Topology and its Applications 348 (2024) 108889

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Translation results for some star-selection games

Christopher Caruvana [a], Jared Holshouser [b,*]

[a] School of Sciences, Indiana University Kokomo, 2300 S. Washington Street, Kokomo, IN 46902, USA
[b] Department of Mathematics, Norwich University, 158 Harmon Drive, Northfield, VT 05663, USA

A R T I C L E I N F O

Article history:
Received 27 February 2023
Received in revised form 17 March 2024
Accepted 18 March 2024
Available online 24 March 2024

MSC:
91A44
54D20
54B20

Keywords:
Star-selection principles
Topological selection games
Pixley-Roy hyperspaces
Uniform spaces

A B S T R A C T

We continue to explore the ways in which high-level topological connections arise from connections between fundamental features of the spaces, in this case focusing on star-selection principles in Pixley-Roy hyperspaces and uniform spaces. First, we find a way to write star-selection principles as ordinary selection principles, allowing us to apply our translation theorems to star-selection games. For Pixley-Roy hyperspaces, we are able to extend work of M. Sakai and connect the star-Menger/Rothberger games on the hyperspace to the ω-Menger/Rothberger games on the ground space. Along the way, we uncover connections between cardinal invariants. For uniform spaces, we show that the star-Menger/Rothberger game played with uniform covers is equivalent to the Menger/Rothberger game played with uniform covers, reinforcing an observation of Lj. Kočinac.

1. Introduction

In this paper, we show how the game-theoretic translation techniques developed in [3–5] can be applied to star-selection principles, as studied in [18,21,19,29,1]. Using a limited-information approach provides a more robust toolkit, and can shed light on various connections between selection games and selections principles. We establish a certain relationship between the star-Rothberger (resp. star-Menger) game for the Pixley-Roy hyperspace of X with the ω-Rothberger (resp. ω-Menger) game for X (see Theorem 3.13); we also establish some equivalences between the uniform-Rothberger and uniform-Menger games on a space to star versions of those games (see Theorem 4.6). We note a game-theoretic characterization of the uniform-Rothberger and uniform-Menger selection principles for paracompact spaces as an application of the well-known Hurewicz/Pawlikowski theorems (see Corollary 4.11). Along the route to Theorem 3.13, we establish Proposition 3.12, a cardinality bound between the so-called k-pseudocharacter of X and the

* Corresponding author.
E-mail addresses: caruvana@gmail.com (C. Caruvana), JHolshou@norwich.edu (J. Holshouser).
URLs: https://chcaru.pages.iu.edu/ (C. Caruvana), https://jaredholshouser.github.io/ (J. Holshouser).

https://doi.org/10.1016/j.topol.2024.108889
0166-8641/

star-Lindelöf degree of the Pixley-Roy hyperspace defined relative to an ideal of compact subsets of X. This follows from Proposition 3.9, which is a generalization of a cardinality bound due to Sakai [28]. We also expand the applicability of the existing game-theoretic translation machinery with Theorem 2.17 and, with Proposition 2.33, we compare selection principle properties relative to sequences and sets.

2. Preliminaries

We use the word *space* to mean *topological space* and, unless otherwise stated, all spaces X are assumed to be Hausdorff. In general, spaces X are assumed to be infinite and, when relevant, non-compact.

2.1. Selection games and cover types

In this section, we introduce the various entities that will be considered in this paper and the general framework for selection games.

Definition 2.1. For a space X, we let $\mathscr{T}_X$ denote the collection of all proper, non-empty open subsets of X.

Definition 2.2. For an open cover $\mathscr{U}$ of a topological space X, we say that $\mathscr{U}$ is *non-trivial* provided that $X \notin \mathscr{U}$. We let $\mathcal{O}_X$ denote the collection of all non-trivial open covers of X.

Definition 2.3. For a space X and a class $\mathcal{A}$ of closed proper subsets of X, a non-trivial open cover $\mathscr{U}$ is an $\mathcal{A}$*-cover* if, for every $A \in \mathcal{A}$, there exists $U \in \mathscr{U}$ so that $A \subseteq U$. We let $\mathcal{O}(X, \mathcal{A})$ denote the collection of all $\mathcal{A}$-covers of X.

Typically, the classes $\mathcal{A}$ that will be considered are bases for ideals or bornologies (see [14]); that is, collections of closed sets $\mathcal{A}$ that cover the space X with the property that, for every $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$. We will also often stipulate that $\{x\} \in \mathcal{A}$ for every $x \in X$. We will refer to such collections as *ideals of closed sets*.

Two such ideals of particular interest are

- $[X]^{<\omega}$, the set of finite subsets of X, and
- $K(X)$, the set of compact subsets of X.

Remark 2.4. Note that,

- if $\mathcal{A} = [X]^{<\omega}$, then $\mathcal{O}(X, \mathcal{A})$ is the collection of all ω-covers of X, which will be denoted by Ω_X.
- if $\mathcal{A} = K(X)$, then $\mathcal{O}(X, \mathcal{A})$ is the collection of all k-covers of X, which will be denoted by $\mathcal{K}_X$.

The notion of ω-covers is commonly attributed to [12], but they were already in use in [22] where they are refereed to as *open covers for finite sets*. The notion of k-covers appears as early as [23] in which they are referred to as *open covers for compact subsets*.

We recall the standard selection principles seen in the literature. For a primer on selection principles and relevant references, the authors recommend [20,31,32].

Definition 2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of sets. Then the single- and finite-selection principles are defined, respectively, to be the properties

$$\mathsf{S}_1(\mathcal{A}, \mathcal{B}) \equiv (\forall A \in \mathcal{A}^\omega) \left(\exists B \in \prod_{n\in\omega} A_n\right) \ \{B_n : n \in \omega\} \in \mathcal{B}$$

and

$$\mathsf{S}_{\text{fin}}(\mathcal{A},\mathcal{B}) \equiv (\forall A \in \mathcal{A}^\omega)\left(\exists B \in \prod_{n\in\omega}[A_n]^{<\omega}\right)\bigcup\{B_n : n \in \omega\} \in \mathcal{B}.$$

For a space X,

- $\mathsf{S}_{\text{fin}}(\mathcal{O}_X,\mathcal{O}_X)$ is known as the *Menger* property.
- $\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$ is known as the *Rothberger* property.
- $\mathsf{S}_{\text{fin}}(\Omega_X,\Omega_X)$ is known as the ω-*Menger* property.
- $\mathsf{S}_1(\Omega_X,\Omega_X)$ is known as the ω-*Rothberger* property.
- $\mathsf{S}_{\text{fin}}(\mathcal{K}_X,\mathcal{K}_X)$ is known as the k-*Menger* property.
- $\mathsf{S}_1(\mathcal{K}_X,\mathcal{K}_X)$ is known as the k-*Rothberger* property.

Selection games arise naturally from the study of selection principles. Selection games originated as topological games; a history of this development is outlined in Telgársky's survey [34] along with a robust list of references.

Definition 2.6. Given a set $\mathcal{A}$ and another set $\mathcal{B}$, we define the *finite-selection game* $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B})$ for $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\begin{array}{c|cccc} \text{I} & A_0 & A_1 & A_2 & \dots \\ \hline \text{II} & \mathcal{F}_0 & \mathcal{F}_1 & \mathcal{F}_2 & \dots \end{array}$$

where $A_n \in \mathcal{A}$ and $\mathcal{F}_n \in [A_n]^{<\omega}$ for all $n \in \omega$. We declare Two the winner if $\bigcup\{\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Definition 2.7. Similarly, we define the *single-selection game* $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ as follows:

$$\begin{array}{c|cccc} \text{I} & A_0 & A_1 & A_2 & \dots \\ \hline \text{II} & x_0 & x_1 & x_2 & \dots \end{array}$$

where each $A_n \in \mathcal{A}$ and $x_n \in A_n$. We declare Two the winner if $\{x_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

The study of strategies for mathematical games like these understandably cannot be separated from the games themselves. Infinite games and corresponding full-information strategies were both introduced in [10]. Some forms of limited-information strategies came along shortly after, like positional (also known as stationary) strategies [7,33]. For more on stationary and Markov strategies, see [11].

Definition 2.8. We define strategies of various strengths below.

- A *strategy for player One* in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is a function $\sigma : (\bigcup\mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for One is called *winning* if whenever $x_n \in \sigma\langle x_k : k < n\rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If player One has a winning strategy, we write $\text{I} \uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- A *strategy for player Two* in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is a function $\tau : \mathcal{A}^{<\omega} \to \bigcup\mathcal{A}$. A strategy τ for Two is *winning* if whenever $A_n \in \mathcal{A}$ for all $n \in \omega$, $\{\tau(A_0,\dots,A_n) : n \in \omega\} \in \mathcal{B}$. If player Two has a winning strategy, we write $\text{II} \uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$.
- A *predetermined strategy* for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves, they are just running

through a pre-planned script. Formally it is a function $\sigma : \omega \to \mathcal{A}$. If One has a winning predetermined strategy, we write $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

- A *Markov strategy* for Two is a strategy which only considers the most recent move of player One and the current turn number. Formally it is a function $\tau : \mathcal{A} \times \omega \to \bigcup \mathcal{A}$. If Two has a winning Markov strategy, we write $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.
- If there is a single element $A_0 \in \mathcal{A}$ so that the constant function with value A_0 is a winning strategy for One, we say that One has a *constant winning strategy*, denoted by $\mathrm{I} \underset{\mathrm{cnst}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

These definitions can be extended to $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ in the obvious way.

Remark 2.9. The following are mentioned in [6, Prop. 15] and [5, Lem. 2.12] for $\square \in \{1, \mathrm{fin}\}$.

- $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A}, \mathcal{B})$ is equivalent to $\mathsf{S}_\square(\mathcal{A}, \mathcal{B})$.
- $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A}, \mathcal{B})$ is equivalent to the property that, for every $A \in \mathcal{A}$, there is $\{x_n : n \in \omega\} \subseteq A$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

Note that the property $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A}, \mathcal{B})$ is a Lindelöf-like principle and falls in the category of what Scheepers [32] refers to as *Bar-Ilan selection principles*.

Definition 2.10. We say that two selection games $\mathcal{G}$ and $\mathcal{H}$ are *equivalent*, denoted $\mathcal{G} \equiv \mathcal{H}$, if the following hold:

- $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathcal{G} \iff \mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathcal{H}$
- $\mathrm{II} \uparrow \mathcal{G} \iff \mathrm{II} \uparrow \mathcal{H}$
- $\mathrm{I} \not\uparrow \mathcal{G} \iff \mathrm{I} \not\uparrow \mathcal{H}$
- $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathcal{G} \iff \mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathcal{H}$

If, in addition, $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathcal{G} \iff \mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathcal{H}$, then we write that $\mathcal{G} \rightleftarrows \mathcal{H}$.

Note that these notions of game equivalence are more robust than the traditional game equivalence, which only depends on full-information strategy equivalence.

We now recall a partial ordering on games introduced in [3,5].

Definition 2.11. Given selection games $\mathcal{G}$ and $\mathcal{H}$, we say that $\mathcal{G} \leq_{\mathrm{II}} \mathcal{H}$ if the following implications hold:

- $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathcal{G} \implies \mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathcal{H}$
- $\mathrm{II} \uparrow \mathcal{G} \implies \mathrm{II} \uparrow \mathcal{H}$
- $\mathrm{I} \not\uparrow \mathcal{G} \implies \mathrm{I} \not\uparrow \mathcal{H}$
- $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathcal{G} \implies \mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathcal{H}$

If, in addition, $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathcal{G} \implies \mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathcal{H}$, then we write that $\mathcal{G} \leq^+_{\mathrm{II}} \mathcal{H}$.

Observe that the partial ordering $\mathcal{G} \leq_{\mathrm{II}} \mathcal{H}$ effectively asserts that, if Two can win the game $\mathcal{G}$ with a particular level of "strength," then Two can win the game $\mathcal{H}$ with the same kind of strength; this is why we chose to use the subscript II. The additional criterion constituting $\mathcal{G} \leq^+_{\mathrm{II}} \mathcal{H}$ asserts that the corresponding

Lindelöf-like property, or Bar-Ilan selection principle, is also transferred; the inequality $\mathcal{G} \leq_{\text{II}} \mathcal{H}$ alone cannot guarantee this since the implications constituting its definition are dependent upon turn number, in general.

Note that, by Remark 2.9, if $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq_{\text{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$ for $\square \in \{1, \text{fin}\}$, then we have that, in particular,

$$\mathsf{S}_\square(\mathcal{A},\mathcal{C}) \implies \mathsf{S}_\square(\mathcal{B},\mathcal{D}).$$

Also, as the reader can readily verify, for any collections $\mathcal{A}$ and $\mathcal{B}$,

$$\mathsf{G}_1(\mathcal{A},\mathcal{B}) \leq^+_{\text{II}} \mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B}).$$

We recall the Translation Theorems, as developed through [3–5].

Theorem 2.12 *([5, Thm. 2.16]). Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions*

- $\overleftarrow{T}_{\text{I},n} : \mathcal{B} \to \mathcal{A}$ *and*
- $\overrightarrow{T}_{\text{II},n} : [\bigcup \mathcal{A}]^{<\omega} \times \mathcal{B} \to [\bigcup \mathcal{B}]^{<\omega}$

for each $n \in \omega$ so that

(P1) If $\mathcal{F} \in \left[\overleftarrow{T}_{\text{I},n}(B)\right]^{<\omega}$, then $\overrightarrow{T}_{\text{II},n}(\mathcal{F}, B) \in [B]^{<\omega}$

(P2) If $\mathcal{F}_n \in \left[\overleftarrow{T}_{\text{I},n}(B_n)\right]^{<\omega}$ for each $n \in \omega$ and $\bigcup_{n\in\omega} \mathcal{F}_n \in \mathcal{C}$, then

$$\bigcup_{n\in\omega} \overrightarrow{T}_{\text{II},n}(\mathcal{F}_n, B_n) \in \mathcal{D}.$$

Then $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{C}) \leq_{\text{II}} \mathsf{G}_{\text{fin}}(\mathcal{B},\mathcal{D})$. If, in addition, $\overleftarrow{T}_{\text{I},1} = \overleftarrow{T}_{\text{I},n}$ for all $n \in \omega$, then $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{C}) \leq^+_{\text{II}} \mathsf{G}_{\text{fin}}(\mathcal{B},\mathcal{D})$.

Corollary 2.13 *([5, Cor. 2.17]). Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions*

- $\overleftarrow{T}_{\text{I},n} : \mathcal{B} \to \mathcal{A}$ *and*
- $\overrightarrow{T}_{\text{II},n} : (\bigcup \mathcal{A}) \times \mathcal{B} \to \bigcup \mathcal{B}$

for each $n \in \omega$ so that the following two properties hold.

($\hat{P}$1) If $x \in \overleftarrow{T}_{\text{I},n}(B)$, then $\overrightarrow{T}_{\text{II},n}(x, B) \in B$.

($\hat{P}$2) If $\mathcal{F}_n \in \left[\overleftarrow{T}_{\text{I},n}(B_n)\right]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{F}_n \in \mathcal{C}$, then

$$\bigcup_{n\in\omega} \left\{ \overrightarrow{T}_{\text{II},n}(x, B_n) : x \in \mathcal{F}_n \right\} \in \mathcal{D}.$$

Then, for $\square \in \{1, \text{fin}\}$, $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq_{\text{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$. If, in addition, $\overleftarrow{T}_{\text{I},1} = \overleftarrow{T}_{\text{I},n}$ for all $n \in \omega$, then $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq^+_{\text{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$.

Remark 2.14. In the situation where $\overleftarrow{T}_{\text{I},1} = \overleftarrow{T}_{\text{I},n}$ for all $n \in \omega$, we may omit the second subscript and simply write $\overleftarrow{T}_{\text{I}}$.

Also, since the conditions in the Translation Theorems relating to the functions $\overrightarrow{T}_{\text{II},n}$ only depend on the range of the corresponding $\overleftarrow{T}_{\text{I},n}$, we will often partially define $\overrightarrow{T}_{\text{II},n}$ on the relevant inputs. In such a case, extending $\overrightarrow{T}_{\mathrm{II},n}$ beyond the given definition to its entire domain is immaterial to the application of the theorem, and thus any mention of such an extension will be omitted.

Remark 2.15. Since, for collections $\mathcal{A}$ and $\mathcal{B}$, and $\square \in \{1, \mathrm{fin}\}$,

$$\begin{aligned} \mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B}) &\implies \mathrm{II} \uparrow \mathsf{G}_\square(\mathcal{A},\mathcal{B}) \\ &\implies \mathrm{I} \not\uparrow \mathsf{G}_\square(\mathcal{A},\mathcal{B}) \\ &\implies \mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B}) \\ &\implies \mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B}), \end{aligned}$$

when applying the Translation Theorems to show

$$\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D}) \text{ or } \mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq^+_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D}),$$

we may assume that $\mathrm{I} \underset{\mathrm{cnst}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{C})$; that is, we may assume that, as in Remark 2.9, for every $A \in \mathcal{A}$, there is some $\{x_n : n \in \omega\} \subseteq A$ with $\{x_n : n \in \omega\} \in \mathcal{B}$.

In a similar way, we may also assume that $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A},\mathcal{B})$ when applying the Translation Theorems to show $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$, which is to say that we may assume $\mathsf{S}_\square(\mathcal{A},\mathcal{C})$ in such a setting.

As we often make use of the Axiom of Choice, we offer a generalization of the Translation Theorem 2.13 to make certain applications more readable. Before we present the generalization, we isolate a technique that will be used.

Remark 2.16. Suppose $R \subseteq X \times Y$ is so that $R[x] = \{y \in Y : (x,y) \in R\} \neq \emptyset$ for each $x \in X$. We can define an equivalence relation $\simeq$ on X by saying that $x_1 \simeq x_2$ if $R[x_1] = R[x_2]$. We can then define, for $\mathbf{x} \in X/{\simeq}$, $\hat{R}[\mathbf{x}] = R[x]$ where $x \in \mathbf{x}$. Now we apply the Axiom of Choice to select, for each $\mathbf{x} \in X/{\simeq}$, some $\hat{\gamma}(\mathbf{x}) \in R[\mathbf{x}]$. This $\hat{\gamma}$ extends to a function $\gamma : X \to Y$ where $\gamma(x_1) = \gamma(x_2)$ whenever $x_1 \simeq x_2$.

Theorem 2.17. *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are collections. Suppose there are functions $\overleftarrow{R}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{P}(\mathcal{A})$ and $\overrightarrow{R}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$ for every $n \in \omega$ so that*

(R1) for all $B \in \mathcal{B}$, $\overleftarrow{R}_{\mathrm{I},n}(B) \neq \emptyset$,
(R2) whenever $x \in A \in \overleftarrow{R}_{\mathrm{I},n}(B)$, then $\overrightarrow{R}_{\mathrm{II},n}(x,A,B) \in B$, and
(R3) whenever $F_n \subseteq A_n \in \overleftarrow{R}_{\mathrm{I},n}(B_n)$ are finite so that $\bigcup_{n\in\omega} F_n \in \mathcal{C}$, then

$$\left\{\overrightarrow{R}_{\mathrm{II},n}(x,A_n,B_n) : n \in \omega \text{ and } x \in F_n\right\} \in \mathcal{D}.$$

Then $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$, where $\square \in \{1, \mathrm{fin}\}$. If, in addition, $\overleftarrow{R}_{\mathrm{I},1} = \overleftarrow{R}_{\mathrm{I},n}$ for all $n \in \omega$, then $\mathsf{G}_\square(\mathcal{A},\mathcal{C}) \leq^+_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B},\mathcal{D})$.

Proof. Define $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ to be a choice function for $\overleftarrow{R}_{\mathrm{I},n}$ as in Remark 2.16; that is, for each $B \in \mathcal{B}$, $\overleftarrow{T}_{\mathrm{I},n}(B) \in \overleftarrow{R}_{\mathrm{I},n}(B)$ and, for $B_1, B_2 \in \mathcal{B}$, $\overleftarrow{R}_{\mathrm{I},n}(B_1) = \overleftarrow{R}_{\mathrm{I},n}(B_2)$ guarantees that $\overleftarrow{T}_{\mathrm{I},n}(B_1) = \overleftarrow{T}_{\mathrm{I},n}(B_2)$. When $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, define $\overrightarrow{T}_{\mathrm{II},n}(x,B)$ by

$$\overrightarrow{T}_{\mathrm{II},n}(x,B) = \overrightarrow{R}_{\mathrm{II},n}\left(x, \overleftarrow{T}_{\mathrm{I},n}(B), B\right).$$

From here, (R2) and (R3) ensure that $\overleftarrow{T}_{\mathrm{I},n}$ and $\overrightarrow{T}_{\mathrm{II},n}$ are as required in Corollary 2.13.

To address the situation in which $\overleftarrow{R}_{\mathrm{I},1} = \overleftarrow{R}_{\mathrm{I},n}$ for all $n \in \omega$, simply ensure that $\overleftarrow{T}_{\mathrm{I},1} = \overleftarrow{T}_{\mathrm{I},n}$ for all $n \in \omega$ in the construction of $\overleftarrow{T}_{\mathrm{I},n}$. □

2.2. Star-selection games

The primary goal in this section is to capture star-selection games as selection games as discussed in Section 2.1. For a systematic treatment of star-selection principles, the authors recommend [21].

Definition 2.18. If $\mathscr{U}$ is an open cover of X and $A \subseteq X$, the star of A relative to $\mathscr{U}$ is $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$. When dealing with singleton sets, we will use $\mathrm{St}(x, \mathscr{U})$ in place of $\mathrm{St}(\{x\}, \mathscr{U})$.

Definition 2.19. For a space X, we define the star-selection principles to be the following:

- $\mathsf{S}^*_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{B})$ is the property that, for each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of open covers, there are finite subsets $\mathscr{V}_n \subseteq \mathscr{U}_n$ for each $n \in \omega$ so that $\{\mathrm{St}(\bigcup \mathscr{V}_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.
- $\mathsf{S}^*_1(\mathcal{O}_X, \mathcal{B})$ is the property that, for each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of open covers, there are open sets $U_n \in \mathscr{U}_n$ for each $n \in \omega$ so that $\{\mathrm{St}(U_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.
- $\mathsf{SS}^*_{\mathcal{J}}(\mathcal{O}_X, \mathcal{B})$ is the property that, for each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of open covers, there are $J_n \in \mathcal{J}$ for each $n \in \omega$ so that $\{\mathrm{St}(J_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.

As introduced in [18], for any space X,

- $\mathsf{S}^*_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ is known as the *star-Menger* property.
- $\mathsf{S}^*_1(\mathcal{O}_X, \mathcal{O}_X)$ is known as the *star-Rothberger* property.

As defined, we cannot apply the Translation Theorems to star-selection principles. We remedy this in the following definition and proposition by finding a way to write star-selection principles as ordinary selection principles on more complicated sets.

In what follows, when discussing collections of subsets of X, we will identify X with $[X]^1$, the set of singletons of X, where the understood bijection is $x \mapsto \{x\}$.

Definition 2.20. Let X be a space.

- If $\mathscr{U}$ is an open cover of X and $\mathcal{J}$ is a collection of subsets of X, then $\mathrm{Cons}(\mathcal{J}, \mathscr{U}) = \{\mathrm{St}(J, \mathscr{U}) : J \in \mathcal{J}\}$. Notice that
$$\mathrm{Cons}(\mathscr{U}, \mathscr{U}) = \{\mathrm{St}(U, \mathscr{U}) : U \in \mathscr{U}\}.$$
- If $\mathscr{C}$ is a collection of covers of X and $f : \mathscr{C} \to \mathcal{P}(\mathcal{P}(X))$, then
$$\mathrm{Gal}(f, \mathscr{C}) = \{\mathrm{Cons}(f(\mathscr{U}), \mathscr{U}) : \mathscr{U} \in \mathscr{C}\}.$$

If f is constantly $\mathcal{A}$, we write $\mathcal{A}$ instead of f.

Notice that
$$\mathrm{Gal}(X, \mathcal{O}_X) = \{\mathrm{Cons}(X, \mathscr{U}) : \mathscr{U} \in \mathcal{O}_X\} = \{\{\mathrm{St}(x, \mathscr{U}) : x \in X\} : \mathscr{U} \in \mathcal{O}_X\},$$
$$\mathrm{Gal}(\mathcal{A}, \mathcal{O}_X) = \{\mathrm{Cons}(\mathcal{A}, \mathscr{U}) : \mathscr{U} \in \mathcal{O}_X\} = \{\{\mathrm{St}(A, \mathscr{U}) : A \in \mathcal{A}\} : \mathscr{U} \in \mathcal{O}_X\},$$

and

$$\mathrm{Gal}(\mathrm{id}, \mathcal{O}_X) = \{\mathrm{Cons}(\mathscr{U}, \mathscr{U}) : \mathscr{U} \in \mathcal{O}_X\} = \{\{\mathrm{St}(U, \mathscr{U}) : U \in \mathscr{U}\} : \mathscr{U} \in \mathcal{O}_X\}.$$

We chose the notation Cons to stand for *constellation*, a collection of stars; we chose the notation Gal to stand for *galaxy*, a collection of constellations, of sorts.

The reader may readily establish the following.

Proposition 2.21. *The following equivalences hold.*

- $\mathsf{S}^*_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{B})$ *is equivalent to* $\mathsf{S}_{\mathrm{fin}}(\mathrm{Gal}(\mathrm{id}, \mathcal{O}_X), \mathcal{B})$.
- $\mathsf{S}^*_1(\mathcal{O}_X, \mathcal{B})$ *is equivalent to* $\mathsf{S}_1(\mathrm{Gal}(\mathrm{id}, \mathcal{O}_X), \mathcal{B})$.
- $\mathsf{SS}^*_{\mathcal{J}}(\mathcal{O}_X, \mathcal{B})$ *is equivalent to* $\mathsf{S}_1(\mathrm{Gal}(\mathcal{J}, \mathcal{O}_X), \mathcal{B})$.

Since we will be interested in selection games, we will let, for notational brevity,

$$\mathsf{G}^*_{\square}(\mathcal{A}, \mathcal{B}) = \mathsf{G}_{\square}(\mathrm{Gal}(\mathrm{id}, \mathcal{A}), \mathcal{B})$$

and

$$\mathsf{SG}^*_{\mathcal{J},\square}(\mathcal{A}, \mathcal{B}) = \mathsf{G}_{\square}(\mathrm{Gal}(\mathcal{J}, \mathcal{A}), \mathcal{B}),$$

where $\square \in \{1, \mathrm{fin}\}$. When $\mathcal{J} = K(X)$, we'll use $\mathsf{SG}^*_{\mathbb{K},\square}(\mathcal{A}, \mathcal{B})$ to denote $\mathsf{SG}^*_{\mathcal{J},\square}(\mathcal{A}, \mathcal{B})$.

Observe that the selection principle $\mathsf{SS}^*_{\mathcal{J}}(\mathcal{A}, \mathcal{B})$ traditionally appears in the single-selection context, as noted in Proposition 2.21. In certain scenarios, like when $\mathcal{J}$ forms an ideal of closed subsets of the ground space, then there is no difference between single- and finite-selections, as we'll make explicit in Proposition 2.22. So the distinction between the selection principles $\mathsf{S}_1(\mathrm{Gal}(\mathcal{J}, \mathcal{A}), \mathcal{B})$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{Gal}(\mathcal{J}, \mathcal{A}), \mathcal{B})$ is generally a bit more subtle, as we illustrate in Examples 2.24 and 4.8. This distinction between the two selection principles is why we have chosen to use the notation $\mathsf{SG}^*_{\mathcal{J},\square}(\mathcal{A}, \mathcal{B})$ for the corresponding games.

Recall that a collection $\mathcal{A}$ is said to *refine* another collection $\mathcal{B}$, also stated as $\mathcal{A}$ *refines* $\mathcal{B}$, if, for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ so that $A \subseteq B$.

Proposition 2.22. *Let X be a space, $\mathcal{J}$ be an ideal of closed sets of X, $\mathcal{Q}$ be a collection of open covers of X, and $\mathcal{B}$ be a collection so that, if $\mathscr{V} \in \mathcal{B}$ and $\mathscr{V}$ refines a collection of open sets $\mathscr{U}$, then $\mathscr{U} \in \mathcal{B}$. Then*

$$\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{B}) \rightleftarrows \mathsf{SG}^*_{\mathcal{J},\mathrm{fin}}(\mathcal{Q}, \mathcal{B}).$$

Proof. Since

$$\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{B}) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{\mathcal{J},\mathrm{fin}}(\mathcal{Q}, \mathcal{B})$$

is evident, we need only show that

$$\mathsf{SG}^*_{\mathcal{J},\mathrm{fin}}(\mathcal{Q}, \mathcal{B}) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{B}).$$

To establish this, we describe how Two can use a winning play in $\mathsf{SG}^*_{\mathcal{J},\mathrm{fin}}(\mathcal{Q}, \mathcal{B})$ to produce a winning play in $\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{B})$. Suppose we have a sequence $\langle \mathscr{U}_n : n \in \omega \rangle \in \mathcal{Q}^\omega$ and $\mathcal{F}_n \in [\mathcal{J}]^{<\omega}$ for each $n \in \omega$ so that

$$\bigcup_{n\in\omega} \{\mathrm{St}(J, \mathscr{U}_n) : J \in \mathcal{F}_n\} \in \mathcal{B}.$$

Since $\mathcal{J}$ is assumed to be an ideal of closed sets, $\bigcup \mathcal{F}_n \in \mathcal{J}$ for each $n \in \omega$ and so we see that $\mathrm{St}\left(\bigcup \mathscr{F}_n, \mathscr{U}_n\right)$ is a legal sequence of plays by Two in $\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{B})$. By our assumption on $\mathcal{B}$ and the fact that

$$\bigcup_{J \in \mathcal{F}_n} \mathrm{St}(J, \mathscr{U}_n) = \mathrm{St}\left(\bigcup \mathcal{F}_n, \mathscr{U}\right)$$

for each $n \in \omega$, we also have that

$$\left\{\mathrm{St}\left(\bigcup \mathscr{F}_n, \mathscr{U}_n\right) : n \in \omega\right\} \in \mathcal{B}.$$

Since the way Two translates the plays between the games is not dependent upon any previous information, it is evident that the strategic strength of Two's play is preserved. □

In a similar vein, we obtain the following.

Proposition 2.23. *For any space X, if $\mathcal{J} = [X]^{<\omega}$ and $\mathcal{Q}$ is a collection of open covers X,*

$$\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{Q}, \mathcal{O}_X) \rightleftarrows \mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{O}_X).$$

Proof. The idea of this proof is similar to the proof of Proposition 2.22.

Let $\langle \mathscr{U}_n : n \in \omega \rangle \in \mathcal{Q}^\omega$ be arbitrary.

To see that

$$\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{Q}, \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{O}_X),$$

suppose we have $\mathcal{F}_n \in [X]^{<\omega}$ so that

$$\bigcup_{n \in \omega} \{\mathrm{St}(x, \mathscr{U}_n) : x \in \mathcal{F}_n\} \in \mathcal{O}_X.$$

Then $\mathcal{F}_n \in \mathcal{J}$ for each $n \in \omega$ and

$$\{\mathrm{St}(\mathcal{F}_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{O}_X.$$

To see that

$$\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{Q}, \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{Q}, \mathcal{O}_X),$$

suppose we have $\mathcal{F}_n \in \mathcal{J}$ so that

$$\{\mathrm{St}(\mathcal{F}_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{O}_X.$$

Since the target set is $\mathcal{O}_X$, it's straightforward to check that

$$\bigcup_{n \in \omega} \{\mathrm{St}(x, \mathscr{U}_n) : x \in \mathcal{F}_n\} \in \mathcal{O}_X$$

is the corresponding winning play by Two. □

Proposition 2.23 does not generalize to other target sets, as witnessed by the following example.

Example 2.24. For $X = \mathbb{R}$ and $\mathcal{J} = [\mathbb{R}]^{<\omega}$,

$$\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{O}_X, \Omega_X) \not\equiv \mathsf{SG}^*_{\mathcal{J},1}(\mathcal{O}_X, \Omega_X).$$

By σ-compactness of $\mathbb{R}$, Two has a winning Markov strategy in $\mathsf{SG}^*_{\mathcal{J},1}(\mathcal{O}_X, \Omega_X)$; indeed, given any open cover $\mathscr{U}_n$, we can find $F_n \in [\mathbb{R}]^{<\omega}$ so that $[-n, n] \subseteq \mathrm{St}(F_n, \mathscr{U}_n)$.

On the other hand, One has a predetermined winning strategy in $\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{O}_X, \Omega_X)$; to see this, let

$$\sigma(n) = \left\{ B\left(x, \frac{1}{2^{n+2}}\right) : x \in \mathbb{R} \right\}.$$

Note that, for any $x \in \mathbb{R}$, $\mathrm{St}(x, \sigma(n)) \subseteq B\left(x, \frac{1}{2^n}\right)$. Therefore, for any sequence $\langle F_n : n \in \omega \rangle$ of finite sets of X and any $n \in \omega$, there cannot be any $x \in F_n$ so that $\{-2, 2\} \subseteq \mathrm{St}(x, \sigma(n))$.

We record the following basic properties of stars, constellations, and galaxies.

Proposition 2.25. *Let X be a space, $\mathcal{A}, \mathcal{B}$ be collections of subsets of X, and $\mathscr{U}, \mathscr{V}$ be open covers of X.*

- *If $A, B \subseteq X$ are so that $A \subseteq B$, then $\mathrm{St}(A, \mathscr{U}) \subseteq \mathrm{St}(B, \mathscr{U})$.*
- *If $\mathscr{V}$ refines $\mathscr{U}$, then, for any $A \subseteq X$, $\mathrm{St}(A, \mathscr{V}) \subseteq \mathrm{St}(A, \mathscr{U})$.*
- *If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathrm{Cons}(\mathcal{A}, \mathscr{U}) \subseteq \mathrm{Cons}(\mathcal{B}, \mathscr{U})$.*
- *If $\mathcal{A}$ refines $\mathcal{B}$, then $\mathrm{Cons}(\mathcal{A}, \mathscr{U})$ refines $\mathrm{Cons}(\mathcal{B}, \mathscr{U})$.*
- *If $\mathscr{V}$ refines $\mathscr{U}$, then $\mathrm{Cons}(\mathcal{A}, \mathscr{V})$ refines $\mathrm{Cons}(\mathcal{A}, \mathscr{U})$.*
- *If $\mathcal{A}$ refines $\mathcal{B}$ and $\mathcal{Q}$ is a collection of open covers of X, then every member of $\mathrm{Gal}(\mathcal{B}, \mathcal{Q})$ is refined by a member of $\mathrm{Gal}(\mathcal{A}, \mathcal{Q})$.*

We now record a fact which can be thought of as a monotonicity law (see [32]).

Proposition 2.26. *Let X be a set. Suppose $\mathcal{A}$ and $\mathcal{B}$ are families so that, for $\mathscr{F} \in \mathcal{B}$, there exists some $\mathscr{G} \in \mathcal{A}$ so that $\mathscr{G}$ refines $\mathscr{F}$. Also suppose that $\mathcal{C}$ and $\mathcal{D}$ are families so that, for every $\mathscr{F} \in \mathcal{C}$ and $\mathscr{G} \subseteq \bigcup \mathcal{B}$, if $\mathscr{F}$ refines $\mathscr{G}$, then $\mathscr{G} \in \mathcal{D}$. Then*

$$\mathsf{G}_\square(\mathcal{A}, \mathcal{C}) \leq^+_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B}, \mathcal{D}).$$

Proof. Define a choice function $\rho : \mathcal{B} \to \mathcal{A}$ so that $\rho(\mathscr{F})$ is a refinement of $\mathscr{F}$ for every $\mathscr{F} \in \mathcal{B}$. Now, given $\mathscr{F} \in \mathcal{B}$, define $\gamma_{\mathscr{F}} : \rho(\mathscr{F}) \to \mathscr{F}$ to be a choice function so that $A \subseteq \gamma_{\mathscr{F}}(A)$ for every $A \in \rho(\mathscr{F})$. We can extend this to a function

$$\gamma : \left(\bigcup \mathcal{A}\right) \times \mathcal{B} \to \bigcup \mathcal{B}$$

so that $\gamma(A, \mathscr{F}) = \gamma_{\mathscr{F}}(A)$ when $A \in \rho(\mathscr{F})$.

Note that, by the definition, $(\hat{P}1)$ of Corollary 2.13 is satisfied; that is, if $A \in \rho(\mathscr{F})$, then $\gamma(A, \mathscr{F}) \in \mathscr{F}$.

To finish the proof, note that, if $\mathbf{F}_n \in [\rho(\mathscr{F}_n)]^{<\omega}$ for all $n \in \omega$ is so that $\bigcup_{n\in\omega} \mathbf{F}_n \in \mathcal{C}$, then

$$\bigcup_{n\in\omega} \{\gamma(A, \mathscr{F}_n) : A \in \mathbf{F}_n\} \in \mathcal{D}$$

by our assumptions on $\mathcal{C}$ and $\mathcal{D}$.

Therefore, Corollary 2.13 applies to yield that $\mathsf{G}_\square(\mathcal{A}, \mathcal{C}) \leq^+_{\mathrm{II}} \mathsf{G}_\square(\mathcal{B}, \mathcal{D})$. $\square$

Corollary 2.27. *Let X be a space and $\mathcal{Q}_1, \mathcal{Q}_2$ be collections of open covers of X. Additionally suppose that $\mathcal{Q}_2$ has the property that, if $\mathscr{U} \in \mathcal{Q}_2$ and $\mathscr{V}$ is an open cover so that $\mathscr{U}$ refines $\mathscr{V}$, then $\mathscr{V} \in \mathcal{Q}_2$. If $\mathcal{A}$ and $\mathcal{B}$ are collections of closed subsets of X with $\mathcal{A} \subseteq \mathcal{B}$, then*

$$\mathsf{SG}^*_{\mathcal{A},\square}(\mathcal{Q}_1, \mathcal{Q}_2) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{\mathcal{B},\square}(\mathcal{Q}_1, \mathcal{Q}_2).$$

Proof. By Proposition 2.25, every element of $\mathrm{Gal}(\mathcal{B}, \mathcal{Q}_1)$ is refined by a member of $\mathrm{Gal}(\mathcal{A}, \mathcal{Q}_1)$. By our assumption on $\mathcal{Q}_2$, Proposition 2.26 applies. □

The following will be used to apply the Translation Theorems under reasonable separation axioms.

Proposition 2.28. *If X is regular, then every open set is a star of the form $\mathrm{St}(V, \mathscr{U})$. If X is T_1, then every open set is a star of the form $\mathrm{St}(x, \mathscr{U})$.*

Proof. Suppose X is regular and let $U \subseteq X$ be open. Let $F \subseteq U$ be closed with non-empty interior $V := \mathrm{int}(F)$. Then define $\mathscr{U} = \{V, U, X \setminus F\}$. Observe that $\mathrm{St}(V, \mathscr{U}) = U$.

Now suppose X is T_1 and let $U \subseteq X$ be open. Let $x \in X$. Then $X \setminus \{x\}$ is open and $\mathscr{U} = \{U, X \setminus \{x\}\}$ is an open cover. Now $\mathrm{St}(x, \mathscr{U}) = U$. □

Corollary 2.29. *If X is regular, then $\bigcup \mathrm{Gal}(\mathrm{id}, \mathcal{O}_X) = \mathscr{T}_X$. If X is T_1, then $\bigcup \mathrm{Gal}(X, \mathcal{O}_X) = \mathscr{T}_X$.*

The following demonstrates why star-selection principles with respect to ω-covers are not, in general, worth investigating.

Proposition 2.30. *If $\mathscr{U} \in \Omega_X$ and $x \in X$, then $\mathrm{St}(x, \mathscr{U}) = X$.*

Proof. Let $y \in X$. Then since $\mathscr{U} \in \Omega_X$, there is a $V_y \in \mathscr{U}$ so that $\{x, y\} \subseteq V_y$. Thus $V_y \cap \{x\} \neq \emptyset$, and so $y \in \mathrm{St}(x, \mathscr{U})$. So $X = \mathrm{St}(x, \mathscr{U})$. □

2.3. Sets versus sequences in selection principles

We devote this section to a technical discussion of the use of sets versus the use of sequences for selection principles. In many instances, there is no need to distinguish between the two, but certain results are only true for sequences, and not for sets. We see this come up with λ-covers, in particular.

Definition 2.31. Recall that a countably infinite collection of open sets $\mathscr{U}$ is a *λ-cover* if, for each $x \in X$, $\{U \in \mathscr{U} : x \in U\}$ is infinite. In a similar way, we say that a (potentially uncountable) sequence of open sets $\langle U_\beta : \beta < \alpha \rangle$ is a *λ-cover* if, for each $x \in X$ and $\beta < \alpha$, there is a $\gamma > \beta$ so that $x \in U_\gamma$. We will let Λ_X denote the collection of all λ-covers of X, leaving the set or sequence formulation to be evident from context.

The main distinction here is that a sequence of open sets may have repeats, especially when the open sets have been created by an operation as we will do later in this paper. In this situation, it is possible for the open sets to form a λ-cover when treated as a sequence, but not when treated as a set.

In light of this issue, we introduce notation for selection principles where the winning condition is in terms of the sequence created by Player Two as opposed to the set created by them.

Definition 2.32. Let $\mathcal{A}$ and $\mathcal{B}$ be collections. Then the sequence-based selection principles are defined as follows:

- $\vec{\mathsf{S}}_1(\mathcal{A}, \mathcal{B})$ holds if and only if, for all sequences $\langle A_n : n \in \omega \rangle$ of elements of $\mathcal{A}$, there are $x_n \in A_n$ so that $\langle x_n : n \in \omega \rangle \in \mathcal{B}$.
- $\vec{\mathsf{S}}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ holds if and only if, for all sequences $\langle A_n : n \in \omega \rangle$ of elements of $\mathcal{A}$, there are $F_n \in A_n^{<\omega}$ so that $F_1 \frown F_2 \frown \cdots \in \mathcal{B}$.

We will overload notation when we write these sequence-based selection principles. For instance, $\vec{\mathsf{S}}_1(\mathcal{O}_X, \mathcal{O}_X)$ will mean that for every sequence of open covers $\langle \mathscr{U}_n : n \in \omega \rangle$, there are $U_n \in \mathscr{U}_n$ with the property that for each $x \in X$ there is an $n \in \omega$ so that $x \in U_n$.

As with regular selection principles, we can also talk about corresponding selection games and strategies. We denote the selection games by $\vec{\mathsf{G}}_1(\mathcal{A}, \mathcal{B})$ and $\vec{\mathsf{G}}_{\text{fin}}(\mathcal{A}, \mathcal{B})$. In the following proposition, we observe some basic relationships between these variations, establishing why the distinction is relevant.

Proposition 2.33. *Let X be a space and $\mathcal{C} \subseteq \mathcal{P}(\mathscr{T}_X)$. Then*

(a) $\vec{\mathsf{G}}_\square(\mathcal{C}, \Lambda_X) \leftrightarrows \mathsf{G}_\square(\mathcal{C}, \mathcal{O}_X)$,
(b) for an ideal $\mathcal{A}$ of closed sets of X, $\vec{\mathsf{G}}_\square(\mathcal{C}, \mathcal{O}(X, \mathcal{A})) \leftrightarrows \mathsf{G}_\square(\mathcal{C}, \mathcal{O}(X, \mathcal{A}))$, *and*
(c) $\vec{\mathsf{G}}_\square(\mathcal{C}, \Lambda_X) \not\equiv \mathsf{G}_\square(\mathcal{C}, \Lambda_X)$.

Proof. To establish (a), first note that

$$\vec{\mathsf{G}}_\square(\mathcal{C}, \Lambda_X) \leq^+_{\text{II}} \mathsf{G}_\square(\mathcal{C}, \mathcal{O}_X)$$

is evident since, assuming Two can play in such a way that they produce a sequence $\langle U_n : n \in \omega \rangle$ which is a λ-cover of X, $\{U_n : n \in \omega\}$ is certainly an open cover of X.

For the reverse inequality, choose $\beta : \omega^2 \to \omega$ to be a bijection so that $\langle \beta(j, k) : k \in \omega \rangle$ is increasing for each $j \in \omega$. Then, assume Two is playing in such a way that they can win $\mathsf{G}_\square(\mathcal{C}, \mathcal{O}_X)$. The main idea here is to apply whatever winning strength Two has on each of the sequences $\langle \beta(j, k) : k \in \omega \rangle$. So consider $\langle \mathscr{U}_n : n \in \omega \rangle \in \mathcal{C}^\omega$ and fix some $n \in \omega$. Let $\langle m, \ell \rangle = \beta^{-1}(n)$ and consider $\langle \langle m, k \rangle : k < \ell \rangle$. Since $\beta(m, k)$ is increasing as a function of k, we see that $\beta(m, k) < n$ for all $k < \ell$. Then we can apply whatever level of strength Two is able to use to win on the initial segment consisting of $\langle \mathscr{U}_{\beta(m,k)} : k < \ell \rangle$. So, for $m \in \omega$, Two produces $\langle U_{\beta(m,k)} : k \in \omega \rangle$ so that $\{U_{\beta(m,k)} : k \in \omega\} \in \mathcal{O}_X$. Consequently, the sequence $\langle U_{\beta(m,k)} : m, k \in \omega \rangle$ is a λ-cover of X.

(b) is evident since any sequence of open sets $\langle U_n : n \in \omega \rangle$ has the property

$$(\forall A \in \mathcal{A})(\exists n \in \omega)\ A \subseteq U_n$$

if and only if $\{U_n : n \in \omega\} \in \mathcal{O}(X, \mathcal{A})$.

For (c), consider ω with the discrete topology. Two can win the Rothberger game, but, if One plays the set of singletons, the winning play can never be a λ-cover; that is, by (a), Two can win $\vec{\mathsf{G}}_1(\mathcal{O}_X, \Lambda_X)$, but Two cannot win $\mathsf{G}_1(\mathcal{O}_X, \Lambda_X)$. □

We record the following corollary to Proposition 2.33 for use later.

Corollary 2.34. *Suppose X is star-Menger and let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers. Then the star-Menger selection principle can be applied to produce a sequence $\langle \mathscr{V}_n : n \in \omega \rangle$ where $\mathscr{V}_n \in [\mathscr{U}_n]^{<\omega}$ for each $n \in \omega$ so that, for every $x \in X$ and $m \in \omega$, there is $n \geq m$ so that $x \in \operatorname{St}\left(\bigcup \mathscr{V}_n, \mathscr{U}_n\right)$.*

3. Star-selection games in Pixley-Roy hyperspaces

Many connections between selection principles involving ω-covers of X and finite powers of X have been established in [12,16,27,30]. By investigating the space of finite sets with the Vietoris topology (see [24]), many analogous results were established in [5]. Inspired by these connections and results of [28,29], we now study Pixley-Roy hyperspaces.

The Pixley-Roy topology was originally defined in [26] on subsets of the real numbers. We will follow the generalized approach to more general classes of sets found in [8].

Definition 3.1. If $\mathcal{A}$ is an ideal of subsets of X, we define the topological space $\mathrm{PR}_{\mathcal{A}}(X)$ to be the set $\mathcal{A}$ with basic open sets of the form

$$[A, U] = \{B \in \mathcal{A} : A \subseteq B \subseteq U\}$$

where $A \in \mathcal{A}$ and $U \subseteq X$ is open. The sets $[A, U]$ form a basis since $\mathcal{A}$ is an ideal; indeed, for $A, B \in \mathcal{A}$ and $U, V \subseteq X$ open,

$$[A \cup B, U \cap V] = [A, U] \cap [B, V].$$

When $\mathcal{A} = [X]^{<\omega}$ (resp. $\mathcal{A} = K(X)$), we use $\mathrm{PR}(X)$ (resp. $\mathrm{PR}_{\mathbb{K}}(X)$) instead of $\mathrm{PR}_{\mathcal{A}}(X)$.

As noted in [8], for any space X and any ideal $\mathcal{A}$ of subsets of X, $\mathrm{PR}_{\mathcal{A}}(X)$ is T_1 and has a basis consisting of clopen sets.

Now we define the cardinal invariants to be discussed. For a general introduction to cardinal invariants, see [13].

Definition 3.2. For a space X, recall that the Lindelöf degree of X, $L(X)$, is the least cardinal κ so that every open cover $\mathscr{U}$ has a subcover of size κ. Below we define various related cardinals.

- $hL(X)$, the *hereditary Lindelöf degree* of X, is the supremum of the Lindelöf degrees of all subspaces (equivalently, all open subspaces) of X.
- $L_{\mathrm{st}}(X)$, called the *star-Lindelöf degree* of X, denotes the least cardinal κ so that every open cover $\mathscr{U}$ of X has a κ-sized subcollection $\{U_\alpha : \alpha < \kappa\}$ so that $\{\mathrm{St}(U_\alpha, \mathscr{U}) : \alpha < \kappa\}$ is an open cover of X.
- $L_\omega(X)$, called the *ω-Lindelöf degree* of X, is the least cardinal κ so that, for every ω-cover of X, there exists a κ-sized sub-ω-cover (a subset which is an ω-cover).
- $hL_\omega(X)$, called the *hereditary ω-Lindelöf degree* of X, is the supremum of the ω-Lindelöf degrees of all open subspaces of X.

Just as we say a space X is Lindelöf if $L(X) = \omega$, we say that

- X is *hereditarily Lindelöf* if $hL(X) = \omega$,
- X is *star-Lindelöf* if $L_{\mathrm{st}}(X) = \omega$,
- X is *ω-Lindelöf* if $L_\omega(X) = \omega$, and
- X is *hereditarily ω-Lindelöf* if $hL_\omega(X) = \omega$.

Proposition 3.3. *For any space X and $n \in \omega$,*

$$L_{\mathrm{st}}(X) \leq L(X) \leq L(X^n) \leq L\left(\bigcup_{k\in\omega} X^k\right) = L_\omega(X).$$

Proof. The star-Lindelöf degree is clearly bounded by the Lindelöf degree of X as $U \subseteq \mathrm{St}(U, \mathscr{U})$ for any open cover $\mathscr{U}$ of X and $U \in \mathscr{U}$. The chain $L(X) \leq L(X^n) \leq L\left(\bigcup_{k\in\omega} X^k\right)$ holds since X can be seen as a closed subspace of X^n and X^n is a closed subspace of $\bigcup_{k\in\omega} X^k$. For the equality $L\left(\bigcup_{k\in\omega} X^k\right) = L_\omega(X)$, see [12,16]. □

Note however, that these cardinals can be different. If X is the Sorgenfrey line, then $L(X) = \omega < L_\omega(X)$, which holds since X is Lindelöf but X^2 is not Lindelöf. On the other hand, ω_1 with the order topology is star-Lindelöf via Fodor's Lemma but is not Lindelöf. So $L_{\mathrm{st}}(\omega_1) = \omega < L(\omega_1)$.

Inspired by the ω-Lindelöf degree, we offer a generalization to other ideals of closed subsets.

Definition 3.4. For a space X and an ideal $\mathcal{A}$ of closed subsets of X, we define

- $L_{\mathcal{A}}(X)$, called the *$\mathcal{A}$-Lindelöf degree* of X, to be the least cardinal κ so that, for every $\mathcal{A}$-cover of X, there exists a κ-sized sub-$\mathcal{A}$-cover (a subset which is an $\mathcal{A}$-cover).
- $hL_{\mathcal{A}}(X)$, called the *hereditary $\mathcal{A}$-Lindelöf degree* of X, to be the supremum of the $\mathcal{A}$-Lindelöf degrees of all open subspaces of X.

Note that, when $\mathcal{A} = [X]^{<\omega}$, $L_{\mathcal{A}}(X) = L_\omega(X)$ and $hL_{\mathcal{A}}(X) = hL_\omega(X)$.

Definition 3.5. Recall that a collection $\mathscr{F}$ of subsets of a topological space X is said to be *discrete* if, for every $x \in X$, there exists an open neighborhood U of X so that U intersects at most one element of $\mathscr{F}$. Let $dc(X)$, called the *discrete cellularity* of X, denote the supremum over all cardinals κ so that there exists a κ-sized discrete family $\mathscr{U}$ of open sets.

Definition 3.6. The *pseudocharacter* $\psi(X)$ of a space X is the least cardinal κ so that every point is the κ-length intersection of open sets. Analogously, the *k-pseudocharacter* $\psi_k(X)$ of a space X is the least cardinal κ so that every compact subset of X is the κ-length intersection of open sets.

The following cardinal inequalities are inspired by various results of [28,29].

Proposition 3.7. *If X has a basis of clopen sets, then $dc(X) \leq L_{\mathrm{st}}(X)$.*

Proof. Let κ be cardinal so that $dc(X) > \kappa$. We will show that $L_{\mathrm{st}}(X) > \kappa$ as well, which in turn means that $L_{\mathrm{st}}(X) \geq dc(X)$. By our assumption that $dc(X) > \kappa$, there is a cardinal $\lambda > \kappa$ and a collection of non-empty open sets $\{U_\alpha : \alpha < \lambda\}$ which is discrete (and so also pairwise disjoint). As we have a basis of clopen sets, we can assume the U_α are clopen. Set $W = X \setminus \bigcup_{\alpha<\lambda} U_\alpha$. To show that W is open, let $y \in W$. Then there is an open set V so that $y \in V$ and V meets at most one U_α. Then $y \in V \setminus U_\alpha$, and $V \setminus U_\alpha \subseteq W$, so W is open.

Let $\mathscr{U} = \{U_\alpha : \alpha < \lambda\} \cup \{W\}$ and note that this forms an open cover of X. Suppose $\mathscr{V}$ is a κ-sized sub-collection of $\mathscr{U}$. Since the U_α are disjoint, $\mathrm{St}(U_\alpha, \mathscr{U}) = U_\alpha$ and $\mathrm{St}(W, \mathscr{U}) = W$ as well. Then

$$\bigcup_{V\in\mathscr{V}} \mathrm{St}\left(V, \mathscr{U}\right) = \mathrm{St}\left(\bigcup \mathscr{V}, \mathscr{U}\right) = \bigcup \mathscr{V} \neq X,$$

which implies that $L_{\mathrm{st}}(X) > \kappa$. □

Corollary 3.8. *For any space X and any ideal $\mathcal{A}$ of compact subsets of X,*

$$dc(\mathrm{PR}_{\mathcal{A}}(X)) \leq L_{st}(\mathrm{PR}_{\mathcal{A}}(X)).$$

Proof. Since $\mathrm{PR}_{\mathcal{A}}(X)$ has a basis of clopen sets, we can apply Proposition 3.7. □

Propositions 3.9 and 3.10 offer a generalization of [28, Thm. 3.10].

Proposition 3.9. *For any space X and any ideal $\mathcal{A}$ of compact subsets of X,*

$$hL_{\mathcal{A}}(X) \leq dc(\mathrm{PR}_{\mathcal{A}}(X)).$$

Proof. We first show that, for open $U \subseteq X$, $\mathrm{PR}_{\mathcal{A}}(U)$ is a clopen subspace of $\mathrm{PR}_{\mathcal{A}}(X)$. $\mathrm{PR}_{\mathcal{A}}(U)$ is clearly open as the union of basic open sets. To see that $\mathrm{PR}_{\mathcal{A}}(U)$ is closed, let $A \notin \mathrm{PR}_{\mathcal{A}}(U)$. That means that $A \not\subseteq U$, so there is some $x \in A \setminus U$. Then $[\{x\}, X]$ is a neighborhood of A that is disjoint from $\mathrm{PR}_{\mathcal{A}}(U)$.

We now show that $L_{\mathcal{A}}(X) \leq dc(\mathrm{PR}_{\mathcal{A}}(X))$. Assume $\mathscr{U} = \{U_\alpha : \alpha < \lambda\}$ is an $\mathcal{A}$-cover of X consisting of non-empty sets. For $\alpha < \lambda$, let

$$V_\alpha = \mathrm{PR}_{\mathcal{A}}(U_\alpha) \setminus \bigcup_{\beta<\alpha} V_\beta.$$

Clearly, the V_α are disjoint. We claim they are clopen as well.

Note that $V_0 = \mathrm{PR}_{\mathcal{A}}(U_0)$, which is clopen. So, given $\alpha > 0$, suppose that V_β is clopen for all $\beta < \alpha$. It is immediate then that V_α is closed. To see that V_α is open, let $A \in V_\alpha$. Then $[A, U_\alpha]$ is a neighborhood of A disjoint from $\bigcup_{\beta<\alpha} V_\beta$. To see this, let $B \in [A, U_\alpha]$ and $\beta < \alpha$. Note that

$$\bigcup_{\gamma\leq\beta} V_\gamma = \bigcup_{\gamma\leq\beta} \mathrm{PR}_{\mathcal{A}}(U_\gamma).$$

Since $A \notin \bigcup_{\gamma<\alpha} V_\gamma$, we see that $A \not\subseteq U_\beta$. Hence, $B \not\subseteq U_\beta$, so $B \notin V_\beta$.

By a similar argument as what's above, $\{V_\alpha : \alpha < \lambda\}$ is a discrete family of clopen sets. So, by the discrete cellularity, there are $\alpha_\xi < \lambda$ for $\xi < \delta := dc(\mathrm{PR}_{\mathcal{A}}(X))$ so that $V_{\alpha_\xi} \neq \emptyset$ for all $\xi < \delta$ and $V_\beta = \emptyset$ for all $\beta \notin \{\alpha_\xi : \xi < \delta\}$. Now, $\{V_{\alpha_\xi} : \xi < \delta\}$ is a cover of $\mathrm{PR}_{\mathcal{A}}(X)$. To see this, let $A \in \mathcal{A}$ and take $\alpha < \lambda$ to be minimal so that $A \subseteq U_\alpha$. Then $A \in V_\alpha$ which means $\alpha \in \{\alpha_\xi : \xi < \delta\}$. Hence, $\{U_{\alpha_\xi} : \xi < \delta\}$ is an $\mathcal{A}$-cover of X.

We finally show that $hL_{\mathcal{A}}(X) \leq dc(\mathrm{PR}_{\mathcal{A}}(X))$. Let $V \subseteq X$ be open. Then $\mathrm{PR}_{\mathcal{A}}(V)$ is clopen, and so $dc(\mathrm{PR}_{\mathcal{A}}(V)) \leq dc(\mathrm{PR}_{\mathcal{A}}(X))$. Thus $L_{\mathcal{A}}(V) \leq dc(\mathrm{PR}_{\mathcal{A}}(X))$ by the same proof as above. □

Proposition 3.10. *For a non-compact space X and any ideal $\mathcal{A}$ of compact subsets of X,*

$$hL(X) \leq hL_{\mathcal{A}}(X).$$

Proof. Suppose $\mathscr{U}$ is an open cover of an open set $V \subseteq X$. Let

$$\mathscr{V} = \left\{\bigcup \mathcal{F} : \mathcal{F} \in [\mathscr{U}]^{<\omega}\right\}.$$

Notice that, since $\mathcal{A}$ consists of compact sets, $\mathscr{V}$ is an $\mathcal{A}$-cover of V. That means we can find a sub-$\mathcal{A}$-cover $\mathscr{V}'$ so that $|\mathscr{V}'| \leq hL_{\mathcal{A}}(X)$. Now for each $W \in \mathscr{V}'$, choose $\mathcal{F}_W \subseteq \mathscr{U}$ which is finite so that $W = \bigcup \mathcal{F}_W$. Let $\mathscr{U}' = \bigcup_{W\in\mathscr{V}'} \mathcal{F}_W$. Then $\mathscr{U}'$ is a cover of V and

$$|\mathscr{U}'| \leq |\mathscr{V}'| \cdot \aleph_0 \leq hL_{\mathcal{A}}(X) \cdot \aleph_0.$$

The rightmost part is equal to $hL_{\mathcal{A}}(X)$ provided that it is infinite. □

Proposition 3.11. *If X is regular, $\psi_k(X) \leq hL(X)$.*

Proof. Let $K \subseteq X$ be compact and consider

$$\mathscr{V} = \{F \subseteq X : K \subseteq \text{int}(F) \text{ and } F \text{ is closed}\}.$$

Notice that $\{X \setminus F : F \in \mathscr{V}\}$ is a cover of $X \setminus K$ by regularity. Using the hereditary Lindelöf degree, there is an $hL(X)$-sized collection $\{X \setminus F_\alpha : \alpha < hL(X)\}$ so that $X \setminus K = \bigcup_{\alpha < hL(X)}(X \setminus F_\alpha)$. Let $U_\alpha = \text{int}(F_\alpha)$ and notice that $K \subseteq U_\alpha$ for all $\alpha < hL(X)$. Clearly, $K \subseteq \bigcap_{\alpha<hL(X)} U_\alpha$. Now if $y \notin K$, then $y \in X \setminus F_\alpha$ for some α. Thus $y \notin U_\alpha$, so $y \notin \bigcap_{\alpha<\kappa} U_\alpha$. Therefore $K = \bigcap_{\alpha<hL(X)} U_\alpha$. □

Proposition 3.12. *If X is a non-compact regular space and $\mathcal{A}$ is an ideal of compact subsets of X, then*

$$\psi_k(X) \leq L_{st}(\text{PR}_{\mathcal{A}}(X)).$$

Proof. Since X is regular, $\psi_k(X) \leq hL(X)$ by Proposition 3.11. Then Propositions 3.10, 3.9 and Corollary 3.8 show that

$$hL(X) \leq hL_{\mathcal{A}}(X) \leq dc(\text{PR}_{\mathcal{A}}(X)) \leq L_{st}(\text{PR}_{\mathcal{A}}(X)).$$

Therefore, $\psi_k(X) \leq L_{st}(\text{PR}_{\mathcal{A}}(X))$. □

The following captures [29, Thm. 4.12] (see also [21, Thm. 3.4 (2)]), but extends it to the other strategy types considered in this paper.

Theorem 3.13. *Assume X is regular. Then, for $\square \in \{1, \text{fin}\}$,*

$$\mathsf{G}^*_\square(\mathcal{O}_{\text{PR}(X)}, \mathcal{O}_{\text{PR}(X)}) \leq_{\text{II}} \mathsf{G}_\square(\Omega_X, \Omega_X).$$

Proof. As noted in Remark 2.15, we may assume that $\text{PR}(X)$ is star-Menger. Then $\text{PR}(X)$ is also star-Lindelöf which, by Proposition 3.12, implies that every singleton of X is a G_δ. So, for $x \in X$, we define open sets $G_{x,n}$ so that $\{x\} = \bigcap\{G_{x,n} : n \in \omega\}$. Without loss of generality, we suppose the sequence $\langle G_{x,n} : n \in \omega\rangle$ is descending. Then, for $F \in [X]^{<\omega}$ and $n \in \omega$, we define $G_{F,n} = \bigcup\{G_{x,n} : x \in F\}$. Note that the sequence $\langle G_{F,n} : n \in \omega\rangle$ is also descending.

We define a choice function $\gamma : \Omega_X \times \text{PR}(X) \to \mathscr{T}_X$ by choosing $\gamma(\mathscr{U}, F) \in \mathscr{U}$ to be so that $F \subseteq \gamma(\mathscr{U}, F)$. For $\mathscr{U} \in \Omega_X$, $n \in \omega$, and $F \in \text{PR}(X)$, let

$$\mathbf{W}(F, \mathscr{U}, n) = [F, \gamma(\mathscr{U}, F) \cap G_{F,n}].$$

Then, for $\mathscr{U} \in \Omega_X$ and $n \in \omega$, define

$$\mathscr{V}_{\mathscr{U},n} = \{\mathbf{W}(F, \mathscr{U}, n) : F \in \text{PR}(X)\}.$$

Then we can define $\overrightarrow{T}_{\text{I},n} : \Omega_X \to \text{Gal}(\text{id}, \mathcal{O}_{\text{PR}(X)})$ by the rule

$$\overrightarrow{T}_{\text{I},n}(\mathscr{U}) = \text{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n}).$$

Fix $\mathscr{U} \in \Omega_X$ and $n \in \omega$. Then, for $W \in \text{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n})$, consider

$$\{F \in \text{PR}(X) : W = \text{St}(\mathbf{W}(F, \mathscr{U}, n), \mathscr{V}_{\mathscr{U},n})\}.$$

We can define a choice function $\chi_{\mathscr{U},n} : \mathrm{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n}) \to \mathrm{PR}(X)$ to be so that $\chi_{\mathscr{U},n}(W) \in \mathrm{PR}(X)$ satisfies

$$W = \mathrm{St}(\mathbf{W}(\chi_{\mathscr{U},n}(W), \mathscr{U}, n), \mathscr{V}_{\mathscr{U},n}).$$

Since $\mathrm{PR}(X)$ is regular, Corollary 2.29 asserts that $\bigcup \mathrm{Gal}(\mathrm{id}, \mathcal{O}_{\mathrm{PR}(X)}) = \mathscr{T}_{\mathrm{PR}(X)}$. So we define $\overrightarrow{T}_{\mathrm{II},n} : \mathscr{T}_{\mathrm{PR}(X)} \times \Omega_X \to \mathscr{T}_X$, in light of Remark 2.14, as follows: if $W \in \mathrm{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n})$, let

$$\overrightarrow{T}_{\mathrm{II},n}(W, \mathscr{U}) = \gamma(\mathscr{U}, \chi_{\mathscr{U},n}(W)).$$

Now, using Corollary 2.34, assume we have sequences $\langle \mathscr{U}_n : n \in \omega \rangle \in \Omega_X^\omega$ and $\langle \mathscr{F}_n : n \in \omega \rangle$ so that $\mathscr{F}_n \in \left[\overrightarrow{T}_{\mathrm{I},n}(\mathscr{U}_n)\right]^{<\omega}$ for each $n \in \omega$ and, for every $F \in \mathrm{PR}(X)$ and every $m \in \omega$, there exist $n \geq m$ and $W \in \mathscr{F}_n$ so that $F \in W$. To show that

$$\bigcup_{n \in \omega} \left\{ \overrightarrow{T}_{\mathrm{II},n}(W, \mathscr{U}_n) : W \in \mathscr{F}_n \right\} \in \Omega_X,$$

let $F \in [X]^{<\omega}$ be arbitrary. Then we can let $\langle \ell_n : n \in \omega \rangle$ be cofinal in ω and, for each $n \in \omega$, choose $W_n \in \mathscr{F}_{\ell_n}$ so that $F \in W_n$ for all $n \in \omega$. To simplify notation, let $F_n = \chi_{\mathscr{U}_{\ell_n},\ell_n}(W_n)$. Then, for each $n \in \omega$, there is $E_n \in \mathrm{PR}(X)$ so that

$$\mathbf{W}(E_n, \mathscr{U}_{\ell_n}, \ell_n) \cap \mathbf{W}\left(F_n, \mathscr{U}_{\ell_n}, \ell_n\right) \neq \emptyset$$

and $F \in \mathbf{W}(E_n, \mathscr{U}_{\ell_n}, \ell_n)$. Observe that $E_n \subseteq F$ for each $n \in \omega$ so, as the map $n \mapsto E_n$, $\omega \to \wp(F)$, is into a finite set, there must be some cofinal $\langle k_n : n \in \omega \rangle$ in ω so that $\langle E_{k_n} : n \in \omega \rangle$ is constant. Hence, let $E \in \mathrm{PR}(X)$ be so that $E = E_{k_n}$ for all $n \in \omega$. Then, observe that $F \subseteq \gamma(\mathscr{U}_{\ell_{k_n}}, E) \cap G_{E,\ell_{k_n}}$ for all $n \in \omega$. Thus, since the $G_{E,n}$ are descending, $F = E$.

Now, behold that, as

$$\mathbf{W}(E_{k_0}, \mathscr{U}_{\ell_{k_0}}, \ell_{k_0}) \cap \mathbf{W}\left(F_{k_0}, \mathscr{U}_{\ell_{k_0}}, \ell_{k_0}\right) \neq \emptyset,$$

there is some $H \in \mathrm{PR}(X)$ so that

$$H \in \left[E_{k_0}, \gamma(\mathscr{U}_{\ell_{k_0}}, E_{k_0}) \cap G_{E_{k_0},\ell_{k_0}}\right] \cap \left[F_{k_0}, \gamma(\mathscr{U}_{\ell_{k_0}}, F_{k_0}) \cap G_{F_{k_0},\ell_{k_0}}\right].$$

Hence,

$$\begin{aligned} F = E = E_{k_0} \subseteq H &\subseteq \gamma(\mathscr{U}_{\ell_{k_0}}, F_{k_0}) \cap G_{F_{k_0},\ell_{k_0}} \\ &\subseteq \gamma(\mathscr{U}_{\ell_{k_0}}, F_{k_0}) \\ &= \overrightarrow{T}_{\mathrm{II},\ell_{k_0}}(W_{k_0}, \mathscr{U}_{\ell_{k_0}}). \end{aligned}$$

Therefore, Corollary 2.13 applies, finishing the proof. □

Let $\mathcal{P}_{\mathrm{fin}}(X)$ represent the set $[X]^{<\omega}$ with the subspace topology induced by the Vietoris topology on compact subsets of X (see [24] for more on the Vietoris topology). As noted in [8], the topology on $\mathrm{PR}(X)$ is finer than the topology on $\mathcal{P}_{\mathrm{fin}}(X)$.

Corollary 3.14. *For any regular space X and $\square \in \{1, \mathrm{fin}\}$,*

$$\mathsf{G}^*_\square(\mathcal{O}_{\mathrm{PR}(X)}, \mathcal{O}_{\mathrm{PR}(X)}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{O}_{\mathcal{P}_{\mathrm{fin}(X)}}, \mathcal{O}_{\mathcal{P}_{\mathrm{fin}(X)}}).$$

Proof. By [5, Thm. 4.8],

$$\mathsf{G}_\square(\Omega_X, \Omega_X) \leftrightarrows \mathsf{G}_\square(\mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}, \mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}).$$

So then Theorem 3.13 applies to yield

$$\mathsf{G}^*_\square(\mathcal{O}_{\mathrm{PR}(X)}, \mathcal{O}_{\mathrm{PR}(X)}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}, \mathcal{O}_{\mathcal{P}_{\text{fin}}(X)}). \quad \square$$

In the next example, we see that the inequality in Theorem 3.13 cannot, in general, be reversed.

Example 3.15. By [29, Cor. 4.8], $\mathrm{PR}(\mathbb{R})$ is not star-Menger. Since $\mathbb{R}$ is ω-Menger (which follows from σ-compactness in every finite power, see [16]), this shows that, even when points are G_δ, the inequality in Theorem 3.13 may not reverse.

Due to the use of Proposition 3.12 in the proof of Theorem 3.13, one may expect an analogous argument to establish an analog to Theorem 3.13 for $\mathrm{PR}_{\mathbb{K}}(X)$. However, an important part in the final steps of the argument for the finite subsets case involving isolating a cofinal sequence on which the E_{k_n} are constant does not seem to have remedy in the compact case. We hence leave this as Question 2 in Section 5.

In general, however,

$$\mathsf{G}_\square(\mathcal{K}_X, \mathcal{K}_X) \leq_{\mathrm{II}} \mathsf{G}^*_\square(\mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)}, \mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)})$$

fails, as witnessed by the following example.

Example 3.16. Let X be the one-point Lindelöfication of discrete ω_1. Then, by [2, Ex. 3.24] and the game duality results of [6], $\mathrm{II} \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$. Note that $\mathrm{PR}_{\mathbb{K}}(X) = \mathrm{PR}(X)$ since the compact subsets of X are exactly the finite subsets. Since $\{\{\alpha\} : \alpha \in \omega_1\}$ is a discrete family of open sets, we see that $\mathrm{PR}_{\mathbb{K}}(X)$ is not star-Lindelöf by Corollary 3.8. In particular, $\mathrm{II} \not\uparrow \mathsf{G}^*_\square(\mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)}, \mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)})$.

4. Star-selection games in uniform spaces

We first recall a few key facts about uniformities. For more on uniformities and uniform spaces, the authors recommend [9,15,17].

Definition 4.1. Given a set X, a collection $\mathcal{E}$ of sets containing the diagonal $\Delta = \{(x, x) : x \in X\}$ is called a *uniformity on* X if,

(1) for all $E \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$;
(2) for all $E, F \in \mathcal{E}$, $E \cap F \in \mathcal{E}$;
(3) for all $E \in \mathcal{E}$, there is an $F \in \mathcal{E}$ so that $F \circ F \subseteq E$; and
(4) for all $E \in \mathcal{E}$, if $E \subseteq F$, then $F \in \mathcal{E}$.

Each $E \in \mathcal{E}$ is called an *entourage*. We refer to $(X, \mathcal{E})$ as a uniform space.

Definition 4.2. The topology generated by a uniformity $\mathcal{E}$ on a set X is

$$\mathscr{T} = \{U \subseteq X : (\forall x \in U)(\exists E \in \mathcal{E})\ E[x] \subseteq U\}$$

where $E[x] = \{y \in X : \langle x, y\rangle \in E\}$.

Definition 4.3. Given a uniform space $(X, \mathcal{E})$ and a collection $\mathscr{U}$ of subsets of X, $\mathscr{U}$ is a *uniform cover* of X (with respect to $\mathcal{E}$) if there exists $E \in \mathcal{E}$ so that $\{E[x] : x \in X\}$ is a refinement of $\mathscr{U}$. We will say a uniform cover is an open uniform cover if it consists of open sets. Let $\mathcal{C}_{\mathcal{E}}(X)$ be the collection of all open uniform covers with respect to $\mathcal{E}$.

As introduced in [19], if $(X, \mathcal{E})$ is a uniform space, then

- $\mathsf{S}_{\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$ is known as the *uniform-Menger* (with respect to $\mathcal{E}$) property and
- $\mathsf{S}_1(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$ is known as the *uniform-Rothberger* (with respect to $\mathcal{E}$) property.

Recall that a collection $\mathscr{U}$ *star-refines* a collection $\mathscr{V}$ if $\{\mathrm{St}(U, \mathscr{U}) : U \in \mathscr{U}\}$ refines $\mathscr{V}$.

The following can be established using the comments in [9, p. 427] along with the fact that entourages that are open in X^2 with the induced topology form a base for the uniformity [17, Thm. 6.6].

Proposition 4.4. *If $\mathcal{E}$ is a uniformity on a non-empty set X, then $\mathcal{C}_{\mathcal{E}}(X)$ has the following properties:*

- *If $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$ and $\mathscr{U}$ is a refinement of an open cover $\mathscr{V}$, then $\mathscr{V} \in \mathcal{C}_{\mathcal{E}}(X)$.*
- *For any $\mathscr{U}_1, \mathscr{U}_2 \in \mathcal{C}_{\mathcal{E}}(X)$, there exists $\mathscr{V} \in \mathcal{C}_{\mathcal{E}}(X)$ so that $\mathscr{V}$ refines both $\mathscr{U}_1$ and $\mathscr{U}_2$.*
- *For every $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$, there exists $\mathscr{V} \in \mathcal{C}_{\mathcal{E}}(X)$ so that $\mathscr{V}$ is a star-refinement of $\mathscr{U}$.*
- *For any $x, y \in X$ with $x \neq y$, there is a $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$ so that, for all $U \in \mathscr{U}$, $x \in U \implies y \notin U$ and $y \in U \implies x \notin U$.*

The following follows from [9, Prop. 8.1.16].

Proposition 4.5. *Suppose X is a topological space. If X admits a collection of open covers $\mathcal{C}$ so that the conditions of Proposition 4.4 hold, then X admits a uniformity $\mathcal{E}$ so that $\mathcal{C} = \mathcal{C}_{\mathcal{E}}(X)$.*

The following captures [19, Thm. 1, Thm. 20], but extends them to the other strategy types considered in this paper.

Theorem 4.6. *Let $(X, \mathcal{E})$ be a uniform space. Then*

$$\mathsf{G}_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{SG}^*_{X,\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{G}^*_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X).$$

Proof. We first show that $\mathsf{G}_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leq^{+}_{\mathrm{II}} \mathsf{SG}^*_{X,\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$. Define $\overleftarrow{R}_{\mathrm{I}} : \mathrm{Gal}(X, \mathcal{C}_{\mathcal{E}}(X)) \to \mathcal{P}(\mathcal{C}_{\mathcal{E}}(X))$ by

$$\overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(X, \mathscr{U})) = \{\mathscr{V} \in \mathcal{C}_{\mathcal{E}}(X) : \mathrm{Cons}(X, \mathscr{U}) = \mathrm{Cons}(X, \mathscr{V})\}.$$

For each $U \in \mathscr{T}_X$, choose $\gamma(U) \in U$ and, for $V \in \mathscr{V} \in \overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(X, \mathscr{U}))$, we define

$$\overrightarrow{R}_{\mathrm{II}}(V, \mathscr{V}, \mathrm{Cons}(X, \mathscr{U})) = \mathrm{St}(\gamma(V), \mathscr{V}).$$

Clearly, $\overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(X, \mathscr{U})) \neq \emptyset$ and is a subset of $\mathcal{C}_{\mathcal{E}}(X)$. When

$$V \in \mathscr{V} \in \overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(X, \mathscr{U})),$$

we know that $\mathrm{Cons}(X, \mathscr{U}) = \mathrm{Cons}(X, \mathscr{V})$ and so $\mathrm{St}(\gamma(V), \mathscr{V}) \in \mathrm{Cons}(X, \mathscr{U})$. Finally, suppose $\mathcal{F}_n \subseteq \mathscr{V}_n \in \overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(X, \mathscr{U}_n))$ are finite so that $\bigcup_n \mathcal{F}_n$ is an open cover of X. For each $n \in \omega$ and each $V \in \mathcal{F}_n$, $\gamma(V) \in V \in \mathscr{V}_n$, so we can conclude that $V \subseteq \mathrm{St}(\gamma(V), \mathscr{V}_n)$. Thus

$$\left\{\overrightarrow{R}_{\mathrm{II}}(V, \mathscr{V}_n, \mathrm{Cons}(X, \mathscr{U}_n)) : n \in \omega \wedge V \in \mathcal{F}_n\right\}$$

forms an open cover as well. Hence, Theorem 2.17 applies.

Next, we show that $\mathsf{SG}^*_{X,\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{G}^*_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$. To see this, define $\overleftarrow{R}_{\mathrm{I}} : \mathrm{Gal}(\mathrm{id}, \mathcal{C}_{\mathcal{E}}(X)) \to \mathcal{P}(\mathrm{Gal}(X, \mathcal{C}_{\mathcal{E}}(X))$ by

$$\overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(\mathscr{U}, \mathscr{U})) = \{\mathrm{Cons}(X, \mathscr{V}) : \mathscr{V} \in \mathcal{C}_{\mathcal{E}}(X) \wedge \mathrm{Cons}(\mathscr{U}, \mathscr{U}) = \mathrm{Cons}(\mathscr{V}, \mathscr{V})\}.$$

For each $x \in X$ and $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$, let $U_{x,\mathscr{U}} \in \mathscr{U}$ be so that $x \in U_{x,\mathscr{U}}$. For

$$\mathrm{St}(x, \mathscr{V}) \in \mathrm{Cons}(X, \mathscr{V}) \in \overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(\mathscr{U}, \mathscr{U})),$$

define

$$\overrightarrow{R}_{\mathrm{II}}(\mathrm{St}(x, \mathscr{V}), \mathrm{Cons}(X, \mathscr{V}), \mathrm{Cons}(\mathscr{U}, \mathscr{U})) = \mathrm{St}(U_{x,\mathscr{V}}, \mathscr{V}).$$

Then if $\mathrm{St}(x, \mathscr{V}) \in \mathrm{Cons}(\mathscr{V}, \mathscr{V}) \in \overleftarrow{R}_{\mathrm{I}}(\mathrm{Cons}(\mathscr{U}, \mathscr{U}))$, we know that

$$\overrightarrow{R}_{\mathrm{II}}(\mathrm{St}(x, \mathscr{V}), \mathrm{Cons}(X, \mathscr{V}), \mathrm{Cons}(\mathscr{U}, \mathscr{U})) \in \mathrm{Cons}(\mathscr{U}, \mathscr{U}).$$

Also, if $\{\mathrm{St}(F_n, \mathscr{V}_n) : n \in \omega\}$ forms a cover, where the $F_n \subseteq X$ are finite, then, as

$$\mathrm{St}(F_n, \mathscr{V}_n) \subseteq \bigcup_{x \in F_n} \mathrm{St}(U_{x,\mathscr{V}_n}, \mathscr{V}_n),$$

we know that

$$\bigcup_{n \in \omega} \left\{\overrightarrow{R}_{\mathrm{II}}(\mathrm{St}(x, \mathscr{V}_n), \mathrm{Cons}(X, \mathscr{V}_n), \mathrm{Cons}(\mathscr{U}_n, \mathscr{U}_n)) : x \in F_n\right\}$$

also forms a cover of X. So Theorem 2.17 applies.

Finally, we show that $\mathsf{G}^*_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{G}_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$. For each $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$, use the properties of uniform covers to choose $\mathscr{U}^\star \in \mathcal{C}_{\mathcal{E}}(X)$ which is a star-refinement of $\mathscr{U}$. As noted in Remark 2.16, we can then define $\alpha_{\mathscr{U}} : \mathscr{U}^\star \to \mathscr{U}$ to be so that $\mathrm{St}(U, \mathscr{U}^\star) \subseteq \alpha_{\mathscr{U}}(U)$ for all $U \in \mathscr{U}^\star$ and $\alpha_{\mathscr{U}}(U) = \alpha_{\mathscr{U}}(V)$ whenever $\mathrm{St}(U, \mathscr{U}^\star) = \mathrm{St}(V, \mathscr{U}^\star)$. We then define $\overleftarrow{T}_{\mathrm{I}} : \mathcal{C}_{\mathcal{E}}(X) \to \mathrm{Gal}(\mathrm{id}, \mathcal{C}_{\mathcal{E}}(X))$ by $\overleftarrow{T}_{\mathrm{I}}(\mathscr{U}) = \mathrm{Cons}(\mathscr{U}^\star, \mathscr{U}^\star)$ and

$$\overrightarrow{T}_{\mathrm{II}}(\mathrm{St}(U, \mathscr{U}^\star), \mathscr{U}) = \alpha_{\mathscr{U}}(U).$$

Note that $\overrightarrow{T}_{\mathrm{II}}$ is well-defined by the conditions on $\alpha_{\mathscr{U}}$. For $W \in \overleftarrow{T}_{\mathrm{I}}(\mathscr{U})$, there is $U \in \mathscr{U}^\star$ so that $W = \mathrm{St}(U, \mathscr{U}^\star)$. Then observe that

$$\overrightarrow{T}_{\mathrm{II}}(W, \mathscr{U}) = \alpha_{\mathscr{U}}(U) \in \mathscr{U}.$$

To finish the proof, suppose we have

$$\mathcal{F}_n \in \left[\overleftarrow{T}_{\mathrm{I}}(\mathscr{U}_n)\right]^{<\omega}$$

so that $\bigcup_{n\in\omega} \mathcal{F}_n$ forms a cover of X. By the fact that applying $\overrightarrow{T}_{\mathrm{II}}$ produces a collection of open sets which is refined by $\bigcup_{n\in\omega} \mathcal{F}_n$, we see that Corollary 2.13 applies. $\square$

The following result can be seen as a generalization of [1, Thm. 3.4 (1)] beyond the paracompact case and to the other strategy types discussed in this paper.

Theorem 4.7. *For any uniform space $(X, \mathcal{E})$,*

$$\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{SG}^*_{\mathbb{K},1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{SG}^*_{\mathbb{K},\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X).$$

Proof. By Proposition 2.22,

$$\mathsf{SG}^*_{\mathbb{K},1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{SG}^*_{\mathbb{K},\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$$

and, by Corollary 2.27,

$$\mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{\mathbb{K},\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X).$$

To finish the proof, we show that

$$\mathsf{SG}^*_{\mathbb{K},\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \leq^+_{\mathrm{II}} \mathsf{SG}^*_{X,\mathrm{fin}}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X).$$

For $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$, let $\gamma(\mathscr{U}) \in \mathcal{E}$ be so that be so that

$$\{(\gamma(\mathscr{U}) \circ \gamma(\mathscr{U}) \circ \gamma(\mathscr{U}))\,[x] : x \in X\}$$

refines $\mathscr{U}$. Then define $\overleftarrow{T}_{\mathrm{I}}(\mathscr{U}) = \{\gamma(\mathscr{U})[x] : x \in X\}$.

We can then define $\eta(K, \mathscr{U})$ for $K \in K(X)$ and $\mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$ to be a finite subset of K so that

$$K \subseteq \bigcup \{\gamma(\mathscr{U})[x] : x \in \eta(K, \mathscr{U})\}.$$

Then, for $\mathcal{F} \in [K(X)]^{<\omega}$ and $\mathscr{V}, \mathscr{U} \in \mathcal{C}_{\mathcal{E}}(X)$, we define

$$\overrightarrow{T}_{\mathrm{II}}\left(\{\mathrm{St}(K, \mathscr{V}) : K \in \mathcal{F}\}, \mathscr{U}\right) = \bigcup_{K \in \mathcal{F}} \{\mathrm{St}(x, \mathscr{U}) : x \in \eta(K, \mathscr{U})\}.$$

To finish this application of Theorem 2.12, we just need to show that $\overrightarrow{T}_{\mathrm{II}}$ takes winning plays to winning plays. So let $\langle \mathscr{U}_n : n \in \omega \rangle \in \mathcal{C}_{\mathcal{E}}(X)^\omega$, $\mathscr{V}_n = \overleftarrow{T}_{\mathrm{I}}(\mathscr{U}_n)$, and suppose we have $\mathcal{F}_n \in [K(X)]^{<\omega}$ so that

$$\bigcup_{n \in \omega} \{\mathrm{St}(K, \mathscr{V}_n) : K \in \mathcal{F}_n\} \in \mathcal{O}_X.$$

Let $x \in X$ be arbitrary and choose $n \in \omega$ and $K \in \mathcal{F}_n$ so that $x \in \mathrm{St}(K, \mathscr{V}_n)$. Then there is some $w_0 \in X$ so that $x \in \gamma(\mathscr{U}_n)[w_0]$ and $K \cap \gamma(\mathscr{U}_n)[w_0] \neq \emptyset$. Let $w_1 \in K \cap \gamma(\mathscr{U}_n)[w_0]$ and $y \in \mathcal{F}_n$ be so that $w_1 \in \gamma(\mathscr{U}_n)[y]$. Then $\langle x, w_0 \rangle, \langle w_0, w_1 \rangle, \langle w_1, y \rangle \in \gamma(\mathscr{U}_n)$ which implies that $\langle x, y \rangle \in \gamma(\mathscr{U}_n) \circ \gamma(\mathscr{U}_n) \circ \gamma(\mathscr{U}_n)$. By the assumption on $\gamma(\mathscr{U}_n)$, there is some $U \in \mathscr{U}_n$ so that $(\gamma(\mathscr{U}_n) \circ \gamma(\mathscr{U}_n) \circ \gamma(\mathscr{U}_n))[x] \subseteq U$ and so we see that $y \in U$. As $x \in U$ as well, we see that $x \in \mathrm{St}(y, \mathscr{U}_n)$. This finishes the proof. □

Theorem 4.7 fails for $\mathsf{SG}^*_{X,1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$, as illustrated by the following example.

Example 4.8. Consider $X = \mathbb{R}$ with the uniformity $\mathcal{E}$ generated by the standard metric; that is, the uniformity generated by

$$\{\{(x, y) \in \mathbb{R}^2 : |x - y| < \varepsilon\} : \varepsilon > 0\}.$$

Then Two has a winning Markov strategy in $\mathsf{SG}^*_{\mathbb{K},1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$ but One has a predetermined winning strategy in $\mathsf{SG}^*_{X,1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$. To see this, note that a winning Markov strategy for Two in $\mathsf{SG}^*_{\mathbb{K},1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$ follows from σ-compactness of $\mathbb{R}$, as illustrated in Example 2.24. A predetermined winning strategy for One in $\mathsf{SG}^*_{X,1}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X)$ is

$$\sigma(n) := \left\{ B\left(x, \frac{1}{2^{n+2}}\right) : x \in \mathbb{R} \right\}.$$

Then, as $\mathrm{St}(x, \sigma(n)) \subseteq B\left(x, \frac{1}{2^n}\right)$, there is no sequence $\langle x_n : n \in \omega \rangle$ or $\mathbb{R}$ so that $\{\mathrm{St}(x_n, \sigma(n)) : n \in \omega\}$ covers $\mathbb{R}$.

Definition 4.9. Given a space X, the *universal* (or *fine*) *uniformity* on X is the finest uniformity on X compatible with its topology (see [9, Exercise 8.1.C] and [15, Thm. 1.20]).

The following, along with Theorem 4.6, extends [1, Thm. 3.2 (1), (2)] and the paracompact case of [1, Thm. 3.3 (1)] to more strategy types.

Theorem 4.10. *If X is paracompact and $\mathfrak{U}$ is the universal uniformity on X, then*

$$\mathsf{G}_\square(\mathcal{C}_{\mathfrak{U}}(X), \mathcal{O}_X) \leftrightarrows \mathsf{G}_\square(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. By [9, Thm 5.1.12], every open cover of a paracompact space has an open star-refinement. Then, by Proposition 4.5, $\mathcal{O}_X$ generates a uniformity on X, the universal uniformity $\mathfrak{U}$, and $\mathcal{O}_X = \mathcal{C}_{\mathfrak{U}}(X)$. □

Using the well-known results of Hurewicz and Pawlikowski (see [25]) along with Theorem 4.10, we obtain the following.

Corollary 4.11. *If X is paracompact and $\mathfrak{U}$ is the universal uniformity on X, then*

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{C}_{\mathfrak{U}}(X), \mathcal{O}_X) \iff \mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{C}_{\mathfrak{U}}(X), \mathcal{O}_X).$$

In general, Theorem 4.10 cannot be generalized beyond paracompact spaces, as the following example illustrates. (Compare to [19, Note 4].)

Example 4.12. Let $\mathcal{E}$ be any uniformity on ω_1 compatible with its order topology. Note that ω_1 is not Menger as it's not even Lindelöf. However, ω_1 is uniform-Rothberger (hence, uniform-Menger) with respect to $\mathcal{E}$.

To establish that ω_1 is uniform-Rothberger, we first show that self compositions of open entourages on ω_1 create open sets that contain tails. So let $U \subseteq \omega_1^2$ be open so that $\Delta \subseteq U$. For each limit ordinal $\lambda < \omega_1$, let $b_\lambda < \lambda$ be so that $(b_\lambda, \lambda]^2 \subseteq U$. By Fodor's Pressing Down Lemma, there is a stationary, thus cofinal, set $S \subseteq \omega_1$ and some $\beta < \omega_1$ so that $b_\lambda = \beta$ for all $\lambda \in S$.

We now show that $[\beta+1, \omega_1) \subseteq (U \circ U)[\beta+1]$. Indeed, let $\gamma > \beta$ and $\lambda \in S$ be so that $\gamma < \lambda$. Notice that $b_\lambda = \beta < \gamma < \lambda$ since $\lambda \in S$; so we obtain that $(\lambda, \gamma) \in U$ and that $(\beta+1, \lambda) \in U$. Hence, $(\beta+1, \gamma) \in U \circ U$, which implies that $\gamma \in (U \circ U)[\beta+1]$.

Now, this means that, for any uniformity $\mathcal{E}$ on ω_1 compatible with its order topology, every uniform cover of ω_1 contains a co-countable element; so ω_1 is uniform-Rothberger.

5. Open questions

Question 1. Are there any other interesting selection principles of the form

$$\mathsf{S}_\square(\mathrm{Gal}(f, \mathcal{A}), \mathcal{B})$$

where $\square \in \{1, \mathrm{fin}\}$, $\mathcal{A}$ and $\mathcal{B}$ are collections, and f is not either constant or the identity?

Question 2. For any space X, is there an analog to Theorem 3.13 for $\mathrm{PR}_{\mathbb{K}}(X)$? That is, is it the case that

$$\mathsf{G}^*_\square(\mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)}, \mathcal{O}_{\mathrm{PR}_{\mathbb{K}}(X)}) \leq_{\mathrm{II}} \mathsf{G}_\square(\mathcal{K}_X, \mathcal{K}_X)?$$

Question 3. Can Corollary 4.11 be generalized beyond the paracompact setting, despite the fact that Theorem 4.10 cannot?

Acknowledgements

The authors would like to thank the referee for their careful reading of this paper and for their comments that led to the improvement of the article.

References

[1] N. Alam, D. Chandra, Some remarks on games of certain star selection principles, Acta Math. Hung. 166 (2) (2022) 373–392.
[2] Christopher Caruvana, Jared Holshouser, Closed discrete selection in the compact open topology, Topol. Proc. 56 (2020) 25–55.
[3] Christopher Caruvana, Jared Holshouser, Selection games on continuous functions, Topol. Appl. 279 (2020) 107253.
[4] Christopher Caruvana, Jared Holshouser, Selection games on hyperspaces, Topol. Appl. 300 (2021) 107771.
[5] Christopher Caruvana, Jared Holshouser, Selection games and the Vietoris space, Topol. Appl. 307 (2022) 107772.
[6] Steven Clontz, Dual selection games, Topol. Appl. 272 (2020) 107056.
[7] Morton Davis, Infinite games of perfect information, Ann. Math. Stud. 52 (1964) 85–101.
[8] Eric K. van Douwen, The Pixley-Roy topology on spaces of subsets, in: George M. Reed (Ed.), Set-Theoretic Topology, Academic Press, 1977, pp. 111–134.
[9] Ryszard Engelking, General Topology, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, 1989.
[10] David Gale, F.M. Stewart, Infinite games with perfect information, in: Contrib. Theory of Games, II, in: Ann. Math. Stud., vol. 28, 1953, pp. 245–266.
[11] Fred Galvin, Rastislav Telgársky, Stationary strategies in topological games, Topol. Appl. 22 (1986) 51–69.
[12] J. Gerlits, Zs. Nagy, Some properties of C(X). I, Topol. Appl. 14 (1982) 151–161.
[13] R. Hodel, Chapter 1 - Cardinal functions I, in: Kenneth Kunen, Jerry E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 1–61.
[14] Henri Hogbe-Nlend, Bornologies and Functional Analysis. Introductory Course on the Theory of Duality Topology-Bornology and Its Use in Functional Analysis, Translated from the French by V.B. Moscatelli, North-Holland Math. Stud., vol. 26, Elsevier, Amsterdam, 1977.
[15] J.R. Isbell, Uniform Spaces, Math. Surv., vol. 12, American Mathematical Society (AMS), Providence, RI, 1964.
[16] Winfried Just, Arnold W. Miller, Marion Scheepers, Paul J. Szeptycki, The combinatorics of open covers. II, Topol. Appl. 73 (3) (1996) 241–266.
[17] John L. Kelley, General Topology, Graduate Texts in Mathematics, Springer, New York, 1975.
[18] Ljubiša D.R. Kočinac, Star-Menger and related spaces, Publ. Math. (Debr.) 55 (3–4) (1999) 421–431.
[19] Ljubiša D.R. Kočinac, Selection principles in uniform spaces, Note Mat. 22 (2) (2003) 127–139.
[20] Ljubiša D.R. Kočinac, Selected results on selection principles, in: Proceedings of the Third Seminar on Geometry and Topology, Tabriz, Iran, July 15–17, 2004, pp. 71–104.
[21] Ljubiša D.R. Kočinac, Star selection principles: a survey, Khayyam J. Math. 1 (1) (2015) 82–106.
[22] R.A. McCoy, K-space function spaces, Int. J. Math. Math. Sci. 3 (1980) 701–711.
[23] R.A. McCoy, Function spaces which are k-spaces, Topol. Proc. 5 (1981) 139–146.
[24] Ernest Michael, Topologies on spaces of subsets, Trans. Am. Math. Soc. 71 (1951) 152–182.

[25] Janusz Pawlikowski, Undetermined sets of point-open games, Fundam. Math. 144 (3) (1994) 279–285.
[26] Carl Pixley, Prabir Roy, Uncompletable Moore spaces, in: Proc. Auburn Topology Conf., 1969, pp. 75–85.
[27] Masami Sakai, Property C'' and function spaces, Proc. Am. Math. Soc. 104 (3) (1988) 917–919.
[28] Masami Sakai, Cardinal functions of Pixley-Roy hyperspaces, Topol. Appl. 159 (13) (2012) 3080–3088.
[29] Masami Sakai, Star versions of the Menger property, Topol. Appl. 176 (2014) 22–34.
[30] Marion Scheepers, Combinatorics of open covers. III: Games, $C_p(X)$, Fundam. Math. 152 (3) (1997) 231–254.
[31] Marion Scheepers, Selection principles in topology: new directions, Filomat 15 (2001) 111–126.
[32] Marion Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2) (2003) 3–41.
[33] Wolfgang M. Schmidt, On badly approximable numbers and certain games, Trans. Am. Math. Soc. 123 (1966) 178–199.
[34] Rastislav Telgársky, Topological games: on the 50th anniversary of the Banach-Mazur game, Rocky Mt. J. Math. 17 (1987) 227–276.