*Whither Russell's Paradox of Predication?**

NINO B. COCCHIARELLA

Indiana University

Russell's paradox has two forms or versions, one in regard to the class of all classes that are not members of themselves, the other in regard to "the predicate: to be a predicate that cannot be predicated of itself."[1] The first version is formulable in the ideography of Frege's *Grundgesetze der Arithmetik* and shows this system to be inconsistent. The second version, however, is not formulable in this ideography, as Frege himself pointed out in his reply to Russell.[2] Nevertheless, it is essentially the second version of his paradox that leads Russell to avoid it (and others of its ilk) through his theory of types.

The first version is of course the relevant version with respect to any formulation of the theory of types in which membership in a class is the fundamental notion, that is, a formulation utilizing 'ϵ' as a primitive binary predicate constant.[3] However, Russell's theory of types (even ignoring its ramification) is essentially concerned with the notion of predication, and only indirectly through the (philosophically questionable) interpretation of predication as the membership relation is the first version of his paradox relevant to this formulation.

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[1] "Letter to Frege," reprinted in [10], p. 125.
[2] "Letter to Russell," *ibid.*, p. 128.
[3] Cf. [5], p. 140 for a specific formulation of this kind of type theory.

Apparently, Russell saw his paradox as generating an aporetic situation in regard to two fundamental "notions," namely, the notion of membership (in a class) and the notion of predication (of an attribute).[4] In regard to the notion of membership, the application of Russell's paradox is not here brought into question. However, in regard to the notion of predication, the applicability of the reasoning grounding Russell's paradox will here be very much brought into question. Indeed, I shall claim that in this case the paradox fails.[5]

Paradoxical reasoning, of course, is inapplicable within any "reasonable" system, since an inconsistent system is *eo ipso* "unreasonable." That is not the point. Rather, it is that the original context of the reasoning grounding Russell's paradox of membership involved deep metaphysical assumptions, some logical, such as the principle of excluded middle, and some ontological, namely the comprehension principle for sets, which really are collectively inconsistent. The joint inconsistency of these assumptions does indeed constitute appropriate motivation for philosophical perplexity (*aporia*) regarding the notion of membership in that original context. On the other hand, the metaphysical assumptions (especially in regard to the "nexus" of copulation) of the original context supposedly grounding Russell's paradox of predication are not, I shall want to argue, after all inconsistent; for once we rigorously specify the essential features of the original context within which the reasoning is supposedly codified, we shall see that the reasoning involves a trivial violation of the restrictions imposed for the *proper substitution* of a formula for a predicate variable in the specification law for predicate variables. These restrictions, I shall argue, are essential not because they preserve consistency—which would be an *ad hoc* justification, a charge, I believe, that it is appropriate to make against the theory of types with its restrictions on grammatical well-formedness—but rather because they are the intuitively natural and appropriate restrictions to make when we are dealing with the notion of predication as opposed to that of membership. (It is assumed throughout that these two "notions" are not the same.) The "loosening" of these restrictions required to "validate"

[4] Gödel (cf. [6], p. 131f.) distinguishes these two forms of Russell's paradox by referring to them as the "extensional" and the "intensional" forms, respectively. For the purposes of the present paper, this distinction is preferable to Ramsey's different but better known distinction between "logical" and "semantical" paradoxes.

[5] With this failure of course goes a primary if not sole motivation for the simple theory of ontological types of third and higher order. The ontological scheme of second-order logic remains unaffected, having as it does a natural motivation of its own. Ramification also has *its* own motivation, and it may be appended to second-order logic (cf. [2], §58.) even though historically it was first appended to the simple theory of types.

the paradox of predication does not, I believe, result in a collection of philosophical assumptions that constitute appropriate motivation for philosophical perplexity.[6]

I

The Logical Context of Russell's Paradox of Predication

Russell's formulation of the set-theoretical version of his paradox is applicable to a certain type of logical context, one formulation of which is Frege's *Grundgesetze*.[7] The central notion of such a context is the membership relation.[8] What, on the other hand, is the type of logical context to which supposedly the paradox of predication is applicable? Obviously, one central notion of this type of context is predication *cum* quantifiable predicate variables, and therefore the context involves at least second-order logic. Equally obvious, the context is not that of (simple or ramified) type theory, for the latter is Russell's proposed replacement for that context, since supposedly by his paradox it was shown to be inconsistent. Accordingly, the syntactical representative of the type of logical context in question is somehow an extension of second-order logic but not an extension in the manner of type theory.[9] We are concerned of course with syntax

[6] This is not to say, however, that the context of Russell's paradox of predication (formalized below as T* and shown to be consistent) is philosophically unobjectionable—at least certainly not simply because it is consistent after all. Since there is no paradox, such philosophical objections cannot, of course, be construed as "lessons" of the paradox. Rather, they will arise at a conceptionally prior stage, as, for example, the nominalistic objection against allowing quantifiable predicate variables, and will, of course, beg the metaphysical issue in question by presupposing their own ontological paradigm. For myself, I have deep misgivings about allowing attributes a "double existence," one expressed through quantifiable predicate variables (but only in the predicate position—a mode of attribute existence which I do accept), the other expressed through additionally allowing predicate variables to occupy subject or nominal positions, thus construing attributes as a special (though perhaps secondary) breed of individual, the two breeds constituting a "two worlds" ontology.

[7] An alternative but more compact formulation is The Ideal Calculus, K, described in [5], Chapter III, Section 1.

[8] By a "logical context" I mean more than an uninterpreted calculus. The context is syntactically represented when formalized by the construction of a calculus. But the calculus alone is not an adequate representative of this context. What remains is the semantical or ontological background motivating the construction of that calculus.

[9] F. P. Ramsey (cf. [9], pp. 118–120) and R. Grossman (cf. [7]) might object here to standard second-order logic since the comprehension principle for complex attributes is here a theorem, whereas both Ramsey and Grossmann deny that there are complex attributes. However, a primary, if not sole, motive for the denial of complex attributes is to avoid Russell's paradox of predication, and in this regard any formulation of predication *cum* quantifiable predicate variables proposed by such a view is, vis-à-vis the paradox of predication,

(though not with syntactical considerations alone) since it is within a syntactical formulation of the logical context that the paradox is to be derived.

Now, although second-order logic is essentially incomplete, there exist formulations which are at least complete in a secondary sense and within the relevant extension of which Russell's paradox should be derivable if it is derivable at all within the intended logical context. Consider, for example, Church's formulation of the pure functional calculus of second order.[10] For convenience I shall refer to this calculus as the system T. Our question now is: How is T to be extended to a system T* which can be construed as an adequate syntactical representative of the type of logical context in which Russell's paradox of predication is formulable?

Obviously, a minimum requirement for the relevant extension of T is that we extend the notion of (well-formed) formula to allow predicate variables to occupy subject (or nominal) positions as well as predicate positions—for how otherwise could we represent "the predicate: to be a predicate that cannot be predicated of itself"?

Our grammar is specified as follows. We shall assume that there are enumerably infinite and pairwise disjoint sets of variables: individual variables and, for each natural number n, n-place predicate variables. (Propositional variables are 0-place predicate variables.) We shall use 'α', 'β', 'γ' to refer to individual variables, also called *subject terms,* and 'π', 'ρ', 'σ', 'τ' to refer to predicate variables (of arbitrary many places), also called *predicate terms.* We understand a *term* to be either an individual variable (subject term) or a predicate variable (predicate term). We shall use 'μ', 'ν' to refer to terms. As logical particles we shall use $\sim$, the negation sign, $\rightarrow$, the conditional sign, and $\bigwedge$, the universal quantifier. Other logical particles, such as $\leftrightarrow$, the biconditional sign, and $\bigvee$, the existential quantifier, are assumed to be defined (as syntactical abbreviations in the metalanguage) in the usual manner. An *atomic formula* is, for some natural number, n, the result of applying an n-place predicate variable π to n terms $\mu_0, \ldots,$ μ_{n-1}:$\pi(\mu_0, \ldots, \mu_{n-1})$. If $n = 0$, this result is understood to be π itself. (Observe that though a predicate term is not a subject term, a predicate term may occupy a subject position in an atomic formula.

on a par with Russell's proposal of type theory. (And so too is my own preferred view disallowing predicate variables in subject or nominal positions!) That is, such a proposal amounts to a replacement of the original type of logical context to which the paradox of predication supposedly is applicable. Our present concern is not the evaluation of alternatives to this type of context but rather a clarification of its characterization and a reassessment of the reasoning supposedly grounding the paradox of predication within it.

[10] [2], Chapter V. In what is to follow, we shall use a notational style variant to Church's.

A subject term, on the other hand, is not allowed to occupy a predicate position.) A *formula* is any member of the intersection of those sets K containing the atomic formulas and such that $\sim\varphi$, $(\varphi \rightarrow \psi)$, $\bigwedge\mu\varphi$ are in K whenever φ, ψ are in K and μ is an individual or predicate variable. We shall use 'φ', 'ψ', 'χ' to refer to formulas. Bondage and freedom of (occurrences of) variables is understood in the usual manner.

We now have the grammatical context for Russell's paradox of predication. Before specifying the transformational context, T*, let us review the informal reasoning grounding the paradox. We quote Russell:

> Let *w* be the predicate: to be a predicate that cannot be predicated of itself. Can *w* be predicated of itself? From each answer its opposite follows. Therefore we must conclude that *w* is not a predicate.[11]

We paraphrase Russell in preferring 'property' to 'predicate'. The statements that there exists and that there does not exist such a property as described by Russell are formulated within the present grammar by

(A) $\bigvee\tau\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$
(B) $\sim\bigvee\tau\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$

respectively, where τ, π are distinct 1-place predicate variables. The presumed argument for expression (B) with respect to the yet-to-be specified transformational system T* seems to be the following:[12]

(1) $\vdash_{T^*} \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)] \rightarrow [\tau(\tau) \leftrightarrow \sim\tau(\tau)]$

by special specification law for predicate variables;

(2) $\vdash_{T^*} \sim [\tau(\tau) \leftrightarrow \sim\tau(\tau)]$

by sentential logic;

(3) $\vdash_{T^*} \sim \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$

by (1), (2) and sentential logic;

(4) $\vdash_{T^*} \bigwedge \tau \sim \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$

[11] *Op. cit.*

[12] We indicate that φ is a theorem of T* by writing '$\vdash_{T^*} \varphi$'. By a theorem of T* we understand any formula terminating a finite sequence of formulas, where each constituent of the sequence is either an axiom of T* or is obtained from preceeding constituents by one of the inference rules of T*.

by (3) and (universal) generalization;

$$(5) \quad \vdash_{T^*} \sim \sim \bigwedge \tau \sim \bigwedge \pi[\tau(\pi) \leftrightarrow \sim \pi(\pi)]$$

that is,

$$\vdash_{T^*} \sim \bigvee \tau \bigwedge \pi[\tau(\pi) \leftrightarrow \sim \pi(\pi)]$$

by (4) and sentential logic.

A minimum basis for T* which recognizes the above reasoning can now be described. As inference rules we retain, without regard for redundancy (in T*), the inference rules of T by understanding the latter now to apply to all formulas (as defined by our present grammar) as well as to those that are wffs of T. These are (1) the rule of *modus ponens,* (2) the rule of generalization (applied to predicate and individual variables), (3) the rule of alphabetic change of bound individual variables, and (4) the rule of substitution for individual variables.

We provide a complete basis for the tautologous formulas by stipulating the following to be axiom schemata of T* (thereby extending the sentential axioms of T to all the formulas):

$$(A1) \quad \varphi \rightarrow (\psi \rightarrow \varphi)$$
$$(A2) \quad (\varphi \rightarrow [\psi \rightarrow \chi]) \rightarrow ([\varphi \rightarrow \psi] \rightarrow [\varphi \rightarrow \chi])$$
$$(A3) \quad (\sim\varphi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \varphi)$$

The confinement laws for the universal quantifier (when applied to both individual and predicate variables) are collapsed into one axiom schema:

$$(A4) \quad \bigwedge \mu(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \bigwedge \mu \psi)$$

where μ is a predicate or individual variable which does not occur free in φ.

What remains to codify the above argument for (B), the sentence of T* denying the existence of "the Russell property," is a special specification law for predicate variables corresponding to the specification axiom +509 (of T) for individual variables.[13] We shall state this law so as to encompass +509 as well. For its formulation, we understand two variables to be of *the same type* if either both are individual variables or, for some natural number n, both are n-place

[13] *Op. cit.*, p. 297.

predicate variables. Where μ, ν are variables, whether of the same type or not, we take

$$\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$$

to be the result of replacing each free occurrence of μ in φ by a free occurence of ν, if such a formula exists; otherwise

$$\varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$$

is understood to be φ itself. The special specification axiom replacing +509 of T is now formulated as:

$$\text{(A5)}\quad \bigwedge\mu\varphi \rightarrow \varphi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$$

where μ, ν are variables of the same type.

By the reasoning in (1) through (5) above, we conclude that so far as T* is concerned, "the Russell property" does not exist:

$$\vdash_{T^*} \sim \bigvee\tau\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$$

II

The Nondefinability of "The Russell Property"

Let us now consider (A), the sentence of T* affirming the existence of "the Russell property." We note first that the truth of (A) is affirmed by Russell through a simple specification of "the predicate: to be a predicate that cannot be predicated of itself."[14] Most writers on the paradox construe such a specification to be a definition in the object language of a (1-place) predicate constant t, read 'is impredicable'[15]:

$$\text{(C)}\quad t(\pi) \leftrightarrow \sim\pi(\pi)$$

Since definitions when construed other than as syntactical abbreviations in the metalanguage amount to additional axioms, expression (C) as an axiom added to T* yields:

$$\text{(D)}\quad \bigwedge\pi[t(\pi) \leftrightarrow \sim\pi(\pi)]$$

[14] Throughout we ignore Russell's use of the modal 'cannot (be)' and read 'is not' in its place. Even were the modal to be distinguished through extending T* to include modality, the same issue of trivial violation of proper substitution remains.

[15] Cf. Carnap [1], p. 83 and Hilbert and Ackermann [8], p. 145.

from which (A) follows by existential generalization. Existential generalization here amounts to extending the special specification axiom (A5) to a form (A5′) which allows specification to constants of the same type as the generalized predicate variable. It would seem then that the reasoning supposedly grounding expression (A) is to be viewed as being codified within that portion of T* described so far in Section 1 [with the replacement of (A5) by (A5′)], which we shall call T_0*, plus the so-called "definitional" axiom (C):

(a) $\vdash_{T_0^*+(C)} \bigwedge\tau \sim \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)] \rightarrow \sim\bigwedge\pi[t(\pi) \leftrightarrow \sim\pi(\pi)]$

by special specification axiom (A5′);

(b) $\vdash_{T_0^*+(C)} \bigwedge\pi[t(\pi) \leftrightarrow \sim\pi(\pi)]$

by "definition" (C) and generalization; and therefore

(c) $\vdash_{T_0^*+(C)} \sim\bigwedge\tau \sim \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$

that is,

$$\vdash_{T_0^*+(C)} \bigvee\tau\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$$

by (a), (b) and sentential logic.

But is (C) a legitimate definition; that is, is T_0* + (C) really a definitional extension of T_0*? One necessary condition for a definition is that it be *noncreative,* and in the present case this means that for any formula φ in which the defined predicate constant t does not occur, if

$$\vdash_{T^*+(C)} \varphi$$

then

$$\vdash_{T_0^*} \varphi$$

Accordingly, if T_0* + (C) really is a definitional extension of T_0*, then expression (A), as well as (B), is already provable in T_0*, that is, T_0* is inconsistent. The following argument, however, shows to the contrary that T_0* is consistent. Moreover, its consistency can be proved by the same "very elementary syntactical argument"[16] which proves the consistency of T, our present version of standard second-order logic.

In the proof of consistency, we shall associate with each formula φ (of our present grammar) a wff $f(\varphi)$ of the extended propositional

[16] [2], p. 306.

calculus, that is, a wff of T in which occur no variables other than propositional (0-place predicate) variables. We observe that the extended propositional calculus is not only consistent but provides an effective test for validity.[17]

We first characterize the following function g defined recursively on the set of formulas:

(i) $g(\pi(\mu_0, \ . \ . \ . \ , \mu_{n-1})) = \pi$

where $\pi(\mu_0, \ . \ . \ . \ , \mu_{n-1})$ is an atomic formula

(ii) $g(\sim\varphi) = \sim g(\varphi)$
(iii) $g(\varphi \rightarrow \psi) = [g(\varphi) \rightarrow g(\psi)]$
(iv) $g(\bigwedge\alpha\varphi) = g(\varphi)$

where α is an individual variable

(v) $g(\bigwedge\pi\varphi) = \bigwedge\pi g(\varphi)$

where π is a predicate variable

Observe that g assigns to each formula an expression which consists only of logical particles and predicate variables. Now since the set of predicate variables is equinumerous with its proper subset of propositional variables (both being of cardinality $\aleph_0$), we have a one-to-one function * correlating different propositional variables with different predicate variables. We define recursively the function h, the domain of which is the range of g:

(i) $h(\pi) = \pi^*$

where π is a predicate variable

(ii) $h(\sim\xi) = \sim h(\xi)$
(iii) $h(\xi \rightarrow \theta) = [h(\xi) \rightarrow h(\theta)]$
(iv) $h(\bigwedge\pi\xi) = \bigwedge\pi^* h(\xi)$

We now take f to be the relative product g/h, that is, $f(\varphi) = h[g(\varphi)]$, for each formula φ. Quite obviously, if φ is an axiom of $T_0{}^*$, then $f(\varphi)$ is a valid formula of the extended propositional calculus. Moreover, each inference rule of $T_0{}^*$ is easily seen to preserve validity under the transformation f. Accordingly, if

$\vdash_{T_0*} \varphi$

[17] *Ibid.*

then $f(\varphi)$ is a valid wff of the extended propositional calculus. Therefore, if $T_0{}^*$ were inconsistent, then the extended propositional calculus would also be inconsistent, which it is not. Accordingly, $T_0{}^*$ is consistent after all. We conclude then that $T_0{}^* + $ (C) is not a definitional extension of $T_0{}^*$, that is, expression (C) cannot be construed as an innocuous definition.

III

THE NONCOMPREHENSIBILITY OF "THE RUSSELL PROPERTY"

Why should it ever have been thought that expression (C), or its closure (D), was a legitimate definitional form? More specifically, at least relative to our present context, where t is a (1-place) predicate constant and φ is a formula in which t does not occur and whose only free variable is μ, what are the exact conditions under which the formula

$$\text{(E)} \quad t(\mu) \leftrightarrow \varphi$$

(or its closure) can be legitimately construed as a definition (relative to T*), that is, under what necessary and sufficient conditions can T* + (E) be construed as a definitional extension of T*? Obviously, T* + (E) is a definitional extension of T* when and only when

$$\vdash_{T^*} \bigvee\tau \bigwedge\mu[\tau(\mu) \leftrightarrow \varphi]$$

where τ is a 1-place predicate variable distinct from μ and not occurring free in φ. Accordingly, it is not the so-called "definition" (C) of "the Russell property" which shows (A) to be provable in T*. Rather conversely, it is the provability of (A) in T* which would show that (C) can reasonably be construed within the type of logical context in question as a definition of a property, specifically "the Russell property."

Of course, if T*, our representative of the type of logical context in question, were inconsistent, then (A) would be provable in it and therefore (C), relative to T*, would be an "acceptable" definition. Moreover, since $T_0{}^*$ is a fragment[18] of T*, (B) is provable in T*; and, accordingly, were we to assume (A) as an axiom of T*, we would immediately have its inconsistency. But there is simply no

[18] According to the observations made in Section 1, T* is to be an extension of T. But T contains as theorems instances of the comprehension principle for complex properties and these are not provable in $T_0{}^*$. Their provability in T requires the general specification laws $*509_0$ and $*509_n$ (*op. cit.*, p. 297), and for these laws some account has yet to be given.

point in assuming the context to be inconsistent only to show that a paradox is provable within it. The logical context of the reasoning supposedly grounding Russell's paradox of predication cannot be based simply on the assumption of (A) as an axiom—not at least without surrendering its philosophical significance—unless that assumption is a consequence of a more general principle comprehending the conditions under which a formula is understood to "determine" or "represent" a property. Clearly, the correct characterization of such a principle is the central difficulty in the specification of the system T*.

We note that in regard to the system T, standard second-order logic, such a principle has already been appropriately characterized. Specifically, it is the principle which concerns the conditions under which a formula may be construed as a substituend of a generalized predicate variable. For, as, and only as, such a substituend does it "represent" a value of that predicate variable. Accordingly, it is the general specification law for predicate variables which comprehends the conditions under which a formula is understood to "determine" or "represent" a property. For the wffs of T, the law is schematized as:

$$\bigwedge\pi\psi \rightarrow \check{S}^{\pi(\alpha_0,\ldots,\alpha_{n-1})}_{\varphi}\ \psi|$$

where π is an n-place predicate variable and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables.[19] Note that from this axiom schema we are able to derive the comprehension principle for properties. For since

$$\check{S}^{\pi(\alpha)}_{\varphi} \sim\bigwedge\alpha[\pi(\alpha) \leftrightarrow \varphi]\,| \;=\; \sim\bigwedge\alpha[\varphi \leftrightarrow \varphi]$$

where π is not free in φ, then

$$\vdash_T \bigwedge\pi \sim\bigwedge\alpha[\pi(\alpha) \leftrightarrow \phi] \rightarrow \sim\bigwedge\alpha[\phi \leftrightarrow \phi]$$

by the general specification axiom for predicate variables, and therefore:

$$\vdash_T \bigvee\pi\bigwedge\alpha[\pi(\alpha) \leftrightarrow \varphi]$$

The question now is: How are we to extend in an intuitive and natural way the notion of the proper substitution of a formula for a predicate variable so that we can apply the notion to all formulas as well as the wffs of T? Needless to say, the extension of this notion is not to violate the distinguishing feature of predication as reflected

[19] Cf. [2], p. 297.

by our grammar, namely, the distinction between subject positions on the one hand and predicate positions on the other. To nullify this distinction by construing predicate positions as "argument" positions on a par with subject positions is to depart from predication as our central concept and to turn rather to membership or a membership type of *relation*.[20]

Because of notational differences, we paraphrase rather than directly quote Church's definition of substitution for T.[21] In doing so, however, we shall understand

$$\varphi\begin{bmatrix}\mu_0 & \cdots & \mu_{n-1}\\ \nu_0 & \cdots & \nu_{n-1}\end{bmatrix}$$

to be the result of *simultaneously* replacing all the free occurrences of $\mu_0, \ldots, \mu_{n-1}$ in φ by free occurrences of $\nu_0, \ldots, \nu_{n-1}$, respectively, if such a formula exists; otherwise it is φ itself.

If π is an n-place predicate variable and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, then

$$\check{\mathrm{S}}^{\pi(\alpha_0,\ldots,\alpha_{n-1})}_{\varphi}\,\psi|$$

shall be ψ unless the following conditions are satisfied: (1) no free occurrence of π (in predicate position) in ψ occurs within a subformula of ψ of the form $\bigwedge\mu\chi$, where μ is a predicate or individual variable distinct from $\alpha_0, \ldots, \alpha_{n-1}$ and occurring free in φ; and (2) for all individual variables $\beta_0, \ldots, \beta_{n-1}$, if $\pi(\beta_0, \ldots, \beta_{n-1})$ occurs in ψ in such a way that the occurrence of π is a free occurrence, then for each $i < n$, there is no subformula of φ of the form $\bigwedge\beta_i\chi$ in which α_i has a free occurrence. If these two conditions are satisfied, then

$$\check{\mathrm{S}}^{\pi(\alpha_0,\ldots,\alpha_{n-1})}_{\varphi}\,\psi|$$

is the result of replacing, for arbitrary individual variables $\beta_0, \ldots, \beta_{n-1}$, each occurrence of $\pi(\beta_0, \ldots, \beta_{n-1})$ in ψ at which π is free by an occurrence of

$$\varphi\begin{bmatrix}\alpha_0, & \ldots, & \alpha_{n-1}\\ \beta_0, & \ldots, & \beta_{n-1}\end{bmatrix}.$$

If we extend this notion of substitution so that it applies to our broader notion of what a formula is, we observe that clause (1)

[20] What Grossmann (*op. cit*) calls "exemplification" is an example of such a membership type of relation.
[21] *Op. cit.*, p. 192 f.

remains quite appropriate as it is and requires no addition. Clause (2), to the contrary, does. We replace clause (2) by:

(2′) for all *terms* (predicate or individual variables) $\mu_0, \ldots, \mu_{n-1}$, if $\pi(\mu_0, \ldots, \mu_{n-1})$ occurs in ψ in such a way that the occurrence of π is a free occurrence, then for each $i < n$, there is no subformula of φ of the form $\bigwedge \mu_i \chi$ in which α_i has a free occurrence.

When conditions (1) and (2′) are satisfied, we take the substitution to be the result of replacing, for arbitrary *terms* (predicate or individual variables) $\mu_0, \ldots, \mu_{n-1}$, each occurrence of $\pi(\mu_0, \ldots, \mu_{n-1})$ in ψ at which π is free by an occurrence of

$$\varphi \begin{bmatrix} \alpha_0, & \ldots, & \alpha_{n-1} \\ \mu_0, & \ldots, & \mu_{n-1} \end{bmatrix}.$$

We observe that the restrictions regarding proper substitution, that is, the two restrictions, (1) and (2′), are required in order to avoid a "clash" of bound and free variables. The first restriction makes the reasonable demand that other than the indicated *subject terms* $\alpha_0, \ldots, \alpha_{n-1}$, no variable free in φ, the substituend, is to become bound upon the substitution of φ (relative to the subject terms $\alpha_0, \ldots, \alpha_{n-1}$) for π in ψ. This restriction is already essential even for standard second-order logic. The second restriction, however, requires that no subject or predicate term occupying, say, the kth subject position of a free occurrence of π in ψ becomes bound when that term replaces a free occurrence in φ of its associated subject term α_k. That this restriction should apply to predicate as well as subject terms occupying subject or "argument" positions of a free occurrence of π is a natural requirement, once we allow predicate terms to occupy subject positions. The restrictions then are quite in order.

Accordingly, we replace axioms $*509_0$ and $*509_n$ of T by a single axiom schema:

$$\text{(A6)} \quad \bigwedge \pi \psi \rightarrow \check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\varphi} \psi \,|$$

where π is a n-place predicate variable and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables

We have now accounted for and replaced every axiom (and inference rule) of T by an analogue appropriate to all the formulas and not just the wffs of T. The system resulting from T by this replacement is $T_0^* +$ (A6). The question now is: Is $T_0^* +$ (A6) an *adequate* syntactical representative of the logical context within which the reasoning supposedly grounding Russell's paradox of predication is to be represented? That is, is $T^* = T_0^* + (A6)$?

In evaluating this question, let us note that the notation:

$$\check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\varphi} \psi \mid$$

is not defined when $\alpha_0, \ldots, \alpha_{n-1}$ are not all of them (distinct) *free subject terms* (individual variables) of φ.[22] In other words, predicate terms are not allowed to indicate the "argument" positions of our formula substituends.

In order to see why this should be so within the context in question consider the attempt to prove (A), the formula of T* affirming the existence of "the Russell property," in a manner analogous to our proof above of the comprehension principle in T for properties. If we were to grant the initial step of this proof, namely,

$$\bigwedge \tau \sim \bigwedge \pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)] \rightarrow \sim\bigwedge \pi[\sim\pi(\pi) \leftrightarrow \sim\pi(\pi)]$$

then expression (A) would follow as a trivial consequence. It is quite obvious, of course, that this initial step cannot be justified by (A5) since the substitution of $\sim\pi$ for τ in $\sim\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$ results in replacing a free occurrence of τ by a bound occurrence of π. In addition, of course, (A5) applies only to terms and $\sim\pi$ is not a term. The relevant law here, if any, is not (A5) but (A6), where supposedly we are to substitute $\sim\pi(\alpha)$ for τ. But

$$\check{S}^{\tau(\alpha)}_{\sim\pi(\alpha)} \sim \bigwedge \pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)] \mid$$

[22] This is not entirely correct in that $\alpha_0, \ldots, \alpha_{n-1}$, on the definition given, are not required to even occur no less occur free in φ. This, however, is merely a matter of logical economy. For in place of φ we can always put $(\varphi \wedge \alpha_0 = \alpha_0 \wedge \ldots \wedge \alpha_{n-1} = \alpha_{n-1})$ and then interchange (as based on provable equivalence) in the substitution result all occurrences of

$$(\wedge \alpha_0 = \alpha_0 \wedge \ldots \wedge \alpha_{n-1} = \alpha_{n-1}) \begin{bmatrix} \alpha_0 & \cdots & \alpha_{n-1} \\ \mu_0 & \cdots & \mu_{n-1} \end{bmatrix}$$

that replaced an occurrence of $\pi(\mu_0, \ldots, \mu_{n-1})$ in ψ by

$$\varphi \begin{bmatrix} \alpha_0 & \cdots & \alpha_{n-1} \\ \mu_0 & \cdots & \mu_{n-1} \end{bmatrix}$$

Identity is understood to be defined as follows:

$$\mu = \nu =df \wedge \sigma[\sigma(\mu) \leftrightarrow \sigma(\nu)]$$

where σ is the first 1-place predicate variable distinct from μ and ν.

is not identical with $\sim\bigwedge\pi[\sim\pi(\pi) \leftrightarrow \sim\pi(\pi)]$, for π occurs free in $\sim\pi(\alpha)$ but becomes bound in the replacement of $\tau(\pi)$ by

$$\sim\pi(\alpha)\begin{bmatrix}\alpha\\ \pi\end{bmatrix}$$

in the formula $\sim\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$. In this case we have a violation of our first restriction, namely, clause (1), which was designed to prevent just such a "clash" of variables. However, because π is not a *subject term* (individual variable), even though in $\sim\pi(\pi)$ it occupies a subject position, we cannot construe the substitution to be:

$$\check{S}^{\tau(\pi)}_{\sim\pi(\pi)} \sim \bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]\,|$$

since substitution in this case is not defined. Note, however, that if it were defined, in this case we would have a conflict between the occurrence of π in the subject position and the occurrence of π in the predicate position in $\sim\pi(\pi)$. For, intuitively the occurrence of π in the subject position would be allowed to become bound upon substitution,[23] whereas the occurrence of π in the predicate position would not. The reason for this last claim is that the replacement of $\tau(\pi)$ by $\sim\pi(\pi)$ by "substituting" $\sim\pi(\pi)$ for τ in $\sim\bigwedge\pi[\tau(\pi) \leftrightarrow \sim\pi(\pi)]$ would be the same replacement of $\tau(\pi)$ by $\sim\pi(\pi)$ except by *improperly* substituting $\sim\pi(\alpha)$ for τ in the same formula. The two substitutions do exactly the same job except that there is readily recognized to be a "clash" of variables in the one, and therefore this substitution is improper, but supposedly (?) there is no "clash" of variables in the other, and therefore this substitution would be said to be proper. Indeed, this is essentially the point of our not defining substitution where the "argument" positions of the substituend are allowed to be occupied by predicate variables. Every substitution effected by allowing predicate terms to indicate the subject positions, construed as the "argument" positions of the substituend, can also be effected by allowing only subject terms to indicate these same "argument" positions. The real difference between the two is that, where some substitutions of the latter kind indicate a clear "clash" of variables and therefore are *improper*, their corresponding substitutions of the former kind fail to indicate the "clash" of variables involved and therefore would be unreasonably recognized as proper. The two substitutions do the same job, except that the one shows itself to be improper, whereas the other fails to do so. Our procedure has been to discriminate in favor of those

[23] Note that in the definition of substitution the variables indicating the "argument" positions in the substituend are allowed to be bound in the formula resulting by proper substitution of that substituend.

that indicate "clashes" of variables that intuitively are really there in the logical context in question.

Now I do not believe that our discrimination here is adventitious. There is, I should like to argue, an ontological basis for it. Furthermore, this basis is to be found in the ontological background implicit in the logical context in question.

IV

The Ontological Background of Russell's Paradox of Predication

In discussing the ontological background of T* let us note that within it there is an explicit ontological distinction being made analogous to that of Aristotle's between primary and secondary being (*ousia*). Retaining this terminology somewhat, we shall call *primary* those "individuals" of the background ontology which can be a value of an individual variable but which cannot be a value of a predicate variable. The remaining "individuals" of the ontology we shall call *secondary individuals*.

Now it may seem dubious to refer to the nonprimary individuals as "individuals" even if they are secondary. Propositions, properties, and attributes, in general, are not normally thought of as "individuals." Propositions are the kind of entity that is asserted, denied, believed, known, conjectured (whether), and so on, in short the kind of entity that is represented only by a complete sentence or formula. Similarly, properties and attributes are the kind of entity that is predicated, ascribed, attributed, that is, the kind of entity that is represented only by predicate expressions (complex or otherwise). All this may be so, and the background ontology of T* agrees up to a point. What this ontology explicitly denies, however, is that propositions, properties, and attributes can be represented only by sentence and predicate forms, respectively. Propositional (0-place predicate) variables are allowed to occur not only in propositional contexts as whole formulas but also in subject or argument positions of predicate variables. Similarly, property (1-place predicate) variables and attribute (n-place predicate) variables, in general, are allowed to occur not only in predicate positions but also in subject positions of themselves as well as of other predicates. Accordingly, predicate variables are being construed as substituends of individual variables, and the values of predicate variables are therefore also values of the individual variables. It is in this fact of the grammar we have constructed as a syntactical representative of this background ontology that we are confronted with an extension of the usual view of the nature of propositions, properties, and attributes. Actually, however, it is more an extension of the notion of *individuality:* the characteristic of being that type of entity for which it is ontologically significant that *it* be a subject of predication, that is, that type of entity which can be *referred to* through the subject expressions of the sentences and formulas of the ontological language in question.

Now it is a noteworthy fact of the constructed grammar that a proposition is entified *qua* proposition and that in general an n-ary attribute is entified *qua* being an n-ary attribute only through quantification binding distinct types of predicate variable, specifically 0-place predicate variables for propositions and n-place predicate variables for n-ary attributes. Consequently, implicit in the background ontology of T* is the assumption that what we have called the secondary individuals do not form a unified ontological category *qua* secondary individuals. Otherwise, the grammar should include a generic type of predicate variable or some such logical element designating or having as its values all and only the secondary individuals. Though less obvious, a similar observation applies to the primary individuals.[24]

However, what is also implicit in the background ontology is that the "individuals," the primary and secondary individuals together, do form just such a unified ontological category. This ontological category is represented in our grammar by allowing any term to occupy any subject position of any predicate variable. The category of being of the values of terms that occupy subject positions is comprehended through quantification binding variables whose essential feature is to represent and to represent no more than the being of a "subject," an individual. What we have called individual variables are exactly such variables as described. No external meaning—especially from the point of view of an ontology construed as an alternative to that grounding T*, for example, the point of view of the theory of types, which, at least for Russell, replaced the point of view of the ontology of T*—should be given to our use of 'individual' here.

If this assessment of T*'s ontological background is correct, then the conditions under which a formula "determines" or "represents" an attribute, that is, the conditions under which the formula might be said to "define" an attribute, must be conditions comprehending

[24] Accordingly, the ontological content of our generic phrase 'secondary individual', as well as that of 'primary individual', is not expressible in T*. Our present perspective, however, is that of general metaphysics and we shall allow ourselves phraseology not permitted in the logistic system T* whose purpose is to be an adequate syntactical representative of the logical context to which supposedly Russell's paradox of predication is applicable.

all the individuals, both primary and secondary. Each attribute must result in a proposition when "applied" or attributed to *any* individual (or n-tuple of individuals if the attribute is n-ary). The resulting proposition will either be or not be the case, though it may be either on ontological grounds.[25] In effect, the relevant "argument" positions of open formulas defining attributes must be occupied only by subject terms (individual variables). Thus "the Russell property," not being "determined" or "represented" by a definiens, which can be significantly predicated in T* of either primary individuals or secondary individuals other than properties, fails to exist as a property in the ontology of T*.

Let us note that every proposition, property, or attribute which can be "determined" or "represented" by a formula of T* is so determined by the comprehension principle for attributes. This principle is already provable in $T_0^* + (A6)$ in exactly the same manner described earlier for T; where φ is a formula and $\alpha_0, \ldots, \alpha_{n-1}$ are all (or even, only some of) the distinct *individual* variables (*subject terms*) occurring free in φ (regardless of the order of their occurrence in φ), then

$$\text{(CP)} \quad \vdash_{T_0^*+(A6)} \bigvee\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi]$$

where π is an n-place predicate variable which does not occur free in φ.

Observe that no instance of this comprehension principle can be construed as affirming the existence of "the Russell property." For, although an individual variable has all the properties among its values, the variable itself still cannot occupy a predicate position; that is, $\bigvee\pi\bigwedge\alpha[\pi(\alpha) \leftrightarrow \sim\alpha(\alpha)]$ is not a formula according to the grammar of the logical context in question. And the reason for this is precisely our respect for the distinction between predication and membership or any membership type of relation. Indeed, predication is not to be construed as a *relation* at all, and the singular way our present grammar has of expressing ("showing forth") this "ontological fact" is through distinguishing *subject terms* (individual variables) from *predicate terms* (predicate variables) and by refusing to allow subject terms to occupy predicate positions. Were prediction to be misconstrued as being a relation, there would be no ontological point to the grammatical

[25] For example, consider the property of being an n-nary attribute:

$$\Pi_n(\alpha) =_{df} \bigvee\pi\bigwedge\sigma[\sigma(\alpha) \leftrightarrow \sigma(\pi)]$$

where π is an n-place predicate variable and σ is the first 1-place predicate variable distinct from π (in case $n = 1$). This property on ontological grounds will be only falsely attributable to primary individuals or secondary individuals other than n-ary attributes.

distinctions we have made, and Russell's paradox of predication would be but a "funny" way of expressing his paradox of membership.[26]

Finally, we may return to our original inquiry regarding whether $T_0^* + (A6)$ is an adequate syntactical representative of the logical context within which the reasoning supposedly grounding Russell's paradox of predication is to be represented, that is, our inquiry whether $T^* = T_0^* + (A6)$.

Now, in so far as $T_0^* + (A6)$ fails to fully express the ontological fact of T^* that every proposition, property, and attribute in general is an *individual* of the background ontology, that is, that it is a value of a bindable individual variable, our response here must be in the negative. One way of supplementing $T_0^* + (A6)$ with this additional ontological content is by stipulating that whatever is true of every individual is therefore true of every secondary individual of any specifiable type as a value of a predicate variable of that type:

$$\text{(A7)} \quad \bigwedge\alpha\phi \rightarrow \bigwedge\pi\phi\begin{bmatrix}\alpha\\ \pi\end{bmatrix}$$

where α is an individual variable and π is any predicate variable which does not occur in ϕ. With the desired additional content expressed in this manner, we may finally identify T^* as the system $T_0^* + (A6) + (A7)$. Russell's paradox of predication, of course, is no longer derivable in its original form since that form involves, as we have seen, a violation of the restrictions imposed for the proper substitution of a formula for a predicate variable in the specification law (A6). Nevertheless, these restrictions, I have argued, are intuitively natural and appropriate from the perspective of the ontology of the type of logical context in question.

Incidentally, it is noteworthy to point out here that relative to the fragment $T_0^* + (A7)$, the comprehension principle (CP) and the

[26] Predication as an ontological category amounts to what I have called the category of *modes of copulation* [3]. In the present grammar only two unary modes, misconstrued nominally as "truth" and "falsity," are represented, the latter by $\sim$, and the former by an implicit interpretation of the concatenation factor between a predicate and its argument expressions. (The addition of a "modal" operator corresponding to the English 'It is true that' would be redundant here. Notice how this redundancy (and the redundancy of any finite iteration of such an operator) along with the view that predication is not a relation answers Bradley's infinite regress argument for predication!) Extensionalists, of course, deny that there are any unary modes of copulation other than truth and falsity and that all n-ary modes, for $n > 1$, are truth-functional (a thesis not being challenged in the present paper). Needless to say, truth and falsity so construed are not to be confused with *semantic* truth and falsity, the latter being *properties of sentences* (and dependent in their analysis on the former).

specification law (A6) for predicate variables determine the same logical conditions; that is, the systems $T_0^* + (A6) + (A7)$ and $T_0^* + (A7) + (CP)$ are equivalent. This shows that the relationship between the comprehension principle (CP) and the specification law (A6) is really about as intimate as I have indicated.[27]

V

The Consistency of T*

It may be thought that all of our formal reconstruction comes to naught, for do we not now have a variant of Russell's argument in which rather than "the predicate: to be a predicate that cannot be predicated of itself" we consider that property: *to be an individual which is identical* (in the sense of having all properties in common) *with a property which that individual does not have?*[28] The existence of such a property is guaranteed in the ontology of T* by the comprehension principle (CP):

$$\vdash_{T^*} \bigvee\tau\bigwedge\alpha[\tau(\alpha) \leftrightarrow \bigvee\pi(\alpha = \pi \wedge \sim\pi(\alpha))]$$

where identity is understood to be as it is defined in footnote 22.

[27] The equivalence can be shown in the manner indicated in [4] where it is proved that (CP) and (A6) as formulated for T, standard second-order logic, determine the same logical conditions relative to the remaining axioms of T. Moreover, by utilizing essentially the same arguments of [4] we can prove T* to be equivalent to a substitution free axiomatization the only inference rule of which is *modus ponens.* The axioms of this substitution free axiom set are all (universal) generalizations of all instances of (A1) through (A3), (CP), and formulas of the following forms (where identity is understood as defined in footnote 22):

$$\bigwedge\mu(\phi \rightarrow \psi) \rightarrow (\bigwedge\mu\phi \rightarrow \bigwedge\mu\psi)$$
$$\phi \rightarrow \bigwedge\mu\phi$$

where μ is a predicate or individual variable which does not occur (free) in ϕ,

$$\bigvee\alpha\mu = \alpha$$

where μ is a predicate or individual variable distinct from α,

$$\bigvee\pi\sigma = \pi$$

where π, σ are distinct n-place predicate variables,

$$\mu = \nu \rightarrow (\phi \rightarrow \psi)$$

where ϕ, ψ are atomic formulas and ψ is obtained from ϕ by replacing an occurrence of ν in *subject position* by an occurrence of μ.

[28] This version of Russell's argument was first suggested to me by Professor Max Zorn.

We note that by (A7), that is, the thesis that every property is an individual,

$$\vdash_{T^*} \bigwedge\alpha[\tau(\alpha) \leftrightarrow \bigvee\pi(\alpha = \pi \wedge \sim\pi(\alpha))] \rightarrow [\tau(\tau) \leftrightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))]$$

and therefore by generalization and distribution over a conditional of a universal into an existential quantifier:

$$\vdash_{T^*} \bigvee\tau\bigwedge\alpha[\tau(\alpha) \leftrightarrow \bigvee\pi(\alpha = \pi \wedge \sim\pi(\alpha))] \rightarrow \bigvee\tau[\tau(\tau) \leftrightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))]$$

That is, within the ontology of T* there is a property which has itself if and only if it is identical with a property which it does not have:

$$\vdash_{T^*} \bigvee\tau[\tau(\tau) \leftrightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))]$$

The claim that such a result is counterintuitive presupposes the principle that identical properties are co-extensive:

(Id*) $\bigwedge\pi\bigwedge\tau(\pi = \tau \rightarrow \bigwedge\alpha[\pi(\alpha) \leftrightarrow \tau(\alpha)])$

For, by (Id*), any property identical with a property which it does not have *eo ipso* does not have itself. That is, since

$$\vdash_{T^*+(Id^*)} \tau = \pi \rightarrow [\tau(\tau) \leftrightarrow \pi(\tau)]$$

by (Id*), (A5), and (A7), then

$$\vdash_{T^*+(Id^*)} \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau)) \rightarrow \sim\tau(\tau)$$

by generalization, (A4) and sentential logic.

But within T*, by law (A5), any property which possesses every property with which it is identical therefore possesses itself:

$$\vdash_{T^*} \bigwedge\pi(\tau = \pi \rightarrow \pi(\tau)) \rightarrow \tau(\tau)$$

and therefore by contraposition

$$\vdash_{T^*} \sim\tau(\tau) \rightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))$$ [29]

[29] We might note as an incidental consequence of this last theorem that the property of being an individual identical with a property which that individual does not have is a property which possesses itself; and therefore since, by (CP), this property exists in T*, we have $\bigvee\tau\tau(\tau)$ as a theorem of T*.

Accordingly, given (Id*), a property is identical with a property which it does not have if and only if that property does not have itself:

$$\vdash_{T^*+(Id^*)} \sim\tau(\tau) \leftrightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))$$

And therefore, by sentential logic, generalization, and quantifier negation, within T* + (Id*) there is no property which has itself if and only if it is identical with a property which it does not have:

$$\vdash_{T^*+(Id^*)} \sim\bigvee\tau[\tau(\tau) \leftrightarrow \bigvee\pi(\tau = \pi \wedge \sim\pi(\tau))]$$

But, as I have shown above, there is such a property within T* and therefore also within T* + (Id*). Consequently, T* + (Id*) is inconsistent. Or, equivalently, (∼Id*) is a theorem of T*.[30]

Although such a result is contradictory within the ontological framework of type theory, it really is very much in accordance with the ontological framework of T*, at least in regard to some of its consequences. For according to the ontological background of T*, *predication is not a relation,* and within T* this significant ontological fact is actually implied by (∼Id*). The proof is as follows, where $\sim\bigvee\rho\bigwedge\pi\bigwedge\alpha[\rho(\pi, \alpha) \leftrightarrow \pi(\alpha)]$, since it denies the existence of any relation which is coextensive with predication, is understood to express in T* the ontological fact that predication is not a relation. We have first by (A6):

$$\vdash_{T^*} \bigwedge\sigma[\sigma(\pi) \leftrightarrow \sigma(\tau)] \rightarrow \check{S}^{\sigma(\beta)}_{\rho(\beta,\alpha)}[\sigma(\pi) \leftrightarrow \sigma(\tau)] \mid$$

that is,

$$\vdash_{T^*} \pi = \tau \rightarrow [\rho(\pi, \alpha) \leftrightarrow \rho(\tau, \alpha)]$$

and therefore

$$\vdash_{T^*} \pi = \tau \rightarrow \bigwedge\alpha[\rho(\pi, \alpha) \leftrightarrow \rho(\tau, \alpha)]$$

by generalization, (A4), and sentential logic; but then

$$\vdash_{T^*} \bigwedge\pi\bigwedge\alpha[\rho(\pi, \alpha) \leftrightarrow \pi(\alpha)] \rightarrow (\pi = \tau \rightarrow \bigwedge\alpha[\pi(\alpha) \leftrightarrow \tau(\alpha)])$$

[30] A similar argument applies to relations of arbitrary many places. For example, in the case of a binary relation (CP) guarantees

$$\vdash_{T^*} \bigvee\rho\bigwedge\alpha\bigwedge\beta[\rho(\alpha, \beta) \leftrightarrow \bigvee\sigma(\alpha = \sigma \wedge \sim\sigma(\alpha, \beta))]$$

But with Id* extended to apply to binary relations as well—call it Id_2*—we can show that no such relation exists in T* + (Id_2*) and, accordingly, that (∼Id_2*) is a theorem of T*.

It is noteworthy, however, that apparently Id_n*, for each natural number *n*, is disprovable in T* only for (*n*-ary) attributes of a rather peculiarly "self-reflexive" and impredicative kind.

by (A5) and elementary quantificational logic, and therefore

$$\vdash_{T^*} \bigwedge\rho\bigwedge\pi\bigwedge\alpha[\rho(\pi, \alpha) \leftrightarrow \pi(\alpha)] \rightarrow (\mathrm{Id}^*)$$

by generalization, distribution over a conditional of a universal into an existential quantifier, and deletion of a vacuous quantifier. But since ($\sim$Id*) is a theorem of T*, it follows that

$$\vdash_{T^*} \sim\bigvee\rho\bigwedge\pi\bigwedge\alpha[\rho(\pi, \alpha) \leftrightarrow \pi(\alpha)]$$ [31]

Aside from yielding such desirable consequences as the above and thereby being of positive significance to T*, the claim that ($\sim$Id*) cannot consistently be a theorem of T* is simply false. For the consistency of T* is easily shown by the same "very elementary syntactical argument" which proves the consistency of T and which earlier we utilized to prove the consistency of T_0*. And this is the case because, by definition of the transformation of any formula of T* into a formula of the extended propositional calculus, the transform of any instance of (A7), being a conditional whose consequent is a vacuous quantification of its antecedent, clearly results in a valid formula. Similarly, the transform of any instance of the comprehension principle for T* results in an instance of the comprehension principle for the extended propositional calculus; and therefore $T_\bullet$* + (A7) + (CP) is consistent. But, because this system is equivalent to T*, it follows that T* is consistent.[32]

In addition, we might note that the following version of the axiom of choice is formulable in T*:

$$(\mathrm{AC}^*) \quad \bigwedge\alpha\bigvee\pi\phi \rightarrow \bigvee\rho\bigwedge\alpha\check{S}^{\pi(\beta_0, \ldots, \beta_{n-1})}_{\rho(\alpha, \beta_0, \ldots, \beta_{n-1})}\phi\,|$$

[31] A similar argument utilizing (Id_n*) shows that $\sim\bigvee\rho\bigwedge\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\rho(\pi, \alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \pi(\alpha_0, \ldots, \alpha_{n-1})]$ is a theorem of T*.

[32] Without going through the consistency of the equivalent system T_0* + (A7) + (CP), a more direct proof of T*'s consistency utilizes the lemma that if

$$\check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\phi}\psi\,| \neq \psi$$

then

$$f(\check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\phi}\psi\,|) = \check{S}^{\pi^*}_{f(\phi)}f(\psi)\,|$$

This lemma is proved by a simple inductive argument on the structure of ϕ. From this lemma it follows that the transform of any instance of (A6) is an instance of the general specification law of the extended propositional calculus.

where α, β_0, . . . , β_{n-1} are distinct individual variables, π is an n-place predicate variable, ρ is an $(n + 1)$-place predicate variable, and ϕ contains no bound occurrences of either ρ or α.[33]

Furthermore, since the transformation into the extended propositional calculus of the antecedent of any instance of (AC*) differs from the transformation of its consequent by an alphabetic change of a bound propositional variable, it follows that the transform of such an instance is valid ; and therefore T* + (AC*) is consistent, but again proved to be so within a decidable fragment of elementary syntax.

In regard, however, to the addition to T* (or T* + (AC*)) of an axiom of infinity, such as

$$\text{(inf)} \quad \bigvee\pi(\bigwedge\alpha\bigvee\beta\pi(\alpha, \beta) \wedge \bigwedge\alpha \sim \pi(\alpha, \alpha) \wedge \bigwedge\alpha\bigwedge\beta\bigwedge\gamma[\pi(\alpha, \beta) \wedge \pi(\beta, \gamma) \rightarrow \pi(\alpha, \gamma)])$$

the situation is somewhat different. For the negation of the transform of (inf) is valid in the extended propositional calculus. Therefore, T* + (~inf), with or without (AC*), is consistent; that is, (inf) is not provable in T*, with or without (AC*). Consequently, establishing the consistency of T* + (inf) requires a different type of proof.

VI

T* is a Conservative Extension of T

One significant syntactical difference between (inf) and (AC*) is that the former but not all instances of the latter is a wff of T, and T + (inf), as is well-known, is consistent. Accordingly, to show that T* + (inf) is consistent it more than suffices to show that T* is a conservative extension of T, that is, that any theorem of T* which is a wff of T is already a theorem of T. For in that case, if T* + (inf) were inconsistent, then (~inf) would be a theorem of T* and therefore a theorem of T, which, by the existence of finite models of T, we know it not to be. Such an argument shows, of course, the much stronger result that any wff of T which is consistent in T is therefore a formula which is consistent in T*.[34]

We note first that the cardinality of the set of terms of T*, that is, of the set of all predicate and individual variables together, is $\aleph_0\aleph_0$,

[33] Cf. [2], p. 131.

[34] The argument that T* is a conservative extension of T was communicated to me by my colleagues Robert Meyer and Max Zorn independent of each other.

which by infinite cardinal arithmetic is $\aleph_0$, and, accordingly, there exists a one-to-one correspondence between all the terms and the individual variables. For convenience, let $\bar{\mu}$, where μ is a predicate or individual variable, be the individual variable associated with μ under such a one-to-one correspondence. We define recursively the function s whose domain is the set of formulas of T* and whose value for any such formula is a wff of T:

(i) $s(\pi(\mu_0, \ldots, \mu_{n-1})) = \pi(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$
(ii) $s(\sim\phi) = \sim s(\phi)$
(iii) $s(\phi \rightarrow \psi) = (s(\phi) \rightarrow s(\psi))$
(iv) $s(\bigwedge\alpha\phi) = \bigwedge\bar{\alpha}s(\phi)$
(v) $s(\bigwedge\pi\phi) = \bigwedge\bar{\pi}\bigwedge\pi s(\phi)$

The claim now is that for any formula ϕ, if $\vdash_{T^*} \phi$, then $\vdash_{T} s(\phi)$. Obviously, if ϕ is an axiom of T* by (A1) though (A4), then $s(\phi)$ is an axiom of T for the same reason. In regard to (A5), we note that

$$s\left(\bigwedge\alpha\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) = \left(\bigwedge\bar{\alpha}s(\phi) \rightarrow s(\phi)\begin{bmatrix}\bar{\alpha}\\ \bar{\beta}\end{bmatrix}\right)$$

$$s\left(\bigwedge\pi\phi \rightarrow \phi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}\right) = \left(\bigwedge\bar{\pi}\bigwedge\pi s(\phi) \rightarrow s(\phi)\begin{bmatrix}\pi\\ \sigma\end{bmatrix}\begin{bmatrix}\bar{\pi}\\ \bar{\sigma}\end{bmatrix}\right)$$

and therefore the s transform of any instance of (A5) is clearly a theorem of T by the specification laws of T. In regard to (A7), observe that

$$s\left(\bigwedge\alpha\phi \rightarrow \bigwedge\pi\phi\begin{bmatrix}\alpha\\ \pi\end{bmatrix}\right) = \left(\bigwedge\bar{\alpha}s(\phi) \rightarrow \bigwedge\bar{\pi}\bigwedge\pi s(\phi)\begin{bmatrix}\bar{\alpha}\\ \bar{\pi}\end{bmatrix}\right)$$

and therefore the s transform of any instance of (A7) is a theorem of T by the rule of alphabetic change of bound individual variables and the insertion of a vacuous quantifier phrase. Finally, since

$$s(\bigvee\pi\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \phi]) = \bigvee\bar{\pi}\bigvee\pi\bigwedge\bar{\alpha}_0 \ldots \bigwedge\bar{\alpha}_{n-1}[\pi(\bar{\alpha}_0, \ldots, \bar{\alpha}_{n-1}) \leftrightarrow s(\phi)]$$

then the s transform of any instance of (CP) is an instance of the comprehension principle of T prefixed by a vacuous quantifier on an individual variable (since π does not occur free in ϕ); and therefore the s transform of any instance of (CP) is a theorem of T. But because s clearly preserves theoremhood in T under the inference rules of T*, and because T* is equivalent to T* + (A7) + (CP), then it follows that for every formula ϕ of T*, if $\vdash_{T^*} \phi$, then $\vdash_{T} s(\phi)$.[35]

Finally, observe that if ϕ is a wff of T, then $s(\phi)$ is an alphabetic variant of ϕ with possibly the insertion of some vacuous quantifiers of the form $\bigwedge\bar{\pi}$ or $\bigvee\bar{\pi}$; and, accordingly, $\vdash_{T} s(\phi)$ if and only if $\vdash_{T} \phi$. We conclude then that if ϕ is a wff of T and a theorem of T*, then ϕ is already a theorem of T. Consequently, T* + (inf) is consistent.[36]

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[35] Retaining (A6) in place of (CP) we note that

$$s(\bigwedge\pi\psi \rightarrow \check{S}^{\pi(\alpha_0,\ldots,\alpha_{n-1})}_{\phi}\psi \mid) = (\bigwedge\bar{\pi}\bigwedge\pi s(\psi) \rightarrow \check{S}^{\pi(\bar{\alpha}_0,\ldots,\bar{\alpha}_{n-1})}_{s(\phi)} s(\psi) \mid)$$

and therefore the s transform of any instance of (A6) is a theorem of T by the specification laws of T.

[36] We perhaps should point out that although T* + (AC*) and T* + (inf) have both been shown to be consistent, the two proofs we have given cannot be joined to show that T* + (AC*) + (inf) is consistent. For as I pointed out above, (AC*) has instances that are not wffs of T, and unless we can show that such instances yield no more consequences in T* than do those that are wffs of T, the above proof does not suffice to establish the consistency of T* + (AC*) + (inf) relative to the consistency of T + (AC) + (inf).