Units Coordination and the Construction of Improper Fractions:

A Revision of the Splitting Hypothesis

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Abstract:

This article communicates findings from a year-long constructivist teaching experiment about the relationship between four 6th grade students' multiplicative structures and their construction of improper fractions. Students' multiplicative structures are the units-coordinations that they can take as given prior to activity—i.e., the units-coordinations that they have *interiorized*. This research indicates that the construction of improper fractions requires having interiorized three levels of units. Students who have interiorized only two levels of units may operate with fractions greater than one, but they do not produce improper fractions. These findings call for a revision in Steffe's (2002) hypothesis that upon the construction of the splitting operation, students' fractional schemes can be regarded as essentially including improper fractions. While the splitting operation seems crucial in the construction of improper fractions, it is not necessarily accompanied by the interiorization of three levels of units.

1. Introduction

Recent research on how children construct fractional knowledge has documented that the construction of improper fractions is no easy accomplishment (e.g., Olive & Steffe, 2002; Steffe, 2002; Tzur, 1999). For example, in Olive & Steffe's research, even after a fourth grade student successfully made fractions like 10/7 he said: "I still don't understand how you could do it. *How can a fraction be bigger than itself?*" (p. 428, italics in the original).

The question appears to be fundamental and puzzling for many students, and it points to the following issue: Although many students may be able to work in some ways with fractions greater than one, they haven't necessarily established such fractions as numbers "in their own right." That is, when asked to draw 10/7 of a length, a student might interpret 10/7 as a mixed number and draw a whole and 3/7 of the whole. So from the student's point of view, 10/7 is larger than one whole. But that interpretation relies on wholes and parts of wholes for the meaning of 10/7. Alone, such a response does not indicate that the student conceives of 10/7 as a number in its own right—an idea I elaborate in this paper. A student who does not consider 10/7 as both a mixed number and as a number in its own right has not, from my point of view, constructed improper fractions. The student may be able to work "successfully" with fractions greater than one in some contexts, as I will demonstrate. But in many situations the student's mathematical activity may be constrained. In fact, without the construction of improper fractions as numbers in their own right, students would not construct fractional number sequences such as "all" of the sevenths (1/7, 2/7, 3/7,…7/7, 8/7, 9/7, …).[i] Not doing so has implications for students "filling in" the number line—for constructing a sense of connectedness and continuity of number.

A great deal of research has been done in the last twenty-five years on the learning of fractions (cf. Pitkethly & Hunting, 1996). Some researchers have analyzed different subconstructs of fractions, such as fraction as ratio and fraction as operator, that might constitute fractional and rational number knowledge (e.g., Behr, Harel, Post, & Lesh, 1993; Behr, Wachsmuth, Post, & Lesh, 1984; Davis, Hunting, & Pearn, 1993; Kieren, 1988, 1995). Other researchers have emphasized the importance of real-world contexts and the informal knowledge that students bring to their activity with fractions in helping students develop fractional knowledge (e.g., Mack, 1990, 1995, 2001; Streefland, 1991). Some researchers have explored students' activity with improper fractions (e.g., Saenz-Ludlow, 1994) or with mixed numbers (e.g., Mack, 1990,1995; Streefland, 1991) as a part of their studies, but these explorations have not tended to be a central focus of the studies. Few researchers have devoted explicit attention to analyzing students' construction of improper fractions. Those researchers who have done so, such as Tzur (1999), Steffe (2002), and Olive and Steffe (2002), have established that basic operations of iterating and partitioning, as well as the construction of composite units, are critical in children's construction of improper fractions.

Iteration generally means repeatedly instantiating an amount in order to produce another amount.[ii] For example, as demonstrated by Tzur (1999), iterating a proper fraction like 6/11 two times (i.e., doubling 6/11) can allow children to produce an amount they can call 12/11. More generally, iteration seems necessary to produce improper fractions because 12/11 cannot gain meaning from being part of a whole. Parts are usually considered to be smaller than (and often embedded in) the whole, and so taking 12 parts out of 11 is puzzling at best (cf. Olive & Steffe, 2002; Thompson & Saldanha, 2003). Instead, a fraction like 12/11 can be made from taking 1/11 12 times, which means that 12/11 has a multiplicative relationship to the part of the whole (1/11) that was used to make it. Twelve-elevenths is still "dependent" on the whole in that it takes its name from a relationship to the whole: Any one of its 12 one-eleventh parts can be iterated 11 times to make the whole. But 12/11 is also *independent* of the whole in that it is not a part of it, and so making improper fractions requires transcending part-whole conceptions of fractions and relying on relationships formed through iteration.

However, Steffe (2002) has demonstrated that iteration alone is not sufficient for a child to produce improper fractions: A child may be able to call two times 6/11 "twelve-elevenths" because the child knows that two times six is twelve. In this case, twelve-elevenths has a whole number meaning of 12 units, but not necessarily a fractional meaning. That is, the child can iterate one-eleventh to produce twelve-elevenths or eleven-elevenths much in the same way that she can iterate 1 to produce 12 or 11. But one-eleventh, by itself and prior to activity, does not imply for the child a whole that is partitioned into 11 equal parts.[iii] In short, eleven-elevenths has yet to have a dual meaning of both 11 items and of *one*—the child has yet to abstract this latter meaning. So, she does not conceive one-eleventh to have an explicit multiplicative relationship to the whole, even though she can make 12/11 by iterating one-eleventh 12 times and call it twelve-elevenths.[iv] In other words, iteration of a unit or proper fractional amount to produce a fraction larger than one does not necessitate that the child has constructed a structural relationship between the part being iterated, the whole, and the result. Without evidence of this structural relationship, it is hard to attribute the construction of improper fractions to the child.

Steffe (2002) has hypothesized that the construction of a new operation, splitting, composed of *simultaneous* partitioning and iterating, opens the way to the construction of improper fractions. Splitting, a fundamental multiplicative operation, is involved in solving a problem like this one:

> *Task 1, Splitting Problem:* Here's a picture of my stick, which is five times the length of yours. Can you make your stick?" (Fig. 1)

(insert Figure 1 here)

Note that Task 1 is stated with language of iteration (my stick is *five times* your stick), and yet a student must invoke partitioning (partitioning a stick into five equal parts) to solve it. [v] More specifically, to solve the problem students need to posit their bar, which is separate from the given bar but also stands in relationship to it: Their bar can be iterated five times to make the given bar, and at the same time the length of their bar is determined by partitioning the given bar into five equal parts. Thus splitting is *more* than just partitioning a bar to find one-fifth of it; splitting a bar means that any part of the partitioned bar can be iterated to make the partitioned bar, so a part is both "inside" the partitioned bar and "outside" of it. Students who solve this problem are aware of at least one multiplicative relationship between their bar and the given bar, namely that their bar taken five times produces the given bar.[vi]

Steffe (2002) places so much importance on splitting for the construction of improper fractions because splitting involves positing a length (or other continuous quantity) that is independent from a given length (or other quantity) and yet stands in relation to it. For example, to make a candy bar that is 7/5 of a given candy bar, a student must posit a hypothetical bar that is independent from the given bar in that it is seven parts, each of which is the length of one-fifth of the original bar. However, the hypothetical bar still stands in relation to the given bar in that iterating any one of those seven parts five times makes a bar identical to the original. Steffe's hypothesis is that upon the construction of the splitting operation, a student has all the operations necessary to construct these relationships and thereby to construct improper fractions. In fact, he has stated that upon the construction of the splitting operation, he regards students' fractional schemes as essentially including improper fractions (p. 299).

In addition, and in amplification of the importance of splitting, Olive and Steffe (2002) have used the construction of splitting *composite units* as a central explanation for why one fourth grade student constructed improper fractions in contrast to two other fourth grade students who did not do so.[vii] An example of a task that involves splitting composite units is the following:

> *Task 2, Splitting Composite Units:* A 33-inch ribbon is three times as long as your ribbon. Draw your ribbon and find its length.

A student who determines that her ribbon is 11 inches may conceive of the 33 inches as a unit of three units, each containing 11 units. If so, then this student is coordinating three levels of units, at least in activity in solving the problem. By emphasizing the importance of splitting composite units, Olive and Steffe have pointed to the notion that coordinating three levels of units is involved in the construction of improper fractions. In other words, a student who can split composite units may have constructed structural relations that are akin to the structural relations necessary to produce a fraction like twelve-elevenths. As I have discussed above, conceiving of twelve-elevenths as an improper fraction means conceiving of it as a unit of 12 units, any of which can be iterated 11 times to produce another unit (the whole), a three-levels-of-units structure.

Based on analysis of data from a longitudinal teaching experiment with four sixth grade students, I agree with Olive and Steffe's (2002) assessment that the coordination of three levels of units is critical in the construction of improper fractions. But my research indicates that the construction of splitting, even of splitting composite units, is not sufficient for students to construct improper fractions. In particular, all four sixth grade students entered the teaching experiment having already constructed splitting (i.e., they could solve Task 1). Three of the four entered the experiment splitting composite units (i.e., they could solve Task 2), and the fourth student learned to split composite units during the first three teaching episodes. However, during the course of the experiment only two of the four students constructed fractional schemes that included improper fractions.

These two students coordinated three levels of units *prior to activity*, which meant that they could project a structure of three levels of units into a situation and then operate further using that structure. In contrast, the other two students coordinated only two levels of units prior to activity, although they could coordinate three levels of units *in activity*. This meant that they could make a structure of three levels of units as the result of operating in a situation, but they didn't produce this structure until they had made it during operative activity. Coordinating only two levels of units prior to activity presented a major constraint in these two students' constructions of improper fractions. So, this research indicates that construction of the splitting operation is not necessarily accompanied by the *a priori* coordination of three levels of units that appears to be necessary for the construction of improper fractions.

The purpose of this article is to elaborate on how the coordination of three levels of units is critical in the construction of improper fractions by examining the relationships between the units coordinations that the students had *interiorized*—the students' multiplicative structures—and their construction of improper fractions. Doing so allows me to propose a revision of Steffe's (2002) hypothesis about splitting.

2. Fractional schemes and multiplicative structures

In this section, I give background on fractional schemes and multiplicative structures in order to set the stage for my analysis of the four sixth grade students' activity with improper fractions and units coordinations.

2.1. The partitive and iterative fractional schemes

A partitive fractional scheme is the first scheme that Steffe (2002) has judged to be a genuine fractional scheme. A student who has constructed a partitive fractional scheme conceives of one-fifth of a candy bar as a part that, when iterated five times, will produce a 5-part bar. To make three-fifths of a candy bar, the student can partition the candy bar into five equal parts, take out one of those parts, and iterate it to make three parts.[viii] Thus the student has transcended part-whole conceptions of fractions in that the student can *disembed* a unit fractional part from the partitioned whole and iterate it to make another fractional part of the partitioned whole. But the student remains tied to part-whole conceptions in that the meaning for three-fifths comes from it being part of a whole, not from it being a fraction that is three times one-fifth. So, students who have constructed only partitive fractional schemes cannot make, for example, seven-fifths of a candy bar by splitting the bar into five equal parts, disembedding one of the parts, and iterating that one-fifth of the bar seven times (cf. Steffe, 2002). In short, even though these students can disembed and iterate fractional parts, it is as if their "mathematical world" is the whole, and going beyond it does not seem to make sense to them. In fact, if these students do iterate one-fifth seven times, they will generally call what they have made "seven-sevenths." Thus, although partitive fractional schemes are crucial for constructing fractional knowledge, they do not include the construction of improper fractions.

Being able to generate *any* fraction by a whole number iteration of a part of it (e.g., to make 9/8 from 1/8 iterated nine times), and yet maintain its relationship to the whole (e.g., simultaneously know that 9/8 is a whole, 8/8, and 1/8 more) means fractions are freed from relying on the whole for meaning even though they stand in relation to the whole. Constructing such a fractional scheme, *an iterative fractional scheme*, is a major advance in students' fractional knowledge (Olive & Steffe, 2002; Steffe, 2002, in press; Tzur, 1999). It also signifies that unit fractions (and possibly non-unit fractions) are iterable units (Steffe, 1992), in that they imply the composite unit from which they were made. For example, for students with iterative fractional schemes, *prior to activity* one-eighth implies a whole consisting of one-eighth iterated eight times. In contrast, even though students who have constructed partitive fractional schemes can iterate one-fifth three times to make three-fifths, one-fifth is not yet an iterable unit for them because one-fifth does not have this a priori multiplicative relationship to the whole.

2.2. Multiplicative structures

I conceive of multiplying *schemes* in the context of goal-directed units-coordinating activity.[ix] The simplest multiplying scheme involves coordinating two levels of units (cf. Steffe, in press). That is, a multiplicative conception of 4*5 involves envisioning four units, each of which contain five units, so the person's concept for five is in some sense "injected" into each of the four units (cf. Kamii & Housman, 2000; Kaput & West, 1994; Steffe, 1992; Thompson & Saldanha, 2003). In coordinating 4 and 5 in this way a student has made at least a *unit of units*, in that each of the four units is a unit consisting of five units.

Multiplicative *structures* are the results of such units-coordinations that students have abstracted and that they can take as given prior to activity (Steffe, 1988, 1992). Students who can coordinate two levels of units *prior* to activity don't have to insert five units into each of four units by enacting this coordination—they have *interiorized* this coordination, so they don't have to make it in activity. Instead, they can take the insertion of five units into each of the four units as something they bring to a situation, and then they can operate further using that coordination.[x] For example, let's say that these students have four strings each containing five beads and they get three more such strings. These students can reason that they have 35 beads in all because they have seven strings total, each containing five beads, and they can determine that counting by five seven times (inserting five units into each of the seven units) produces 35 beads. Alternately, the students may consider that they can add 20 beads in the first set of strings with 15 beads in the second set of strings, but without visual cues it will likely be *another* problem for them to determine that the result, 35, is made from seven strings, each containing 5 beads.

Coordinating *three* levels of units prior to activity is even more sophisticated and involves conceiving of 20, say, as a *unit* of five units each containing four units, where the unit of units of units (20) can itself be used in further activity like iterating or uniting. [xi] Students who can take a *unit-of-units-of-units* structure as given can, in the bead example above, reason that they have 35 beads in all by combining 20 beads from the first four strings with 15 beads from the additional three strings, and *simultaneously* know that they are uniting four units of five with three units of five to make seven units of five. Thus for them, the result, 35, is “automatically” a unit of seven units each containing five units, and to a situation involving 35 they bring this structural, three-levels-of-units conception of 35.

3. Methodology and methods

3.1. The teaching experiment

The data analyzed for this article is drawn from a year-long constructivist teaching experiment (Confrey & Lachance, 2000; Steffe & Thompson, 2000) in which I taught two pairs of sixth-grade students at a rural middle school in north Georgia from October 30, 2003 to May 12, 2004.[xii] The students were invited to participate after selection interviews in September and early October of 2003, which I will discuss later in this section. The students and I met twice weekly in 30-minute episodes for two to three weeks, followed by a week off.

Most sessions included the use of JavaBars (Biddlecomb & Olive, 2000), a computer software program, and all sessions were videotaped with two digital cameras for on-going and retrospective analysis. One camera captured the interaction between the pair of students and myself, and the other camera recorded the students' written or computer work. Two witness-researchers were present at all sessions to assist in videotaping and to provide other perspectives during all three phases of the experiment: the actual teaching episodes, on-going analysis that occurred between episodes during the course of the experiment, and retrospective analysis of the videotapes (cf. Steffe & Thompson, 2000).

3.2. Participants

In September and October of 2003 I and two other researchers conducted 20-minute selection interviews with 20 sixth-grade students out of a pool of approximately 100 students, all of whom had the same classroom mathematics teacher.[xiii] I observed four of this teacher's five mathematics classes, which were organized by level of achievement, and I consulted with this teacher to identify students to interview. My intention was to select a pair of students who had interiorized three levels of units and a pair who had interiorized two levels of units. During the one-on-one interviews, which were not tape-recorded or video-recorded, I used fractional tasks and tasks involving the splitting operation to make initial assessments of students' multiplicative structures and their fractional schemes.

My students ended up being a pair of girls, Bridget and Deborah, and a pair of boys, Carlos and Michael. The girls, both Caucasian, were in the top two mathematics classes in the sixth grade. The boys, one Asian-American and one Caucasian, were in the middle-level mathematics class. In their selection interviews, all four students demonstrated that they had constructed at least a partitive fractional scheme. I made this inference because, given a cardboard rectangle that represented a candy bar, all four students marked an estimate for the fair share for *one* out of six people sharing the bar. They indicated that to test whether the share was fair they could cut off their estimate and repeat it six times; if the share was fair, the result should be the whole candy bar. They all used fractional language, calling the fair share one-sixth of the candy bar. All four students also solved a splitting task (cf. Task 1), indicating that they had constructed a splitting operation. I conjectured that the girls had interiorized three levels of units because they split composite units by solving problems that required them to reverse the operations of their fractional schemes with composite units.[xiv] In contrast, Carlos experienced difficulty in splitting composite units (cf. Task 2), and Michael solved a problem like Task 2 but did not appear to be as facile as the girls. Thus I conjectured that the boys had interiorized at least two levels of units, but probably not three. I was to learn that these conjectures, based on one interaction with each student, were not entirely accurate.

3.3. JavaBars

The computer program I used throughout the experiment, JavaBars (Biddlecomb & Olive, 2000), allows students to draw rectangles (bars) of variable dimension. The program was developed to allow students to enact key mental actions such as partitioning, iterating, and disembedding, in establishing a fractional microworld.[xv] Using the PARTS menu, students can partition a bar into some number of equal parts (e.g., they can dial to 10 and mark the bar into 10 equal parts). Students can partition a partition (e.g., partition each of the 10 parts into, say, three equal parts), CLEAR a partition (erase all marks on the bar), BREAK a bar into its parts, or PULL OUT a piece or part from a bar. When students use the function PULL OUT, the part simultaneously remains in the bar and is disembedded from the bar. Students can COPY bars or parts of bars, JOIN parts or bars together, or use REPEAT to iterate a part or bar some number of times. Students can FILL a part of a bar, or a whole bar, to change it to a different color. JavaBars also has a MEASURE function: If students set a bar as the unit of measure, the computer will give the fractional "measure" of any bar made from that unit bar.

4. Analysis

In this section, I present analysis of data from both pairs of students to demonstrate three points. First, differences *within* each pair of students with respect to their activity with improper fractions were apparent early in the teaching experiment. Second, throughout the year Bridget and Carlos exhibited persistent lacunas in their making of and reasoning with fractions greater than one that their respective partners, Deborah and Michael, did not exhibit. Third, in solving problems that required them to coordinate two fractions in the same bar, Bridget and Carlos also exhibited units-coordinating activity that was qualitatively different from Deborah and Michael. I will use these differences in units-coordinating activity to account for Bridget and Carlos *not* constructing an iterative fractional scheme during their sixth grade year, even though they *had* constructed a splitting operation and could split composite units. Finally, although I am highlighting Bridget's and Carlos's schemes and activity in contrast to their partners, Bridget and Carlos were not entirely similar. To help distinguish the two of them and to maintain a sense of chronology in the presentation, I organize the analysis around the pairs (Bridget and Deborah, Carlos and Michael).

4.1. Bridget and Deborah

4.1.1. No improper fractions initially. During the first teaching episode on October 30th I asked Bridget and Deborah to draw seven-fifths of their candy bar, if the drawing of the rectangle on their paper represented a candy bar. Both expressed some dismay, but each drew a picture. Bridget marked the given rectangle into five parts before she drew her new bar (see top two bars in Fig. 2). She explained her drawing of the new bar as making fifths and then "adding

(insert Figure 2 here)

two more pieces" for a total of seven. Deborah said she knew that seven-fifths was "one and two-fifths," so she drew a whole bar and didn't close it until she had determined how long two-fifths of the original bar was; then she added on that much (see bottom two bars in Fig. 2). However, when both a witness-researcher and I asked the girls about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. As Bridget explained, "they turned into seven pieces instead of five pieces." Thus both girls demonstrated "typical" conceptions of students with partitive fractional schemes, and I inferred that for them seven-fifths was yet to mean one-fifth iterated seven times.

At this point, a witness-researcher asked the girls to start with a new drawing of a candy bar and draw three-fifths of it. Each girl marked a fresh copy of the rectangle into five parts and drew a new bar below that spanned three of those parts, marking the three equal parts in the new bars when prompted. Both readily stated, when asked, that any one of those parts was one-fifth of the candy bar. So the witness-researcher asked them to make a new bar that was seven of those parts. Both girls drew 7-part bars and marked in the parts (Fig. 3). When asked to shade

(insert Figure 3 here)

the first part in their new bars and tell how much it was of the candy bar, without hesitation—but with a slight frown—Bridget said it was one-fifth. Deborah smiled broadly in agreement. Bridget explained that the shaded part was equal to the first part of the original candy bar. The witness-researcher asked them to shade in another piece of their 7-part bars, and again both girls agreed that the piece was one-fifth of the candy bar, and that the 7-part bar was seven-fifths. So the girls appeared to make meaning for seven-fifths of a candy bar as seven parts, each part being one-fifth of the candy bar. Thus this intervention by the witness-researcher *potentially* marked both girls' initial construction of an iterative fractional scheme.

4.1.2. Iterating beyond the whole. After the first episode, the girls and I began to use JavaBars. During the fourth teaching episode with them on November 11th, I asked Deborah to make a bar in the microworld that was 2/15 longer than a 13/13-bar.[xvi] After 37 seconds, Deborah made a 13/13-bar, copied it, erased the marks showing the 13 parts, and partitioned the bar into 15 parts. Then she pulled out one-fifteenth, repeated it to make a 2/15-bar, and placed the 2/15-bar at the right end of the 15/15-bar (Fig. 4). I asked Bridget how long Deborah's bar was.

(insert Figure 4 here)

After a 5-second pause during which she appeared to count the number of parts in Deborah's bar, Bridget called the result seventeen-seventeenths "'cause there's seventeen pieces." Deborah said the bar was seventeen-fifteenths because "it's more than one," and Bridget quickly agreed. When asked why, Bridget said that the whole (unit) bar had 15 pieces and Deborah had added on two of those pieces.

Based on Deborah's pause before initiating activity in this situation, I infer that she did not immediately know what to do to solve this problem. However, once she pulled out a 1/15-part from the 15/15-bar and used it to make the 15/15-bar longer by two-fifteenths, it seemed unproblematic for her that the new bar was seventeen-fifteenths. Her justification, that seventeen-fifteenths is more than one, allows me to infer that at that point, a bar two-fifteenths longer than a unit bar was *both* a whole unit bar and two-fifteenths more *and* seventeen-fifteenths of the unit bar. Holding this "dual view" of a bar two-fifteenths longer than a unit bar was a change from Deborah's view of seven-fifths of a candy bar on October 30th. I account for this change by inferring that on October 30th, Deborah began to construct a unit fraction as a unit that could be disembedded from and iterated beyond the whole to make a new unit (of unit fractions) that still stood in relation to the whole, but in a way that transcended part-whole relationships. She seemed to solidify that construction on November 11th: One-fifteenth of a unit bar could be iterated 17 times to produce a bar that was a unit made of 17 units, each of which gained its identity from its relationship to the unit bar. So I can conclude that at that point, Deborah was coordinating three levels of units: Seventeen-fifteenths was a unit of 1/15-units, any one of which could be iterated 15 times to make the whole, fifteen-fifteenths, another unit both "within" the 17/15 and independent of it. Thus I can attribute an iterative fractional scheme to her.

In contrast, Bridget's meaning for fractions remained basically partitive: Parts of a bar gained their identity from their relationship to the "current whole" of which they had just become a part, rather than in relation to the whole *from which they had been a part*. Thus Bridget's unit fractions did not yet seem to be iterable outside of the whole, and she did not seem to independently coordinate three levels of units in this context. However, as soon as she heard Deborah's response, Bridget made sense of it and could explain why the bar was seventeen-fifteenths. Her comment may mean that Bridget knew, just as when she made seven-fifths on October 30th, that there were 17 parts in the new bar and each one was the same size as each of the 15 parts in the unit bar.

Still, based on this interaction with Bridget I can't attribute an iterative fractional scheme to her, primarily because she did not independently produce the name of the bar as seventeen-fifteenths. If she went on to do so in other situations, that might confirm an initial construction of an iterative fraction scheme here. Making sense of Deborah's response likely indicates that

Bridget was, when prompted, coordinating three levels of units *in activity*. But it doesn't indicate that she was taking this coordination as a given, *prior to* making an improper fraction or interpreting her partner's activity.

Since Bridget exhibited activity with fractions greater than one that was similar to students who have constructed only partitive fractional schemes, during this same November 11th episode I posed problems designed to free her from feeling confined to the material of whole. For example, I posed this problem:

> *Task 3: Magic Cake Problem.* This rectangular cake is magic in that the cake fills right back in when you take out a piece. It's also magic in that it can only be divided into five pieces. Seventeen people come to the party and they each want a slice from the magic cake. Can you show how much cake they get all together?

The magic refilling of the cake is intended to open opportunities for students to iterate beyond the whole by having more "material" available. So in posing a magic cake problem to Bridget, I was harmonizing with my inference that she had not yet constructed an iterative fractional scheme while at the same time testing out whether solving such a problem might engender this construction. However, magic cake problems did not appear to be problematic for Bridget. She solved them with ease by disembedding a unit fraction of the cake (in this case, 1/5), iterating the unit fraction however many times she needed beyond the whole (in this case, 17 times), and calling the result seventeen-fifths. So even though improper fractions still seemed problematic for Bridget in a way that they did not seem to be for Deborah (as I will continue to demonstrate), iteration beyond the whole did *not* seem to be the main issue. Thus I established Bridget was *unlike* students who have constructed only a partitive fractional scheme and generally do not iterate beyond the whole, although I could not yet attribute an iterative fractional scheme to her.

4.1.3. Coordinating two fractions in the same bar. In December, the girls and I worked on problems that involved making a fraction bar into another fraction bar, such as:

> *Task 4, Making a 4/4-bar into a 6/5-bar:* Make a 4/4-bar. Make that bar into a 6/5-bar without erasing the fourths marks.

These problems concern coordinating two fractions in the same bar, which requires students to coordinate units within units.[xvii] Differences in the girls' units-coordinating activity to solve these problems were quickly apparent.

On December 2nd, I asked the girls to make a 2/2-bar into a 7/6-bar without clearing the half mark. Bridget was the first to solve the problem by partitioning each half of the 2/2-bar into three parts. She noted that there were six sixths in the bar, so she just needed to add on one more part to make seven-sixths. Both girls swiftly solved another problem like this one, pronouncing it "easy." Then I posed Task 4 (above), which met with a great deal of concern and surprise.

Protocol 4.1: Making a 4/4-bar into 6/5 of the unit bar on December 2nd 2003.[xviii]

[The teacher has just asked the girls to make a 4/4-bar into a 6/5-bar without erasing the fourths marks.]
D: Uh oh. [pause] What!? I don't *think* so!
T [to Deborah]: You don't think so?!
B [staring intently at the 4/4-bar on the screen]: Six-fifths.
D: Can we clear it?
T: Nope! [Deborah and the teacher laugh.]
B: Wait a minute! You couldn't divide them in twenty and then take away, like, something, something like that…Okay, you wanted six-fifths. Well, you could pull out [one 1/4-part] and make it a fifths bar instead of a fourths bar [smiling a little, as if she is kidding. Her intention is to join a 1/4-part onto the 4/4-bar to make a 5-part bar. Everyone, including Bridget, laughs.]
D: That'd still be fourths!
T: Yeah, we want it to be six-fifths of the unit bar.

After 10 seconds, Deborah announced that she had an idea, but Bridget remained stumped. Deborah solved the problem by partitioning each fourth into five parts, pulling out four of those parts, and joining them onto the 4/4-bar (which was now a 20/20-bar, Fig. 5). Bridget

(insert Figure 5 here)

agreed with Deborah about making 20 parts total and about the need to join on more parts. As Deborah completed her solution, I asked Bridget whether those four little parts were one-fifth. She hesitated, and when Deborah confirmed that the four parts were indeed one fifth, Bridget said, “Yeah, ‘cause five and four is twenty and then you’ve made the whole bar.”

In this situation, I claim that Deborah began to modify her multiplying and iterative fractional schemes so that she could coordinate any two fractions in the same bar, and I discuss her accommodation at greater length elsewhere (Hackenberg, 2005a). In short, using her multiplying scheme, Deborah knew that inserting five parts into each fourth gave a total of 20 parts, which she could use to view the bar as a unit of two *different* units-of-units structures. For Deborah, the 20-part bar was a unit of four units each containing five units, which certainly allowed her to show the 4/4-bar with which she had begun. However, she also viewed the resulting 20-part bar as a unit of five units each containing four units, which allowed her to identify one-fifth of the bar (four-twentieths), and thus also six-fifths (by pulling out and adding on four-twentieths).

Yet Bridget was the first to have an idea of “dividing in twenty,” which indicates that multiplying four and five was relevant for her in the situation, and which may have helped Deborah formulate her solution. But Bridget retreated from her idea, joking about pulling out one 1/4-part from the 4/4-bar and then joining the 1/4-part onto the 4/4-bar to make it “a fifths bar,” a bar consisting of five equal parts. Her joking manner suggests that she knew the parts of that bar could be identified as fifths only because the bar would consist of five equal parts, not because the 5-part bar was five-fifths of the unit bar. Thus she did not seem to know how to use “twenty” to solve the problem. Furthermore, when Deborah began to explain her solution, Bridget did not seem to know why *specifically* 20 parts would be useful—she seemed uncertain of the number of twentieths that would constitute one-fifth. I infer that she had not yet constructed a way to shift flexibly between viewing the bar as a unit consisting of two different units-of-units structures. I conjecture that the reason she could not make this shift was because she had not yet interiorized three levels of units. So, even though she could produce three levels of units in activity, she could not take one three-levels-of-units view of the bar as given in producing the other three-levels-of-units view of the bar.

Corroboration of my inference and conjecture occurred with the next task I posed to the girls, making a 3/3-bar into a 9/8-bar without erasing the thirds. Deborah solved this problem by mentally determining how to partition each eighth of the target 9/8-bar in order to be able to show the 3/3-bar with which we began. Her solution path became evident when she asked, “What’s sixteen divided by three?” By her question I infer that she was thinking about partitioning each eighth of the target bar into two parts, which would yield sixteenths. Then she was trying to determine whether having sixteen parts would allow her to show the thirds with which we had started. After a few moments, she said, “It’s [16 divided by 3] not even is it. I mean, it’s not a whole number.” Fifteen seconds later she announced that she had solved it and started to explain that she would divide each of the thirds into eight parts.

Protocol 4.2: Making a 3/3-bar into a 9/8-bar on December 2nd 2003.

B: That’s *exactly* what I was thinking about. I was going to say something about twenty-four, but I got…
[Deborah nods at Bridget. The teacher suggests that Bridget make it. Bridget partitions each third into eight parts.]
T: Now how does that help us?
B: That’s twenty-fourths. And then, nine-eighths.
D: Three of these little twenty-fourths [points to the bar] equal one-eighth.
B: Yeah, so you would—like, if you could clear it [erase the marks on the bar], you could make, um, eight more, so you’d have nine-eighths.
D: Actually, you just take three of those twenty-fourths.
T: Well let’s see, so where are the eighths? [Both girls stare intently at the screen.]

D: The eighths are three twenty-fourths [points to the first three parts of the 24/24-bar. The girls identify the first few eighths by pointing.]
T: How many eighths do we have here?
B: Twenty-four—wait a minute, eight.
D: Eight.
[The teacher encourages Bridget to finish making nine-eighths of the unit bar. Bridget pulls out one twenty-fourth and repeats it to make three of them.]
T: How much is that [the 3-part bar Bridget has just made]?
D and B: [nearly simultaneously] One-eighth.
T: How do you know that?
B: Because eight and three is twenty-four, and then three is in one bar.
D [simultaneously with Bridget's explanation]: They're twenty-fourths.

Bridget's activity after she made the 24 parts corroborates that she did not know exactly how to use them to make nine-eighths. That is, she said, "if you could clear it, you could make, um, eight more, so you'd have nine-eighths." This comment demonstrates an intention to iterate the entire bar, erased of marks, nine times. So at that moment, nine-eighths was nine equal parts, regardless of their relationship to the unit bar. Thus she did not appear to be coordinating the three levels of units involved in making nine-eighths, which corroborates that she was not coordinating three levels of units prior to activity. In November she had often made improper fractions like nine-eighths without conflation. But in those cases, the eighths were "singular" units, while in this case, the eighths themselves were composite units. I infer that under the cognitive load of this units-coordination Bridget dropped back to conceiving of nine-eighths as a unit of nine units, without forming eight of those units as equal to the unit bar.

Nevertheless, Bridget *could* make sense of Deborah's explanation that three twenty-fourths made one-eighth. But her justification for why, "Because eight and three is twenty-four, and then three is in one bar," is a bit ambiguous, in part because she dropped any fractional language that might have clarified her statement. Since she responded correctly to my question about the number of eighths in the 24/24-bar, I infer that in her explanation she meant that three *twenty-fourths* are in one *1/8*-bar, since eight times three *twenty-fourths* is a "whole" 24/24-bar.

So her activity at the end of the protocol allows me to confirm that Bridget could make three levels of units in activity, in retrospective interpretation of her partner's responses. However, I cannot conclude that she had made an accommodation similar to Deborah's or that she was coordinating three levels of units prior to activity.

4.1.4. Two views of Bridget's lacuna. Because Bridget did not seem to independently solve these problems that Deborah was solving, at the start of the next teaching episode on December 4th I posed a problem more like the coordination problems with which we had begun:

> *Task 5, Making a 6/6-bar into a 23/18-bar*: Make a 6/6-bar. Make that bar into 23/18 of the unit bar, without clearing the sixths.

My goal was for Bridget to once again operate independently—to reestablish coordinating units in a problem situation similar to one in which she had previously been successful. That is, because 18 is a multiple of 6, Task 5 is similar to making a 2/2-bar into a 7/6-bar, the first problem of this kind that Bridget successfully solved on December 2nd. Therefore, I expected that Bridget would not have difficulty making sixths into eighteenths (by partitioning each sixth into three equal parts), and that then she would pull out and join on the number of those parts that she needed to make a 23-part bar.

I asked Bridget to explain how she would solve Task 5, and she said "You'd copy that [the 6/6-bar] three times and you'd have eighteen pieces, and then you would just add until you got to twenty-three." I didn't understand that Bridget had said something different from what I expected until I asked her to make the 23/18-bar. Bridget repeated her idea, making two copies of the 6/6-bar so that there were now three 6/6-bars on the screen. "There's eighteen pieces," she said, explaining again that she could pull out pieces and add however many more she needed to get 23. When Deborah said she thought those pieces were sixths, Bridget retreated from her idea.

Bridget's activity confirms that she could iterate beyond the unit bar in creating a fraction larger than one: For her, 23/18 was larger than a unit bar and was a unit of 23 units. However, it was almost as if any particular unit would do—here she expressed her intention to iterate one-sixth 23 times, while in Protocol 4.2 she suggested that she could iterate the whole unit bar nine times to make nine-eighths. So Bridget did not seem to be making iterable units *in relation to the unit bar*. As demonstrated in Protocol 4.2, for her nine-eighths was a unit of nine units, but she did not form eight of those units as equal to the unit bar. Similarly, 23/18 was a unit of 23 units, but it did not seem to be a logical necessity for her that iterating one of those units 18 times would constitute the unit bar. This interaction with her demonstrates well the central lacuna in her thinking: She was not coordinating all levels of units involved in making improper fractions, even though she iterated units to make amounts that were, for her, larger than a whole unit bar. This lacuna recurred throughout the rest of the teaching experiment.

Another brief example may illuminate Bridget's experience of improper fractions as "odd" or "bothersome" numbers, even through May. At the start of the May 3rd teaching episode I posed the following problem: "An apple costs 75 cents. How much does 5/3 of an apple cost?" "Oh my gosh!" Bridget exclaimed, with some annoyance in her voice, while Deborah said the problem was easy. Bridget then said, "Two and one-third?" as Deborah solved the problem (nearly instantly). Bridget seemed to be trying to make sense of five-thirds, while Deborah could take five-thirds as given and operate with it to solve the problem. So five-thirds did not seem to be a fractional number for Bridget, and she endeavored to interpret it as a mixed number. But it was not automatic for her to do so—she knew that five-thirds was more than one, but she did not know how much more. In contrast, Deborah did not need to think of five-thirds as a mixed number in her operative activity—five-thirds was one-third five times, and she could use that to swiftly determine that five-thirds of an apple cost five times 25 cents, or one dollar and 25 cents.

4.2. Carlos and Michael

Just as with the girls, differences in activity with fractions greater than one and with units coordinations were soon apparent between the boys. However, because of differences I had observed in selection interviews between the girls and the boys, the boys and I followed a somewhat different trajectory than the girls and I did. In particular, in the first few teaching episodes with the boys I did not focus on making improper fractions. Instead, since Carlos had demonstrated difficulty with splitting composite units in his interview, I posed problems like Task 2. I also posed fraction comparison problems with *proper* fractions, where each boy would make a fraction in the microworld, compare them, and explain the comparison. Yet soon the boys and I were working on reasoning with fractions greater than one.

4.2.1. Comparing improper fractions. During the fifth teaching episode with the boys on November 18th, Michael demonstrated that he had constructed an iterative fractional scheme, while whether Carlos had done so remained questionable. The boys had just successfully compared an 8/7-bar with a 12/11-bar by assessing (visually) that the parts in the 8/7-bar were bigger. Then I asked them to predict which would be bigger, 15/14 or 14/13. Both boys thought 15/14 would be bigger, and Carlos was particularly swift with and certain of his answer. So I asked the boys to make the two fractions.

Protocol 4.3: Comparing improper fractions on November 18th 2003.

T: So, who wants to make fifteen-fourteenths?
C [points to Michael]: He can.
M [simultaneously with Carlos, raising his hand]: Me.
T: Okay, Michael, you make fifteen-fourteenths. And then Carlos, you'll make—which one will you make?
C: Fourteen-thirteenths.

T: Okay. And then we'll see.
[Michael draws a new unit bar and makes a copy. He partitions the copy into 14 parts, pulls out one part, and places the part adjacent to the right end of the 14/14-bar. Then Michael rests his chin in his hands. Carlos confirms that he is to make fourteen-thirteenths. He copies the unit bar and partitions the copy into 14 parts.]
M: No—no—no! [Michael gestures with his hands toward the screen.] You made fourteenths—[looks at Carlos] yours is thirteenths [gives a little laugh].
C [to the teacher]: Didn't you say fourteen-thirteenths?
T: Fourteen-thirteenths. So how many—
C: Oh yeah [as if he understands now].
T: Oh, you can always clear [erase the marks on the bar] if you want to.
C: Yeah, I keep on forgetting. [Carlos erases the marks and partitions the bar into 13 parts.]
T: Okay, what'd you have now?
C: Now I have thirteen-thirteenths. [He pulls out one of the 13 parts and places it adjacent to the right end of the 13/13-bar. Michael says that the bars are almost the same length, but both boys confirm visually that Carlos's bar is longer. When the teacher asks why that is, Carlos says his bar has bigger pieces, and Michael notes that it's just like the problem of comparing eight-sevenths and twelve-elevenths.]

This protocol shows that not only could Michael make fifteen-fourteenths without incident, he could also critique Carlos's production of fourteen-thirteenths. Thus Michael gave every indication of having constructed an iterative fraction scheme. As shown in this protocol by how Michael made his fraction, for him fifteen-fourteenths certainly meant one-fourteenth more than fourteen-fourteenths. But based on his response to Carlos, I infer that for Michael fifteen-fourteenths simultaneously meant one-fourteenth iterated 15 times. I make this inference because Michael said to Carlos "yours is thirteenths." I take this comment to mean that Michael thought of fourteen-thirteenths as consisting of thirteenths, specifically as 14 one-thirteenths, or one-thirteenth iterated 14 times. Thus I can attribute to Michael, at that point, the a priori coordination of three levels of units in the formation of improper fractions: For him, fifteen-fourteenths was a unit of 15 units, any of which could be iterated 14 times to make the unit bar.

Carlos had made improper fractions such as eight-sevenths from a 7/7-bar by pulling out one-seventh and joining it to the 7/7-bar. But to make fourteen-thirteenths, Carlos partitioned the unit bar into the largest number stated in the fraction, 14 parts instead of 13 parts. Michael's intervention provoked Carlos to enter a state of perturbation, which in turn engendered his need to clarify the goal (what he was to make) with me. It is likely that Carlos resolved the perturbation by realizing (he would say "remembering") that *fourteen*-thirteenths meant to make a total of 14 parts, each of which was one-thirteenth of the unit bar. Carlos's comment that he "kept on forgetting" confirms that he tended to interpret fractions like fourteen-thirteenths as thirteen-fourteenths unless someone pointed him toward making thirteenths. At the time, I conjectured that his tendency to interpret fourteen-thirteenths as thirteen-fourteenths might have involved that common characteristic of a student with a partitive fractional scheme: a reluctance to extend beyond the unit bar. Yet as shown in protocol 4.3, when prompted Carlos *could* extend beyond the unit bar in making bars representing improper fractions, which is not characteristic of a student with only a partitive fractional scheme.

Still, to test my conjecture, on December 4th I posed the same Magic Cake Problem (Task 3) to Carlos as I had to Bridget. At first, Carlos partitioned the cake into five parts and then partitioned the first of the five parts into 17 parts. Then he said he messed up, erased the marks on his cake, partitioned it into 17 parts, and pulled out one part as the share for one person at the party. I told him he would be right if we split the cake among 17 people, but I reminded him that the cake could only be cut into five pieces. Carlos erased the marks and partitioned the cake back into five parts. "So since it keeps on refilling I can just take out one piece and make seventeen," he said. He did so, pulling out one of the five parts and copying it to make 12 of them. He then joined the 12 parts onto the original cake. When I asked him for a fraction name for the amount of cake he'd made, he said, without hesitation, "seventeen-fifths." I asked him how many unit cakes were in seventeen-fifths of the cake, and he said, “three…and two more,” which he indicated, upon probing, meant three unit cakes and two-fifths more.

I infer that in solving the Magic Cake Problem, Carlos tried to figure out a way to coordinate “seventeen” and “fifths.” When he rejected his first try, he then ignored the fifths and tried to make sure all 17 people got a share of the cake by partitioning it into 17 parts. Unlike Bridget, he commented specifically about the cake refilling, and his awareness of this “magic” seemed to free him from the need to stay within the amount of cake given (the unit cake) so that he could make 17 shares that were *more* cake than a single cake. Furthermore, his response that seventeen-fifths of the cake was three unit cakes “and two more” showed that when requested he could break seventeen-fifths of a unit bar into whole unit bars and fractional parts of unit bars.

In subsequent episodes, Carlos seemed to make fractions just larger than one with less frequent conflation. For example, during the next teaching episode on December 9th he made nine-sevenths without first partitioning the bar into nine parts. However, I still could not attribute an iterative fractional scheme to him at this point because he operated as he had in Protocol 4.3 intermittently throughout the teaching experiment.

4.2.2. Not enough candy. One of the best views of Carlos’s lacuna with improper fractions occurred relatively late in the teaching experiment, in the teaching episode on March 24th. The boys were working on problems that involved a distributive operation, where they were to make different-sized candy bars and take a fractional amount of the whole collection. Carlos took one-fifth of three different bars by taking one-fifth of each of the three bars, and Michael took five-sevenths of four different bars by taking one-seventh of each of the four bars and repeating that amount five times. Both seemed convinced that taking the same fractional amount of each of the bars produced that fractional amount of all the bars. So it seemed like each boy had a distributive operation in this context. Then I asked Carlos to make five different-sized candy bars and make seven-fifths of that whole collection.

Carlos made and colored five different candy bars. Then he partitioned each bar into *seven* parts (see Fig. 6).

(insert Figure 6 here)

Protocol 4.4: Carlos's reluctance to extend beyond the whole on March 24th.

T [to Carlos, who is pulling out one part from his largest bar]: Mmm. So let's see, how many—what did you make—what's this part [points to the first part in his largest bar] of the whole bar?
C: One-seventh.
T: Oh.
C: Oh [smiles].
T: I said to make seven-fifths, right?
C: Yeah [smiles].
T: Hmm. So if you're going to make seven-fifths, [pause] do you want one-seventh?
C: No, I need more. But I thought that all these are sevenths, right?
T: Well, we want to make seven-fifths.
C: Oh—just out of one of them?
T: Seven-fifths of the whole thing.
M: Clear them [smiles; his suggestion is to erase the marks on the bars].
T: Seven-fifths of the whole thing.
C: Oh! Okay! [Carlos clears the marks on all five bars and erases the 1/7-part that he had pulled out of the largest bar. He partitions each bar into five parts. Whispers] All right. [Carlos breaks all the bars into all of their parts—there are 25 parts in all on the screen. Then he starts to pull one part away from each of the five bars, arranging four of the five parts on top of the largest part, so that part forms a "placemat," see bottom left of Fig. 7.]

(insert Figure 7 here)

T: So what have you made right here [points to the five parts Carlos is arranging]?
C: This is one-seventh.
T: One-seventh?
C: No wait—one-fifth. [Carlos continues to rearrange the parts so they fit.]
M: You need to make six more of those.
C: But I don't have enough pieces, right?
T: Oh—that's a good point.
M: That's why you need to—[stops, glances at the teacher, and gives a little laugh].
C: That's why I split them into sevenths.
T: Oh, I see.
C: 'Cause I was just going to make them five.

T: Oh I see, I see. But what would that make, if you had split it into sevenths and just made five?
C: That would be…five-sevenths!
T: It's a good point that we don't have enough [pieces]. What are you going to do?
M [whispering]: Copy!
C: Yeah we've got to copy some of them. [Carlos continues making his arrangements. He uses the fifth part from each of the five bars to make two more parts for each of the five bars. Then he continues to make his arrangements, making seven arrangements in all, each consisting of one of five parts from each of the five bars.]

Carlos's activity with this problem allowed me to better understand his ways of operating with fractions greater than one. Prior to that episode he had not explicitly verbalized his concern that he didn't have enough pieces. I told him he had a good point because at that moment I understood more clearly how problematic it was for him to go beyond the whole in making an improper fraction. That is, he seemed to know that seven-fifths was more than one. But making something more than one required more material (candy) than he had available, and this situation with multiple bars seemed to highlight that problem. So he tried to coordinate seven and five in a way that remained within the given material. This way of operating differed from Bridget's ways of operating in that I never saw Bridget make the reciprocal of an improper fraction when she was intending to make an improper fraction. Thus Carlos's activity with fractions greater than one was, in a sense, more "typically partitive" than Bridget's.

However, Carlos's activity in Protocol 4.4 was also similar to Bridget's in that seven-fifths was not a three-levels-of-units structure for him, which confirms my conjecture that he was not yet coordinating three levels of units prior to activity in making fractions greater than one. Carlos could make seven-fifths if he "heard" the fraction as seven one-fifths, because he could iterate one-fifth seven times and call the result seven-fifths. But, a priori, the relationship between seven-fifths, the one-fifth part, and the whole was not available to him. In other words, prior to making seven-fifths, Carlos did not seem to "imagine" that two-fifths more than five-fifths, and one-fifth iterated seven times, were the same amount. In this sense, making five-sevenths instead of seven-fifths was a reasonable response to this problem because Carlos *could* anticipate, prior to activity, how five-sevenths was related to the whole: The whole was a unit of seven units and five-sevenths was made from one of those units iterated five times.[xix]

This interpretation also explains why Carlos was able to solve magic cake problems (cf. Task 3). Since the “magic” of the cake ensured that there was plenty of cake available, he could continue to pull out parts until he had the number he needed. He could also retrospectively analyze the 17/5-bar that he’d made, determining the number of unit bars and number of fifths it contained. But doing so did not mean that Carlos had constructed these relationships *prior* to making the bar. So I claim that Carlos’s scheme for making fractions greater than one was an extension of his partitive fractional scheme. It was not an iterative fractional scheme because Carlos was not really making improper fractions—he was making long fraction bars that did not seem to have an a priori relationship with the whole. The lack of a priori relationship to the whole was a central similarity between his lacuna and Bridget’s lacuna.

4.2.3. Coordinating two fractions in the same bar. Because my work with the girls on coordinating two fractions in the same bar preceded my work in this area with the boys, I made some changes in working with them. As demonstrated, long before March I knew that improper fractions were problematic for Carlos. I also knew that my main goal in posing problems that required coordinating two fractions in the same bar was for students to use their multiplying schemes in service of their fractional schemes. So, I eliminated the improper fraction component of the problems when I worked in this area with the boys in January.

On January 14th I began by posing problems like making a 4/4-bar into an 8/8-bar without clearing the fourths. Michael pronounced problems like these “easy,” and both boys solved them swiftly. Then I asked them to make a 2/2-bar into a 3/3-bar without erasing the half mark. This problem evoked a different response! After five seconds, Carlos smiled and announced he was stumped. Yet after about 35 seconds, both Carlos and Michael had an idea about six. Michael noted that “two times three is six.” He partitioned each half into six parts and then he and Carlos colored the bar to show the thirds. Both boys determined the size of the little parts Michael had made (twelfths) and the number of twelfths in one-third, although Michael was the first to do so.

To find out to what extent the boys were coordinating their multiplying schemes with their fractional schemes, I posed some similar problems, such as making a 5/5-bar into a 4/4-bar. On this problem, Carlos proposed partitioning each fifth into eight parts, a suggestion likely based on his immediately preceding success in partitioning each third of a 3/3-bar into eight equal parts in order to make the bar into a 4/4-bar. I asked the boys if they could use something smaller or easier to solve the problem. While Carlos proposed using two, Michael said, “Twenty. No, no wait. Something that would equal twenty.” But Carlos insisted on his idea. “Try two’s Michael!” he said. So Michael partitioned each fifth into two parts to make 10 parts total (middle bar of Fig. 8). When I asked where the fourths were, Michael pointed to the middle of the third

(insert Figure 8 here)

and eighth parts,[xx] and he partitioned each of the 10 parts again into two parts to make 20 parts total (bottom bar of Fig. 8). “So that would be just like using four, like my way!” he exclaimed. Both boys identified that five twentieths made one-fourth, although again Michael was the first to do so.

Carlos’s way of operating in these situations seemed to involve partitioning the bar into more parts by partitioning each part (in this case, each fifth) into some number of parts. His suggestion of partitioning each fifth into eight equal parts would indeed have worked, but his suggestion of partitioning each fifth into two parts would not. His latter suggestion leads me to infer that Carlos was suggesting a number of parts into which to partition each fifth without anticipating whether the partitioning activity would produce the desired result—so he was experimenting, without carrying out his experiment in thought prior to enacting it. Thus his way of operating did not seem to include anticipating either the total number of parts in the bar or the number of those parts in one part of the "target" fraction. I make this claim because had Carlos anticipated making 10 total parts *with a goal to make fourths* of the bar, he likely would have realized that partitioning each fifth into two equal parts would not work. This inference about Carlos's experimental or enactive activity was corroborated over the next few episodes.

In contrast, Michael's goal to make "something that would equal twenty" was likely based on coordinating two three-level-of-units views of the bar. He seemed to engage in a *thought experiment* (versus experimenting, as Carlos did) to test his idea before enacting it. I infer that he could anticipate that 20 parts would "work" because four parts in each fifth would make 20 parts in all, and 20 divided by 4 was 5, so five parts would be in each fourth of the bar. I conjecture that this way of reasoning was possible for him because he could take the 5/5-bar as a unit of units, and he could anticipate inserting units into each of those units toward a goal of reorganizing the bar as unit of another number of units of units. So like Deborah, he could view the bar as a unit of two different units-of-units structures. Michael could take the whole bar as a unit of five units into each of which he could insert four units, and he could anticipate that the 20 units created could be considered as a unit of four units each containing five units.

In future episodes, Michael continued to demonstrate that he was determining how to partition the bar into the smallest number of parts necessary to show the two fractions in the same bar. In contrast, Carlos continued to demonstrate "experimental" partitioning of partitions to solve these problems. I contend that the experimental nature of Carlos's scheme came from him not yet taking the whole bar as a unit of units, into each of which he could anticipate inserting units, to accomplish a goal of viewing the whole bar as a unit of a different number of units of units. For example, to make the 5/5-bar into the 4/4-bar, Carlos could certainly conceive of the 5/5-bar as a unit of five units. He could propose inserting units of eight into each fifth and he could use his multiplying scheme to determine that 40 parts would result. So he could make the bar into a unit of units of units in activity. But because Carlos could not yet coordinate three levels of units prior to activity, I conjecture that, like Bridget, he could not hold in mind the view of the bar as a unit of five units each containing eight units and flexibly "switch" to viewing the 40 parts in the whole bar as a unit consisting of four units, each of which contained of 10 units.

5. Discussion

5.1. Summary of Bridget's and Carlos's lacunas

Even at the end of the teaching experiment in May, I cannot conclude that either Bridget or Carlos had constructed an iterative fractional scheme, and therefore that either student was operating with improper fractions. They *were* operating with fractions greater than one. Bridget, as I have discussed, could iterate units beyond the unit bar and knew that she was making something that was larger than a unit bar. Nevertheless, for her the iterated unit did not have an a priori (or even a distinctly determined) relationship with the unit bar. So the central lacuna in her thinking was that it was not a logical necessity for her to form her iterable units in a specific, structural relationship to the unit bar. Carlos's lacuna was similar, but the evidence for it was somewhat more obvious and more "partitive" in nature, in that he often made the reciprocal of the improper fraction he was intending to make in order to avoid the problem of not having enough material. When released from that constraint either by the magic cake or some questioning from me or his partner Michael, Carlos would readily iterate beyond the unit bar and seemed to know he was making something larger than a unit bar. But as with Bridget, I cannot claim that the resulting fraction represented a three-levels-of-units structure for him, and I certainly cannot claim that he used such a structure prior to making the fraction.

With Carlos, not coordinating three levels of units prior to activity was evident in the differences between him and Michael in making two fractions in the same bar, as I have described. A key difference between the boys was the degree to which they could anticipate the total number of parts that would allow them to solve these tasks, let alone anticipate the *smallest* total number of parts that they could use. Similarly, Bridget's activity was different from Deborah's in coordinating two fractions in the same bar, although Bridget was not as willing to experiment in trying to solve the problems. One reason for her lack of experimentation is that the tasks I posed to the girls involved making improper fractions with composite units, which seemed to be overwhelming for someone who was not yet coordinating three levels of units prior to activity.[xxi] Thus, even though Bridget, like Carlos, seemed to think that multiplication was relevant in these situations, she did not seem to know what to do with the total number of parts she considered making (e.g., 20 parts in Protocol 4.1 and 24 parts in Protocol 4.2).

5.2. Splitting fractions

Yet, in October both Carlos and Bridget demonstrated that they had constructed a splitting operation. So based on Steffe's (2002) hypothesis about splitting, that neither Carlos nor Bridget constructed an iterative fractional scheme during the following seven months was a surprise. My interpretation of their surprising ways of operating with fractions greater than one involves examining how each of them used their splitting operation in other contexts, in contrast to how a fourth grade student named Jason constructed and used his splitting operation in Steffe's research.

As I noted earlier, Bridget entered the teaching experiment splitting composite units. That is, in the first teaching episode on October 30th she solved the following two problems independently and drew pictures to show her solutions:

> *Task 6:* Sara's stack of CDs is 65 cm tall, which is 5 times the height of Roberto's stack; what would you do to find out how tall Roberto's stack is?
>
> *Task 7:* Tanya has $16, which is 4/5 of what David has; how much does David have?

Note that Task 6 is a lot like Task 2 (a 33-inch ribbon is three times as long as another ribbon; determine the length of the other ribbon), and that Task 7 is even more complex in that fractional relationships are explicitly involved. To solve Task 6, I infer that Bridget split a composite unit, the 65-centimeter height, by positing another height that was independent of the 65-centimeter height and yet stood in relation to it: The other height iterated 5 times would produce the 65-centimeter height, and the 65-centimeter height partitioned into 5 equal parts would be the same length as the posited height. To solve Task 7, Bridget divided 16 by four to get four dollars. Then she added 16 and 4 to get 20. I infer that she split a fractional quantity, four-fifths of David's amount of money, into four equal parts in order to determine one-fifth of David's amount of money. Then she added one-fifth of the money and four-fifths of the money to determine all of David's money.[xxii] It is important to note that she did *not* iterate one-fifth of David's money five times to produce all of his money, as her partner Deborah did. Bridget agreed with Deborah's more explicitly multiplicative solution—that is, Bridget seemed to be able to interpret it even though she did not produce it independently. In any case, for Bridget, four-fifths was one-fifth four times, and she could use her splitting operation on four-fifths by partitioning it into four equal parts.

Although Carlos entered the teaching experiment with a splitting operation, in early teaching episodes he had difficulty solving problems like Task 6, which shows he did not enter the teaching experiment splitting composite units. But in the second teaching episode with the boys on November 6th he learned to do so (Hackenberg, 2005a). In addition, soon he split fractional quantities in reasoning reversibly with his partitive fraction scheme to solve problems like Task 7, and he solved them similarly to Bridget. Thus, early in the teaching experiment both Bridget and Carlos used their splitting operations on composite units and on proper fractions consisting of composite units.

My current hypothesis is that both students could split composite units to solve problems like Tasks 6 and 7 because of units coordinations they could make *in activity*. For example, they both could make 65 into a unit of 5 units each containing 13 units in the process of solving Task 6; they did not have to take 65 as this three-levels-of-units structure prior to mentally producing the structure in solving the problem, and they did not have to use 65 as a three-levels-of-units structure in further operating. Explaining their splitting of fractions without having interiorized three levels of units is somewhat more problematic, but my current hypothesis that they could coordinate *two different* two-levels-of-units structures in activity. That is, for both students, four-fifths was a unit of four units. Furthermore, five-fifths was four of those units and one more of those units, i.e., a unit of five of those units. So, both students could split the four-fifths into four equal parts, one of which could be united with the original four parts to produce five-fifths. This activity is operationally *different* from viewing four-fifths as a unit of four units, any of which can be iterated five times to make the whole.

The abilities of Bridget and Carlos to operate in these ways were different from Jason (Steffe, 2002), a fourth-grade student who could not solve problems like Task 6 before he constructed his splitting operation, even though he had constructed a partitive fractional scheme. In fact, before he had constructed a splitting operation, given a rectangle that represented two-fifths of a candy bar, Jason could not make the whole candy bar. So at that point, for Jason, two-fifths did not mean one-fifth two times. He could not split two-fifths into two one-fifths, which is not a surprise given that he had not yet constructed splitting (i.e., if he was given a bar that was two times longer than another bar, he could not make the other bar). Once he could split, at the start of his fifth grade, he could use his splitting operation in constructing commensurate fractional schemes and fractional composition schemes, which indicated he was coordinating three levels of units prior to activity (Steffe, 2003, 2004). So for Jason, the construction of splitting seemed to have been accompanied by the interiorization of three levels of units. This also seemed to be the case for other students participating in the Olive and Steffe's research project (1990) who, upon constructing a splitting operation, also constructed an iterative fractional scheme (Leslie P. Steffe, personal communication, May 5, 2006). In contrast, Carlos and Bridget could use their splitting operations to reverse their partitive fractional schemes without simultaneously constructing improper fractions, since for them the construction of splitting did not seem to be accompanied by the interiorization of three levels of units.

5.3. Hypothesis revision and implications

Although the splitting operation still seems to be instrumental in the construction of an iterative fractional scheme, it does not appear to be sufficient for it. Thus I suggest a revision to Steffe's (2002) hypothesis about splitting. Steffe has stated, "Upon the emergence of the splitting operation, I regard the partitive fractional scheme as an *iterative fractional scheme*" (2002, p. 299, italics in the original). My revision of that hypothesis is that students can construct the splitting operation without also interiorizing the coordination of three levels of units, and this interiorized coordination appears to be necessary for constructing improper fractions, and therefore an iterative fractional scheme. Two other sub-hypotheses follow from this revision. The first is that constructing a splitting operation is what allows students with partitive fractional schemes to reverse the operations of their partitive fractional schemes, as shown by Bridget's and Carlos's solutions to problems like Task 7. The second is that constructing a splitting operation *with fractional parts* is possible when students have interiorized only two levels of units, and doing so allows students to iterate beyond the whole in making fractions greater than one, as Bridget and Carlos did.

This hypothesis revision is, in one sense, of concern primarily to researchers who are studying how students learn fractions. That is, it can be viewed as a "finding" with somewhat narrow appeal. However, one of the main goals in doing research of this kind—i.e., in formulating models of students' mathematical ways and means of operating referred to as the *mathematics of students* (Steffe & Tzur, 1994)—is to learn how to better communicate with students mathematically. Since two students from two different pairs in the teaching experiment operated similarly in producing fractions greater than one, but not improper fractions, it may be that this aspect of the mathematics of Bridget and of Carlos is quite common among middle school students. As noted at the beginning of the paper, constructing improper fractions is crucial for filling in the number line—for constructing fractional numbers that open the way to developing a sense of connectedness or continuity of number. So, not yet operating with improper fractions can be quite limiting for students if they are to engage in proportional reasoning and reasoning with rates, two central areas of mathematical thinking that involve fractional numbers and that are typically part of middle school mathematics classes.

In this case, then, important questions emerge for researchers and teachers about the expectations for these students and the mathematical activity in which these students might productively engage—i.e., curricular issues. This research indicates that for students to construct improper fractions and take them as given in further operating, bringing forth the interiorization of three levels of units is critical. But such learning is not easy to engender![xxiii] So, formulating problems that fall *within* the current ways of operating of students like Bridget and Carlos is equally critical. In particular, these students *can* use their current schemes to make interpretations of others' units-coordinating activity while not yet being able to generate similar activity independently. Recognizing and honoring this distinction means designing mathematical activity for these students that is different from the mathematical activity designed for students who have interiorized three levels of units. Thus this research may help researchers and teachers to understand the significance, for some students, of the perplexing nature of improper fractions, and it calls for using this understanding to foster better mathematical communication with students who are grappling with fractions greater than one.

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[i] A fractional number is the result of abstracting a program of operations of all the ways a student has to make the fraction. For example, to have made a fractional number like 5/3 involves abstracting all the ways a student has to make it, such as iterating 1/3 five times as well as joining 2/3 and 3/3, as a program of operations.

[ii] Note that I will use iteration to refer to the *mental action* of repeating an amount to produce another amount. Iteration is often accompanied by physical repetition, if a child is drawing or using a computer tool. However, physical repetition does not always indicate the mental action of iteration.

[iii] Here I am distinguishing between using 1/11 as an *iterating* unit (i.e., iterating 1/11) and having constructed 1/11 as an *iterable* unit (Steffe, 1992, in press). An iterating unit can be repeatedly instantiated to produce a larger amount, while an iterable unit *implies* the composite unit that can be made from it in iteration.

[iv] Note that such a child can use part-whole meanings of fractions to distinguish between iterating 1/11 three times to produce 3/11, and iterating 1 three times to produce 3. However, this way of producing fractions is not available in the case of 12/11, since part-whole meanings for 12/11 are problematic.

[v] My thanks to Andy Norton for this illuminating comment at a recent conference.

[vi] They may not necessarily be aware that one-fifth times the length of the given bar produces a length equal to their bar, depending on the state of their fractional language and fractional schemes.

[vii] I use *composite unit* to mean a unit consisting of units. For example, 10 is a unit of 10 units.

[viii] They can use iterating in this way because their units of one are iterable.

[ix] A scheme consists of an assimilated situation, an activity, and a result (von Glasersfeld, 1995).

[x] Note that these students have constructed at least an explicitly nested number sequence (Steffe, 1988). For these students, five consists of one iterated five times, rather than just being the result of a count of five items. Thus *one* is an iterable unit for these students.

[xi] Note that for these students, 20 consists not just of one iterated twenty times but of four iterated five times or five iterated four times, etc. That is, these students have constructed *composite units* as iterable units.

[xii] The purpose of the experiment was to investigate students' construction of algebraic reasoning based on their evolving quantitative reasoning in the context of a teacher who endeavors to create a certain quality of interaction with the students that I call *mathematical caring relations* (MCRs) (Hackenberg, 2005a, 2005b, 2005c). This article only focuses on a portion of the analysis and does not explicitly address algebraic reasoning or MCRs.

[xiii] Another witness-researcher was also planning to be a teacher-researcher with four other students, so a total of eight students were being selected.

[xiv] That is, each girl solved a problem like this one: Twenty-one dollars is three-sevenths of my money; how much money do I have?

[xv] JavaBars is a Java version of TIMA: Bars, one of the three computer microworlds developed as part of the teaching experiment *Children's Construction of the Rational Numbers of Arithmetic* (Steffe & Olive, 1990) which took place from 1990-1995 at The University of Georgia. For more information on the design of JavaBars, see Biddlecomb (1994).

[xvi] The rationale for posing a task like this one, as opposed to asking a student to make a bar 2/15 longer than a unit bar, is that a student can solve the latter task without having to think about the total number of fifteenths necessary to complete it. That is, the student can rely on partitive fractional meanings of 2/15 in conjunction with a whole bar, 1 and 2/15, without coordinating these ideas with 17/15.

[xvii] My main goal in posing these problems involved the larger purpose of the experiment, investigation of the students' construction of algebraic reasoning.

[xviii] In the protocols, B stands for Bridget, D for Deborah, C for Carlos, M for Michael, T for the teacher-researcher (the author), and W for a witness-researcher. Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher's perspective.

[xix] Five-sevenths can be viewed or taken as three-levels-of-units structure: as a unit of five units, any of which can be iterated seven times to produce the whole (another unit). However, this view is not necessary to make five-sevenths, at least in a partitive sense. As noted previously, prior to interiorizing three levels of units, students can use their iterable units of one to iterate one-seventh five times and then rely on part-whole relationships to give meaning to the result.

[xx] Visual material on the screen may have helped him determine these locations because a same-sized bar above had been colored into fourths on a previous problem.

[xxi] I discuss other reasons for her lack of experimentation elsewhere (Hackenberg, 2005a, 2005b).

[xxii] In solving Task 7, Bridget used her splitting operation to reason reversibly with her partitive fractional scheme. Discussion of the splitting operation as pivotal in reasoning reversibly is beyond the scope of this paper but is addressed elsewhere (Hackenberg, 2005a).

[xxiii] In particular, development is likely involved in the interiorization of three levels of units, and so its construction may not be engendered solely by instruction. By development I am referring to maturation of the nervous system (cf. Piaget, 1964).

References

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis – emphasis on the operator construct. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum Associates.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Biddlecomb, B. D. (1994). Theory-based development of computer microworlds. *Journal of Research in Early Childhood Education, 8*(2), 87-98.

Biddlecomb, B., & Olive, J. (2000). JavaBars [Computer software]. Retrieved June 4, 2002 from http://jwilson.coe.uga.edu/olive/welcome.html

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In R. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-265). Hillsdale, NJ: Erlbaum.

Davis, G., Hunting, R. P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior, 12*, 63-76.

Hackenberg, A. J. (2005a). *Construction of algebraic reasoning and mathematical caring relations*. Unpublished doctoral dissertation, University of Georgia.

Hackenberg, A. J. (2005b). Mathematical caring relations as a framework for research and learning. In Lloyd, G. M., Wilson, M., Wilkins, J. L. M., Behm, S. L. (Eds.), *Proceedings of the Twenty-seventh Annual Meeting of the North American Chapter of the*

International Group for the Psychology of Mathematics Education [CD-ROM]. Roanoke, VA: Virginia Polytechnic Institute and State University.

Hackenberg, A. J. (2005c). A model of mathematical learning and caring relations. *For the Learning of Mathematics, 25*(1), 45-51.

Kamii, C., & Housman, L. B. (2000). *Young children reinvent arithmetic* (2nd ed.). New York: Teachers College Press.

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In J. Confrey & G. Harel (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany, NY: State University of New York Press.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In M. J. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31-65). Albany, NY: State University of New York Press.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*(5), 422-441.

Mack, N. K. (2001). Building on informal knowledge through a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-295.

Olive, J., & Steffe, L. P. (2002). The construction of an iterative fractional scheme: The case of Joe. *Journal of Mathematical Behavior, 20*, 413-437.

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), *Piaget rediscovered: A report of the conference on cognitive studies and curriculum development* (pp. 7-20). Ithaca, NY: School of Education, Cornell University.

Pitkethly, A., & Hunting, R. P. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*, 5-38.

Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education, 25*(1), 50-85.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In M. J. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*, 267-307.

Steffe, L. P. (2003). Fraction commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior, 22*, 237-295.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129-162.

Steffe, L. P. (in press). Fraction monograph. *Journal of Mathematical Behavior*.

Steffe, L. P., & Olive, J. (1990). *Children's construction of the rational numbers of arithmetic*. National Science Foundation. The University of Georgia. Athens, GA.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Hillsdale, NJ: Erlbaum.

Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (Vol. 4, pp. 8-32). London: Falmer.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht: Kluwer Academic.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.