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Plane algebraic curves of arbitrary genus via Heegaard Floer homology

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Abstract. Suppose C is a singular curve in $\mathbb{C}P^2$ and it is topologically an embedded surface of genus g; such curves are called cuspidal. The singularities of C are cones on knots K_i. We apply Heegaard Floer theory to find new constraints on the sets of knots $\{K_i\}$ that can arise as the links of singularities of cuspidal curves. We combine algebro-geometric constraints with ours to solve the existence problem for curves with genus one, $d > 33$, that possess exactly one singularity which has exactly one Puiseux pair $(p;q)$. The realized triples (p,d,q) are expressed as successive even terms in the Fibonacci sequence.

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1. Singular curves

Let C be an irreducible algebraic curve in $\mathbb{C}P^2$, defined as the zero set of a homogeneous polynomial f of degree d. Such a curve is called *cuspidal* if the singular points of C are all unibranched; that is, the singular points are isolated and the link of each singularity is a knot in S^3 (such knots are often called *algebraic knots*). Cuspidal curves form a natural family of algebraic curves that are topologically embedded surfaces.

The theory of cuspidal curves of higher genus has not drawn as much attention as the case of rational cuspidal curves, those of topological genus zero. One of the inherent difficulties in the higher genus setting is that the complement of a curve is not a rational homology ball (in the language of algebraic geometry, a $\mathbb{Q}$–acyclic surface, see [12]). The effect of this is that one of the main tools in studying rational cuspidal curves, namely the "semicontinuity of the spectrum," which is a main ingredient of

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a classification result in [10], becomes considerably less restrictive if the genus is greater than zero. Section 10 presents a more detailed discussion of these issues. Notice, however, that in [20] the classification result of [10] was proved using only the semigroup distribution property of [4,11].

The goal of this paper is to investigate the singularities of cuspidal curves. To state the main results, we need some background. To each singular point one associates the δ-invariant and the Milnor number μ, which for unibranched singular points are related by $\mu = 2\delta$. By definition, δ is the 3-genus of the associated linking circle K. The genus g of C and the δ-invariants, $\{\delta_i\}$, of the set of singular points, $\{z_i\}_{i=1}^n$, are related by the *genus formula*:

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{n} \delta_i.$$

Our goal is to find constraints on the possible sets of singularities beyond those given by the genus formula. The basic idea is topological. We suppose that a curve $C \subset \mathbb{C}P^2$ with some collection of singularities exists. Let $Y(C)$ denote the three-manifold which arises as the boundary of a closed regular neighborhood $N(C)$ of C. The homeomorphism type of this three-manifold depends only on the degree, genus, and the links of the singularities of C. The complement $\mathbb{C}P^2 - \mathrm{Int}(N(C))$ is a smooth four-manifold with particularly simple algebraic topology; in particular, its intersection form is identically zero. To show that C cannot exist, then, it suffices to show that $Y(C)$ cannot bound a four-manifold with trivial intersection form. For this, we use invariants derived from the Heegaard Floer homology of $Y(C)$ [29]. These invariants are a generalization of the influential "correction terms" associated to rational homology three-spheres used in the study of homology cobordism groups and knot concordance. Among the many important references, we mention just one, [14], which informed our original work on this paper.

In the present situation, two particularly useful invariants derived from the Heegaard Floer homology complexes associated to $Y(C)$ present themselves, which we refer to as the "bottom" and "top" correction terms. They depend on a choice of Spin^c structure whose Chern class is torsion. The key feature of these invariants is that their values bound the characteristic numbers of smooth negative semi-definite four-manifolds bounded by $Y(C)$, where negative semi-definite means that the self-intersection of any closed surface is non-positive. Notice that $\mathbb{C}P^2 - \mathrm{Int}(N(C))$ is negative semi-definite with *either* of its orientations.

One must have means to compute the correction terms. As a first step, we show that $Y(C)$ has a simple description as surgery on a knot in the connected sum of copies of $S^1 \times S^2$. With this, along with the fact that the links of the singularities are all L-space knots [15] (in particular, their knot Floer homology complexes are determined by their Alexander polynomials), the computation of the correction terms becomes algorithmic by way of a surgery formula [30].

Our main result, Theorem 1 can be seen as a higher genus generalization of [4, Theorem 6.5], or as a proof of an analogue of [11, Conjecture 2]. Its statement uses the notion of the *semigroup of a singularity*. This semigroup of $\mathbb{Z}_{\geq 0}$ is defined precisely in Section 7.1; in the special case that the link of the singularity is a (p, q)-torus knot, the semigroup is generated by p and q. If we are given a finite collection of semigroups $S_1, \ldots, S_n$, we can define a function $R : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ as follows:

$$R(u) = \min_{\substack{k_1+\cdots+k_n=u \\ k_i \geq 0}} \sum_{i=1}^{n} \#\{S_i \cap [0, k_i)\}.$$

Note that if $k_i = 0$ for some i, then the number of elements in $\{S_i \cap [0, k_i)\}$ is 0.

Theorem 1. *Suppose C is a cuspidal curve of genus g and degree d in $\mathbb{C}P^2$. Let $z_1, \ldots, z_n$ denote its singular points, $S_1, \ldots, S_n$ the corresponding semigroups, and R the function defined above. Then for any $j = 1, \ldots, d-2$ and any $b = 0, \ldots, g$, we have*

$$0 \leq R(jd - 2b + 1) - \frac{(j+1)(j+2)}{2} + b \leq g. \tag{1.1}$$

Unpacking the left inequality in (1.1) yields that for any $j = 1, \ldots, d-2$ and $b = 0, \ldots, g$, and for any non-negative $k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = jd + 1 - 2b$

$$\sum_{i=1}^{n} \#\{S_i \cap [0, k_i)\} \geq \frac{(j+1)(j+2)}{2} - b. \tag{1.2}$$

Inequality (1.2) depends on the genus g of the curve only through the possible range b. Stated this way, (1.2) bears a strong resemblance to [11, Proposition 2]; in fact, the case $b = 0$ of (1.2) is exactly [11, Proposition 2], which was proved using an elementary dimension counting argument for projective curves. Indeed, the expression $\sum_{i=1}^{n} \#\{S_i \cap [0, k_i)\}$ can be interpreted as a number of linear constraints which is sufficient to ensure that an algebraic curve, viewed as an element in a vector space of homogenous polynomials, intersects C at z_i with multiplicity k_i. This interpretation leads directly to the left-hand inequality in the case $b = 0$. It would be interesting to know if algebro-geometric techniques could be used to prove (1.2) for any other values of b (the argument of [11] would need to be altered to incorporate the genus) or, for that matter, if algebraic geometry could shed light on the right-hand inequality in our theorem. Regardless, it is important to stress that while Theorem 1 is stated for algebraic curves, our techniques lie in the realm of smooth topology; that is, our inequalities are satisfied for C^∞ maps $f : C \hookrightarrow \mathbb{C}P^2$ of surfaces which are topological embeddings, and for which there are a finite collection of points $z_i \in C$ satisfying $df(z_i) = 0$ near which f appears holomorphic (that is, within local charts). It is also worth pointing out that our result can be generalized to surfaces in any smooth 4-manifold with the rational homology of $\mathbb{C}P^2$. In particular, there are direct analogues of Theorem 1 which restrict the cuspidal curves in fake projective

planes (slight care is needed to account for the image of the inclusion map on the first homology of C); for a description of the 50 distinct complex algebraic surfaces with the same Betti numbers as $\mathbb{C}P^2$, see [8,24,34]).

Theorem 1 appears to be a useful tool for studying cuspidal curves and can effectively obstruct many configurations of singularities from arising on curves of fixed genus and degree (we give some examples in Section 8). Combining Theorem 1 with tools from algebraic geometry yields even better results. For instance, we can classify singularities of simple form that can occur on a genus one curves possessing a single singularity. From the perspective of algebraic geometry, the theorem is most naturally stated in terms of *Puiseux pairs*; we note that a singularity has one Puiseux pair (p,q) precisely when its link is a (p,q)-torus knot (or, equivalently, it is equisingular to $z^p + w^q = 0$). Here p and q are positive, coprime integers.

Theorem 2. *Suppose that C is a cuspidal curve of degree $d > 33$, genus 1, possessing a single singularity with one Puiseux pair (p,q). Then there exists $j > 0$ such that $d = \phi_{4j}$ and $(p,q) = (\phi_{4j-2}, \phi_{4j+2})$, where $\phi_0, \phi_1, \ldots$ are the Fibonacci numbers (normalized so $\phi_0 = 0$, $\phi_1 = 1$).*

In fact, the above is a simplified statement of Theorem 9.1, which additionally provides a finite list of possible triples $(p,q;d)$ with $d \leq 33$. The proof of this result uses Theorem 1 in conjunction with a *multiplicity bound*, expressed in Theorem 9.9. The latter bounds from above the degree of a cuspidal curve under consideration by a linear function of the multiplicity of its singular point (here, the multiplicity is the minimum of p and q). The multiplicity bound, in turn, comes from a general bound on certain numerical invariants of the singular points, the so-called Orevkov $\overline{M}$-numbers. These numbers are derived from the cohomology of a minimal good resolution of the singular points, and the bound which they satisfy is a consequence of the Bogomolov–Miyaoka–Yau inequality. Note that Theorem 2 is only an obstruction: it says nothing about whether the triples $(\phi_{4j-2}, \phi_{4j+2}; \phi_{4j})$ are realized by algebraic curves. As counterpoint, however, we can explicitly construct genus one curves of degree ϕ_{4j} with one cusp and one Puiseux pair using a technique of Orevkov [26]:

Theorem 3 (Proposition 9.14 below; cf. [26, Theorem C]). *For any $j = 1, 2, \ldots$ there exists a curve of genus 1 and degree ϕ_{4j} having a unique singularity with one Puiseux pair $(\phi_{4j-2}, \phi_{4j+2})$.*

One can also produce curves realizing some of the exceptional cases (all of which satisfy $d \leq 33$) described in Theorem 9.1. Taken together, we solve the geography problem for cuspidal curves with one singularity and one Puiseux pair (modulo a few low degree cases where curves have yet to be constructed).

Computer experiments suggest that the only instances of Puiseux pairs (p,q) and degrees d that satisfy all the criteria from Theorem 1 but fail the BMY multiplicity bound are those in the family $(p,q) = (a, 9a+1)$ and $d = 3a$. It would be interesting to know whether these are indeed the only additional cases passing the

criteria of Theorem 1, and whether they can be realized by embedded surfaces in the C^∞ category.

In [2] Bodnár and Némethi restated [4, Theorem 6.5] in the language of lattice homology. A natural question to ask is the following:

Question 4. *Is there a reformulation of Theorem 1 in the language of lattice homologies?*

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2. Overview and Notation

Let $N(C)$ be a closed regular neighborhood of C, having three-manifold boundary Y. The complement of the interior of $N(C)$ in $\mathbb{C}P^2$ is a smooth four-manifold X with boundary $-Y$. In the next section we study the algebraic topology of X. In particular, we verify that the intersection form on $H_2(X)$ is identically zero, and study the restriction map $H^2(X) \to H^2(-Y)$. Taken together, this information serves as the topological input for the analytic obstructions we consider.

2.1. *d*-invariants. Heegaard Floer homology provides obstructions to a Spin^c three-manifold bounding a negative semi-definite Spin^c four-manifold. These obstructions are often referred to as d-invariants. To define them, recall that if $\mathfrak{s}$ is a Spin^c structure on Y, then Heegaard Floer theory yields a chain complex $CF^\infty(Y,\mathfrak{s})$, freely generated as a module over $\mathbf{F}[U,U^{-1}]$ (we use $\mathbf{F}=\mathbb{Z}_2$ throughout). The complex is equipped with a $\mathbb{Z}$ filtration, and the filtered homotopy type of $CF^\infty(Y,\mathfrak{s})$ is an invariant of the pair $(Y,\mathfrak{s})$. In the case that $\mathfrak{s}$ has torsion first Chern class, the complex has a grading by rational numbers. Acting by U in the base ring lowers the filtration level by one and the grading by two. See [31] for the definition of $CF^\infty(Y,\mathfrak{s})$ (as a relatively $\mathbb{Z}$-graded complex), and [29] for the definition of its absolute $\mathbb{Q}$-grading.

The complex $CF^\infty(Y,\mathfrak{s})$ supports an action by $H_1(Y)/\text{Torsion}$ which is well defined up to filtered chain homotopy, and therefore the homology $HF^\infty(Y,\mathfrak{s})$ inherits an action by $H_1(Y)/\text{Torsion}$ (in fact the action on homology extends to the exterior algebra on $H_1(Y)/\text{Torsion}$ [31, Section 4.2.5]). Using this action, we can define two associated groups, $HF^\infty(Y,\mathfrak{s})_{\mathrm{b}}$ and $HF^\infty(Y,\mathfrak{s})_{\mathrm{t}}$; the "b" and "t" are shorthand for "bottom" and "top." To define them, one simply considers the kernel and cokernel, respectively, of the $H_1(Y)/\text{Torsion}$ action. In the case that Y is a rational homology sphere, the action is zero, so that both groups equal $HF^\infty(Y,\mathfrak{s})$. In the case that all triple cup products on $H^1(Y)$ vanish (a necessary and sufficient

condition that $HF^\infty(Y)$ be "standard" [19]) $HF^\infty(Y,\mathfrak{s})_{\mathrm{b}}$ and $HF^\infty(Y,\mathfrak{s})_{\mathrm{t}}$ are isomorphic to $\mathbf{F}[U,U^{-1}]$.

There is a subcomplex $CF^-(Y,\mathfrak{s}) = CF^\infty(Y,\mathfrak{s})_{\{i<0\}}$ consisting of elements of filtration level less than 0, and the corresponding homology group $HF^-(Y,\mathfrak{s})$ inherits an $H_1(Y)/$ Torsion action. Thus there are groups $HF^-(Y,\mathfrak{s})_{\mathrm{b}}$ and $HF^-(Y,\mathfrak{s})_{\mathrm{t}}$ and homomorphisms induced by inclusion: $HF^-(Y,\mathfrak{s})_* \to HF^\infty(Y,\mathfrak{s})_*$, where $* = \mathrm{b}$ or t. These top and bottom complexes were first defined in [29]. One useful reference is [33]. Since then a general theory has been developed in [18]. Using these complexes, invariants can be defined as follows.

Definition 2.1. The top and bottom d-invariants of the pair $(Y,\mathfrak{s})$, denoted $d_{\mathrm{b}}(Y,\mathfrak{s})$ and $d_{\mathrm{t}}(Y,\mathfrak{s})$, are defined by the property that $(d_*(Y,\mathfrak{s})-2)$ is the maximal grading among all elements in $HF^-(Y,\mathfrak{s})_*$ that map nontrivially into $HF^\infty(Y,\mathfrak{s})_*$, where $* = \mathrm{b}$ or t.

The analysis of the restriction map $H^2(X) \to H^2(Y)$ in the next section determines the Spin^c structures on Y whose d-invariants we must compute (to ultimately obstruct the existence of X). To enumerate these Spin^c structures, we will use the following notation.

Definition 2.2. Suppose q is a positive integer. Define $\mathcal{S}_q$ to be the set of numbers

$$-(q-1)/2, -(q-1)/2+1, \ldots, (q-1)/2.$$

So, for example, $\mathcal{S}_5 = \{-2,-1,0,1,2\}$ and $\mathcal{S}_6 = \{-5/2,-3/2,-1/2,1/2,3/2,5/2\}$. The following theorem is a restatement of a Theorem 4.2, proved in Section 4. It is a consequence of the fact that $X = \mathbb{C}P^2 - \mathrm{Int}(N(C))$ is negative semi-definite with either orientation, together with the fact that the d-invariants of the boundary of such a four-manifold are bounded by a function determined by its intersection form.

Theorem 2.3. *Suppose that C is a cuspidal curve in $\mathbb{C}P^2$ with $Y = \partial N(C)$. Then there is an enumeration of torsion* Spin^c *structures on Y, $\{\mathfrak{s}_m\}$, by integers in the range $\lfloor \frac{-d^2+1}{2} \rfloor \le m \le \lfloor \frac{d^2-1}{2} \rfloor$. With respect to this enumeration, for all $k \in \mathcal{S}_d$, the following inequalities are satisfied.*

$$d_{\mathrm{b}}(Y,\mathfrak{s}_{dk}) \ge -g$$

and

$$d_{\mathrm{t}}(Y,\mathfrak{s}_{dk}) \le g.$$

In order to compute the invariants $d_{\mathrm{b}}(Y,\mathfrak{s})$ and $d_{\mathrm{t}}(Y,\mathfrak{s})$ we need to understand the geometry of Y. Perhaps the most elegant description of Y is as a graph manifold obtained by splicing the circle bundle over the surface of genus $g(C)$ with Euler number d^2 to the complements of the links of the singularities of C. For the purposes of computing its Floer homology, however, it is more useful to have a description of Y as obtained by d^2 surgery on a knot K_C in $Y_{2g} := \#^{2g} S^1 \times S^2$. We provide such a description in Theorem 3.1. Indeed, K_C can be described as the

connected sum $B\#K_1\#K_2\#\cdots\#K_n$, where $B \subset Y_{2g}$ is a simple knot which depends only on the genus of C, and $K_i \subset S^3$, $i = 1, \ldots, n$ are the links of the singular points of C.

2.2. Computing d-invariants. Let K be a null-homologous knot in a three-manifold M and let $M_k(K)$ denote the manifold constructed by k surgery on K. For each Spin^c structure $\mathfrak{s}$, the complex $CF^\infty(M_k(K), \mathfrak{s})$ is determined by a $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain complex $CFK^\infty(M, K, \mathfrak{t})$ called the *knot Floer homology chain complex*, associated to K and some Spin^c structure $\mathfrak{t}$ on M. In general, the "surgery formula" relating the knot Floer complex to the complexes of the surgered manifolds can be rather complicated. For the manifolds arising in this article, however, it will simplify considerably due to the fact that the surgery coefficient is large with respect to the genus of the knot. Indeed, for our purposes it will suffice to understand the homology of subcomplexes of a single doubly filtered chain complex $CFK^\infty(Y_{2g}, K_C, \mathfrak{s}_0)$ associated to K_C and the unique Spin^c structure on $\#^{2g} S^1 \times S^2$ having trivial first Chern class.

A key to efficiently understanding this latter complex is that the knots K_i that occur as links of singularities are so-called *L-space knots*. For such knots the complexes $CFK^\infty(S^3, K_i)$ are determined by the Alexander polynomials, $\Delta_{K_i}(t)$. Moreover, knot Floer complexes obey a Künneth principle under connected sums: $CFK^\infty(M\#N, K\#J) \simeq CFK^\infty(M, K) \otimes CFK^\infty(N, J)$; see [30, Section 7]. Using this, and the relation $(Y_{2g}, K_C) = (Y_{2g}, B)\#(S^3, K_1)\#\cdots\#(S^3, K_n)$ established in Theorem 3.1 below, we have

$$CFK^\infty(Y_{2g}, K_C) \simeq CFK^\infty(Y_{2g}, B) \otimes_{i=1}^{n} CFK^\infty(S^3, K_i),$$

and $CFK^\infty(Y_{2g}, B)$ has been fully described [30, Proposition 9.2]. Making the connections between these complexes, the Alexander polynomials, and the d-invariants, leads to the following result. Details are presented in Section 6.

Theorem 2.4. *There exist invariants γ_m determined by the Alexander polynomials $\Delta_{K_i}(t)$ with the following property. If $\lfloor \frac{-d^2+1}{2} \rfloor \leq m \leq \lfloor \frac{d^2-1}{2} \rfloor$, then*

$$d_b(Y, \mathfrak{s}_m) = \frac{(q-2m)^2 - q}{4q} + g - 2 \max_{\substack{a,b\geq 0 \\ a+b=g}} \{\gamma_{m+a-b} + a\}$$

and

$$d_t(Y, \mathfrak{s}_m) = \frac{(q-2m)^2 - q}{4q} + g - 2 \min_{\substack{a,b\geq 0 \\ a+b=g}} \{\gamma_{m+a-b} + a\}.$$

The Alexander polynomial of an algebraic knot can be interpreted in terms of the semigroup of the associated singularity. Transferring this interpretation to the invariants γ_m and combining it with Theorem 2.3 and some algebraic manipulation yields Theorem 1.

3. Properties of a neighborhood of C, its boundary, and its complement

We continue to let $N(C)$ denote a closed regular neighborhood of C. Let $Y = \partial N(C)$, a closed oriented three-manifold. The complement of $\mathrm{Int}(N(C))$ in $\mathbb{C}P^2$ is a smooth four-manifold X with boundary $-Y$. In this section we provide a surgery description of Y and homological properties of the pair $(X, -Y)$.

3.1. A geometric description of $N(C)$ and Y. To describe $N(C)$, we begin with a surface of genus g having a single boundary component. We denote this surface by F_g. The product $F_g \times D^2$ has boundary $\#^{2g} S^1 \times S^2$. Contained in its boundary is the knot $B = \partial F_g \times \{0\}$. Notice that B is null homologous in $\#^{2g} S^1 \times S^2$.

Theorem 3.1. *If a cuspidal curve C of degree d has singular points with links K_i, then $N(C)$ is built by adding a two handle to $F_g \times D^2$ along the knot $B \#_i K_i$ with framing d^2. In particular, $\partial N(C) = Y$ is built from $\#^{2g} S^1 \times S^2$ by performing d^2 surgery on $B \#_i K_i$.*

Proof. The neighborhood $N(C)$ is constructed in steps as follows. Let D_i denote a ball neighborhood of the singular point z_i. Joining D_1 to each D_i, $i > 1$ with a one-handle, each a tubular neighborhood of an arc on C, yields a four-ball D. The boundary of D is a three-sphere S with $S \cap C = \#_i K_i$. The complementary region $C - D$ is diffeomorphic to the surface F_g with neighborhood $D' \cong F_g \times D^2$ having the knot B in its boundary. Thus, we have $N(C) = D \cup D'$, with the union identifying a neighborhood of $\#_i K_i$ with a neighborhood of B.

The union $D \cup D'$ can be formed in two steps. First, neighborhoods of a point on $\#_i K_i$ and a point on B are identified. Since D is a ball, this produces a manifold D'' diffeomorphic to D'. The union of the two knots becomes $B\#_i K_i$. The remainder of the identification is completed by adding a 2–handle to D'' along $B\#_i K_i$. The framing is d^2, that is, the self-intersection of C. □

Corollary 3.2. *$H_1(Y) = \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$ and $H_2(Y) = \mathbb{Z}^{2g}$.* □

3.2. The complement $X = \mathbb{C}P^2 - \mathrm{Int}(N(C))$. The following theorem summarizes elementary homological calculations.

Theorem 3.3.

(1) *$H_1(X) \cong \mathbb{Z}_d$ and $H_2(X) \cong \mathbb{Z}^{2g}$.*

(2) *The image of the map $\mathrm{Tors}(H^2(X)) \to H^2(Y)$ is isomorphic to $\mathbb{Z}_d \subset \mathbb{Z}_{d^2}$.*

(3) *The map $H^2(X)/\mathrm{Torsion} \to H^2(Y)/\mathrm{Torsion}$ is an isomorphism.*

(4) $\mathrm{Image}(H^2(\mathbb{C}P^2) \to H^2(X)) = \mathrm{Tors}(H^2(X))$.

(5) *The intersection form on $H_2(X)$ is identically 0.*

Proof. The map $\mathbb{Z} \cong H_2(C) \to H_2(\mathbb{C}P^2) \cong \mathbb{Z}$ is given by multiplication by d. Using this, the long exact sequence of the pair $(\mathbb{C}P^2, C)$ and excision yields $H_1(X,Y) = 0$, $H_2(X,Y) \cong \mathbb{Z}_d \oplus \mathbb{Z}^{2g}$ and $H_3(X,Y) = 0$.

Applying Poincaré duality and the universal coefficient theorem yields: $H_1(X) \cong \mathbb{Z}_d$, $H_2(X) \cong \mathbb{Z}^{2g}$, and $H_3(X) = 0$. In particular, we have part (1) of the theorem.

The long exact sequence of the pair (X,Y) includes the exact sequence

$$\begin{aligned} &\longrightarrow H_3(X,Y) \xrightarrow{\partial_3} H_2(Y) \xrightarrow{\iota_2} H_2(X) \xrightarrow{p_2} H_2(X,Y) \xrightarrow{\partial_2} \\ &\xrightarrow{\partial_2} H_1(Y) \xrightarrow{\iota_1} H_1(X) \xrightarrow{p_1} H_1(X,Y) \longrightarrow 0 \end{aligned}$$

which can be written as

$$0 \to \mathbb{Z}^{2g} \xrightarrow{\iota_2} \mathbb{Z}^{2g} \xrightarrow{p_2} \mathbb{Z}_d \oplus \mathbb{Z}^{2g} \xrightarrow{\partial_2} \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g} \xrightarrow{\iota_1} \mathbb{Z}_d \to 0.$$

We next observe that the map ∂_2 must be nonzero on the $\mathbb{Z}_d$ summand. If not, there would be an exact sequence

$$\mathbb{Z}^{2g} \to \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g} \to \mathbb{Z}_d \to 0.$$

Clearly this is impossible: the image of the initial $\mathbb{Z}^{2g}$ would have to be of rank $2g$. This implies that no element in the image of $\mathbb{Z}^{2g}$ is torsion. The quotient would then contain elements of order d^2. It immediately follows that the map p_2 is the 0 map.

Observe also that ι_1 must be nontrivial on the $\mathbb{Z}_{d^2}$ summand: there is no element in $\mathbb{Z}_d \oplus \mathbb{Z}^{2g}$ that ∂_2 could map to an element of order d^2. Given an element of infinite order in $\mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$, by adding an element from $\mathbb{Z}_{d^2}$ to it we can assume it is in the kernel of ι_1, and thus in the image of ∂_2.

By duality, the map $H^2(X) \to H^2(Y)$ corresponds to the map ∂_2: $\mathbb{Z}_d \oplus \mathbb{Z}^{2g} \to \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$, which we have now seen is nontrivial on torsion and injective on the free summand. Statements (2) and (3) follow quickly.

To prove (4), we consider a portion of the long exact sequence for the pair $(\mathbb{C}P^2, X)$:

$$H^2(\mathbb{C}P^2) \xrightarrow{\nu_1} H^2(X) \xrightarrow{\nu_2} H^3(\mathbb{C}P^2, X).$$

We have $H^2(\mathbb{C}P^2) \cong \mathbb{Z}$ and $H^2(X) \cong H_2(X,Y) \cong \mathbb{Z}_d \oplus \mathbb{Z}^{2g}$. For the last term we have by excision and Lefschetz duality, $H^3(\mathbb{C}P^2, X) \cong H^3(N(C), Y) \cong H_1(N(C)) \cong H_1(C) \cong \mathbb{Z}^{2g}$. Thus, our sequence becomes

$$\mathbb{Z} \xrightarrow{\nu_1} \mathbb{Z}_d \oplus \mathbb{Z}^{2g} \xrightarrow{\nu_2} \mathbb{Z}^{2g}.$$

Clearly ν_2 vanishes on the $\mathbb{Z}_d$ summand, so this summand must be contained in the image of ν_1. Since the domain of ν_1 is of rank one, the $\mathbb{Z}_d$ summand is precisely the image of ν_1. The proof of (4) is complete.

For statement (5), we recall that the intersection form on $H_2(X)$ is given by a composition $H_2(X) \to H_2(X,Y) \to H^2(X) \to \mathrm{Hom}(H_2(X), \mathbb{Z})$. But the map $H_2(X) \to H_2(X,Y)$ (previously called p_2) has already been shown to equal zero. □

4. Bounds on the d-invariant

Bounds on the d-invariants of Y depend on the relationship between Spin^c structures on Y and those on the complementary space X. We begin with an examination of this relationship and then apply results of [29] to attain our desired bounds on the d-invariants.

4.1. Spin^c structures on X and Y.

Theorem 4.1. *If C is a curve of degree d and $X = \mathbb{C}P^2 - \mathrm{Int}(N(C))$, then the torsion* Spin^c *structure $\mathfrak{s}_m$ on ∂X extends to X if $m = kd$ for $k \in \mathcal{S}_d$. Here $\mathfrak{s}_m$ is the* Spin^c *structure on ∂X which extends to a structure $\mathfrak{t}_m$ on $N(C)$ satisfying $\langle c_1(\mathfrak{t}_m), [C]\rangle + d^2 = 2m$.*

Proof. This result is proved in [5] in the case that C is rational. Here is an outline of the argument, identifying why it generalizes to the nonrational case.

There is a Spin^c structure $\mathfrak{t}$ on $\mathbb{C}P^2$ having $c_1(\mathfrak{t})$ the generator of $H^2(\mathbb{C}P^2)$. Denote its restriction to X by $\mathfrak{t}'$. By Theorem 3.3, $c_1(\mathfrak{t}')$ is a torsion class in $H^2(X)$ mapping to an element of order d in $H^2(Y)$. (In the rational case, $H^2(Y)$ is torsion, so the work of Theorem 3.3 was not required.)

We have seen that $H^2(X) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_d$. Since this cohomology group acts effectively on the set of Spin^c structures, the orbit of $\mathfrak{t}'$ under the action of the torsion in $H^2(X)$ is a set of Spin^c structures on X with d elements, all that restrict to give torsion Spin^c structures on Y. The map $\mathrm{Tors}(H^2(X)) \to H^2(Y)$ is injective, so these structures are distinct.

The enumeration of Spin^c structures as the $\mathfrak{s}_m$ is described in more detail in [5]. □

4.2. Bounds. The following result provides bounds on the bottom and top d-invariants.

Theorem 4.2. *If the complex curve C has degree d and topological genus g, then for $k \in \mathcal{S}_d$,*

$$d_{\mathrm{b}}(Y, \mathfrak{s}_{dk}) \geq -g$$

and

$$d_{\mathrm{t}}(Y, \mathfrak{s}_{dk}) \leq g.$$

Proof. This is an application of [29, Proposition 9.15], which says that if W is a negative semi-definite four-manifold for which the restriction map $H^1(W) \to H^1(\partial W)$ is trivial, then we have the inequality:

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4d_{\mathrm{b}}(\partial W, \mathfrak{s}|_{\partial W}) + 2b_1(\partial W),$$

where $b_2^-(W)$ is the dimension of the maximal subspace of $H_2(W)$ on which the intersection form is non-degenerate and $b_1(\partial W)$ is the rank of the first cohomology.

We apply this proposition to $-X$. The restriction map $H^1(-X) \to H^1(Y)$ is trivial since $H^1(X) = 0$ and, just as for X, the intersection form on $-X$ is zero. Hence $-X$ is negative semi-definite. Now triviality of the intersection form implies $b_2^-(-X) = 0$ and $c_1(\mathfrak{s})^2 = 0$ for any $\mathfrak{s} \in \mathrm{Spin}^c(-X)$. Note that $c_1(\mathfrak{s})^2$ is defined by lifting a multiple of $c_1(\mathfrak{s}) \in H^2(-X)$ to $H^2(-X, Y)$ where the intersection form is defined. Such a lift exists only when $c_1(\mathfrak{s}|_Y)$ is torsion, but the Spin^c structures we consider on Y all satisfy this assumption. Thus the left hand side of the inequality is zero for all $\mathfrak{s} \in \mathrm{Spin}^c(-X)$. Since $b_1(Y) = 2g$, the inequality becomes:

$$0 \leq 4d_{\mathrm{b}}(Y, \mathfrak{s}|_Y) + 2(2g).$$

This says that $d_{\mathrm{b}}(Y, \mathfrak{s}) \geq -g$ for any Spin^c structure on Y that extends to $-X$. But Theorem 4.1 determined exactly which Spin^c structures on Y extend: they are those of the form $\mathfrak{s}_{dk}$ where $k \in \mathcal{S}_d$. This proves the first inequality of the theorem.

To prove the second inequality, we apply the same analysis to the pair $(X, -Y)$, arriving at

$$d_{\mathrm{b}}(-Y, \mathfrak{s}) \geq -g.$$

Now it suffices to show that $d_{\mathrm{b}}(-Y, \mathfrak{s}) = -d_{\mathrm{t}}(Y, \mathfrak{s})$. But this follows easily by observing that the filtration and grading reversing duality isomorphism [32, Proposition 2.5]:

$$CF^\infty(-Y, \mathfrak{s}) \simeq (CF^\infty(Y, \mathfrak{s}))^*$$

is compatible with the $H_1(Y)/\,\mathrm{Torsion}$ action, in the sense that if $\gamma \in H_1$ acts on $CF^\infty(Y)$ by the chain endomorphism a_γ, then γ acts on $CF^\infty(-Y)$ by the adjoint a_γ^*. Thus the kernel of the H_1 action on $HF^\infty(-Y)$ is identified, by a filtration and grading reversing isomorphism, with the cokernel of the action on $HF^\infty(Y, \mathfrak{s})$. The stated relationship between d_{b} and d_{t} follows immediately. □

5. The Heegaard Floer homology of $Y_{2g} = \#^{2g} S^1 \times S^2$

Given that Y is built as surgery on a knot in Y_{2g}, we begin by reviewing the structure of the complex $CF^\infty(Y_{2g})$. In particular, in this section we describe an explicit basis for this complex and its homology, and describe the $H_1(Y_{2g})/\,\mathrm{Torsion}$ module structure in terms of this basis. We then describe the "top" and "bottom" Floer homology groups. This description will be used in the next section in conjunction with the knot Floer homology filtration of K_C to compute the Floer homology of Y.

5.1. Case of Y_1. For $Y_1 = S^1 \times S^2$ and Spin^c structure $\mathfrak{s}_0$ with first Chern class $c_1(\mathfrak{s}_0) = 0$, we have $CF^\infty(Y_1, \mathfrak{s}_0) \simeq \mathbf{F}[U, U^{-1}] \oplus \mathbf{F}[U, U^{-1}]$, where the element 1 has grading $1/2$ in the first summand and $-1/2$ in the second. The boundary operator on the complex is trivial, and thus we can identify $CF^\infty(Y_1)$ with $HF^\infty(Y_1)$. Let $x^* \in H_1(S^1 \times S^2) \cong \mathbb{Z}$ be a generator. Then x^* acts $\mathbf{F}[U, U^{-1}]$-equivariantly

on $HF^\infty(Y_1)$, taking the element 1 in the first $\mathbf{F}[U,U^{-1}]$ to the element 1 in the second. Thus we can identify

$$CF^\infty(Y_1) \simeq \Lambda^*(H^1(Y_1)) \otimes \mathbf{F}[U,U^{-1}],$$

where classes in $H_1(Y_1)/$ Torsion act, via the duality pairing between $H_1/$ Torsion and H^1, on elements in the exterior algebra $\Lambda^*(H^1(Y_1))$.

5.2. From Y_1 to Y_n. There is a Künneth principle for the Floer homology of connected sums of three-manifolds [32, Theorem 6.2], stating that:

$$CF^\infty(M\#N,\mathfrak{s}_M\#\mathfrak{s}_N) \simeq CF^\infty(M,\mathfrak{s}_M) \otimes_{\mathbf{F}[U,U^{-1}]} CF^\infty(N,\mathfrak{s}_N). \tag{5.1}$$

This homotopy equivalence respects the $\Lambda^*(H_1/\text{ Torsion})$ module structure, in the following sense: there is a natural isomorphism

$$H_1(M\#N) \cong H_1(M) \oplus H_1(N)$$

with which a class $\gamma \in H_1(M\#N)$ can be identified with $\gamma_M \oplus \gamma_N \in H_1(M) \oplus H_1(N)$. Then γ acts on $CF^\infty(M\#N)$ as $\gamma_M \otimes \mathrm{Id}_N + \mathrm{Id}_M \otimes \gamma_N$ under the homotopy equivalence (5.1).

Using this, together with our description of the Floer homology of Y_1 above, allows us to conclude that

$$CF^\infty(Y_n) \simeq \Lambda^*(H^1(Y_n)) \otimes \mathbf{F}[U,U^{-1}]$$

as $\Lambda^*(H_1/\text{ Torsion}) \otimes \mathbf{F}[U,U^{-1}]$–modules, where $H_1/$ Torsion classes act by the duality pairing, as above.

5.3. A useful change of basis for Y_2. While the module structure on $CF^\infty(Y_n)$ is completely described above, it will be useful to have an alternate description for $CF^\infty(Y_{2g})$ which will be compatible with the filtration of $CF^\infty(Y_{2g})$ induced by the knot B and, ultimately, K_C. Our description is determined by a change of basis for the Heegaard Floer homology of $Y_2 = Y_1\#Y_1$, and the Künneth principle above. Thus we begin with Y_2. Denote the generators of the first cohomology of the two connect summands of $Y_2 = Y_1\#Y_1$ by x and y. Thus, $\Lambda^* H^1(Y_2)$ has basis

$$\{1, x, y, x \wedge y\}.$$

We denote the hom-dual generators of $H_1(Y_2)$ as x^*, y^*. We have the following alternative description [30, Theorem 9.3] of the action of $H_1(Y_2)$ on the chain complex; recall that the action of $H_1(Y_2)$ commutes with the action of U. We will call the complex equipped with this action the *knot adapted complex*.

Theorem 5.1. *$CF^\infty(Y_2, \mathfrak{s}_0) \simeq \Lambda^* H^1(Y_2) \otimes \mathbf{F}[U, U^{-1}]$ as a module over $\mathbf{F}[U, U^{-1}]$. The rational gradings of $1, x, y$, and $x \wedge y$ are $-1, 0, 0$, and 1, respectively. All these elements are at filtration level 0. The $\mathbf{F}[U, U^{-1}]$-equivariant action of $H_1(Y_2)$ on $CF^\infty(Y_2, \mathfrak{s}_0)$ is given by:*

- $x^*(x \wedge y) = y$
- $x^*(x) = 1 + U(x \wedge y)$
- $x^*(y) = 0$
- $x^*(1) = Uy$.

The action of y^* is analogous; see Figure 1 for a graphical presentation of the action of x^*.

Proof. As a graded module over $\mathbf{F}[U, U^{-1}]$, the above description is clearly isomorphic to our previous description. To obtain the non-standard (i.e. not induced by the hom-pairing) action of $H_1/\text{Torsion}$, we perform the (equivariant, filtered) change of basis

$$1 \to 1 + Ux \wedge y, \quad x \to x, \quad y \to y, \quad x \wedge y \to x \wedge y. \qquad \square$$

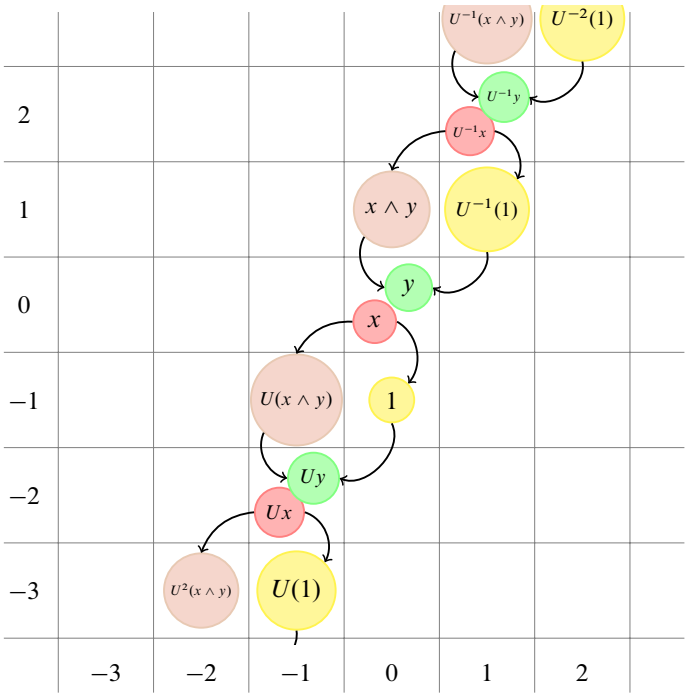

Figure 1. The action of x^* on the $CF^\infty(Y_2, \mathfrak{s}_0)$. The horizontal coordinate shows the filtration level, the vertical coordinate shows the grading.

Remark 5.2. Recall that the action of U lowers filtration levels by one and gradings by two. Thus, the grading of $U^i(x \wedge y)$ is $1-2i$. The gradings of $U^i x$ and $U^i y$ are $-2i$. The grading of $U^i(1)$ is $-1-2i$.

5.4. Case of Y_{2g}. Applying the Künneth principle to the knot adapted complex gives rise to a model for $CF^\infty(Y_{2g})$ which we will use throughout the article. In terms of this complex, we will now compute the "bottom" and "top" knot Floer homology groups. That is, we will find all possible chains in the knot adapted complex which are homologous to generators for these groups (both of which are *a priori* isomorphic to $\mathbf{F}[U,U^{-1}]$). First, we recall the following definition.

Definition 5.3. Suppose an abelian group G acts on a second abelian group H; that is, there is a homomorphism $G \to \mathrm{Hom}(H,H)$. We define H_{bottom} to be the kernel of the action; that is, all elements $h \in H$ such that $g(h)=0$ for all $g \in G$. We define H_{top} to be the cokernel of the action; that is, the quotient of H by the subgroup generated by elements of the form $g(h)$ for some $g \in G$ and $h \in H$. Usually we will abbreviate "bottom" and "top" by "b" and "t," respectively.

We establish some notation for elements in the complex $CF^\infty(Y_{2g})$:

Definition 5.4. Denote the generators of $H^1(Y_{2g}) = H^1(\#^g Y_2)$ in their natural order by $\{x_1,y_1,x_2,\dots,y_g\}$, and let $w_i = x_i \wedge y_i$. Let $\mathcal{A}$ denote the set of subsets of $\{1,2,\dots,g\}$. For each $\alpha \in \mathcal{A}$ we set $w_\alpha = \wedge_{i\in\alpha} w_i$. For $\alpha \in \mathcal{A}$ we let $n(\alpha)$ denote the number of elements in α.

Theorem 5.5. *$CF^\infty(Y_{2g},\mathfrak{s}_0)$ is isomorphic to the $\mathbf{F}[U,U^{-1}]$ module $\Lambda^* H^1(Y_{2g}) \otimes \mathbf{F}[U,U^{-1}]$. The bottom homology is given by*

$$HF^\infty(Y_{2g},\mathfrak{s}_0)_{\mathrm{b}} = \mathbf{F}\big[U,U^{-1}\big]\left[\sum_{\alpha\in\mathcal{A}} U^{n(\alpha)-g} w_\alpha\right].$$

Furthermore, the top homology $HF^\infty(Y_{2g},\mathfrak{s}_0)_{\mathrm{t}}$ is generated by any of the $U^{n(\alpha)-g}w_\alpha$, and any two such $U^{n(\alpha)-g}w_\alpha$ are equivalent in the quotient. These elements are all of grading level g.

In this statement, the brackets around the summation indicate the homology class represented by the cycle.

Proof. For Y_2, the statement is easily verified from our description of the knot adapted complex given in Theorem 5.1. The general case follows immediately from the Künneth principle for connected sums. □

Example 5.6. The group $HF^\infty_{\mathrm{b}}(Y_4,\mathfrak{s}_0)$ is generated over $\mathbf{F}[U,U^{-1}]$ by

$$(x_1\wedge y_1\wedge x_2\wedge y_2) + U^{-1}(x_1\wedge y_1) + U^{-1}(x_2\wedge y_2) + U^{-2}(1).$$

The group $HF_{\mathrm{t}}^{\infty}(Y_{2g}, \mathfrak{s}_0)$ is generated over $\mathbf{F}[U, U^{-1}]$ by either

$$(x_1 \wedge y_1 \wedge x_2 \wedge y_2),\ U^{-1}(x_1 \wedge y_1),\ U^{-1}(x_2 \wedge y_2) \quad \text{or} \quad U^{-2}(1),$$

which are equal in the quotient group.

6. Knot Floer homology and d-invariants of surgery

6.1. Description of $CFK^{\infty}(Y_{2g}, B)$. A null homologous knot in a three-manifold M induces a second filtration on $CF^{\infty}(M, \mathfrak{s}_0)$, called the *knot filtration* or *Alexander filtration*. In our case we have the doubly filtered complex $CFK^{\infty}(Y_{2g}, B, \mathfrak{s}_0)$. This complex was computed in [30] and is described as follows:

(1) As a graded, $\mathbb{Z}$–filtered chain complex $CFK^{\infty}(Y_{2g}, B, \mathfrak{s}_0) \cong CF^{\infty}(Y_{2g}, \mathfrak{s}_0)$.
(2) The knot filtration of an element $U^i \cdot \zeta$, with $\zeta \in \Lambda^k(H^1(Y_{2g}))$ is given by $-g + k - i$.
(3) The $H_1(Y)$/ Torsion action is given as in the knot adapted complex.

The following is immediate.

Theorem 6.1. *If w_α is a product of distinct w_i (according to our labeling convention from Definition 5.4) then $w_\alpha U^{g-n(\alpha)} \in CFK^{\infty}(Y_{2g}, B, \mathfrak{s}_0)$ has bifiltration level $(g - n(\alpha), n(\alpha))$, where the value of the second coordinate, $n(\alpha)$, represents the knot filtration level.*

6.2. Homology of $(Y_{2g}, B\#K)$. Given a knot $K \subset S^3$, we can form the knot $B\#K \subset Y_{2g}$. We will denote this knot by K_C, since the case of primary interest will be that arising from a cuspidal curve, whereby K is given as the connected sum of the links of the singular points. Much of what we say here, however, applies to a general knot in S^3. Like the Heegaard Floer complexes of closed three-manifolds, the knot Floer complexes behave naturally with respect to connected sums, see [30, Section 7]. We have the following.

Theorem 6.2. $CFK^{\infty}(Y_{2g}, K_C, \mathfrak{s}_0) \simeq CFK^{\infty}(K) \otimes CFK^{\infty}(Y_{2g}, B, \mathfrak{s}_0)$, *where the bifiltration is additive under tensors. Moreover, under this equivalence, a class $\gamma \in H_1(Y_{2g})$ acts on the knot complex of K_C by* $\mathrm{Id} \otimes a_\gamma$ *where a_γ is the action of γ on the complex for B.*

Recall that $H_*(CFK^{\infty}(K)) \cong \mathbf{F}[U, U^{-1}]$. From the previous theorem along with Theorem 5.5 we have the following.

Theorem 6.3.

(a) *We have $HFK^{\infty}(Y_{2g}, K_C, \mathfrak{s}_0)_{\mathrm{b}} \cong \mathbf{F}[U, U^{-1}]$. Furthermore the generators of grading g are represented by sums*

$$\sum_{\alpha \in \mathcal{A}} a_\alpha \otimes U^{n(\alpha)-g} w_\alpha,$$

where the a_α are arbitrary cycles of grading 0 in $CFK^{\infty}(K)$, each representing a generator of $HFK^{\infty}(K)$.

(b) *Similarly,* $HFK^\infty(Y_{2g}, K_C, \mathfrak{s}_0)_t \cong \mathbf{F}[U, U^{-1}]$*, where generators of grading* g *are represented by elements of the form*

$$a_\alpha \otimes U^{n(\alpha)-g} w_\alpha.$$

Here a_α *is an arbitrary cycle of grading* 0 *in* $CFK^\infty(K)$ *representing a generator of* $HFK^\infty(K)$.

6.3. Computing d-invariants of d^2-surgery on (Y_{2g}, K_C). We consider d^2 surgery on K_C in Y_{2g}. The resulting manifold, $Y_{2g,d^2}(K_C)$, has $H_1(Y_{2g,d^2}(K_C)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{d^2}$. There are thus d^2 torsion Spin^c structures on $Y_{2g,d^2}(K_C)$; these come with a natural enumeration by integers m, $\{\mathfrak{s}_m\}_{-d^2/2<m\le d^2/2}$, as given in [30, Section 3.4] and described below. We now present a surgery formula describing the Heegaard Floer homology of these surgered manifolds in terms of the knot Floer complex of K_C.

Recall, for a manifold M with Spin^c structure $\mathfrak{s}$ we define

$$CF^-(M, \mathfrak{s}) = CF^\infty(M, \mathfrak{s})_{\{i<0\}},$$

the elements of filtration level less than 0. The homology of this complex is denoted $HF^-(M, \mathfrak{s})$. There is a natural map $HF^-(M, \mathfrak{s}) \to HF^\infty(M, \mathfrak{s})$.

Theorem 6.4 (see [30, Section 4]). *For* $d^2 \ge 2g(K) + 2g - 1$ *and for* $-d^2/2 < m < d^2/2$*, there is an isomorphism of pairs of* $\mathbf{F}[U]$ *modules,*

$$\begin{aligned}\big(CF^\infty(Y_{2g,d^2}(K_C), \mathfrak{s}_m), CF^-(Y_{2g,d^2}(K_C), \mathfrak{s}_m)\big)\\ \cong \big(CFK^\infty(Y_{2g}, K_C, \mathfrak{s}_0), CFK^\infty(Y_{2g}, K_C, \mathfrak{s}_0)_{\{i<0, j<m\}}\big)[s].\end{aligned}$$

The grading shift $[s]$ *is given by*

$$s = \frac{(2m-n)^2 - n}{4n}.$$

If a class is at bi-filtration level (i, j) *in* $CFK^\infty(Y_{2g}, K_C, \mathfrak{s}_0)$ *then it represents a class at filtration level* $\max(i, j-m)$ *in* $CFK^\infty(Y_{2g,d^2}(K_C), \mathfrak{s}_m)$.

Remark 6.5. In [30, Remark 4.3] the bound given would be presented as $d^2 \ge 2g(K_C) - 1$. We used here the fact that $g(K_C) = g(K) + g$. Notice that for the knots we are considering, $g(K) + g = \frac{(d-1)(d-2)}{2}$. Thus, the inequality $d^2 \ge 2g(K) + 2g - 1$ becomes $d^2 \ge (d-1)(d-2) - 1$, which holds for all $d \ge 1$.

Let $\mathcal{C} \to \mathcal{D}$ be a map of graded $\mathbf{F}[U]$ modules. We denote by $\gamma(\mathcal{C}, \mathcal{D})$ the maximal grading of an element in $\mathcal{C}$ that maps nontrivially to $\mathcal{D}$, if defined.

Our principal example is the following. For a manifold M with Spin^c structure $\mathfrak{s}$, there is a natural map $HF^-(M, \mathfrak{s}) \to HF^\infty(M, \mathfrak{s})$.

Definition 6.6. For M a rational homology sphere and $\mathfrak{s}$ a Spin^c structure, we define $d(M,\mathfrak{s}) = \gamma(HF^-(M,\mathfrak{s}), HF^\infty(M,\mathfrak{s})) + 2$.

Remark 6.7. The d-invariant is often defined in terms of $HF^+(M,\mathfrak{s})$. The equivalence with our definition is elementary.

For general M, a similar definition applies to define bottom and top d-invariants.

Definition 6.8. For general M and $\mathfrak{s}$ a torsion Spin^c structure, we define

$$d_{\mathrm{b}}(M,\mathfrak{s}) = \gamma\big(HF^-(M,\mathfrak{s})_{\mathrm{b}}, HF^\infty(M,\mathfrak{s})_{\mathrm{b}}\big) + 2$$

and

$$d_{\mathrm{t}}(M,\mathfrak{s}) = \gamma\big(HF^-(M,\mathfrak{s})_{\mathrm{t}}, HF^\infty(M,\mathfrak{s})_{\mathrm{t}}\big) + 2.$$

Note that while our definition makes sense for any manifold, it is not as clear what the geometric meaning of d_{t} and d_{b} are when the three-manifold has non-trivial triple cup products.

For any knot K for which $CFK^\infty(K)$ is well understood, Theorem 6.4 provides sufficient information to compute $d_{\mathrm{b}}(Y_{2g,d^2}(K_C))$ and $d_{\mathrm{t}}(Y_{2g,d^2}(K_C))$. The result is best described in terms of an auxiliary function.

Definition 6.9. Let $\mathcal{T}$ be a set of ordered pairs of integers. For any integer m we define

$$\gamma_m(\mathcal{T}) = \min_{(i,j)\in\mathcal{T}} \big(\max\{i, j-m\}\big).$$

In brief, γ_m measures the minimum diagonal distance from an element in $\mathcal{T}$ to the lower left quadrant with top right vertex $(0,m)$. The following result is essentially a corollary of Theorem 6.4.

Theorem 6.10. *For the complex $\mathcal{C} = CFK^\infty(K)$, let $\mathcal{T}(\mathcal{C})$ be the set of all filtrations levels (ordered pairs) of cycles of grading 0 that represent generators of $HFK^\infty(K)$. For large surgery,*

$$d\big(S^3_n(K), \mathfrak{s}_m\big) = -2\gamma_m\big(\mathcal{T}(\mathcal{C})\big) + \big((2m-n)^2 - n\big)/4n.$$

Example 6.11.

(a) If K is a negative trefoil, then $\mathcal{T} = \{(1,1)\}$; if K is a positive trefoil, then $\mathcal{T} = \{(1,0),(0,1)\}$.

(b) More generally, suppose that K is a positive L–space knot. Then the complex $CFK^\infty(K)$ is what is called a staircase complex, generated by so-called type A elements of grading 0 and type B elements of grading one. The set $\mathcal{T}$ is the set of all type A vertices of the staircase complex of K. Details are presented, for instance, in [5].

(c) If K is a connected sum of L-space knots $K_1,\dots,K_n$ and $\mathcal{T}_1,\dots,\mathcal{T}_n$ are the corresponding sets $\mathcal{T}_i = \mathcal{T}(CFK^\infty(K_i))$, then $\mathcal{T}$ is a set of sums $t_1+\cdots+t_n$, where $t_i \in \mathcal{T}_i$; see also [5].

Items (b) and (c) of the above example are the most important in our applications.

To state the corresponding result for the bottom and top d-invariants, we introduce additional notation. For a set $\mathcal{T}$ of ordered pairs of integers, we let $\mathcal{T}\{a,b\}$ be the same set shifted by (a,b). Applying Theorem 6.3 we have the following.

Theorem 6.12. *For the complex $\mathcal{C} = CFK^\infty(K)$, let $\mathcal{T}(\mathcal{C})$ be the set of all filtrations levels (ordered pairs) of cycles of grading 0 that represent generators of $HFK^\infty(K)$. Let n be a large integer.*

(1) $d_b\big(Y_{2g,n}(K_C),\mathfrak{s}_m\big) = -2\max\limits_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_m(\mathcal{T}(\mathcal{C})\{a,b\})\big\} + g + s,$

(2) $d_t\big(Y_{2g,n}(K_C),\mathfrak{s}_m\big) = -2\min\limits_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_m(\mathcal{T}(\mathcal{C})\{a,b\}\big\} + g + s,$

where $s = ((2m-n)^2-n)/4n$.

An elementary calculation restates Theorem 6.12 in somewhat simpler terms, using the same notation as in Theorem 6.12.

Theorem 6.13. *For the complex $\mathcal{C} = CFK^\infty(K)$,*

(1) $d_b\big(Y_{2g,n}(K_C),\mathfrak{s}_m\big) = -2\max\limits_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_{m-b+a}(\mathcal{T}(\mathcal{C})) + a\big\} + g + s,$

(2) $d_t\big(Y_{2g,n}(K_C),\mathfrak{s}_m\big) = -2\min\limits_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_{m-b+a}(\mathcal{T}(\mathcal{C})) + a\big\} + g + s,$

where $s = ((2m-n)^2-n)/4n$.

Combining Theorems 4.2 and 6.13, we have the following.

Theorem 6.14. *If C is a cuspidal curve of degree d, genus g, then for all $k \in \mathcal{S}_d$, and $\mathcal{C} = CFK^\infty(K)$,*

$$d_b\big(Y_{2g,d^2}(K_C),\mathfrak{s}_{kd}\big) = -2\max_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_{kd-b+a}(\mathcal{T}(\mathcal{C})) + a\big\} + g + s \geq -g$$

and

$$d_t\big(Y_{2g,d^2}(K_C),\mathfrak{s}_{kd}\big) = -2\min_{\substack{a,b\geq 0\\ a+b=g}}\big\{\gamma_{kd-b+a}(\mathcal{T}(\mathcal{C})) + a\big\} + g + s \leq g,$$

where $s = \frac{(2kd-d^2)^2-d^2}{4d^2}$ and K is the connected sum of the links of the singularities of C.

7. Semigroups, Alexander polynomials, and the d-invariant

The computation of the obstructions to a set of knots occurring as the links of singularities has been reduced to computing $\gamma_m(CFK^\infty(K))$ for particular knots K. We will now summarize an interpretation of the value of $\gamma_m(K)$ in terms of classical

invariants of the singular points and in terms of the Alexander polynomial of K. The material is presented in greater detail in [5]; further references include [16] for a discussion of the Heegaard Floer theory and [38] for the relationship between semigroups and Alexander polynomials.

7.1. Semigroup of a singular point. Suppose that z is a cuspidal singular point of a curve C and B is a sufficiently small ball around z. There exists a local parameterization ψ of C; that is, a holomorphic map $\psi(t) = (x(t), y(t))$ mapping a neighborhood of $0 \in \mathbb{C}$ bijectively to a neighborhood of $z \in C$, with $\psi(0) = z$. For any holomorphic function $F(x, y)$ defined near z we define the order of the zero of F at z to be the order of the zero of the analytic map $t \mapsto F(x(t), y(t)) \in \mathbb{C}$ at 0. Let S be the set integers which can be realized as the order for some F. Then S is clearly a semigroup of $\mathbb{Z}_{\geq 0}$, which we call the *semigroup of the singular point*. The *gap sequence*, $G := \mathbb{Z}_{\geq 0} \setminus S$, has precisely $\mu/2$ elements; the largest is $\mu - 1$, where μ is the Milnor number.

The following two lemmas appear in Lemma 2.4 and a subsequent discussion in [5]. Further detail can be found in [38].

Lemma 7.1. *The Alexander polynomial of the link of a singular point can be written as* $\Delta_K(t) = 1 + (t-1)\sum_{j=1}^{k} t^{g_j}$, *where* $g_1, \ldots, g_k$ *is the gap sequence of the semigroup of the singular point. In particular* $k = \#G = \mu/2 = g_k(K)$.

If one expands the Alexander polynomial further, the following arises.

Lemma 7.2. *If K is the link of an isolated singularity of a curve C and $\Delta_K(t)$ is expanded as*

$$\Delta_K(t) = 1 + (t-1)g(K) + (t-1)^2 \sum_{j=0}^{\mu-2} k_j t^j,$$

then $k_j = \#\{m > j : m \notin S\}$.

Example 7.3. Consider the knot $T(4, 7)$. Its Alexander polynomial is

$$\begin{aligned}
\frac{(t^{28}-1)(t-1)}{(t^3-1)(t^7-1)} &= 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18} \\
&= 1 + (t-1)\big(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}\big) \\
&= 1 + 9(t-1) + (t-1)^2\big(9 + 8t + 7t^2 + 6t^3 + 6t^4 \\
&\qquad + 5t^5 + 4t^6 + 4t^7 + 4t^8 + 3t^9 + 2t^{10} + 2t^{11} \\
&\qquad + 2t^{12} + t^{13} + t^{14} + t^{15} + t^{16}\big).
\end{aligned}$$

The semigroup is $(0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, \ldots)$. The gap sequence is $1, 2, 3, 5, 6, 9, 10, 13, 17$.

Definition 7.4. For any finite increasing sequence of positive integers G, we define

$$I_G(m) = \#\{k \in G \cup \mathbb{Z}_{<0} : k \geq m\}, \tag{7.1}$$

where $\mathbb{Z}_{<0}$ is the set of the negative integers. We call I_G the *gap function*, because in most applications G will be a gap sequence, that is the complement of some semigroup.

Notice that $k_m = I_G(m+1)$, where k_m is defined in Lemma 7.2.

7.2. Expressing $\gamma_m(K)$ in terms of the semigroup. We now wish to restate Theorem 6.14 in terms of the coefficients of the Alexander polynomial, properly expanded. For the gap sequence for the knot K_i, denoted G_{K_i}, let

$$I_{K_i}(s) = \#\{k \geq s : k \in G_{K_i} \cup \mathbb{Z}_{<0}\}.$$

Earlier we defined for a Heegaard Floer complex $\mathcal{C} = CFK^\infty(K)$ the set of integer pairs $\mathcal{T}(\mathcal{C})$ of filtration levels of cycles in $\mathcal{C}$ which represent generators of $HFK^\infty(K)$. By definition we have $\gamma_m(\mathcal{T}(\mathcal{C})) = \min_{(i,j)\in\mathcal{T}}(\max\{i, j-m\})$. We have already seen that computing γ_m is the main step in computing d-invariants of manifolds built by surgery on K (or by surgery on $K \# B \subset Y_{2g}$). We have the following results.

Theorem 7.5 ([4, Proposition 4.6]). *If the knot K is the link of a singularity on a cuspidal curve, then $\gamma_m(K) = I_{G_K}(m+h)$, where G_K is the gap sequence of K and h is its genus.*

For two functions $I, I' : \mathbb{Z} \to \mathbb{Z}$ bounded below we define the following operation:

$$I \diamond I'(s) = \min_{m\in\mathbb{Z}}\{I(m) + I'(s-m)\}. \tag{7.2}$$

Theorem 7.6 ([4, Theorem 5.6]). *For $K = \#K_i$ with the K_i the links of the singularities on a cuspidal curve, we have $\gamma_m(CFK^\infty(K)) = I(m+h)$, where $I = I_1 \diamond \cdots \diamond I_n$, and h is the genus of K.*

7.3. Proof of Theorem 1. We need some preliminaries. For a semigroup $S \subset \mathbb{Z}_{\geq 0}$ we introduce another function.

$$R(m) := \#\{S \cap [0, m)\}. \tag{7.3}$$

The function R is closely related to $I(m)$ defined above, in fact in [5, Lemma 6.2] it is proved that

$$R(m) = m - h + I(m),$$

where $h = \#(\mathbb{Z}_{\geq 0} \setminus S)$. If S is a semigroup of a (unibranched) singular point, then h is the genus of the link of the singularity.

Given two semigroups S_1 and S_2, we can consider two gap sequences G_1, G_2 and the corresponding gap functions I_1 and I_2. Then

$$R_1 \diamond R_2 = m - h + I_1 \diamond I_2,$$

where $h = \#(\mathbb{Z}_{\geq 0} \setminus S_1) + \#(\mathbb{Z}_{\geq 0} \setminus S_2)$.

The proof of Theorem 1 is now a direct application of the above results. By Theorem 3.1, the manifold Y is a surgery on $B \# K$, where K is a connected sum of the links of singular points of C. We use now Theorem 6.14 together with Theorem 7.5 to see that for $k \in \mathcal{S}_d$ we have

$$\begin{aligned} \max_{a,b\geq 0, a+b=g} \{I(kd - b + a + h) + a\} - \frac{s}{2} &\leq g \\ \min_{a,b\geq 0, a+b=g} \{I(kd - b + a + h) + a\} - \frac{s}{2} &\geq 0, \end{aligned} \tag{7.4}$$

where

$$s = \frac{(2kd - d^2)^2 - d^2}{4d^2} = \frac{(d - 2k - 1)(d - 2k + 1)}{4}$$

and h is the genus of the connected sum of links of singularities; that is

$$h = \frac{(d-1)(d-2)}{2} - g = 1 + d\frac{(d-3)}{2} - g.$$

Substituting $a = g - b$ yields

$$kd - b + a + h = \left(k + \frac{d-3}{2}\right) d - 2b + 1.$$

We write $j = k + \frac{d-3}{2}$ and notice that $k \in \mathcal{S}_d$ if and only if $j = -1, \ldots, d-2$. Then (7.4) takes the following form

$$0 \leq I(jd + 1 - 2b) + g - b - \frac{s}{2} \leq g, \text{ for all } b = 0, \ldots, g.$$

Expressing $s/2$ in variables j and d yields $\frac{(j-d+1)(j-d+2)}{2}$. Now we replace I by R. After straightforward simplifications, we obtain

$$0 \leq R(jd + 1 - 2b) + b - \frac{(j+1)(j+2)}{2} \leq g.$$

Note that the cases $j = -1, 0$ are excluded in the statement of our theorem, as they contain no information.

8. Examples

We will present here several applications of the results of the previous sections, along with detailed computations. More substantial applications in algebraic geometry will be presented in Section 9 and especially in Section 9.4.

8.1. A degree 21, genus one example. Consider the case of $d = 21$. If a degree d curve is of genus one and has a single singularity of type (p, q), then one would have

$$\frac{(d-1)(d-2)}{2} - \frac{(p-1)(q-1)}{2} = 1.$$

This simplifies to $(p-1)(q-1) = 378$. There are eight relatively prime pairs (p, q) that satisfy this equation: $(2, 379)$, $(3, 190)$, $(4, 127)$, $(7, 64)$, $(8, 55)$, $(10, 43)$, $(15, 28)$, and $(19, 22)$.

For each possibility, Theorem 1 provides 38 two-sided inequalities that must be satisfied by the associated function R. (The value of j ranges from 1 to $d-2$ and b ranges from 0 to $g = 1$.) The first of these inequalities, with $j = 1$ and $b = 0$, is:

$$3 \leq R(22) \leq 4.$$

The semigroup generated by $\{2, 379\}$ contains 11 elements in the interval $[0, 22)$, and thus $R(22) = 11$ does not satisfy this inequality. Similarly, the semigroup generated by $\{3, 190\}$ contains eight elements in the interval $[0, 22)$, and thus $R(22) = 8$ does not satisfy the inequality. The semigroup generated by $\{4, 127\}$ contains six elements in the interval $[0, 22)$, and thus $R(22) = 6$ does not satisfy the inequality.

In the next two cases, $(7, 64)$ and $(8, 55)$, all these inequalities are satisfied. In Section 9 we will discuss the realization of these curves and place the example $d = 21$, $(p, q) = (8, 55)$ in a general sequence of realizable curves, related to the fact that 8, 21, and 55 are the Fibonacci numbers ϕ_6, ϕ_8, and ϕ_{10}.

For the pair $(10, 43)$, we need to consider a different value of j to find the first obstruction. Here we let $j = 2$ and $b = 0$, giving the inequality

$$6 \leq R(43) \leq 7.$$

The semigroup generated by $(10, 43)$ contains five elements in the interval $[0, 43)$, and thus $R(43)$ does not satisfy the inequality. Finally, we can rule out the possibilities of $(15, 28)$ and $(19, 22)$ by returning to the inequality $3 \leq R(22) \leq 4$. In both cases, $R(22) = 2$.

8.2. A degree seven, genus three example. As a second example, we consider a singular curve of genus 3, showing that there is no degree seven curve with one singular point, whose Puiseux pair is $(4, 9)$.

A generic curve of degree $d = 7$ has genus 15 and the $(4, 9)$-torus knot has genus 12. Thus a degree seven curve with one singular point, whose Puiseux pair is $(4, 9)$ would have genus 3. Theorem 1 provides 20 two-sided inequalities; the value of j is between 1 and 5 and the value of b is between 0 and 3. Of these inequalities, exactly two provide obstructions. For $j = 1, b = 0$ and $j = 3, b = 3$, we have the constraints:

$$3 \leq R(8) \leq 6,$$
$$7 \leq R(16) \leq 10.$$

The semigroup generated by $\{4, 9\}$ has two element in $[0, 8)$, so $R(8) = 2$ does not satisfy the first inequality. This semigroup contains six elements in the interval $[0, 16)$ (these elements are $\{0, 4, 8, 9, 12, 13\}$) and thus $R(16) = 6$ does not satisfy the second inequality.

8.3. A degree nine, genus eight example. The obstructions given by Theorem 1 become weaker as the genus increases, necessarily so, since more singularity types can be realized. We present here one more example, one in which the obstruction remains effective despite the genus being large relative to d. We consider the case of $d = 9$ and the curve having a singular point, whose Puiseux pair is $(5, 11)$.

Since the generic genus of a degree nine curve is 28 and the $(5, 11)$-torus knot has genus 20, a curve of degree nine and having one singular point whose Puiseux pair is $(5, 11)$ would have genus eight. Thus, Theorem 1 provides 63 two-sided inequalities, as j ranges from 1 to 7 and b ranges from 0 to 8. Precisely one of these provides an obstruction. In the case $j = 5, b = 8$ we get inequalities

$$13 \leq R(30) \leq 21.$$

The semigroup generated by $\{5, 11\}$ contains 12 elements in the interval $[0, 30)$, and thus the inequality is not satisfied.

8.4. A singularity $T(4, 7)$ on a degree six curve. The singularity was discussed briefly in Example 7.3. Since a generic degree six curve has genus 10 and the $(4, 7)$-torus knot has genus nine, a degree six curve having one singular point, whose Puiseux pair is $(4, 7)$, is of genus one. There are eight constraints given by Theorem 1. Two of these are

$$3 \leq R(7) \leq 4$$
and
$$5 \leq R(11) \leq 6.$$

Since for $(4, 7)$, $R(7) = 2$ and $R(11) = 4$, these inequalities are violated.

This example is of special interest. We will see in Example 10.1 that another important criterion, *semicontinuity of the spectrum*, is insufficient to obstruct this case.

9. Genus one curves with one singularity having one Puiseux pair

In the section we prove our classification result for genus one curves with a single singular point with one Puiseux pair. The bulk of the work lies in the obstruction of curves, for which we use Theorem 1 together with the multiplicity bound from Section 9.3 below. For the sake of exposition, we introduce some (non-standard) terminology. Throughout the section, we will call a curve C of *type* (p,q), if it has precisely one singularity and that singularity has one Puiseux pair (p,q). Let $\phi_0,\phi_1,\dots$ be the sequence of Fibonacci numbers such that $\phi_0 = 0$, $\phi_1 = 1$ and $\phi_{n+1} = \phi_n + \phi_{n-1}$. The main theorem of this section is the following.

Theorem 9.1. *Suppose $C \subset \mathbb{C}P^2$ is an algebraic curve of genus one, degree d, and of type (p,q). Then either*: (A) $d = \phi_{4n}$, $p = \phi_{4n-2}$, $q = \phi_{4n+2}$ for some $n > 0$; or (B) the values of (p,q) and d are on the following list.

(a) $(p,q) = (2,5)$, $d = 4$;

(b) $(p,q) = (2,11)$, $d = 5$;

(c) $(p,q) = (3,10)$, $d = 6$;.

(d) $(p,q) = (6,37)$, $d = 15$;.

(e) $(p,q) = (9,64)$, $d = 24$;.

(f) $(p,q) = (10,73)$, $d = 27$;

(g) $(p,q) = (12,91)$, $d = 33$;

(h) $(p,q) = (p, 9p+1)$, $d = 3p$ for $p = 2,\dots,10$.

Remark 9.2. (a) Theorem 9.1 does not state that any of these cases can be realized as an algebraic curve, nor does it state in how many ways each case can be realized if some realization exists. In Proposition 9.14 and Proposition 9.15 we clarify that cases (a)–(d) can be realized by an algebraic curve and that the main case $(\phi_{4n-2},\phi_{4n+2})$ can be realized.

(b) All the special cases have degree at most 33.

We begin with the following simple result.

Proposition 9.3. *If a degree d curve is of genus one and has one singularity with one Puiseux pair (p,q), then $(p-1)(q-1) = d(d-3)$.*

Proof. This is an immediate corollary of the genus formula, restating the condition that $(d-1)(d-2)/2 = (p-1)(q-1)/2 + 1$. □

The rest of this section is devoted to proving Theorem 9.1.

9.1. Preliminary bounds. Assuming that C satisfies the assumptions of Theorem 9.1 and the d, p, and q are as in the statement of that theorem, we begin by developing some basic bounds.

Lemma 9.4. *If $d \geq 5$, then $p \leq d-3$ and $q \geq d+3$.*

Proof. First observe that $p \leq d-1$: since $p < q$ and $(p-1)(q-1) = (d-3)d$, it immediately follows that $p - 1 \leq d - 2$.

We now improve this to show that $p \leq d-2$. By Theorem 1, setting $j = 1$ and $b = 1$, we find

$$2 \leq R(d-1) \leq 3.$$

If $p = d-1$, then 0 is the only element of the semigroup generated by p and q that is in the interval $[0, d-1)$, in which case $R(d-1) = 1$, giving a contradiction.

Finally, we consider the case that $p = d-2$, which we write as $p - 1 = d - 3$. Clearly $q - 1 = d$ and $q = d + 1$. By Theorem 1, setting $j = 2$ and $b = 1$ we find

$$5 \leq R(2d-1) \leq 6.$$

The following integers are the first six elements in the semigroup generated by $d-2$ and $d+1$ in increasing order:

$$\begin{aligned}\{0, d-2, d+1, 2(d-2), (d-2)+(d+1), 2(d+1)\}& \\ = \{0, d-2, d+1, 2d-4, 2d-1, 2d+2\}&.\end{aligned}$$

Thus, $R(2d-1) \leq 4$, with equality whenever $d \geq 5$, giving the desired contradiction. (If $d = 4$, then the element $3d - 6$ would also be an element in the semigroup that is less than $2d - 1$.)

For the lower bound on q, we observe that the minimum value of q would occur if $p = d - 3$. Solving for q yields $q = d + 2 + \frac{4}{d-4}$. Since q is an integer and $d > 4$, it follows that $q \geq d + 3$. □

We now place a stronger upper bound on p and a lower bound on q.

Lemma 9.5. *Suppose C satisfies the conditions of Theorem 9.1. If $d > 6$, then $p < \frac{1}{2}d$ and $q \geq 2d - 1$.*

Proof. First observe that by Theorem 1 with $j = 1$ and $b = 0$,

$$3 \leq R(d+1) \leq 4.$$

If $p > \frac{1}{2}d$ then there are at most two elements (0 and p) in the semigroup generated by p and q in the interval $[0, d+1)$, giving a contradiction.

If $p = \frac{1}{2}d$, then one computes $q - 1 = \frac{2(d-1)(d-2)-2}{d-2}$, which is not an integer since $d > 4$.

Given that $p < \frac{1}{2}d$, elementary algebra shows that $q > 2d - 1 - \frac{4}{d-2}$. Since $d > 6$ and q is an integer, $q \geq 2d - 1$ as desired. □

For bounds in the reverse direction, we have the following lemma.

Lemma 9.6. *If $d > 8$, then $p > \frac{d-2}{3}$ and $q \leq 3d + 16$.*

Proof. Again applying Theorem 1 with $j = 1$ and $b = 1$, we have

$$R(d-1) \leq 3.$$

That is, at most two positive multiples of p are in the interval $[0, d-2]$. Thus, $p > \frac{d-2}{3}$ as desired.

Simple algebra now yields that $q < 3d + 7 + \frac{30}{d-5} < 3d + 17$. □

9.2. A Bogomolov–Miyaoka–Yau based inequality. We begin with a summary of a result of Orevkov [26] which is based on the Bogomolov–Miyaoka–Yau inequality [23].

Associated to each singular point on a curve C there is an *Orevkov $\overline{M}$-number*, defined in full generality in [26]; we note here that in the case of singularities having link a (p,q)-torus knot, that is, having a single Puiseux pair (p,q), the value is $\overline{M} = p + q - [q/p] - 3$.

We have the following consequence of the Bogomolov–Miyaoka–Yau inequality [23]; because the details are fairly technical, we delay presenting them until Section 9.6.

Theorem 9.7. *If $C \subset \mathbb{C}P^2$ is a cuspidal curve of genus $g > 0$ and degree d with singular points $z_1 \ldots, z_n$ and corresponding $\overline{M}$ numbers $\overline{M}_1, \ldots, \overline{M}_n$, then*

$$\sum_{i=1}^{n} \overline{M}_i \leq 3d + 4g - 5. \tag{9.1}$$

Example 9.8. Theorem 1 does not prohibit the existence of a curve of degree $3p$ ($p = 1, 2, \ldots$) with genus 1 and a singularity $(p; 9p+1)$. One can indeed check that this case satisfies (1.1) for all j. Nevertheless it does not satisfy (9.1) if $p \geq 11$. In fact, we have $\overline{M} = p + 9p + 1 - 9 - 3 = 10p - 11$ and the bound is $\overline{M} \leq 3d - 1 = 9p - 1$. This is satisfied only when $p \leq 10$.

9.3. The multiplicity bound. We will now prove a multiplicity bound similar to one given by Orevkov in [26, Theorem A]. We restrict to the case of interest, $g = 1$ and one singular point, but with care the argument extends to arbitrary genus and multiple singularities. In the case of a singular point with one Puiseux pair (p,q), the multiplicity is the minimum of p and q which, since we assume throughout that $p < q$, is given by p.

Theorem 9.9. *Suppose that C is a cuspidal curve of degree d, genus $g = 1$, and with one singular point of multiplicity m. Then $d < \alpha m + \beta$, where $\alpha = \frac{1}{2}(3 + \sqrt{5})$ and $\beta = \frac{3}{2} + \frac{11}{10}\sqrt{5}$.*

Proof. By [6, Proposition 2.9] we have the Milnor number μ and $\overline{M}$ number satisfy $\mu \leq m(\overline{M} - m + 2)$ (this is immediate for a singularity with one Puiseux pair). Therefore, by the genus formula

$$(d-1)(d-2) - 2g \leq m\big(\overline{M} - m + 2\big).$$

Using the assumption that $g = 1$ and Theorem 9.7, it follows that

$$d^2 - 3d \leq m(3d - 1 - m + 2).$$

This can be rewritten as

$$d^2 - 3(1+m)d + (m^2 - m) \leq 0.$$

Viewing this as a quadratic polynomial in d yields

$$2d \leq 3(1+m) + \sqrt{9(1+m)^2 - 4(m^2 - m)}.$$

This simplifies to

$$2d \leq 3 + 3m + \sqrt{5}m\sqrt{1 + \frac{22}{5m} + \frac{9}{5m^2}},$$

which we can rewrite as

$$2d \leq 3 + (3+\sqrt{5})m + \sqrt{5}m\left(\sqrt{1 + \frac{22}{5m} + \frac{9}{5m^2}} - 1\right).$$

The proof is completed by showing that for $m \geq 2$,

$$m\left(\sqrt{1 + \frac{22}{5m} + \frac{9}{5m^2}} - 1\right) < \frac{11}{5}.$$

This is an elementary exercise in calculus, perhaps most easily solved for substituting $m = \frac{1}{x}$ to consider

$$\frac{\sqrt{1 + \frac{22}{5}x + \frac{9}{5}x^2} - 1}{x}$$

on the interval $(0, \frac{1}{2}]$. The first derivative of this function is easily seen to be negative, and L'Hôpital's rule determines the limit at 0 to be $\frac{11}{5}$. □

9.4. Classification theorem. Theorem 9.1 will be deduced from the multiplicity bound (Theorem 9.9) along with a technical result, Lemma 9.11, which follows the proof of a sequence of simpler lemmas. Throughout the rest of this section we assume C is a curve of degree d and genus one, with exactly one singular point, and that singular point is of type (p, q). We remind the reader that $p < q$.

We need to introduce some notation. Let $\phi_0, \phi_1, \ldots$ be the sequence of Fibonacci numbers such that $\phi_0 = 0$, $\phi_1 = 1$ and $\phi_{n+1} = \phi_n + \phi_{n-1}$. Most elementary texts on number theory include the necessary background, for instance regarding such facts as $\gcd(\phi_n, \phi_{n+1}) = 1 = \gcd(\phi_n, \phi_{n+2})$ as well as nonlinear relations, such as Cassini's Identity $\phi_{n-1}\phi_{n+1} - \phi_n^2 = (-1)^n$, and its generalization $\phi_{n-r}\phi_{n+r} - \phi_n^2 = (-1)^{n-r+1}\phi_r^2$.

Our next step is to rule out some special cases of possible values of p.

Lemma 9.10. *Suppose C is as in the assumptions of Theorem 9.1 and is not one of the exceptional cases $(p,q) = (2,5), (2,11), (3,10)$, or $(6,37)$. Then $p \neq \frac{\phi_{2j-1}}{\phi_{2j+1}} d$ for all $j > 0$.*

Proof. Suppose that $p = \frac{\phi_{2j-1}}{\phi_{2j+1}} d$ for some j. Since ϕ_{2j-1} is coprime to ϕ_{2j+1}, we see that $\frac{d}{\phi_{2j+1}}$ is an integer.

Since $(p-1)(q-1) = d(d-3)$, we have

$$\big(\phi_{2j-1}d - \phi_{2j+1}\big)(q-1) = \phi_{2j+1}d(d-3).$$

The left hand side can be rewritten using the identity $\phi_{2j+1} = 3\phi_{2j-1} - \phi_{2j-3}$ to give

$$\big[(d-3)\phi_{2j-1} + \phi_{2j-3}\big](q-1) = \phi_{2j+1}d(d-3).$$

Taking these equalities modulo d and $d-3$, respectively, we arrive at

$$\phi_{2j+1}(q-1) = 0 \bmod d$$

and

$$\phi_{2j-3}(q-1) = 0 \bmod (d-3).$$

Thus d divides $\phi_{2j+1}(q-1)$, $d-3$ divides $\phi_{2j-3}(q-1)$ and so $\operatorname{lcm}(d, d-3)$ divides $\operatorname{lcm}(\phi_{2j+1}, \phi_{2j-3})(q-1)$. The value of $\operatorname{lcm}(d, d-3)$ is either $d(d-3)$ or $d(d-3)/3$, depending upon whether or not d is divisible by 3. In either case, we have

$$d(d-3) | 3\phi_{2j-3}\phi_{2j+1}(q-1).$$

Since $(p-1)(q-1) = d(d-3)$, if follows that

$$(p-1) | 3\phi_{2j-3}\phi_{2j+1}. \tag{9.2}$$

Notice that in the case that d is not divisible by 3, we have the stronger constraint $(p-1) | \phi_{2j-3}\phi_{2j+1}$.

Denote $x = \phi_{2j-1}$ and $y = \frac{d}{\phi_{2j+1}} \in \mathbb{Z}$, so that $xy = p$. Notice that $x^2 + 1 = \phi_{2j-1}^2 + 1 = \phi_{2j-3}\phi_{2j+1}$, which follows from the basic identities satisfied by the Fibonacci numbers. By (9.2) $xy - 1$ divides $3(x^2+1)$; that is, there exist $c > 0$ such that

$$c(xy - 1) = 3x^2 + 3.$$

Taking both sides modulo x, we infer that $c = kx - 3$ for some integer $k > 0$. Substituting this, after simplifications we obtain

$$(ky - 3)x = 3y + k. \tag{9.3}$$

This equation has only a finite number of positive integral solutions, which we now enumerate. First, if $x = 1$, then

$$y = 1 + \frac{6}{k-3}$$

and the only solutions for the triple (x, y, k) are $\{(1, 2, 9), (1, 3, 6), (1, 4, 5), (1, 7, 4)\}$. Similarly, the only solutions with $y = 1$ are $\{(2, 1, 9), (3, 1, 6), (4, 1, 5), (7, 1, 4)\}$. If $x \geq 2$ and $y \geq 2$, we write

$$k = 3\left(\frac{x+y}{xy-1}\right).$$

An easy calculus exercise shows that on the domain $\{x \geq 2, y \geq 2\}$ the maximum of the right hand side is achieved at $(2, 2)$, with value $k = 4$. For $k = 1, 2, 3$, and 4, one finds the only solutions for (x, y, k) are

$$\big\{(4, 13, 1), (5, 8, 1), (8, 5, 1), (13, 4, 1), (2, 8, 2), (8, 2, 2), \\ (2, 3, 3), (3, 2, 3), (1, 7, 4), (2, 2, 4), (7, 1, 4)\big\}.$$

Thus, the values of (x, y) to consider are

$$\big\{(1, 2), (1, 3), (1, 4), (1, 7), (2, 2), (2, 3), (2, 8), (4, 13), (5, 8)\big\}$$

and their symmetric pairs.

Recall that we have $x = \phi_{2j-1}$, $y = \frac{d}{\phi_{2j+1}}$, and $p = xy$. The only possibilities for x are $x = 1, 2, 5$, and 13, in which case $y = \frac{d}{2}, \frac{d}{5}, \frac{d}{13}$, and $\frac{d}{34}$, respectively. The possible pairs (x, y) are thus

$$\{(1, 2), (1, 3), (1, 4), (1, 7), (2, 1), (2, 2), (2, 3), (2, 8), (5, 8), (13, 4)\}.$$

An immediate calculation yields the following possibilities for (p, d):

$$\big\{(2, 4), (3, 6), (4, 8), (7, 14), (2, 5), (4, 10), (6, 15), (16, 40), (40, 104), (52, 136)\big\}.$$

For most of these, the corresponding value of $q = 1 + \frac{d(d-3)}{p-1}$ is not an integer. The values of (p, q, d) that can arise as *integer* triples are

$$\big\{(2, 5, 4), (3, 10, 6), (2, 11, 5), (6, 37, 15)\big\}. \qquad \square$$

Lemma 9.11. *Suppose C is as in the assumptions of Theorem 9.1 and is not one of the exceptional cases. If $d \geq \phi_{2k-1} + 2$ and $d \geq 6$, then $p < \frac{\phi_{2k-1}}{\phi_{2k+1}} d$.*

Proof of Lemma 9.11. We proceed by induction over k. The base case, $k = 1$, is that $p < \frac{1}{2}d$, which is the statement of Lemma 9.5.

Notice that the sequence

$$a_k = \frac{\phi_{2k-1}}{\phi_{2k+1}}$$

is decreasing, converging to $\frac{2}{3+\sqrt{5}}$. So suppose we have already proved that $p < \frac{\phi_{2k-3}}{\phi_{2k-1}}d$ for some k and assume that $p > \frac{\phi_{2k-1}}{\phi_{2k+1}}d$ (by Lemma 9.10 we do not have to consider the possibility that $p = \frac{\phi_{2k-1}}{\phi_{2k+1}}d$).

Assume momentarily that

$$q > \frac{\phi_{2k-1}}{\phi_{2k-3}}d.$$

Then the number of the elements in the semigroup generated by p and q in the interval $[0, \phi_{2k-1}d]$ is the number of lattice points in the triangle

$$\left\{(x, y) \in \mathbb{R}^2_{\geq 0} : x\frac{p}{\phi_{2k-1}d} + y\frac{q}{\phi_{2k-1}d} \leq 1\right\}.$$

This is at most the number of the lattice points in the triangle

$$T = \left\{(x, y) \in \mathbb{R}^2_{\geq 0} : x\frac{1}{\phi_{2k+1}} + y\frac{1}{\phi_{2k-3}} < 1\right\};$$

notice that since $p > \frac{\phi_{2k-1}}{\phi_{2k+1}}d$ and $q > \frac{\phi_{2k-1}}{\phi_{2k-3}}d$, we replaced the inequality ≤ 1 with the strict inequality < 1, essentially deleting the hypothenuse of the triangle.

Counting lattice points in a polygon with lattice points as vertices can be done using Pick's theorem. In our situation, though, the triangle is especially simple so we can use an elementary argument, which can be found for instance in [35, p. 64], to conclude that the number of lattice points in T is

$$R = \frac{(\phi_{2k+1} + 1)(\phi_{2k-3} + 1)}{2} - 1.$$

Finally, elementary properties of Fibonacci numbers permit us to rewrite this as

$$R = \frac{(\phi_{2k-1} + 1)(\phi_{2k-1} + 2)}{2} - 1.$$

To summarize, under the assumptions that $p > \frac{\phi_{2k-1}}{\phi_{2k+1}}d$ and $q > \frac{\phi_{2k-1}}{\phi_{2k-3}}d$, we have that the number of elements in the semigroup generated by p and q in the interval $[0, \phi_{2k-1}d]$ is at most R. However, Theorem 1 with $j = \phi_{2k-1}$ and $b = 0$ states that

$$0 \leq R(\phi_{2k-1}d + 1) - \frac{(\phi_{2k-1} + 1)(\phi_{2k-1} + 2)}{2} \leq 1,$$

and in particular,

$$R(\phi_{2k-1}d+1) \geq \frac{(\phi_{2k-1}+1)(\phi_{2k-1}+2)}{2}.$$

Noting that $R = R(\phi_{2k-1}d+1)$ yields a contradiction. (We have used here that $d \geq \phi_{2k-1}+2$, since Theorem 1 requires that $j \leq d-2$.) Thus, (1.1) is not satisfied for $j = \phi_{2k-1}$ and $b = 0$.

With this contradiction, we can now conclude that under the assumption $p > \frac{\phi_{2k-1}}{\phi_{2k+1}}d$ we must have $q \leq \frac{\phi_{2k-1}}{\phi_{2k-3}}d$.

Recall that our induction hypothesis is that $p < \frac{\phi_{2k-3}}{\phi_{2k-1}}d$. That p is an integer implies that

$$p \leq \frac{\phi_{2k-3}}{\phi_{2k-1}}d - \frac{1}{\phi_{2k-1}}.$$

We can use these inequalities to conclude

$$(p-1)(q-1) \leq \left(\frac{\phi_{2k-3}}{\phi_{2k-1}}d - \frac{1}{\phi_{2k-1}} - 1\right)\left(\frac{\phi_{2k-1}}{\phi_{2k-3}}d - 1\right).$$

This can be written as

$$(p-1)(q-1) \leq d^2 - \left(\frac{\phi_{2k-3}}{\phi_{2k-1}} + \frac{\phi_{2k-1}}{\phi_{2k-3}}\right)d + 1 - \frac{d}{\phi_{2k-3}} + \frac{1}{\phi_{2k-1}}. \tag{9.4}$$

The term in parenthesis can be rewritten as

$$\frac{\phi_{2k-3}}{\phi_{2k-1}} + \frac{\phi_{2k-1}}{\phi_{2k-3}} = \frac{\phi_{2k-3}}{\phi_{2k-1}} + \frac{\phi_{2k-1}}{\phi_{2k-3}} + \frac{\phi_{2k+1}}{\phi_{2k-1}} - \frac{\phi_{2k+1}}{\phi_{2k-1}}.$$

Using the facts that $\phi_{2k-3}+\phi_{2k+1} = 3\phi_{2k-1}$ and $\phi_{2k-3}\phi_{2k+1} - \phi_{2k-1}^2 = 1$, the first and the third term yield 3, while the second and the last give $-\frac{1}{\phi_{2k-1}\phi_{2k-3}}$, so (9.4) can be rewritten as

$$(p-1)(q-1) \leq d^2 - \left(3 - \frac{1}{\phi_{2k-3}\phi_{2k-1}}\right)d + 1 - \frac{d}{\phi_{2k-3}} + \frac{1}{\phi_{2k-1}}, \tag{9.5}$$

which can be rewritten as

$$(p-1)(q-1) \leq d^2 - 3d + \frac{d}{\phi_{2k-1}\phi_{2k-3}} + 1 - \frac{d}{\phi_{2k-3}} + \frac{1}{\phi_{2k-1}}.$$

Since $(p-1)(q-1) = d^2 - 3d$, this implies that

$$\frac{d}{\phi_{2k-1}\phi_{2k-3}} + 1 - \frac{d}{\phi_{2k-3}} + \frac{1}{\phi_{2k-1}} \geq 0.$$

This is equivalent to

$$d \leq \phi_{2k-3}\left(\frac{\phi_{2k-1}+1}{\phi_{2k-1}-1}\right).$$

The conditions of the theorem include $d \geq \phi_{2k-1} + 2$, so this equality implies

$$\phi_{2k-1} + 2 \leq \phi_{2k-3}\left(\frac{\phi_{2k-1}+1}{\phi_{2k-1}-1}\right).$$

Clearing the denominator and writing $\phi_{2k-3} = \phi_{2k-1} - \phi_{2k-2}$ yields

$$(\phi_{2k-1} - 1)(\phi_{2k-1} + 2) \leq (\phi_{2k-1} - \phi_{2k-2})(\phi_{2k-1} + 1).$$

Expanding, this becomes

$$\phi_{2k-1}^2 + \phi_{2k-1} - 2 \leq \phi_{2k-1}^2 + \phi_{2k-1} - \phi_{2k-2}\phi_{2k-1} - \phi_{2k-2}.$$

Finally, this can be rewritten as

$$\phi_{2k-2}\phi_{2k-1} + \phi_{2k-2} \leq 2,$$

which is false for $k \geq 1$.

With this final contradiction we see that $p \leq \frac{\phi_{2k-1}}{\phi_{2k+1}} d$ must hold, so the induction step is accomplished. □

We shall need another result.

Lemma 9.12. *If $d > 6$ and $p + q > 3d$, then $p \leq \frac{3}{8}d$; equivalently, if $p > \frac{3}{8}d$ then $p + q \leq 3d$.*

Proof. Suppose that $p + q > 3d$ and $8p > 3d$. The eight multiples ap with $a = 0, \ldots, 7$, are possibly in $[0, 3d]$, but $8p$ is not. The conditions imply that $p + aq$ is not in the interval for any $a > 0$. It is possible that q is in the interval, but $2q$ is not, since in Lemma 9.5 we showed that $q \geq 2d - 1$. This gives a maximum of 9 elements in $S \cap [0, 3d]$, while (1.1) for $j = 3$ and $b = 0$ implies that there must be at least 10. □

Proof of Theorem 9.1. We suppose $d > 6$; for $d \leq 6$ the result is a straightforward computation.

We consider three cases. The first case is $p \leq \frac{3}{8}d$. Combining this with the result of Theorem 9.9, which states that $d < \alpha p + \beta$, yields $(\frac{8}{3} - \alpha)p < \beta$. This places an upper bound on d; performing the arithmetic and using simple bounds on α and β yields $d \leq 300$. All of these can be analyzed with a computer search, which yields the exceptional cases (a)–(h) of Theorem 9.1. Notice that the only examples having degree more than 33 are in item (h) of that list, but the BMY inequality rules these out: see Example 9.8 following Theorem 9.7.

Suppose now that $p > \frac{3}{8}d$. By Lemma 9.12, $p + q \leq 3d$. The second case is that $p + q \leq 3d - 1$, so $q \leq 3d - 1 - p$. Substituting this into (1.1) we obtain

$$(p - 1)(3d - p - 2) \geq d(d - 3).$$

We proceed as in the proof of Theorem 9.9. The inequality can be rewritten in terms of a quadratic polynomial in d:

$$d^2 - 3pd + (p^2 + p - 2) \leq 0.$$

Applying the quadratic formula yields

$$2d \leq 3p + \sqrt{5p^2 - 4p + 8} = 3p + \sqrt{5}p\sqrt{1 - \frac{4}{5p} + \frac{8}{5p^2}}.$$

This in turn can be written

$$2d \leq 3p + \sqrt{5}p + \sqrt{5}p\left(\sqrt{1 - \frac{4}{5p} + \frac{8}{5p^2}} - 1\right).$$

The term in parenthesis equals 0 for $p = 2$ and is negative for $p > 2$. Thus, in general, we have $d < \alpha p$.

For some k, $d \in [\phi_{2k-1} + 2, \phi_{2k+1} + 1]$. By Lemma 9.11 we have $p < \frac{\phi_{2k-1}}{\phi_{2k+1}} d$. Combined with $d < \alpha p$ we find

$$\frac{\phi_{2k+1}}{\phi_{2k-1}} < \frac{d}{p} < \alpha. \tag{9.6}$$

The sequence $\frac{\phi_{2k+1}}{\phi_{2k-1}}$ are the even convergents of the continued fraction expansion of α. As such, they form an increasing sequence converging to α, offering the closest lower approximations for given denominators. More precisely, (9.6) implies that $p > \phi_{2k-1}$. (See, for instance, [25], for these results concerning continued fractions. In particular, Theorem 7.13 of [25] states the required result concerning the sense in which convergents of continued fractions provide the best rational approximations to an irrational number.)

We now have $\phi_{2k-1} < p < \frac{\phi_{2k-1}}{\phi_{2k+1}} d$, implying that $d > \phi_{2k+1}$, and thus $d = \phi_{2k+1} + 1$. We now have

$$\frac{\phi_{2k+1}}{\phi_{2k-1}} < \frac{\phi_{2k+1} + 1}{p},$$

so

$$p < \phi_{2k-1} + \frac{\phi_{2k-1}}{\phi_{2k+1}}.$$

Since p is an integer,

$$p \leq \phi_{2k-1},$$

a contradiction.

The last case is $p + q = 3d$. Expanding $(p-1)(q-1) = d(d-3)$, we find $p + q = pq + 3d - d^2 + 1$. Combining these gives that $pq = d^2 - 1$. Writing $(q-p)^2 = (q+p)^2 - 4pq$ yields $(q-p)^2 = 5d^2 + 4$.

We need to find all the integers d such that $5d^2+4$ is a square. This problem is a case of Pell's equation. One accessible reference is [10, Section 5.1], where this precise problem is solved. The result states that $5d^2+4$ is a square if and only if $d=\phi_{2i}$ for some integer i. For a more general number theoretic discussion, a good reference is the discussion of the *Unit Theorem* in [21].

As a brief aside, we include here a summary of the argument.

Lemma 9.13. *If (x,y) is an integer solution to $5y^2+4=x^2$ then $y=\pm\phi_{2n}$ for some n.*

Proof. This equation can be rewritten as $(\frac{x}{2}+\frac{y}{2}\sqrt{5})(\frac{x}{2}-\frac{y}{2}\sqrt{5})=1$. Notice that x and y must have the same parity, and thus solutions correspond precisely to units of norm one in the algebraic number ring $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. (Units have norm either plus or minus one.) For a real quadratic number ring, the set of units forms an abelian group isomorphic to $\mathbb{Z}_2\oplus\mathbb{Z}$ (see, for instance, [21] or [9]). In our case, the infinite summand is generated by an element of the form $\frac{a}{2}+\frac{b}{2}\sqrt{5}$, where a and b are positive and have the same parity. Clearly, a generator of this form will have the minimum possible value of a. Since $\gamma=\frac{1}{2}+\frac{1}{2}\sqrt{5}$ does have norm -1, this is the generator of the set of units modulo torsion.

The first five powers of γ are $\frac{1}{2}+\frac{1}{2}\sqrt{5}$, $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, $\frac{4}{2}+\frac{2}{2}\sqrt{5}$, $\frac{7}{2}+\frac{3}{2}\sqrt{5}$ and $\frac{11}{2}+\frac{5}{2}\sqrt{5}$. Notice the numerators of the coefficients of $\sqrt{5}$ in γ^n are the Fibonacci numbers, ϕ_n and the numerator of the rational parts can be expressed as $3\phi_{n-1}+\phi_{n-2}$. For instance, $11=3\cdot 3+2$. That this pattern continues is an easy inductive argument using the defining recursion relation for Fibonacci numbers. Finally, since γ has norm -1, only even powers of γ have norm one, and thus only the Fibonacci numbers ϕ_{2n} appear as solutions for y in our original equation $5y^2+4=x^2$. □

Solving the pair of equations $p+q=3d$ and $(p-1)(q-1)=d(d-3)$ for p and q, with $q>p$, yields

$$p=\frac{3}{2}d-\frac{1}{2}\sqrt{5d^2+4}=\phi_{2i-2}$$
$$q=\frac{3}{2}d+\frac{1}{2}\sqrt{5d^2+4}=\phi_{2i+2}.$$

Notice, that $\gcd(p,q)=\gcd(\phi_{2i-2},\phi_{2i+2})=\phi_{\gcd(2i-2,2i+2)}$. If i is odd, then p and q are both divisible by 3, so they are not coprime, and the case is ruled out. We are left with the case $p=\phi_{4i-2}$, $q=\phi_{4i+2}$ and $d=\phi_{4i}$. □

9.5. Construction of curves. We will now use Orevkov's argument (see [26, Section 6]) to construct curves with $(p,q)=(\phi_{4j-2},\phi_{4j+2})$ and degree ϕ_{4j}.

Proposition 9.14. *For any $j=1,2,\ldots$ there exists a curve of genus 1 and degree ϕ_{4j} having a unique singularity with of type $(\phi_{4j-2},\phi_{4j+2})$.*

Proof. Fix a curve N of degree 3 with one node. Let $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ be a Cremona transformation as in [26]. Let F_1 be a cubic that passes through the node of N and is tangent to one branch with the tangency order eight. Such curve exists by a parameter counting argument; that is, the space of all cubics has dimension $\binom{3+2}{2} - 1 = 9$, we need one parameter to make F_1 pass through the node, and each order of tangency is one more condition, so we need altogether eight conditions. Notice that F_1 has genus one and does not intersect N away from the node.

We define inductively $F_j = f(F_{j-1})$. Since f is biregular away from $\mathbb{C}P^2 \setminus N$, each curves F_j has genus one and a single cusp. The characteristic sequence of the point of F_1 which is tangent to N is $(1, 8)$ (by this we mean that it is a smooth point of F_1 and the order of tangency is eight), and $(1, 8) = (\phi_2, \phi_6)$. The image of this point under the composite $f \circ f \circ \cdots \circ f$ is the singular point of F_j and the characteristic sequence becomes $(\phi_{4j-2}, \phi_{4j+2})$ by the same argument as in [26]. The degree of F_j is computed via the genus formula and the relation

$$(\phi_{4j-2} - 1)(\phi_{4j+2} - 1) = \phi_{4j}(\phi_{4j} - 3). \qquad \square$$

Proposition 9.15. *Cases (a), (b), (c) and (d) from Theorem 9.1 can be realized; that is, there exists a curve of degree* 4 *with singularity* $(2; 5)$*, a curve of degree* 5 *with singularity* $(2; 11)$*, a curve of degree* 6 *with singularity* $(3; 10)$ *and a curve of degree* 15 *with singularity* $(6; 37)$.

Proof. In [36, Example 1] there is given a construction of curves (b) and (d). A degree 4 curve with a $(2; 5)$ singularity can be given by an explicit equation. We are thankful to Karoline Moe for giving us an explicit construction of curve (c); we will not present her proof here. $\square$

The following result is well known to experts, we refer to [1] for a modern approach.

Proposition 9.16. *There exists a curve of degree* 6 *with a singularity* $(2; 19)$*. This is case (h) from the list with* $p = 2$.

9.6. The BMY inequality. Here we provide background for the proof of Theorem 9.7. Our approach closely follows [7,26]. The Bogomolov–Miyaoka–Yau inequality, see [17,23], is one of the main tools in studying curves in algebraic surfaces.

To formulate the BMY inequality we need some preliminaries. We let X be a (closed) algebraic surface. Recall that a *divisor* on X is a formal sum $\sum \alpha_i D_i$, where $\alpha_i \in \mathbb{Z}$ and D_i are closed algebraic curves on X. One of the main examples is the *canonical divisor* K. This is a divisor which represents a class in $H_2(X;\mathbb{Z})$ that is Poincaré dual to the first Chern class of the cotangent bundle of X.

Let D be a reduced effective divisor (that is, each irreducible component of D, which is a reduced algebraic curve, has coefficient one) with the property that $X \setminus D$ is of log-general type. We refer to [22, Section I.1] for the definition of log-general type and note that in our applications $X \setminus D$ will always be of this type. There exists a so called Zariski–Fujita decomposition of the divisor $K + D$; this is a unique decomposition $K + D = H + N$, where H and N are rational divisors and H is the *numerically effective* (in [22] this is called "arithmetically effective") part and N is the negative part of $K + D$; see [13] or [22, Section I.3].

The two fundamental proprieties of this decomposition are that $H \cdot N = 0$ and $N^2 \leq 0$. The BMY inequality as given in [17] or [26, Theorem 2.1] says that $H^2 \leq 3\chi(X \setminus D)$, where χ is the Euler characteristic; for our purpose the following formulation is sufficient.

Theorem 9.17 (BMY inequality). *Suppose X is an algebraic surface, K its canonical divisor, and D a divisor on X such that $X \setminus D$ is of log-general type (see [13]). Then*

$$(K + D)^2 \leq 3\chi(X \setminus D). \tag{9.7}$$

If, in addition, in the Zariski–Fujita decomposition $K + D = H + N$ we have $N \neq 0$, then we cannot have an equality in (9.7).

Suppose that $C \subset \mathbb{C}P^2$ is a cuspidal curve of positive genus $g > 0$ with singular points $z_1, \dots, z_n$, $n > 0$. For some $m > 0$, appropriately blowing up m points resolves the singularities, providing what is called a *good resolution* (also known as an SNC resolution, where SNC stays for "simple normal crossings"); in particular, it constructs a curve C', *the strict transform of* C, in a manifold X diffeomorphic to $\mathbb{C}P^2 \# _m \overline{\mathbb{C}P}^2$.

The steps of forming the good resolution of C build a sequence of divisors in X, $E_1, \dots, E_m$, each of multiplicity one (they corresponds to the exceptional divisors of the blow-ups constituting the good resolution). The *reduced exceptional divisor* E is the sum $\sum E_i$; see [38, Section 8.1]. We set

$$D = C' + E.$$

This is a reduced effective divisor on X.

A result of Wakabayashi [37] states that the complement of a positive genus algebraic curve in $\mathbb{C}P^2$ of degree $d \geq 4$ is of log-general type. By the genus formula, any curve of degree 3 or less is either nonsingular or genus 0. In particular, C is of degree four or more and Wakabayashi's result implies that the complement $\mathbb{C}P^2 \setminus C$

is of log-general type [37]. Since $\mathbb{C}P^2 \setminus C \cong X \setminus D$, we have $X \setminus D$ is log-general type, so Theorem 9.17 applies.

In order to show that the inequality in (9.7) is sharp, we use the following result proved in [27]; see [28] for more detailed exposition.

Lemma 9.18. *If C has n cuspidal singular points and $K + D = H + N$ is the Zariski–Fujita decomposition, then $N^2 < -\frac{n}{2}$. In particular, if C has at least one cuspidal singular point, then N is not trivial.*

Since $X \setminus D \cong \mathbb{C}P^2 \setminus C$, we have $\chi(X \setminus D) = (2g + 1)$. Thus, (9.7) becomes

$$(K + D)^2 < 3(2g + 1).$$

This can be written as

$$K(K + D) + D(K + D) < 6g + 3.$$

By the adjunction formula $D(K + D) = 2g - 2$; see [37, Section 7.6]. Substituting this, we obtain

$$K(K + D) < 4g + 5.$$

The homology of X splits as an orthogonal sum, with one summand spanned by L (representing a generator of $H_2(\mathbb{C}P^2)$) and separate summands, one for each singular point. Details are presented in [26, Section 2]. Accordingly, we write $K = K_0 + K_1 + \cdots + K_n$ and $D = D_0 + D_1 + \cdots + D_n$. Here K_0 and D_0 belong to the summands spanned by L and K_i, and the D_i belong to the summands corresponding to the singular points z_i. Note that $K_0 = -3L$ and $D_0 = dL$.

Using this decomposition, we can write the inequality as a summation:

$$K_0(K_0 + D_0) + \sum_{i=1}^{n} K_i(K_i + D_i) < 4g + 5.$$

Substituting the values of K_0 and D_0 we obtain.

$$9 - 3d + \sum_{i=1}^{n} K_i(K_i + D_i) < 4g + 5.$$

According to [7, Proposition 4.1], $K_i(K_i + D_i)$ can be identified with the Orevkov $\overline{M}$-number (where it was called the codimension). Thus,

$$\sum_{i>0} \overline{M}_i < 3d + 4g - 4.$$

As both sides of the above inequality are integers, we have

$$\sum_{i>0} \overline{M}_i \leq 3d + 4g - 5.$$

Theorem 9.7 is proved.

10. The semicontinuity of the spectrum

The spectrum Σ of a singular point of a plane curve is a collection of rational numbers from the interval $(0,2)$, where each rational number can occur multiple times. The count with multiplicity, $\#\Sigma$, is the Milnor number of the singularity. It is one of the strongest invariants of singularities. From a topological point of view, the spectrum can be (almost) recovered from the Tristram–Levine signatures of the link. For a singularity $x^p - y^q = 0$ (that is a singularity whose link is $T(p,q)$), the spectrum is the set

$$\Sigma_{p,q} = \left\{ \frac{i}{p} + \frac{j}{q},\ 1 \le i \le p-1,\ 1 \le j \le q-1 \right\},$$

where if a number x can be presented in ν different ways as a sum $\frac{i}{p} + \frac{j}{q}$, it means that x appears in $\Sigma_{p,q}$ with multiplicity ν.

There is a property of semicontinuity of spectra. Following [10, Section 2.4] we will formulate it as follows.

Suppose C is an algebraic curve in $\mathbb{C}P^2$ of arbitrary genus and not necessarily cuspidal. Suppose $\deg C = d$. Let $z_1, \dots, z_n$ be the singular points and $\Sigma_1, \dots, \Sigma_n$ the corresponding *spectra*. Let

$$\Sigma_{d,d} := \left\{ \frac{i}{d} + \frac{j}{d},\ 1 \le i, j \le d-1 \right\}$$

be the spectrum of the singularity $x^d - y^d = 0$. Then for any $x \in \mathbb{R}$ we have

$$\#\big(\Sigma_{d,d} \cap (x, x+1)\big) \ge \sum_{j=1}^{n} \#\big(\Sigma_j \cap (x, x+1)\big). \tag{10.1}$$

Equation (10.1), the spectrum semicontinuity property, is one of the strongest obstructions to the existence of curves in $\mathbb{C}P^2$ with prescribed singularities. It is most effective if the total number of elements of the spectra $\sum \#\Sigma_j$ is close to $\#\Sigma_{d,d} = (d-1)^2$, that is, if the (geometric) genus of C is small. The spectrum was effectively used in [10] to classify rational cuspidal curves with one cusp and one Puiseux pair at that cusp. We will show it is of limited effectiveness in case of curves of genus one.

Substituting $x = -1 + \frac{l}{d}$. into (10.1), where $l = 1, \dots, d-1$ we obtain

$$\sum_{j=1}^{n} \#\left(\Sigma_j \cap \left(0, \frac{l}{d}\right) \right) \le \frac{1}{2}(l-1)(l-2). \tag{10.2}$$

This equation in [10] is referred to as (SS_l). We shall examine how these inequalities apply to the classification problem of cuspidal curves of genus one with one singular point and one Puiseux pair, as in Theorem 9.1.

Example 10.1. The case where C has degree 6 and is of type $(4, 7)$ (so its genus is 1) satisfies all the SS_l inequalities, but cannot occur by discussion in Section 8. In fact there is only one singular point with spectrum

$$\Sigma = \left\{\frac{11}{28}, \frac{15}{28}, \frac{18}{28}, \frac{19}{28}, \frac{23}{28}, \frac{25}{28}, \frac{26}{28}, \frac{27}{28}, \dots\right\}$$

(the spectrum is symmetric around 1, so we give only elements in spectrum in the interval $[0, 1]$). The values of $\#\Sigma \cap (0, \frac{l}{d})$ for $l = 1, \dots, 6$ are $0, 0, 1, 3, 6, 9$, which are less than or equal to $0, 0, 1, 3, 6, 10$, as given by the right hand side of (10.2). Theorem 1, however, obstructs the existence of such curve, see Section 8.4.

Similarly, one can show that the property (SS_l) admits, for example, a genus one curve of degree 75 and with a singularity of type $(28, 201)$.

Example 10.2. If C is a curve of type (p, q), then, according to [10, Example 2.4], (SS_{d-1}) reads as

$$\frac{(p-1)(q-1)}{2} + \left\lfloor \frac{q}{d} \right\rfloor - \left\lfloor \frac{pq}{d} \right\rfloor \leq \frac{(d-2)(d-3)}{2}.$$

Since $(p-1)(q-1) = d(d-3)$, this gives

$$\left\lfloor \frac{pq}{d} \right\rfloor \geq d - 3 + \left\lfloor \frac{q}{d} \right\rfloor.$$

Writing $pq = d(d-3) + p + q - 1$ we arrive at

$$\left\lfloor \frac{p+q-1}{d} \right\rfloor \geq \left\lfloor \frac{q}{d} \right\rfloor.$$

This inequality is trivially satisfied whenever $p \geq 1$.

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