Topology and its Applications 338 (2023) 108670

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

The Hurewicz property and the Vietoris hyperspace

Christopher Caruvana

School of Sciences, Indiana University Kokomo, 2300 S. Washington St., Kokomo, IN 46902, United States of America

ARTICLE INFO

Article history:
Received 3 June 2023
Received in revised form 7 August 2023
Accepted 11 August 2023
Available online 19 August 2023

MSC:
primary 54D20
secondary 54B20, 54C35

Keywords:
Hurewicz selection principle
Vietoris topology
ω-covers
k-covers
Groupable covers
Reznichenko property
Countable fan-tightness
C_p-theory
C_k-theory

ABSTRACT

In this note, we characterize when the Vietoris space of compact subsets of a given space has the Hurewicz property in terms of a selection principle on the given space itself using k-covers and the notion of groupability introduced by Kočinac and Scheepers. We comment that the same technique establishes another equivalent condition to a space being Hurewicz in each of its finite powers. We end with some characterizations involving spaces of continuous functions and answer a question posed by Kočinac.

1. Introduction

The Hurewicz property was introduced in [11] and was conjectured by Hurewicz to characterize the property of being σ-compact. Though, as discussed in [27], this conjecture is false, investigations about the Hurewicz property continue. In [15], Kočinac and Scheepers introduce the notion of groupability to characterize the Hurewicz property using other types of selection principles that mirror Menger's property. An important ingredient in this line of investigation is the game-theoretic characterization of the Hurewicz property established in [23]. Also in [15], a selection principle which is equivalent to every finite power of a space having the Hurewicz property is established. This adds to the list of characterizations of certain

E-mail address: chcaru@iu.edu.

https://doi.org/10.1016/j.topol.2023.108670

properties of X in every finite power; for other characterizations of this flavor involving traditional selection principles, see [9,12,22].

Characterizations of the Hurewicz property involving hyperspaces have been obtained in recent years. Characterizations in terms of certain properties on $C_p(X)$ for certain classes of spaces X have been done in [25] and [21]. Following generalizations of π-networks in [16], a characterization of the Hurewicz property on certain hyperspaces of X in terms of further generalized notions of π-networks of X can be found in [5]. Other characterizations can be found in [7] and [26]. For generalized versions of groupability, see [6].

In this note, we continue this work by characterizing the Hurewicz property on certain hyperspaces in terms of selective covering properties on X using the notion of groupability, extending on the family of relationships established in [3]. The first result is a general criterion which guarantees the equivalence of the Hurewicz property on a hyperspace with a certain selective property on the space X. As a consequence, we characterize the Hurewicz property on the Vietoris space of compact subsets of X in terms of k-covers of X and groupability.

The original characterization involving finite powers of X was done using ω-covers, which cover finite subsets of X. Since finite powers of X clearly cover finite subsets, relationships between properties of covers of finite powers and ω-covers are not surprising. However, the finite powers of X carry more information than finite subsets, alone. Here, we establish a similar equivalence involving ω-covers with the set of finite subsets of X adequately topologized.

We end by showing that the selective properties discussed above are also characterized by the property that the space $C_p(X)$ or $C_k(X)$ is Reznichenko and has countable (strong) fan-tightness.

2. Preliminaries

Throughout, by *space* we mean *topological space* and follow definitions and notation of [8]. For a space X, we will use $[X]^{<\omega}$ to denote the set of all non-empty finite subsets of X and $K(X)$ to denote the set of non-empty compact subsets of X. To avoid trivialities, we will assume our spaces X to be infinite or non-compact, wherever appropriate. We also assume all spaces are Hausdorff.

As usual, for sets A and B, we will use the notation A^B to be the set of all functions $B \to A$. So, for example, X^ω is the set of all sequences in X. When dealing with a finite power ω^{n+1} of ω where $n \in \omega$, we may use the notation m_j in place of $m(j)$ for $j \in n+1 = \{0, 1, \ldots, n\}$. The symbol $\langle\rangle$ denotes the empty sequence. We will also use $X^{<\omega}$ to denote the set $\bigcup_{n\in\omega} X^n$ of finite sequences of X and, if the base X is written with a few symbols, we will use $(X)^{<\omega}$ in place of $X^{<\omega}$ to avoid confusion.

By a *topological operator* we mean a function on the class of topological spaces that produces a collection of sets related to the topological structure of the input. Some well-known topological operators that we'll be using in this paper are recalled below.

- $\mathscr{T}$: For a space X, $\mathscr{T}_X$ is the collection of all open sets $U \subseteq X$ where $U \neq \emptyset$ and $U \neq X$.
- $\mathcal{O}$: For a space X, $\mathcal{O}_X$ is the collection of non-trivial open covers of X; that is, all $\mathscr{U} \subseteq \mathscr{T}_X$ with $X = \bigcup \mathscr{U}$.
- Ω: For a space X, Ω_X is the collection of ω-covers of X; that is, all $\mathscr{U} \in \mathcal{O}_X$ so that,

$$(\forall F \in [X]^{<\omega})(\exists U \in \mathscr{U})\ F \subseteq U.$$

- $\mathcal{K}$: For a space X, $\mathcal{K}_X$ is the collection of k-covers of X; that is, all $\mathscr{U} \in \mathcal{O}_X$ so that

$$(\forall K \in K(X))(\exists U \in \mathscr{U})\ K \subseteq U.$$

Definition 1. For a space X, let $\mathbb{K}(X)$ denote the set $K(X)$ endowed with the Vietoris topology; that is, the topology generated by sets of the form $\{K \in \mathbb{K}(X) : K \subseteq U\}$ and $\{K \in \mathbb{K}(X) : K \cap U \neq \emptyset\}$ for $U \subseteq X$ open. For $U_1, \ldots, U_n$ open in X, define

$$[U_1, \ldots, U_n] = \left\{ K \in \mathbb{K}(X) : K \subseteq \bigcup_{j=1}^{n} U_j \text{ and } (\forall j)\, [K \cap U_j \neq \emptyset] \right\}.$$

These sets form a basis for the topology on $\mathbb{K}(X)$.

When working with a fixed subspace $\mathbb{Y} \subseteq \mathbb{K}(X)$, we will use $[U_1, \ldots, U_n]$ to mean $[U_1, \ldots, U_n] \cap \mathbb{Y}$.

For a detailed treatment of the Vietoris topology, see [19].

Definition 2. We will let $\mathcal{P}_{\text{fin}}(X)$ denote the set $[X]^{<\omega}$ with the topology it inherits as a subspace of $\mathbb{K}(X)$.

Definition 3. For a space X, by an *ideal of closed sets*, we mean a collection $\mathcal{A}$ of proper closed subsets of X with $[X]^{<\omega} \subseteq \mathcal{A}$ so that the following two properties hold:

- For $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$ as long as $A \cup B \neq X$.
- For $A \in \mathcal{A}$, if $B \subseteq A$ is closed and non-empty, then $B \in \mathcal{A}$.

Such collections are often referred to as bases for ideals or bornologies (see [10]).

The two ideals of closed sets we'll be most interested in are $[X]^{<\omega}$ and $K(X)$.

Definition 4. For a space X and an ideal $\mathcal{A}$ of closed sets, we define the set of $\mathcal{A}$-covers, denoted by $\mathcal{O}_X(\mathcal{A})$, to be all $\mathscr{U} \in \mathcal{O}_X$ so that

$$(\forall A \in \mathcal{A})(\exists U \in \mathscr{U})\ A \subseteq U.$$

Note that $\Omega_X = \mathcal{O}_X([X]^{<\omega})$ and $\mathcal{K}_X = \mathcal{O}_X(K(X))$. Moreover, since $[X]^{<\omega}$ and $K(X)$ can be defined topologically, one could hope for a convenient operator to be included in our list of operators above. However, not all ideals of closed sets can necessarily be defined without referencing the particular topology they are coming from, so we will not be able to drop the subscript for general $\mathcal{O}_X(\mathcal{A})$.

Remark 5. Observe that, since we define $\mathcal{O}_X(\mathcal{A})$ to consist of non-trivial open covers and $\mathcal{A}$ is an ideal of closed sets, each $\mathscr{U} \in \mathcal{O}_X(\mathcal{A})$ has the property that $\mathscr{U} \setminus \mathscr{F} \in \mathcal{O}_X(\mathcal{A})$ for every $\mathscr{F} \in [\mathscr{U}]^{<\omega}$. Moreover, every member of $\mathcal{O}_X(\mathcal{A})$ is infinite.

Definition 6. We say that a space X has the *Hurewicz property* (or that X is *Hurewicz*) if, for any sequence $\langle \mathscr{U}_n : n \in \omega \rangle \in \mathcal{O}_X^\omega$, there exists a sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ so that $\mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$ for each $n \in \omega$ and

$$(\forall x \in X)(\exists m \in \omega)(\forall n \geqslant m)\ x \in \bigcup \mathscr{F}_n.$$

Note that we are avoiding the use of the commonly used notation $\mathsf{U}_{\text{fin}}(\mathcal{O}, \Gamma)$ to avoid complications in which $\bigcup \mathscr{F}_n = X$ for some $n \in \omega$ or the set $\{\bigcup \mathscr{F}_n : n \in \omega\}$ is finite.

We will also generally be careful to distinguish between sequences and countable sets, as can be observed in our treatment of the following definition.

Definition 7 *([23]).* For topological operators or collections $\mathcal{A}$ and $\mathcal{B}$,

- $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ denotes the following selection principle: for each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of elements of $\mathcal{A}$, there is a sequence $\langle \mathscr{F}_n : n \in \omega\rangle$, where each $\mathscr{F}_n$ is a finite subset of $\mathscr{U}_n$, so that $\bigcup_{n\in\omega} \mathscr{F}_n \in \mathcal{B}$.
- $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the following selection principle: for each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of elements of $\mathcal{A}$, there is a sequence $\langle A_n : n \in \omega\rangle$, where $A_n \in \mathscr{U}_n$ for each $n \in \omega$, so that $\{A_n : n \in \omega\} \in \mathcal{B}$.

Observe that the selection principles can be concisely captured using quantifiers:

$$\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B}) \equiv (\forall \mathscr{U} \in \mathcal{A}^\omega)\left(\exists \mathscr{F} \in \prod_{n\in\omega} [\mathscr{U}_n]^{<\omega}\right) \bigcup_{n\in\omega} \mathscr{F}_n \in \mathcal{B}$$

and

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \equiv (\forall \mathscr{U} \in \mathcal{A}^\omega)\left(\exists A \in \prod_{n\in\omega} \mathscr{U}_n\right) \{A_n : n \in \omega\} \in \mathcal{B}.$$

We will use the notation $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ or $X \models \mathsf{S}_1(\mathcal{A},\mathcal{B})$ to denote that the corresponding selection principle holds for X.

Note that $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is the *Menger property* and that $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is the *Rothberger property*. We will call the selection principles $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ and $\mathsf{S}_{\mathrm{fin}}(\mathcal{K},\mathcal{K})$ the *ω-Menger* and *k-Menger* properties, respectively. Likewise, we call the selection principles $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{S}_1(\mathcal{K},\mathcal{K})$ the *ω-Rothberger* and *k-Rothberger* properties, respectively. In any scenario in which we say X has the $\mathfrak{P}$ property, we may also say that X is $\mathfrak{P}$.

Remark 8. Note that the Hurewicz property implies the Menger property.

As discussed in [15], the property $(*)$ defined by Gerlits and Nagy [9] is shown in [20] to be equivalent to the property of being both Hurewicz and Rothberger.[1] So, following [15], we take this as our definition.

Definition 9. We say that a space X is *Gerlits-Nagy* if X is both Hurewicz and Rothberger.

The Hurewicz property has a corresponding game.

Definition 10. The *Hurewicz game* on a space X is played as follows. In the n^{th} inning, One chooses $\mathscr{U}_n \in \mathcal{O}_X$ and Two responds with some $\mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$. Two is declared the winner if

$$(\forall x \in X)(\exists m \in \omega)(\forall n \geqslant m)\ x \in \bigcup \mathscr{F}_n.$$

Otherwise, One wins.

Like above, we are avoiding the use of a general game schema, $\mathsf{G}_{\mathrm{ufin}}(\mathcal{A},\mathcal{B})$, as defined in [1], in which case the game $\mathsf{G}_{\mathrm{ufin}}(\mathcal{O},\Gamma)$ could be seen as the Hurewicz game. As before, such a formulation would require care, as the authors observe, to account for finite selections which cover the space.

Definition 11. A *strategy* for One in the Hurewicz game on X is a function

$$\sigma : \left([\mathcal{O}_X]^{<\omega}\right)^{<\omega} \to \mathcal{O}_X.$$

The strategy σ is said to be *winning* if, for any sequence $\langle \mathscr{F}_n : n \in \omega\rangle$ with

[1] The results are stated relative to subsets of the real line, but the proofs lend themselves to greater generality.

$$\mathscr{F}_n \in [\sigma(\langle \mathscr{F}_j : j < n\rangle)]^{<\omega}$$

for every $n \in \omega$, then

$$(\exists x \in X)(\forall m \in \omega)(\exists n \geqslant m)\ x \notin \bigcup \mathscr{F}_n.$$

Though [23, Thm. 27] is stated for sets of reals, the proof extends to arbitrary spaces, so we state it in that level of generality here. Recall that the main idea of the proof is to use the fact that the Hurewicz property necessitates that a space be Lindelöf and then to use countable sub-covers of open covers corresponding to a given strategy for One to have Two respond with initial segments of those countable covers. Looking at the entire tree of partial plays in this scenario, we can apply the Hurewicz property on the tree to produce a play which beats the given strategy for One since the Hurewicz property generates a cover that covers every point by all but finitely members of the selection.

Theorem 12 *([23, Thm. 27]). A space X has the Hurewicz property if and only if One does not have a winning strategy in the Hurewicz game on X.*

There are also corresponding games for the standard selection principles $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ that are widely studied in the literature. We will recall here only the elements of selection game theory relevant to this paper.

Definition 13. Let $\mathcal{A}$ and $\mathcal{B}$ be collections. The *finite-selection game* $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ is played as follows. In the n^{th} inning, One chooses $A_n \in \mathcal{A}$ and Two responds with $\mathscr{F}_n \in [A_n]^{<\omega}$. Two is declared the winner if $\bigcup_{n\in\omega} \mathscr{F}_n \in \mathcal{B}$. Otherwise, One wins.

Definition 14. Let $\mathcal{A}$ and $\mathcal{B}$ be collections. The *single-selection game* $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is played as follows. In the n^{th} inning, One chooses $A_n \in \mathcal{A}$ and Two responds with $x_n \in A_n$. Two is declared the winner if $\{x_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

In this paper, we will only be studying two strategy types for One.

Definition 15. We define two strategy types for One.

- A *strategy for One* in $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is a function $\sigma : (\bigcup \mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for One is called *winning* if whenever $x_n \in \sigma\langle x_k : k < n\rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If One has a winning strategy, we write $\mathrm{I} \uparrow \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.
- A *predetermined strategy* for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves, they are just running through a script. Formally, it is a function $\sigma : \omega \to \mathcal{A}$. If One has a winning predetermined strategy, we write $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

These definitions can be extended to $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ in the obvious way.

Remark 16. In general, $\mathsf{S}_\square(\mathcal{A}, \mathcal{B})$ holds if and only if $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{A}, \mathcal{B})$ where $\square \in \{1, \mathrm{fin}\}$. See [4, Prop. 15].

Definition 17 *([15]).* Let $\mathfrak{C}$ be a class. We say that $\mathscr{C} \in \mathfrak{C}$ is *groupable* if there exists $\phi : \mathscr{C} \to \omega$ so that $\phi^{-1}(n)$ is finite for each $n \in \omega$ and, for any infinite $J \subseteq \omega$, $\bigcup\{\phi^{-1}(n) : n \in J\} \in \mathfrak{C}$. Then, we set

$$\mathfrak{C}^{\mathrm{gp}} = \{\mathscr{C} \in \mathfrak{C} : \mathscr{C} \text{ is groupable}\}.$$

If $\mathfrak{O}$ is a topological operator, we will let $\mathfrak{O}^{\mathrm{gp}}$ be the topological operator which generates the groupable elements of $\mathfrak{O}_X$ for a space X.

Remark 18. Note that any groupable element must be countable.

Originally, the following theorem appeared as a longer list of equivalent conditions, all assuming that X is an ϵ-space (a space in which every ω-cover has a countable subset which is an ω-cover). However, such a hypothesis isn't necessary for the following theorem since both conditions of the equivalence imply that the space X is an ϵ-space.

Theorem 19 *([15, Thm. 16]). For a space X, every finite power of X has the Hurewicz property if and only if $X \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.*

3. Preparations

The following is an adaption of [15, Lemma 9 and Cor. 10] using the same zig-zag technique.

Lemma 20. *For any sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of $\mathcal{O}_X^{\mathrm{gp}}(\mathcal{A})$, there exists a sequence $\langle \mathscr{V}_n : n \in \omega \rangle$ of $\mathcal{O}_X^{\mathrm{gp}}(\mathcal{A})$ where the set $\{\mathscr{V}_n : n \in \omega\}$ is pairwise disjoint and $\mathscr{V}_n \subseteq \mathscr{U}_n$ for every $n \in \omega$.*

Proof. Let $\phi_n : \mathscr{U}_n \to \omega$ witness the groupability of $\mathscr{U}_n$. Fix a bijection $\beta : \omega \to \omega^2$ and use the following notation: $\beta(j) = \langle \beta(j)_1, \beta(j)_2 \rangle$. Let

$$\mathscr{W}_{\beta(0)_1,\beta(0)_2} = \mathcal{A}_0 = \phi^{-1}_{\beta(0)_1}(0).$$

For $m \in \omega$, suppose we have $\mathcal{A}_m \in \left[\bigcup_{n\in\omega} \mathscr{U}_n\right]^{<\omega}$ and $\mathscr{W}_{\beta(m)_1,\beta(m)_2} \subseteq \mathscr{U}_{\beta(m)_1}$ defined. Let

$$M = \min\left\{n \in \omega : (\forall k \geqslant n)\ \phi^{-1}_{\beta(m+1)_1}(k) \cap \mathcal{A}_m = \emptyset\right\}.$$

Such an M must exist since $\mathcal{A}_m$ is finite and $\phi_{\beta(m+1)_1}$ is finite-to-one. Then define

$$\mathscr{W}_{\beta(m+1)_1,\beta(m+1)_2} = \phi^{-1}_{\beta(m+1)_1}(M)$$

and $\mathcal{A}_{m+1} = \mathcal{A}_m \cup \mathscr{W}_{\beta(m+1)_1,\beta(m+1)_2}$.

This recursively defines $\{\mathscr{W}_{n,m} : n, m \in \omega\}$ where $\mathscr{W}_{n,m} \subseteq \mathscr{U}_n$ for every $n, m \in \omega$. Let

$$\mathscr{V}_n = \bigcup\{\mathscr{W}_{n,m} : m \in \omega\}$$

and observe that $\mathscr{V}_n \subseteq \mathscr{U}_n$. Also note that $\mathscr{V}_n \in \mathcal{O}_X^{\mathrm{gp}}(\mathcal{A})$ for each $n \in \omega$ by the groupability of $\mathscr{U}_n$ and that the family $\{\mathscr{V}_n : n \in \omega\}$ is pairwise disjoint by construction. □

The following is a generalization of [15, Lemma 15].

Lemma 21. *Suppose $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X^{\mathrm{gp}}(\mathcal{A}))$. Then, for any sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of $\mathcal{O}_X(\mathcal{A})$, there exists a sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ so that the following properties hold:*

- $(\forall n \in \omega)\ \mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$,
- *$\{\mathscr{F}_n : n \in \omega\}$ is pairwise disjoint, and*
- $(\forall A \in \mathcal{A})\,(\exists m \in \omega)(\forall n \geqslant m)(\exists V \in \mathscr{F}_n)\ A \subseteq V$.

Proof. By $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X^{\text{gp}}(\mathcal{A}))$ we can assume that each $\mathscr{U}_n \in \mathcal{O}_X^{\text{gp}}(\mathcal{A})$. Then, by Lemma 20, we can assume that $\{\mathscr{U}_n : n \in \omega\}$ is pairwise disjoint. Fix a bijective enumeration $\mathscr{U}_n = \{U_{n,m} : m \in \omega\}$ for each $n \in \omega$ and let

$$\mathscr{W}_n = \left\{ \bigcap_{j=0}^{n} U_{j,m_j} : m \in \omega^{n+1} \right\} \setminus \{\emptyset\}.$$

Also, for $n \in \omega$, let $\gamma_n : \mathscr{W}_n \to \prod_{j=0}^{n} \mathscr{U}_j$ be a choice function[2] so that, for each $W \in \mathscr{W}_n$,

$$W = \bigcap_{j=0}^{n} \text{proj}_j(\gamma_n(W)),$$

where proj_j is the usual projection mapping. Notice that $\mathscr{W}_n \in \mathcal{O}_X(\mathcal{A})$ for each $n \in \omega$ so we can apply $\mathsf{S}_{\text{fin}}(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X^{\text{gp}}(\mathcal{A}))$ to produce $\mathscr{G}_n \in [\mathscr{W}_n]^{<\omega}$ so that

$$\mathscr{W} := \bigcup_{n \in \omega} \mathscr{G}_n \in \mathcal{O}_X^{\text{gp}}(\mathcal{A}).$$

Let $\phi : \mathscr{W} \to \omega$ witness the groupability of $\mathscr{W}$ and note that

$$(\forall A \in \mathcal{A})\, (\exists m \in \omega)(\forall n \geqslant m)\, \left(\exists W \in \phi^{-1}(n)\right)\ A \subseteq W.$$

Then, for each $n \in \omega$, let $\chi_n : \phi^{-1}(n) \to \omega$ be so that, for each $W \in \phi^{-1}(n)$,

$$\chi_n(W) = \min\{k \in \omega : W \in \mathscr{G}_k\}.$$

Finally, set

$$\mathscr{F}_n = \left\{\text{proj}_n\left(\gamma_{\chi_n(W)}(W)\right) : W \in \phi^{-1}(n)\right\} \in [\mathscr{U}_n]^{<\omega}.$$

Since $\{\mathscr{U}_n : n \in \omega\}$ is pairwise disjoint, the collection $\{\mathscr{F}_n : n \in \omega\}$ is also pairwise disjoint. To finish the proof, let $A \in \mathcal{A}$ be arbitrary and let $m \in \omega$ be so that, for each $n \geqslant m$, there is some $W \in \phi^{-1}(n)$ with $A \subseteq W$. Note that, for $n \geqslant m$ and $W \in \phi^{-1}(n)$ with $A \subseteq W$, we have that

$$A \subseteq W \subseteq \text{proj}_n(\gamma_{\chi_n(W)}(W)) \in \mathscr{F}_n.$$

This finishes the proof. □

4. The general hyperspace correspondence

We will now restrict our attention to ideals of compact sets; an *ideal of compact sets* of a space X is any ideal $\mathcal{A}$ of closed sets of X so that $\mathcal{A} \subseteq K(X)$. We will also use $[X]^{\omega}$ to denote the set of countably infinite subsets of a set X.

Definition 22. We say that a pair $(\mathfrak{T}, \mathfrak{A})$ is an *adequate context* if $\mathfrak{T}$ is a class of topological spaces and $\mathfrak{A}$ is a topological operator so that the following properties hold:

[2] For each $W \in \mathscr{W}_n$, the set $\left\{\langle U_0, U_1, \ldots U_n\rangle \in \prod_{j=0}^{n} \mathscr{U}_j : W = \bigcap_{j=0}^{n} U_j\right\}$ is non-empty, so we can use the Axiom of Choice to select an element from each of these sets and assign them to the corresponding $W \in \mathscr{W}_n$.

- For every space $X \in \mathfrak{T}$, $\mathfrak{A}_X$ is an ideal of compact subsets of X.
- For every space $X \in \mathfrak{T}$, $\mathbb{A}(X) \in \mathfrak{T}$ where $\mathbb{A}(X)$ is the set $\mathfrak{A}_X$ with the topology it inherits as a subspace of $\mathbb{K}(X)$.

As alluded to in the preliminaries, we will be considering two contexts, which will be shown to be adequate below: the class of non-compact spaces along with the operator that generates all compact subsets, and the class of all infinite spaces along with the operator that generates all finite subsets. When the space X has infinite compact subsets and $\mathfrak{A}$ is the operator that generates the finite subsets, then $\mathbb{A}(X) = \mathcal{P}_{\text{fin}}(X)$ is a proper subset of $\mathbb{K}(X)$.

We now offer a modification to [3, Def. 4.2].

Definition 23. Suppose $(\mathfrak{T}, \mathfrak{A})$ is an adequate context. We will say that the operator $\mathfrak{A}$ is *closed under* $\mathfrak{A}$*-unions* if the following properties hold:

- For each $K_0 \in \mathfrak{A}_X$, $\{K \in \mathbb{A}(X) : K \subseteq K_0\} \in \mathfrak{A}_{\mathbb{A}(X)}$.
- For each $\mathbf{K} \in \mathfrak{A}_{\mathbb{A}(X)}$, $\bigcup \mathbf{K} \in \mathfrak{A}_X$.

As pointed out by [3, Lemma 4.3], when $\mathfrak{T}$ is all infinite (resp. non-compact) spaces, the operator $\mathfrak{A}$ which produces finite subsets (resp. compact subsets) of a given space X is closed under $\mathfrak{A}$-unions. In the case of the compact subsets, it follows from results of [19].

Definition 24. Suppose we have an adequate context $(\mathfrak{T}, \mathfrak{A})$. We let $\mathcal{O}(\mathfrak{A})$ be the topological operator defined by $X \mapsto \mathcal{O}_X(\mathfrak{A}_X)$. We will say that the operator $\mathfrak{A}$ is *suitable* if all of the following properties hold:

(S1) $\mathfrak{A}$ is closed under $\mathfrak{A}$-unions.
(S2) For every $X \in \mathfrak{T}$ and every $\mathscr{U} \in \mathcal{O}_X(\mathfrak{A}_X)$, $\{[U] : U \in \mathscr{U}\} \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$.
(S3) For every $X \in \mathfrak{T}$ and every $\mathscr{U} \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$, $\{V \in \mathscr{T}_X : (\exists U \in \mathscr{U})\ [V] \subseteq U\} \in \mathcal{O}_X(\mathfrak{A}_X)$.
(S4) For every $X \in \mathfrak{T}$, if $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$, then X is Hurewicz.
(S5) For every $X \in \mathfrak{T}$, the following are equivalent:

- $\mathbb{A}(X)$ is Menger.
- $\mathbb{A}(X) \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$.
- $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$.

Note that, when $\mathfrak{T}$ is the class of all non-compact spaces and $\mathfrak{A}$ is the operator that generates compact subsets, then $\mathbb{A}(X) = \mathbb{K}(X)$ and so the set $\mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$ is the set of k-covers of $\mathbb{K}(X)$. Analogously, when $\mathfrak{T}$ is the class of all infinite spaces and $\mathfrak{A}$ is the operator that generates finite subsets, then $\mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$ is the set of ω-covers of $\mathcal{P}_{\text{fin}}(X)$.

Theorem 25. *Suppose we have an adequate context* $(\mathfrak{T}, \mathfrak{A})$ *where* $\mathfrak{A}$ *is suitable. Then, for any* $X \in \mathfrak{T}$*, the following are equivalent.*

(i) $\mathbb{A}(X)$ *is Hurewicz.*
(ii) $\mathbb{A}(X) \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.
(iii) $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.
(iv) $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$ *and* $\mathcal{O}^{\text{gp}}_X(\mathfrak{A}_X) = \mathcal{O}_X(\mathfrak{A}_X) \cap [\mathscr{T}_X]^\omega$.

Proof. Note that the implication (ii) $\Longrightarrow$ (i) follows from (S4).

(i) $\Longrightarrow$ (iv): Suppose $\mathbb{A}(X)$ is Hurewicz and notice that $\mathbb{A}(X)$ is Menger, which implies that $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$ by (S5). So we need only show that every countable member of $\mathcal{O}_X(\mathfrak{A}_X)$ is groupable. So let $\mathscr{U} = \{U_n : n \in \omega\} \in \mathcal{O}_X(\mathfrak{A}_X)$. We define a strategy for One in the Hurewicz game relative to $\mathbb{A}(X)$ in the following way. Let

$$\sigma(\langle\rangle) = \{[U_n] : n \in \omega\}.$$

For $k \in \omega$, suppose we have $\langle \mathscr{G}_j : j < k\rangle$, $\langle M_j : j < k\rangle$, and $\sigma(\langle \mathscr{G}_j : j < k\rangle)$ defined. For any

$$\mathscr{G}_k \in [\sigma(\langle \mathscr{G}_j : j < k\rangle)]^{<\omega},$$

let

$$M_k = \max\{m \in \omega : [U_m] \in \mathscr{G}_k\}.$$

Then set

$$\sigma(\langle \mathscr{G}_j : j \leqslant k\rangle) = \{[U_n] : n > M_k\}.$$

Notice that, by (S2), $\sigma(\langle \mathscr{G}_j : j \leqslant k\rangle) \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$ since $\{U_n : n > M_k\} \in \mathcal{O}_X(\mathfrak{A}_X)$. So this σ is a strategy for One in the Hurewicz game on $\mathbb{A}(X)$.

By Theorem 12, σ cannot be a winning strategy. So there is a sequence $\langle \mathscr{G}_j : j \in \omega\rangle$ of finite selections and a corresponding increasing sequence $\langle M_j : j \in \omega\rangle$ of naturals as defined above that win against σ. We can then define $\phi : \mathscr{U} \to \omega$ by the rule

$$\phi(U_n) = \min\{k \in \omega : n \leqslant M_k\}.$$

Since ϕ is evidently finite-to-one, we need only show it has the required covering property. So let $J \subseteq \omega$ be infinite and $K \in \mathfrak{A}_X$. By our construction, there must be some $\ell \in J$, $\ell > 0$, and some $W \in \mathscr{G}_\ell$ for which $K \in W$. Hence, there must be some $n \in (M_{\ell-1}, M_\ell]$ so that $W = [U_n]$. Note that $\phi(U_n) = \ell$ and that $K \subseteq U_n$. That is, $K \subseteq U_n \in \phi^{-1}(\ell)$, which establishes that $\{U_n : n \in \omega\}$ is groupable.

The implication (iv) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (i): Suppose $X \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$ and let $\langle \mathscr{U}_n : n \in \omega\rangle$ be a sequence of open covers of $\mathbb{A}(X)$. Note that

$$\mathscr{U}_n^* := \left\{\bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{U}_n]^{<\omega}\right\} \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$$

since $\mathfrak{A}_{\mathbb{A}(X)}$ consists of compact sets. Fix a choice function $\gamma_{u,n} : \mathscr{U}_n^* \to [\mathscr{U}_n]^{<\omega}$ to be so that $\bigcup \gamma_{u,n}(U) = U$. By (S3),

$$\mathscr{V}_n := \{V \in \mathscr{T}_X : (\exists U \in \mathscr{U}_n^*)\ [V] \subseteq U\} \in \mathcal{O}_X(\mathfrak{A}_X).$$

Fix a choice function $\gamma_{v,n} : \mathscr{V}_n \to \mathscr{U}_n^*$ so that $[V] \subseteq \gamma_{v,n}(V)$. By Lemma 21, we can choose $\mathscr{G}_n \in [\mathscr{V}_n]^{<\omega}$ with the properties guaranteed by the lemma. Define

$$\mathscr{F}_n = \bigcup\{\gamma_{u,n} \circ \gamma_{v,n}(V) : V \in \mathscr{G}_n\}$$

and note that $\mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$.

We now show that the selection $\{\mathscr{F}_n : n \in \omega\}$ witnesses the Hurewicz property for $\mathbb{A}(X)$. So let $K \in \mathbb{A}(X)$. By our use of Lemma 21, we can find $m \in \omega$ so that, for every $n \geqslant m$, there is some $V \in \mathscr{G}_n$ with

$$K \subseteq V \implies K \in [V] \subseteq \gamma_{v,n}(V) = \bigcup \gamma_{u,n} \circ \gamma_{v,n}(V) \subseteq \bigcup \mathscr{F}_n.$$

Hence, $\mathbb{A}(X)$ is Hurewicz.

(i) $\implies$ (ii): Suppose $\mathbb{A}(X)$ is Hurewicz. By the equivalence of (i) and (iii) established above, we know that $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\mathrm{gp}}(\mathfrak{A}))$. It follows that $\mathbb{A}(X) \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$ by (S5). So we need only show every countable member of $\mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$ is groupable. So let $\mathscr{U} = \{U_n : n \in \omega\} \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$. We define a strategy for One in the Hurewicz game on $\mathbb{A}(X)$.

Throughout this portion of the proof, we will be using the fact that $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\mathrm{gp}}(\mathfrak{A}))$ to find countable subsets of members of $\mathcal{O}_X(\mathfrak{A}_X)$ which are in $\mathcal{O}_X(\mathfrak{A}_X)$. We will also use (S3) to take members of $\mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$ and refine them with sets of the form $[V]$ for $V \in \mathscr{T}_X$ that form elements of $\mathcal{O}_X(\mathfrak{A}_X)$.

To begin, let $\{V_m : m \in \omega\} \in \mathcal{O}_X(\mathfrak{A}_X)$ be so that, for every $m \in \omega$, there is $n \in \omega$ with $[V_m] \subseteq U_n$. Then we let

$$\sigma(\langle\rangle) = \mathscr{V}_{\langle\rangle} := \{[V_m] : m \in \omega\}.$$

For $\mathscr{G}_0 \in [\mathscr{V}_{\langle\rangle}]^{<\omega}$, define

$$M_0 = \min\{N \in \omega : (\forall W \in \mathscr{G}_0)(\exists n \leqslant N)\ W \subseteq U_n\}.$$

Note that $\{U_n : n > M_0\} \in \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$. So we can let $\{V_{0,m} : m \in \omega\} \in \mathcal{O}_X(\mathfrak{A}_X)$ be so that, for every $m \in \omega$, there is some $n > M_0$ with $[V_{0,m}] \subseteq U_n$. Define

$$\sigma(\langle \mathscr{G}_0 \rangle) = \mathscr{V}_{\langle M_0 \rangle} := \{[V_{0,m}] : m \in \omega\}.$$

Now, let $k \in \omega$ and suppose we have $\mathscr{G}_j$, M_j, for $j \leqslant k$, and $\mathscr{V}_{\langle M_j : j \leqslant k\rangle}$ defined where $\mathscr{V}_{\langle M_j : j \leqslant k\rangle}$ refines $\mathscr{U}$. For $\mathscr{G}_{k+1} \in [\mathscr{V}_{\langle M_j : j \leqslant k\rangle}]^{<\omega}$, let

$$M_{k+1} = \min\{N > M_k : (\forall W \in \mathscr{G}_{k+1})(\exists n \leqslant N)\ W \subseteq U_n\}.$$

Let $\{V_{k+1,m} : m \in \omega\} \in \mathcal{O}_X(\mathfrak{A}_X)$ be so that, for every $m \in \omega$, there is some $n > M_{k+1}$ so that $[V_{k+1,m}] \subseteq U_n$. Define

$$\sigma(\langle \mathscr{G}_j : j \leqslant k+1 \rangle) = \mathscr{V}_{\langle M_j : j \leqslant k+1\rangle} := \{[V_{k+1,m}] : m \in \omega\}.$$

This defines a strategy σ for One in the Hurewicz game on $\mathbb{A}(X)$.

By Theorem 12, we know that σ cannot be winning. So there is a sequence $\langle \mathscr{G}_j : j \in \omega\rangle$ and a corresponding increasing sequence $\langle M_j : j \in \omega \rangle$ of naturals so that, for any $K \in \mathbb{A}(X)$, there exists some $m \in \omega$ so that, for every $n \geqslant m$, $K \in \bigcup \mathscr{G}_n$. We can define $\phi : \mathscr{U} \to \omega$ by the rule

$$\phi(U_n) = \min\{k \in \omega : n \leqslant M_k\}.$$

Note that ϕ is clearly finite-to-one. So we need only show it satisfies the covering criterion for groupability of $\mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)})$. Let $J \subseteq \omega$ be infinite and $\mathbf{K} \in \mathfrak{A}_{\mathbb{A}(X)}$. Observe that $\bigcup \mathbf{K} \in \mathfrak{A}_X$ by (S1). There must be some $\ell \in J$, $\ell > 0$, and some $W \in \mathscr{T}_X$ with $\bigcup \mathbf{K} \in [W] \in \mathscr{G}_\ell$. Note that $\bigcup \mathbf{K} \in [W]$ means that $\bigcup \mathbf{K} \subseteq W$. By construction, there is some $n \in (M_{\ell-1}, M_\ell]$ so that $[W] \subseteq U_n$. Hence, for any $K \in \mathbf{K}$, $K \subseteq \bigcup \mathbf{K} \subseteq W$. That is, $K \in [W] \subseteq U_n$. Since $K \in \mathbf{K}$ was arbitrary, $\mathbf{K} \subseteq U_n \in \phi^{-1}(\ell)$. This establishes that $\mathscr{U}$ is groupable, and thus, that $\mathbb{A}(X) \models \mathsf{S}_{\mathrm{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\mathrm{gp}}(\mathfrak{A}))$. □

We now offer a game-theoretic characterization.

Theorem 26. *Let $\square \in \{1, \text{fin}\}$. Suppose we have an adequate context $(\mathfrak{T}, \mathfrak{A})$ where $\mathfrak{A}$ is suitable and, for every $X \in \mathfrak{T}$,*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X(\mathfrak{A}_X)) \iff \mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X(\mathfrak{A}_X)).$$

Then, for any $X \in \mathfrak{T}$,

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X)) \iff \mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X)).$$

Proof. We need only show that, for $X \in \mathfrak{T}$,

$$\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X)) \implies \mathrm{I} \not\uparrow \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X)).$$

So suppose $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X))$. By Remark 16 and Theorem 25, $X \models \mathsf{S}_\square(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$ and $\mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X) = \mathcal{O}_X(\mathfrak{A}_X) \cap [\mathscr{T}_X]^\omega$. By the hypothesis, we see that $\mathrm{I} \not\uparrow \mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X(\mathfrak{A}_X))$. Now, since any strategy σ employed by One in the game cannot be winning, Two can always beat σ and, since every countable member of $\mathcal{O}_X(\mathfrak{A}_X)$ is groupable, Two's winning play is always groupable. Hence, One does not have a winning strategy in $\mathsf{G}_\square(\mathcal{O}_X(\mathfrak{A}_X), \mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X))$. □

To address single-selections, we need a bit more in our hypotheses.

Definition 27. Suppose we have an adequate context $(\mathfrak{T}, \mathfrak{A})$. We will say that the operator $\mathfrak{A}$ is *super suitable* if it is suitable and, for all $X \in \mathfrak{T}$, the following are equivalent:

- $\mathbb{A}(X)$ is Rothberger.
- $\mathbb{A}(X) \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$.
- $X \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$.

Theorem 28. *Suppose we have an adequate context $(\mathfrak{T}, \mathfrak{A})$ where $\mathfrak{A}$ is super suitable. Then, for any $X \in \mathfrak{T}$, the following are equivalent.*

(i) *$\mathbb{A}(X)$ is Gerlits-Nagy.*
(ii) *$\mathbb{A}(X) \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.*
(iii) *$X \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.*

Proof. (i) $\implies$ (iii): Since $\mathbb{A}(X)$ is Hurewicz, Theorem 25 asserts that $\mathcal{O}_X^{\text{gp}}(\mathfrak{A}_X) = \mathcal{O}_X(\mathfrak{A}_X) \cap [\mathscr{T}_X]^\omega$. Since $\mathbb{A}(X)$ is Rothberger, by the criterion in the definition of super suitable, $X \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$. Hence, $X \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.

(iii) $\implies$ (ii): By Theorem 25, $\mathbb{A}(X) \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$. Moreover, also by an application of Theorem 25,

$$\mathcal{O}_{\mathbb{A}(X)}^{\text{gp}}(\mathfrak{A}_{\mathbb{A}(X)}) = \mathcal{O}_{\mathbb{A}(X)}(\mathfrak{A}_{\mathbb{A}(X)}) \cap [\mathscr{T}_{\mathbb{A}(X)}]^\omega.$$

By the criterion in the definition of super suitable, $\mathbb{A}(X) \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}(\mathfrak{A}))$, and, by the above identity, we see that $\mathbb{A}(X) \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$.

(ii) $\implies$ (i): Since $\mathbb{A}(X) \models \mathsf{S}_1(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$, $\mathbb{A}(X) \models \mathsf{S}_{\text{fin}}(\mathcal{O}(\mathfrak{A}), \mathcal{O}^{\text{gp}}(\mathfrak{A}))$ which establishes that $\mathbb{A}(X)$ is Hurewicz by Theorem 25. Also, by the criterion in the definition of super suitable, $\mathbb{A}(X)$ is Rothberger. □

5. Applications of the hyperspace correspondence

5.1. The compact sets

It was shown in [3] that $\mathbb{K}(X)$ captures the properties of k-Lindelöf and k-Menger on X. Notably, k-Rothberger isn't in this list. Indeed, [3, Ex. 4.19] points out that $\mathbb{R}$ is k-Rothberger but not Rothberger. Since the property of Rothberger is hereditary with respect to closed subsets and $\mathbb{R}$ embeds as a closed subspace of $\mathbb{K}(\mathbb{R})$, $\mathbb{K}(\mathbb{R})$ is not Rothberger. However, Theorem 25 can be used to establish a relationship between X and $\mathbb{K}(X)$ relative to the Hurewicz property.

We start by recalling the following fact regarding k-covers.

Lemma 29 *([3, Cor. 4.13]). Let X be a space.*

- *If $\mathscr{U}$ is a k-cover of X, then $\{[U] : U \in \mathscr{U}\}$ is a k-cover of $\mathbb{K}(X)$.*
- *If $\mathscr{U}$ is a k-cover of $\mathbb{K}(X)$, then*

$$\mathscr{V} := \{V \in \mathscr{T}_X : (\exists U \in \mathscr{U})\ [V] \subseteq U\}$$

is a k-cover of X.

Theorem 30 *([3, Cor. 4.17]). For any space X, the following are equivalent:*

- *$\mathbb{K}(X)$ is Menger.*
- *$\mathbb{K}(X)$ is k-Menger.*
- *X is k-Menger.*

Lemma 31. *For any space X, if $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{K}, \mathcal{K}^{\mathrm{gp}})$, then X is Hurewicz.*

Proof. Suppose $X \models \mathsf{S}_{\mathrm{fin}}(\mathcal{K}, \mathcal{K}^{\mathrm{gp}})$ and let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. Note that

$$\mathscr{V}_n := \left\{ \bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{U}_n]^{<\omega} \right\} \in \mathcal{K}_X.$$

By Lemma 21, we can produce $\mathscr{G}_n \in [\mathscr{V}_n]^{<\omega}$ for each $n \in \omega$ with the properties guaranteed in the lemma. Let $\gamma_n : \mathscr{G}_n \to [\mathscr{U}_n]^{<\omega}$ be a choice function so that $V = \bigcup \gamma_n(V)$ for each $V \in \mathscr{G}_n$. Then define

$$\mathscr{F}_n = \bigcup_{V \in \mathscr{G}_n} \gamma_n(V)$$

and notice that $\mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$. To finish the proof, we need only show that $\langle \mathscr{F}_n : n \in \omega \rangle$ witnesses the Hurewicz criterion. So let $x \in X$ and observe that $\{x\} \in \mathbb{K}(X)$. By the use of Lemma 21, there is some $m \in \omega$ so that, for every $n \geqslant m$, $\{x\} \subseteq \bigcup \mathscr{G}_n$. Hence, for every $n \geqslant m$, there is some $V \in \mathscr{G}_n$ for which $x \in V$. Observe that, for $V \in \mathscr{G}_n$,

$$V = \bigcup \gamma_n(V) \subseteq \bigcup_{V \in \mathscr{G}_n} \bigcup \gamma_n(V) = \bigcup \mathscr{F}_n.$$

Therefore, for every $n \geqslant m$, $x \in \bigcup \mathscr{F}_n$. $\square$

We will also use the following game-theoretic results.

Theorem 32 *([3, Cor. 4.18 and Thm. 4.21]). For any space X and $\square \in \{1, \text{fin}\}$,*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_\square(\mathcal{K}_X, \mathcal{K}_X) \iff \mathrm{I} \uparrow \mathsf{G}_\square(\mathcal{K}_X, \mathcal{K}_X).$$

Corollary 33. *For any space X, the following are equivalent:*

(i) $\mathbb{K}(X)$ *is Hurewicz.*
(ii) $\mathbb{K}(X) \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K}^{\text{gp}})$.
(iii) $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$ *and* $\mathcal{K}_X^{\text{gp}} = \mathcal{K}_X \cap [\mathscr{T}_X]^\omega$.
(iv) $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K}^{\text{gp}})$.
(v) *One does not have a winning strategy in* $\mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X^{\text{gp}})$.

Proof. Let $\mathfrak{T}$ be the class of all non-compact spaces and $\mathfrak{A}$ be the operator that generates the compact subsets of a given space X. Then $(\mathfrak{T}, \mathfrak{A})$ is an adequate context since it is well-known that X is compact if and only if $\mathbb{K}(X)$ is compact [19]. As mentioned above, by results of [19], $\mathfrak{A}$ is closed under $\mathfrak{A}$-unions. Lemmas 29 and 31 and Theorem 30 then finish showing that $\mathfrak{A}$ is suitable. Thus, Theorem 25 applies to give the equivalence of (i)–(iv). The equivalence of (iv) and (v) follows from Theorems 26 and 32. □

As mentioned above, $\mathbb{R}$ is an example of a space which is k-Rothberger but where $\mathbb{K}(X)$ is not Rothberger. So, we cannot apply Theorem 28 in this setting. In fact, we can even say more.

Example 34. The real line $\mathbb{R}$ has the property that $\mathbb{R} \models \mathsf{S}_1(\mathcal{K}, \mathcal{K}^{\text{gp}})$ but $\mathbb{K}(\mathbb{R})$ is not Gerlits-Nagy.

Indeed, for any sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of k-covers of $\mathbb{R}$, choose $U_n \in \mathscr{U}_n \setminus \{U_k : k < n\}$ to be so that $[-2^n, 2^n] \subseteq U_n$. Then define $\phi : \{U_n : n \in \omega\} \to \omega$ by $\phi(U_n) = n$. Thus, $\{U_n : n \in \omega\}$ is groupable.

On the other hand, as noted previously, $\mathbb{K}(\mathbb{R})$ is not Rothberger, so $\mathbb{K}(\mathbb{R})$ is not Gerlits-Nagy.

Therefore, in light of Example 34, we see that a single-selection analog to Corollary 33 is not obtained. Nevertheless, we still obtain the following as a direct application of Theorems 26 and 32.

Corollary 35. *For any space X,*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X^{\text{gp}}) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X^{\text{gp}}).$$

5.2. The finite sets

In [3], it was shown that $\mathcal{P}_{\text{fin}}(X)$ captures the properties of ω-Lindelöf, ω-Menger, and ω-Rothberger on X, just as the finite powers of X do (see [9,12,22]). We use Theorem 25 to extend this relationship to the Hurewicz property.

The following can be seen as a sort of analog of The Wallace Theorem [8, Thm. 3.2.10] in the context of finite subsets instead of finite powers.

Lemma 36 *([3, Cor. 4.7]). Let X be a space.*

- *If $\mathscr{U}$ is an ω-cover of X, then $\{[U] : U \in \mathscr{U}\}$ is an ω-cover of $\mathcal{P}_{\text{fin}}(X)$.*
- *If $\mathscr{U}$ is an ω-cover of $\mathcal{P}_{\text{fin}}(X)$, then*

$$\mathscr{V} := \{V \in \mathscr{T}_X : (\exists U \in \mathscr{U})\ [V] \subseteq U\}$$

is an ω-cover of X.

Theorem 37 *([3, Cor. 4.10]). For any space X, the following are equivalent:*

- $\mathcal{P}_{\mathrm{fin}}(X)$ *is Menger.*
- $\mathcal{P}_{\mathrm{fin}}(X)$ *is ω-Menger.*
- X *is ω-Menger.*

Theorem 38 *([24]). For any space X and $\Box \in \{1, \mathrm{fin}\}$,*

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_{\Box}(\Omega_X, \Omega_X) \iff \mathrm{I} \uparrow \mathsf{G}_{\Box}(\Omega_X, \Omega_X).$$

Corollary 39. *For any space X, the following are equivalent:*

(i) $\mathcal{P}_{\mathrm{fin}}(X)$ *is Hurewicz.*
(ii) $\mathcal{P}_{\mathrm{fin}}(X) \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.
(iii) $X \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.
(iv) Every finite power of X is Hurewicz.
(v) One does not have a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Omega_X, \Omega_X^{\mathrm{gp}})$.

Proof. Let $\mathfrak{T}$ be the class of all infinite spaces and $\mathfrak{A}$ be the operator that generates the finite subsets of a given space X. Then $(\mathfrak{T}, \mathfrak{A})$ is an adequate context. As mentioned above, $\mathfrak{A}$ is closed under $\mathfrak{A}$-unions. To finish showing that $\mathfrak{A}$ is suitable, use Lemma 36 and Theorems 19 and 37. Thus, Theorem 25 applies to give us the equivalence of (i)–(iii). The equivalence of (iii) and (iv) is the content of Theorem 19. The equivalence of (iii) and (v) follows from Theorems 26 and 38. □

In this case, we can appeal to Theorem 28 since we have access to the following result.

Theorem 40 *([3, Cor. 4.11]). For any space X, the following are equivalent:*

- $\mathcal{P}_{\mathrm{fin}}(X)$ *is Rothberger.*
- $\mathcal{P}_{\mathrm{fin}}(X)$ *is ω-Rothberger.*
- X *is ω-Rothberger.*

Corollary 41. *For any space X, the following are equivalent:*

(i) $\mathcal{P}_{\mathrm{fin}}(X)$ *is Gerlits-Nagy.*
(ii) $\mathcal{P}_{\mathrm{fin}}(X) \models \mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$.
(iii) $X \models \mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$.
(iv) Every finite power of X is Gerlits-Nagy.
(v) One does not have a winning strategy in $\mathsf{G}_1(\Omega_X, \Omega_X^{\mathrm{gp}})$.

Proof. Let $\mathfrak{T}$ be the class of infinite spaces and $\mathfrak{A}$ be the operator that produces the finite subsets of a given space X. It was shown above that $\mathfrak{A}$ is suitable. Theorem 40 establishes that $\mathfrak{A}$ is super suitable, so Theorem 28 applies to provide the equivalence of (i)–(iii). The equivalence of (iii) and (iv) is established in [15, Thm. 19]. The equivalence of (iii) and (v) follows from Theorems 26 and 38. □

We end this section with a question.

Question 1. Are there other natural examples of (super) suitable operators $\mathfrak{A}$ other than those listed above?

6. Relationships with spaces of continuous functions

For a space X, let $C(X)$ denote the set of all continuous real-valued function on X. Given an ideal of closed sets $\mathcal{A}$ of X, let $C_{\mathcal{A}}(X)$ denote the set $C(X)$ endowed with the topology of uniform convergence of elements of $\mathcal{A}$; this topology has as a basis sets of the form

$$[f; A, \varepsilon] := \{g \in C(X) : \sup\{|f(x) - g(x)| : x \in A\} < \varepsilon\}$$

for $f \in C(X)$, $A \in \mathcal{A}$, and $\varepsilon > 0$. When $\mathcal{A} = [X]^{<\omega}$ (resp. $\mathcal{A} = K(X)$), we use $C_p(X)$ (resp. $C_k(X)$) instead of $C_{\mathcal{A}}(X)$. Indeed, $C_p(X)$ coincides with the ring of continuous real-valued functions on X with the topology of point-wise convergence. Likewise, $C_k(X)$ coincides with the ring of continuous real-valued functions on X with the topology of uniform convergence on compacta, which is equivalent to the compact-open topology.

Arhangel'skii, in [2], introduced the notion of countable fan-tightness and gave a characterization of a space X being Hurewicz in all of its finite powers in terms of $C_p(X)$.

Definition 42. For a space X and $x \in X$, let $\Omega_{X,x} = \{A \subseteq X : x \in \mathrm{cl}(A) \setminus A\}$ where $\mathrm{cl}(A)$ is the closure of A in X. Then the space X has *countable fan-tightness at* x provided that $X \models \mathsf{S}_{\mathrm{fin}}(\Omega_{X,x}, \Omega_{X,x})$. We say that X has *countable fan-tightness* if X has countable fan-tightness at x for every $x \in X$.

We will also address the single-selection analog.

Definition 43. For a space X and $x \in X$, X has *countable strong fan-tightness at* x if $X \models \mathsf{S}_1(\Omega_{X,x}, \Omega_{X,x})$. If X has countable strong fan-tightness at x for every $x \in X$, then X is said to have *countable strong fan-tightness.*

Kočinac and Scheepers, in [15], extend the characterization of [2] to include the so-called Reznichenko property, which, according to [18], was introduced by Reznichenko at a seminar in 1996.

Definition 44. For a space X and $x \in X$, X has the *Reznichenko property at* x if $\Omega_{X,x} \cap [X]^{\omega} = \Omega^{\mathrm{gp}}_{X,x}$. If X has the Reznichenko property at x for every $x \in X$, we say that X is *Reznichenko.*

As $C_{\mathcal{A}}(X)$ is homogeneous, we can address the properties listed above at any particular point and establish them for the entire space; for convenience, we focus on $\mathbf{0}$, the function which is constantly 0.

Combining Corollary 39 and [15, Thm. 21], we obtain:

Theorem 45. *For any Tychonoff space X, the following are equivalent:*

- $\mathcal{P}_{\mathrm{fin}}(X)$ *is Hurewicz.*
- $\mathcal{P}_{\mathrm{fin}}(X) \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.
- $X \models \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.
- *Every finite power of X is Hurewicz.*
- *One does not have a winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega_X, \Omega_X^{\mathrm{gp}})$.
- $C_p(X) \models \mathsf{S}_{\mathrm{fin}}(\Omega_{C_p(X),\mathbf{0}}, \Omega^{\mathrm{gp}}_{C_p(X),\mathbf{0}})$.
- $C_p(X)$ *is Reznichenko and has countable fan-tightness.*

Similarly, combining Corollary 41, [14], and [15, Thm. 26], we obtain:

Theorem 46. *For any Tychonoff space X, the following are equivalent:*

- *$\mathcal{P}_{\text{fin}}(X)$ is Gerlits-Nagy.*
- *$\mathcal{P}_{\text{fin}}(X) \models \mathsf{S}_1(\Omega, \Omega^{\text{gp}})$.*
- *$X \models \mathsf{S}_1(\Omega, \Omega^{\text{gp}})$.*
- *Every finite power of X is Gerlits-Nagy.*
- *One does not have a winning strategy in $\mathsf{G}_1(\Omega_X, \Omega_X^{\text{gp}})$.*
- *$C_p(X) \models \mathsf{S}_1(\Omega_{C_p(X),\mathbf{0}}, \Omega^{\text{gp}}_{C_p(X),\mathbf{0}})$.*
- *$C_p(X)$ is Reznichenko and has countable strong fan-tightness.*

We now turn our attention to $C_k(X)$. For a set X, we will let $\wp(X)$ denote the power set of X.

Definition 47. Let X be a Tychonoff space. We define $\mathfrak{u} : C(X) \times \omega \to \mathscr{T}_X \cup \{X\}$ by the rule $\mathfrak{u}(f, n) = f^{-1}((-2^{-n}, 2^{-n}))$. We then define $\mathbb{U} : \Omega_{C_k(X),\mathbf{0}} \times \omega \to \wp(\mathscr{T}_X)$ by the rule

$$\mathbb{U}(\mathscr{F}, n) = \{U \in \mathscr{T}_X : (\exists K \in K(X))(\exists f \in \mathscr{F})\ K \subseteq U \subseteq \mathfrak{u}(f, n)\}.$$

Also define $\mathscr{D} : \mathcal{K}_X \to \wp(C(X))$, by the rule

$$\mathscr{D}(\mathscr{U}) = \{f \in C(X) : (\exists U \in \mathscr{U})\ f[X \setminus U] = \{1\}\}.$$

Lemma 48. *Let X be a Tychonoff space.*

(i) If $\mathscr{F} \in \Omega_{C_k(X),\mathbf{0}}$, then, for any $n \in \omega$, $\mathbb{U}(\mathscr{F}, n) \in \mathcal{K}_X$.
(ii) If $\mathscr{U} \in \mathcal{K}_X$, then $\mathscr{D}(\mathscr{U}) \in \Omega_{C_k(X),\mathbf{0}}$.

Proof. (i): Let $K \in K(X)$ and consider the neighborhood $[\mathbf{0}; K, 2^{-n}]$ of $\mathbf{0}$. We can choose

$$f \in \mathscr{F} \cap [\mathbf{0}; K, 2^{-n}].$$

For each $x \in K$, $|f(x)| < 2^{-n}$ so we see that

$$\sup\{|f(x)| : x \in K\} < 2^{-n}.$$

So $K \subseteq f^{-1}((-2^{-n}, 2^{-n})) = \mathfrak{u}(f, n)$. If $\mathfrak{u}(f, n) = X$, we can let $U \in \mathscr{T}_X$ be so that $K \subseteq U \subseteq \mathfrak{u}(f, n)$. Then $U \in \mathbb{U}(\mathscr{F}, n)$. Otherwise, $\mathfrak{u}(f, n) \in \mathbb{U}(\mathscr{F}, n)$.

(ii): Consider any neighborhood $[\mathbf{0}; K, \varepsilon]$ of $\mathbf{0}$ in $C_k(X)$. Since $\mathscr{U} \in \mathcal{K}_X$, we can find $U \in \mathscr{U}$ so that $K \subseteq U$. As X is Tychonoff, we can find $f \in C(X)$ so that $f[K] = \{0\}$ and $f[X \setminus U] = \{1\}$. Now,

$$f \in \mathscr{D}(\mathscr{U}) \cap [\mathbf{0}; K, \varepsilon],$$

which completes the proof. □

We remark that, as reported in [13], it was shown in [17] that $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$ if and only if $C_k(X)$ has countable fan-tightness. The single-selection analog of this equivalence was established in [13]. We now establish the groupable analogues of these equivalences to include the Reznichenko property on $C_k(X)$ which answers Problem 4.3 of [13] in the affirmative. We provide the complete proof (in the finite-selection context) for the convenience of the reader.

Theorem 49. *For any Tychonoff space, the following are equivalent:*

(i) $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K}^{\text{gp}})$

(ii) One does not have a winning strategy in $\mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X^{\text{gp}})$.
(iii) $C_k(X)$ *is Reznichenko and has countable fan-tightness.*
(iv) $C_k(X) \models \mathsf{S}_{\text{fin}}(\Omega_{C_k(X),\mathbf{0}}, \Omega^{\text{gp}}_{C_k(X),\mathbf{0}})$.

Proof. The equivalence of (i) and (ii) is established in Corollary 33.

(i) $\Longrightarrow$ (iii): We first show that $C_k(X)$ has countable fan-tightness. So let $\langle \mathscr{F}_n : n \in \omega \rangle$ be a sequence of $\Omega_{C_k(X),\mathbf{0}}$. We then let $\mathscr{U}_n = \mathbb{U}(\mathscr{F}_n, n)$. Note that $\mathscr{U}_n \in \mathcal{K}_X$ by Lemma 48. As $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K}^{\text{gp}})$, we can choose $\mathscr{G}_n \in [\mathscr{U}_n]^{<\omega}$ so that

$$\bigcup_{n\in\omega} \mathscr{G}_n \in \mathcal{K}_X^{\text{gp}}.$$

Let $\phi : \bigcup_{n\in\omega} \mathscr{G}_n \to \omega$ be as guaranteed by the definition of groupability.

For each $n \in \omega$, define $\mathfrak{f}_n : \mathscr{G}_n \to \mathscr{F}_n$ to be so that, for $U \in \mathscr{G}_n$, $U \subseteq \mathfrak{u}(\mathfrak{f}_n(U), n)$. Let $\mathscr{H}_n = \mathfrak{f}_n[\mathscr{G}_n]$ and observe that $\mathscr{H}_n \in [\mathscr{F}_n]^{<\omega}$.

To see that $\bigcup_{n\in\omega} \mathscr{H}_n \in \Omega_{C_k(X),\mathbf{0}}$, consider any basic neighborhood $[\mathbf{0}; K, \varepsilon]$ of $\mathbf{0}$. Let $m \in \omega$ be so that $2^{-m} < \varepsilon$ and consider

$$M := \max\left\{\phi(U) : U \in \bigcup_{k<m} \mathscr{G}_k\right\}.$$

By the groupability criterion, $\bigcup\{\phi^{-1}(n) : n > M\} \in \mathcal{K}_X$. So we can find $\ell_0 > M$ and $U \in \phi^{-1}(\ell_0)$ so that $K \subseteq U$. Since $\ell_0 > M$, we know that $U \notin \bigcup_{k<m} \mathscr{G}_k$. So we can find $\ell \geqslant m$ so that $U \in \mathscr{G}_\ell$. It follows that

$$K \subseteq U \subseteq \mathfrak{u}(\mathfrak{f}_\ell(U), \ell) = \mathfrak{f}_\ell(U)^{-1}((-2^{-\ell}, 2^{-\ell})).$$

As $\ell \geqslant m$ and $2^{-m} < \varepsilon$, we see that

$$\mathfrak{f}_\ell(U) \in [\mathbf{0}; K, \varepsilon] \cap \bigcup_{n\in\omega} \mathscr{H}_n.$$

Now, we show that $C_k(X)$ is Reznichenko. So let $\mathscr{F} = \{f_n : n \in \omega\} \in \Omega_{C_k(X),\mathbf{0}}$. By Corollary 33, we know that $\mathbb{K}(X)$ is Hurewicz and that, by Theorem 12, One has no winning strategy in the Hurewicz game on $\mathbb{K}(X)$. So we define a strategy σ for One in the Hurewicz game on $\mathbb{K}(X)$.

Set $\mathscr{F}_0 = \mathscr{F}$. For $K \in K(X)$, let $\mathfrak{f}_0(K) \in [\mathbf{0}; K, 1] \cap \mathscr{F}_0$. Then let $U_0(K) \in \mathscr{T}_X$ be so that

$$K \subseteq U_0(K) \subseteq \mathfrak{f}_0(K)^{-1}((-1, 1)).$$

Let $\sigma_0(\langle\rangle) = \{U_0(K) : K \in K(X)\}$ and $\sigma(\langle\rangle) = \{[U] : U \in \sigma_0(\langle\rangle)\}$, and notice that $\sigma(\langle\rangle)$ is an open cover of $\mathbb{K}(X)$. Also fix a choice function $\gamma_0 : \sigma_0(\langle\rangle) \to \omega$ so that $U \subseteq f^{-1}_{\gamma_0(U)}((-1, 1))$ and observe that $U \mapsto [U]$, $\sigma_0(\langle\rangle) \to \sigma(\langle\rangle)$, is a bijection.

For $k \in \omega$, suppose we have $\langle \mathscr{V}_j^* : j < k \rangle$, $\langle \mathscr{V}_j : j < k \rangle$, $\langle M_j : j < k \rangle$, $\sigma_0(\langle \mathscr{V}_j : j < k \rangle)$, $\sigma(\langle \mathscr{V}_j : j < k \rangle)$, $\langle \gamma_j : j \leqslant k \rangle$, and $\langle \mathscr{F}_j : j \leqslant k \rangle$ defined. For any $\mathscr{V}_k \in [\sigma(\langle \mathscr{V}_j : j < k \rangle)]^{<\omega}$, let $\mathscr{V}_k^* \in [\sigma_0(\langle \mathscr{V}_j : j < k \rangle)]^{<\omega}$ be so that $\{[U] : U \in \mathscr{V}_k^*\} = \mathscr{V}_k$. Also, let $M_k = \max \gamma_k[\mathscr{V}_k^*]$ and $\mathscr{F}_{k+1} = \{f_n : n > M_k\}$.

For $K \in K(X)$, let $\mathfrak{f}_{k+1}(K) \in [\mathbf{0}; K, 2^{-k-1}] \cap \mathscr{F}_{k+1}$. Then let $U_{k+1}(K) \in \mathscr{T}_X$ be so that

$$K \subseteq U_{k+1}(K) \subseteq \mathfrak{f}_{k+1}(K)^{-1}((-2^{-k-1}, 2^{-k-1})).$$

Define

$$\sigma_0(\langle \mathscr{V}_j : j \leqslant k \rangle) = \{U_{k+1}(K) : K \in K(X)\}$$

and $\sigma(\langle \mathscr{V}_j : j \leqslant k \rangle) = \{[U] : U \in \sigma_0(\langle \mathscr{V}_j : j \leqslant k \rangle)\}$. Note that $U \mapsto [U]$, $\sigma_0(\langle \mathscr{V}_j : j \leqslant k \rangle) \to \sigma(\langle \mathscr{V}_j : j \leqslant k \rangle)$, is a bijection. Observe also that $\sigma(\langle \mathscr{V}_j : j \leqslant k \rangle)$ is an open cover of $\mathbb{K}(X)$ and fix a choice function $\gamma_{k+1} : \sigma_0(\langle \mathscr{V}_j : j \leqslant k \rangle) \to \{n \in \omega : n > M_k\}$ so that

$$U \subseteq f^{-1}_{\gamma_{k+1}(U)}((-2^{-k-1}, 2^{-k-1})).$$

This defines a strategy σ for One in the Hurewicz game on $\mathbb{K}(X)$. Since σ is not winning, there is a sequence $\langle \mathscr{V}_n : n \in \omega \rangle$ of choices with a corresponding increasing sequence $\langle M_n : n \in \omega \rangle$ of naturals as described above so that

$$(\forall K \in \mathbb{K}(X))(\exists m \in \omega)(\forall n \geqslant m)\ K \in \bigcup \mathscr{V}_n.$$

Define $\phi : \mathscr{F} \to \omega$ by $\phi(f_n) = \min\{k \in \omega : n \leqslant M_k\}$. We show that this meets the criterion for groupability. Let $J \subseteq \omega$ be infinite and consider an arbitrary neighborhood $[\mathbf{0}; K, \varepsilon]$ of $\mathbf{0}$. We can find $\ell \in J$ so that $K \in \bigcup \mathscr{V}_\ell$, and $2^{-\ell} < \varepsilon$. So there is some $U \in \mathscr{V}^*_\ell$ with $K \in [U]$, which is to say that $K \subseteq U$. By construction, $M_{\ell-1} < \gamma_\ell(U) \leqslant M_\ell$, so $\phi(f_{\gamma_\ell(U)}) = \ell$. Observe further that

$$K \subseteq U \subseteq f^{-1}_{\gamma_\ell(U)}((-2^{-\ell}, 2^{-\ell})).$$

That is, that $f_{\gamma_\ell(U)} \in [\mathbf{0}; K, \varepsilon]$. Therefore, $\bigcup\{\phi^{-1}(n) : n \in J\} \in \Omega_{C_k(X),\mathbf{0}}$.

The implication (iii) $\Longrightarrow$ (iv) is evident.

(iv) $\Longrightarrow$ (i): We will show that $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$ and that $\mathcal{K}^{\text{gp}}_X = \mathcal{K}_X \cap [\mathscr{T}_X]^\omega$, which, by Corollary 33, establishes the implication.

To show that $X \models \mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$, let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of k-covers of X. By Lemma 48, $\langle \mathscr{D}(\mathscr{U}_n) : n \in \omega \rangle$ is a sequence of $\Omega_{C_k(X),\mathbf{0}}$. We can then choose, for each $n \in \omega$, $\mathscr{G}_n \in [\mathscr{D}(\mathscr{F}_n)]^{<\omega}$ so that $\bigcup_{n\in\omega} \mathscr{G}_n \in \Omega_{C_p(X),\mathbf{0}}$. Now, for each $\mathscr{G}_n$, let $\mathscr{H}_n \in [\mathscr{U}_n]^{<\omega}$ be so that, for $f \in \mathscr{G}_n$, there is some $U \in \mathscr{H}_n$ so that $f[X \setminus U] = \{1\}$. We show that $\bigcup_{n\in\omega} \mathscr{H}_n \in \mathcal{K}_X$. So let $K \in K(X)$ and consider the neighborhood $[\mathbf{0}; K, 1/2]$ of $\mathbf{0}$. We can find $n \in \omega$ and $f \in \mathscr{G}_n$ so that $f \in [\mathbf{0}; K, 1/2]$. Then, for $U \in \mathscr{H}_n$ with $f[X \setminus U] = \{1\}$, we see that $K \subseteq U$. Hence, $\bigcup_{n\in\omega} \mathscr{H}_n \in \mathcal{K}_X$.

To show that every countable k-cover of X is groupable, let $\mathscr{U} = \{U_n : n \in \omega\} \in \mathcal{K}_X$. We will define a strategy for One in $\mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X)$.

For each $K \in K(X)$ and $U \in \mathscr{U}$ so that $K \subseteq U$, let $f_{K,U} \in C(X)$ be so that $f_{K,U}[K] = \{0\}$ and $f_{K,U}[X \setminus U] = \{1\}$. It follows that

$$\mathscr{F}_0 := \{f_{K,U} : K \in K(X), U \in \mathscr{U}, K \subseteq U\} \in \Omega_{C_k(X),\mathbf{0}}.$$

By $C_k(X) \models \mathsf{S}_{\text{fin}}(\Omega_{C_k(X),\mathbf{0}}\Omega^{\text{gp}}_{C_k(X),\mathbf{0}})$, we can find $\{g_n : n \in \omega\} \subseteq \mathscr{F}_0$ and an increasing sequence $\langle M_n : n \in \omega \rangle$ of naturals so that $\psi : \{g_n : n \in \omega\} \to \omega$ defined by

$$\psi(g_n) = \min\{\ell \in \omega : n \leqslant M_\ell\}$$

satisfies the condition of groupability. In particular, note that, for any $K \in K(X)$, there exists some $m \in \omega$ so that, for all $n \geqslant m + 1$,

$$[\mathbf{0}; K, 1/2] \cap \{g_\ell : M_{n-1} < \ell \leqslant M_n\} \neq \emptyset.$$

Define $\gamma : \omega \to \omega$ to be so that

$$\gamma(n) = \min\{k \in \omega : g_n[X \setminus U_k] = \{1\}\}$$

and let

$$\sigma(\langle\rangle) = \mathscr{W} := \{U_{\gamma(n)} : n \in \omega\}.$$

Observe that $\sigma(\langle\rangle) \in \mathcal{K}(X)$.

For $k \in \omega$, suppose we have $\langle \mathscr{V}_j : j < k\rangle$, $\langle \nu_j : j < k\rangle$, $\langle \mu_j : j < k\rangle$, and $\sigma(\langle \mathscr{V}_j : j < k\rangle)$ defined. For $\mathscr{V}_k \in [\sigma(\langle \mathscr{V}_j : j < k\rangle)]^{<\omega}$, consider $\Lambda \in [\omega]^{<\omega}$ so that $\mathscr{V}_k = \{U_{\gamma(n)} : n \in \Lambda\}$. Let $\mu_k = \max\{\psi(g_n) : n \in \Lambda\}$ and $\nu_k = \max\{\gamma(n) : n \leqslant M_{\mu_k}\}$. Now, define

$$\sigma(\langle \mathscr{V}_j : j \leqslant k\rangle) = \{U_n : n > \nu_k\} \cap \mathscr{W} \in \mathcal{K}_X.$$

This completes the definition of σ. By Theorem 32, σ cannot be a winning strategy. So there is a play $\langle \mathscr{V}_n : n \in \omega\rangle$ by Two that beats σ. We also have the auxiliary sequences $\langle \nu_n : n \in \omega\rangle$ and $\langle \mu_n : n \in \omega\rangle$ as defined above.

We show that the sequences $\langle \nu_n : n \in \omega\rangle$ and $\langle \mu_n : n \in \omega\rangle$ are strictly increasing. Let $k \in \omega$ and note that

$$\sigma(\langle \mathscr{V}_j : j \leqslant k\rangle) = \{U_n : n > \nu_k\} \cap \mathscr{W}.$$

Now, consider $\Lambda_0, \Lambda_1 \in [\omega]^{<\omega}$ so that $\mathscr{V}_k = \{U_{\gamma(n)} : n \in \Lambda_0\}$ and $\mathscr{V}_{k+1} = \{U_{\gamma(n)} : n \in \Lambda_1\}$. For $\ell \in \Lambda_1$, notice that $\gamma(\ell) > \nu_k$. Since $\nu_k = \max\{\gamma(n) : n \leqslant M_{\mu_k}\}$, we see that $\ell > M_{\mu_k}$. Hence,

$$\psi(g_\ell) > \max\{\psi(g_n) : n \in \Lambda_0\} = \mu_k$$

and so $\mu_{k+1} > \mu_k$. Moreover, the inequalities $\nu_k < \gamma(\ell)$ and $\ell \leqslant M_{\psi(g_\ell)} \leqslant M_{\mu_{k+1}}$ imply that $\nu_k < \nu_{k+1}$.

So we define $\phi : \mathscr{U} \to \omega$ by the rule $\phi(U_n) = \min\{k \in \omega : n \leqslant \nu_k\}$. We show that ϕ satisfies the criterion of groupability. So let $J \subseteq \omega$ be infinite, let $K \in K(X)$ be arbitrary, and consider $[\mathbf{0}; K, 1/2]$. By our definition of $\{g_n : n \in \omega\}$, we can let $j \in J$, $j > 0$, be so that

$$[\mathbf{0}; K, 1/2] \cap \{g_\ell : M_{\mu_{j-1}} < \ell \leqslant M_{\mu_j}\} \neq \emptyset.$$

So let $\ell \in (M_{\mu_{j-1}}, M_{\mu_j}]$ be so that $g_\ell \in [\mathbf{0}; K, 1/2]$ and observe that $K \subseteq U_{\gamma(\ell)}$. Since $\ell \leqslant M_{\mu_j}$, we see that $\gamma(\ell) \leqslant \nu_k$. Moreover, since the interval $(M_{\mu_{j-1}}, M_{\mu_j}]$ was defined in the j^{th} inning according to σ, we know that $\gamma(\ell) > \nu_{j-1}$. Therefore, $\phi(U_{\gamma(\ell)}) = j$ and the proof is complete. □

We end with the single-selection analog to Theorem 49. The proof is omitted as it is nearly identical to the proof of Theorem 49. The key thing to notice is that the transferal of finite-selections above allows one to transfer single-selections in the same way. Also, the use of Corollary 33 in the proof needs to be replaced with Corollary 35.

Theorem 50. *For any Tychonoff space, the following are equivalent:*

(i) $X \models \mathsf{S}_1(\mathcal{K}, \mathcal{K}^{\text{gp}})$
(ii) *One does not have a winning strategy in* $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X^{\text{gp}})$.
(iii) $C_k(X)$ *is Reznichenko and has countable strong fan-tightness.*
(iv) $C_k(X) \models \mathsf{S}_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}}^{\text{gp}})$.

References

[1] N. Alam, D. Chandra, Some remarks on games of certain star selection principles, Acta Math. Hung. 166 (2) (2022) 373–392.

[2] A.V. Arhangel'skii, Hurewicz spaces, analytic sets, and fan tightness of function spaces, Sov. Math. Dokl. 33 (1986) 396–399.

[3] Christopher Caruvana, Jared Holshouser, Selection games and the Vietoris space, Topol. Appl. 307 (2022) 107772.

[4] Steven Clontz, Dual selection games, Topol. Appl. 272 (2020) 107056.

[5] Ricardo Cruz-Castillo, Alejandro Ramírez-Páramo, Jesús F. Tenorio, Hurewicz and Hurewicz-type star selection principles for hit-and-miss topology, Filomat 37 (4) (2023) 1143–1153.

[6] Pratulananda Das, Upasana Samanta, Debraj Chandra, On certain generalized versions of groupability, Topol. Appl. 258 (2019) 47–57.

[7] Pratulananda Das, Upasana Samanta, Debraj Chandra, Some observations on Hurewicz and $\mathcal{I}$-Hurewicz property, Topol. Appl. 258 (2019) 202–214.

[8] Ryszard Engelking, General Topology, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, 1989.

[9] J. Gerlits, Zs. Nagy, Some properties of C(X). I, Topol. Appl. 14 (1982) 151–161.

[10] Henri Hogbe-Nlend, Bornologies and Functional Analysis. Introductory Course on the Theory of Duality Topology-Bornology and Its Use in Functional Analysis. Translated from the French by V.B. Moscatelli, North-Holland Math. Stud., vol. 26, Elsevier, Amsterdam, 1977.

[11] W. Hurewicz, Über eine Verallgemeinerung des Borelschen Theorems, Math. Z. 24 (1925) 401–421.

[12] Winfried Just, Arnold W. Miller, Marion Scheepers, Paul J. Szeptycki, The combinatorics of open covers. II, Topol. Appl. 73 (3) (1996) 241–266.

[13] Ljubiša D.R. Kočinac, Closure properties of function spaces, Appl. Gen. Topol. 4 (2) (2003) 255–261.

[14] Ljubiša D.R. Kočinac, Marion Scheepers, Function spaces and a property of Reznichenko, Topol. Appl. 123 (1) (2002) 135–143.

[15] Ljubiša D.R. Kočinac, Marion Scheepers, Combinatorics of open covers. VII: Groupability, Fundam. Math. 179 (2) (2003) 131–155.

[16] Zuquan Li, Selection principles of the Fell topology and the Vietoris topology, Topol. Appl. 212 (2016) 90–104.

[17] Shou Lin, Chuan Liu, Hui Teng, On fan tightness and strong Fréchet property for $C_k(X)$, Adv. Math., Beijing 23 (3) (1994) 234–237 (Chinese).

[18] V.I. Malykhin, G. Tironi, Weakly Fréchet-Urysohn and Pytkeev spaces, Topol. Appl. 104 (2000) 181–190.

[19] Ernest Michael, Topologies on spaces of subsets, Trans. Am. Math. Soc. 71 (1951) 152–182.

[20] Andrej Nowik, Marion Scheepers, Tomasz Weiss, The algebraic sum of sets of real numbers with strong measure zero sets, J. Symb. Log. 63 (1) (1998) 301–324.

[21] Alexander V. Osipov, A functional characterization of the Hurewicz property, Iran. J. Math. Sci. Inform. 17 (1) (2022) 99–109.

[22] Masami Sakai, Property C'' and function spaces, Proc. Am. Math. Soc. 104 (3) (1988) 917–919.

[23] Marion Scheepers, Combinatorics of open covers. I: Ramsey theory, Topol. Appl. 69 (1) (1996) 31–62.

[24] Marion Scheepers, Combinatorics of open covers. III: Games, $C_p(X)$, Fundam. Math. 152 (3) (1997) 231–254.

[25] Marion Scheepers, A sequential property of $C_p(X)$ and a covering property of Hurewicz, Proc. Am. Math. Soc. 125 (9) (1997) 2789–2795.

[26] Boaz Tsaban, The Hurewicz covering property and slaloms in the Baire space, Fundam. Math. 181 (3) (2004) 273–280.

[27] Boaz Tsaban, Menger's and Hurewicz's problems: solutions from "The Book" and refinements, in: Set Theory and Its Applications, Annual Boise Extravaganza in Set Theory, Boise, ID, USA, 1995–2010, American Mathematical Society (AMS), Providence, RI, 2011, pp. 211–226.