On static equilibrium of a hemispheroid

In the course of a coffee-table conversation with my friends regarding the nature of static equilibrium of different solid objects the situation involving a uniform hemisphere came up. Intuition (and perhaps experience) tells that a uniform hemisphere as shown in Fig. 1 resting on a flat surface will be at stable equilibrium, and so will an oblate hemispheroid as shown in Fig. 2. Things get complicated when we move to a prolate hemispheroid like the one shown in Fig. 3, for which the nature of its equilibrium is less obvious. The intuition does come to mind though that if the prolate hemispheroid is made indefinitely taller, keeping its equatorial radius fixed, then beyond a point the equilibrium should become unstable. Intrigued, we decided to probe into the matter quantitatively.

Fig. 1

Fig. 2

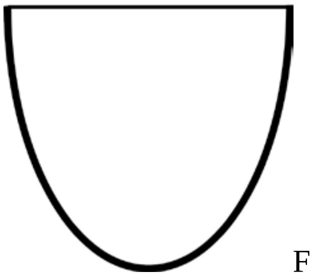

Fig. 3

Before we get into details of a spheroid, let us first look at the uniform hemisphere, and also briefly go over the understanding of different kinds of equilibrium.

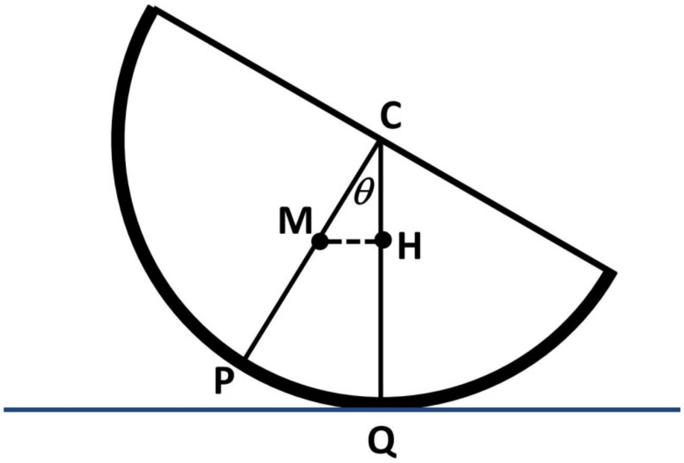

Fig. 4. A uniform hemisphere tilted at an angle θ with respect to its equilibrium configuration.

Fig. 4 shows a uniform hemisphere resting on a horizontal plane, tilted at an angle θ with respect to its equilibrium position. The point C is the center of the corresponding sphere, the straight line CP the hemisphere's symmetry axis, and the point P is where the hemisphere would touch the horizontal plane at equilibrium. Presently, at the tilted position, the hemisphere touches the plane at the point Q. Both the straight line segments CP and CQ have the length R, the radius of the hemisphere. The point M is the hemisphere's center of mass, the length of MP being h_0, which is the vertical height of the center of mass from the plane at equilibrium. Hence, the length of CM will be $(R-h_0)$. Now, since Q is on a surface of a hemisphere and C the corresponding sphere's center, the straight line CQ will be perpendicular to the horizontal plane touching at Q, and will therefore be vertical in this situation. MH is the perpendicular dropped from M on CQ, which makes the length of QH the height of the center of mass from the plane at the tilted position, which we call h. Now, from the right angled triangle CHM we can see that the length of CH is equal to $(R-h_0)\cos\theta$. Hence, the height h of the center of mass will be given by

$$h = R-(R-h_0)\cos\theta \quad . \tag{1}$$

Defining Δh as $\Delta h = h-h_0$, the change in vertical height of the center of mass relative to that at the equilibrium position, we obtain from (1)

$$\Delta h = (R-h_0)(1-\cos\theta) \quad . \tag{2}$$

At this juncture we use the fact that for a uniform hemisphere, $h_0 = \frac{5}{8}R$ [1], which lets us write from (1)

$$\Delta h = \tfrac{3}{8}R(1-\cos\theta) \quad . \tag{3}$$

From the above relation we observe that for infinitesimal values of θ, both positive and negative, Δh is positive, implying that at $\theta = 0$, h has a local minimum, confirming that $\theta = 0$ is a configuration of stable equilibrium for the hemisphere. Incidentally, starting from (1) we can easily derive that $\left(\frac{dh}{d\theta}\right)_{\theta=0} = 0$ and $\left(\frac{d^2h}{d\theta^2}\right)_{\theta=0} > 0$ which also confirm the fact that h has a local minimum at $\theta = 0$.

We now attempt an extension of the above argument to a hemispheroid.

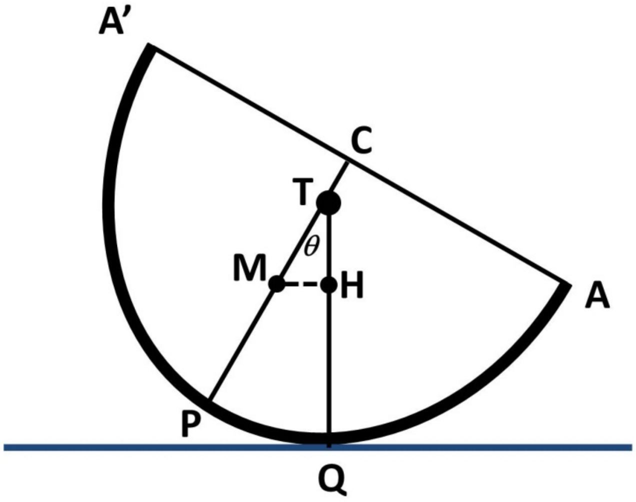

Fig. 5. A uniform hemispheroid tilted at an angle θ with respect to its equilibrium configuration.

We consider a uniform hemispheroid tilted at an infinitesimal angle θ with respect to its equilibrium position, as shown in Fig. 5 in which the angle θ is exaggerated for clarity. From now on we denote the equatorial radius (the length of CA and CA' in the figure) of the hemispheroid by a and the length of the symmetry axis CP by b, so that if $a > b$ we have an oblate hemispheroid and if $b > a$ we have a prolate one. At the tilted position the hemispheroid touches the horizontal plane at the point Q, and QT is the straight line normal to the horizontal plane and hence is vertical itself. QT intersects CP at the point T as shown, M is the position of the center of mass of the hemispheroid and MH is the perpendicular dropped on QT. As in the case of the sphere, we denote the present vertical height of the center of mass, namely the length of QH by h, and the corresponding vertical height at the equilibrium position, namely the length of MP by h_0. Now, since θ is infinitesimal, and PT and QT are normal to the hemispheroid surface at the points P and Q respectively, by definition the length of both PT and QT is equal to r_0, the radius of curvature of the generating ellipse at the point P. Beyond this juncture, following the same line of argument as in the case of the hemisphere, we can derive in a fairly straightforward way

$$\Delta h = (r_0 - h_0)(1 - \cos\theta) \quad . \tag{4}$$

We see from (4) that if $r_0 > h_0$, for an infinitesimal θ, both positive and negative, $\Delta h > 0$, implying that $\theta = 0$ is a configuration of stable equilibrium.

On the other hand, if $r_0 < h_0$, for an infinitesimal θ, both positive and negative, $\Delta h < 0$, implying that $\theta = 0$ is a configuration of unstable equilibrium.

Our newly derived principle turns out to be quite simple! *All we need to do is to compare the height of the center of mass from the contact point at equilibrium and the radius of curvature of the generating curve at the contact point, and infer the nature of the equilibrium.* The principle does fail to infer the nature of the equilibrium though if the said height of the center of mass and the radius of curvature happen to be equal.

Let us hold on still, and first see what the above principle tells for the equilibrium of a uniform hemispheroid for given values of *a* and *b*. We define the aspect ratio μ as $\mu = \frac{b}{a}$. We use the facts that $r_0 = \frac{a^2}{b}$ [1] and $h_0 = \frac{5}{8}b$ [2]. Using these expressions and our newly derived principle at hand we can derive through a couple of straightforward steps that for values of μ less than the critical value of $\mu_c = \sqrt{8/5} \approx 1.2649$, the equilibrium of the hemispheroid will be stable, and above that critical value the equilibrium will be unstable.

But what happens when μ is exactly equal to μ_c? In order to answer that and also verify whether our principle indeed leads to the correct inference we consider the exact expression of h as a function of θ for a uniform hemispheroid for finite values of θ. We present the expression below, leaving the derivation to the reader as a challenge on the side.

$$h = \frac{(b^2 \cos\theta + a^2 \sin\theta \tan\theta)}{\sqrt{b^2 + a^2 \tan^2\theta}} - \frac{3}{8} b \cos\theta \quad . \qquad (5)$$

It is worth noting that for *a*=*b*, the above expression becomes

$$h = \frac{5}{8}b + \frac{3}{8}b(1-\cos\theta) \quad ,$$

which is the indeed the one expected for a hemisphere of radius *b*.

To reduce some degree of cumbersomeness, we cast equation (5) in terms of the dimensionless variable $w = h/a$ and the previously defined aspect ratio μ:

$$w = \frac{(\mu^2 \cos\theta + \sin\theta \tan\theta)}{\sqrt{\mu^2 + \tan^2\theta}} - \frac{3}{8} \mu \cos\theta \quad . \qquad (6)$$

From (6), we can show that

$\left(\frac{dw}{d\theta}\right)_{\theta=0} = 0$ (confirming equilibrium), and

$$\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} = \frac{8-5\mu^2}{8\mu} \quad . \tag{7}$$

From (7), we see that if indeed $\mu < \sqrt{8/5}$ then $\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} > 0$, implying stable equilibrium, and if instead $\mu > \sqrt{8/5}$ then $\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} < 0$, implying unstable equilibrium just as inferred before using our simple principle!

Now, to investigate what happens when $\mu = \mu_c = \sqrt{8/5}$, starting from (6) we enumerate the third and fourth order derivatives of w with respect to θ for $\mu = \mu_c$, and obtain

$$\left(\frac{d^3w}{d\theta^3}\right)_{\theta=0} = 0 \text{ , and}$$

$$\left(\frac{d^4w}{d\theta^4}\right)_{\theta=0} = \mu_c + \frac{2}{\mu_c} - \frac{3}{\mu_c{}^3} \approx 1.36 \quad . \tag{8}$$

Since $\left(\frac{d^4w}{d\theta^4}\right)_{\theta=0} > 0$ for $\mu = \mu_c$, we infer that for $\mu = \mu_c$ the equilibrium of the uniform hemispheroid is actually stable!

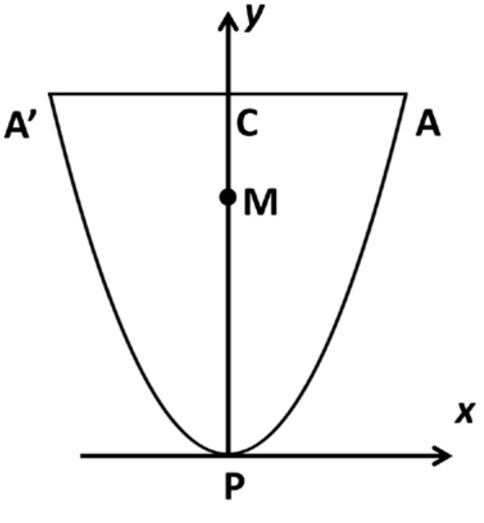

Fig. 6. The uniform paraboloid under consideration.

At this point, we consider the fact that there is no reason our simple principle should not work for solids of revolution in general. Since we have not presented a rigorous proof of our principle, we would like to verify its validity for at least one more solid of revolution. We choose a paraboloid for that purpose. Fig. 6 shows a uniform paraboloid obtained by revolving a parabola of the form $y = x^2 / c$ about the y axis. We denote the height of the paraboloid, namely the length of CP by *b* and the radius of the paraboloid, namely the length of CA and CA' by *a*. In this case constant *c* turns out to be a^2 / b. M is the center of mass of the paraboloid, and its distance h_0 from the point P for a uniform paraboloid is given by [2]

$$h_0 = \tfrac{2}{3} b \quad . \tag{9}$$

Also, the radius of curvature r_0 of the parabola at the tip P is given by [1]

$$r_0 = \tfrac{1}{2} c = \frac{a^2}{2b} \quad . \tag{10}$$

Let us remember our simple principle, and once again compare r_0 and h_0. Defining the aspect ratio of the paraboloid as $\mu = \frac{b}{a}$, using (9) and (10) we can show through a few straightforward steps that for values of μ less than the critical value of $\mu_c = \frac{\sqrt{3}}{2} \approx 0.866$, the equilibrium of the paraboloid will be stable, and above that critical value the equilibrium will be unstable.

In order to verify the above deduction and investigate what happens when μ is exactly equal to μ_c, we once again consider the exact expression of h as a function of θ for a uniform paraboloid for finite values of θ. We present the expression below, leaving the detailed derivation to the reader.

$$h = \frac{2}{3} b \cos\theta + \frac{a^2}{4b} \sin\theta \tan\theta \quad . \tag{11}$$

In terms of the dimensionless variable $w = h / a$ and the aspect ratio μ, equation (11) reads as,

$$w = \frac{2}{3} \mu \cos\theta + \frac{1}{4\mu} \sin\theta \tan\theta \quad . \tag{12}$$

From (12), we can show that

$\left(\frac{dw}{d\theta} \right)_{\theta=0} = 0$ (confirming equilibrium), and

$$\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} = \frac{3-4\mu^2}{6\mu} \quad . \tag{13}$$

From (13), we see that if indeed $\mu < \sqrt{3}/2$ then $\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} > 0$, implying stable equilibrium, and if instead $\mu > \sqrt{3}/2$ then $\left(\frac{d^2w}{d\theta^2}\right)_{\theta=0} < 0$, implying unstable equilibrium. Once again this turns out to be the same as what we had already inferred for the paraboloid using our principle involving the radius of curvature and the height of the center of mass that presents us with a way to circumvent all the tedious calculation, except right at the point of transition.

Now, to investigate what happens at that point of transition for the paraboloid when $\mu = \mu_c = \sqrt{3}/2$, starting from (12) we enumerate the third and fourth order derivatives of w with respect to θ for $\mu = \mu_c$ and obtain

$$\left(\frac{d^3w}{d\theta^3}\right)_{\theta=0} = 0 \ , \ \text{and}$$

$$\left(\frac{d^4w}{d\theta^4}\right)_{\theta=0} = \frac{2\mu_c}{3} + \frac{1}{\mu_c} \approx 1.73 \quad .$$

Since $\left(\frac{d^4w}{d\theta^4}\right)_{\theta=0} > 0$ for $\mu = \mu_c$, we infer that for $\mu = \mu_c$ the equilibrium of the uniform paraboloid is also stable!

References

[1] Wolfram Mathworld: http://mathworld.wolfram.com/

[2] ‘Elements of the Differential and Integral Calculus with Applications’, William Shaffer Hall, Bibliobazaar, 2008.