The Fractional Knowledge and Algebraic Reasoning of Students with the First Multiplicative Concept

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Abstract

To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning, a clinical interview study was conducted with 18 middle and high school students. Six students with each of 3 different multiplicative concepts participated. This paper reports on the fractional knowledge and algebraic reasoning of 6 students with the most basic multiplicative concept. The fractional knowledge of these students was found to be consistent with prior research, in that the students had constructed partitioning and iteration operations but not disembedding operations, and that the students conceived of fractions as parts *within* wholes. The students' iterating operations facilitated their work on algebra problems, but the lack of disembedding operations was a significant constraint in writing algebraic equations and expressions, as well as in generalizing relationships. Implications for teaching these students are discussed.

Key Words: fractional knowledge; algebraic reasoning; struggling students; first multiplicative concept; iterating operation; disembedding operation

1. Introduction

The school mathematics topics of fractions and algebra are persistently challenging for students (Kaput, 1998; Kastberg & Norton, 2007; National Mathematics Advisory Panel [NMAP], 2008). Some researchers have suggested that students' whole number knowledge interferes with building ideas about fractions (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Mack, 1995; Streefland, 1991), and that there is a cut-point between arithmetical and algebraic ways of knowing (e.g., Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Hercovics & Linchevski, 1994). Other researchers have investigated how students can reorganize their whole number schemes to construct fraction schemes (e.g., Biddlecomb, 2002; Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Tzur, 2004), as well as how students can use their whole number knowledge to make and justify generalizations of arithmetical and quantitative reasoning—often called "early" algebraic reasoning (e.g., Carpenter, Franke, & Levi, 2003; Kaput, Carraher, & Blanton, 2008; Russell, Bastable & Schifter, 2011). Although these two bodies of research are relatively separate, they are similar in that researchers seek to help students use their ways of operating with whole numbers to construct knowledge in persistently challenging domains.

More recently, researchers have begun to explore how students' fractional knowledge and algebraic reasoning may be related (e.g., Empson, Levi, & Carpenter, 2011; Kilpatrick & Izsak, 2008), with the goal of augmenting students' understanding in both domains. This focus is sensible since fractional knowledge is implicated in many aspects of algebraic reasoning, including reasoning with structures of numerical relationships (Bastable & Schifter, 2008; Empson, et al., 2011), proportional reasoning (Nabors, 2003), reciprocal reasoning when solving equations (Hackenberg & Lee, under review), and co-variational reasoning (Ellis, 2007b). However, since this research is in its infancy, there are numerous unexplored issues.

One issue is how knowledge in these two domains is related for students with a particular multiplicative concept called the *first* multiplicative concept (MC1) (Hackenberg & Tillema, 2009; Steffe, 1992, 1994). Some research has been completed on the fractional knowledge of MC1 students. These students have yet to construct fractions as measurable extents (Biddlecomb, 2002; Olive & Vomvoridi, 2006; Steffe & Olive, 2010), so they fall within the 80% of fourth grade students and 40% of eighth grade students who did not demonstrate this meaning of fractions on the 2003 National Assessment of Educational Progress (Kastberg & Norton, 2007, pp. 89-90). Instead, MC1 students conceive of fractions primarily as one part (or several parts) *within* a number of parts, regardless of the relative size of the parts. Although MC1 students are perhaps most common in elementary school, Steffe (2007) has estimated that they constitute 30-50% of the student population at the end of fifth grade. Furthermore, for some students the first multiplicative concept is rather enduring in that making a transition to more advanced multiplicative concepts can take two or more years (Steffe & Cobb, 1988). So, even the optimistic estimate of 30% means a substantial portion of incoming sixth grade students are operating in ways that significantly constrain their fractional knowledge, and little is known about exactly what this implies for their algebraic reasoning.

To understand relationships between students' fractional knowledge and their algebraic reasoning in the areas of equation writing and generalizing activity, a clinical interview study was conducted with 18 middle and high school students. Six students with each of three different multiplicative concepts were invited to participate. The purpose of this paper is to investigate relationships between the fractional knowledge, equation writing, and generalizing activity of the six MC1 students in the study. The research questions are:

1) What fractional knowledge is attributable to the MC1 students?

2) How do the MC1 students solve algebra problems that involve writing equations to represent relationships among unknowns, as well as generalizing?
3) How are the MC1 students' fractional ways of operating related to their equation writing and generalizing activity?

To set up the data and analysis I first outline my conceptual orientation to students' mathematical thinking, multiplicative concepts, and fractional knowledge. Then I review research on relationships between students' fractional knowledge and algebraic reasoning.

2. Students' Mathematical Thinking, Multiplicative Concepts, and Fractional Knowledge

2.1. Operations, Schemes, and Concepts

I view students' mathematical thinking in terms of mental actions, or *operations* (Piaget, 1970; von Glasersfeld, 1995). Operations critical for fractional knowledge include *partitioning*, or marking a quantity into equal parts (Pothier & Sawada, 1983; Steffe & Olive, 2010); *iterating*, or repeatedly instantiating a fractional part to make a larger fraction (Steffe & Olive, 2010; Tzur, 1999); and *disembedding*, or taking a part out of a whole without mentally destroying the whole (Steffe & Olive, 2010). Operations such as these are *interiorized* physical actions—that is, they arise from re-processing physical actions in such a way that they can be performed mentally, without having to be carried out materially. Interiorizing an action means the action is available to a person in visualized imagination, prior to carrying out activity in a situation.

Operations are the components of *schemes*, goal-directed ways of operating that consist of three parts: an assimilated situation, activity, and a result (von Glasersfeld, 1995). For example, if a student has constructed a *partitive fraction scheme*, then a situation of the scheme is a request to make a new length that is three-fifths of a foot. The activity of the scheme involves partitioning the foot into five equal parts, disembedding one of those parts, and iterating the part to make three such parts. The student then assesses the result of her activity, a 3-part length, as three parts out of five parts. Even though students use their part-whole ideas to evaluate the results of this scheme (Steffe & Olive, 2010), this scheme marks the first time fractions begin to become measurable extents for students. For this reason, Steffe (2002) refers to the partitive fraction scheme as the first "genuine" fraction scheme (p. 305).

In turn, a *concept* is the result of a scheme that people have interiorized. For example, a student who has interiorized the result of her partitive fraction scheme can take that result, a length of three-fifths consisting of three one-fifths, as a given in further activity. This student could engage in problems such as how to take one-fourth of three-fifths, or how to re-make the whole if a given length is three-fifths of the whole. I take students' *reasoning* to be the functioning of their operations, schemes, and concepts in on-going interaction in their experiential worlds.

2.2. Three Multiplicative Concepts

During elementary and middle school, students construct three qualitatively distinct multiplicative concepts that involve how students produce and coordinate units of units, or composite units (Hackenberg & Tillema, 2009; Steffe, 1992, 1994). These concepts are the interiorized results of students' multiplying schemes. The most advanced of these concepts, the third multiplicative concept (MC3), has been found to be required for constructing a great deal of fractional knowledge, including schemes for adding, subtracting, multiplying, and dividing fractions, as well as schemes that underlie equivalent fractions and conceiving of improper fractions as numbers (Hackenberg, 2007, 2010; Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Tunc-Pekkan, 2008). MC3 students coordinate three levels of units prior to operating, which means that they can view a length as, say, a unit of four units each containing six units

(Figure 1, top). Being able to create this structure prior to operating means that students can take it as a given and can switch to view the length as a different three-levels-of-units structure—as, say, a unit of six units each containing four units (Figure 1, bottom).

The more basic second multiplicative concept (MC2) has been found to be required for starting to conceive of fractions as measurable extents (Steffe & Olive, 2010). For example, students who have constructed the second multiplicative concept (MC2 students) can learn to construct a partitive fraction scheme. MC2 students coordinate two levels of units prior to operating, which means that they can view a length as a unit of units, or composite unit, prior to activity. For example, they can imagine a length partitioned into four equal parts without actually having to make the partitions (Figure 2, top). In addition, in the process of working on a problem they can insert more parts into each of these parts, creating a three-levels-of-units structure *in activity*. However, unlike their MC3 peers, MC2 students do not maintain this structure in further operating. For example, if they partitioned each of the fourths of the length into six equal parts, in further operating the length would become a unit of 24 units for them (Figure 2, bottom).

In contrast with MC2 students, MC1 students can coordinate two levels of units *in activity*—they cannot take a unit of units, or a composite unit, as given. This means that they are "in" the activity of partitioning a whole; they cannot imagine a whole as partitioned ahead of operating. So, MC1 students tend to construct *parts-within-wholes fraction schemes* (PWWFS) (Steffe & Olive, 2010). A situation of a PWWFS is a request to make a fraction, say three-fifths, of a given quantity represented by a segment or rectangle. The activity of the scheme involves the student in partitioning the given length into five parts (not necessarily equal) and identifying three of those parts in some way—for example, by shading them. So, these students conceive of three-fifths as three parts embedded within what an observer sees as the whole (five parts). I emphasize that this scheme does *not* include a disembedding operation, in contrast with the partitive fraction scheme. In other words, MC1 students do not mentally take the three shaded parts out of the five parts in some way while maintaining the five parts as a unit. In fact, if they did take the three parts in some way, they would only see two parts left—they would have no sense that the three parts could be considered in comparison to the original five parts. In general, MC1 students can learn to partition, but they have not yet constructed disembedding operations (Steffe, 1992; Steffe & Olive, 2010).

2.3. Fraction Schemes and Operations

Now that I have introduced two fraction schemes, a partitive fraction scheme and a PWWFS, I lay out the landscape of fraction schemes and operations more fully in order to situate students with each of the three multiplicative concepts. Although not all of these schemes will be used in the analysis for this paper, I give this overview to position the fractional knowledge of MC1 students in relation to the fractional knowledge of other students. A PWWFS is indeed the most basic scheme, as described above. Since it does not involve a disembedding operation, it is in the province of MC1 students. A disembedding operation is required for all other fraction schemes that I discuss in this section. See Table 1 for a summary of these schemes.

A *part-whole fraction scheme* involves both partitioning and disembedding, in that students not only partition a length into some number of equal parts, but they also disembed those parts from the whole while not mentally destroying the whole (Steffe & Olive, 2010). With this scheme students think of fractions as parts *out* of wholes. That is, when students construct a part-whole fraction scheme they can remove parts from the whole (mentally or materially) and maintain the whole as a unit of units to which the parts can be compared or named. Doing so requires a disembedding operation, and it is equivalent to saying that the student can take a units of units as given—that the student is an MC2 student.

However, students with only part-whole fraction schemes students do not view fractions as measurable extents—as lengths or quantities in their own right (Kerslake, 1986; Lamon, 2007; Norton & Wilkins, 2010; Steffe & Olive, 2010; Tzur, 1999). When students construct a *partitive **unit** fraction scheme* (PUFS), they go beyond solely part-whole ideas. In fact, researchers have found that students who construct a PUFS go on to construct more advanced fraction schemes and operations relatively soon (Wilkins & Norton, 2011); so, a PUFS appears to function like a "gateway scheme" to more advanced fractional knowledge.

A situation of a PUFS is a request to make a unit fraction, say one-fifth, of a given length. The activity of this scheme is to mark a part with the intention to create that part so that there are five such parts in the whole length. The student marks a part, takes the part out, and repeats it to see if repeating it five times re-creates the whole. In other words, students who construct this scheme have an idea that one-fifth is an amount they can take five times to make the whole. So, partitioning, disembedding, *and iterating* are all operations of this scheme. A PUFS is often investigated by assessing the reverse of it (Norton & Wilkins, 2010; Tzur, 2004). That is, students are posed a task like this: "Given this segment that represents 1/6 of a whole, can you draw the whole?" Students who solve this problem by iterating the segment six times and take the iterated length as a unit of units demonstrate the construction of a *reversible* PUFS, which implies the construction of a PUFS. These schemes are in the province of MC2 students.

As discussed earlier, a partitive fraction scheme also involves the operations of partitioning, disembedding, and iterating, and it is an advance over the PUFS (Steffe & Olive, 2010). However, with only a partitive fraction scheme students produce only proper fractions as results; they do not produce improper fractions (Steffe, 2002). Constructing a scheme that includes improper fractions as results, an *iterative fraction scheme*, relies on disembedding as well as an operation that is the composition of partitioning and iterating, called *splitting* (Steffe, 2002; Norton, 2008). A basic task used to assess a splitting operation is one such as the following: "This stick (segment) is five times the length of another stick; can you make the other stick?" Students with a splitting operation solve this problem by positing a stick independent from the given stick. The length of the new stick is determined by partitioning the given into five equal parts, and at the same time, the new stick can be iterated exactly five times to make the given. So upon the construction of splitting, partitioning and iterating become "unified."

A situation of an iterative fraction scheme is a request to make a proper or improper fraction, say seven-fifths, of a length. Students who have constructed an iterative fraction scheme partition the length into five equal parts and iterate one of those parts seven times. The students view the resulting seven parts as a unit of units, any one of which could be iterated five times to create another composite unit (a unit of five units) from which the seven-fifths is independent but in relation to which the seven-fifths is named. Coordinating these two, related composite units has been found to require the third multiplicative concept (Hackenberg, 2007; Steffe & Olive, 2010). For students who have constructed an iterative fraction scheme, any fraction is a whole number multiple of a unit fraction (e.g., 7/5 = 1/5 x 7), so this scheme is multiplicative and yields what Steffe and Olive (2010) refer to as fractional numbers.

3. Relationships between Students' Fractional Knowledge and Algebraic Reasoning

Given this overview of students' fractional knowledge, I now review literature on relationships between students' fractional knowledge and algebraic reasoning. I highlight studies with middle school students since all MC1 students in this study were middle school students.

Yet because MC1 students demonstrate ways of reasoning similar to some elementary school students, I also draw upon early algebra research. Based on this research (e.g., Kaput, 2008; Russell et al., 2011), I view students' beginning algebraic reasoning to be about (1) generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations in some way, not necessarily with standard algebraic notation; and (2) learning to reason with algebraic notation in lieu of quantities. These two points correspond to Kaput's two core aspects of algebra, with a focus on the first of his three strands of algebra.[1]

This view of beginning algebraic reasoning is more encompassing than some views about when students can be considered to be reasoning algebraically (e.g., Filloy & Rojano, 1989, 2010; Hercovics & Linchevski, 1994). For example, Filloy and Rojano have stated that students are not reasoning algebraically until they are engaged in solving an equation in one unknown with unknowns on both sides of the equation (e.g., $ax + b = cx + d$), because they believe that this activity involves explicitly operating on unknowns, a hallmark of algebraic (v. arithmetic) thinking. However, early algebra researchers have questioned more restrictive views because of evidence that elementary school students can use algebraic notation to represent generalizations and can learn to operate on this notation to some degree (e.g., Brizuela & Schilemann, 2004; Carraher, Schilemann, & Schwartz, 2008; Russell et al., 2011). These researchers have championed generalizing as a key feature of early algebraic reasoning, along with systematically representing generalizations. This emphasis on generalizing is consistent with Kieran's (2007)

[1] Kaput's (2008, p. 11) two core aspects are (A) Algebra as systematically symbolizing generalizations of regularities and constraints, and (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems. These core aspects are embodied in three strands: (1) Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning; (2) Algebra as the study of functions, relations, and joint variation; (3) Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

use of global/meta-level activities, such as problem solving and generalizing, to describe one of three main categories of algebra research at the secondary level.[2]

I take *generalizing* to be "an activity in which people in specific sociomathematical contexts engage in at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases" (Ellis, 2011, p. 311). I also follow Ellis (2007c) in distinguishing between *generalizing actions*, such as relating, searching, and extending, and *reflection generalizations*, which are declarations, such as verbal or written statements. I take *abstracting* to be a process that underlies making general ideas, that involves a stripping away of particulars and an ability to view particulars as representing a phenomenon broader than the particulars (Kaput, Blanton, & Moreno, 2008; von Glasersfeld, 1991, 1995).

3.1. Generalizations of Numerical Structure

Some researchers have focused on connections between fractional knowledge and algebraic reasoning by viewing students' fractional knowledge as an entrée into reasoning about mathematical properties and numerical structures (Empson & Levi, 2011; Empson, et al., 2011; Russell, et al., 2011). In particular, Empson and colleagues have found that students can develop reasoning strategies that mutually support their understanding of fractions and mathematical properties. For example, in a quest to determine 20 times ¾, a sixth grade student reasoned as follows: 20 times ¾ is the same as 10 times 1½, which is the same as 5 times 3, which is 15 (Empson & Levi, p. 85). This reasoning is based on the Associative Property of Multiplication, although students certainly may use this reasoning prior to awareness of this mathematical property. The argument is that instruction that helps students to become aware of these properties

[2] Kieran's (2007) other two categories are generational activities, which include writing algebraic equations based on quantitative situations, and transformational activities, which include but are not limited to algebraic manipulations.

in their reasoning with fractions—i.e., to identify commonalities in their thinking across various instances of it, one aspect of generalizing (Ellis, 2011)—also helps students learn to reason algebraically. This research is significant, but it does not expressly attend to MC1 students, and it does not feature students' use of algebraic notation to represent unknowns.

3.2. Quantitative Reasoning as a Basis for Making Generalizations

Other researchers have looked to quantitative situations as a basis for students to construct quantitative relationships that involve fractions (or decimals), make generalizations, and use algebraic notation to symbolize those generalizations (Ellis, 2007a, 2007b; Olive & Caglayan, 2008). For example, Ellis conducted a two-part study to investigate students' generalizing activity and reasoning about linear relationships, a teaching experiment with seven middle school students and observations of a middle school classroom. Teaching experiment students reasoned about two quantitative situations, one involving gear ratios and the other involving speed. During the experiment students created emergent ratios as quantities in their own right, which meant that they used two quantities that were directly measured, such as distance and time, to create a third, intensive quantity, such as speed (Schwartz, 1988). For example, if a character walking at a constant speed went 15 cm in 12 s, students reasoned that for every 5 cm he walked, 4 s would elapse, and ultimately that 1 cm corresponded to 4/5 s. Students used these emergent ratios to make reflection generalizations about slopes of lines. In contrast, in the classroom study students reasoned primarily with patterns of differences in two direct measures (e.g., x- and y-values in a table) without ever creating emergent ratios.

Ellis's study (2007a, 2007b) represents important findings about a critical area of algebraic reasoning: co-variation of quantities as a basis for developing functional understanding (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012; Lobato & Thanheiser, 2002;

Thompson & Thompson, 1996). Approaching the teaching of algebra courses from the perspective of functions has been popular in the last two decades (e.g., Chazan, 2000; Heid, 1996; Kieran, Boileau, & Garacon, 1996; Yerushalmy, 2006), in part because this approach can help unify aspects of algebra (Chazan & Yerushalmy, 2003). However, based on Ellis's analysis, it is likely that constructing emergent ratios and reasoning about co-variation of quantities requires the third multiplicative concept. Little is known about how MC1 students might engage in this complex domain. Indeed, Ellis worked only with middle- and high-achieving students in her study. Although MC1 middle school students may not always be low-achieving, they are less likely to be middle- or high-achieving, given that their multiplicative concepts significantly constrain their mathematical reasoning.

3.3. The Development and Use of Algebraic Notation with Fractions

In addition to studying how students create generalizations of numerical structure and quantitative relationships involving fractions, researchers have studied how middle school students use algebraic notation (e.g., Johanning, 2004; Olive & Caglayan, 2008; MacGregor & Stacey, 1997; Swafford & Langrall, 2000). However, few if any studies have addressed how students understand fractions as multipliers of unknowns—e.g., conceive of and symbolize one-fifth of an unknown length as $(1/5)x$.[3] For example, Swafford and Langrall asked 10 sixth grade students who had not received formal algebra instruction to work on six algebraic problems that involved whole numbers as multipliers of unknowns. Similarly, Johanning engaged 31 seventh and eighth grade students who had not taken a formal algebra course in two algebraic problems in which whole numbers were the only multipliers of unknowns. In Olive & Caglayan's study, four eighth grade students worked on one problem involving decimals (values of coins) as

[3] Researchers have found that conceiving of fractions as multipliers on *known* quantities is a significant challenge for students (e.g., Behr, Harel, Post, & Lesh, 1993; Hackenberg, 2010).

multipliers of unknowns, but not with fractions. Since multiplying unknowns by fractions is critical for reciprocal reasoning in solving equations, more needs to be known how students conceive of fractions as multipliers on unknowns and variables.

One finding across these studies is that students often prefer to solve problems that adults consider algebraic by using methods that circumvent manipulation of algebraic notation. For example, Swafford and Langrall (2000) found that when students did write equations, they did not use them to solve problems. Similarly, in Johanning's (2004) study no student voluntarily used algebraic notation to solve problems; most students used systematic guess and check or "unwinding" methods (cf. Nathan & Koedinger, 2000). These findings are consistent with those of MacGregor and Stacey (1997), who found that the majority of Australian students through age 15—even those who had taken an algebra course—did not interpret letters as generalized numbers (variables) or specific unknowns. Instead, students often thought of letters as shorthand names, replaced letters with numbers, or ignored them (Booth, 1984; Kuchemann, 1981). In contrast, Olive and Caglayan (2008) found that the students in their study did conceive of letters as unknown quantities, but only the MC3 students could formulate a complete equation in one unknown, because only the MC3 students could construct a sophisticated enough quantitative structure. Taken together, these studies highlight the challenges of learning to use algebraic notation while also indicating that students without traditional algebraic skills may be able to solve algebra problems more adeptly than previously thought. This could be good news for MC1 students, given that their progress in learning algebraic notation is currently uncharted.

3.4. Characteristics of MC1 Students' Fractional and Algebraic Knowledge

With the exception of Yerushalmy (2006), none of the algebra studies cited in this literature review have focused explicitly on students who persistently struggle with mathematics, or who exhibit low achievement. This omission is not surprising, since low-achieving students are often left out of mathematics education research (Baxter, Woodward, & Olsen, 2001; Yerushalmy). Although some researchers question whether new approaches to algebra—and the era of reform in mathematics education in general—serve these students well (Baker, Gersten, & Lee, 2002; Baxter, Woodward, & Olsen; Kroesbergen, Van Luit, & Mas, 2004), others express optimism that low-achieving or persistently struggling students can experience success with reform-oriented mathematics instruction (Empson, 2003; Henderson & Landesman, 1995; Schoenfeld, 2002). That optimism must be supported by more research, however, due to the constraints faced by low-achieving and persistently struggling students.

Based on studies about MC1 students (e.g., Olive & Vomvoridi, 2006; Steffe & Olive, 2010; Ulrich, 2012), my hypothesis is that after about third grade, most of them are low-achieving or persistently struggling. For example, Olive and Vomvoridi have analyzed the case of a sixth grade MC1 student, Tim, who had constructed a PWWFS. One feature of Tim's scheme was that both a unit fraction and the whole referred to the same partitioned image: One-sixth meant a whole partitioned into six equal parts, and six-sixths meant the same partitioned whole (Figure 3). Another feature of Tim's fraction scheme was that the number of parts could determine a unit fraction—e.g., if there were four equal sevenths shown, Tim called that entire amount "one-fourth." These ideas about fractions led Tim to add up parts regardless of size. So for example, in adding one-half and one-fourth, Tim said the answer would be one-fifth because one-half was one part and one-fourth was four parts, for a total of five parts (Figure 4).

As already noted, MC1 students like Tim have not constructed a disembedding operation, and so they do not see inclusions—parts within wholes and also separate from them—that are apparent to observers (Olive & Vomvoridi, 2006). MC1 students can learn to partition a quantity into equal parts; they do view those parts constituting the whole quantity; and they do form a goal to exhaust the whole quantity when partitioning (Steffe & Olive, 2010), which are three basic criteria for the construction of fractions according to Piaget and colleagues (Piaget, Inhelder, & Szeminska, 1981). However, MC1 students sometimes have difficulty balancing the dual goals of making equal parts and exhausting the whole—that is, if they are focused on one of these goals, the other may slip away (Steffe & Olive). More importantly, since they do not view the partitioned whole in terms of part-to-part and part-to-whole relations, they do not satisfy a fourth criterion for the construction of fractions (Piaget, et al).

The case of Tim (Olive & Vomvoridi, 2006) is a happy one, however. Through a combination of targeted interview interactions and revised classroom teaching, Tim did construct a PUFS over a period of about 4 weeks. The authors state, “Without these modifications of his fractional schemes, Tim would have joined the many students for whom fractions remain a mystery and, consequently, have little chance of reasoning algebraically” (p. 44). So, the case of Tim is meant to demonstrate that MC1 students are challenged to construct robust fractional knowledge and may struggle with classroom instruction. However, the case also shows that these students can make progress with concerted, targeted intervention. In fact, Tim’s progress is not often the norm for MC1 students. In other studies, MC1 students have operated with the first multiplicative concept for two years or longer, despite on-going teaching interactions with trained researchers (Steffe & Cobb, 1988).

4. Methods

4.1. Participants

The clinical interview study that is the basis for this paper occurred at a school district in a small Midwestern town. The middle school contained grades seven and eight. Seven seventh grade students, 10 eighth grade students, and one tenth grade student participated in the study.[4] Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based interviews to assess students' multiplicative concepts (See Appendix A for selection interview protocol). Six students with each of three multiplicative concepts were invited to participate; this paper focuses on the six MC1 students, three seventh grade and three eighth grade students. The three seventh grade MC1 students were enrolled in a seventh grade mathematics class for struggling students; the three eighth grade MC1 students were taking an eighth grade pre-algebra class. The three seventh grade students and one of the eighth grade students received special education support for one period per day. All MC1 students had received some instruction in their mathematics classes on the use of algebraic notation and equation solving.

4.2. Data Collection

4.2.1. Interviews.

Students participated in two 45-minute, task-based interviews (Goldin, 2000), a fractions interview and an algebra interview. All interviews except four were conducted by the author; a mathematics education doctoral student conducted the two interviews for each of two MC3 students. When not acting as interviewer, the author and the doctoral student assisted with and served as a witness to the interviews. All interviews occurred in a quiet conference room at the schools during lunch periods or study halls. Each interview was video-recorded with two cameras, one focused on the interaction between the interviewer and student, and one focused on the student's written work. The work video was digitally mixed into the interaction video for analysis. All students completed the fractions interview prior to the algebra interview, but the

[4] The tenth grade student was selected because the original intent of the study was to include both middle and high school students. However, due to scheduling constraints it was not possible to engage other high school students in the study. So more middle school students were interviewed to achieve the targeted total number of participants, 18.

time between interviews varied from 3 weeks to 4 months, with an average time of just less than 2 months.[5] A main reason for the length of time between interviews was that interviews with middle school students could occur only one day during the school week, on the day with scheduled study halls around lunch periods. Interviewing at this time was recommended by the principal in order to lessen class disruptions, not interfere with sports practices after school, and not burden parents with picking up children who rode the bus.

The interview protocols were refined in a prior pilot study (Hackenberg, 2009) and were designed so that the reasoning involved in the fractions interview could be drawn upon for solving problems in the algebra interview. For example, one fractions interview task was the following: “A 65-cm stack of CDs is 5 times the height of another stack. Can you make a drawing of the situation and determine the height of the other stack?” The algebra interview included a similar situation with both heights unknown. Students were asked to make a drawing of the situation, write equations to represent the situation, and explain the meaning of the equations in relation to the situation and their drawing. In this example, the fractions interview task is intended to elicit a splitting operation, although it may be solved by trialing a height for the other stack and iterating that height five times—i.e., without partitioning. In turn, students may use a splitting operation to begin the algebra interview task, which is also intended to elicit the use of algebraic notation. In general, in the fractions interview unknowns could remain implicit in students’ thinking, while students were asked to be more explicit about how they represented and operated on unknowns in the algebra interview. (See Appendices B and C for interview protocols.)

4.2.2. Nature of interviewing interactions.

[5] Four months between interviews was unusual and occurred for only one student who switched schools in the middle of the year. It took some time to locate this student and schedule a second interview. The median length of time between interviews was also a little under 2 months, and the modes were 3 weeks and 1 month.

During the interviews the interviewers aimed to follow the interview protocol, which included regularly requesting students to make drawings of quantitative situations, solve problems and write equations, and describe their solutions and equations in relation to their drawings. However, the interviewers were also free to depart from the protocol when necessary to investigate conjectures about students' thinking (Clement, 2000). For example, if a student did not appear to make equal parts when drawing fractions, an interviewer might pose additional questions to establish whether this lack of making equal parts was a persistent feature of the student's thinking. In fact, such departures occurred frequently with MC1 students. For example, for MC1 students the interviewer (the author) often adjusted numbers to be smaller than those in the protocols, since MC1 students often do not readily know multiplication facts. I also circled back to basic fractions problems with these students, sometimes even in the algebra interview.

For example, with four of the six MC1 students, I used a set of questions intended to investigate the partitioning, iterating, and disembedding operations of these students more closely than allowed for in the original interview protocols.[6] In these problems students were given a candy bar (rectangle) cut from card stock and four possible fair shares of that bar (all cut from card stock), based on different numbers of people sharing the bar. The four possible fair shares (a 1/3-piece, ¼-piece, 1/6-piece, and 1/8-piece) had hairlines running down the center of them horizontally in order to show their orientation. An initial task with these materials was to select the fair share for 6 people and to justify the choice. I will refer to these supplemental questions as the Cut-Out Bar Problems (see Appendix D).

[6] I intended to use the problems with all MC1 students but ran short on time with two students. These two eighth grade MC1 students, Wanda and Cynthia, each had shorter interviews (about 30 minutes each). Wanda's fraction interview was short because she had a class responsibility to return to, and her algebra interview was short because she stated that she did not want to work on the problems. Cynthia's algebra interview was short because she switched schools and had a different schedule.

Overall, the interviewers sought to harmonize with student thinking (Steffe & Thompson, 2000) to the extent possible in the short time that they interacted with the students, and they aimed for their questions of students to be open-ended rather than funneling (Wood, 1998). For example, the interviewers rephrased questions to students as requested or as the interviewers deemed necessary, but they tried to keep rephrasing in the spirit of using language or ideas that might elicit further mathematical activity from the student, and they avoided guiding students to particular answers. Above all, the interviewers aimed to communicate an attitude of respect to the students and genuine interest in the students' ways of thinking (Ginsburg, 1997).

4.2.3. Inclusion of the Border Problem.

All questions in the algebra interview involved fractions[7] except for one, the Border Problem (Figure 5), which has been used to introduce ideas of unknowns and variables to middle school students (Boaler & Humphreys, 2005; Fendel, Resek, Alper, & Fraser, 1997; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002; cf. Swafford & Langrall, 2000). I included the Border Problem in the algebra interview specifically because I suspected that MC1 students might be significantly challenged by the other problems (Ginsburg, 1997; Goldin, 2000). Since the Border Problem has been found to be particularly accessible to a wide range of thinkers (Boaler & Humphreys; Swafford & Langrall), I thought that it could give insight into MC1 students' generalizing activity, even if they were not able to work productively on the other problems. Although the Border Problem does not involve fractions, it is still possible to analyze how students used operations like iterating or disembedding when working on the problem.

4.2.4. Written fractions assessment.

[7] That is, the problems involved fractions from the researchers' perspective; I make no claims that students saw the problems in this way.

In addition to the interviews, in their regular mathematics classes students completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge.[8] This assessment confirmed that the MC1 students had yet to construct a PUFS or any more advanced fraction schemes at the time of completing the assessment. Results from the assessment are reported more fully in Hackenberg and Lee (under review).

4.3. Data Analysis

Analysis occurred in two phases. The first phase involved formulating a second-order model (Steffe & Thompson, 2000) of each student's fraction operations, schemes, and concepts; equation writing and solving; and generalizing activity, to the extent possible over two interactions. A second-order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Researchers generate second-order models out of their repertoire of tools, which for this study included theoretical constructs (see §2.1, 2.3); models from prior research (e.g., Biddlecomb, 2002; Olive & Vomvoridi, 2006; Steffe & Olive, 2010); and a commitment to use theoretical constructs in an orienting but not deterministic way (Clement, 2000). That is, we aimed to construct second-order models that were consistent with models from prior research, which lends explanatory power and believability to the findings (Clement), while expecting students to surprise us with novel ways of operating.

To accomplish this first phase, the researchers engaged in repeated viewing of videofiles and took detailed analytic notes (Cobb & Gravemeijer, 2008), which included transcriptions, data summaries, and memos (Corbin & Strauss, 2008). In this process, the researchers made conjectures about students' ways of operating. Then the researchers wrote a narrative for each

[8] Two students out of 18 (an MC1 student and an MC3 student) did not return this assessment to the researchers despite repeated attempts to obtain it.

student that represented the initial second-order model for that student. The researchers read all of the narratives, discussed questions about them, and viewed data together to resolve differences in interpretations of data excerpts, refine conjectures, and revise the models (Clement, 2000). These models provided the basis for determining the fractional knowledge attributable to the MC1 students (the first research question), as well as how the MC1 students solved algebra problems that involved writing equations to represent relationships among unknowns and generalizing activity (the second research question).

In the second phase of analysis, the researchers looked across the students to articulate differences in how students with different multiplicative concepts solved the problems in each interview, as well as similarities and differences among students with the same multiplicative concept. The author of this paper wrote syntheses of the fractional and algebraic ways of operating of students with each of the three multiplicative concepts, which she discussed with the other researcher. These syntheses allowed the researchers to examine how the operations, schemes, and concepts that constituted students' fractional knowledge were involved in students' equation writing and generalizing activity (the third research question).

5. Analysis and Findings: MC1 Students' Fractional Knowledge

In this section I discuss the fraction operations and schemes I could attribute to the MC1 students; see Table 2 for an overview.

5.1. Students' Partitioning Operations

To all six MC1 students I could attribute a partitioning operation in the following sense: All MC1 students marked a rectangle or circle into a number of relatively equal parts in the process of solving a problem that asked them to make fair shares. However, when the researcher used fraction language (e.g., asked a student to draw one-third of a given rectangle), the partitioning operations of only three of the MC1 students were activated. Indeed, the nature of students' partitioning operations was not identical. One eighth grade student, Zachary, seemed to have no necessity to exhaust the whole when sharing a bar fairly with more than two people. For example, he did not make three parts in aiming to share a rectangle fairly among three people: He made four parts, whether marking vertically or cross-partitioning (Figure 6). He seemed undisturbed by a fourth part, referring to it as "extra."

The other five MC1 students usually tried to exhaust the whole, at least when working on equal sharing problems, and they also usually tried to make equal parts. However, when probed about equal parts, students did not always indicate an obvious logical necessity for them. For example, at one point seventh grade student Courtney shared a bar fairly among five people, but the parts were not equal. When the researcher asked whether she could do it again but a little more accurately, Courtney asked, "What do you mean?" The interviewer agreed it was a good question and shaded two nonconsecutive parts in the bar different colors (Figure 7, top). The interviewer said that she got one and Courtney got the other; did they each get the same amount? Courtney said no, and she identified that the interviewer got more. Then Courtney made two further tries to make the five parts more equal, ultimately cutting off an "extra" part of the bar (Figure 7, bottom). So in her focus on making equal-sized parts, Courtney ended up not exhausting the whole.

One seventh grade student, Henry, had an interesting way of demonstrating his dual goals to exhaust the whole and make equal-sized parts. On problems that involved drawing a fraction of a bar, as well as equal sharing problems, he drew very closely spaced hash marks along an entire edge of a rectangle, counted these marks, and then tried to divide the number of marks appropriately. For example, to draw a bar that was one-half of a given bar, Henry used this technique, drawing 51 marks along one edge. He ultimately arrived at 26 hash marks along one half of the bar and 25 hash marks along the other (Figure 8). Throughout his fraction interview Henry seemed consumed by this technique: He appeared deeply engaged in the activity of making the marks and would not mark estimates for a part of a bar without the hash marks.

5.2. Students' Iterating Operations

As with partitioning, I could attribute some kind of iterating operation to all six MC1 students. However, three students did not demonstrate an iterating operation until working on the Cut-Out Bar Problems (see Appendix D). I give seventh grade student Laura's work as an example because it was the most advanced of these three. When Laura worked on these problems, the interviewer asked her to select the fair share for six people and to justify her answer. Laura picked out the appropriate piece with no hesitation. When asked to check by testing her choice, she placed all four different size parts on the given candy bar (Figure 9). The interviewer asked if that tested it, and she said she guessed so.

Then the interviewer suggested that she use the piece she selected "on here [the bar]" to see whether it was fair. Laura moved the part along the bar, marking each successive placement, and she found that approximately six fit. When asked how many parts she was able to fit, she said five. The interviewer interpreted that comment to mean that the sixth placement was not quite a full part (due to errors in placing and marking positions). When asked to imagine that the parts fit exactly and to give the fraction name for one part, Laura said "one-fifth or one-fourth, probably." The interviewer said: "Is that what you would call one out of six pieces?" Laura nodded but then said, "no, one-sixth." So, this activity shows the possible emergence of an iterating operation in testing the fairness of a share.

The remaining three students showed evidence of an iterating operation in solving tasks intended to assess a splitting operation, or during work on a task intended to assess a PUFS. Because I will discuss the students' splitting operations separately, here I discuss evidence for eighth grade student Wanda's iterating operation in the context of a task intended to assess a PUFS. During the fractions interview with Wanda, the interviewer posed the following problem: "I have $5 and that's one-third of your money. How much money do you have?" The interviewer drew a rectangle to represent the $5. Wanda's initial solution was "two" because "three plus two is five." The interviewer asked whether Wanda had more or less money than the interviewer, and Wanda said that she had less money but she was not sure why—it had to do with the one-third.

Later in Wanda's interview, after working on the problem of sharing two identical bars fairly among three people, the interviewer returned to this problem. Wanda drew a new rectangle consisting of three copies of the given rectangle: That is, she repeated the length of the rectangle so that there were three copies of this length, and then she drew in the rest of the rectangle (Figure 10, top). When asked how much her new rectangle was if the given rectangle was $5, Wanda eventually said 15 because "5, 10, 15." This solution suggests the construction of an iterating operation in that Wanda iterated both the length and 5 three times[9] to produce a longer rectangle and 15, respectively. However a caveat is in order: Wanda did not appear to view the 15 as her money, as I describe when I discuss disembedding next.

5.4. Students' Disembedding Operations

None of the six MC1 students demonstrated evidence of a disembedding operation. I describe two ways in which the lack of disembedding emerged, one of which involved problems

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[9] When I refer to Wanda or any MC1 student iterating a number like 5, I do not mean to imply that the student is necessarily iterating a composite unit (a units of units). For MC1 students, segments of their number sequences, such as 5, can be considered numerical composites, which are sequences of abstract unit items that contain records of counting (Steffe, 1992). Steffe also demonstrates that MC1 students may construct composite units where their units of one are not iterable, which means that a composite unit for these students is not a unit of units.

intended to assess a PUFS. Wanda's work on the "$5 is one-third of your money problem" is one such example. After Wanda had created the longer rectangle made from repeating a length of the given rectangle three times, the interviewer asked her to "show the pieces." Wanda drew a mark to indicate a part the same length as the given $5-rectangle. She shaded in this part. "That's yours," said Wanda. "And this (gesturing to the rest of the rectangle, the unshaded part) is mine" (Figure 10, bottom). Although Wanda did eventually make another mark in the rectangle to show three equal parts, and although she did determine that the whole rectangle was 15, she did not appear to think the 15 represented *her* money. Instead, for her the unshaded part was her money, which indicates that Wanda was not conceiving of the $5 as both embedded in her money and separate from her money. Once she identified the $5 as the interviewer's money, it was as if that part had been cut off from the rectangle, leaving the rest (unshaded) for her amount of money.

The above interpretation is made more credible by the fact that two other students demonstrated similar conflations. For example, Laura worked on a version of this problem in which Tyrone had $8, and that was one-third of Cammy's money. Laura drew a rectangle partitioned horizontally into three equal parts. She shaded the lowest part and identified that shaded part as Tyrone's money and the two upper unshaded parts as Cammy's money. Thus Laura did not see Tyrone' money, $8, as both embedded in and separate from Cammy's money. When asked, Laura said Cammy had more money than Tyrone. She was not sure how much Cammy had—she thought it might be 64 (i.e., she was considering multiplying the two 8s).

The other three students demonstrated a lack of disembedding in problems in which they were to draw a fraction of a bar or in fair sharing problems. For example, the interviewer asked Courtney to share a bar fairly among four people. Courtney made four relatively equal parts. She drew out one share from this bar when requested, of reasonable length but about half as wide.

When asked, she said that part was one-third. Courtney explained, “you got three pieces left,” and she said that she had taken one part away. This activity shows the lack of a disembedding operation because the whole consisting of four parts appeared to be destroyed when Courtney drew out one part from the bar. So, she mentally cut off one of the four parts, but she did not disembed it. To check this interpretation, the interviewer stated that there were four pieces in the whole bar and asked, “Is it fair to call this [the drawn out part] one out of four pieces?” Courtney paused. “Or you’re not sure,” said the interviewer. “I’m not sure,” said Courtney.

5.5. Students’ Splitting Operations

None of the six MC1 students had constructed a splitting operation. This conclusion was made initially from the students’ selection interviews and written fractions assessment, and it was not contradicted during the other interviews. The most basic task used to assess students’ splitting operations involved a given length of a string that was seven times the length of another string; students were to draw the other string (see Appendix B, F1). The students worked on this and similar tasks in these ways: By taking some amount away from the given length (these students did not partition or iterate), by partitioning the given length into a number of parts unrelated to the number in the problem (e.g., partitioning into four parts when the relationship was seven times), or by using only their iterating operations on the whole given segment (these students made no partitions). I focus on eighth grade student Cynthia because she initially used a method of taking some amount away, and later she iterated the whole given segment.

Cynthia was first posed the splitting task with the relationship seven times. We called the given segment Theo’s string; it was seven times the length of the other string, which we referred to as Sam’s. Cynthia said, “Okay, Sam’s is seven times longer than this one (Theo’s)?” After she heard the problem again, she said Sam’s would have to be shorter and moved her finger over seven finger spaces from the right end of the bar, making a mark on the bar at this point (Figure 11). When asked, she drew out Sam's segment separately; it was about two-thirds the size of the given segment (Theo's). The interviewer asked, "Is Theo's string seven times the size of Sam's?" Cynthia paused and then said, "To me, yes. Because seven times means, like, it's bigger because [of] the 'times.' But um, so then Sam's is not a times. So I just think of it [as] smaller." She solved the splitting task with the relationship three times in the same way.

When Cynthia was posed the splitting task with the relationship two times between the lengths, she paused and reviewed her prior work. "I think I messed up on this," she said. She appeared to resolve the situation by noting that she had two ways to do it: One way was what she had done already, and now she would do another way. She drew a copy of the given segment (Theo's) below the given, and then another copy to the right of it—i.e., she appeared to iterate the given segment twice to produce the new segment for this problem, which we called Joe's (Figure 12, top). When the interviewer asked if Theo's was two times Joe's, Cynthia paused and asked for another piece of paper. She drew a copy of Theo's segment below and called that Joe's. Then she drew two more copies of Theo's to the right of Theo's and called all three segments Theo's (Figure 12, bottom). She said that Theo's was two times the size of Joe's because "Theo and Joe had equal at first, but then Theo added on to it, so that would make two times, bigger." So, Cynthia's second solution was to use Theo's length for the other segment and then to extend Theo's length in order to demonstrate a length that was two times another length. When the interviewer emphasized that Theo's length could not increase, Cynthia went back to her original method of taking away some of the given length to produce the new length.

5.6. Parts-within-wholes Fraction Scheme (PWWFS)

To five of the six MC1 students I could attribute at most a PWWFS. This conclusion follows fairly easily from their construction of partitioning but not disembedding and is covered by some examples of this scheme in the prior sections. For example, Zachary not exhausting the whole when drawing one-third, Henry's hash mark technique for determining fractional amounts within bars, Wanda's work on the "$5 is one-third problem," and Courtney's work on fair sharing problems all demonstrate at most a PWWFS. Furthermore, all six students showed contraindication of the construction of a PUFS. Evidence from Wanda and Laura has already been presented in the section on disembedding.

5.7. Partitive Unit Fraction Scheme (PUFS)

However, there is also evidence toward the end of the fractions interview and in the algebra interview that Laura was constructing a PUFS. If so, this would imply that she also was in the process of constructing a disembedding operation—and that she was becoming an MC2 student. Near the end of the fractions interview Laura was posed a problem where a given bar was ¼ of another bar, and she was to draw the other bar. Laura drew a copy of the given bar beneath the given bar and then drew on three more parts that were slightly smaller than the given bar. In interaction with the interviewer, Laura revealed that she intended for these parts to all be the same size. She solved a second problem where a given bar was 1/5 of another bar in exactly the same way. She also solved this same problem in the algebra interview, confirming the construction of a reversible PUFS, and therefore of a PUFS. However, she did not demonstrate the use of this scheme with composite units, as has already been presented in the section on disembedding. This scheme was the most advanced fraction scheme I could attribute to any of the MC1 students. See Table 3 for a summary of the students' fraction schemes and operations.

6. Analysis and Findings: MC1 Students' Algebraic Reasoning

In this section I discuss the MC1 students' algebraic reasoning, focusing on their equation writing with multiplicative relationships, their writing of algebraic expressions, and their generalizing activity on the Border Problem. In the analysis I examine how their fraction operations and schemes facilitated or constrained their work on the algebra problems.

6.1. Equation Writing with Multiplicative Relationships

Two of the MC1 students, with interviewer support, wrote equations to represent multiplicative relationships between unknowns that were correct from the researchers' perspectives. The analysis suggests that the students' iterating operations were one reason these students were successful, but that the lack of disembedding operations was a major source of difficulties. To substantiate these claims, I present Henry's work on the first problem in the algebra interview (see Appendix C, A1) because it represented the most advanced work on equation writing out of all MC1 students.

6.1.1. Henry's solution to the Cord Problem (A1).

The first problem in the algebra interview was:

> A1. *Cord Problem.* Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord. Could you draw a picture of this situation? Can you write an equation for this situation? Can you write another equation?

Initially all MC1 students made a drawing for A1 in which one of the lengths (represented by a segment or rectangle) was a little more than half of the other. Two students refined their pictures to make a more accurate representation by iterating a shorter segment five times to make a longer segment. Only one of these two students, Henry, wrote a multiplicative equation for A1 that was correct from the researchers' perspectives.

In Henry's initial picture for A1, the segment representing Rebecca's cord length was longer than the segment representing Stephen's, and Stephen's segment was just over half of Rebecca's (Figure 13, top). *Without any intervention from the interviewer*, Henry reinitiated his activity and drew a small segment. Then he drew a copy of that segment below. He spanned that copy with his right thumb and forefinger and moved this distance out to the right. Then he extended the lower segment by that much. He repeated this process, beginning to use the thumb and forefinger of his left hand to hold the distance and his right hand to draw (the third time). Then he did that again (the fourth time). He pointed to each part as if to check the number of parts. Then he did it again, the fifth time. "Okay," he said, signaling being finished. So, Henry repeated one cord length five times, and this new segment represented the other cord length (Figure 13, bottom). Henry called the long segment Rebecca's and the short segment Stephen's. However, when the interviewer restated the problem, Henry switched these meanings.

Henry's initial equation for A1 was "$S \cdot O = R$cord," which he said meant "Stephen's cord times what Rebecca's cord is, equals Rebecca's cord." He said that he wrote an "O" to "leave it open," since he did not know the length of Rebecca's cord. This exchange followed:

Data Excerpt 1: Henry examines the relationships in his picture of A1.[10]

I: So do you think the five times has anything to do with the equation?
H: Yeah [questioning, like he is not sure how but is certain that it does]. But Stephen's cord is like probably really long. And her cord we don't know. But if we find out Stephen's really, real cord, then we can find out, uh, Rebecca's cord.
I: Okay, so you said Stephen's cord is real long [drawing finger along the segment representing Stephen's cord length in the picture]. How many of Rebecca's cords would you need to make Stephen's?
H: Four.
I: Four more?
H: Mm-hmm [yes].
I: So how many total is that?
H: All together?

[10] In the data excerpts, H stands for Henry, L for Laura, C for Courtney, and I for the interviewer (the author). Comments enclosed in brackets describe students' nonverbal action or interaction from the researcher's perspective.

I: Yeah. How many total cords of Rebecca's length would you need to make Stephen's?
H [slight pause]: Five.
I: So that means if we take five of these [drawing finger along Rebecca's length] we'd have the same length [as Stephen's cord], right?

Henry agreed and then generated a correct equation, "$R \cdot 5$ = Stephens cord." In the process of explaining this equation he changed the 5 to a 4, saying, "No, Rebecca's times four equals Stephen's cord, 'cause she already has one."

The researcher then posed a numerical example in order to test the equation: "Let's say Stephen's cord length is 15 feet; how long is Rebecca's cord?" Henry spent nearly 6 minutes determining Rebecca's cord length. He initially thought it would be 10 feet. Then he said "5 feet" and arrived at 15 feet for her length. He appeared to be iterating an amount three times, because then he said, "three, 9 feet." To explain his answer of 9 feet, Henry counted by threes along the first four parts of Stephen's segment, arriving at 12, and then counted by ones ("13, 14, 15") along the fifth part. When the interviewer repeated back to Henry how he had counted along Stephen's segment, Henry changed his mind. "Yeah, hers is like three feet," he said.

Finally, the interviewer asked Henry about his equation—whether he wanted to use 4 or 5. Henry said four, although under questioning he agreed that 3 x 4 was not 15. Following that exchange, Henry changed his equation back to "$R \cdot 5$ = Stephens cord." When asked for any other equations he could write for the situation, he wrote "$5 \cdot 3 = 15$, $3 \cdot 5 = 15$, and $3 \div 15 = 5$."

6.1.2. Analysis of Henry's solution to the Cord Problem (A1).

My interpretation of Henry's work relies on the operations I could attribute to him. Henry's iterating operation was an asset for him in at least three ways. First, Henry used his iterating operation spontaneously and independently to modify his drawing for A1. Doing so allowed him to create a quantitative foundation that was a key reference during the rest of his algebraic work. Although Henry's use of iterating was similar to Cynthia's use of iterating to work on splitting tasks (see §5.5), only one other MC1 student in the study, Laura, made a similar drawing on A1. Thus I infer that creating this kind of drawing is not a trivial achievement for an MC1 student. Second, although Henry did not write a correct equation independent of interaction with the interviewer, I suggest that his iterating operation allowed him to make sense of the support that the interviewer offered in terms of questions about his picture. In contrast, none of the other MC1 students wrote a similar equation with similar questions—not even Laura. Third, also similar to Cynthia, Henry used his iterating operation alone to determine Rebecca's cord length when Stephen's cord length was 15 feet. That is, Henry did not partition the known length into five equal parts, which reflects the lack of a splitting operation. Instead, he appeared to make an estimate and test it by iterating it, which eventually led to a correct solution with support from the interviewer.

However, I suggest that Henry's lack of a disembedding operation was a constraint for him, as shown by him initially stating that four of Rebecca's lengths were needed to make Stephen's length, and then later by changing the 5 to a 4 in his equation. Henry appeared to think of Stephen's length as five parts, one of which had to be Rebecca's, leaving Stephen with only four parts. Although with the support of the interviewer's questions Henry did return to a correct equation, it's not clear whether using 5 was a logical necessity for him. Indeed, without a disembedding operation it would be unlikely for Henry to make sense of a segment that is five times another, because that relationship requires thinking about the shorter segment as both embedded in and disembedded from the longer segment. So, without that operation, it would be more natural for Henry to think of the longer segment as "four more" than the shorter segment (cf. Steffe & Olive, 2010). This analysis indicates that writing equations to represent multiplicative relationships between quantities is quite challenging for students who have not yet constructed disembedding operations.

6.2. Expression Writing with Fractions as Multipliers

Four MC1 students worked on a version of this problem during the algebra interview:[11]

A3. There are 5 identical candy bars (rectangles) and each candy bar weighs some number of ounces. Let's say that h = the weight of one bar. Could you draw a picture of 1/7 of all the candy? Can you write an expression for the weight of 1/7 of all the candy?

For all MC1 students I changed the problem to be about finding the weight of ½ of 2 identical bars. Because Laura did the most advanced work on this problem out of the four students, I present her work here. In fact, the other three students made very little progress on the problem.

6.2.1. Laura's work on A3.

In her work on A3, Laura said that she needed another number, so the interviewer asked her to *draw* half the candy. Laura drew two bars, each of which looked like one-half of a given bar, and she identified them by writing "half of both bars" (Figure 14). This exchange followed:

Data Excerpt 2: Laura determines the weight of two half bars.

I: What would be the weight of one of these pieces [circling a half bar with finger]?"
L: Half of the, half of however much that weighs [tapping a whole bar with her pen].
I [nodding]: Uh-huh. Which it's h, right? So have you done things in math like write half, half of h?
[Laura shakes her head no.]
I: No? But you agree that that [her drawn half bar] weighs half of whatever that [original bar] weighs. And if it's h we could maybe call it half of h?
[Laura nods.]
I: So if that's [pointing to her half bar] half of what that [whole bar] weighs and that's [pointing to the other half bar] half of what that [whole bar] weighs, what would they weigh together [referring to the two half bars together]?
L: Those two put together is the same as a whole bar.
I: Oh, okay. Say that one more time.
L: Both of the pieces [half bars] would equal h.

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[11] Each of Wanda and Cynthia worked only on A1, A2, and A7 in their algebra interviews (see §4.2.2, footnote 6).

Then the interviewer asked her to draw one-third of all the candy. Laura drew two copies of the original bar, each partitioned into three equal parts (Figure 15). She shaded the last part of each bar, drew a circle and bracket around them, and wrote "1/3 of both bars." When asked about the weight of this, she said it would be a little less than before. Then interviewer asked if she could write an expression for the weight of one of the pieces, and Laura wrote "$1/3 \cdot h$ = ?" She said the other part would also be "one-third." When asked if she would write an expression for "what they would be together," Laura wrote "1/3 + 1/3 = 2/3 of *H*". In explanation she said, "One-third plus one-third equals two-thirds, and two-thirds is part of *h*."

6.2.2. Analysis of Laura's work on A3.

Laura's work was representative of at least two features of the MC1 students' work on their version of A3. First, the students did not spontaneously use the given letter (*h*) to represent the weight of one bar, even though in their math classes all had received some instruction on the use of algebraic notation and equation solving. Second, none of the students independently used notation to show multiplying *h* by a fraction—in fact, Laura denied that she had done this in math class. It's possible that in their math classes Laura and the other MC1 students had only been exposed to using whole numbers as multipliers on unknowns, although this is somewhat unlikely if students had engaged in multiplying by reciprocals to solve equations. In any case, Laura articulated a fractional part of an unknown only after the interviewer made a fairly heavy suggestion about it to her, and her work on this problem was the most advanced of all MC1 students.

Evidence that Laura conceived of fractions as multipliers on unknowns would be more compelling if she had written $1/3h + 1/3h$ in her final expression. As written, her expression does not indicate the use of fractions as multipliers. It may show an awareness of dealing with fractional parts of the unknown weight, and it may show the beginnings of a partitive fraction scheme. However, in this study, there was little to no evidence that MC1 students conceived of fractions as multipliers of unknowns. This finding is consistent with prior research that has suggested an iterative fraction scheme is required to conceive of fractions as multipliers in this way (Hackenberg, 2010), putting this concept in the realm of MC3 students.

6.3. Engaging in Generalizing Activity: Solving the Border Problem

6.3.1. Overview of MC1 students' work.

In contrast to their work on A1 and A3, five MC1 students made progress on the Border Problem (Figure 5). In fact, all six MC1 students initially thought that there were 40 squares on the border of the 10 by 10 grid. They indicated that they arrived at this result by repeating ten four times—which implies the use of their iterating operation. Upon counting to check, five students adjusted their initial ideas based on observations about counting the corner squares of the grid twice. Two students adjusted by subtracting 4 from 40. Three students adjusted by adding 10 and 10 for the top and bottom sides, and then adding 8 and 8 for the left and right sides, eliminating both corner squares from these sides. So, these five students generated a method for counting the number of squares on the border (part (a) of the Border Problem, Figure 5). When asked to use their method on a 6 by 6 grid (part (b)), all five students first drew this new grid and then successfully used their method on it. In addition, all five students verbalized, to some degree, how to use their method on any grid (part (d)).

Due to time constraints, MC1 students were not asked to generate another method for counting squares (part (c)), and only Henry and Courtney were asked to "use algebra" (use algebraic notation) to communicate their methods (part (e)). Henry had a discussion about this part of the problem with the interviewer, who suggested that x could represent the number of unit squares in one row. After asking Henry what *x* was in each of the first two grids (the 10 by 10 and 6 by 6), the interviewer asked if Henry could use *x* to write down an expression for the number of squares on the border. Henry wrote "*x* = top row 10" and then underneath "*x* = top row 6." Then he added the 10 and the 6 to get 16. Courtney also had a discussion with the interviewer about using a letter to represent the number of unit squares in one row of the grid. However, after that discussion she said she still did not know how to use algebra to communicate her method. So, no student made a correct solution to part (e) from the perspectives of the researchers.

Yet, the five students who solved parts (a), (c), and (d) engaged in two forms of generalizing activity (Ellis, 2011). In using their method on a grid of different size they extended their reasoning beyond the range in which it originated, and in verbalizing patterns they observed they began to identify commonalities across cases. However, I propose that the students' lack of disembedding operations constrained the nature of their generalizing activity and prevented them from writing algebraic expressions (one kind of reflection generalization). I use Henry and Courtney as examples for explaining these conclusions because these two students demonstrated the most detailed verbalizations of their generalizing activity (another kind of reflection generalization) on this part of the problem (part (d)).

6.3.2. Henry's generalizing activity.

In describing how to use his method on any size grid (part (d)), Henry said, "I'd tell them to do the top first, see how much in a row it would be [pointing at the top row]. And then do the bottom, and that'd be the same. And then after that, like, whatever number's at the end [points to a corner], go to the next box [down] on the other side and put, like, put how much it is. Don't use the same number two times." When asked if he meant to not count a corner square again if it's already been counted, Henry agreed. Then the following exchange took place:

Data Excerpt 3: Henry's comments on the relationship between the side lengths of the grid.

I: And then how do you know this [drawing finger along the right vertical side of the 6 by 6 grid] has to be four? Does that have any relationship to knowing that the side [drawing finger across top row] is six?
H: Top, you mean.
I: The top, sorry, the top is six.
H: Yeah. [He says something about getting mixed up which is not fully audible.]
I: Uh-huh. So here [pointing to the 10 by 10 grid] it was 10 and 10 [drawing fingers along the top and bottom rows], and 8 and 8 [drawing finger along the two side rows].
H: Mm-hmm [yes, nodding].
I: So, does the eight have any relationship to the ten?
H [3-s pause, frowning]: Huh? Or you can just do it like this [and he labels each edge of the 10 by 10 grid with a number: 10 and 10 on top and bottom, 8 and 8 on the vertical sides]. But usually, they have just an 8 [pointing to the left side], but you have to do the 10 [pointing to the bottom row].

In this data except the interviewer was trying to ask Henry about how the numbers of squares in two adjacent sides in his method were related. These questions did not seem to be sensible to Henry. That is, he did not really respond to the question about how the 6 and 4 were related in the 6 by 6 grid, and then he responded to the question about the 10 and 8 by labeling the 10 by 10 grid and talking about the different sides. He did not show evidence even of seeing a numerical relationship unrelated to the grid—for example, he did not indicate that 4 was 2 less than 6, or 8 was 2 less than 10. And he did not demonstrate any relationship about the side lengths in the representation, such as that you take off two squares from the 6 squares to leave 4 squares, or similarly for the 10 squares and 8 squares.

My account of Henry's activity is that he did not view the relationship between the 4 and the 6 and the 8 and the 10 structurally. In other words, he did not appear to see 4 unit squares as embedded in the 6 unit squares on one side of the square, and also separate from the 6 unit squares in the sense of being an independent row or column (and similarly for 8 and 10). This means that in thinking about the grid he did not appear to disembed 4 unit squares from 6 unit squares while leaving the 6 unit squares intact—and I infer he did not do so because he had not constructed a disembedding operation. His comments do provide evidence that he knew two different numbers should be involved—that a person can't just add the same number four times as he initially did. But the lack of a disembedding operation contributed to Henry's reflection generalization (a verbalization of his generalizing activity) about adding the number of unit squares in the top and bottom rows, and then adding a different pair of numbers for the other sides of the grid. This statement might lead to writing something like $x + x + y + y$ as an algebraic expression, but it would not lead to something like $x + x + x - 2 + x - 2$.

6.3.3. Courtney's generalizing activity.

In contrast with Henry, Courtney subtracted 4 from 40 in solving the Border Problem. After she had used her method on the 6 by 6 grid, the following exchange took place:

Data Excerpt 4: Courtney explains her method.

I: So if you were going to say it [your method] in words, what would you say?
C: I would say, since there's, since a square has ten sides, on each one, I'd add ten plus ten four times and then I subtracted four 'cause I counted all four ends, and I counted them twice. So I subtracted 4 since there's 4 sides. For the 10 by 10 I got 40 and then I subtracted 4 and I got 36.
I: Okay, and so tell me again why is it that you do the 10 four times [pointing to Courtney's notation, which was 10 + 10 = 20 + 10 = 30 + 10 = 40]?
C: 'Cause you have four sides, and there's four 10s.

At this point the interviewer did not probe for clarity about reasons for subtracting four. However, prior to this data excerpt and within it, Courtney said she subtracted four in order to not count corner squares twice. In fact, except for stating that she subtracted four due to there being four sides, her reflection generalization does not seem problematic in any obvious way.

Yet recall that Courtney did not write an algebraic expression for her method. To try to account for why she did not, I look to what she did write. Her numerical notation of 10 + 10 = 20

+ 10 = 30 + 10 = 40 indicates that she was “in” the activity of uniting and absorbing tens, rather than viewing the 10s as both separate from and embedded in the 40. Indeed, when MC1 students like Courtney take a number (10) and repeat it, they do not consider these numbers (four 10s) as both embedded in and disembedded from the result (40). Instead, it’s as if the tens disappear after they have been used. In short, Courtney’s method really was not 10 + 10 + 10 + 10 – 4, structurally; I infer she did not maintain awareness of the 10s as segments of the 40 after computation. This analysis indicates that it was quite reasonable for Courtney *not* to know how to use the interviewer’s suggestion to let *x* represent the number of squares in one row, since the relationship between the number of unit squares on each side and the total number of border squares was rather ephemeral for her.

7. Discussion and Concluding Thoughts

7.1. Summary

I begin this section by summarizing my responses to my three research questions on the MC1 students’ fractional knowledge, solutions to algebra problems, and relationships between their fractional knowledge and these solutions. All six MC1 students had constructed partitioning and iterating operations of some kind, but not disembedding or splitting operations. They had all constructed PWWFS, and one (Laura) appeared to construct a PUFS during the course of the study, which means she also constructed a disembedding operation (see Tables 2 and 3). So, the fractional knowledge of these students was consistent with what has been found about these students in prior research (Olive & Vomvoridi, 2006; Steffe & Olive, 2010; Ulrich, 2012).

Table 4 summarizes the algebraic ideas attributable to the MC1 students and highlights connections between the students’ fraction operations and work on the algebra problems. In short, the MC1 students’ fraction operations did influence how they worked on the algebra problems and demonstrated algebraic ideas. Iterating operations used on segments were helpful in creating drawings to show multiplicative relationships between two quantities that were correct from the interviewers' perspectives. Iterating operations also likely made it possible for students to generate a method for the Border Problem, in that students could iterate 10 four times and then retroactively adjust after counting squares. However, the lack of disembedding operations was a significant constraint in writing multiplicative equations, writing algebraic expressions, and generalizing relationships in the Border Problem.[12] Prior research has not identified how MC1 students' multiplicative concepts and fraction operations might facilitate or limit their algebraic work, so these findings are the first contribution of the study.

However, since MC1 students in this study wrote few equations or expressions overall, and none in their work on the Border Problem, some researchers may dismiss their work in the algebra interview as not algebraic. I agree that under a more restrictive view of algebraic reasoning (e.g., Filloy & Rojano, 1989, 2010; Herscovics & Linchevski, 1994) these students would not be considered algebraic reasoners. However, I also agree that the dividing line between what is considered pre-algebraic and algebraic is difficult to draw (Carraher & Schliemann, 2007). So, although there may be disagreements, I am in favor of more inclusive categorizations because they are consistent with viewing students' mathematical knowledge as a sequence of modifications and reorganizations of prior knowledge, rather than as consisting of strict cut-points between less sophisticated and more sophisticated ways of knowing. Since the definition of beginning algebraic reasoning that I adopt means that students can be considered to be reasoning algebraically even if they are not using standard algebraic notation to systematically represent generalizations and abstractions, I can claim that Henry and Courtney were reasoning

[12] I am not suggesting that the lack of disembedding operations was the only constraint—just that it was indeed one.

algebraically in their work on the Border Problem, where both students engaged in generalizing activity and produced reflection generalizations. In addition, I can claim that Henry was reasoning algebraically in his work on the Cord Problem, where indeed he did use algebraic notation to represent his ideas about the relationship between two unknowns. Of course, I also want to recognize that the students' algebraic reasoning was primitive in the sense that there were constraints on the scope of their generalizations and their representations of them.

7.2. Contributions

In addition to identifying how MC1 students' multiplicative concepts and fraction operations facilitates or limits their algebraic work, this study makes three contributions to the knowledge base on relationships between students' fractional knowledge and algebraic reasoning. First, it indicates that students' fractional knowledge and algebraic reasoning are connected even when fractional knowledge is relatively basic. Thus, this study supports the perspective that working to develop strategic reasoning with fractions can positively influence students' algebraic knowledge (Empson & Levi, 2011; Empson, et al., 2011; Russell et al., 2011). However, there is a cautionary note here: Some MC1 students may not easily construct more advanced multiplicative concepts that will allow them to develop more sophisticated strategic reasoning with fractions. These students need to be able to do work that is sensible to them and that will not be the same as their MC2 and MC3 peers. For example, MC1 students may be able to use iterating operations to represent some multiplicative relationships. But these students' lack of disembedding operations will be a consistent constraint in both their fractional and algebraic ways of operating.

Second, this study reveals that students' fractional operations influence the generalizing activity in which students engage and the reflection generalizations that students make. This finding is compatible with Ellis's (2007c) generalization taxonomy, which comes from a position of actor-oriented generalizing. However, it informs the taxonomy in that it specifies how the content of students' generalizing activity will vary across mathematical thinkers with different multiplicative concepts. For example, searching to identify commonality across cases, as both Henry and Courtney did in working on the Border Problem with 10 by 10 and 6 by 6 grids, is different for MC1 students than their MC2 and MC3 peers. MC1 students literally do not see the 8 (or 4) as both embedded in and separate from the 10 (or 6). So, these students can identify that two different numbers are involved in their method, but they don't identify this relationship structurally as MC2 and MC3 students can do.

Third, this study demonstrates that the fractional operations that MC1 students have constructed—and have not constructed—impact their use of algebraic notation. In particular, the lack of a disembedding operation means using algebraic notation to represent basic multiplicative relationships between quantities as in the Cord Problem, or embedded additive relationships as in the Border Problem, is a significant challenge for them. This issue is critical, given that many algebraic situations involve disembedding a part from a whole while maintaining the whole, and further, being able to represent these relationships with algebraic notation. In fact, the high numbers of students in MacGregor and Stacey's (1997) study who did not conceive of letters as variables or unknowns even after algebra instruction may be due in part to the presence of MC1 students. For example, after a 20-lesson algebra instructional unit with 42 11- and 12-year-old students, only 63% viewed letters as stand-ins for unknown quantities. MacGregor and Stacey aimed to provide explanations for this result other than cognitive development, but it seems unwise to ignore cognitive factors such as students' multiplicative concepts, which clearly impact how students at that age conceive of and use algebraic notation.

Although it is certainly possible for MC1 students to use standard algebraic notation to some degree, as demonstrated by Henry and Laura, it is also clear that doing so is a significant challenge that is unlikely to be fully overcome until MC1 students construct more advanced multiplicative concepts.

7.3 Implications

An open question from this study has to do with exactly this issue: Since the first multiplicative concept is so limiting for developing fractional knowledge and algebraic reasoning, what can be done to help students progress beyond it, especially in middle school? If Steffe's (2007) estimates are correct and MC1 students make up 30-50% of incoming sixth grade students, then much more attention needs to be paid to this question. There is evidence that some MC1 middle school students can progress relatively quickly to the second multiplicative concept: Tim in Olive and Vomvoridi's (2006) study made significant changes in a month, and in this study Laura also made changes in about that amount of time.[13] Yet as previously stated, this kind of rapid progress is not the norm: MC1 students may take two years or longer to construct the second multiplicative concept, despite focused teaching interventions (Steffe & Cobb, 1988).

In either case, more needs to be known about how to help MC1 middle school students advanced their multiplicative concepts and work productively where they are—i.e., curricular materials based on learning trajectories (Simon, 1995; Steffe, 2004) for these students are needed (Daro, Mosher, & Corcoran, 2011). In Tim's case, the teacher's instructional changes and interview interactions likely supported his construction of a PUFS; these changes included (1) the use of iterated unit segments to reason about multiplicative relationships between a unit fraction and its related whole and (2) many opportunities to reason with fraction bars (Olive &

[13] Laura's fractions interview was on April 7, and her algebra interview was on May 3.

Vomvoridi, 2006). In Laura's case, her change was likely more independent of interview interaction, given that she and I met only three times. Yet our interactions did involve both of the findings from Olive and Vomvoridi. Research studies that build on these findings to develop a knowledge base on MC1 students will make great strides toward the aim of an equitable mathematics education for all learners (National Council of Teachers of Mathematics [NCTM], 2000).

Acknowledgements

The research reported in this manuscript was supported by a Proffitt Grant in the School of Education at Indiana University. The findings and statements in the paper do not necessarily represent the views of the School of Education at Indiana University. An earlier version of this paper was presented at the annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education in November, 2012. The author gratefully acknowledges helpful comments on earlier drafts from Erik Tillema and Mi Yeon Lee.

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Figure Captions

Figure 1. A length structured as a unit of four units each containing six units (top) or a unit of six units each containing four units (bottom).

Figure 2. A length structured as a unit of units (top); when MC2 students insert six parts into each of the four units, the length becomes a unit of 24 units for them (bottom).

Figure 3. Sixth grade MC1 student Tim's images of one-sixth (left) and six-sixths (right).

Figure 4. A rendition of sixth grade MC1 student Tim's idea that ½ + ¼ = 1/5.

Figure 5. The Border Problem.

Figure 6. Eighth grade student Zachary's efforts to share a rectangle fairly among three people.

Figure 7. Seventh grade student Courtney's first attempt to share a bar fairly among five people (top) and a later attempt with an extra part crossed off (bottom).

Figure 8. Seventh grade student Henry's hash mark technique to make halves.

Figure 9. Seventh grade student Laura placed all four different cut-out parts (1/3, ¼, 1/6/ 1/8) on the whole candy bar.

Figure 10. Eighth grade student Wanda's drawing of three a rectangle whose length consists of three segments of the $5 rectangle (top); Wanda's marking and shading to show the interviewer's money ($5) and her money (the rest of the bar).

Figure 11. The curved mark shows the result of eighth grade student Cynthia taking away seven finger spaces to solve a splitting task; the new string (Sam's) is drawn below.

Figure 12. Cynthia iterating segments to solve splitting tasks (two tries). The top represents iterating a copy of Theo's segment twice to make Joe's; the bottom represents copying Theo's to be Joe's and then iterating Theo's segment to make Theo's longer.

Figure 13. Henry's initial drawing for A1 (top two segments) and his self-initiated revision (lower two segments). Hash marks were added to his revised picture during our interactions.

Figure 14. Laura's drawing of one-half of two identical bars.

Figure 15. Laura's drawing of one-third of all the candy.

Figure D.1. The whole bar and one fair share when sharing with 3, 4, 6, and 8 people.

Appendix A: Unrecorded Selection Interview Questions

S1. There are 6 cans of juice in a package and 8 packages in a box. A crate contains 5 boxes. How can you figure out how many cans of juice are in a crate? Can you draw a picture to show how you know?

S2. There are 96 students going on the field trip and 6 buses. The same number of students is to ride on each bus. Draw a picture to show how you will determine how many students will ride on each bus. How many students will ride on each bus?

  a. *If this problem is hard, pose this problem:* In a classroom, there are 6 rows with 4 seats in each row. The teacher adds 12 more seats to the classroom. How many total rows can be made with four seats in each row? How many seats will be in the classroom?

S3. This segment represents a picture of a piece of licorice. Theo's piece of licorice is five times longer than this one. Can you imagine Theo's piece? Can you draw it? Show me how you know that this piece of licorice is five times longer than Theo's. What fraction of this piece of licorice is Theo's piece? (If this is hard, pose the problem with two times or three times.)

S4. The Giant Soda at the convenience store is 24 ounces. That's 8 times the amount of soda that Stephanie drank. What would you do to find out how much Stephanie drank? Draw a picture to show.

S5. Camika has $21. That's 1/7 the size of the amount of money that Rickard has. What would you do to find out how much money Rickard has? Draw a picture to show.

S6. At a small party, five friends split the submarine sandwich shown (a rectangle) fairly. Can you draw that? You take one of these pieces and share it fairly among 4 people, you and three friends. How much of the whole sandwich do 3 of these people get? (If this is hard, ask about the share for just 1 of the three people.)

S7. The drawing below (a rectangle) is a picture of a candy bar. Draw a candy bar that is seven-fifths of that bar.

S8. The two rectangles represent two identical candy bars. Show how you'd share them equally among three people. How much of one candy bar would one person get?

S9. The picture (a rectangle partitioned into three equal parts) represents a sub sandwich that is 3 feet long. That's five times the amount of sandwich Sara wants to eat. Draw how much she will get. How much of a foot does she eat?

Appendix B: Fractions Interview Questions

F1. The drawing (a line segment) below shows my piece of string. Think of *your* piece of string so that mine is five times longer than yours. Can you draw what you're thinking of? Can you show for sure that mine is 5 times longer than yours?

F2. Sara has a stack of CDs that is 65 centimeters tall. That's 5 times the height of Roberto's stack. Draw a picture of this situation. How tall is Roberto's stack of CDs? How did you solve this problem?

F3. Tyrone has $21. That's 1/7 the size of Cammy's amount of money. Draw a picture of this situation. How much money does Cammy have? How did you solve this problem?

F4. Share 3 identical sub sandwiches (shown by 3 identical rectangles) fairly among 5 people. Show the share for one person. Probe their picture and how they made it. How much of a sandwich does each person get?

   a. If difficult, try 2 sandwiches among 3 people.
   b. If they solve the problem, try 5 sandwiches shared fairly among 7 people.

F5. *Improper Fraction Problem.* The drawing (a rectangle) is a picture of a candy bar. Draw a separate candy bar that is 9/7 of that bar. If the student completes this problem, ask her or him to shade one piece and tell how much it is of the original bar.

F6. *Making The Whole Problem.* This candy bar (a rectangle) is 3/5 the size of another candy bar. Make a separate drawing of the other candy bar. Probe to ask how they made their drawing.

   a. If difficult, try: This candy bar (a rectangle) is 1/5 of another candy bar. Make a separate drawing of the other candy bar.
   b. If they solve the problem, try: This candy bar (a rectangle) is 4/3 of another candy bar. Make a separate drawing of the other candy bar.

F7. Tanya has $84, which is 4/7 the size of David's amount of money. Draw a picture of this situation. How much does David have?

   a. If difficult, try: Tanya has $15, which is 1/5 the size of David's amount of money. Draw a picture of this situation. How much does David have?
   b. If they solve the problem, try: Cassie earned $48 babysitting. That's 4/3 the amount of money Serena earned. Draw a picture of this situation. How much money did Serena earn?

F8. This candy bar (a rectangle) is partitioned into three equal parts, so it's a 3/3-bar. Make that bar into a 5/5-bar but don't erase the half mark. Probe their reasoning.

F9. Draw a picture of 1/3 of 1/7 of this rectangular cake (a skinny rectangle). How much is that piece of the whole cake? How do you know?

   a. If they solve the problem, try 3/5 of 1/7 of the cake.
   b. If they solve (a), try 1/7 of 3/5 of the cake.

Appendix C: Algebra Interview Questions

A1. Do you have a cord with earplugs for listening to music? How long do you think it is? It's not a value that we know exactly, right? But we could measure it to find the exact value. Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord.

a. Could you draw a picture of this situation? Describe what your picture represents.
b. Can you write an equation for this situation? Can you tell me in words what your equation means?
c. *As necessary.* Can you check your equation with your picture?
d. *As necessary*. Check your equation using this question: Who has a longer cord, Stephen or Rebecca?
e. Can you write more than one equation? *As necessary* (if they have only written something like $t = 5*q$, where t represents Stephen's cord length and q represents Rebecca's cord length): Can you write an equation to express Rebecca's cord length in terms of Stephen's?
f. *As necessary* (if they have written something like $t = q \div 5$): Can you write this equation using multiplication?
g. Let's say that Stephen's cord is 15 feet long. Explain how to find the length of Rebecca's cord.

A2. *If A1 is difficult, try this:* How tall do you think your math teacher (or the principal) is? It's not a value that we know exactly, right? But we could ask him or have him stand by a measuring tape and find out the value. Let's say we know that he is three times the height of a little boy who's one-year-old. We don't know the one-year old's height either.

a. Could you draw a picture of this situation? Describe to me what your picture represents.
b. Can you write an equation for this situation? What does your equation mean in words?
c. *As necessary.* Can you check your equation with your picture?
d. Can you write more than one equation? *As necessary* (if they have written something like $t = 3*b$, where t represents the height of the teacher and b represents the height of the baby): Can you write an equation to determine the height of the baby in terms of the height of the adult?
e. *As necessary* (if they have written something like $b = t \div 3$): Can you write this equation using multiplication?
f. Let's say that the teacher/principal is 6 feet tall. How tall is the baby?
g. Let's say that the teacher/principal is 7 feet tall. How tall is the baby?
h. Let's say that the baby is 2 and a half feet tall. How tall is the principal/teacher?

A3. There are 5 identical candy bars (show picture) and each candy bar weighs some number of ounces. Let's say that h = the weight of one bar. How much does 1/7 of all the candy weigh?

a. If this question is hard, start with 2 or 3 bars and ask about 1/3 or 1/5.
b. If still hard, use sharing language to find out about whether the student can make fair shares.

c. Could you draw a picture of 1/7 of all the candy?
d. Can you write down an expression for the weight of 1/7 of all the candy?

A4. *Optional, depending on time.* There are three candy bars on the table, each of different weight. The first weighs *a* ounces, the second weighs *b* ounces, the third weighs *c* ounces.

a. How much does 1/5 of all the candy weigh?
b. Could you draw a picture of 1/5 of all the candy?
c. Can you write down an expression for the weight of 1/5 of all the candy?

A5. Theo has a stack of CDs some number of cm tall. Sam's stack is two-fifths of that height.

a. Draw a picture to represent this situation.
b. Can you write an expression for how tall the height of Sam's stack is? Can you tell me in words what your expression means?
c. Can you write an equation based on your expression in (b)? Can you tell me in words what your equation means?
d. Can you write another equation for the situation? Can you tell me in words what your equation means?
e. Ask them to test their equations with particular numbers, such as 10 cm as the height of Theo's stack.

A6. *Optional, depending on progress on A5.* Christina earned some money babysitting. That's 4/3 of what Serena earned.

a. Draw a picture to represent this situation.
b. Can you write an equation that relates the amount of money Christina earned to the amount of money Serena earned? Can you tell me in words what your equation means?
c. Can you write another equation? Can you tell me in words what your equation mean(s)?
d. Ask them to test their equations with particular numbers, such as $36 for Christina's money.

A7. Here is a picture of a 10 by 10 grid with the squares on the border highlighted.

a. Without counting one-by-one, and without writing anything down, can you find a way to determine how many squares are on the border? Elicit and probe reasoning.
b. Can you find another method?
c. Now, let's say your square was 6 by 6. Could you use your first method to determine the number of squares on the border?
d. How would you describe in words how to use your first method on any grid?
e. How would you use algebra to write an expression to communicate your first method to someone? Probe the meaning of letters and what the expression means in words.
f. Do parts (c) through (e) for the second method, if time.

Appendix D: Cut-Out Bar Problems

[insert Figure D.1]

Figure D.1. The whole bar and one fair share when sharing with 3, 4, 6, and 8 people.

Here is a candy bar and several separate pieces that were made from identical candy bars. (The interviewer places the pieces on the table, not in order of size.) The candy bar is to be shared fairly among six people. Which one of these pieces would be one share? How do you know it's going to be fair?