullet 03 = \bullet 97,$$

so τ^2 sends $3 \to 9$, $4 \to 7$ on E. There are only two elements of L with these properties

$$\tau_1 = (01427)(3\mathrm{X}968) \quad \text{and} \quad \tau_2 = (014\mathrm{X}7832956)$$

While these yield two different elements of order seven, $\sigma_1 = \tau_1 81$ and $\sigma_2 = \tau_2 81$, they give the same extensions of $\langle a, c, 0, \epsilon_{146}\rangle$, as $\sigma_1^0 = \sigma_2$. Therefore $\langle a, c, 0, \epsilon_{146}, \sigma_1\rangle \simeq 2^3 : 7 : 3$.

4.2. A complex of cohexagons. Suppose we are given vertices $u \leftrightsquigarrow v \leftrightsquigarrow w \in V(\Gamma)$, with $u \neq w$. It is not too difficult to show that there are exactly five ways to complete this to a *copentagon*: $u \leftrightsquigarrow v \leftrightsquigarrow w \leftrightsquigarrow r \leftrightsquigarrow s \leftrightsquigarrow u$, and that all of these are equivalent under J_1. Furthermore, each such copentagon *rotates in* Γ, that is, there exists an element of J_1 that sends $u \to v \to w \to r \to s \to u$.

If we try to do this with six points, starting from $u \leftrightsquigarrow v \leftrightsquigarrow w$, an interesting thing occurs. Without loss of generality we start with $\mathrm{X}\bullet \leftrightsquigarrow \bullet \leftrightsquigarrow \infty\bullet$. The only way to extend this is to do

$$\mathrm{X} \bullet a \leftrightsquigarrow \mathrm{X}\bullet \leftrightsquigarrow \bullet \leftrightsquigarrow \infty\bullet \leftrightsquigarrow \infty \bullet b.$$

If we want to wrap this into a *cohexagon*, we will need that $a = b$ and that the last point be $\bullet a$.

$$\bullet a \leftrightsquigarrow \mathrm{X} \bullet a \leftrightsquigarrow \mathrm{X}\bullet \leftrightsquigarrow \bullet \leftrightsquigarrow \infty\bullet \leftrightsquigarrow \infty \bullet a \leftrightsquigarrow \bullet a.$$

You can see two examples in Figure 5.

However, these cohexagons do not all rotate in Γ. Notice that $\{\bullet, \bullet a\}$ are opposite in the cycle and adjacent in Γ. However, there is but a single choice of a that would make this true for the other pairs of opposite points: $\{\mathrm{X}\bullet, \infty \bullet a\}$ and $\{\infty\bullet, \mathrm{X} \bullet a\}$, namely $a = \mathrm{X}$. In general, then, there is a unique way to complete a partial cohexagon $u \leftrightsquigarrow v \leftrightsquigarrow w$ such that opposite vertices are adjacent in Γ. Furthermore, we will show that these cohexagons rotate in Γ. In the other cohexagons, opposite points do not have consistent distances between them, and so the cycle cannot rotate.

Let us now imagine starting with $\bullet$ cojoined to three points: $\mathrm{X}\bullet$, $\infty\bullet$, and $1\bullet$. This gives us a starting point for three cohexagons. By completing each of these, we get Figure 6. However, this process can be continued (uniquely!) along the outside of this object. If we identify points that recur, the result is a hexagonal graph of cojoins among 38 points, as shown in Figure 7. In the figure, the cohexagons are labeled such that moving left-to-right, starting at the $\mathrm{X}\bullet \leftrightsquigarrow \bullet \leftrightsquigarrow \infty\bullet$ cohexagon, you get $A \to B \to C \to \cdots \to R \to S \to A$. The image has been recentered about cohexagon C. Call this graph Γ_{38}.

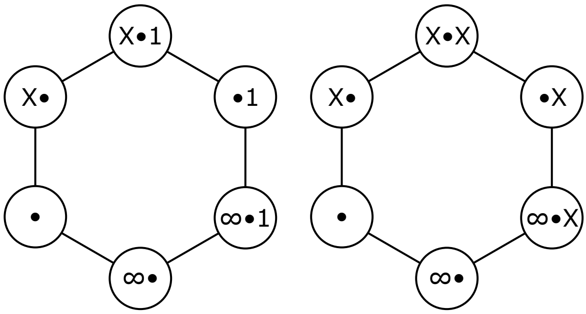

Figure 5. The left cohexagon cannot rotate, whereas the right one can.

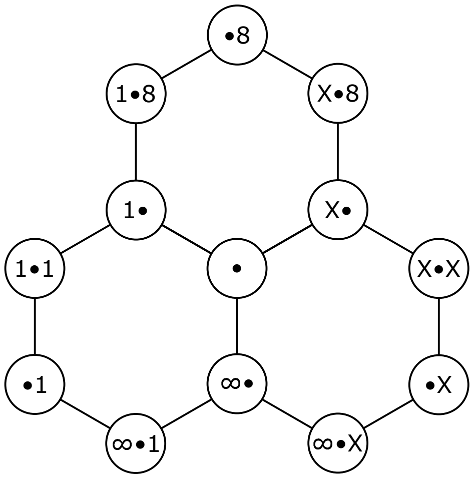

Figure 6. A triple of completed cohexagons.

We contend there is an element, π_C, of order 6 that rotates Γ_{38} about cohexagon C. Now π_C sends $\bullet \to \bullet 94$, so $\pi_C = \tau 94$, where $\tau \in L$. But $\bullet 94 \to \bullet$ as well, so we must have

$$\bullet 94 \tau 94 = \bullet \Longrightarrow \bullet 94 \tau = \bullet 49.$$

and so τ switches 4 and 9 in E. Now using that $\bullet \mathrm{X} \to 8 \bullet 6$, we get

$$\bullet \mathrm{X} \tau 94 = 8 \bullet 6 \Longrightarrow \bullet \mathrm{X} \tau = 8 \bullet 649 = \bullet 2.$$

and so $\tau : \mathrm{X} \to 2$ in E. Since $\{4, 9, \mathrm{X}\}$ is a collinear triple, this completely determines τ as $\pi_{075} = (\underline{075})(1683\mathrm{X}2)(49)$. Finally, we have identified

$$\pi_C = [0\mathrm{X}8\infty 31][257469] \bullet (075)(1683\mathrm{X}2)(49)94$$

and that this element really does complete this rotation can be checked directly.

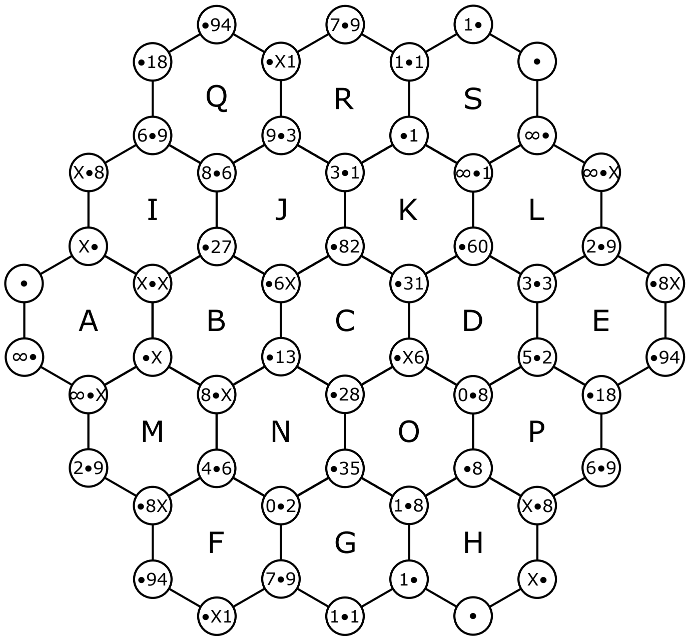

Figure 7. The 38-point graph starting from a triple of cojoins of •.

We now show there is an element, σ, of order 19, that sends cohexagons along in a line, $A \to B \to C \to \cdots$. We have that $\bullet \to \mathrm{X} \bullet \mathrm{X} = \bullet 374$, so $\sigma = \mu 374$, with $\mu \in L$:

$$\bullet 6\mathrm{X} \to \bullet 31 \implies \bullet 6\mathrm{X}\mu 374 = \bullet 31 \implies \bullet 6\mathrm{X}\mu = \bullet 31473 = \bullet 31.$$

$$\bullet 13 \to \bullet \mathrm{X}6 \implies \bullet 13\mu 374 = \bullet \mathrm{X}6 \implies \bullet 13\mu = \bullet \mathrm{X}6473 = \bullet 58.$$

Now that we have the action of μ on four points of E, it is completely determined. To see how this can be found, let us show the action of μ on the augmented lines **0**(2786X)∞ and **6**(70412)**7**, although we will ignore the action on the outside T elements initially:

0	2	7	8	6	X	∞	**6**	7	0	4	1	2	**7**
	?	?	?	3	1			?	2	?	5	?	
	?	?	?	8	5			?	?	?	?	?	
	?	?	?	?	?			?	?	?	?	?	

Looking at the 2786X action, the second line cannot be 82315 (in any order), since 5 could only drop down from a 1, which is not in the top line. This forces the second line to be 601X3, in some order, whose image must contain 3?518. The only option is that the ? is 2, and so $0 \to 2$. The 1 and 3 in line 3 gives a 5 and 8 in line 4, and as it cannot be 32518 again, it must be 05648. Finally, the point 2 is in the first and third lines, so its image must be in the second and fourth lines, which means it is either 0 or 6.

Now look at the 70412 action. The second line there must then contain 250 or 256. The latter is a noncollinear triple, so it is 2509X and $2 \to 0$. Since X appears on both second lines, its preimage must be in both first lines. As it cannot be 2, it must be 7.

This allows us to complete these to as many lines as we wish. Furthermore, we can now match the inner 5-cycles with the appropriate outside T elements to find the action of μ on T. (This is helpful in verifying that the produced element does in fact carry out the described action on all 38 points.)

0	2	7	8	6	X	∞		**6**	7	0	4	1	2	**7**
X	0	X	6	3	1	**7**		**5**	X	2	9	5	0	**4**
2	2	1	3	8	5	**4**		**0**	1	0	7	4	2	**8**
3	0	5	8	6	4	**8**		**X**	5	2	X	9	0	**1**
6	2	4	6	3	9	**1**		**2**	4	0	1	7	2	**9**
5	0	9	3	8	7	**9**		**3**	9	2	5	X	0	∞

Therefore

$$\mu = \pi_{386} = [0X2365][19\infty748] \bullet (02)(15497X)(386)$$

and $\sigma = \pi_{386}374$.

Looking at their actions on the cohexagons, it is easy to see that $\sigma^{\pi_C} = \sigma^{12}$, and so $\langle \sigma, \pi_C \rangle \simeq 19:6$, a maximal subgroup of J_1. These are all conjugate within J_1, and so each has such a 38-point graph. By counting, this correspondence must be one-to-one, as there are $175560/114 = 1540$ conjugates of $19:6$, and there are $266 \cdot \binom{12}{3}/38 = 1540$ different graphs produced in this manner. We have proven the following.

Theorem 35. *Let J_1 act on Γ and $H \leq J_1$ be a subgroup isomorphic to $19:6$. Then H is the stabilizer of a 38-point graph of cohexagons. Such a graph can be generated by choosing a point and three of its cojoins in Γ, and completing cohexagons so that opposite vertices are adjacent.*

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