Making "Concreteness Fading" More Concrete as a Theory of Instruction for Promoting Transfer

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Abstract (word count = 149; max = 250)

To promote learning and transfer of abstract ideas, contemporary theories advocate that teachers and learners make explicit connections between concrete representations and the abstract ideas they are intended to represent. *Concreteness fading* is a theory of instruction that offers a solution for making these connections. As originally conceived, it is a three-step progression that begins with enacting a physical instantiation of a concept, moves to an iconic depiction, and then fades to the more abstract representation of the same concept. The goals of this paper are: (1) to improve the theoretical framework of concreteness fading by defining and bringing greater clarity to the terms abstract, concrete, and fading, and (2) to describe several testable hypotheses that stem from concreteness fading as a theory of instruction. Making this theory of instruction more "concrete" should lead to an optimized concreteness fading technique with greater promise for facilitating both learning and transfer.

Key words: Concreteness Fading, Learning Activities, Cognitive Development, Abstract Reasoning, Concrete Representations, Transfer of Training

Making "Concreteness Fading" More Concrete as a Theory of Instruction for Promoting Transfer

In many learning settings, concepts can be presented using a variety of representations, some of which are more concrete than others. For example, in mathematics, a teacher could present the concept of equality using a concrete balance scale or a symbolic equation. In science, a teacher could present the concept of a hydrogen bond using a molecule model set or written symbolic notation. In reading, a teacher could present the concept of the phoneme /t/ using colorful photographs of animals that start with the letter "t" or simple black and white drawings of those same animals. One recommendation for teachers is to make explicit connections between these different representations, particularly between those that are more concrete and those that are more abstract. For example, the Institute of Education Sciences published a practice guide for *Organizing Instruction and Study to Improve Student Learning* in a variety of domains, and one of the key recommendations is to "connect and integrate abstract and concrete representations of a concept" (Pashler et al. 2007, 15). While this recommendation is supported by several peer-reviewed experimental studies, there is a need for a broader range of evidence that demonstrates external validity, such as those conducted in real world learning settings.

Concreteness fading is a theory of instruction that offers a solution for making these connections in a variety of settings – by beginning with physical interactions with concrete instantiations of a target concept, and gradually and explicitly fading toward more abstract representations (e.g., Fyfe, McNeil, Son, and Goldstone 2014; see Figure 1). However, despite increasing empirical and theoretical support (e.g., Bruner 1966; Fyfe et al. 2014; Fyfe et al. 2015; Goldstone and Son 2005; Gravemeijer 2002; Koedinger and Anderson 1998; McNeil and Fyfe 2012; Nathan 2012), concreteness fading remains an underspecified theory of instruction with few explicit design principles for how to implement it as an instructional technique. The goal of this paper is to better specify the central components of concreteness fading as a theory of instruction that has broad application. We aim (1) to expand the theoretical framework of concreteness fading by defining and bringing greater clarity to the terms abstract, concrete, and fading, and (2) to describe six testable hypotheses that stem from concreteness fading as a theory of instruction, which elaborate on the theory and enhance the fidelity of application of the theory for classroom implementation. First, we provide background information on the origins of concreteness fading to highlight gaps in the literature, then we proceed to address our two aims.

Brief Background on Concreteness Fading

Concreteness fading is a theory of instruction intended to facilitate connections among multiple representations of a target concept along a progression from more concrete to more abstract, with the goal of supporting transfer of the concept. For example, the quantity two could first be represented by physical apples, then by a pictorial representation of the apples, then by the symbolic numeral two (Figure 1). Concreteness fading is intended to accomplish at least three goals: (1) facilitate learning by starting with a grounded, meaningful concrete instantiation of the concept that affords physical interactions, (2) promote transfer by ending with a decontextualized, generic representation that can be used across contexts, and (3) draw connections between the representations to help learners interpret them as mutual referents.

A general notion of concreteness fading was put forth by Bruner in 1966. He proposed that learning occurs by going through three stages of representation: an enactive stage, which is action based (e.g. acting on a physical concrete model), an iconic stage, which is image-based (e.g., using a graphic or pictorial model), and a symbolic stage, which is notation-based (e.g., using a word or symbol model; see Figure 1). Bruner’s proposal, among others, inspired mathematics education research on children with learning disabilities using the concrete-representational-abstract (CRA) sequence of instruction (see Miller & Hudson, 2007). The CRA sequence begins with physical objects, transitions to pictorial representations, and ends with math symbols. It is typically conceived of as a developmental learning progression that spans multiple lessons; transitions only occur once students master the material in the previous stage.

More recent work on concreteness fading has not been tied directly to Bruner's three stages; rather, it has been used more broadly as a term that encompasses *any* progression from concrete to abstract, without requiring the initial form to be a physical object, without requiring three distinct stages, and without requiring mastery at each stage (e.g., Braithwaite and Goldstone 2013; Ding and Li 2014; Johnson et al. 2014). For example, Goldstone and Son (2005, 70) define concreteness fading as "the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models." They operationalized this process with only two steps and the transition between steps occurred within a lesson, rather than across multiple lessons over time.

Critical Gaps in the Literature

Importantly, this work (stemming from both developmental psychology and cognitive science) has inspired key educational insights as well as the production of instructional materials that have been shown to support student learning (e.g., Ding and Li 2014; Fyfe et al. 2015; Goldstone and Son 2005). However, its impact in the field of education overall has been somewhat limited because concreteness fading remains an underspecified theory of instruction. There are at least four critical gaps in the literature. First, there is a need to better specify the key properties of "concrete" representations versus "abstract" representations as they apply to the fading progression. Although Bruner (1966) focused on the physicality of the enactive stage, there are additional dimensions on which they vary. Second, there is a need to specify the cognitive experience from the learner's perspective. How does the fading progression change the learner's cognitive interpretation of the representations? Third, there is a need to outline the practical solutions that the fading progression offers from the teacher's perspective. What instructional dilemmas could potentially be resolved with concreteness fading? Fourth, there is a need to describe and test principles for how to implement concreteness fading as an instructional technique. Prior research has tended to compare one possible version of concreteness fading to business-as-usual controls without considering different versions of the fading technique.

The goal of this paper is to help address these gaps via two overarching aims. First, we aim to expand the theoretical framework of concreteness fading by defining and bringing greater clarity to the terms abstract, concrete, and fading. Second, we aim to describe six testable hypotheses that stem from concreteness fading as a theory of instruction. Our first aim should have theoretical implications by specifying key properties of the progression as well as how those properties influence the learner's experience of and the teacher's use for concreteness fading. Our second aim should have practical and theoretical implications. The hypotheses with sufficient empirical support can serve as design principles for practitioners to implement in educational environments. The hypotheses with little or no empirical support can serve as theory-driven questions for future research. This work should help lead to an optimized version of concreteness fading that can facilitate learning and transfer for a broad range of learning settings.

Aim 1: Define Key Terms Relevant for Concreteness Fading

Concreteness fading starts with a representation of a target concept that is more concrete and ends with a representation of the concept that is more abstract. But what does it mean for a representation to be concrete or abstract? To be clear, there are numerous existing attempts to discuss this broad notion of "concrete versus abstract" using varying approaches and terms (e.g., Goldstone and Son, 2005; Kaminski, Sloutsky, and Heckler 2013; Koedinger, Alibali, and Nathan 2008). Our goal is not to add new terms to the mix, but rather to unify this literature by reviewing the terms used and providing clear definitions as they apply to concreteness fading.

What is an Abstract Representation?

Typically, in the context of learning and education, the term *abstract* is associated with formalisms, which are "heavily regulated notational systems that have no inherent meaning except those that are established by convention" (Nathan, 2012, 125). For example, the Arabic numerals (e.g., 1, 2, 3) and operational symbols (e.g., +, –, ÷) used in mathematics are often referred to as abstract symbols; they are often perceptually sparse, two-dimensional, arbitrarily related to their assigned meaning, and used to represent quantities and operations across a wide array of contexts (e.g., in an equation, at the grocery, on a scoreboard).

Indeed, previous work has often equated abstract representations with formalisms or defined abstract representations only as they relate to concrete representations. For example, in Koedinger and Nathan (2004), the term abstract is used to refer to symbolic equations that contain formalisms (e.g., $6x + 66 = 81.90$). Fyfe et al. (2014, 9) define the term abstract by describing typical features of abstract representations, stating that they "eliminate extraneous perceptual properties, represent structure efficiently, and are more arbitrarily linked to their referents." Kaminski and colleagues (2009) define the term abstract only indirectly via comparison with concrete representations. They state that "concrete instantiations communicate more information than their abstract, generic counterparts" (151). Here, we attempt to unify this body of research by outlining several defining features of abstract representations and offering an explicit definition. Note that we are referring specifically to external representations of abstract ideas, not “internal” mental representations or the abstract ideas themselves.

Koedinger and colleagues (2008) provided a systematic contrast between abstract representations and “grounded” representations (i.e., those that are most similar to physical objects and everyday experiences). Abstract representations, as compared to grounded representations, have fewer benefits for improving access to long-term memory (less familiar), have lower reliability (less redundancy), greater support for working memory (higher externalizability), and greater efficiency (more concise). The authors suggest that while using abstract representations during math problem solving is often more error prone than grounded representations, concise presentation of abstract representations makes them more efficient to use, and more externalizable (we can easily write them down), which together, places fewer demands on working memory than grounded representations.

However, at their core, external abstract representations present a conundrum, since they are, in actuality, concrete, necessarily depicted in this physical world (e.g., Nathan 2012). For example, even purely symbolic equations must be made concrete so they can be depicted with a certain font of a certain size in a way that takes up space, offers visual contrast, and is registered by our perceptual system in a multitude of ways. Indeed, Landy and Goldstone (2007) have shown that something as concrete as the visual spacing of an equation (e.g., $2 + 3 \quad \times 4$ versus $2 + \quad 3 \times 4$) can influence learners’ solutions, despite being mathematically irrelevant. Similarly, Lupyan (2016) has shown that the spatial orientation of very basic images of triangles – often considered the epitome of abstract representations – influences people’s judgments of them as triangles. Thus, in a sense, the term abstract representation is an oxymoron as all representations have concrete, perceptual features that influence the ways we think about them.

When we refer to an abstract representation, what exists is a concrete representation that is more idealized or more generic than others. That is, there is an abstract*ion* – something that exists only as an idea (e.g., triangle-ness) – that needs to be depicted in some way. This is at the heart of distinction in linguistics (de Saussure 1959) between the *signifier*, which presents the perceivable signs used for representation (e.g., a printed word), and the *signified*, which specifies the underlying referential meaning (e.g., the meaning of the word). While certain representations get "close" to the abstraction or idea (i.e., they are more "*idea*lized"), they are still inherently concrete, with the mapping between the representation in the form of the signifier and the object of signification mediated by processes performed by the *interpretant* (Peirce 1998). Because the relation between the signifier and signified object is often completely arbitrary, the connection must be socially mediated and scaffolded for the interpretant to infer the appropriate translation.

To avoid the conundrum posed by abstract representations, from this point forward we abandon the term "abstract representation" as, we believe, no such externalized representation exists; instead, we adopt the term idealized representation (see also Goldstone and Son 2005) to refer to the final stage of the concreteness fading progression. We highlight the fact that the term "idealized" is relative, not dichotomous (see also Belenky and Schalk 2014; Fyfe et al. 2014; Kaminski et al. 2009), as some representations can be more idealized than others.

What is a more idealized representation? The key notion is that there is an underlying idea or relation – the signified object – that needs to be depicted, and even though it can be depicted or represented in various ways, the idea itself remains common across these varying representations. Thus, even if the external representation, or signifier, itself changes (e.g., from a black 2 to a green II), there is an invariant relation that does not get lost as you move across dramatically different surface features (Nathan et al. 2013). An idealized representation is just a form of concrete representation that makes this invariant relation salient across a broad class of contexts. That is, a highly idealized representation makes it easier to see, infer, and focus on this invariant relation relative to a less idealized representation. It is in this way that idealized representations can support and foster generalization of deep structure relationships and promote transfer. Consequently, *we define an idealized representation as a depiction that involves the least effort of an agent to infer the invariant relation as part of generalization and transfer*. In other words, an idealized representation makes it easy to think past its specific depiction (see also Palmer 1978; Rosch 1978) and transfer the invariant relation to new depictions and contexts. By operationalizing this for generalization and transfer, this definition applies to any interpretant, be it an expert or a novice, and underscores the well-documented finding that novices will typically do worse with idealized representations even when they are capable of thinking about the underlying ideas being signified (e.g., Koedinger and Nathan 2004).

This definition makes two contributions. First, it expands Bruner's (1966) conception of the symbolic stage to think past what the depiction *is* (e.g., words and mathematical symbols) to include what the depiction *does* (e.g., makes it easy to infer the invariant relation). Second, this definition suggests there are certain qualities that will commonly be associated with an idealized representation. For example, for any given depiction, one can ask what type of information *can* be inferred and what type of information can be inferred *easily*. An idealized representation will have few pieces of information that can be inferred and the most easily inferred piece will often be the central, invariant relation of interest. By "most easily," we mean to acknowledge that ease is a complex interaction of the agent, the external representation, and the socio-historical context in which the encounter occurs. This means it will likely be: familiar to the agent, contextually relevant, perceptually apprehensive (and often sparse) to reduce the type of information available, and generic, in the sense that it can be used across several different contexts. For example, imagine depicting an equivalence relation using a physical balance scale or a symbolic equation. A lot of information can be inferred from the balance scale (e.g., color, weight, shape, size, equivalence, quantities, distance). Further, the equivalence relation may not be the most salient for every learner in every context. In contrast, less information can be inferred from the equation (e.g., equivalence, font, quantities), and, for the knowledgeable person operating with a social need for interpreting the symbol structure, the equivalence relation is salient. Thus, the symbolic equation is considered a more idealized representation of equivalence.

What is a Concrete Representation?

As all external representations are concrete, by virtue of their externalization, what do we intend the term “concrete representation” to signify? How should it distinguish itself? As noted, concrete representations vary in terms of how idealized or generic they are. A more idealized representation gets “closer” to the core idea or invariant relation. A less idealized representation adds information in a way that increases the effort needed to generalize the invariant relation. Koedinger and colleagues (2008, 367) referred to these less idealized representations as “grounded,” because they “refer to physical objects and everyday events.”

Importantly, the type of information or concreteness that is added to the idealized representation can vary. That is, depictions can be considered less idealized on several dimensions. As such, there is not a single continuum, but many. Here, we attempt to expand on Bruner’s (1966) original attention to the physicality of the depiction to specify the four most common types of information on which concrete representations vary (Figure 2).

One important distinction is whether the information is perceptual or conceptual in nature, as discussed in Kaminski, Sloutsky, and Heckler (2013). Perceptual information is related to the physical properties of the depiction registered by our perceptual system. Two common types of perceptual information that vary across representations include physicality and perceptual richness. Conceptual information is related to the knowledge activated by the learner. Two common types of conceptual information that vary across representations include the familiarity of the representation to the learner and the narrative context in which it is embedded.

Physicality refers to whether the representation is two-dimensional (e.g., a drawing on paper) or three-dimensional (e.g., counting blocks). Three-dimensional objects are often referred to as manipulatives and include most physical objects used to help students understand key concepts (e.g., balance scale, molecule model set, base-ten blocks). For example, Martin and Schwartz (2005) compared students' abilities to understand fractions using actual 3D physical tiles or just 2D pictures of tiles. In mathematics, meta-analyses continue to show that instruction with manipulatives is often more effective than instruction without manipulatives, though there is variability across studies (Carbonneau, Marley, and Selig 2012). The primary advantage of 3D representations is the affordance of real or imagined action (e.g., Glenberg et al. 2004) and the ability to interact with the materials in a way that facilitates the construction of knowledge.

Perceptual richness often refers to the visual surface features of the signifier, such as the color or pattern of the depiction or how "real" it looks. For example, McNeil et al. (2009) compared students' abilities to solve word problems using perceptually rich bills and coins designed to look like real money (i.e., green bills with a face in the center and words/signatures in various places) or perceptually blank bills and coins (i.e., a white piece of paper with a 1 written in the center in plain black ink). Using perceptually rich materials alone is often not a very effective technique and can sometimes reduce learning and performance relative to less perceptually rich materials (e.g., Berends and van Lieshout 2009; Kaminski et al. 2013; Scheiter et al. 2009; Son, Smith, and Goldstone 2011). The potential drawback is related to the seductive details effect (e.g., Harp and Mayer 1998) in which captivating, but irrelevant features draw attention away from the to-be-learned idea or invariant relation.

Other information is more conceptual in nature and often more dependent on the individual learner's knowledge and the context. For example, familiarity can increase the effort it takes to generalize the underlying idea or principle. Consider reading a story problem about measuring the length of "an object" versus measuring the length of "a pen." A person who uses pens may activate prior knowledge that may be relevant or irrelevant (e.g., pens are discrete objects, pens are around six inches long, pens come in lots of colors, my red pen exploded yesterday). For example, Petersen and McNeil (2013) examined the effect of familiarity by comparing children's abilities to count with objects they had experience with (e.g., toy animals, pencils) versus objects that were novel (e.g., small blue discs, pinwheel tops). Familiarity can make it difficult to think about the depiction symbolically – not just as a toy for example, but also a representation of a math concept (e.g., Uttal, Scudder, and DeLoache 1997).

Another type of conceptual information is the narrative context in which the representation is embedded. For example, Koedinger and Nathan (2004) compared students' performance on arithmetic and algebraic word problem with and without a narrative context that were matched on their underlying invariant quantitative relations. Contextual problems included a full narrative (e.g., "When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66.00 he received in tips.), while the non-contextual "word equation" problems did not (e.g., "Starting with 81.9, subtract 66."), but both were similar in perceptual concreteness (i.e., printed sentences on paper). Contextual information can sometimes be helpful by promoting more intuitive strategies and reasoning as opposed to the application of memorized algorithms (e.g., Carraher, Carraher, and Schliemann 1985). Koedinger and Nathan (2004) were also able to show that the familiarity of language itself was beneficial *above and beyond* the context afforded by the narrative of story problems. Students solving story problems *and* word-equations were more likely to engage in successful "invented" solution strategies than when they solved symbolic equations matched on their invariant relations (Koedinger and Nathan 2004). Further, when students were given story and word-equation problems, they made fewer sense-making errors, apparently because of the familiar nature of the language-based formats.

As these cases illustrate, representations cannot easily be binned into "concrete" or "abstract." Rather, all external representations are concrete representations, but vary in terms of how idealized they are. Further, the information they convey can be either perceptual or conceptual. Teasing apart these different types of information (Figure 2) is key to thinking about the types of external representations we use during instruction, how we connect these different representations, and ultimately, how we learn and generalize from them.

Moving forward, we use the term concrete representation to refer to any external representation. We distinguish types of concrete representations by referring to them as more or less idealized. A fully "idealized representation" (Goldstone and Son 2005) is optimized for apprehending invariant relations when stripped of added information. We acknowledge that this is still somewhat unsatisfactory (e.g., one representation could be *more* idealized on a perceptual dimension but *less* idealized on a conceptual dimension), but we believe this highlights the true nature of external representations as concrete representations that vary in the amount of information they convey to a person interacting with them for some contextualized purpose.

What is Concreteness Fading?

Concreteness fading is a theory of instruction intended to facilitate connections among multiple representations along a progression. It represents one specific technique that falls under the broader notion of progressive formalization (e.g., Freudenthal 1983; Romberg 1998), which encompasses pedagogical methods that draw on students' initial, informal experiences with specific situations and objects to support development of a grounded interpretation of formal representations and notation for later generalization (Nathan 2012). We *define concreteness fading as the three-step progression by which a concrete representation of a concept is explicitly faded into a generic, idealized representation of that same concept.* "Fading" is accomplished by removing perceptual and conceptual information either within or across lessons until one arrives at the representation that involves the least effort to infer and generalize the invariant relation. This definition maintains the three-stage nature of Bruner's (1966) sequence, but is somewhat more expansive. For example, it does not equate concrete with enactive, physical models, but leaves open the possibility that the initial representation is concrete on several different dimensions. It also specifies how to "fade" from one representation to the next, similar to the "decontextualization" process discussed in Goldstone and Son (2005). Finally, it incorporates our definition of an idealized representation, which specifies the final representation's function.

Different concrete representations offer varying advantages. For example, concrete representations that contain more perceptual and conceptual information often support initial learning because they provide a grounded, meaningful context in which to reason (e.g., Glenberg et al. 2004; Schliemann and Carraher 2002). In contrast, more idealized representations reduce the number of extraneous surface features and, because of their generic applicability, promote transfer across a range of novel contexts (e.g., Kaminski et al. 2008; Son and Goldstone 2009). Concreteness fading is offered as a progression that capitalizes on the initial advantages of concrete materials and explicitly fades to more idealized materials to promote transfer.

We contend that the key fading stage has critical implications for the learner's cognitive interpretations of the representations. Specifically, the goal of the fading process is not to explicate the similarities and differences across the varying representation, but rather to link the representations as mutual referents by aligning them on the continuum and presenting them in a specified order. Because of this, there is no need to hold the representations in mind simultaneously. Cognitively, the experience should be of one representation fading or morphing into the next one, as if they were one and the same, simply donning and shedding new clothing.

Consider a potentially parallel scenario in cognitive development. DeLoache (2000) ran a series of experiments investigating children's use of symbols. In the symbolic task, children watch as an experimenter hides "big snoopy" in a room. The experimenter explains that "little snoopy" is hiding in the same place in a scale model room, which is smaller but otherwise identical. Children use the scale model as a symbol of the big room to determine the location of "little snoopy." In a clever nonsymbolic version of the task (DeLoache, Miller, and Rosengren 1997), children were convinced that the big room became the scale room via a magical shrinking machine. In this case, children did not need to use the scale model as a symbol for the big room; instead, it *was* the big room. The invariant relation of relative spatial locations of objects in the two models became clear. Two-and-a-half-year-old children who completed this nonsymbolic version were much more successful than same-aged children who completed the symbolic version even though the task itself was identical. The shrinking machine offers a progression of the scale model from object to symbol, rather than its coexistence as both object and symbol.

In a similar way, concreteness fading is an attempt to fade or morph a less idealized representation of a concept into a more idealized or generic representation of the same concept.

The goal is to think of them as one and the same, each an instantiation of a common invariant set of relations, rather than hold the different representations in mind at the same time.

Given this experience of the learner, concreteness fading has the potential to address both sides of a key instructional dilemma: the "concreteness" side and the "abstractness" side. To understand the *concreteness side of the dilemma*, imagine one teacher who frequently uses physical manipulatives in her classroom (e.g., counting bears, fraction tiles). Her dilemma is that her students do well working with the manipulatives, but they often fail to solve problems of similar difficulty when the manipulatives are not available. In other words, her students fail to transfer the knowledge they learned to novel situations – their knowledge is inert. The teacher decides to try and use the manipulatives more effectively by making explicit connections with more idealized, symbolic representations. Concreteness fading is a solution to the concreteness side of the dilemma because it is one way to make manipulatives more effective and connected to more idealized, transferrable representations.

To understand the *abstractness side of the dilemma*, imagine a second teacher who frequently has her students practice solving symbolic arithmetic problems. Her dilemma is that her students can easily detect a similar problem and even name the appropriate procedure, but they often do not understand why the procedure works and they routinely make illogical errors – their knowledge is not meaningfully grounded. Wertheimer (1959) described this as "unproductive thinking," as illustrated with his infamous example of the solution to the area of a parallelogram. Many students exhibit unproductive thinking when asked to find the area of a parallelogram with a novel shape or orientation because the standard procedure does not work. To address students' needs, the teacher decides to try and make the symbols and procedures more meaningful. Concreteness fading is a solution to the abstractness side of the dilemma because it offers a way to ground or ascribe meaning to arbitrary, idealized symbols.

In sum, in line with our first aim, we expanded the theoretical framework of concreteness fading by defining and bringing greater clarity to the terms abstract, concrete, and fading. Our definitions both build on and expand some of the original work on concreteness fading, particularly the work of Bruner (1966). In doing so, we specified the key properties of the representations that comprise the fading progression, we theorized the ways in which the progression of representations influences the learner's cognitive experience, and we outlined the practical solutions that the progression can offer to teachers facing instructional dilemmas.

Aim 2: Describe Six Testable Hypotheses for Researchers

It is one thing to state what concreteness fading is, but another to prescribe how to use it to promote generalization and transfer of complex ideas. Currently, there are few specific, theory-driven recommendations for implementing it as an instructional technique. Given the definition of concreteness fading presented above, we outline six testable hypotheses for the research community that should support future theory-driven research and inform principles for faithfully implementing concreteness fading during instruction (see Table 1).

Hypothesis 1

We hypothesize a three-step progression will be more effective than any two-step fading progression, and in turn, a two-step progression will be more effective than using a single representation. To truly fade or decontextualize from a less idealized representation to a more idealized representation, the invariant relation should be depicted in three stages. With three steps, one experiences the key "fading" stage, which maintains some correspondence to the initial concrete representation, but also starts to strip away the extraneous perceptual detail. In this way, a three-stage progression allows for a more gradual, explicit decontextualization, which can help the learner better see the “morph” from one representation to the next. By strengthening the learner’s awareness of what is changing along the progression, it also helps to make salient that which is invariant – and critical for promoting generalization of key information.

Several experiments suggest that a two- or three-step progression is more effective than using only a concrete or only an idealized representation (e.g., Fyfe et al. 2015; McNeil and Fyfe 2012; Peterson, Mercer, and O’Shea 1988). For example, Goldstone and Son (2005) had college students learn about a key scientific concept, competitive specialization, from computer simulations that varied in perceptual richness. The concrete representations were images of colorful ants foraging for fruit, whereas the idealized representations were images of black dots and green shapes. Using a two-stage progression from perceptually rich to generic resulted in better transfer than using only perceptually rich representations or only generic representations.

However, no study that we are aware of has experimentally compared a three-stage to a two-stage progression. An anecdotal comparison across two studies suggests three stages may be more effective (Figure 3). Kaminski and colleagues (2008) had participants learn modular arithmetic using a generic representation (solid black shapes), a perceptually rich representation (images of measuring cups), or a rich representation followed by a generic representation (a two-stage version of concreteness fading). Participants who learned only with generic representations exhibited better transfer than those in the two-stage fading group. McNeil and Fyfe (2012) conducted a follow-up study using identical materials; the key exception was that the fading condition included all three stages (see Figure 3). They found that the three-step fading condition resulted in the best transfer immediately and three weeks later providing empirical support for the full, three-step progression. However, experimental research is needed to directly test this hypothesis, particularly with younger students in typical classroom settings.

Hypothesis 2

We hypothesize the order of the progression will matter: a concreteness fading progression that becomes more idealized will be more effective than a progression that becomes less idealized. By definition, the concreteness fading progression should begin with a representation that contains the most information and end with a representation that is most idealized. In theory, the goal is to start with a representation that facilitates initial access into the learning progression. That way, the first experience is grounded and meaningful and the target concept or relation is understood in a specific (often familiar) context. Then, the goal is to end with a more generic representation that is portable across different contexts. That way, the initial understanding can be transferred across those different contexts as well.

Note, this contrasts with an effort-based account, which suggests minimizing the effort required to generalize the concept right away. Several researchers argue that generic representations should be used to introduce concepts because they reduce the amount of extraneous information and therefore reduce the effort needed to generalize the core concept (e.g., Kaminski et al. 2008; Uttal et al. 1997). Many math teachers, for example, believe that students will perform better when new topics are introduced via symbolic representations, arguing that these idealized representations are “pure” and lack distractions and ambiguity (Nathan and Koedinger 2000). Textbooks also propagate this view by emphasizing symbol-based problems prior to verbal or pictorial scenarios (Sherman, Walkington, and Howell, 2016). However, we argue that there are certain types of information (e.g., physicality, familiarity) that will add effort, but are beneficial to include initially to activate students' knowledge of the target concept in a meaningful way.

Some direct evidence for this hypothesis comes from Fyfe et al. (2015), in which elementary school students learned about math equivalence problems using a progression from physical objects to pictorial worksheets to equations (the concreteness fading treatment, see Figure 4) or a reverse progression (from equations to worksheets to objects). Students' exposure to the materials was identical except for the order in which they were presented. Students in the concreteness fading condition solved more transfer problems correctly than did children in the other condition. This hypothesis has additional support from several studies that have demonstrated the benefits of conditions that start with less idealized materials relative to conditions that start with more idealized materials (e.g., Goldstone and Son 2005; Koedinger and Anderson 1998; Ottmar and Landy 2017; Sherman and Bisanz 2009). However, there is a need for a broader range of evidence that demonstrates external validity, such as those conducted in real world learning settings. Although still tentative, this hypothesis has some support to consider it an instructional principle for teachers. That is, when using multiple representations, teachers should use a progression that becomes more idealized, rather than less idealized.

Hypothesis 3

We hypothesize a concreteness fading progression that includes a physical manipulative in the first stage will be more effective than one that does not. As recommended by Bruner (1966), one should include actual physical objects in the initial stage of the concreteness fading progression. This is in line with embodied accounts of cognition, which suggest that high-level cognitive processes stem from action and perception (e.g., Barsalou 2008; Wilson 2002). There are several theoretical benefits of using physical materials. Physical materials can activate real-world knowledge, allowing learners to draw on their prior experiences and practical knowledge of the materials (e.g., Baranes, Perry, and Stigler 1989; Carraher et al. 1985), they can induce physical action, which can improve memory and understanding (e.g., Glenberg et al. 2004; Martin and Schwartz 2005), and they can stimulate the development of abstract reasoning (e.g., Piaget 1970). If it is not possible or feasible to include a physical object, one should select an initial concrete representation that resembles a physical model in order to induce imagined action (Glenberg et al. 2004) or at least remind learners of the affordances of the object.

To date, only one study has compared a fading progression with and without a physical model. Butler and colleagues (2003) investigated the effects of teaching middle school students with mathematics disabilities about fractions over 10 lessons. For half of the students, the first three lessons included physical manipulatives, the next three lessons included pictorial representations, and the final four lessons focused on symbolic fraction problems. For the other half of the students, the first and second set of three lessons included pictorial representations and the final four lessons focused on symbolic problems. Students who worked with manipulatives during the first stage outperformed the other group. However, the comparison group was only exposed to two types of representations (pictorial and symbolic) and the experimental group was exposed to three (physical, pictorial, and symbolic). To truly test this hypothesis, future research should compare groups who each engage with three distinctive representations that become more idealized, but only one group begins with a physical model. This novel test of the unique effects of physicality would provide a new contribution linking kinesthetic and conceptual processes.

Another consideration is if the use of physical manipulatives encourages interactions among peers and with the teacher, or if the manipulatives are used in a constructive (intrapersonal) but non-interactive manner. Studies that compare manipulative use in classroom settings have not always been attentive to the role that social interactions with the manipulatives can serve in promoting understanding. Chi and Wylie (2014, 228) note, however, that interactive (inter-personal) modes are likely to show greater learning benefits than individual constructive modes because "novel and innovative ideas and perspectives may emerge that neither peer originally knew and could not have generated while working independently." Thus, we suggest as part of the first stage, that future investigations also consider whether the physical manipulatives are used constructively (but independently), or interactively (and collaboratively).

Hypothesis 4

We hypothesize learners will remember different aspects of the original representation based on the idealized representation to which it was faded. Any given representation will inevitably convey more information than just the target concept and could be faded to or from in numerous ways. Palmer (1978) provides an illustrative example. Imagine you have four boxes of varying heights and widths (see Figure 5). There are numerous ways you could decide to represent those boxes in a more idealized way (i.e., in a way that is more generic and could be applied beyond those specific boxes). One may preserve the height differences, one may preserve the width differences, and one may preserve the total area differences. Although this is not a comprehensive example of the entire fading progression, it does highlight the fact that a representation may be "faded to" or "faded from" in several different ways.

Imagine one learner engages with a fading progression that focuses on height and the resulting idealized representation strips away all information other than height (width, area, perimeter, etc.). Another learner engages with a fading progression that focuses on width. We would expect the first learner to better recall the height differences of the original rectangle stimuli than their width differences. But, we would expect the opposite pattern for the second learner. Generally, we expect that the learner's experience with the original representation will most profoundly influence their encoding of later representations (Tulving & Thomson 1973). This hypothesis offers a novel cognitive claim about the retrospective impact that idealization can have on encoding and recall of previously presented stimuli.

Hypothesis 5

We hypothesize that presenting the three stages one at a time in the specified order will be more effective than presenting the stages all at once or presenting any two of the three simultaneously. This is one of the features of concreteness fading that differentiates it from other approaches. For example, one common approach to connecting multiple representations is to engage in direct comparison or coordination, which involves the simultaneous presentation of examples and an explicit mapping process that highlights similarities and differences (e.g., Nathan et al. 2013; Rittle-Johnson and Star 2011; Son, Smith, and Goldstone 2011). Concreteness fading may support spontaneous comparison across representations, but the real goal is to think of the representations as one and the same, each an instantiation of a common invariant set of relations. If the different representations were presented simultaneously, it would reinforce their individuality and their distinctiveness (Arnheim 1954). By presenting them sequentially, it should reinforce the idea that they are mutual referents of the same concept. Although this hypothesis contrasts with several pieces of evidence that suggest simultaneous viewing and comparison can promote learning more than sequential viewing of the same stimuli (e.g., Rittle-Johnson and Star 2011; Son et al. 2011), none of these studies have been conducted using a concreteness fading paradigm in which the representations vary systematically on the dimension of concreteness. Establishing this hypothesis within a concreteness fading progression would be a novel contribution to our understanding of cognitive development.

Hypothesis 6

We hypothesize a fading sequence in which each successive stage explicitly references the previous stage will be more effective than a fading sequence in which no stage references the previous one. Although the stages should be presented sequentially, they should still be linked in some way. By linked, we mean referenced. For example, in Fyfe et al. (2015), children first learned about equivalence via a sharing game with monkey puppet and frog puppet (Figure 4). Then, they were told that they were going to do the same problem, but this time on paper with pictures of monkey and frog. Thus, participants were explicitly made aware that their experience with the preceding representation was relevant for their experience with the next one.

One of the difficulties in working with multiple representations is not knowing or understanding when two non-identical representations are depicting the same construct (e.g., Ainsworth, Bibby, and Wood 2002), particularly if they are not presented simultaneously. Although the gradual decontextualization should provide clues as to the relation between the representations in concreteness fading, it should be made explicitly clear to the learners that the representations are related and that they represent the same idea. By being explicit about the links, this should reduce or eliminate students' search for relevant prior experiences or representations and therefore reduce cognitive load (e.g., Sweller et al. 1998). When teachers do this organically during classroom instruction, it has been referred to as "projection" (Engle 2006; Nathan et al. 2013), thus it has some external validity. However, this hypothesis has yet to be tested under carefully controlled experimental settings with strong internal validity.

In sum, in line with our second aim, we put forth six hypotheses regarding the concreteness fading progression, which are intended to serve as theory-driven questions for future research and to contribute to potential design principles for implementing concreteness fading in practice. It is critical to note that these hypotheses center on the components of concreteness fading and ways to optimize them, not necessarily on how concreteness fading compares to other instructional theories and methods. We certainly acknowledge that there are multiple, effective ways to connect representations during instruction and that concreteness fading may not be the most beneficial in all cases (see for example Jaakkola and Veermans 2017; Johnson, Reisslein and Reisslein 2014; Tapola, Veermans and Niemivirta 2013). Indeed, we agree with Bruner (1963, 530) when he wrote, “with respect to the sequence in which material is presented, different sequences are obviously needed to achieve different objectives. The idea of one right sequence is a myth.” Thus, although concreteness fading is a good technique for fostering transfer, it is certainly not the only one.

Even within concreteness fading, there is a need to consider the generalizability of our predictions across age groups, outcomes, and topics of instruction as well as the role of individual differences and social versus independent learning contexts. For example, most research on concreteness fading has been in the domains of math and science, both of which rely heavily on traditional mathematical symbols. Although it is certainly the case that learning in most domains includes multiple representations and examples that vary in concreteness, the fading progression may be more relevant for domains that require learners to master an arbitrary, but conventional system of symbolic notation. Also, the fading progression may be more relevant for young learners who have not yet mastered symbolic notation. For example, preschool children may benefit immensely by starting with physical, grounded materials that help ascribe meaning to the arbitrary symbols; but, high schoolers who have worked with the symbolic notation for years may have a lower need for the initial grounding. Similarly, within an age group, individual differences in knowledge and experience may influence whether and how concreteness fading is most effective. Potentially, those with higher prior knowledge may only need a two-stage progression whereas novices may benefit more from a comprehensive three-stage progression. Most of the research has also emphasized independent learning experiences and assessment, though interactions with different representations can be very different in a collaborative setting. Future research should continue to evaluate these and other hypotheses regarding the role of concreteness fading in learning and transfer, paying attention to the generalizability of the findings across domains, ages, social setting, and learner differences.

Conclusion

Concreteness fading is a promising theory of instruction for fostering transfer by making connections between multiple representations. The theory posits *the three-step progression by which a concrete representation of a concept is explicitly faded into a generic, idealized representation of that same concept.* Concreteness fading is intended to facilitate initial learning by starting with a grounded, meaningful representation and to support transfer by moving toward competency with a decontextualized, idealized representation. In this paper, we expanded the theoretical framework of concreteness fading by defining relevant terms, theorizing how the fading progression influences the learner's cognitive experience, outlining the practical solutions that the fading progression can offer, and generating six testable hypotheses to motivate theory-driven research and to inform potential principles for implementation by practitioners.

Given the recommendation to foster connections during learning, it is imperative to design and test instructional techniques for doing so. This work has the potential to contribute to an optimized, theory-driven version of concreteness fading that enables teachers and designers to more thoughtfully integrate concreteness fading in their learning environments. These definitions and hypotheses are also offered to clarify how concreteness fading theory sheds light on the nature of learning and transfer, thus making concreteness fading more concrete.

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Table 1

Six testable hypotheses that should support research on concreteness fading

1	A three-step progression will be more effective than any two-step fading progression, and in turn, a two-step progression will be more effective than using a single representation.
2	The order of the progression will matter. A concreteness fading progression that becomes more idealized will be more effective than a progression that becomes less idealized.
3	A concreteness fading progression that includes a physical manipulative in the first stage will be more effective than one that does not.
4	Learners will remember different aspects of the original representation based on the idealized representation to which it was faded.
5	Presenting the three stages one at a time in the specified order will be more effective than presenting the stages all at once or presenting any two of the three simultaneously.
6	A fading progression in which each stage explicitly references the previous stage will be more effective than a fading progression in which no stage references the previous one.

Figure 1

An example of the three-step concreteness fading progression

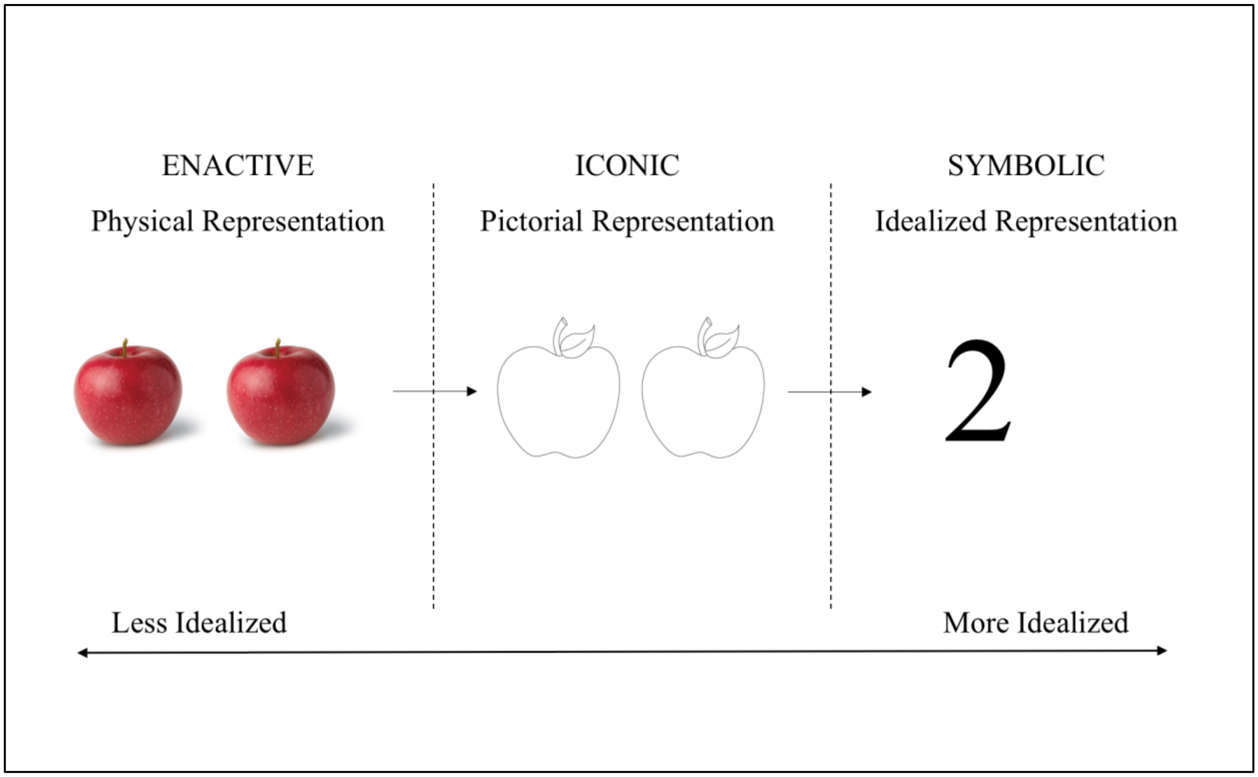


Figure 2

External representations that vary in the amount of perceptual and conceptual information

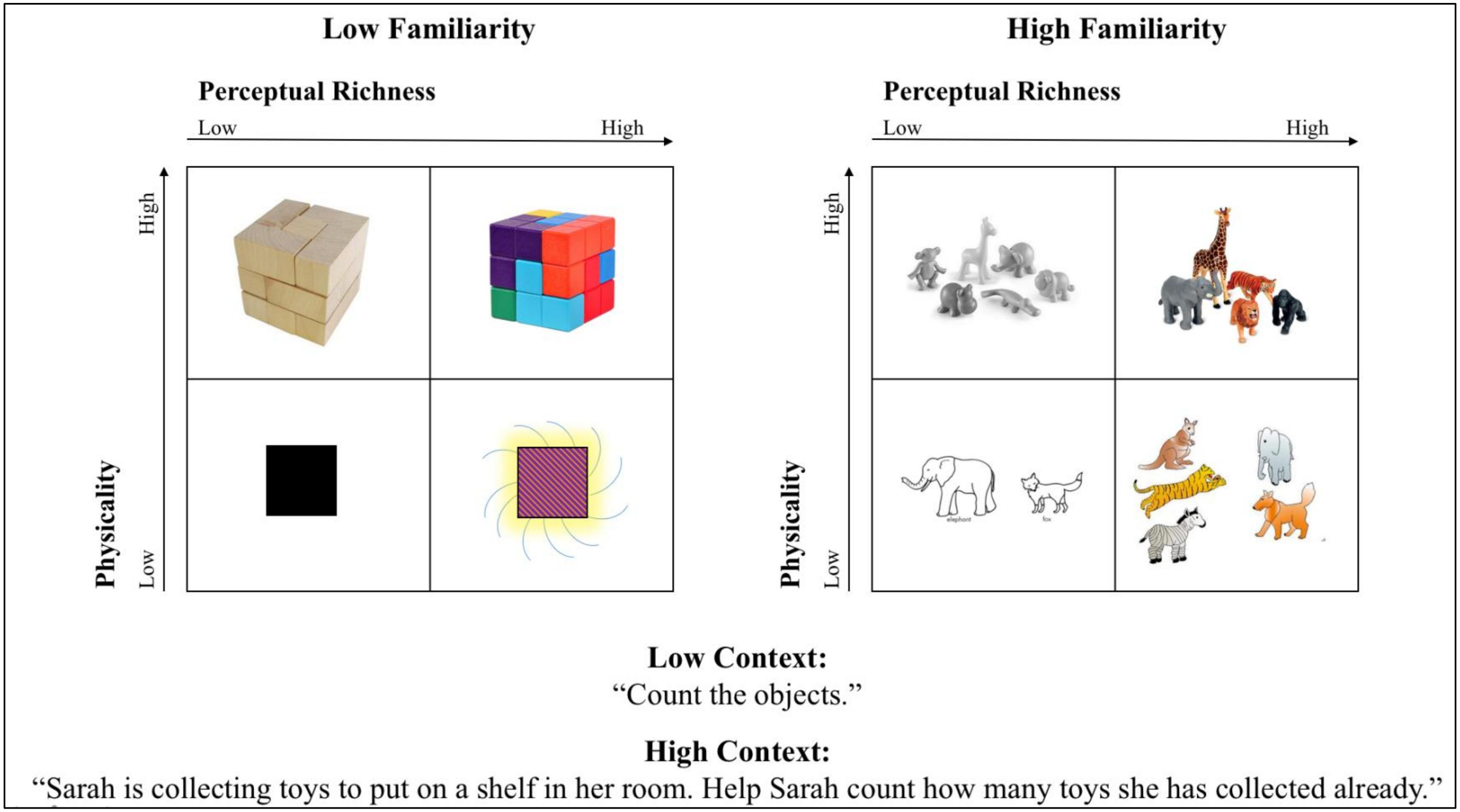


Figure 3

A comparison of the representations used in Kaminski, Sloutsky, and Heckler (2008) and in McNeil and Fyfe (2012)

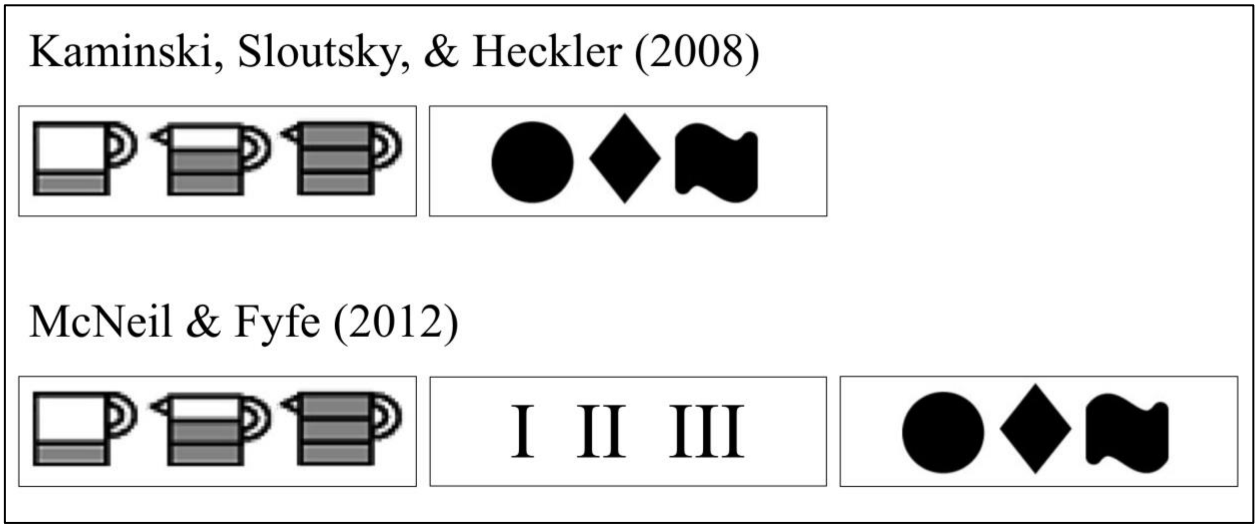


Figure 4

Representations used in the concreteness fading condition in Fyfe, McNeil, and Borjas (2015)

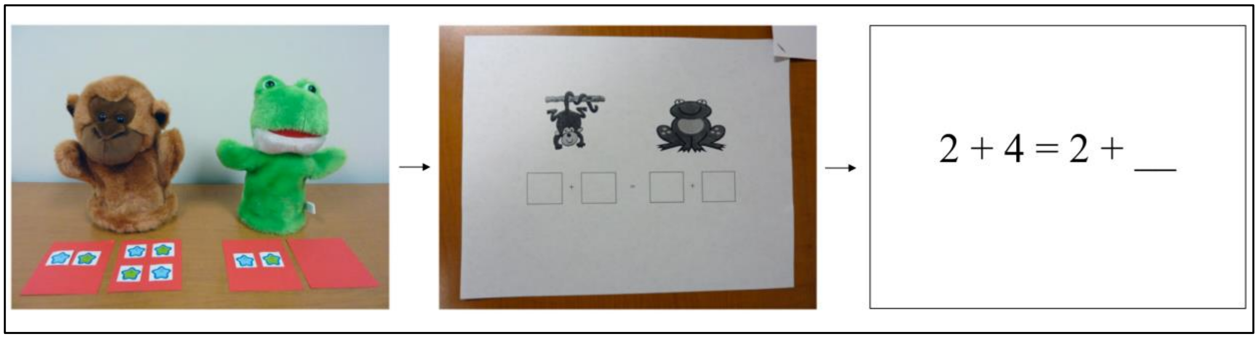
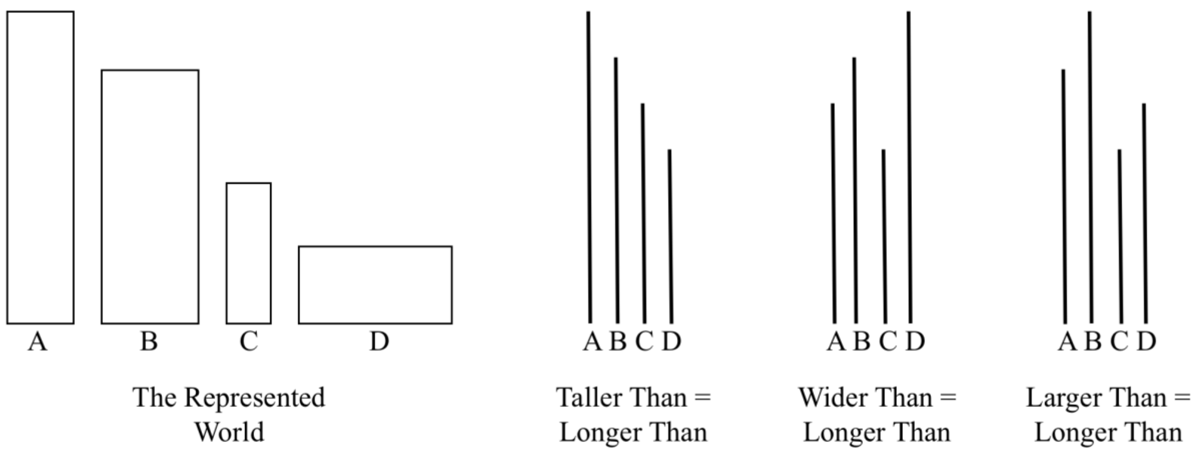


Figure 5

Figure adapted from Palmer (1978)



From Palmer (1978, p. 263): “The represented world in each case consists of the objects shown [on the left]. For each representing world, the correspondence of objects is indicated by the letters beneath them, and the correspondence of relations is indicated by the expression beneath each world.”