PARAMETRIC MODELING OF INTRINSIC STRUCTURE COVARIANCE FUNCTIONS FOR NON-HOMOGENEOUS AND NON-STATIONARY SPATIO-TEMPORAL RANDOM PROCESSES ON THE SPHERE

Jongwook Kim

Submitted to the faculty of the Graduate School
in partial fulfillment of the requirements
for the degree
Doctor of Philosophy
in the Department of Statistics,
Indiana University Bloomington

July 2025

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Doctoral Committee

Chunfeng Huang, Ph.D.

Daniel Manrique-Vallier, Ph.D

Michael Trosset, Ph.D.

Scott Robeson, Ph.D.

July 2025

Jongwook Kim

PARAMETRIC MODELING OF INTRINSIC STRUCTURE COVARIANCE FUNCTIONS FOR NON-HOMOGENEOUS AND NON-STATIONARY SPATIO-TEMPORAL RANDOM PROCESSES ON THE SPHERE

Identifying appropriate models for random processes and their associated covariance functions is one of the primary goals in spatial and spatio-temporal statistics, as it enables researchers to analyze the dependence structure within the data. For this purpose, assumptions of spatial homogeneity and temporal stationarity are commonly used, and many models have been developed under these conditions. However, these assumptions are often overly strong and unrealistic in practical applications.

Moreover, when working on the sphere, standard approaches from Euclidean space may not be appropriate due to the unique geometric and topological properties of the spherical domain. Despite this, relatively fewer studies have addressed random process modeling and covariance function development specifically for the sphere.

In this research, we introduce a parametric modeling framework for intrinsic structure covariance functions (ISCFs), designed to address non-homogeneous and non-stationary spatio-temporal random processes and their covariance functions on the sphere. To alleviate the assumption of spatial homogeneity while accounting for the spherical domain, we apply the theory of intrinsic random functions (IRFs) on the sphere. Similarly, to address temporal non-stationarity, we use the concept of random processes with stationary increments, exploring their relationship with intrinsic random functions on the real line.

We also provide a methodology for estimating the parameters associated with the ISCF model. This is demonstrated through a simulation study and an application to a real-world

dataset, highlighting the advantage of the model's interpretable parameters.

Chunfeng Huang, Ph.D.

Daniel Manrique-Vallier, Ph.D.

Michael Trosset, Ph.D.

Scott Robeson, Ph.D.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

Many methodologies for modeling of random processes and their associated covariance functions in spatio-temporal data analysis have been developed under the assumption of Euclidean space with spatial homogeneity and temporal stationarity. Although a wide range of techniques and theoretical apporaches have been proposed to relax these strong assumptions within Euclidean domains, analogous developments for spatio-temporal processes on the sphere remain relatively sparse. Spatial stochastic processes on the spherical surface require different statistical and mathematical tools compared to those used in Euclidean settings. Porcu et al. (2016) [12] emphasized that the choice of distance metric can significantly affect parameter estimation, especially over large regions on the sphere. Huang et al. (2019) [8] addressed the modeling of non-homogeneous spatial data on the sphere by extending the idea of intrinsic random functions—a flexible class of non-homogeneous models originally introduced by Matheron (1973) [10].

Introducing appropriate parameters is another important aspect of spatio-temporal data modeling. Yadrenko and Balakrishnan (1983) showed [15] how to construct parametric homogeneous covariance functions on the sphere using generating functions of Legendre polynomials. Porcu et al. (2016) [12] introduced diverse parametric covariance models for homogeneous and stationary spatio-temporal random processes on the sphere, comparing two different types of models: adaptive models from the Euclidean case and models constructed directly on the sphere. The adaptive models are modifications of models designed for Euclidean space, while the models constructed directly on the sphere are only valid for the great circle distance. According to their findings, direct construction is a more intuitive and flexible approach whereas the adaptive model requires stronger assumptions.

In this research, we develop parametric models of intrinsic structure covariance functions (ISCFs) through direct construction on the sphere and demonstrate how these models can be used to analyze the dependency structure of spatially non-homogeneous and temporally non-stationary processes on the sphere. Chapter 2 provides a literature review covering background knowledge on spatially homogeneous and temporally stationary random processes on the sphere, spatially non-homogeneous random processes on the sphere, and temporally non-stationary random processes. Chapter 3 explores the connection between random processes with stationary increments and intrinsic random functions on the real line. This relationship provides valuable insight for modeling temporally non-stationary stochastic processes and serves as an effective tool for addressing temporal non-stationarity in spatio-temporal random processes on the sphere. In Chapter 4, we introduce our intrinsic structure covariance functions (ISCFs) and explore how spatial non-homogeneity and temporal non-stationarity can be simultaneously incorporated into spatio-temporal random processes on the sphere using the ISCF framework. Chapter 5 demonstrates a methodology for estimating the parameters of ISCFs via numerical optimization, demonstrated through both a simulation study and an application to a real-world dataset on global temperature anomalies. Finally, Chapter 6 provides a conclusion to the study and discusses potential directions for future research.

Chapter 2

Background

This chapter provides a review of the literature and background understanding of spatially homogeneous and temporally stationary random processes on the sphere, as well as non-homogeneous processes on the sphere and non-stationary processes on the real line.

We start by introducing a spatio-temporal random process defined on the sphere. Denote this process by $X(P,t)$, where $P \in \mathbb{S}^2$ represents a location on the unit sphere and $t \in \mathbb{R}$ denotes time on the real line. Assume that the process is continuous in quadratic mean. We can expand such a random process through its spectral representation (Roy 1969) [13]:

$$X(P,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P) \tag{2.1}$$

$$\text{where } Z_{\ell,m}(t) = \int_{\mathbb{S}^2} X(P,t) Y_\ell^m(P) dP$$

and the $Y_\ell^m(\cdot)$ are the real spherical harmonics such that

$$\begin{cases} Y_\ell^m(P) = \sqrt{\frac{2\ell+1}{2\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\zeta)\cos(m\theta) \\ Y_\ell^0(P) = \sqrt{\frac{2\ell+1}{4\pi}} P_l^0(\cos\zeta) \\ Y_\ell^{-m}(P) = \sqrt{\frac{2\ell+1}{2\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\zeta)\sin(m\theta). \end{cases}$$

Note that each coefficient $Z_{\ell,m}(t)$ in Equation (2.1) is a function of time t and is independent of the spatial location P. Here, $\theta \in (0, 2\pi]$ represents the longitude and $\zeta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the latitude; hence, a location on the sphere is denoted by $P = (\zeta, \theta)$. The function $P_\ell^m(\cdot)$ denotes the associated Legendre polynomials.

The spectral decomposition in Equation (2.1) serves as a foundational tool for analyzing spatio-temporal random processes on the sphere especially in the context of spatial homogeneity and temporal stationarity assumptions.

2.1 Homogeneous and Stationary Random Processes on the Sphere

In this section, we assume that a spatio-temporal random process $X(P, t)$ is (weakly) homogeneous and (weakly) stationary, and then provide the background information about the random process on the sphere with the relevant assumptions for the covariance modeling. Even though the terms homogeneity and stationarity are often interchangeably used, in this paper, we reserve homogeneity as a spatial term and stationarity as a temporal term.

Definition 2.1.1. (Obukhov 1947)[11] Suppose that $X(\cdot)$ is a spatial random process on $\mathbb{S}^2$, a unit sphere. Then, the process is called (spatially) *homogeneous* if for any $P, Q \in \mathbb{S}^2$ and $g \in G$ where G is a group of rotation operators,

$$E\big(X(P)\big) = E\big(X(gP)\big)$$

and

$$Cov\big(X(P), X(Q)\big) = cov\big(X(gP), X(gQ)\big)$$

where

$$P = (\zeta_1, \theta_1) \text{ and } Q = (\zeta_2, \theta_2)$$

$$\text{Latitude: } \zeta_1, \zeta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad \text{Longitude: } \theta_1, \theta_2 \in (0, 2\pi]$$

Proposition 1. (Obukhov 1947, Yaglom 1961) [11] [17] A spatial random process $X(\cdot)$ is homogeneous if and only if

$$E\big(X(P)\big) \text{ is a constant}$$

and

$$Cov\big(X(P), X(Q)\big) \text{ is a function of } \overrightarrow{PQ}$$

for any $P, Q \in \mathbb{S}^2$, and $\overrightarrow{PQ} = \psi(P, Q)$ is the great circle distance between P and Q defined as

$$\psi(P, Q) = \arccos(\sin\zeta_1 \sin\zeta_2 + \cos\zeta_1 \cos\zeta_2 \cos(\theta_1 - \theta_2)).$$

Proof. ($\Leftarrow$) It is trivial that for any $P \in \mathbb{S}^2$, if $E\big(X(P)\big)= c$ where c is a constant, then for any rotation g, $E\big(X(gP)\big)= c$ as well; thus, $E\big(X(P)\big)= E\big(X(gP)\big)$. Also, a great circle distance is rotation invariant; that is, $\psi(P, Q) = \psi(gP, gQ)$ for any $P, Q \in \mathbb{S}^2$. Therefore, if a covariance function of the random process is a function of a great circle distance $\psi(P, Q)$, it is rotation invariant. In conclusion, $Cov\big(X(P), X(Q)\big)= Cov\big(X(gP), X(gQ)\big)$.

($\Rightarrow$) It is obvious if $X(P)$ has the same expected value regardless of any rotation of g, then its expected value should be a constant. Obukhov (1947) [11] and Yaglom(1987) [18] showed that if a random process $X(P)$ in $\mathbb{S}^2$ is homogeneous, then

$$Cov\big(X(P), X(Q)\big)= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\cos\overrightarrow{PQ})$$

where $P_\ell(\cdot)$ is a Legendre polynomial, $a_\ell \geq 0$, and $\sum_\ell (2\ell+1)a_\ell < \infty$. $\square$

Definition 2.1.2. Suppose that $X(\cdot)$ is a temporal random process defined on $\mathbb{R}$. Then, the process is called (temporally) *stationary* if

$$E\big(X(t)\big) = E\big(X(\tau_h t)\big)$$

and

$$Cov\big(X(t), X(s)\big) = Cov\big(X(\tau_h t), X(\tau_h s)\big)$$

for any $t, s, h \in \mathbb{R}$ and $\tau_h \in \mathrm{T}$ where T is a group of shift operators.

Proposition 2. (Hannan 1970) [6] A temporal random process $X(\cdot)$ is stationary if and only if

$$E\big(X(t)\big) \text{ is a constant}$$

and

$$Cov\big(X(t), X(s)\big) \text{ is a function of } t - s$$

for any $t, s \in \mathbb{R}$.

Proof. ($\Rightarrow$) It is obvious that $E\big(X(t)\big)$ should have a constant value for any t since

$$E\big(X(t)\big) = E\big(X(t+h)\big) \quad \text{for any } h \in \mathbb{R}.$$

By the definition of stationarity,

$$Cov\big(X(t), X(s)\big) = Cov\big(X(t+h), X(s+h)\big) \quad \text{for any } h \in \mathbb{R}.$$

If we choose $h = -s$, then,

$$Cov\big(X(t), X(s)\big) = Cov\big(X(t-s), X(0)\big) =: C(t-s)$$

($\Leftarrow$) Denote $Cov\big(X(t), X(s)\big) = C(t-s)$. Then, by the given condition, $Cov\big(X(t+h), X(s+h)\big) = C(t-s)$. Thus,

$$Cov\big(X(t), X(s)\big) = Cov\big(X(t+h), X(s+h)\big) \quad \text{for any } h \in \mathbb{R}.$$

It is obvious that $E\big(X(t)\big) = E\big(X(t+h)\big)$ for any $h \in \mathbb{R}$, since $E\big(X(t)\big)$ is a constant for any $t \in \mathbb{R}$.

$\square$

Now, we can simultaneously consider the concepts of the spatial homogeneity and the temporal stationarity for spatio-temporal random processes on the sphere.

Definition 2.1.3. A spatio-temporal random process $X(\cdot,\cdot)$ is (spatially) *homogeneous* and (temporally) *stationary* if the process is

$$E\big(X(P,t)\big) = E\big(X(gP, \tau_h t)\big)$$

and

$$Cov\big(X(P,t), X(Q,s)\big) = Cov\big(X(gP, \tau_h t), X(gQ, \tau_h s)\big)$$

for any $P, Q \in \mathbb{S}^2$; $t, s, h \in \mathbb{R}$, $g \in G$, and $\tau_h \in \mathrm{T}$. In other words, the process is spatially rotation invariant and temporally shift invariant.

Remark 2.1.1. (Roy 1969) [13]

A spatio-temporal process $X(P,t)$ is homogeneous and stationary if and only if

$$\mathbb{E}\big(X(P,t)\big) \text{ is a constant}$$

and

$$\text{Cov}\left(X(P,t), X(Q,s)\right) \text{ is a function of } \overrightarrow{PQ} \text{ and } t-s.$$

for any $P, Q \in \mathbb{S}^2$ and $t, s \in \mathbb{R}$, where $\overrightarrow{PQ}$ denotes the great circle distance between P and Q.

2.1.1 Homogeneous and Stationary Covariance Functions on the Sphere

Note that a spatio-temporal random process on the sphere, $X(P,t)$, can be expanded using its spectral representation (Yaglom 1961Roy 1969) [17] [13], as shown in Equation (2.1):

$$X(P,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P)$$

where

$$Z_{\ell,m}(t) = \int_{\mathbb{S}^2} X(P,t) Y_\ell^m(P) dP$$

Since the coefficient $Z_{\ell,m}(t)$ is stochastic, we must first compute its covariance function to obtain the covariance of the random process $X(P,t)$. Roy (1969) [13] showed that if the process is spatially homogeneous and temporally stationary, then the coefficients $Z_{\ell,m}(t)$ are uncorrelated for different values of ℓ or m. This independency of the coefficients across spatial indices enables the expression of the covariance structure of the spatio-temporal process in a stationary form that depends solely on the time lag. Consequently, the covariances of the coefficient and the original process can be expressed as follows (Roy 1969) [13]:

$$Cov\big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\big) \quad = \quad a_\ell(h) := \delta_\ell^{\ell'} \delta_m^{m'} \int_{-\infty}^{\infty} e^{i\omega h} dF_\ell(\omega) \tag{2.2}$$

$$Cov\big(X(P,t),X(Q,s)\big) \quad = \quad \xi_0(\overrightarrow{PQ},h) := \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell(h) P_\ell(\cos \overrightarrow{PQ}) \tag{2.3}$$

where h is $t-s$, $P_\ell(\cdot)$ is a Legendre polynomial, and $F_\ell(\omega)$ is a non-negative measure bounded on $[0,\infty)$. The function $\delta_\ell^{\ell'}$ equals 1 if $\ell=\ell'$ and 0 otherwise.

The term $\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(h)$ must be finite, as it represents the second moment of the random process, which is required to be finite. Additionally, each $a_\ell(h)$ must be a positive semi-definite function with respect to the temporal lag h, as it serves as the covariance function of the temporal coefficients $Z_{\ell,m}(t)$. A more detailed discussion of these properties, along with the derivations of Equations (2.2) and (2.3), is provided in Appendix A.1.

Note that the spatio-temporal covariance function of the process $X(P,t)$ in Equation (2.3) depends on $a_\ell(h)$, which captures temporal stationarity. Thus, one can parameterize the covariance structure of $X(P,t)$ by specifying an appropriate functional form for $a_\ell(h)$ in Equation (2.2).

Remark 2.1.2. (Roy 1969) [13] For a homogeneous and stationary random process $X(P,t)$ on the sphere, if a spatial term $P \in \mathbb{S}^2$ is fixed as P_0, then $\{X(P_0,t) : t \in \mathbb{R}\}$ is a stationary stochastic process, and its covariance function is given by:

$$\xi_0(0,h) = \int_{-\infty}^{\infty} e^{i\omega h} dG_\ell(\omega)$$

$$\text{where} \qquad G_\ell(\omega) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} F_\ell(\omega), \qquad \int_{-\infty}^{\infty} dG_\ell(\omega) < \infty$$

Remark 2.1.3. (Roy 1969) [13] For a homogeneous and stationary random process $X(P,t)$ on the sphere, if a temporal term $t \in \mathbb{R}$ is fixed as t_0, then $\{X(P,t_0) : P \in \mathbb{S}^2\}$ is a homogeneous random field on the sphere, and its covariance function is given by:

$$\xi_0(\psi, 0) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\cos\psi)$$

$$\text{where} \quad a_\ell = a_\ell(0) \geq 0 \text{ for all } \ell \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{\ell=0}^{\infty} a_\ell < \infty$$

2.1.2 Parameterization of the Homogeneous and Stationary Covariance Functions on the Sphere

There are several approaches for parameterizing covariance functions, such as separable models and adaptations of models originally developed for Euclidean space to the sphere. However, our objective is to construct a covariance function that is not only intuitive and interpretable but also avoids relying on overly restrictive assumptions.

For instance, under the separability assumption, the temporal and spatial components of the covariance function can be expressed independently. Specifically, we assume that $a_\ell(h) = a_\ell C(h)$, where $a_\ell \geq 0$ for $\ell \in \mathbb{Z}_{\geq 0}$, $h \in \mathbb{R}$ is the temporal lag, and ψ denotes the great-circle distance. This results in a product form of the spatio-temporal covariance function (Stein 2005) [14], given by

$$R(\psi, h) = C(h) \sum_{\ell=0}^{\infty} a_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\psi) := C(h)B(\psi),$$
$$\text{where} \quad B(\psi) = \sum_{\ell=0}^{\infty} a_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\psi).$$

That is, we can separate the time term $C(h)$ from the spatial term $B(\psi)$. This separable covariance may be desirable for its relatively cheap computation. However, this model is physically unrealistic and requires strong assumptions. Stein (2005) [14] strongly criticized the idea of the separable model. He insisted that separable covariance functions tend to be sensitive and vulnerable to change in the locations. Even small changes can cause huge differences in the correlations between certain linear combinations of observations.

Another approach can be adapted models from Euclidean space to the sphere. For example, we can build a model by some transformation from $\mathbb{R}^3$ to $\mathbb{S}^2$. If $R_0(h,t)$ is valid in $\mathbb{R}^3 \times \mathbb{R}$, then $R(2\sin(\psi = 2), t)$ is valid on $\mathbb{S}^2 \times \mathbb{R}$ (Yadrenko and Balakrishnan 1983, Huang et al. 2011) [15] [7]. Therefore, we can find some valid covariance function on the three dimensional Euclidean space and then transfer it to the spherical space. However, this method is not intuitive and does not provide any physical interpretation.

Thus, more direct construction considering the spherical domain and its geodesic distance is more realistic and interpretable. To achieve such a goal, diverse homogeneous and stationary covariance models have been developed considering the spherical domain and its geodesic distance. For instance, by using the generating function of Legendre polynomials, we can derive a valid parametric covariance model (Yadrenko and Balakrishnan 1983) [15] such that

$$\begin{aligned}
\xi_0(\psi, h) &= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell(h) P_\ell(\cos\psi) \\
&= \frac{1}{4\pi} \frac{1 - \alpha^2 g^2(h)}{(1 - 2\alpha g(h)\cos\psi + \alpha^2 g^2(h))^{3/2}}
\end{aligned} \tag{2.4}$$

where $a_\ell(h) = \alpha^\ell g^\ell(h)$, and $g(\cdot)$ is a stationary temporal covariance function.

Proof. For $\xi_0(\psi, h)$ to have finite variance, one must have

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(0) < \infty.$$

By the generating function of Legendre polynomials,

$$\begin{aligned}
\frac{1}{\sqrt{1-2xt+t^2}} &= \sum_{\ell=0}^{\infty} P_\ell(x)t^\ell && (2.5)\\
-\frac{1}{2}\frac{-2x+2t}{(1-2tx+t^2)^{3/2}} &= \sum_{\ell=0}^{\infty} \ell P_\ell(x)t^{\ell-1} \quad \text{(By differentiating with respect to } t\text{)} \\
-t\frac{-2x+2t}{(1-2tx+t^2)^{3/2}} &= \sum_{\ell=0}^{\infty} 2\ell P_\ell(x)t^\ell \quad \text{(By multiplying } 2t\text{)} && (2.6)\\
\frac{1-t^2}{(1-2tx+t^2)^{3/2}} &= \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(x)t^\ell \quad \text{(By adding (2.5) and (2.6)).}
\end{aligned}$$

Now let $t = \alpha g(h)$, $\alpha^\ell g^\ell(h) = a_\ell(h)$, and $x = \cos\psi$. Then,

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} P_\ell(\cos\psi)\alpha^\ell g^\ell(h) \\
&= \frac{1}{4\pi}\frac{1-\alpha^2 g^2(h)}{(1-2\alpha g(h)\cos\psi + \alpha^2 g^2(h))^{3/2}}.
\end{aligned}$$

□

The covariance function $\xi_0(\psi, h)$ in Equation (2.4) is a function of spherical distance ψ and time lag h, indicating homogeneity and stationarity as desired. Porcu et al. (2016) [12] introduced several parametric covariance models—such as the Negative Binomial, Multiquadric, Sine Series, and Sine Power—for spatially homogeneous and temporally stationary random processes on the sphere. These models are summarized in Table 2.1. For these homogeneous and stationary covariance models, we assume the form $a_\ell(h) = \alpha^\ell g^\ell(h)$ in Equations (2.2) and (2.3), where $g(\cdot)$ is a temporal covariance function and $\alpha \in (0, 1)$.

Table 2.1: Parametric homogeneous and stationary covariance models on the sphere. A function $g(\cdot)$ denotes any valid covariance function on the real line. (Yadrenko and Balakrishnan 1983 [15], Porcu et al. 2016 [12])

Model name	Analytic expression	Parameters range
Generating function of Legendre polynomials	$\xi_0(\psi,h)=\frac{1}{4\pi}\frac{1-\alpha^2g^2(h)}{(1-2\alpha g(h)\cos\psi+\alpha^2g^2(h))^{3/2}}$	$\alpha\in(0,1)$
Negative binomial	$\xi_0(\psi,h)=\frac{1}{4\pi}\{\frac{1-\alpha}{1-\alpha g(h)\cos\psi}\}^{\tau}$	$\alpha\in(0,1)$, $\tau>0$
Multiquadric	$\xi_0(\psi,h)=\frac{1}{4\pi}\frac{(1-\alpha)^{2\tau}}{\{1+\alpha^2-2\alpha g(h)\cos\psi\}^{\tau}}$	$\alpha\in(0,1)$, $\tau>0$
Sine series	$\xi_0(\psi,h)=\frac{1}{4\pi}e^{\alpha g(h)\cos\psi-1}\{1+\alpha g(h)\cos\psi\}/2$	$\alpha\in(0,1)$
Sine power	$\xi_0(\psi,h)=\frac{1}{4\pi}\{1-2^{-\eta}(1-\alpha g(h)\cos\psi)\}^{\eta/2}$	$\alpha\in(0,1)$, $\eta\in(0,2]$
Adapted multiquadric	$\xi_0(\psi,h)=\frac{1}{4\pi}\{\frac{\{1+\alpha g^2(h)\}(1-\alpha)}{1+\alpha^2g^2(h)-2\alpha g(h)\cos\psi}\}^{\tau}$	$\alpha\in(0,1)$, $\tau>0$
Poisson	$\xi_0(\psi,h)=\exp[\lambda\{\cos(\psi)\alpha g(h)-1\}]$	$\alpha\in(0,1)$, $\lambda>0$

2.2 Non-Homogeneous Spatial Random Processes on the Sphere

In this section, we relax the assumption of the homogeneous spatial terms by applying the concept of intrinsic random functions (Matheron 1973) [10], a flexible family that can be used for the representation of non-homogeneous properties.

2.2.1 Intrinsic Random Functions on the Sphere

In order to define an intrinsic random function, it is first required to understand an allowable measure.

Definition 2.2.1. Let Λ denote the dual of $C(\mathbb{S}^2)$, that is, the set of all finite regular signed Borel measures on $\mathbb{S}^2$. For $f \in C(\mathbb{S}^2)$, we define that

$$
\begin{aligned}
f(\lambda^s) &= \int_{\mathbb{S}^2} f(x)\lambda^s(dx) \\
f(g\lambda^s) &= \int_{\mathbb{S}^2} f(gx)\lambda^s(dx)
\end{aligned}
$$

where $g\lambda^s$ is a rotated measure and $g \in G$.

Then, $\lambda^s \in \Lambda^s$ is called an *allowable measure of order* κ if it annihilates the spherical harmonics with orders less than κ (Huang et al. 2019) [8]. That is, we can states that

$$
Y_\ell^m(\lambda^s) = \int_{\mathbb{S}^2} Y_\ell^m(P)\lambda^s(dP) = 0 \quad \text{for} \quad 0 \leq \ell \leq \kappa, \quad |m| \leq \ell.
$$

We define Λ_κ^s as the class containing all allowable measures of order κ. Then, obviously, $\Lambda_{\kappa+1}^s \subset \Lambda_\kappa^s \subset \Lambda^s$.

Example 2.2.1. Define $\lambda_{P_1,P_2}^s := \delta_{P_1} - \delta_{P_2}$ for $P_1, P_2 \in \mathbb{S}^2$. Then λ_{P_1,P_2}^s is an allowable measure of order 1 on the sphere; that is, $\lambda_{P_1,P_2}^s \in \Lambda_1^s$, since

$$
\begin{aligned}
Y_0^0(\lambda_{P_1,P_2}^s) &= \int_{\mathbb{S}^2} Y_0^0(P)\,\lambda_{P_1,P_2}^s(dP) \\
&= Y_0^0(P_1) - Y_0^0(P_2) \\
&= \frac{1}{\sqrt{4\pi}} - \frac{1}{\sqrt{4\pi}} = 0.
\end{aligned}
$$

By using the properties of an allowable measure, we can easily show that λ^s is the rotated measure $g\lambda^s$ invariant.

Proposition 3. (Huang et al. 2019) [8] An allowable measure is rotation invariant. That is, if $\lambda^s \in \Lambda_\kappa^s$, then $g\lambda^s \in \Lambda_\kappa^s$.

Proof. We can express the rotation operation as a linear transformation such that

$$Y_\ell^m(gP) = \sum_{m'=-\ell}^{\ell} S_\ell^{mm'}(g) Y_\ell^{m'}(P) \quad \text{for } P \in \mathbb{S}^2.$$

In addition, since $\lambda^s \in \Lambda_\kappa^s$, it is true that $Y_\ell^m(\lambda^s) = 0$ for any $\ell < \kappa$ and $|m| \leq \ell$. Hence,

$$\begin{aligned} Y_\ell^m(g\lambda^s) &= \int_{\mathbb{S}^2} Y_\ell^m(gP)\lambda^s(dx) = \sum_{m'=-\ell}^{\ell} S_\ell^{mm'}(g) \int_{\mathbb{S}^2} Y_\ell^{m'}(P)\lambda^s(dP) \\ &= \sum_{m'=-\ell}^{\ell} S_\ell^{mm'}(g) Y_\ell^{m'}(\lambda^s) = 0. \end{aligned}$$

□

By the definition of an allowable measure and its rotation invariant property, we can finally establish the connection between an intrinsic random function on the sphere and the low-frequency truncation as presented in Definition 2.2.2 and Proposition 4.

Definition 2.2.2. (Huang et al. 2019) [8] A spatial random process $X(P)$ is an *intrinsic random function with order κ on the sphere* ($\mathrm{IRF}^s(\kappa)$) where $\kappa \in \mathbb{Z}$ and $\kappa \geq 0$ if the process $X(\lambda^s)$ is rotation invariant for any $\lambda^s \in \Lambda_\kappa^s$. That is, for any $\lambda^s \in \Lambda_\kappa^s$ and $g \in G$, it satisfies that

$$\begin{aligned} &E\big(X(\lambda^s) = E(X(g\lambda^s)\big) \\ &Cov\big(X(\lambda_1^s), X(\lambda_2^s)\big) = Cov\big(X(g\lambda_1^s), X(g\lambda_2^s)\big) \quad \text{where} \quad \lambda_1^s, \lambda_2^s \in \Lambda_\kappa^s \end{aligned}$$

If $\lambda^s \in \Lambda_\kappa^s$, then $g\lambda^s \in \Lambda_\kappa^s$; in other words, $\lambda^s \in \Lambda_\kappa^s$ is invariant for a rotation operator $g \in G$.

Remark 2.2.1. A homogeneous random process is denoted as an IRF$^s(0)$. Note that this notation differs slightly from Matheron's (1973) [10], where a homogeneous process is represented by IRF$^s(-1)$. In this paper, $\kappa \geq 1$ is used to describe non-homogeneous random processes, instead of $\kappa \geq 0$.

Proposition 4. (Huang et al. 2019) [8] For $\kappa \geq 1$ and $\kappa \in \mathbb{Z}$, a continuous spatial random process X(P) on the sphere with finite second moments is an intrinsic random function of order κ (IRF$^s(\kappa)$) if and only if its low frequency-truncated process $X_\kappa(P)$ is homogeneous where

$$X_\kappa(P) = \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P).$$

We can see that when $\kappa = 0$ (i.e., IRK$^s(0)$), this indicates a spatially homogeneous random process. The proof of Proposition 4 can be found in the Appendix A.2.

Definition 2.2.3. A homogeneous covariance function of $X_\kappa(P)$ is called an *intrinsic covariance function of order κ on the sphere* (ICF$^s(\kappa)$) (Huang et al. 2019) [8], which is also referred to as a generalized covariance function (Matheron 1973) [10].

Since the intrinsic covariance function of order κ (IRF$^s(\kappa)$) is homogeneous, we can apply the covariance structure of Equation (2.3) with $h = 0$ and $a_\ell(0) = a_\ell$. That is:

$$\begin{aligned} \phi_\kappa(\overrightarrow{PQ}) &:= \mathrm{Cov}\left(\sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P), \quad \sum_{\ell'=\kappa}^{\infty} \sum_{m'=-\ell'}^{\ell'} Z_{\ell',m'} Y_{\ell'}^{m'}(Q) \right) \\ &= \sum_{\ell=\kappa}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\cos \overrightarrow{PQ}) \quad \text{where} \quad a_\ell = \mathrm{cov}(Z_{\ell,m}, Z_{\ell',m'}) \\ &= \xi_0(\overrightarrow{PQ}, 0) - \sum_{\ell=0}^{\kappa-1} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\cos \overrightarrow{PQ}) \end{aligned} \tag{2.7}$$

where $\xi_0(\cdot,\cdot)$ is a covariance function in Equation (2.3).

2.2.2 Non-Homogeneous Covariance Functions on the Sphere

As a way to derive the closed-form non-homogeneous covariance function of IRF$^s(\kappa)$, Huang et al. (2019) [8] introduced a covariance function of an IRF$^s(\kappa)$ $X(P)$ on the sphere by applying a reproducing kernel Hilbert space suggested by Levesley et al. (1999) [9].

We define a function space such that:

$$X_\kappa = \left\{ X(P) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P), \quad P \in \mathbb{S}^2 \quad : \quad \sum_{\ell=\kappa}^{\infty} \frac{1}{a_\ell} \sum_{m=-\ell}^{\ell} |Z_{\ell,m}|^2 < \infty \right\}$$

Then, nil space can be defined that

$$N = \mathrm{span}\{Y_\ell^m(P) : \quad 0 \le \ell < \kappa, \quad -\ell \le m \le \ell\}.$$

By introducing the new basis of the nil space, $q_1(\cdot), q_2(\cdot), \dots, q_{\kappa^2}(\cdot) \in N$ and a set of distinct points $\{\tau_1, \tau_2, \dots, \tau_{\kappa^2}\} \in \mathbb{S}^2$ such that $q_\nu(\tau_\mu) = I(\nu, \mu)$ for $1 \le \nu, \mu \le \kappa^2$, we can define an inner product for $f, g \in X_\kappa$:

$$< f, g >= \sum_{\nu=1}^{\kappa^2} f(\tau_\nu) g(\tau_\nu) + < f, g >_\kappa \qquad \text{where} \qquad < f, g >_\kappa = \sum_{\ell=\kappa}^{\infty} \frac{1}{a_\ell} \sum_{-\ell}^{\ell} Z_{\ell,m,f} Z_{\ell,m,g}$$

Then, there exists a reproducing kernel such that:

$$\begin{aligned} R(P,Q) = \phi_\kappa(\overrightarrow{PQ}) &+ \sum_{\nu=1}^{\kappa^2} \sum_{\mu=1}^{\kappa^2} \phi_\kappa(\overrightarrow{\tau_\nu \tau_\mu}) q_\nu(P) q_\mu(Q) + \sum_{\nu=1}^{\kappa^2} q_\nu(P) q_\nu(Q) \\ &- \sum_{\nu=1}^{\kappa^2} \phi_\kappa(\overrightarrow{Q\tau_\nu}) q_\nu(P) - \sum_{\nu=1}^{\kappa^2} \phi_\kappa(\overrightarrow{P\tau_\nu}) q_\nu(Q). \end{aligned} \tag{2.8}$$

The function $\phi_\kappa(\cdot)$ is an intrinsic covariance function presented in Equation (2.7). By Kolmogorov existence theorem, there exists a Gaussian random process having Equation (2.8) as its covariance function; hence, we can construct a covariance function of an intrinsic random function of its order κ with this reproducing kernel and its intrinsic covariance function.

2.3 Non-Stationary Temporal Random Processes

Similar to the assumption of spatial homogeneity, the assumption of temporal stationarity is often very restrictive in practical applications. This assumption can be relaxed by employing the concept of random processes with stationary increments, as introduced by Yaglom (1958) [16]. Therefore, a thorough understanding of this concept is beneficial in effectively addressing temporal non-stationarity in stochastic modeling.

2.3.1 Random Processes with Stationary Increments and Structure Functions

Definition 2.3.1. (Yaglom 1958, Chen et al. 2024) [16] [3]

A stochastic process $\{X(t), t \in \mathbb{R}\}$ is called a *random process with stationary increments of order* d, denoted by $I(d)$, if $\Delta_\iota^d X(t)$ is stationary where

$$\Delta_\iota^d X(t) = \sum_{k=0}^{d} (-1)^k \binom{d}{k} X(t - k\iota), \quad t, \iota \in \mathbb{R} \quad d \in \mathbb{Z}_{\geq 0}$$

Additionally, $D^d(h; \iota_1, \iota_2)$ is its structure function where

$$D^d(h; \iota_1, \iota_2) = E\Bigg([\Delta_{\iota_1}^d X(t+h)][\Delta_{\iota_2}^d X(t)]\Bigg), \quad h, \iota_1, \iota_2 \in \mathbb{R}$$

This is positive semi-definite.

Note that when $\iota = \iota_1 = \iota_2$ and $h = 0$, $D(\iota) = E\Bigg([X(t) - X(t-\iota)]^2\Bigg)$ is a variogram. If $d = 1$, then $\Delta_\iota X(t) = X(t) - X(t-\iota)$; thus, $D(\iota_1, \iota_2) = E\Bigg([X(t) - X(t-\iota_1)] \cdot [X(t) - X(t-\iota_2)]\Bigg)$. Likewise, if $d = 2$, then we can say that $\Delta_\iota^2 X(t) = X(t) - 2X(t-\iota) + X(t-2\iota) = \{X(t) - X(t-\iota)\} - \{X(t-\iota) - X(t-2\iota)\}$. A stationary process corresponds to the case when $d = 0$. In the case of a discrete time series, that is, $t \in \mathbb{Z}$, the random process with stationary increments of order d is also called an integrated process. A random process with stationary increments with order 1 ($d = 1$) is also called intrinsic stationary in that $Y(t) = X(t + 1) - X(t)$ is stationary.

Random Processes with Stationary Increments with Order 1

Let $X(t)$ be a random process with stationary increments with order 1 and $Y(t)$ be a stationary random process such that $Y(t) = X(t + 1) - X(t)$ where $t \in \mathbb{Z}$. That is, $X(t)$ is a discrete time series. Since $Y(t)$ is stationary (Yaglom 1987) [18],

$$Y(t) = \int_{-\pi}^{\pi} e^{i\omega t} dZ_y(\omega), \quad C_y(h) = E\big(Y(t+h)\cdot Y(t)\big) = \int_{-\pi}^{\pi} e^{i\omega h} dF_y(\omega), \tag{2.9}$$

where $h \in \mathbb{Z}$, $F_y(\omega)$ is a bounded non-decreasing function and $E(|dZ_y(\omega)|^2) = dF_y(\omega)$ for $Z_y(\omega)$, a random function with uncorrelated increments. Then, we can achieve the following spectral representations:

$$X(t) = \int_{-\pi}^{\pi} (e^{i\omega t} - 1) dZ(\omega) + X_0 \tag{2.10}$$

$$D(h; \iota_1, \iota_2) = \int_{-\pi}^{\pi} e^{i\omega h}(1 - e^{-i\omega\iota_1})(1 - e^{i\omega\iota_2}) dF(\omega) \tag{2.11}$$

Note that $X_0 = X(0)$ is a random variable, $Z(\omega_2) - Z(\omega_1) = \int_{\omega_1}^{\omega_2} \frac{1}{e^{i\omega}-1} dZ_y(\omega)$, $F(\omega_2) - F(\omega_1) = \int_{\omega_1}^{\omega_2} \frac{1}{|e^{i\omega}-1|^2} dF_y(\omega)$.

The derivations of the spectral representations in Equation (2.10) and (2.11) are provided in Appendix A.3.

Similarly, we can also generalize this for a continuous time series $X(t)$ (Yaglom 1987) [18]. Suppose that a stationary random process $Y(t)$ is a derivative of $X(t)$; that is, $Y(t) =$

$X'(t)$ if it exists when $t, h \in \mathbb{R}$. Therefore,

$$Y(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y(\omega), \quad C_y(h) = E\big(Y(t+h) \cdot Y(t)\big) = \int_{-\infty}^{\infty} e^{i\omega h} dF_y(\omega), \tag{2.12}$$

Then,

$$X(t) = \int_{-\infty}^{\infty} (e^{i\omega t} - 1) dZ(\omega) + X_0 \tag{2.13}$$

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})(1 - e^{i\omega\iota_2}) dF(\omega) \tag{2.14}$$

Note that a random variable $X_0 = X(0)$, $\quad Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{i\omega}$, $\quad F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^2}$.

The derivations of Equation (2.13) and (2.14) are also provided in Appendix A.4.

Equations (2.10)-(2.14) reveal that these spectral representations exhibit a jump discontinuity at $\omega = 0$.To address this issue, the spectral representation of a random process with stationary increments, as described in Equation (2.13), can alternatively be written as (Yaglom 1987) [18]:

$$X(t) = \int_{-\infty}^{\infty} (e^{i\omega t} - 1) dZ(\omega) + X_0 + X_1 t \tag{2.15}$$

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})(1 - e^{i\omega\iota_2}) dF(\omega) + A_1^2 h_1 h_2 \tag{2.16}$$

Note that $X_0 = X(0)$, $\quad Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{i\omega}$, $\quad F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^2}$ where $0 \notin (\omega_1, \omega_2)$ and the limit is defined as $\int_{-\infty}^{\infty} = \lim_{T\to\infty, \epsilon\to 0}\{\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\}$. A random variable X_1, which can be a constant, is introduced in order to deal with the jump discontinuity at $\omega = 0$. i.e., $\lim_{\epsilon\to 0}[Z_y(\epsilon) - Z_y(-\epsilon)] = X_1 \neq 0$ and $\lim_{\epsilon\to 0}[F_y(\epsilon) - F_y(-\epsilon)] = \langle |X_1| \rangle = A_1^2 > 0$.

Remark 2.3.1. (Yaglom 1987) [18]

If the correlation function $C_y(h)$ in Equation (2.9) or (2.12) decays rapidly with respect to h, that is,

$$\int_{-\infty}^{\infty} |C_y(h)|^2 \, dh < \infty, \tag{2.17}$$

then the random variables X_1 and A_1 in (2.15) and (2.16), respectively, are zero, and the spectral density function $f(\omega)$ of the process $X(t)$ exists. Consequently, the spectral distribution function $dF(\omega)$ in (2.16) can be written as $f(\omega)\, d\omega$, where $f(\omega) = F'(\omega) \geq 0$.

In the later parts of this research, we assume that the condition in (2.17) is satisfied. We also assume that the random variable X_0 in (2.15) is zero.

Higher Order Random Processes with Stationary Increments

Analogously to the works shown above, the spectral representation of second order random process with stationary increments and its structure function can be derived. Suppose $X(t)$ is a random process with stationary increment with order 2, denoted by $I(2)$, for $t \in \mathbb{R}$. Then, $\Delta_1 X(t)$ is $I(1)$ and $Y(t) = \Delta_1^2 X(t)$ is a stationary random process $I(0)$. Assuming the conditions of remark 2.3.1 and $X(0) = \Delta_1 X(0) = 0$, the spectral representation of $X(t)$ and its structure function are given:

$$X(t) = \int_{-\infty}^{\infty} (e^{i\omega t} - 1 - i\omega t) dZ(\omega) \tag{2.18}$$

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})^2(1 - e^{i\omega\iota_2})^2 dF(\omega) \tag{2.19}$$

Note that $X_0 = X(0)$, $Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{(i\omega)^2}$, and $F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^4}$ where $0 \notin (\omega_1, \omega_2)$.

The derivations of Equations (2.18) and (2.19) are presented in Appendix A.5.

According to Brockwell and Davis (1991) [1], the order d of stationary increments usually stops at 1 or 2 in practice. However, we can still take a further step for the random processes with stationary increments of any finite order $d \in \mathbb{Z}_{\geq 0}$.

Let $X(t)$ be a random process with stationary increments with order d, and let $Y(t)$ be a stationary random process such that $Y = X^{(d)}$, the d-th derivative of X, where $t, h \in \mathbb{R}$. Since $Y(t)$ is stationary,

$$Y(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y(\omega), \quad E\big(Y(t+h) \cdot Y(t)\big) = C_y(h) = \int_{-\infty}^{\infty} e^{i\omega h} dF_y(\omega),$$

where $F_y(\omega)$ is a bounded non-decreasing function and $E(|dZ_y(\omega)|^2) = dF_y(\omega)$ for $Z_y(\omega)$, a random function with uncorrelated increments. Then, we can achieve the following spectral representations (Yaglom 1987) [18]:

$$X(t) = \int_{-\infty}^{\infty} \left(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!} \right) dZ(\omega) + X_0 + X_1 t + \cdots + X_d t^d \qquad (2.20)$$

Where $\int_{-\infty}^{\infty} = \lim_{T\to\infty, \epsilon\to 0} \{\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\}$ and $Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{(i\omega)^d}$. A random variable $X_0 = X(0)$ and $X_1, X_2, \cdots, X_{d-1}$ are some random variables. A random variable X_d is introduced in order to deal with the jump discontinuity at $\omega = 0$. These random variables $X_0, X_1, \cdots, X_d$ can, in particular, be constants.

Remark 2.3.2. (Yaglom 1987) [18] As a result of (2.20) and by the assumption of remark 2.3.1 and $X_0, \cdots, X_{d-1} = 0$, we can get such spectral representations for $I(d)$:

$$
\begin{aligned}
X(t) &= \int_{-\infty}^{\infty} \left(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!} \right) dZ(\omega) \\
\Delta_{\iota}^{d} X(t) &= \int_{-\infty}^{\infty} e^{i\omega t}(1 - e^{-ih\omega})^{d} dZ(\omega) \\
D(h; \iota_1, \iota_2) &= \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})^{d}(1 - e^{i\omega\iota_2})^{d} dF(\omega)
\end{aligned}
$$

Where $Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{(i\omega)^d}$, and $F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^{2d}}$.

Chapter 3

Connection between Random Processes with Stationary Increments and Intrinsic Random Functions on $\mathbb{R}$

Although random processes with stationary increments are closely related to intrinsic random functions (IRFs) on the real line, the connection between these two concepts has not been clearly articulated in the literature. In this chapter, we investigate the relationship between random processes with stationary increments and IRFs, and establish their equivalence on the real line under specific conditions. This equivalence enhances our understanding of non-stationary temporal or spatio-temporal random processes by allowing for the interchangeable use of the theoretical frameworks associated with these two models. As an initial step, we define the intrinsic random functions on the real line.

3.1 Intrinsic Random Functions on the real line

In Section 2.2.1, we discussed the definition and application of intrinsic random functions (IRFs) on the sphere. These concepts can be naturally extended to random processes with stationary increments on the real line. Before introducing the definition and key properties of IRFs in this setting, we first identify the class of allowable measures on the real line associated with such processes.

3.1.1 Allowable Measures for Temporal Random Processes

Definition 3.1.1. (Chiles and Delfiner 1999 [4]) A discrete measure λ^t is an *allowable measure of order d on* $\mathbb{R}$ if it annihilates polynomials of degree less than d. That is, a set of λ_i^t

to n points x_i on $\mathbb{R}$ defines a discrete measure

$$\lambda^t = \sum_{i=1}^{n} \lambda_i^t \delta_{x_i} \quad \text{where } \delta_{x_i} \text{ is the Dirac measure at } x_i$$

and

$$f(\lambda^t) = \int_{\mathbb{R}} f(x)\lambda^t(dx) = \sum_{i=1}^{n} \lambda_i^t f(x_i)$$

Then,

$$p(\lambda^t) = \int_{\mathbb{R}} p(x)\lambda^t(dx) = \sum_{i=1}^{n} \lambda_i^t x_i^\ell = 0$$

where $\ell = 0, 1, 2, \cdots, d-1$ and $p(\cdot)$ is a polynomial function at the degree less than d. We denote that Λ_d^t is a class of such allowable measures, and then it is clear that $\Lambda_{d+1}^t \subset \Lambda_d^t$.

As noted by Chiles and Delfiner (1999) [4], any concept related to the discrete allowable measure $\lambda^t \in \Lambda_d^t$ in Definition 3.1.1 can be extended to the continuous case by considering

$$f(\mu) = \int f(x)\,\mu(dx),$$

where μ belongs to the broader class $\mathcal{M}_d$ of measures with compact support that satisfy the condition

$$\int \mu(dx)x^\ell = 0, \quad \text{for } \ell = 0, \ldots, d.$$

In this paper, we primarily work with the discrete allowable measure $\lambda^t \in \Lambda_d^t$ as introduced in Definition 3.1.1.

Proposition 5. (Chiles and Delfiner 1999) [4] An allowable measure $\lambda \in \Lambda_d^t$ is closed under translations of shift. That is,

$$p_{d-1}(\tau_h \lambda^t) = \sum_i \lambda_i^t p_{d-1}(x_i + h) = 0$$

where $p_{d-1}(\cdot)$ is any polynomial function of degree $d-1$ and τ_h is a shift operator by $h \in \mathbb{R}$.

Proof. This one can be shown by the closeness of polynomials under the translations. By the binomial theorem,

$$(x+h)^\ell = \sum_{s=0}^{\ell} \binom{\ell}{s} x^s h^{\ell-s}$$

By the fact that $\lambda^t \in \Lambda_d^t$,

$$\sum_{i=1}^{n} \lambda_i^t (x_i + h)^\ell = \sum_{s=0}^{\ell} \binom{\ell}{s} h^{\ell-s} \sum_{i=1}^{n} \lambda_i^t x_i^s = 0 \quad \text{for any } \ell \in \mathbb{Z} \text{ such that } 0 \leq \ell \leq d-1.$$

□

By using these concepts and properties of the allowable measures, intrinsic random functions on $\mathbb{R}$ can now be formally defined.

3.1.2 Intrinsic Random Functions for Temporal Random Processes

Definition 3.1.2. A temporal random process $X(t)$ is called an *intrinsic random function of order d on* $\mathbb{R}$ (IRF$^t(d)$), where $t \in \mathbb{R}$ and $d \in \mathbb{Z}_{\geq 0}$, if the process $X(\lambda^t)$ is shift-invariant. That is, for any allowable measure $\lambda^t \in \Lambda_d^t$ and any shift operator $\tau_h \in \mathrm{T}$, where T is a group of shift transformations, the following conditions hold:

$$\mathbb{E}\left[X(\lambda^t)\right] = \mathbb{E}\left[X(\tau_h \lambda^t)\right],$$

$$\mathrm{Cov}\left(X(\lambda_1^t), X(\lambda_2^t)\right) = \mathrm{Cov}\left(X(\tau_h \lambda_1^t), X(\tau_h \lambda_2^t)\right) \quad \text{for all } \lambda_1^t, \lambda_2^t \in \Lambda_d^t.$$

Remark 3.1.1. A stationary random process is denoted as an $\mathrm{IRF}^t(0)$.

Definition 3.1.3. (Chiles and Delfiner 1999) [4] Let $X(t)$ be a random process that is an intrinsic random function of order d on $\mathbb{R}$ ($\mathrm{IRF}^t(d)$). Define

$$\lambda^t_{\epsilon_x} := \delta_x(dt) - \sum_n x^n \lambda_n(dt),$$

where the functions λ_n satisfy

$$\int \lambda_n(dx)\, x^m = \delta^m_n \quad \text{with } \delta^m_n = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } |n| \leq d.$$

Then $\lambda^t_{\epsilon_x} \in \Lambda^t_d$, because

$$\lambda^t_{\epsilon_x}(t^m) = \int \delta_x(dt)\, t^m - \sum_n x^n \int \lambda_n(dt)\, t^m = x^m - \sum_n x^n \delta^m_n = 0.$$

A stationary covariance function of the process $X(\lambda^t_{\epsilon_x})$ is called an *intrinsic covariance function of order d on* $\mathbb{R}$ ($\mathrm{ICF}^t(d)$).

3.2 Random Processes with Stationary Increments and Their Connection to Intrinsic Random Functions on $\mathbb{R}$

We now examine the connection between random processes with stationary increments and intrinsic random functions on the real line. First, we demonstrate—using Definitions 3.1.1 and 3.1.2—that the difference operator commonly used in the context of stationary increments corresponds to an allowable measure in the definition of an $\mathrm{IRF}^t(d)$.

Lemma 3.2.1. (Chiles and Delfiner 1999) [4] The differencing operator Δ_ι in Definition 2.3.1 reduces the degree of the polynomial by one level. That is, if we suppose any polynomial function with any degree $n \in \mathbb{N}$ such as

$$p_n(t) = \sum_{i=0}^{n} a_i t^i$$

Then,

$$\Delta_\iota p_n(t) = \sum_{i=1}^{n} a_i p^*_{i-1}(t) \quad \text{where} \quad p^*_{i-1}(t) = -\sum_{j=0}^{i-1} \binom{i}{j} t^j (-\iota)^{i-j}$$

Proof. Let

$$p_n(t) := \sum_{i=0}^{n} a_i t^i = a_0 + a_1 t + \cdots + a_n t^n$$

Then, for Δ_ι in Definition 2.3.1,

$$\begin{aligned}\Delta_\iota p_n(t) = p_n(t) - p_n(t-\iota) &= \sum_{i=0}^{n} a_i t^i - \sum_{i=0}^{n} a_i (t-\iota)^i \\ &= \sum_{i=0}^{n} a_i \{t^i - (t-\iota)^i\} = \sum_{i=1}^{n} a_i \{t^i - (t-\iota)^i\}\end{aligned}$$

According to the binomial theorem,

$$\begin{aligned}(t-\iota)^i = \sum_{j=0}^{i} \binom{i}{j} t^j (-\iota)^{i-j} &= \binom{i}{0}(-\iota)^i + \binom{i}{1} t(-\iota)^{i-1} + \cdots \\ &+ \binom{i}{i-1} t^{i-1}(-\iota) + \binom{i}{i} t^i \\ &= t^i - p^*_{i-1}(t)\end{aligned}$$

where

$$p^*_{i-1}(t) = -\left[\binom{i}{0}(-\iota)^i + \binom{i}{1} t(-\iota)^{i-1} + \cdots + \binom{i}{i-1} t^{i-1}(-\iota)\right],$$

which is a polynomial function of x with order $i-1$.

Hence,

$$\Delta_\iota p_n(t) = \sum_{i=1}^{n} a_i p^*_{i-1}(t)$$

In other words, $\Delta_\iota p_n(t)$ is a polynomial of order $n-1$. That is, Using Δ_ι reduces the degree of the polynomial by one level. $\square$

Lemma 3.2.2. Define

$$\lambda^t_{\Delta^d_{\iota,t}} := \sum_{k=0}^{d} (-1)^k \binom{d}{k} \delta_{t-k\iota}, \quad t, \iota \in \mathbb{R}.$$

Suppose that $X(t)$ is an $\text{IRF}^t(d)$. Then, $\lambda^t_{\Delta^d_{\iota,t}} \in \Lambda^t_d$, and $X(\lambda^t_{\Delta^d_{\iota,t}}) = \Delta^d_\iota X(t)$. That is, the finite difference operator Δ^d_ι, defined in Definition 2.3.1, corresponds to an allowable measure for an $\text{IRF}^t(d)$ and annihilates all polynomials of degree up to $d-1$.

Proof. First, suppose $p_{d-1}(t)$ is polynomial of degree $d-1$. Since $p_{d-1}(\lambda^t_{\Delta_{\iota,t}}) = \Delta_\iota p_{d-1}(t)$, by Lemma 3.2.1, $\Delta_\iota p_{d-1}(t)$ becomes a polynomial of order $d-2$. Then, it is obvious that $\Delta^{d-1}_\iota p_{d-1}(t)$ is a polynomial with degree 0, which is a constant.

Since

$$p_{d-1}(\lambda^t_{\Delta^d_{\iota,t}}) = \Delta^d_\iota p_{d-1}(t) = \Delta_\iota \Big\{ \Delta^{d-1}_\iota p_{d-1}(t) \Big\},$$

we can conclude that

$$\Delta^d_\iota p_{d-1}(x) = 0$$

To sum up, Δ^d_ι annihilates polynomials of degree $d-1$; thus, it is an allowable measure of order d. $\square$

Lemma 3.2.3. Suppose $Y(t)$ is stationary for $t \in \mathbb{R}$ and $\lambda^t \in \Lambda^t$ is any allowable measure with any order. Then, $Y(\lambda^t)$ is stationary. That is,

$$E\big(Y(\lambda^t)\big) = E\big(Y(\tau_h \lambda^t)\big)$$

$$Cov\big(Y(\lambda_1^t), Y(\lambda_2^t)\big) = Cov\big(Y(\tau_h \lambda_1^t), Y(\tau_h \lambda_2^t)\big)$$

for any $\lambda^t, \lambda_1^t, \lambda_2^t \in \Lambda^t$, and for any $h \in \mathbb{R}$, let τ_h be a shift operator.

Proof. Suppose a random process $Y(t)$ is stationary for $t \in \mathbb{R}$ and $\lambda^t \in \Lambda^t$ is any allowable measure with any order.

$$E\big(Y(\lambda^t)\big) = E\Big(\sum_{i=1}^{n} \lambda_i^t Y(t_i)\Big)$$

$$= \sum_{i=1}^{n} \lambda_i^t E\big(Y(t_i)\big)$$

By stationarity of $Y(t)$ in terms of t, for any $h \in \mathbb{R}$,

$$= \sum_{i=1}^{n} \lambda_i^t E\big(Y(t_i + h)\big)$$

Since $\tau_h \lambda^t$ is a finite measure,

$$= E\Big(\sum_{i=1}^{n} \lambda_i^t Y(t_i + h)\Big) = E\big(Y(\tau_h \lambda^t)\big)$$

Similarly, for any $\lambda_1^t, \lambda_2^t \in \Lambda^t$,

$$Cov\big(Y(\lambda_1^t), Y(\lambda_2^t)\big) = Cov\Big(\sum_{i=1}^{n} \lambda_{1i}^t Y(t_i), \sum_{j=1}^{n} \lambda_{2j}^t Y(t_j)\Big)$$

$$= \sum_{i=1}^{n} \lambda_{1i}^{t} \sum_{j=1}^{n} \lambda_{2j}^{t} Cov\big(Y(t_i), Y(t_j)\big)$$

By stationarity of $Y(t)$ in terms of t, for any $h \in \mathbb{R}$,

$$= \sum_{i=1}^{n} \lambda_{1i}^{t} \sum_{j=1}^{n} \lambda_{2j}^{t} Cov\big(Y(t_i + h), Y(t_j + h)\big)$$

$$= Cov\big(\sum_{i=1}^{n} \lambda_{1i}^{t} Y(t_i + h), \sum_{j=1}^{n} \lambda_{2j}^{t} Y(t_j + h)\big)$$

$$= Cov\big(Y(\tau_h \lambda_1^t), Y(\tau_h \lambda_2^t)\big)$$

Therefore, if $Y(t)$ is stationary for t, then $Y(\lambda)^t$ is stationary for λ^t, an allowable measure with any order. $\square$

With the preceding definitions, remarks, and lemmas, we are now equipped to establish the connection between intrinsic random functions and random processes with stationary increments on the real line.

Proposition 6. For $d \geq 1$ and $d \in \mathbb{Z}$, a stochastic process $X(t)$ is an intrinsic random function of order d (IRF$^t(d)$) on the real line if and only if $X(t)$ is a random process with stationary increments of order d, provided that the conditions in Remarks 2.3.1 and 2.3.2 hold.

Proof. ($\Rightarrow$) Let a random process $X(\cdot)$ be an intrinsic random function with order d on the real line, and define

$$\lambda_{\Delta_{\iota,t}^{d}}^{t} := \sum_{k=0}^{d} (-1)^k \binom{d}{k} \delta_{t-k\iota}, \quad t, \iota \in \mathbb{R}.$$

Then, $X(\lambda^t_{\Delta^d_{\iota,t}}) = \Delta^d_\iota X(t)$ and by Lemma 3.2.2, $\lambda^t_{\Delta^d_{\iota,t}}$ is allowable with its degree d on $\mathbb{R}$.

By Definition 3.1.2 of an intrinsic random function, for an allowable measure $\lambda^t_{\Delta^d_{\iota,t}}, \lambda^t_{\Delta^d_{\iota,s}} \in \Lambda^t_d$ and any $t, s, x \in \mathbb{R}$,

$$E\Big(X(\lambda^t_{\Delta^d_{\iota,t}})\Big) = E\Big(X(\tau_x \lambda^t_{\Delta^d_{\iota,t}})\Big) \tag{3.1}$$

$$Cov\Big(X(\lambda^t_{\Delta^d_{\iota,t}}), X(\lambda^t_{\Delta^d_{\iota,s}})\Big) = Cov\Big(X(\tau_x \lambda^t_{\Delta^d_{\iota,t}}), X(\tau_x \lambda^t_{\Delta^d_{\iota,s}})\Big) \tag{3.2}$$

Since $X(\lambda^t_{\Delta^d_{\iota,t}}) = \Delta^d_\iota X(t)$ and $X(\lambda^t_{\Delta^d_{\iota,s}}) = \Delta^d_\iota X(s)$, Equation (3.1) and (3.2) can also be expressed as

$$E\Big(\Delta^d_\iota X(t)\Big) = E\Big(\Delta^d_\iota X(\tau_x t)\Big)$$

$$Cov\Big(\Delta^d_\iota X(t), \Delta^d_\iota X(s)\Big) = Cov\Big(\Delta^d_\iota X(\tau_x t), \Delta^d_\iota X(\tau_x s)\Big)$$

Hence, $\Delta^d_\iota X(t)$ is stationary, which implies that $X(t)$ is a random process with stationary increments of order d.

($\Leftarrow$) Suppose $X(t)$ is a random process with stationary increments of order d. Then, by its spectral representation given the condition in Remark 2.3.1 and 2.3.2,

$$X(t) = \int_{-\infty}^{\infty} \Big(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!}\Big) dZ(\omega)$$

Then, for any $\lambda^t \in \Lambda^t_d$,

$$
\begin{aligned}
X(\lambda^t) &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} \left(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!} \right) dZ(\omega) \lambda^t(dt) \\
&= \sum_{j=1}^{n} \lambda_j^t \int_{-\infty}^{\infty} \left(e^{it_j\omega} - 1 - it_j\omega - \cdots - \frac{(it_j\omega)^{d-1}}{(d-1)!} \right) dZ(\omega) \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{n} \lambda_j^t \left(e^{it_j\omega} - 1 - it_j\omega - \cdots - \frac{(it_j\omega)^{d-1}}{(d-1)!} \right) dZ(\omega)
\end{aligned}
$$

By the definition of an allowable measure, any $\lambda^t \in \Lambda_d^t$ annihilates polynomials of its order $d-1$. Hence,

$$
= \sum_{j=1}^{n} \lambda_j^t \int_{-\infty}^{\infty} e^{it_j\omega} dZ(\omega)
$$

By defining $Y^*(t) := \int_{-\infty}^{\infty} e^{it\omega} dZ(\omega)$, which is a stationary process for $t \in \mathbb{R}$,

$$
= \sum_{j=1}^{n} \lambda_j^t Y^*(t_j) = Y^*(\lambda^t)
$$

Since $Y^*(t)$ is stationary with respect to t, $Y^*(\lambda^t)$ is stationary with respect to λ^t by Lemma 3.2.3. Therefore, $X(\lambda^t)$ is stationary for λ^t as well. In conclusion, $X(t)$ is an intrinsic random function of order d on the real line.

□

As a result of Proposition 6, we can conclude that $\mathrm{IRF}^t(d)$ and I(d) processes are equivalent under the conditions of Remark 2.3.1 and 2.3.2. This proposition also highlights that the differencing operator can be viewed as an allowable measure that annihilates the polynomial components of an intrinsic random function. Furthermore, it enables the use of the spectral representation of I(d) processes, as described in Remark 2.3.2, to explore the underlying structure of an intrinsic random function on $\mathbb{R}$. We can also investigate the relationship between structure functions and intrinsic covariance functions.

Remark 3.2.1. Let $X(\cdot)$ be an intrinsic random function of order d on $\mathbb{R}$, denoted as $\text{IRF}^t(d)$. Let $K(\cdot)$ be its intrinsic covariance function of $X(\cdot)$ in Definition 3.1.2. Then, the structure function $E\Big(\Delta^d X(t) \cdot \Delta^d X(s)\Big)$ can be expressed in terms of the intrinsic covariance function as

$$E\Big(\Delta^d X(t) \cdot \Delta^d X(s)\Big) = \sum_{k_1=0}^{d} (-1)^{k_1} \binom{d}{k_1} \sum_{k_2=0}^{d} (-1)^{k_2} \binom{d}{k_2} K\Big((t-k_1)-(s-k_2)\Big)$$

For example, for $d = 1$,

$$E\Big(\Delta^1 X(t) \cdot \Delta^1 X(s)\Big) = 2K(t-s) - K(t-s+1) - K(t-s-1)$$

With these foundations established, we are now prepared to address non-homogeneous and non-stationary random processes by simultaneously employing the concepts of intrinsic random functions across the different spaces.

Chapter 4

Non-Homogeneous and Non-Stationary Random Processes on the Sphere

As noted in previous studies, the assumptions of spatial homogeneity and temporal stationarity are implausible in many cases, and each of these assumptions can be relaxed by applying the concepts of intrinsic random functions on $\mathbb{S}^2$ and $\mathbb{R}$.

4.1 Intrinsic Random Function for spatio-temporal random processes on the Sphere

Definition 4.1.1. A spatio-temporal random process $X(\cdot,\cdot)$ on $\mathbb{S}^2$ is an *intrinsic random function of order* (κ, d) *on* $\mathbb{S}^2 \times \mathbb{R}$, i.e. IRF$(\kappa, d)$ if

$$E\Big(X(\lambda^s, \lambda^t)\Big) = E\Big(X(g\lambda^s, \tau_h\lambda^t)\Big)$$

$$Cov\Big(X(\lambda_1^s, \lambda_1^t), X(\lambda_2^s, \lambda_2^t)\Big) = Cov\Big(X(g\lambda_1^s, \tau_h\lambda_1^t), X(g\lambda_2^s, \tau_h\lambda_2^t)\Big)$$

for any allowable measures $\lambda^s, \lambda_1^s, \lambda_2^s \in \Lambda_\kappa^s$ and $\lambda^t, \lambda_1^t, \lambda_2^t \in \Lambda_d^t$. The classes Λ_κ^s and Λ_d^t are of all possible allowable measures on $\mathbb{S}^2$ and $\mathbb{R}$ with orders κ and d respectively, g is any rotation operator, and τ_h is a shift operator for any $h \in \mathbb{R}$.

Remark 4.1.1. An IRF$(0, 0)$ is a spatially homogeneous and temporally stationary random process on the sphere.

Theorem 1. A spatio-temporal random process on the sphere $X(P,t)$ is an intrinsic random function with orders (κ, d) if and only if its truncated and differenced process

$$\Delta_\iota^d X_\kappa(P,t) = \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} \Delta_\iota^d Z_{\ell,m}(t) Y_\ell^m(P), \quad P \in \mathbb{S}^2, \quad t, \iota \in \mathbb{R}$$

is spatially homogeneous and temporally stationary.

Proof. ($\Rightarrow$) Suppose that a spatio-temporal random process $X(\cdot,\cdot)$ on $\mathbb{S}^2$ is an IRF(κ, d). Denote allowable measures

$$\lambda_Q^s(dP) = \delta_Q(dP) - \sum_{\ell<\kappa} \sum_{m=-\ell}^{\ell} Y_\ell^m(Q) Y_\ell^m(dP)$$

$$\lambda_{\Delta_{\iota,s}^d}^t = \sum_{k=0}^{d} (-1)^k \binom{d}{k} \delta_{s-k\iota}, \quad s, \iota \in \mathbb{R}$$

where $P, Q \in \mathbb{S}^2; \quad s, h \in \mathbb{R};$

$\delta_Q(dP)$ and $\delta_{s-k\iota}$ are the Dirac measures on $\mathbb{S}^2$ and $\mathbb{R}$, respectively.

Huang et al. (2019) [8] show that $\lambda_Q^s \in \Lambda_\kappa^s$, and the proof is presented in Appendix A.2. Likewise, by Lemma 3.2.2, it is shown that $\lambda_{\Delta_{\iota,s}^d}^t \in \Lambda_d^t$. Therefore,

$$\begin{aligned} X(\lambda_Q^s, \lambda_{\Delta_{\iota,s}^d}^t) &= \int_{\mathbb{R}} \int_{\mathbb{S}^2} X(P,t) \lambda_Q^s(dP) \lambda_{\Delta_{\iota,s}^d}^t(dt) \\ &= \int_{\mathbb{R}} \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P) \lambda_Q^s(dP) \lambda_{\Delta_{\iota,s}^d}^t(dt) \end{aligned}$$

Since λ_Q^s and $\lambda_{\Delta_{\iota,s}^d}^t$ are finite measures and $Z_{\ell,m}(t)$ is integrable,

by Fubini's theorem,

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{\mathbb{R}} Z_{\ell,m}(t) \lambda_{\Delta_{\iota,s}^d}^t(dt) \int_{\mathbb{S}^2} Y_\ell^m(P) \lambda_Q^s(dP)$$

By the definitions and properties of λ_Q^s as explained Appendix A.2,

$$= \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(Q) \int_{\mathbb{R}} Z_{\ell,m}(t) \lambda_{\Delta_{\iota,s}^d}^t(dt)$$

By the definitions of $\lambda_{\Delta_{\iota,s}^d}^t$ and Lemma 3.2.2,

$$= \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(Q) \Delta_\iota^d Z_{\ell,m}(s)$$

$$= \Delta_\iota^d X_\kappa(Q,s)$$

Likewise,

$$X(g\lambda_Q^s, \tau_h \lambda_{\Delta_{\iota,s}^d}^t) = \Delta_\iota^d X_\kappa(gQ, \tau_h s)$$

By the definition of IRF(κ,d),

$$E\Big(X(\lambda_Q^s, \lambda_{\Delta_{\iota,s}^d}^t) \Big) = E\Big(X(g\lambda_Q^s, \tau_h \lambda_{\Delta_{\iota,s}^d}^t) \Big)$$

$$Cov\Big(X(\lambda_P^s, \lambda_{\Delta_{\iota,t}^d}^t), X(\lambda_Q^s, \lambda_{\Delta_{\iota,s}^d}^t) \Big) = Cov\Big(X(g\lambda_P^s, \tau_h \lambda_{\Delta_{\iota,t}^d}^t), X(g\lambda_Q^s, \tau_h \lambda_{\Delta_{\iota,s}^d}^t) \Big)$$

Therefore,

$$E\Big(\Delta_\iota^d X_\kappa(Q,s) \Big) = E\Big(\Delta_\iota^d X_\kappa(gQ, \tau_h s) \Big)$$

$$Cov\Big(\Delta_\iota^d X_\kappa(P,t), \Delta_\iota^d X_\kappa(Q,s) \Big) = Cov\Big(\Delta_\iota^d X_\kappa(gP, \tau_h t), \Delta_\iota^d X_\kappa(gQ, \tau_h s) \Big)$$

That is, $\Delta_\iota^d X_\kappa(Q,s)$ is spatially homogeneous and temporally stationary.

($\Leftarrow$) Suppose $X(P,t)$ is a spatio-temporal random process on the sphere, and its truncated and differenced process $\Delta_\iota^d X_\kappa(P,t)$ is homogeneous and stationary. We can have spectral representations such that

$$\Delta_\iota^d X_\kappa(P,t) = \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} \Delta_\iota^d Z_{\ell,m}(t) Y_\ell^m(P)$$

$$X(P,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P)$$

Then, for any $\lambda^s \in \Lambda_\kappa^s$ and $\lambda^t \in \Lambda_d^t$,

$$E\Big(X(\lambda^s,\lambda^t)\Big) = E\int_{\mathbb{R}}\int_{\mathbb{S}^2} X(P,t)\lambda^s(dP)\lambda^t(dt)$$

$$= E\int_{\mathbb{R}}\int_{\mathbb{S}^2}\Big\{\sum_{\ell=0}^{\kappa-1}\sum_{m=-\ell}^{\ell} Z_{\ell,m}(t)Y_\ell^m(P)\Big\}\lambda^s(dP)\lambda^t(dt) + E\int_{\mathbb{R}}\int_{\mathbb{S}^2} X_\kappa(P,t)\lambda^s(dP)\lambda^t(dt)$$

where $X_\kappa(P,t) = \sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} Z_{\ell,m}(t)Y_\ell^m(P)$

Since $\lambda^s \in \Lambda_\kappa^s$,

$$= E\int_{\mathbb{R}}\int_{\mathbb{S}^2} X_\kappa(P,t)\lambda^s(dP)\lambda^t(dt)$$

Since λ^s and λ^t are finite measures and $X(P,t)$ is integrable, by Fubini's theorem,

$$= \int_{\mathbb{R}}\int_{\mathbb{S}^2} E\Big(X_\kappa(P,t)\Big)\lambda^s(dP)\lambda^t(dt)$$

By the homogeneity of $X_\kappa(P,t)$,

$$= \int_{\mathbb{R}}\int_{\mathbb{S}^2} E\Big(X_\kappa(gP,t)\Big)\lambda^s(dP)\lambda^t(dt)$$

By Fubini's theorem again,

$$= \int_{\mathbb{S}^2} E\Big(\int_{\mathbb{R}} X_\kappa(gP,t)\lambda^t(dt)\Big)\lambda^s(dP) = \int_{\mathbb{S}^2} E\Big(\sum_{i=1}^{n}\lambda_i X_\kappa(gP,t_i)\Big)\lambda^s(dP)$$

$$= \int_{\mathbb{S}^2} E\Big(\sum_{i=1}^{n}\lambda_i\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} Z_{\ell,m}(t_i)Y_\ell^m(gP)\Big)\lambda^s(dP)$$

$$= \int_{\mathbb{S}^2}\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(gP)E\Big(\sum_{i=1}^{n}\lambda_i Z_{\ell,m}(t_i)\Big)\lambda^s(dP)$$

$$= \int_{\mathbb{S}^2} \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) E\bigg(Z_{\ell,m}(\lambda^t) \bigg) \lambda^s(dP)$$

By Proposition 6, given ℓ and m, a random process with stationary increments,

$Z_{\ell,m}(t)$ is an IRF(d) with respect to $t \in \mathbb{R}$.

Therefore,

$E\bigg(Z_{\ell,m}(\lambda^t) \bigg) = E\bigg(Z_{\ell,m}(\tau_h \lambda^t) \bigg)$ for any allowable $\lambda^t \in \Lambda_d^t$ and any shift operator τ_h.

Hence,

$$= \int_{\mathbb{S}^2} \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) E\bigg(Z_{\ell,m}(\tau_h \lambda^t) \bigg) \lambda^s(dP)$$
$$= \int_{\mathbb{S}^2} \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) E\bigg(\sum_{i=1}^{n} \lambda_i Z_{\ell,m}(\tau_h t_i) \bigg) \lambda^s(dP)$$
$$= \int_{\mathbb{S}^2} \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) E\bigg(\int_{\mathbb{R}} Z_{\ell,m}(\tau_h t) \lambda^t(dt) \bigg) \lambda^s(dP)$$

By Fubini's theorem,

$$= E\bigg(\int_{\mathbb{R}} \int_{\mathbb{S}^2} \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) Z_{\ell,m}(\tau_h t) \lambda^s(dP) \lambda^t(dt) \bigg)$$
$$= E\bigg(\int_{\mathbb{R}} \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(gP) Z_{\ell,m}(\tau_h t) \lambda^s(dP) \lambda^t(dt) \bigg)$$
$$= E\bigg(\int_{\mathbb{R}} \int_{\mathbb{S}^2} X(gP, \tau_h t) \lambda^s(dP) \lambda^t(dt) \bigg) = E\bigg(X(g\lambda^s, \tau_h \lambda^t) \bigg)$$

As a result, $E\bigg(X(\lambda^s, \lambda^t) \bigg) = E\bigg(X(g\lambda^s, \tau_h \lambda^t) \bigg)$. In the same manner, we can show that

$$Cov\bigg(X(\lambda_1^s, \lambda_1^t) \bigg) = Cov\bigg(X(g\lambda_2^s, \tau_h \lambda_2^t) \bigg)$$

for any $\lambda_1^s, \lambda_2^s \in \Lambda_\kappa^s$ and $\lambda_1^t, \lambda_2^t \in \Lambda_d^t$. Thus, $X(P,t)$ is an IRF(κ,d). □

Theorem 1 implies that, given their spatial components, IRF(κ, d) processes can also be interpreted as random processes with stationary increments in the temporal domain.

Therefore, we can use their associated structure functions to analyze the temporal dependence and variability of the processes, particularly in non-stationary settings. Likewise, to analyze spatial dependence, we can employ the framework of intrinsic random functions and their corresponding intrinsic covariance functions on the sphere, which are well-suited for capturing spatial non-homogeneity.

4.2 Intrinsic Structure Covariance Functions of IRF(κ, d) Processes on the Sphere

Let $X(P,t)$ be an IRF(κ, d) on $\mathbb{S}^2 \times \mathbb{R}$. In this case, the process is both spatially non-homogeneous and temporally non-stationary. By applying the structure of the spectral representation in Equation (2.1), we can extend its truncated process $X_\kappa(P,t)$ to

$$X_\kappa(P,t) = \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P)$$

where $P \in \mathbb{S}^2$ and $t \in \mathbb{R}$. Here, the coefficient $Z_{\ell,m}(t)$ for $\ell \geq \kappa$ is an IRF$^t(d)$ on the real line. Therefore, by Proposition 6, it can be represented using the spectral form of random processes with stationary increments of order d. Consequently, by Remark 2.3.2, we obtain the following spectral representation of $Z_{\ell,m}(t)$ for $\kappa \leq \ell < \infty$ and $-\ell \leq m \leq \ell$:

$$Z_{\ell,m}(t) = \int_{-\infty}^{\infty} \left(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!} \right) \frac{1}{(i\omega)^d} dZ_\ell^m(\omega) \tag{4.1}$$

$$\Delta^d Z_{\ell,m}(t) = \int_{-\infty}^{\infty} e^{i\omega t}(1 - e^{-i\omega})^d \frac{1}{(i\omega)^d} dZ_\ell^m(\omega) \tag{4.2}$$

where $Z_\ell^m(\omega)$ is a random function with uncorrelated increments with respect to $t \in \mathbb{R}$.

Then, using the fact that the truncated process $X_\kappa(P,t)$ is spatially homogeneous (as noted in Proposition 4) and that the coefficients $Z_{\ell,m}(t)$ are random processes with stationary increments of order d, the covariance functions of $Z_{\ell,m}(t)$ and $\Delta^d Z_{\ell,m}(t)$ given in equations (4.1) and (4.2) can be expressed when $\ell, \ell' \geq \kappa$ and $-\ell \leq m, m' \leq \ell$:

$$
\begin{aligned}
b_\ell(t,s) &:= Cov\Big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\Big) \\
&= \int_{-\infty}^{\infty} \Big(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!}\Big) && (4.3) \\
&\qquad \Big(e^{is\omega} - 1 - is\omega - \cdots - \frac{(is\omega)^{d-1}}{(d-1)!}\Big) \frac{1}{\omega^{2d}} dF_\ell(\omega) \delta_\ell^{\ell'} \delta_m^{m'} && (4.4) \\
a_\ell(t-s) &:= Cov\Big(\Delta^d Z_{\ell,m}(t), \Delta^d Z_{\ell',m'}(s)\Big) \\
&= \int_{-\infty}^{\infty} e^{i\omega(t-s)} (1-e^{-i\omega})^d (1-e^{i\omega})^d \frac{1}{\omega^{2d}} dF_\ell(\omega) \delta_\ell^{\ell'} \delta_m^{m'} && (4.5)
\end{aligned}
$$

where $F_\ell(\omega)$ is a bounded non-decreasing function and $E(|dZ_\ell^m(\omega)|^2) = dF_\ell(\omega)$.

Definition 4.2.1. Let $X(P,t)$ be an IRF(κ, d) on $\mathbb{S}^2 \times \mathbb{R}$. We call the covariance function of a truncated and differenced random process $\Delta^d X_\kappa(P,t)$ on the sphere an *Intrinsic Structure Covariance Function of order* (κ, d), abbreviated as ISCF(κ, d). By Equation (4.4) and (4.5), we can derive the spectral representation of an ISCF(κ, d) as following:

$$
\begin{aligned}
\Phi_{(\kappa,d)}(\overrightarrow{PQ}, t-s) &:= Cov\Big(\Delta^d X_\kappa(P,t), \Delta^d X_\kappa(Q,s)\Big) \\
&= \sum_{\ell=\kappa}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos \overrightarrow{PQ}) \int_{-\infty}^{\infty} e^{i\omega(t-s)} (1-e^{-i\omega})^d (1-e^{i\omega})^d \frac{1}{\omega^{2d}} dF_\ell(\omega) \\
&= \xi_0\Big(\overrightarrow{PQ}, t-s\Big) - \sum_{\ell=0}^{\kappa-1} \frac{2\ell+1}{4\pi} a_\ell(h) P_\ell(\cos \overrightarrow{PQ}) && (4.6)
\end{aligned}
$$

where $P, Q \in \mathbb{S}^2$, $\quad t, s \in \mathbb{R}$, and $dF_\ell(\omega)$ is a non-decreasing and bounded measure. The function $a_\ell(h)$ is a stationary time series covariance function in (4.5) and $\xi_0(\cdot,\cdot)$ is a homogeneous and stationary spatio-temporal covariance function on the sphere. Therefore, the covariance functions presented in Table 2.1 can be used for the structure of $\xi_0(\cdot,\cdot)$.

Remark 4.2.1. The Intrinsic Structure Covariance Function (ISCF) (κ, d), is an intrinsic covariance function of order κ with respect to the spatial components on the sphere, and a structure function of order d with respect to the temporal components.

Remark 4.2.2. The intrinsic structure covariance function, ISCF(κ, d), exhibits spatial homogeneity and temporal stationarity, which facilitate its estimation using the method of moments, as demonstrated in Chapter 5.

Remark 4.2.3. Different parametric models of ISCFs can be obtained by varying the functions $a_\ell(\cdot)$ and $\xi_0(\cdot,\cdot)$ in Equation (4.6), as illustrated in Table 2.1.

As a result, the ISCF provides valuable insights into the homogeneous and stationary dependency structure of the data. Furthermore, this framework can be extended to explore the structure of non-homogeneous and non-stationary covariance functions associated with intrinsic random functions, IRF(κ, d)s, for spatio-temporal data analysis on the sphere.

4.3 Non-Homogeneous and Non-Stationary Covariance Functions of IRF(κ, d)s on $\mathbb{S}^2 \times \mathbb{R}$

Let $X(P,t)$ be an IRF(κ, d). Then, it can be expressed as

$$X(P,t) = X_N(P,t) + X_\kappa(P,t)$$

where

$$X_N(P,t)=\sum_{\ell=0}^{\kappa-1}\sum_{m=-\ell}^{\ell}Z_{\ell,m}(t)Y_\ell^m(P),\quad X_\kappa(P,t)=\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}Z_{\ell,m}(t)Y_\ell^m(P)$$

Thus, the non-homogeneous and non-stationary covariance function of the IRF(κ,d) can be decomposed as:

$$\begin{aligned}
Cov\big(X(P,t),X(Q,s)\big) &= Cov\Big(X_N(P,t)+X_\kappa(P,t),X_N(Q,s)+X_\kappa(Q,s)\Big)\\
&= Cov\Big(X_\kappa(P,t),X_\kappa(Q,s)\Big)+Cov\Big(X_\kappa(P,t),X_N(Q,s)\Big)\\
&\quad + Cov\Big(X_N(P,t),X_\kappa(Q,s)\Big)+Cov\Big(X_N(P,t),X_N(Q,s)\Big)\\
&= \phi_\kappa\big(\overrightarrow{PQ},(t,s)\big)\\
&\quad+\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\kappa-1}\sum_{m'=-\ell}^{\ell}Cov\Big(Z_{\ell,m}(t),Z_{\ell',m'}(s)\Big)Y_\ell^m(P)Y_{\ell'}^{m'}(Q)\\
&\quad+\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\kappa-1}\sum_{m'=-\ell}^{\ell}Cov\Big(Z_{\ell',m'}(t),Z_{\ell,m}(s)\Big)Y_{\ell'}^{m'}(P)Y_\ell^m(Q)\\
&\quad+\sum_{\ell=0}^{\kappa-1}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\kappa-1}\sum_{m'=-\ell}^{\ell}Cov\Big(Z_{\ell,m}(t),Z_{\ell',m'}(s)\Big)Y_\ell^m(P)Y_{\ell'}^{m'}(Q)
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
\phi_\kappa\big(\overrightarrow{PQ},(t,s)\big) &:= Cov\big(X_\kappa(P,t),X_\kappa(Q,s)\big)\\
&=\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=\kappa}^{\infty}\sum_{m'=-\ell'}^{\ell'}Cov\big(Z_{\ell,m}(t),Z_{\ell',m'}(s)\big)Y_\ell^m(P)Y_{\ell'}^{m'}(Q)\\
&=\sum_{\ell=\kappa}^{\infty}\frac{2\ell+1}{4\pi}b_\ell(t,s)P_\ell(\cos\overrightarrow{PQ})
\end{aligned}\tag{4.8}$$

Here, $b_\ell(t,s)$ is the non-stationary covariance function of a process with stationary increments of order d, as defined in Equation (4.4).

The function $\phi_\kappa\big(\overrightarrow{PQ},(t,s)\big)$ in (4.8) is the spatial intrinsic covariance function of order κ with respect to the spatial components on the sphere, as it represents the covariance function of the truncated process $X_\kappa(P,t)$. Therefore, $\phi_\kappa\big(\overrightarrow{PQ},(t,s)\big)$ is spatially homogeneous. In contrast, the remaining terms in (4.7) involving the spherical harmonics $Y_{\ell'}^{m'}(\cdot)$ for $\ell' < \kappa$ reflect the spatially non-homogeneous characteristics of the intrinsic random function on the sphere. These non-homogeneous properties render inference of these components from a single realization of the process infeasible (Chiles and Delfiner 1999) [4].

Although $\phi_\kappa\big(\overrightarrow{PQ},(t,s)\big)$ is spatially homogeneous as the intrinsic covariance function on the sphere, this function also serves as the non-stationary temporal covariance function of a random process with stationary increments as represented by $b_\ell(t,s)$ in Equation (4.4) and (4.8). This temporal non-stationarity can be addressed by applying differencing operations, allowing one to derive intrinsic structure covariance functions (ISCF)s, as described in Definition 4.2.1, that capture the underlying stationary behavior of the process.

Example

Equation (4.7) indicates that, depending on the structure of the covariance functions of the coefficients $Z_{\ell,m}(t)$, different non-homogeneous and non-stationary covariance functions of IRF(κ,d) can be derived. For instance, we can express IRF(κ,d) as follows:

$$X(P,t) = \sum_{\nu=1}^{\kappa^2} Z_\nu(t)q_\nu(P) + \sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} Z_{\ell,m}(t)Y_\ell^m(P),$$

where $q_\nu(\cdot)$ is a new basis function of the nil space explained in Equation 2.2.2.

If we assume that

$$Z_\nu(t) = -\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} Z_{\ell,m}(t)Y_\ell^m(\tau_\nu) + A_\nu,$$

where $Z_\nu(t)$ is expressed as a linear combination of $Z_{\ell,m}(t)$ with a coefficient of $Y_\ell^m(\tau_\nu)$, defined for $\ell \geq \kappa$ and $|m| < \ell$. The random variable A_ν is assumed to be independent of all other $Z_{\ell,m}(t)$ and $Z_\mu(t)$ for $\mu \neq \nu$, with variance $\mathrm{Var}(A_\nu) = \gamma_\nu^2 \in \mathbb{R}$. Under these assumptions, we can derive the following:

$$\begin{aligned}
&Cov\Big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\Big) = b_\ell(t,s)\\
&Cov\Big(Z_\nu(t), Z_{\ell,m}(s)\Big) = Cov\Big(Z_{\ell,m}(t), Z_\nu(s)\Big) = -\sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell} b_\ell(t,s)Y_\ell^m(\tau_\nu)\\
&Cov\Big(Z_\nu(t), Z_\mu(s)\Big) = \gamma_\nu^2 I(\nu,\mu) + \sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=\kappa}^{\infty}\sum_{m'=-\ell'}^{\ell'} b_\ell(t,s)Y_\ell^m(\tau_\nu)Y_{\ell'}^{m'}(\tau_\mu)
\end{aligned}$$

where $b_\ell(t,s)$ is an non-stationary covariance function of a random process with stationary increments, as presented in Equation 4.4.

As a result, we can derive the non-homogeneous and non-stationary covariance function of IRF(κ, d) as

$$\begin{aligned}
&R(P,Q,t,s) := Cov\Big(X(P,t), X(Q,s)\Big)\\
&= \phi_\kappa\Big(\overrightarrow{PQ},(t,s)\Big) + \sum_{\nu=1}^{\kappa^2}\sum_{\mu=1}^{\kappa^2}\phi_\kappa\Big(\overrightarrow{\tau_\nu\tau_\mu},(t,s)\Big)q_\nu(P)q_\mu(Q) + \sum_{\nu=1}^{\kappa^2}\gamma_\nu^2 q_\nu(P)q_\nu(Q)\\
&-\sum_{\nu=1}^{\kappa^2}\phi_\kappa\Big(\overrightarrow{Q\tau_\nu},(t,s)\Big)q_\nu(P) - \sum_{\nu=1}^{\kappa^2}\phi_\kappa\Big(\overrightarrow{P\tau_\nu},(t,s)\Big)q_\nu(Q) \qquad (4.9)
\end{aligned}$$

where

$$\phi_\kappa\left(\overrightarrow{PQ},(t,s)\right)= Cov\left(X_\kappa(P,t),X_\kappa(Q,s)\right)$$
$$=\sum_{\ell=\kappa}^{\infty}\frac{2\ell+1}{4\pi}P_\ell(\cos\overrightarrow{PQ})$$
$$\int_{-\infty}^{\infty}\left(e^{it\omega}-1-it\omega-\cdots-\frac{(it\omega)^{d-1}}{(d-1)!}\right)\left(e^{-is\omega}-1+is\omega-\cdots-\frac{(-is\omega)^{d-1}}{(d-1)!}\right)\frac{1}{\omega^{2d}}dF_\ell(\omega)$$

where $P,Q\in\mathbb{S}^2,\quad t,s\in\mathbb{R}$, and $dF_\ell(\omega)$ is a non-decreasing and bounded measure.

By the Kolmogorov existence theorem, there exists a Gaussian random process having such covariance function; hence, we can derive a covariance function of an intrinsic random function of order (κ,d). We can verify that, given $t,s\in\mathbb{R}$ and $\gamma_\nu^2=1$, the covariance function in Equation (4.7) corresponds to the reproducing kernel of the reproducing kernel Hilbert space (RKHS) on the sphere suggested by Leversely et al. (1999) [9] in Section 2.2.2. The positive semi-definiteness of Euquation (4.9) is provided in the appendix A.6.

Note that the function $\phi_\kappa(\cdot,\cdot)$ in Equation (4.7) and (4.9) represents the covariance of the truncated process $X_\kappa(\cdot,\cdot)$, while the intrinsic structure covariance function (ISCF) $\Phi_{(\kappa,d)}(\cdot,\cdot)$ in Equation (4.6) represents the covariance function of the truncated and differenced process $\Delta^d X_\kappa(\cdot,\cdot)$. Hence, $\phi_\kappa(\cdot,\cdot)$ is not temporally stationary, whereas $\Phi_{(\kappa,d)}(\cdot,\cdot)$ is.

Chapter 5

Simulation Study and Data Analysis

In this section, a methodology for parameter estimation is presented for the proposed intrinsic structure covariance function using its homogeneity and stationarity. This is illustrated using a simulation study and the analysis of a real-world dataset on global temperature anomalies, which was collected by the National Space Science and Technology Center (NSSTC) at the University of Alabama, Huntsville.

5.1 Simulation Study

In the simulation, we adopt the intrinsic structure covariance function (ISCF) with the generating function of Legendre polynomials with $\alpha g(h) = \alpha e^{-\beta|h|}$ from Table 2.1. Consequently, the corresponding homogeneous and stationary function with a scale parameter $\gamma_0^2 \in \mathbb{R}_{\geq 0}$ can be expressed as

$$\begin{aligned}\gamma_0^2 \cdot \xi_0(\psi, h) &= \gamma_0^2 \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos\psi) \int_{-\infty}^{\infty} e^{i\omega h}(1-e^{-i\omega})(1-e^{i\omega}) \frac{1}{\omega^2} dF_\ell(\omega) \\ &= \frac{\gamma_0^2}{4\pi} \frac{1-\alpha^2 g^2(h)}{(1-2\alpha g(h)\cos\psi + \alpha^2 g^2(h))^{3/2}} \quad \text{where} \quad \alpha g(h) = \alpha e^{-\beta|h|} \end{aligned} \tag{5.1}$$

By Remark 2.3.1, we assume the existence of a spectral density such that $f_\ell(\omega)d\omega = dF_\ell(\omega) = \frac{\alpha^\ell \beta\ell}{\pi(\omega^2+\beta^2\ell^2)} d\omega$, which is non-negative since $\ell, \alpha, \beta \geq 0$, corresponding to $\alpha^\ell g^\ell(h) = \alpha^\ell \beta^{-\beta\ell|h|}$.

This homogeneous and stationary covariance function in Equation (5.1) is flexible and practical, as the decay rate of the covariance with respect to both the time lag and the spherical distance can be controlled by adjusting the values of the parameters. To illustrate this, we conduct an analysis to examine the roles of the parameters α and β in $a_\ell(h) = \alpha^\ell e^{-\beta\ell|h|}$ within the covariance function given in Equation (5.1). In Figure 5.1, we observe that correlations decrease as both great circle distances and temporal lags increase, as expected. Figure 5.1 also demonstrates how the parameters α and β control the decay rates of the covariance with respect to the time lag and spherical distance. The top two plots compare covariance functions with different α values while keeping $\beta = 0.1$ fixed. The left plot shows how the covariance values change with the time lag, while holding the great circle distance $\psi = 0$ constant. The right plot illustrates how covariance values vary with different great circle distances while fixing the time lag at $h = 0$. Similarly, the two plots at the bottom compare covariance values for different β values, with $\alpha = 0.8$ fixed.

From the visualization in Figure 5.1, we conclude that the flexibility of the decay rate can be controlled by adjusting α and β. Additionally, we observe that larger values of α and β lead to a faster decay in the covariance. This property is important because it suggests that we do not need to consider every observation for parameter estimation. Specifically, when calculating the estimate, we only need to consider pairs of data whose spatial or temporal distances are small enough for their correlation values to be significantly different from zero. For example, as shown in Figure 5.1, when $\alpha = 0.8$ and $\beta = 0.3$ (bottom green curve), pairs with spherical distances greater than 1.5 and time lags greater than 4 have correlations that are nearly zero. Thus, these pairs can be ignored in the estimation process.

In this simulation study, we aim to select reasonable values for α and β to demonstrate their estimation, ensuring that the correlations decay at an appropriate rate—not too slowly and not too quickly. For instance, when $\alpha = 0.8$ and $\beta = 0.9$, the correlation values at a time lag $h = 1$ or at a great circle distance near zero are already close to zero, indicating that the process is nearly independent of its spatial and temporal distances. On the other hand, when $\alpha = 0.1$ and $\beta = 0.1$, correlations decay so slowly that they do not approach zero even for pairs with large time lags and great circle distances. This slowly decaying function makes the model unrealistic, increases computational cost, and leads to unreliable inference. Finally, the value of the scale parameter γ_0^2 has little effect on the demonstration of the estimation using the simulation data, so we fix $\gamma_0^2 = 1$.

By utilizing this flexible homogeneous and stationary covariance function, the simulation primarily demonstrates parameter estimation for the ISCF(κ, d) models of the IRF(κ, d), considering $\kappa = 0, 1, 2$ and $d = 0, 1$.

5.1.1 Covariance Function of IRF(0,0)

As mentioned in Remark 4.1.1, an IRF$(0,0)$ model represents a spatially homogeneous and temporally stationary random process on the sphere. Therefore, since its covariance function is already an intrinsic structure covariance function, we can adopt the homogeneous and stationary covariance function $\gamma_0^2 \cdot \xi_0(\cdot, \cdot)$, given in Equation (5.1), as the covariance function for IRF$(0,0)$.

5.1.2 Covariance Function of IRF(1,1)

Let $X(P,t)$ be an IRF$(1,1)$ on the sphere. For this simulation, we use the non-homogeneous and non-stationary covariance model in Equation 4.9. Then, by setting $\gamma_\nu^2 = 1$ and $q_1(P) = Y_0^0(P) = \frac{1}{\sqrt{4\pi}}$, the non-homogeneous and non-stationary covariance functions of $X(P,t)$ can be derived as follows.

$$
\begin{aligned}
R(P,Q,t,s) &= Cov\big(X(P,t), X(Q,s)\big) \\
&= \phi_1\big(\overrightarrow{PQ}, (t,s)\big) + \frac{1}{4\pi}\phi_1\big(0, (t,s)\big) + \frac{1}{4\pi} \\
&\quad - \frac{1}{\sqrt{4\pi}}\phi_1\big(\overrightarrow{Q\tau}, (t,s)\big) - \frac{1}{\sqrt{4\pi}}\phi_1\big(\overrightarrow{P\tau}, (t,s)\big)
\end{aligned} \tag{5.2}
$$

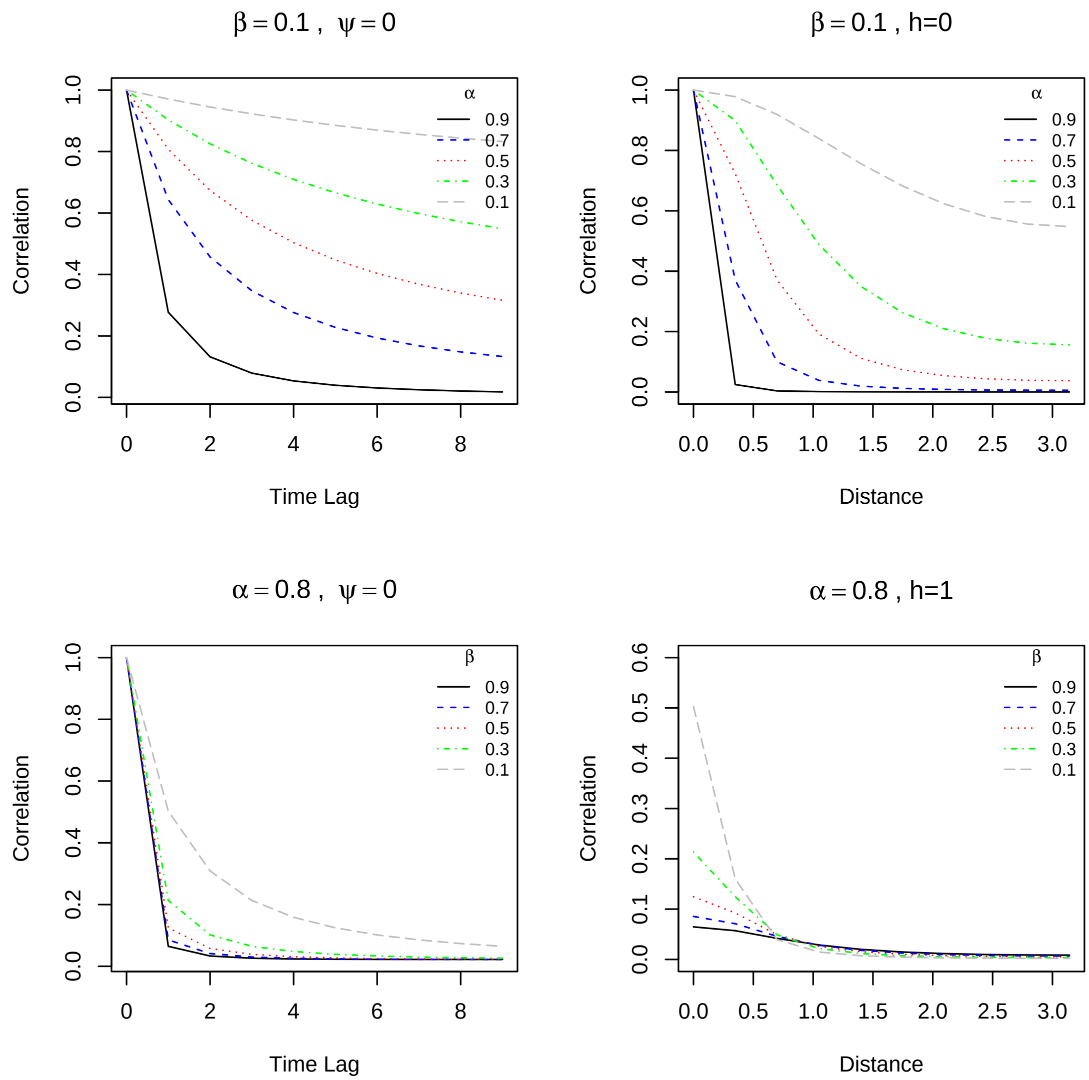

Figure 5.1: Correlations based on different parameter values for spatially homogeneous and temporally stationary model with the generating function of Legendre polynomials with $ag(h) = \alpha e^{-\beta\ell|h|}$ in Table 2.1.

where

$$\begin{aligned}\phi_1\big(\overrightarrow{PQ},(t,s)\big) &= Cov\big(X_1(P,t), X_1(Q,s)\big)\\ &= \gamma_0^2 \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos\overrightarrow{PQ}) \int_{-\infty}^{\infty} (e^{i\omega t}-1)(e^{-i\omega s}-1)\frac{1}{\omega^2} dF_\ell(\omega),\end{aligned}$$

and $\gamma_0^2 \in \mathbb{R}_{\geq 0}$ is a scale parameter.

Note that the function $\phi_1\big(\cdot,(\cdot,\cdot)\big)$ is the spatially homogeneous but temporally non-stationary. The temporal stationairity can be achieved by deriving the intrinsic structure covariance function, ISCF$(1,1)$. The corresponding intrinsic structure covariance function, ISCF$(1,1)$, can be computed as

$$\begin{aligned}
\Phi_{(1,1)}\big(\overrightarrow{PQ},(t,s)\big) &:= Cov\Big(\Delta_1 X_1(P,t), \Delta_1 X_1(Q,s)\Big) \\
&= \gamma_0^2 \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos \overrightarrow{PQ}) \int_{-\infty}^{\infty} e^{i\omega(t-s)}(1-e^{-i\omega})(1-e^{i\omega})\frac{1}{\omega^2} dF_\ell(\omega) \\
&= \gamma_0^2 \cdot \xi_0\big(\overrightarrow{PQ}, t-s\big) - \frac{\gamma_0^2}{4\pi} \qquad (5.3)
\end{aligned}$$

where the function $\gamma_0^2 \cdot \xi_0(\cdot,\cdot)$ is given in (5.1).

5.1.3 Covariance Function of IRF(2,1)

Now, let $X(P,t)$ be an IRF$(2,1)$ on the sphere. Following a similar approach and based on Equation (4.9), we set $\gamma_\nu^2 = 1$, and choose the basis functions of Nil space as $\big\{Y_0^0(P), Y_1^{-1}(P), Y_1^0(P), Y_1^1(P)\big\}$ instead of $\big\{q_1(P), q_2(P), q_3(P), q_4(P)\big\}$. Then, the corresponding non-homogeneous and non-stationary covariance functions are given by:

$$\begin{aligned}
R(P,Q,t,s) &:= Cov\Big(X(P,t), X(Q,s)\Big) \\
&= \phi_2\big(\overrightarrow{PQ},(t-s)\big) + \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{1}\sum_{m'=-\ell'}^{\ell'} \phi_2\big(\overrightarrow{\tau_{\ell,m}\tau_{\ell',m'}},(t,s)\big) Y_\ell^m(P) Y_{\ell'}^{m'}(Q) \\
&\quad + \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{1}\sum_{m'=-\ell'}^{\ell'} Y_\ell^m(P) Y_{\ell'}^{m'}(Q) - \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell} \phi_2\big(\overrightarrow{Q\tau_{\ell,m}},(t,s)\big) Y_\ell^m(P) \\
&\quad - \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell} \phi_2\big(\overrightarrow{P\tau_{\ell,m}},(t,s)\big) Y_\ell^m(Q) \qquad (5.4)
\end{aligned}$$

where

$$
\begin{aligned}
\phi_1\big(\overrightarrow{PQ},(t,s)\big) &= Cov\big(X_1(P,t),X_1(Q,s)\big) \\
&= \gamma_0^2 \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos\overrightarrow{PQ}) \int_{-\infty}^{\infty} \big(e^{i\omega t}-1-i\omega t\big)\big(e^{-i\omega s}-1-(-iws)\big)\frac{1}{\omega^4} dF_\ell(\omega)
\end{aligned}
$$

where $\gamma_0^2 \in \mathbb{R}_{\geq 0}$ is a scale parameter.

For the intrinsic structure covariance function, ISCF$(2,1)$,

$$
\begin{aligned}
\Phi_{(2,1)}\left(\overrightarrow{PQ},t-s\right) &:= Cov\left(\Delta_1 X_2(P,t),\Delta_1 X_2(Q,s)\right) \\
&= \gamma_0^2 \sum_{\ell=2}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\cos\overrightarrow{PQ}) \int_{-\infty}^{\infty} e^{i\omega(t-s)}(1-e^{-i\omega})^2(1-e^{i\omega})^2 \frac{1}{\omega^4} dF_\ell(\omega) \\
&= \gamma_0^2 \cdot \xi_0\left(\overrightarrow{PQ},t-s\right) - \frac{\gamma_0^2}{4\pi} - \frac{3\gamma_0^2}{4\pi} P_1(\cos\overrightarrow{PQ})\alpha g(t-s) \qquad (5.5)
\end{aligned}
$$

where γ_0^2 is a scale parameter. The functions $\gamma_0^2 \cdot \xi_0(\cdot,\cdot)$ and $\alpha g(t-s)$ are presented in (5.1).

5.1.4 Simulation Procedure

The procedure for the simulation and parameter estimation of the intrinsic structure covariance function for the IRF$(1,1)$ model is outlined here. Simulations for other levels of the IRF models can be carried out in a similar manner. For instance, in the case of IRF$(2,1)$, the covariance function in Equation (5.2) is replaced by Equation (5.4), and the intrinsic structure covariance function in Equation (5.3) is replaced by Equation (5.5).

1. Generate $X(P,t)$, an IRF$(1,1)$, from multivariate normal MVN$\big(0, R(P,Q,t,s)\big)$. The non-homogeneous and non-stationary covariance function R(P,Q,t,s) is given in Equation (5.2). To generate this temporally non-stationary random process, the property of I(d) that $X(P,t) = \sum_{s=1}^{t} \Delta_1^d X(P,s)$, assuming $X(P,0) = 0$, can be utilized. Therefore, to obtain the dataset for an IRF$(1,1)$ process $X(P,t)$, we first generate the temporally stationary process $\Delta_1 X(P,t)$ using the covariance function in Equation (5.3). After that, we reconstruct the desired non-stationary IRF$(1,1)$ process via the summation $X(P,t) = \sum_{s=1}^{t} \Delta_1 X(P,s)$.

2. Compute the homogeneous and stationary random process $\Delta_1 X_1(P,t)$ by truncating and differencing the process $X(P,t)$ generated in Step 1. For the truncation, $\Delta_1 X_\kappa(P,t)$ is approximated by the residual process $\Delta_1 X_{\kappa,r}(P,t)$, obtained by regressing $\Delta_1 X(P,t)$ on $Y_\ell^m(P)$ for $\ell < \kappa$, with the time component t treated as a factor (Bussberg 2020) [2].

3. Estimate the parameters of an intrinsic structure covariance function in Equation (5.3) through Method of Moments (MoM) estimates and numerical optimization with respect to the sum of squares between their theoretical values and MoM estimates. For the numerical optimization, the quasi-Newton method is employed via the R function *nlminb* (Gay 1990) [5].

To get the MoM estimates, let $\Delta_1 X_1(P,t)$ be a spatio-temporal random process with its average $\mu = 0$. Then, its MoM estimator is (Bussberg 2020) [2]:

$$\hat{\Phi}_{(1,1)}(\psi_i, h_j) = \frac{1}{|N_{\psi_i,h_j}|} \sum_{\{(P,t),(Q,s)\} \in N_{\psi_i,h_j}} \Delta_1 X_{1,r}(P,t) \Delta_1 X_{1,r}(Q,s) \tag{5.6}$$

where

$$N_{\psi_i,h_j} = \Big\{\{(P,t),(Q,s)\} \quad | \quad \psi_i - \epsilon_s \leq \psi(P,Q) \leq \psi_i + \epsilon_s, \quad h_j - \epsilon_t \leq |t-s| \leq h_j + \epsilon_t\Big\} \tag{5.7}$$

The terms ϵ_s and ϵ_t are tolerances introduced because, in practice, there may be relatively few pairs of points with distances exactly equal to ψ_i or h_j. For this simulation study, we set $\epsilon_s = 0.05$ while $\epsilon_t = 0$ since the temporal terms are discrete. As a result, the condition $h_j - \epsilon_t \leq |t-s| \leq h_j + \epsilon_t$ in Equation (5.7) can be replaced by the simpler form, $|t-s| = h_j$.

We then seek $\hat{p}$ and $\hat{\gamma_0}$ that minimize the following loss function.

$$L(p,\gamma_0) = \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\Big[\hat{\Phi}_{(1,1)}(\psi_i,h_j) - \Phi_{(1,1)}(\psi_i,h_j,p,\gamma_0^2)\Big]^2.$$

Here, p denotes the parameters of $\alpha g(\cdot)$ in the temporally stationary function given in Equation (5.1). Since we use $a_\ell(h) = \alpha^\ell g^\ell(h) = \alpha^\ell e^{-\beta\ell|h|}$, which decays exponentially with the time lag h, the parameters of the intrinsic structure covariance function to estimate are $p = (\alpha, \beta)$. The term γ_0^2 is a scale parameter appearing in Equation (5.3).

5.1.5 Simulation Result

In this simulation setting, each simulated dataset consists of 1,000 locations and 20 discrete temporal points. Figure 5.2 and Figure 5.3 compare the MoM estimates, fitted values, and true values of the covariance, given that either time lags or great circle distances are fixed with $\alpha = 0.80$, $\beta = 0.10$, and $\gamma_0^2 = 1$. These parameters are chosen because they yield reasonable decay rates in the covariance functions, as illustrated in Figure 5.1 at the beginning of this section. The "True" value refers to the actual values of the intrinsic structure covariance function computed with the true parameter values. The "Fitted" values are obtained by plugging the parameter estimates into the intrinsic structure covariance function formula in Equation (5.3), while the "MoM" values come from Equation (5.6). Since the simulation uses discrete temporal points, the tolerance ϵ_t in Equation (5.7) is unnecessary and can therefore be set to $\epsilon_t = 0$ when deriving the MoM values. From this comparison, we conclude that our estimation of the intrinsic structure covariance function performs well, as the fitted values closely match the true values across varying spatial distances and time lags. Furthermore, as expected, the values of the intrinsic structure covariance function exhibit decay with increasing time lags and great-circle distances.

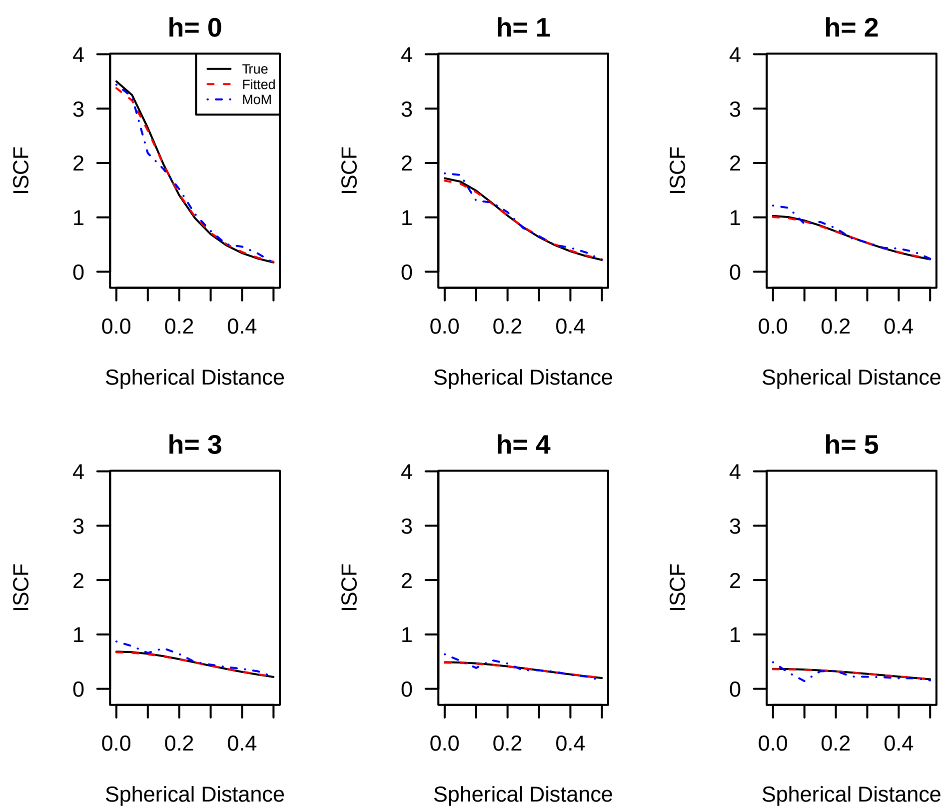

Figure 5.2: MoM estimates, fitted covariance values, and true ISCF values for IRF$(1,1)$ with $\alpha = 0.8$, $\beta = 0.1$, and $\gamma_0^2 = 1$.

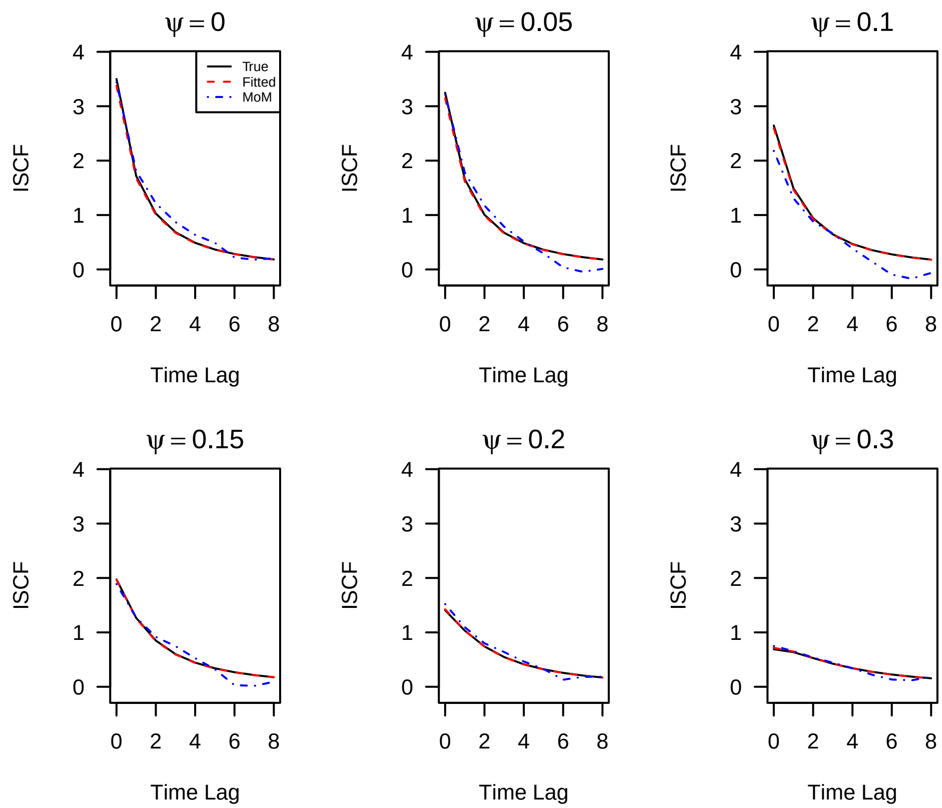

Figure 5.3: MoM estimates, fitted covariance values, and true ISCF values for IRF$(1, 1)$ with $\alpha = 0.8$, $\beta = 0.1$, and $\gamma_0^2 = 1$.

In order to access the robustness of the parameter estimates, we conduct the same simulation multiple times. Figure 5.4 shows the distributions of the 500 parameter estimates with the true parameter values $\alpha = 0.8$, $\beta = 0.1$, and $\gamma_0^2 = 1$. "Avg" denotes the average value of the 500 estimates, while "True" refers to the true parameter values used in the simulation dataset. By examining the shape of the distributions and the average values of these estimates, we can assess the robustness of the estimation. Table 5.1 presents the average values and standard errors of the parameter estimates for the intrinsic structure covariance function, based on 500 simulation runs of IRF(κ, d) processes with varying levels of non-homogeneity κ and non-stationarity d. From these results, we conclude that the parameter estimation performs consistently well across different levels of $\kappa = 0, 1, 2$ and $d = 0, 1$.

The performance of parameter estimation can vary depending on the true values of the parameters, as these values influence the decay behavior of the covariance or intrinsic structure covariance functions. The simulation study shows that estimates of α are relatively robust, closely aligning with the true value and exhibiting low variability. sufficiently large to cause the covariance function to decay too rapidly. This behavior arises because β appears in the exponential term of $a_\ell(h) = \alpha^\ell e^{-\beta\ell|h|}$ in the intrinsic structure covariance function. If β is too large, the covariance—or intrinsic structure covariance function value—between time-lagged pairs decays so rapidly that it approaches zero even for small lags (e.g., $h \geq 1$), thereby making accurate estimation difficult. For example, when $\alpha = 0.8$ and $\beta = 0.9$ as shown in Figure 5.1 (bottom left), the covariance value is already almost zero for $h \geq 1$. As a result, values of very big βs become indistinguishable in that their covariance values are always zero unless their time lags h are near 0. The estimation of the scale parameter γ_0^2 also performs correctly depending on the other two parameter values. In general, the results of the simulation study allow us to verify the validity of the suggested parameter estimation method.

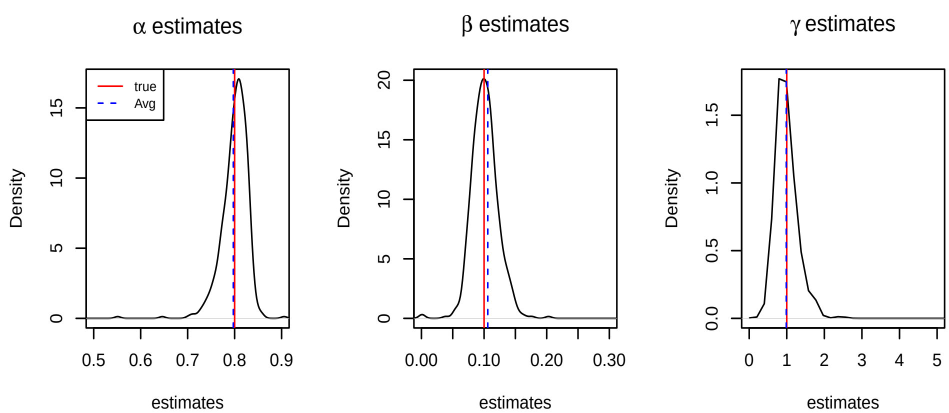

Figure 5.4: Distributions of the parameter estimates with $\alpha = 0.8$, $\beta = 0.1$, and $\gamma_0^2 = 1$ are shown for the model based on the generating function of Legendre polynomials, where $a_\ell(h) = \alpha^\ell e^{-\beta\ell|h|}$, as summarized under $IRF(1,1)$ in Table 5.1. The "true" value (solid red line) represents the actual parameter value, while "Avg" (dotted blue line) denotes the average of the 500 estimates.

Table 5.1: Averages and standard errors of the parameter estimates from 500 simulations with true parameter values: $\alpha = 0.8$, $\beta = 0.1$, and $\gamma_0^2 = 1$. Each simulation includes 1,000 spatial locations and 20 temporal points. The function used is $a_\ell(h) = \alpha^\ell e^{-\beta\ell|h|}$.

	Avg($\hat{\alpha}$) (se($\hat{\alpha}$))	Avg($\hat{\beta}$) (se($\hat{\beta}$))	Avg($\hat{\gamma_0^2}$) (se($\hat{\gamma_0^2}$))
IRF(0,0)	0.7898 (0.0032)	0.1099 (0.0025)	1.0939 (0.0310)
IRF(1,1)	0.7970 (0.0012)	0.1240 (0.0199)	1.0126 (0.0137)
IRF(2,1)	0.7816 (0.0042)	0.1565 (0.0060)	1.1368 (0.0136)

5.2 Analysis of Temperature Anomaly

The National Space Science and Technology Center (NSSTC) at the University of Alabama Huntsville provides data of monthly temperature anomalies from 1979 to 2024. Each anomaly (measured in degrees Celsius) was computed by subtracting each grid point's baseline average from each month's temperature in the lower troposphere of the Earth. The baseline average was measured from 1979 to 1990. Therefore, if some location has a positive temperature anomaly, then it implies that the location at that time point had a higher temperature than the base line average. On the other hand, if it shows a negative value, then the temperature of that location was lower than the long-term average. Figure 5.5 shows examples of January and July in 2015 and 2016 to explain how the data set appears. For the data analysis, 10,368 grid points are used with temporal terms of 24 months from 2015 to 2016.

In this analysis, similar to the simulation study, we consider a model that uses the generating function of Legendre polynomials, with $\alpha g(h) = \alpha e^{-\beta|h|}$ (as presented in Table 2.1), for the intrinsic structure covariance function (ISCF) in Equation (5.3). For the distance bins used in the method of moments (MoM) estimation in Equation (5.7), we set $\epsilon_t = 0.05$ and $\epsilon_s = 0$, following the same rationale as in the simulation study. Building on this setup,

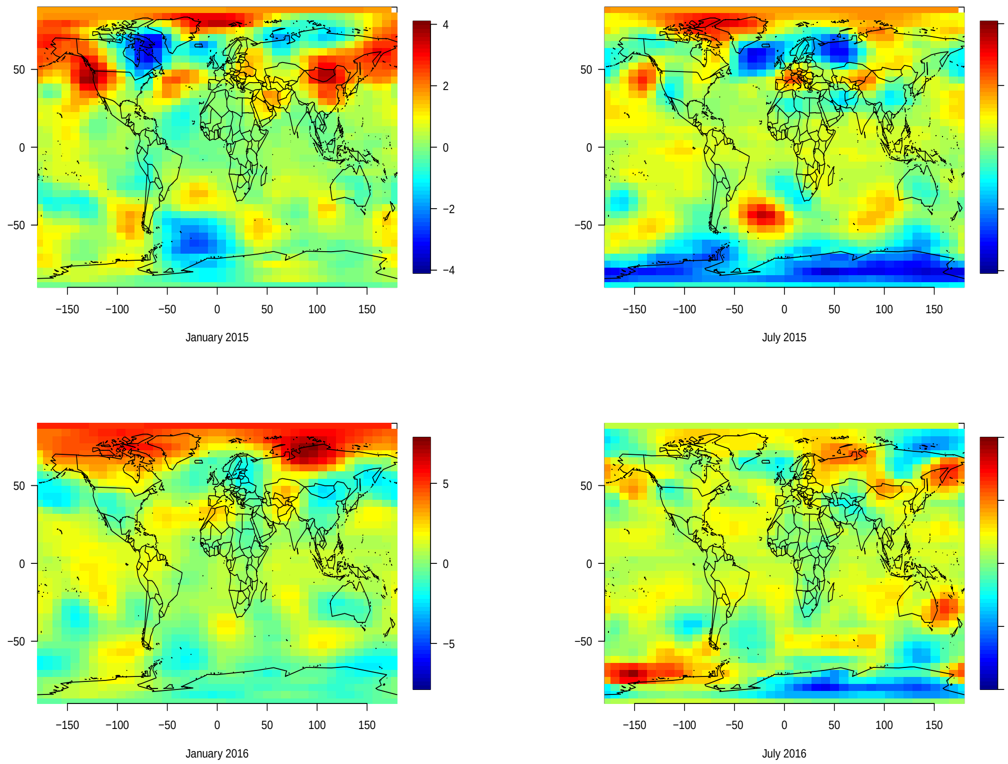

Figure 5.5: Heat maps of the temperature anomaly data for January and July in 2015 and 2016

we compare the results obtained using the $IRF(0,0)$ model and the $IRF(1,1)$ model on the dataset. Figures 5.6 and 5.7 show how the covariance values of $IRF(0,0)$ and the intrinsic structure covariance function (ISCF) values of $IRF(1,1)$ change with time lags when the great circle distance is fixed. These two figures illustrate that the effect of the time lag h differs substantially between the two models. In Figure 5.6, we observe that the influence of time lags persists for $IRF(0,0)$ even with relatively large values of h. The covariance values of $IRF(0,0)$ decay slowly and do not approach zero, even as time lags increase, as shown in the figure. In time series analysis, graphical methods, such as plotting the covariance or correlation function, are commonly used to detect non-stationarity. If the covariance does not decay rapidly with increasing time lags, it suggests that the process is non-stationary. A characteristic feature of an integrated process— a discrete case of a random process with

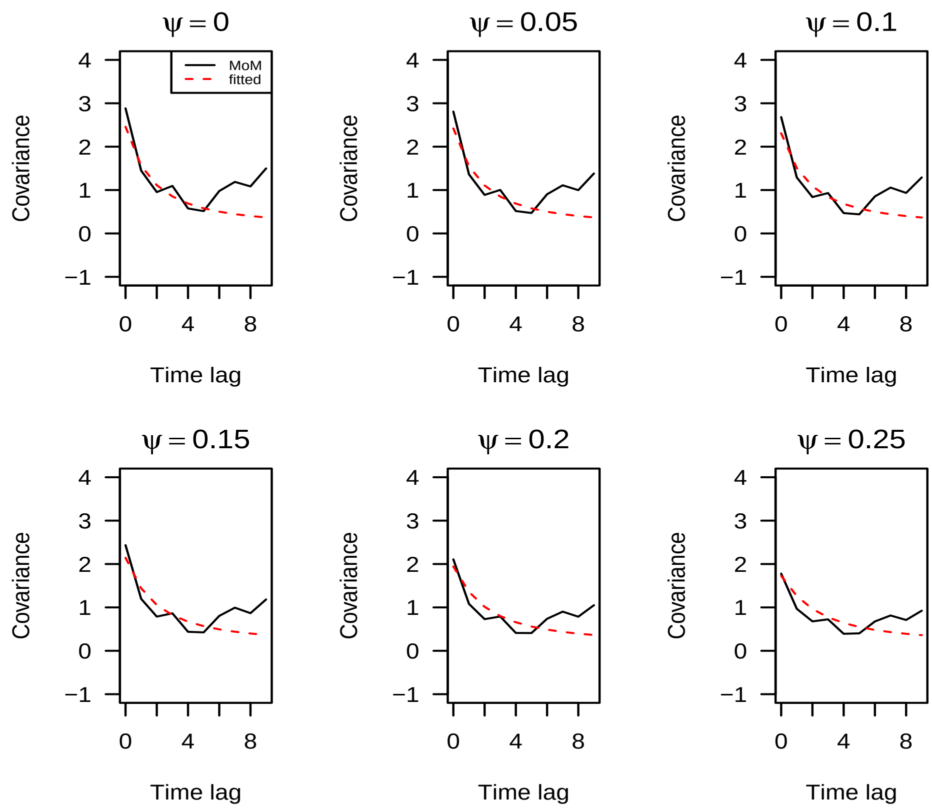

Figure 5.6: MoM estimates (black solid line) and fitted covariance values (red dashed line) of the $IRF(0,0)$ model for the 2015–2016 temperature anomaly data, given fixed great circle distances (spherical distances) ψ.

stationary increments— is its slowly decaying correlation function (Brockwell and Davis 1991) [1]. In contrast, for $IRF(1,1)$, as shown in Figure 5.7, the ISCF values decay exponentially and approach zero as the time lag increases. This rapid decay may indicate that an integrated process is a suitable model for the dataset. As a result, we conclude that the appropriate level of non-stationarity for this dataset is $d = 1$.

To choose an appropriate order of non-homogeneity κ, we can apply the graphical methodology suggested by Bussberg (2020) [2]. Given a value of d in $IRF(\kappa, d)$, it has been shown that the intrinsic structure covariance function, $\Phi_{(\kappa,d)}(\psi, h)$, is spatially homogeneous and temporally stationary. Therefore, for an appropriate estimate, $\hat{\Phi}_{(\kappa,d)}(\overrightarrow{PQ}, |t-s|)$, one would expect that

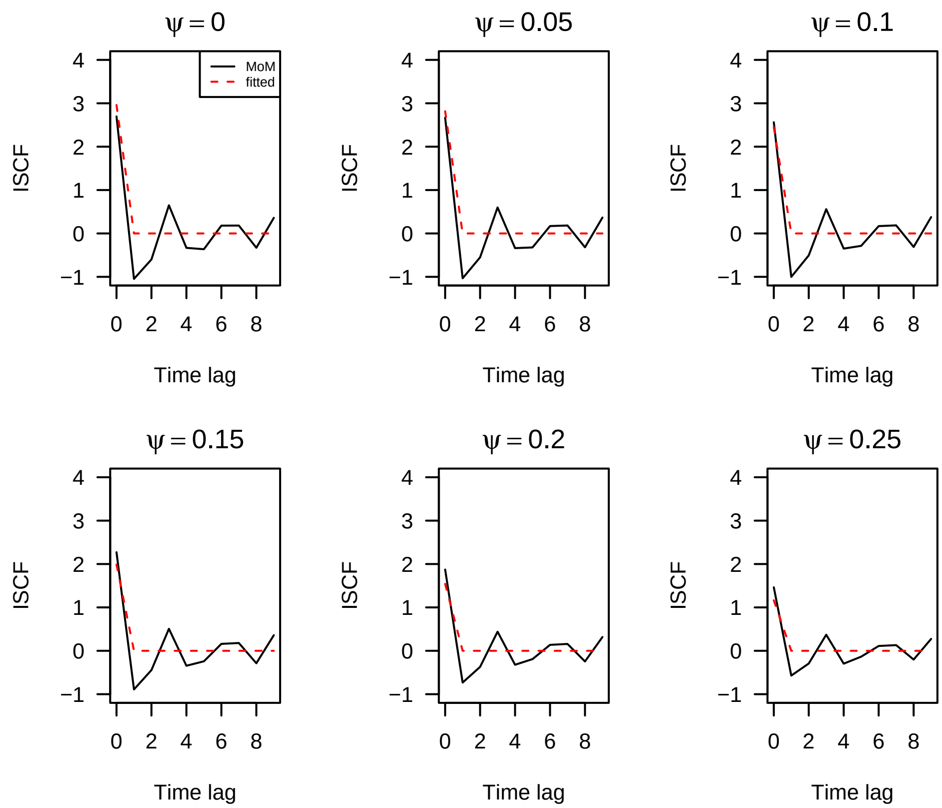

Figure 5.7: MoM estimates (black solid line) and fitted covariance values (red dashed line) of the $IRF(1,1)$ model for the 2015–2016 temperature anomaly data, given fixed great circle distances (spherical distances) ψ.

$$M(n) = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\Big(\hat{\Phi}_{(n,d)}(\psi_i, h_j) - \hat{\Phi}_{(n+1,d)}(\psi_i, h_j) \\ - \{\hat{\Phi}_{(n,d)}(0, h_j) - \hat{\Phi}_{(n+1,d)}(0, h_j)\}P_k(\cos\psi_i)\Big)^2 \tag{5.8}$$

to be small for $n \geq \kappa$. The method moment estimator can be used to get $\hat{\Phi}_{(n,d)}(\psi_i, h_j)$. That is,

$$\hat{\Phi}_{(n,d)}(\psi_i, h_j) = \frac{1}{N_{\psi_i,h_j}} \sum_{\{(P,t),(Q,s)\}\in N_{\psi_i,h_j}} \Delta_1^d X_{n,r}(P,t)\Delta_1^d X_{n,r}(Q,s)$$

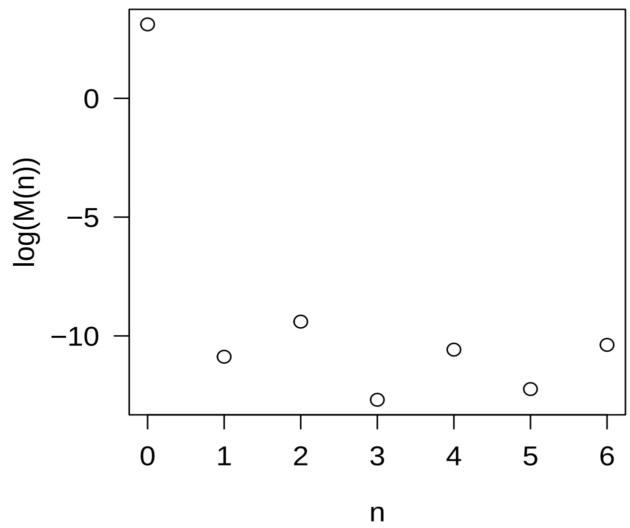

Figure 5.8: Plot of the criterion $\log M(n)$ as a function of n, representing the level of non-homogeneity, for the temperature anomaly data from 2015–2016.

where $\Delta_1^d X_{n,r}(P,t)$ is an estimator of the truncated and differenced process, $\Delta_1^d X_n(P,t)$. It is obtained as a residual process from regressing $\Delta_1^d X(P,t)$ on $\{Y_\ell^m(P)\}_{\ell<n}$ as described in Equation (5.6). Likewise, the neighborhood N_{ψ_i,h_j} can be defined in the same manner as in Equation (5.7). Thus, the graphical analysis can be derived to estimate the order of non-homogeneity κ by using the criterion $M(n)$ in Equation (5.8). Figure 5.8 shows that, for $d = 1$, the criterion becomes small when $n \geq 1$, suggesting that $\kappa = 1$ is a suitable level of non-homogeneity.

Even though the true ISCF values are unknown for this real-world dataset, we can still compare our estimated ISCF values with the corresponding MoM estimates to assess the performance of the estimation from a data-driven perspective. As shown in Figures 5.7, the fitted values and MoM estimates for the $IRF(1,1)$ model are generally closer to each other than those for the $IRF(0,0)$ model, presented in Figures 5.6. However, as illustrated in Figure 5.10, the $IRF(1,1)$ model provides a less accurate representation of ISCF variation with respect to great circle distance at fixed time lags, compared to the $IRF(0,0)$ model

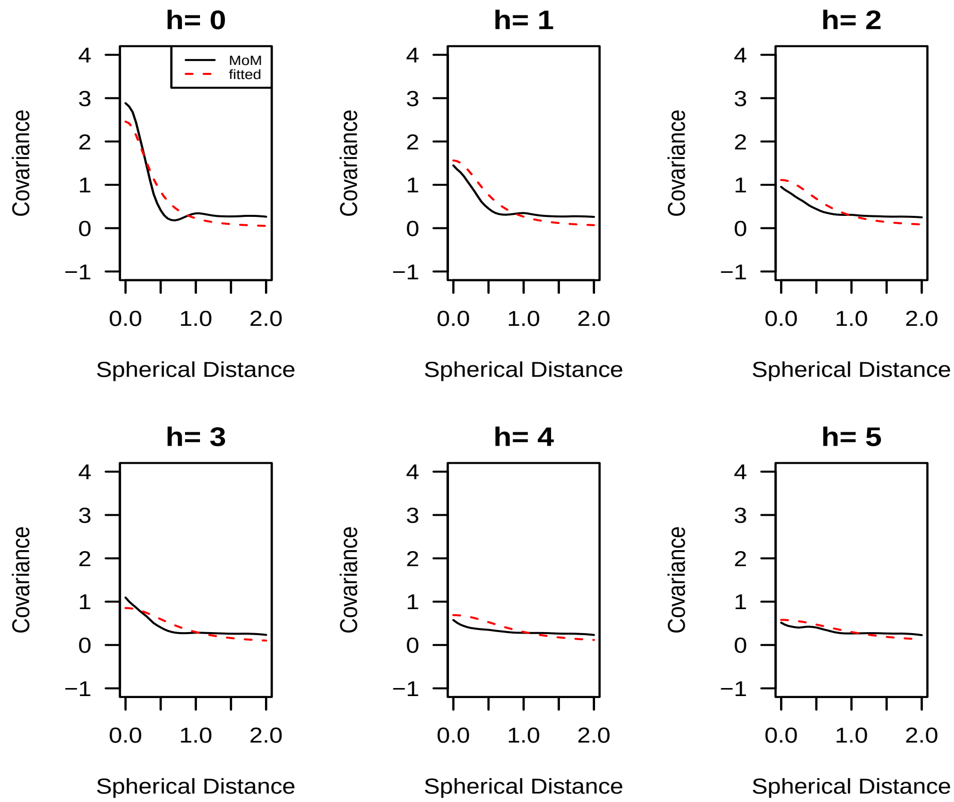

Figure 5.9: MoM estimates (black solid line) and fitted covariance values (red dashed line) of the $IRF(0,0)$ model for the 2015–2016 temperature anomaly data, given fixed time lags h.

in Figure 5.9. This limitation arises from the inability of our parametric $IRF(1,1)$ model to capture the negative ISCF values observed at certain time lags—for instance, when $h = 1, 2, 4, 5$. The presence of negative ISCF values also contributes to the oscillatory behavior seen in the MoM estimates in Figures 5.7, which is also not adequately captured by our parametric model. Therefore, developing novel models of ISCF that account for these negative or oscillatory values could be a valuable direction for future research.

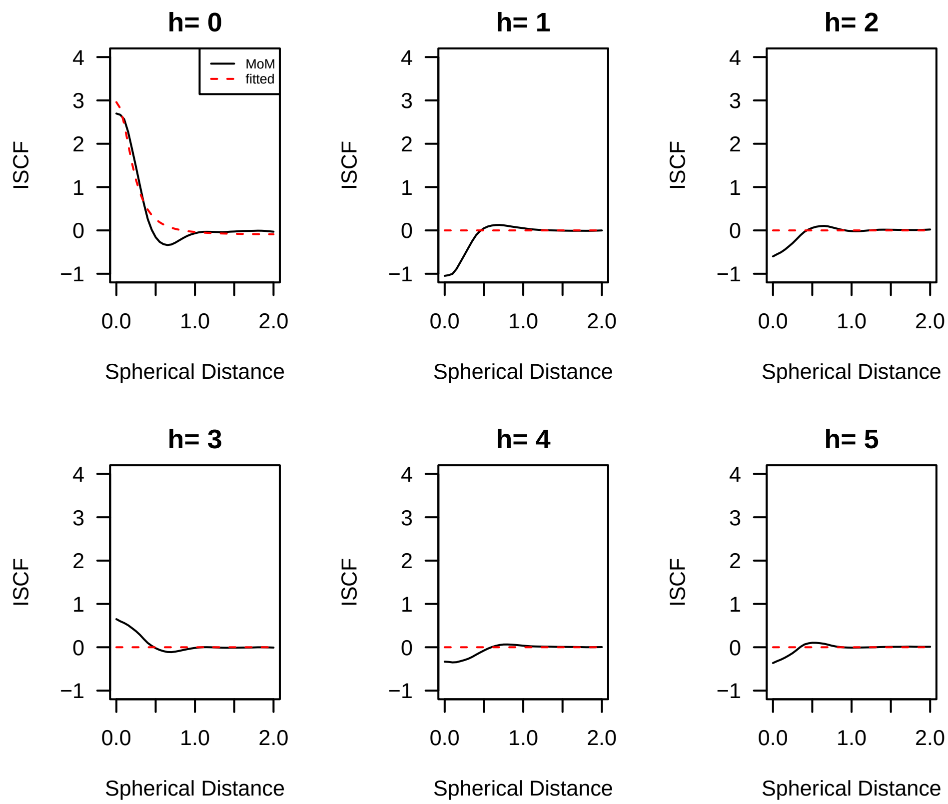

Figure 5.10: MoM estimates (black solid line) and fitted covariance values (red dashed line) of the $IRF(1,1)$ model for the 2015–2016 temperature anomaly data, given fixed time lags h.

Chapter 6

Conclusions and Future Directions

In this research, we introduced spatially non-homogeneous and temporally non-stationary models for random processes on the sphere by developing intrinsic structure covariance functions (ISCFs). These ISCFs combine spatial intrinsic covariance functions with temporal structure functions.

To address spatially non-homogeneous random processes, we adopted the framework of intrinsic random functions (IRFs) and intrinsic covariance functions (ICFs) on the sphere. The allowable measure of IRFs on the sphere truncates the low-frequency components in the spectral representation of the random process (Huang et al. 2019) [8]. This property allows us to construct homogeneous ICFs and subsequently derive spatially non-homogeneous covariance functions for IRFs on the sphere.

Analogously, to handle temporal non-stationarity, the concepts of random processes with stationary increments and their structure functions were applied. In this case, temporal stationarity can be attained by taking differences of the original processes (Yaglom 1958) [16]. The resulting stationary covariance function is referred to as the structure function. We also demonstrated the connection between random processes with stationary increments and intrinsic random functions on the real line.

Therefore, by simultaneously applying the concepts of IRFs on the sphere and random processes with stationary increments, we proposed an ISCF model that integrates a spatial intrinsic covariance function with a temporal structure function. ISCFs provide a useful framework for exploring covariance structures in non-homogeneous and non-stationary spatio-temporal random processes on the sphere. Additionally, we showed that ISCFs can be expressed in a parametric form, with parameters that characterize the decay of the process over temporal lags and geodesic distances.

Finally, we investigated the proposed model through a simulation study. The results demonstrate that the ISCF estimation performs well. Furthermore, we also applied this model to the real-world data about the temperature anomaly.

Several directions remain open for extending and enhancing the proposed model of non-homogeneous and non-stationary random processes on the sphere and their intrinsic structure covariance functions. One promising approach is the development of various parametric or nonparametric models of the ISCF that take advantage of the flexibility of its modeling framework. As illustrated, the flexible ISCF framework allows for the modeling of diverse dependency structures across different datasets, thereby enhancing its applicability to a wide range of real-world problems. Hence, by utilizing the flexibility of the proposed modeling framework, we can develop various new ISCF models to accommodate different dependency structures in data, such as oscillatory or negative ISCF values, as discussed in Section 5.2.

Another important direction is to extend the proposed ISCF model to develop optimal prediction methods for spatio-temporal data on the sphere. In particular, constructing a universal kriging framework based on the ISCFs would enable efficient prediction at unobserved locations and future times while accounting for both spatial non-homogeneity and temporal non-stationarity. Further theoretical development and practical implementation of such predictive methods would enhance the applicability of ISCF models to real-world problems in spatio-temporal data analysis.

Appendix A

APPENDIX

A.1 Proofs of Equation (2.2) and (2.3)

Note that we can express the process:

$$X(P,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P), \quad P \in \mathbb{S}^2, \quad t \in \mathbb{R}$$

$$Z_{\ell,m}(t) = \int X(P,t) Y_\ell^m(P) dP$$

Roy (1969) [13] stated that if the process is homogeneous and stationary, then

$$Cov(Z_{\ell,m}(t), Z_{\ell',m'}(s)) = \delta_\ell^{\ell'} \delta_m^{m'} a_\ell(h), \quad h = t - s \tag{A.1}$$

where

$$\delta_\ell^{\ell'} = \begin{cases} 1, & \ell = \ell' \\ 0, & \text{otherwise} \end{cases}, \qquad \delta_m^{m'} = \begin{cases} 1, & m = m' \\ 0, & \text{otherwise} \end{cases}$$

To prove (A.1), we first need to show $Cov\big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\big)$ is stationary if $X(P,t)$ is stationary. The corresponding proof is below:

$$E\big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\big) = E\Big[\int_{\mathbb{S}^2} X(P,t) Y_\ell^m(P) dP \int_{\mathbb{S}^2} X(Q,s) Y_\ell^m(Q) dQ\Big]$$

(by Fubini's theorem)

$$= \int_{\mathbb{S}^2 \times \mathbb{S}^2} E\Big[X(P,t)X(Q,s)\Big] Y_\ell^m(P) Y_\ell^m(Q) d(P \times Q)$$

We know $E\big[X(P,t)X(Q,s)\big]$ is stationary; therefore, $E\big(Z_{\ell,m}(t), Z_{\ell',m'}(s)\big)$ is also stationary. Now, we look into the condition of $\delta_\ell^{\ell'} \delta_m^{m'}$ in Equation (A.1). According to the definition of homogeneity,

$$Cov\big(X(P,t), X(Q,s)\big) = Cov\big(X(gP,t), X(gQ,s)\big) \tag{A.2}$$

where g is any rotation operator. First, we can extend the left side of Equation (A.2):

$$\begin{aligned} Cov(X(P,t), X(Q,s)) &= Cov\left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(t) Y_\ell^m(P), \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(s) Y_\ell^m(Q)\right) \\ &= \sum_{\ell_1=0}^{\infty} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{\ell_2=0}^{\infty} \sum_{m_2=-\ell_2}^{\ell_2} Y_{\ell_1}^{m_1}(P) Y_{\ell_2}^{m_2}(Q) Cov\big(Z_{\ell_1,m_1}(t) Z_{\ell_2,m_2}(s)\big) \end{aligned} \tag{A.3}$$

In equation (A.3), the only random parts are from the coefficients $Z_{\ell_1,m_1}(t)$ and $Z_{\ell_2,m_2}(s)$ containing $X(P,t)$ and $X(Q,s)$. On the other hand, the spherical harmonics $Y_{\ell_1}^{m_1}(P)$ and $Y_{\ell_2}^{m_2}(Q)$ are deterministic. Likewise, for the right side of Equation (A.2),

$$\begin{aligned} Cov(X(gP,t), X(gQ,s)) &= Cov\left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{l,m}(t) Y_\ell^m(gP), \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_{\ell,m}(s) Y_\ell^m(gQ)\right) \\ &= \sum_{\ell_1=0}^{\infty} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{\ell_2=0}^{\infty} \sum_{m_2=-\ell_2}^{\ell_2} Y_{\ell_1}^{m_1}(gP) Y_{\ell_2}^{m_2}(gQ) Cov\big(Z_{\ell_1,m_1}(t) Z_{\ell_2,m_2}(s)\big) \end{aligned} \tag{A.4}$$

Moreover, by Roy (1969) [13], we can express the rotation operation as a linear transformation of the spherical harmonics of the same order such that

$$Y_\ell^m(gP) = \sum_{m'=-\ell}^{\ell} S_\ell^{mm'}(g) Y_\ell^{m'}(P) \tag{A.5}$$

with respect to the matrix form:

$$\begin{bmatrix} Y_\ell^{-\ell}(gP) \\ Y_\ell^{-\ell+1}(gP) \\ \vdots \\ Y_\ell^{\ell}(gP) \end{bmatrix}_{(2\ell+1)\times 1} = \begin{bmatrix} S_\ell^{-\ell-\ell}(g) & \cdots & S_\ell^{-\ell\ell}(g) \\ \vdots & \ddots & \vdots \\ S_\ell^{\ell-\ell}(g) & \cdots & S_\ell^{\ell\ell}(g) \end{bmatrix}_{(2\ell+1)\times(2\ell+1)} \begin{bmatrix} Y_\ell^{-\ell}(P) \\ Y_\ell^{-\ell+1}(P) \\ \vdots \\ Y_\ell^{\ell}(P) \end{bmatrix}_{(2\ell+1)\times 1}$$

where $S_\ell(g) = \left(S_\ell^{mm'}(g)\right) = \begin{bmatrix} S_\ell^{-\ell-\ell}(g) & \cdots & S_\ell^{-\ell\ell}(g) \\ \vdots & \ddots & \vdots \\ S_\ell^{\ell-\ell}(g) & \cdots & S_\ell^{\ell\ell}(g) \end{bmatrix}_{(2\ell+1)\times(2\ell+1)}$

is a $(2\ell+1)\times(2\ell+1)$ unitary matrix. Therefore, by plugging Equation (A.5) into Equation (A.4), we can find

$$\begin{aligned}
&Cov\big(X(gP,t), X(gQ,s)\big) \\
&= \sum_{\ell_1=0}^{\infty}\sum_{m_1=-\ell_1}^{\ell_1}\sum_{\ell_2=0}^{\infty}\sum_{m_2=-\ell_2}^{\ell_2} Cov\big(Z_{\ell_1,m_1}(t)Z_{\ell_2,m_2}(s)\big) \\
&\quad \sum_{m_1'=-\ell_1}^{\ell_1} S_{\ell_1}^{m_1m_1'}(g)Y_{\ell_1}^{m_1'}(P) \sum_{m_2'=-\ell_2}^{\ell_2} S_{\ell_2}^{m_2m_2'}(g)Y_{\ell_2}^{m_2'}(Q)
\end{aligned}$$

$$= \sum_{\ell_1=0}^{\infty} \sum_{m_1'=-\ell_1}^{\ell_1} \sum_{\ell_2=0}^{\infty} \sum_{m_2'=-\ell_2}^{\ell_2} Y_{\ell_1}^{m_1'}(P) Y_{\ell_2}^{m_2'}(Q)$$

$$\sum_{m_1=-\ell_1}^{\ell_1} S_{\ell_1}^{m_1 m_1'}(g) \sum_{m_2=-\ell_2}^{\ell_2} S_{\ell_2}^{m_2 m_2'}(g) Cov\big(Z_{\ell_1,m_1'}(t) Z_{\ell_2,m_2'}(s)\big) \tag{A.6}$$

Define $B_{\ell_1,\ell_2,h} := \Big(\mathrm{Cov}\big(Z_{\ell_1,m_1}(t), Z_{\ell_2,m_2}(s)\big)\Big)$, which is a $(2\ell_1+1)\times(2\ell_2+1)$ rectangular matrix, where $h = t - s$. Then, the coefficients of $Y_{\ell_1}^{m_1'}(P)Y_{\ell_2}^{m_2'}(Q)$ in Equation A.6 become

$$S_{\ell_1}(g) B_{\ell_1,\ell_2,h} S_{\ell_2}(g)^t \tag{A.7}$$

In the same manner, the coefficients of $Y_{\ell_1}^{m_1}(P)Y_{\ell_2}^{m_2}(Q)$ in Equation (A.3) can be represented by

$$B_{\ell_1,\ell_2,h} \tag{A.8}$$

Since (A.7) = (A.8) by Equation (A.2), by Schur decomposition, we can conclude that

$$Cov\big(Z_{\ell_1,m_1}(t), Z_{\ell_2,m_2}(s)\big) = B_{\ell_1,\ell_2,h} = \begin{bmatrix} a_\ell(h) & & \\ & \ddots & \\ & & a_\ell(h) \end{bmatrix}_{(2\ell+1)\times(2\ell+1)}$$

where $\ell = \ell_1 = \ell_2$

$$= a_\ell(h) I_{(2\ell+1)\times(2\ell+1)}$$

$$= \delta_{\ell_1}^{\ell_2} \delta_{m_1}^{m_2} a_\ell(h)$$

This is the end of the proof of Equation (A.1).

Now that we have Equation (A.1), we can apply this covariance function of the coefficients to Equation A.3, the covariance function of the process. Hence,

$$\begin{aligned}
Cov(X(P,t), X(Q,s)) &= \sum_{\ell_1=0}^{\infty} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{\ell_2=0}^{\infty} \sum_{m_2=-\ell_2}^{\ell_2} Cov\big(Z_{\ell_1,m_1}(t) Z_{\ell_2,m_2}(s)\big) Y_{\ell_1}^{m_1}(P) Y_{\ell_2}^{m_2}(Q) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell(h) Y_\ell^m(P) Y_\ell^m(Q) \qquad \text{(by (A.1))} \\
&= \sum_{\ell=0}^{\infty} a_\ell(h) \sum_{m=-\ell}^{\ell} Y_\ell^m(P) Y_\ell^m(Q) \\
&= \sum_{\ell=0}^{\infty} a_\ell(h) \frac{2\ell+1}{4\pi} P_\ell(\cos \overrightarrow{PQ}) \quad \text{(by Additivity theorem)}
\end{aligned}$$

where $a_\ell(h)$ is positive semidefinite, and $\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(0) < \infty$

The function $a_\ell(h)$ should be positive semidefinite since it is a time series covariance function. In addition, $\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(0) < \infty$ since the second moment should be finite. Because $Cov\big(X(P,t), X(P,t)\big) = Var\big(X(P,t)\big) < \infty$, it should be $\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(0) P_\ell(0) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} a_\ell(0) < \infty$. In conclusion, the covariance function, $Cov\big(X(P,t), X(Q,s)\big)$, only depends on its spherical distance $\psi(P,Q) = \overrightarrow{PQ}$ and the time difference $h = t - s$. This indicates that $Cov\big(X(P,t), X(Q,s)\big)$ is homogeneous and stationary.

$\square$

A.2 Proof of Proposition 4

($\Rightarrow$) Define

$$\lambda_Q^s(dP) := \delta_Q(dP) - \sum_{\ell<\kappa} \sum_{m=-\ell}^{\ell} Y_\ell^m(Q) Y_\ell^m(dP)$$

where $P, Q \in \mathbb{S}^2$ and $\delta_Q(dP)$ is the Dirac measures on $\mathbb{S}^2$. First, we need to prove that λ_Q^s is the allowable measure on $\mathbb{S}^2$ that truncates spherical harmonics with orders less than κ. That is, any spatial random process on the sphere $X(P)$,

$$\begin{aligned} X(\lambda_Q^s) &= \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(P) \lambda_Q^s(dP) \\ &= \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(Q) \\ &=: X_\kappa(Q) \end{aligned}$$

Proof. For $\ell \le \kappa$, $|m_0| \le \ell_0$,

$$\begin{aligned} Y_{\ell_0}^{m_0}(\lambda_Q^s) &= \int_{\mathbb{S}^2} Y_{\ell_0}^{m_0}(P)\lambda_Q^s(dP) = \int_{\mathbb{S}^2} Y_{\ell_0}^{m_0}(P)\Big\{\delta_Q(dP) - \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Y_\ell^m(Q)Y_\ell^m(dP)\Big\} \\ &= Y_{\ell_0}^{m_0}(Q) - \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Y_\ell^m(Q)\int_{\mathbb{S}^2} Y_{\ell_0}^{m_0}(P)Y_\ell^m(dP) \end{aligned}$$

(by the orthogonality of the associated Legendre polynomials and its derivative,)

$$= Y_{\ell_0}^{m_0}(Q) - Y_{\ell_0}^{m_0}(Q) = 0$$

Therefore, $\lambda_Q^s \in \Lambda_\kappa^s$ in that it annihilates associated Legendre polynomials with order less than κ. $\square$

Now, suppose that $X(P)$ is an IRF$_\kappa$ for $\kappa \geq 1$ and $\lambda_Q^s(dP) = \delta_Q(dP) - \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Y_\ell^m(Q)$. By the proceeding proof, $\lambda_Q^s \in \Lambda_\kappa^s$ that annihilates associated Legendre polynomials with order less than κ. As a result, $X(\lambda_Q^s) = X_\kappa(Q)$, and likewise, $X(g\lambda_Q^s) = X_\kappa(gQ)$. Note that $E(X(\lambda_P^s) = E(X(g\lambda_P^s))$ and $Cov(X(\lambda_P^s), X(\lambda_Q^s)) = Cov(X(g\lambda_P^s), X(g\lambda_Q^s))$ by the condition that $X(P)$ is an IRF$_\kappa$. In conclusion, we can get

$$E(X_\kappa(P)) = E(X_\kappa(gp)) \quad \text{and} \quad Cov(X_\kappa(P), X_\kappa(Q)) = Cov(X_\kappa(gP), X_\kappa(gQ)).$$

This indicates that $X_\kappa(P)$ is homogeneous.

($\Leftarrow$) Let assume that the truncated process $X_\kappa(P)$ is homogeneous for any $\kappa \geq 1$ and any $P \in \mathbb{S}^2$. Since

$$X(P) = \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P) + X_\kappa(P) \qquad \text{and} \qquad \lambda^s \in \Lambda_\kappa^s,$$

We can say that

$$\begin{aligned}
X(\lambda^s) &= \int_{\mathbb{S}^2} X(P)\lambda^s(dP) \\
&= \int_{\mathbb{S}^2} \left\{ \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P) + X_\kappa(P) \right\} \lambda^s(dP) \\
&= \int_{\mathbb{S}^2} \left\{ \sum_{\ell<\kappa}\sum_{m=-\ell}^{\ell} Z_{\ell,m} Y_\ell^m(P) \right\} \lambda^s(dP) + \int_{\mathbb{S}^2} X_\kappa(P)\lambda^s(dP) \\
&= \int_{\mathbb{S}^2} X_\kappa(P)\lambda^s(dP) \\
&= X_\kappa(\lambda^s)
\end{aligned}$$

The second last equality can be achieved because the finite sum in the first term vanishes by the definition of the allowable measure $\lambda^s \in \Lambda_\kappa^s$. Then, the expectation value of $X_\kappa(\lambda^s)$ can be presented as

$$E\big(X_\kappa(\lambda^s)\big)= E\int_{\mathbb{S}^2} X_\kappa(P)\lambda^s(dP) = \int_{\mathbb{S}^2} E\big(X_\kappa(P)\big)\lambda^s(dP)$$

since Fubini's Theorem can be applied by the fact that λ^s is a finite measure and $X_\kappa(P)$ is integrable. Likewise, for $g \in G$,

$$X(g\lambda^s) = X_\kappa(g\lambda^s) \quad \text{and} \quad E\big(X_\kappa(g\lambda^s)\big)= \int_{\mathbb{S}^2} E\big(X_\kappa(gP)\big)\lambda^s(dP).$$

Since $X_\kappa(P)$ is homogeneous, $E(X_\kappa(P)) = E(X_\kappa(gP))$. Therefore, we can conclude that

$$E(X_\kappa(\lambda^s)) = E(X_\kappa(g\lambda^s))$$

Similarly, by Fubini's theorem, for $P, Q \in \mathbb{S}^2$ and $\lambda_1^s, \lambda_2^s \in \Lambda_\kappa^s$,

$$Cov\big(X(\lambda_1^s), X(\lambda_2^s)\big)= \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} Cov(X_\kappa(P), X_\kappa(Q))\lambda_1^s(dP)\lambda_2^s(dQ)$$

and

$$Cov\big(X(g\lambda_1^s), X(g\lambda_2^s)\big)= \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} Cov\big(X_\kappa(gP), X_\kappa(gQ)\big)\lambda_1^s(dP)\lambda_2^s(dQ) \quad \text{for} \quad P, Q \in \mathbb{S}^2$$

We know that $Cov\big(X_\kappa(P), X_\kappa(Q)\big)= Cov\big(X_\kappa(gP), X_\kappa(gQ)\big)$ by its assumption of homogeneity. As result, we get the desired conclusion that

$$Cov\big(X(\lambda_1^s), X(\lambda_2^s)\big)= Cov\big(X(g\lambda_1^s), X(g\lambda_2^s)\big).$$

Hence, $X(P)$ is an IRF$^s(\kappa)$. $\square$

A.3 Proofs of Equation (2.10) and (2.11)

Proof.

Note that $Y(t) = X(t+1) - X(t)$. Hence,

$$\Delta_\iota X(t) = X(t) - X(t-\iota) = \sum_{s=t-\iota}^{t-1} Y(s) = \sum_{s=t-\iota}^{t-1} \int_{-\pi}^{\pi} e^{i\omega s} dZ_y(\omega)$$

$$= \int_{-\pi}^{\pi} \sum_{s=t-\iota}^{t-1} e^{i\omega s} dZ_y(\omega) = \int_{-\pi}^{\pi} e^{i\omega(t-\iota)} \frac{(e^{i\omega\iota} - 1)}{e^{i\omega} - 1} dZ_y(\omega)$$

$$= \int_{-\pi}^{\pi} e^{i\omega t} \frac{(1 - e^{-i\omega\iota})}{e^{i\omega} - 1} dZ_y(\omega)$$

Therefore, $X(t) - X(0) = \int_{-\pi}^{\pi} \frac{e^{i\omega t} - 1}{e^{i\omega} - 1} dZ_y(\omega)$

Let $E(|dZ_y(\omega)|^2) = dF_y(\omega)$. Then,

$$D(h; \iota_1, \iota_2) = \int_{-\pi}^{\pi} e^{i\omega h} \frac{(1 - e^{-i\omega\iota_1})(1 - e^{i\omega\iota_2})}{|e^{i\omega} - 1|^2} dF_y(\omega)$$

Let $Z(\omega_2) - Z(\omega_1) = \int_{\omega_1}^{\omega_2} \frac{1}{e^{i\omega} - 1} dZ_y(\omega)$,

$$F(\omega_2) - F(\omega_1) = \int_{\omega_1}^{\omega_2} \frac{1}{|e^{i\omega} - 1|^2} dF_y(\omega), \quad \text{and} \quad X(0) = X_0.$$

Then,

$$X(t) = \int_{-\pi}^{\pi} (e^{i\omega t} - 1) dZ(\omega) + X_0$$

$$D(h; \iota_1, \iota_2) = \int_{-\pi}^{\pi} e^{i\omega h} (1 - e^{-i\omega\iota_1})(1 - e^{i\omega\iota_2}) dF(\omega)$$

□

A.4 Proofs of Equation (2.13) and (2.14)

Proof. In the case of a continuous time term, we can see a stationary process $Y(t)$ as a derivative of $X(t)$; thus, $Y(t) = X'(t)$ if it exists where $t, h \in \mathbb{R}$.

In other words,

$$Y(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y(\omega), \quad E\big(Y(t+h) \cdot Y(t)\big) = C_y(h) = \int_{-\infty}^{\infty} e^{i\omega h} dF_y(\omega)$$

Then,

$$\Delta_\iota X(t) = X(t) - X(t-\iota) = \int_{t-\iota}^{t} Y(s)ds = \int_{-\infty}^{\infty}\int_{t-\iota}^{t} e^{i\omega s} ds dZ_y(\omega)$$
$$= \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega(t-\iota)}}{i\omega} dZ_y(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \frac{(1-e^{-i\omega\iota})}{i\omega} dZ_y(\omega)$$

Therefore,

$$X(t) - X(0) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega} dZ_y(\omega)$$
$$X(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega} dZ_y(\omega) + X_0 \quad \text{where } X_0 = X(0)$$

If we can assume $X(0) = 0$,

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h} \frac{(1-e^{-i\omega\iota_1})(1-e^{i\omega\iota_2})}{\omega^2} dF_y(\omega)$$

Therefore,

$$X(t) = \int_{-\infty}^{\infty} (e^{i\omega t} - 1) dZ(\omega)$$

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})(1 - e^{-i\omega\iota_2}) dF(\omega)$$

Note that $Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{i\omega}, \quad F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^2}.$

$\square$

A.5 Proofs of Equation (2.18) and (2.19)

Proof.

Since $X(\cdot)$ is a random process with stationary increments of order 2,

by the definition of $I(2)$ and the spectral representation in Equation (2.3.2),

$$\Delta_\iota X(t) = X(t) - X(t-\iota) = \int_{t-\iota}^{t} \int_{-\infty}^{\infty} \frac{e^{i\omega s} - 1}{i\omega} dZ_y(\omega) ds$$

$$= \int_{-\infty}^{\infty} \left[\frac{e^{i\omega t} - e^{i\omega(t-\iota)}}{(i\omega)^2} - \frac{\iota}{i\omega} \right] dZ_y(\omega) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}(1 - e^{-i\omega\iota}) - i\omega\iota}{(i\omega)^2} dZ_y(\omega)$$

Therefore,

$$X(t) - X(0) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1 - i\omega t}{(i\omega)^2} dZ_y(\omega)$$

By assuming $X(0) = 0$,

$$X(t) = \int_{-\infty}^{\infty} (e^{i\omega t} - 1 - i\omega t) dZ(\omega)$$

Also,

$$\Delta_\iota^2 X(t) = \Delta_\iota X(t) - \Delta_\iota X(t-\iota)$$

$$= \int_{-\infty}^{\infty} \left[\frac{e^{i\omega t} - e^{i\omega(t-\iota)}}{(i\omega)^2} - \frac{\iota}{i\omega}\right] dZ_y(\omega) - \int_{-\infty}^{\infty} \left[\frac{e^{i\omega(t-\iota)} - e^{i\omega(t-2\iota)}}{(i\omega)^2} - \frac{\iota}{i\omega}\right] dZ_y(\omega)$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\omega t}(1 - e^{-i\omega\iota})^2}{(i\omega)^2} dZ_y(\omega)$$

As a result, we can get the structure function such that

$$D(h; \iota_1, \iota_2) = \int_{-\infty}^{\infty} e^{i\omega h}(1 - e^{-i\omega\iota_1})^2(1 - e^{i\omega\iota_2})^2 dF(\omega)$$

Note that $Z(\omega_2) - Z(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dZ_y(\omega)}{(i\omega)^2}, \quad F(\omega_2) - F(\omega_1) = \int_{-\omega_1}^{\omega_2} \frac{dF_y(\omega)}{\omega^4}.$

□

A.6 Proof of the positive definiteness of Equation (4.9)

The function $R(P, Q, t, s)$ in Equation (4.9) is positive semidefinite.

We can rewrite (4.9) as :

$$\begin{aligned}
&Cov\Big(X(P,t), X(Q,s)\Big) \\
&= \sum_{\ell=\kappa}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \left(e^{it\omega} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!}\right) \\
&\quad \left(e^{-is\omega} - 1 + is\omega - \cdots - \frac{(-is\omega)^{d-1}}{(d-1)!}\right) \frac{1}{\omega^{2d}} dF_\ell(\omega) \\
&\quad \Bigg[Y_\ell^m(P) Y_\ell^m(Q) - \sum_{\nu=1}^{\kappa^2} Y_\ell^m(P) Y_\ell^m(\tau_\nu) q_\nu(Q) - \sum_{\nu=1}^{\kappa^2} Y_\ell^m(\tau_\nu) Y_\ell^m(Q) q_\nu(P) \\
&\quad + \sum_{\nu=1}^{\kappa^2} \sum_{\mu=1}^{\kappa^2} Y_\ell^m(\tau_\nu) Y_\ell^m(\tau_\mu) q_\nu(P) q_\mu(Q)\Bigg] + \sum_{\nu=1}^{\kappa^2} \gamma_\nu^2 q_\nu(P) q_\nu(Q)
\end{aligned}$$

Therefore, for any $n \in \mathbb{N}$, $c_i, c_j \in \mathbb{C}$, $x_i, x_j \in \mathbb{S}^2$ and $t_i, t_j \in \mathbb{R}$ where $i, j = 1, 2, 3 \ldots,$

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \bar{c}_j Cov\Big(X(x_i, t_i), X(x_j, t_j)\Big) \\
&= \sum_{\ell=\kappa}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\left|\sum_{i=1}^{n} c_i\left(e^{i\omega t_i} - 1 - it\omega - \cdots - \frac{(it\omega)^{d-1}}{(d-1)!}\right)\right. \\
&\qquad\qquad \left.\left\{Y_\ell^m(x_i) - \sum_{\nu=1}^{\kappa^2} Y_\ell^m(\tau_\nu) q_\nu(x_i)\right\}\right|^2 dF_\ell(\omega) + \sum_{\nu=1}^{\kappa^2}\left|\gamma_\nu \sum_{i=1}^{n} c_i q_\nu(x_i)\right|^2 \geq 0
\end{aligned}$$

where $F_\ell(\omega)$ is a non-negative measure.

$\square$

BIBLIOGRAPHY

[1] Brockwell, P. J. and R. A. Davis (1991). *Time Series: Theory and Methods, 2nd edition.* Springer.

[2] Bussberg, N. W. (2020). *Environmental Applications of Temporal, Spatial, and Spatio-Temporal Statistics.* Ph. D. thesis, Indiana University Bloomington.

[3] Chen, W., C. Huang, H. Zhang, and M. Schaffer (2024). Spectral density estimation for random processes with stationary increments. *Applied Stochastic Models in Business and Industry 40*, 960–978.

[4] Chiles, J.-P. and P. Delfiner (1999). *Geostatistics.* John Wiley & Sons.

[5] Gay, D. (1990). *nlminb: Optimization using PORT routines.* R port: Douglas Bates and Deepayan Sarkar. Underlying Fortran code by David M. Gay.

[6] Hannan, E. (1970). *Multiple Time Series.* WILEY SERIES in PROBABILITY and STATISTICS: PROBABILITY and STATISTICS SECTION Series. Wiley.

[7] Huang, C., H. Zhang, and S. M. Robeson (2011). On the validity of commonly used covariance and variogram functions on the sphere. *Mathematical Geosciences 43*(6), 721–733.

[8] Huang, C., H. Zhang, S. M. Robeson, and J. Shields (2019). Intrinsic random functions on the sphere. *Statistics and Probability Letters 146*, 7–14.

[9] Levesley, J., W. Light, D. Ragozin, and X. Sun (1999). A simple approach to the variational theory for interpolation on spheres. In M. W. Müller, M. D. Buhmann, D. H. Mache, and M. Felten (Eds.), *New Developments in Approximation Theory*, Basel, pp. 117–143. Birkhäuser Basel.

[10] Matheron, G. (1973). The intrinsic random functions and their applications. *Advances in Applied Probability 5*(3), 439–468.

[11] Obukohov, A. (1947). Statistical homogenous random fields on a sphere. *Uspekhi Mathematicheskikh 2*, 196–198.

[12] Porcu, E., M. Bevilacqua, and M. G. Genton (2016). Spatio-temporal covariance and cross-covariance functions of the great circle distance on a sphere. *Journal of the American Statistical Association 111*(514), 888–898.

[13] Roy, R. (1969). *Processus stochastiques sur la sphère.* Ph. D. thesis, Université de Montréal.

[14] Stein, M. L. (2005). Space–time covariance functions. *Journal of the American Statistical Association 100*(469), 310–321.

[15] Yadrenko, M. I. and A. V. Balakrishnan (1983). *Spectral Theory of Random Fields (Spektral'naja Teorija Sluchajnykh Polej).* Optimization Software, Publications Division.

[16] Yaglom, A. M. (1958). Correlation theory of processes with random stationary nth increments. *American Mathematical Society*, 87–141.

[17] Yaglom, A. M. (1961). Second-order homogeneous random fields. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Contributions to Probability Theory*, pp. 593–622. University of California Press.

[18] Yaglom, A. M. (1987). Correlation theory of stationary and related random functions. *Volume I: Basic Results. 526.*

CURRICULUM VITAE

Jongwook Kim

EDUCATION

- **Ph.D. in Statistical Science**, Indiana University, Bloomington, IN, 2025
  - **Advisor:** Chunfeng Huang
  - **Dissertation:** Parametric Modeling of Intrinsic Structure Covariance Functions for Non-Homogeneous and Non-Stationary Spatio-Temporal Random Processes on the Sphere
  - **Ph.D. Minor:** Mathematics
- **M.S. in Applied Statistics**, Loyola University Chicago, Chicago, IL, 2017
  - **Advisor:** Adriano Zambom
- **M.S. in Integrated Marketing Communications**, Northwestern University, Evanston, IL, 2015
  - **Specialization:** Marketing Analytics
- **B.S. in Business Management**, Hongik University, Seoul, South Korea, 2014

PROFESSIONAL EXPERIENCE

- **Associate Instructor**, Indiana University, Bloomington, IN, Aug 2023 – May 2025
- **Teaching Assistant**, Indiana University, Bloomington, IN, Jun 2019 – Aug 2023
- **Research Assistant**, Loyola University Chicago, Chicago, IL, Aug 2016 – Dec 2017

PUBLICATIONS

- Kim, J., Huang, C., Bussberg, N. (2025). Intrinsic Random Functions and Parametric Covariance Models of Spatio-Temporal Random Processes on the Sphere. Preprint: arXiv:2502.00579

- Kim, J., Huang, C. (2025). Random Processes with Stationary Increments and Intrinsic Random Functions on the Real Line. Preprint: arXiv:2501.15680
- Zambom, A.Z., Kim, J. (2020). New Algorithms and Software for Significant Controlled Variable Selection. *Statistics, Optimization & Information Computing*
- Zambom, A.Z., Kim, J. (2018). Testing for Additivity in Nonparametric Heteroscedastic Regression Models. *Journal of Nonparametric Statistics*
- Zambom, A.Z., Kim, J. (2018). Consistent Significance Controlled Variable Selection in High-Dimensional Regression. *STAT*, 7(1)

SOFTWARE

- Kim, J., Zambom, A.Z. (2017). `SignifReg`: An R package for consistent significance-controlled variable selection.

TALKS AND PRESENTATIONS

- **Intrinsic Random Functions and Parametric Covariance Models of Spatio-Temporal Random Processes on the Sphere**
  - Department of Mathematical Sciences, Ball State University, 2025
- **Parametrized Covariance Modeling for Non-Homogeneous and Non-Stationary Spatio-Temporal Random Processes on the Sphere**
  - Joint Statistical Meetings, Toronto, Canada, 2023

HONORS AND AWARDS

- Graduate and Professional Student Government Travel Award, Indiana University Bloomington, 2025
- Department of Statistics Scholarship, Indiana University Bloomington, 2023
- College of Arts and Sciences Travel Award, Indiana University Bloomington, 2023
- College Graduate Fellowship, Indiana University Bloomington, 2020

- Merit Award, Loyola University Chicago, 2016–2017
- Merit-Based Scholarship, Hongik University, 2011–2012

TECHNOLOGIES

- **Certificates:** SAS Certified Base Programmer for SAS 9; SAS Certified Advanced Programmer for SAS 9
- **Languages and Software:** R, SAS, SQL, C++, LaTeX, SPSS, Excel

OTHER ACTIVITIES

- **Korean International Statistical Society (KISS)**, Member since 2023
- **Volunteer Activities**
  - Medill Cause Marketing Initiative (2015): Provided pro-bono marketing support for Chicago-area nonprofits
  - Volunteer Abroad: Cambodia (2005), Mongolia (2006), India (2011)
- **Military Service**
  - Republic of Korea Air Force
  - Sergeant – Air Defense Missile Operator