THE LOW-INTENSITY LIMIT OF BERNOULLI-VORONOI AND POISSON-VORONOI MEASURES

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With love and affection, I dedicate this thesis to my parents.

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Sandeep Bhupatiraju

THE LOW-INTENSITY LIMIT OF BERNOULLI-VORONOI AND POISSON-VORONOI MEASURES

On a graph G the Bernoulli-Voronoi measure is an edge percolation process obtained as a result of the following procedure; first select vertices by a Bernoulli site percolation,with intensity p, as nuclei for a Voronoi tessellation and then delete edges whose end vertices lie in different Voronoi tiles. A natural question which arises in this context is the existence of the weak limit as the intensity of the nuclei tends to zero. In this work, we study the existence and properties of this limit on d-regular trees and other infinite Cayley graphs. A related model in the continuous setting is the Poisson-Voronoi model, which can be defined on any metric space. In this setting, the nuclei of the Voronoi tiles are obtained by a Poisson process, with intensity lambda, and the random set we study is the boundary of the Voronoi tiles. We study various properties of the low-intensity limit, as lambda tends to zero, of this measure in the hyperbolic space.

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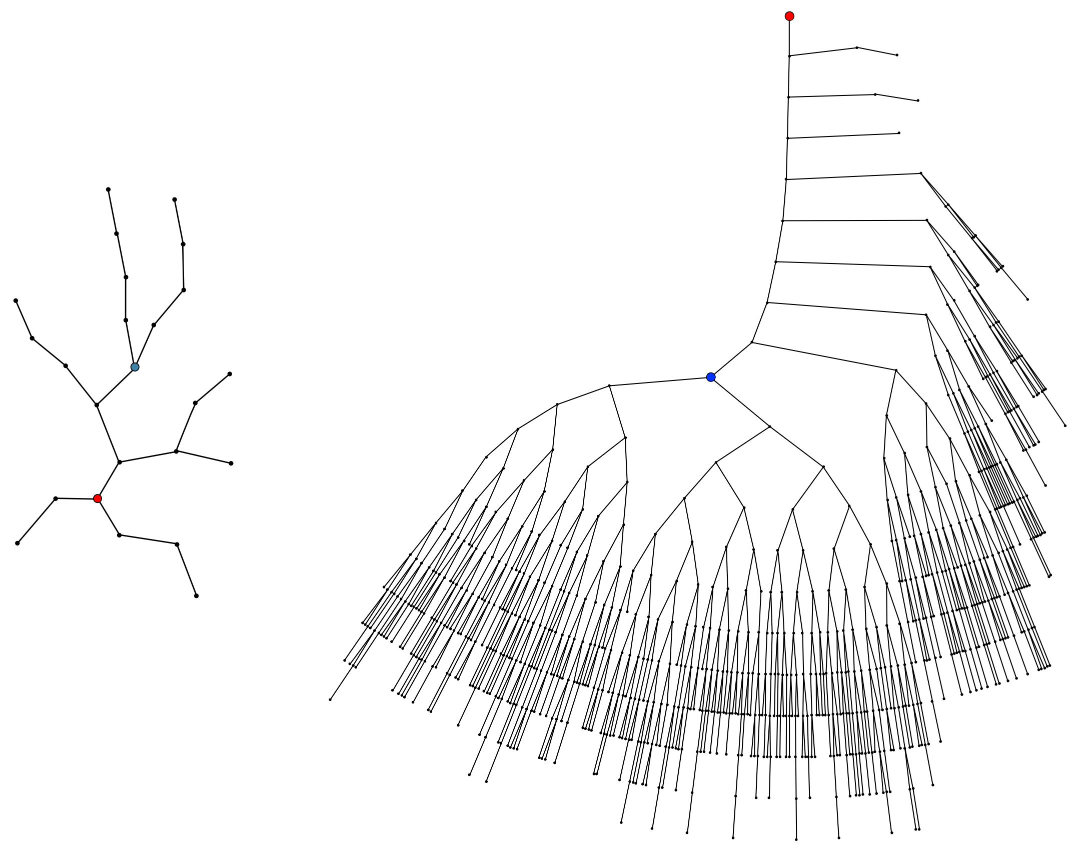
Chapter 1

Introduction

For a set of points P in a metric space, we have a natural partition of the space into cells, called the *Voronoi tessellation* associated to P. Each cell contains exactly one point from P, which we call the nucleus of the cell, and all points of the space closest to that nucleus. In other words, we decompose the space into 'regions of influence' of the points in P. The boundaries of these cells consist of points that are equidistant to two or more nuclei.

If there was a random procedure for producing these nuclei, we would obtain a random tiling of the space, or more generally a partition of the space into random convex polytopes. In this thesis, the main object of study is the associated random tiling produced by a natural point process, with some density parameter p, of the space. This set-up prompts natural questions about the limit object obtained when the density of the nuclei is reduced to zero. The bulk of this thesis explores the existence and properties of such limit tessellations on some graphs and continuous metric spaces.

In discrete metric spaces, such as graphs, a natural method of producing nuclei is to use an independent Bernoulli coin toss, with parameter $p \in [0,1]$ to decide if a vertex is a nucleus, i.e., Bernoulli site percolation, and then obtain the resulting Voronoi cells. Here, if a point is equidistant to multiple nuclei, we use a rule to break ties and as a result, we have a partition of the vertex set. In other words, we obtain an equivalence relation on the vertex set where two vertices are in the same equivalence class if they are closest to the same nucleus. The Voronoi decomposition is obtained by retaining edges whose end vertices are in the same cell, in which case we call the edge *open*, and deleting edges whose end vertices are in different cells, in which case we call them *closed*. This results in an edge percolation, i.e., a measure on the space of subgraphs of the given graph, which we denote by μ_p and call the *Bernoulli-Voronoi measure* with intensity p. Figure 1.1 below depicts the connected component of a fixed vertex, also called a *Voronoi cluster*, for different intensities p. As far as we know this model has not received any attention in the literature.

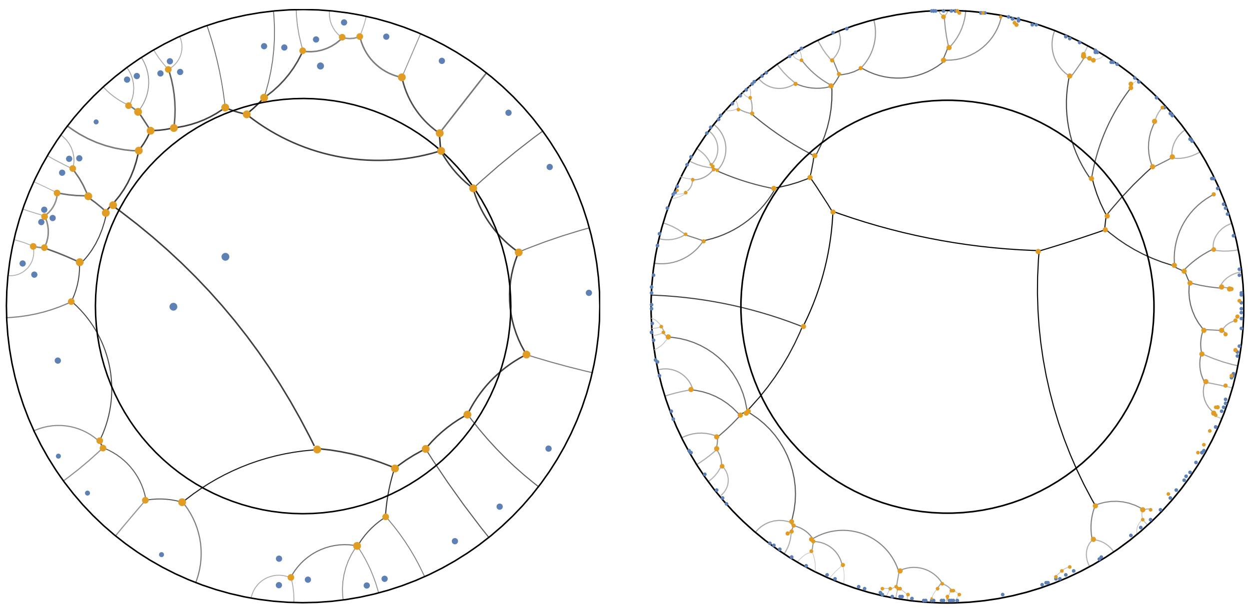


(a) $p = 0.05$

(b) $p = 0.005$

Figure 1.1: The Voronoi cell of a fixed vertex in the 3-regular tree, where the nuclei are generated by a Bernoulli(p) process. The fixed vertex is shown in red and the nucleus of its cell is shown in black.

In continuous metric measure spaces, one natural method of producing random nuclei is the *Poisson point process*. The Voronoi diagram arising from the Poisson points is called the Poisson-Voronoi tessellation. Figure 1.2 below depicts samples of the tessellation for different intensities.



(a) $\lambda = 10^{-2}$ **(b)** $\lambda = 10^{-7}$

Figure 1.2: Poisson-Voronoi diagram, for various λ, where the simulation is complete within the Euclidean ball of radius 0.7, centered at $o \in \mathbb{D}$. The nuclei are depicted by the blue points and the vertices of the cells by orange points. For small values of λ, typically, the nuclei are sparse and when depicted in the Poincaré disk they tend to be very close to the unit circle.

This model has received intense scrutiny as it provides a natural partition of the space, while still capturing much of the geometry and making it computationally amenable. As a result, it is widely used in various fields outside of mathematics and goes by the names *Thiessen polygons* in meteorology, *Wigner-Seitz zones* in chemistry and *domain of action* in crystallography. As an object of study within stochastic geometry, it is most widely studied in Euclidean spaces $\mathbb{R}^d$, and much attention has been devoted to various statistics relating to the shape of the Poisson-Voronoi cells; for instance see [KKBS92], [Cal03] and the monograph [Ml94] by Møller. In computational geometry, the main objective has been to find efficient algorithms for computing the Voronoi boundaries for a given set of points in $\mathbb{R}^d$, the most famous of which is Fortune's sweep line algorithm; see [AK00], and [Aur87]. Interest in extending the study of Voronoi diagrams to different metric spaces has sparked interest in efficient computation of the Voronoi diagram in the Poincare disk and this continues to be an active research problem; see [NN10], [OT96] and [IIM85].

Benjamini and Schramm [BS01] laid the foundation for the study of percolation models on Poisson-Voronoi tilings in the Hyperbolic setting. More recenty, Benjamini, Paquette and Pfeffer [BPP+18] have studied the properties of the simple random walk on the graph obtained from the Poisson-Voronoi boundaries and have shown that this graph has a positive anchored expansion constant and as a consequence the random walk is ballistic. Amid this activity, questions about the low-intensity limit of the Poisson-Voronoi tiling, analogous to those mentioend earlier, remains unexplored. Here, we explore the low-intensity limit of the Poisson-Voronoi tessellation on the Poincare disk and explicitly compute some limiting probabilities.

This thesis is roughly divided into two parts. The first part is devoted to the study of the Bernoulli-Voronoi model on $d-$regular trees and infinite vertex-transitive graphs. The next is devoted to the study of the limit Poisson-Voronoi tessellation in the Poincare disk.

Chapter 2

Bernoulli–Voronoi model

For any graph $G = (V, E)$, we can define the following natural collection of measures $\{\mu_p\}$, for $p \in (0, 1]$, on the set of subgraphs of G. Let $\{X_v\}$, $v \in V$, be a collection of i.i.d. $U[0, 1]$ random variables and fix $0 < p \leq 1$. Call X_v the *label* of the vertex v and call v a *nucleus* if $X_v < p$. We associate to every vertex its closest nucleus, in the graph metric. If there is a tie, associate the vertex to the nucleus with the smallest label. Using this procedure we obtain a partition of the vertex set V. Subsequently, we obtain an edge percolation μ_p by deleting the edges whose end vertices are associated to different nuclei. Call the edges *closed* if they are deleted in this procedure, and *open* if they are not deleted. This procedure involves a Bernoulli site percolation with intensity p, followed by a Voronoi partition of the vertices, along with a tie-breaking rule, and for this reason we refer to this measure μ_p as the Bernoulli–Voronoi measure with intensity p.

If G is finite and $p < 1$, there is a positive probability that there are no nuclei, and hence a Voronoi decomposition is not well defined. We could, in this case define all edges to be open, but here we are interested in low intensity measures, and hence restrict ourselves to the class of infinite graphs, where the construction of μ_p is unambiguous.

We note that μ_1, is the Dirac measure supported on the null subgraph $H = (V, \phi)$, since every vertex is a nucleus almost surely. When $0 < p < 1$, μ_p is supported in the set of subgraphs all of whose connected components are finite. A natural question which arises is whether we can define μ_0. Note that μ_0 cannot be defined by the above procedure, as there would be no nuclei to define the percolation. However, if it exists, a natural candidate for such a measure is the weak limit $\lim_{p \to 0} \mu_p$.

We note that the existence of the limit measure is considerably simpler in the case of

$\mathbb{Z}$. In this case the measure is trivial, i.e., it is supported only on the graph $\mathbb{Z}$. Since the measure is automorphism invariant, it suffices to show that the probability of a having a closed edge adjacent to the root o tends to zero, as $p \to 0$. We construct a sample from μ_p by generating the labels of the vertices in the following order; first generate X_0 at time 0, then at time $2n-1$ generate X_n and at time $2n$ generate X_{-n}. Continue this until the first time a nucleus appears. If the nucleus appears at time $2n$, then at times $2n+1, 2n+2, 2n+3$ generate $X_{-n}, X_{-n-1}, X_{-n-2}$, followed by all other labels in any order. Similarly, if the first nucleus appears at time $2n-1$, then generate X_n, X_{n+1}, X_{n+2}, followed by all other labels. We observe that for an edge adjacent to 0 to be closed, we need to have at least one nucleus amongst the three labels generated after the first nucleus. This probability is bounded above by $1-q^3$, where $q = 1-p$. Hence as $p \to 0$ the probability of observing a closed edge adjacent to o under μ_p vanishes. This shows that the limit measure exists and is trivial.

Existence of μ_0 on $\mathbb{T}_d$

It is not a priori clear that $\lim_{p\to 0} \mu_p$ exists on a given infinite graph. We restrict our attention to the class of vertex-transitive infinite graphs and in particular to the class of d-regular trees $\mathbb{T}_d$, where explicit calculations are possible. The main result in this section is the existence of the weak limit of the Bernoulli–Voronoi measure μ_0 on d-regular trees as stated below.

Theorem 2.1. *Let $\mathbb{T}_d$ denote the d-regular tree with $d \geq 3$. Then the measures μ_p converge weakly to a non-trivial limiting measure μ_0, in the sense that $\mu_p(C) \to \mu_0(C)$ as $p \to 0$ for every cylinder event C.*

Proof of the existence of μ_0 on $\mathbb{T}_d$

For the existence of the weak limit it suffices to show that for any elementary cylinder set C, $\lim_{p\to 0} \mu_p(C)$ exists. Elementary cylinder sets in the space $\Omega = \{0,1\}^E$ are of the form $C = C(F,K) = \big\{\omega \in \Omega \;\big|\; \omega(f) = 1,\ \omega(k) = 0,\ \forall f \in E(F),\ \ k \in E(K)\big\}$, for any fixed finite subgraphs F and K of $\mathbb{T}_d$. Here we are declaring the status of a finite set of edges, where the edges in F are open and those in K are closed. Using Kolmogorov's consistency theorem, this would imply the existence of the measure μ_0.

For fixed finite sets F and K the probability of finding nuclei in $F \cup K$ decays to zero as the intensity p decreases. In addition, from the definition of the model, every vertex is associated to a nucleus and consequently the path from any vertex to its associated nucleus is open. Hence, the probability of finding finite open components decays to zero in the limit. By the inclusion-exclusion principle it suffices to consider cylinder sets where $F \cup K$ is connected. Furthermore, we can assume that the connected components $\{F_i\}_1^k$ of F satisfy the property that for any given F_i there is another component F_j such that $d(F_i, F_j) = 1$. Here, K is given by the collection of edges separating the components of F, and we allow components F_i to be singleton vertices. We show that the low-intensity limit of such cylinder sets exists.

Let $\mathcal{G}_i$ be the component of $\mathbb{T}_d \setminus K$ containing the component F_i. Let $B(v, n)$ denote the ball of radius n centered at v, and let ∂H denote the external vertex boundary of the subgraph H and $\partial_i(H) := \mathcal{G}_i \cap \partial(H)$. From the discussion above, we can assume $\mathcal{G}_i$ is infinite for all $1 \leq i \leq k$. We note that the nucleus c_i associated to F_i must lie in $\mathcal{G}_i$, since otherwise the components will not all be associated to distinct nuclei, and thereby contradicting the condition that all edges of K are closed.

The strategy here is as follows. First, we find a finite disjoint collection of $(d-1)$-ary subtrees in $\mathcal{G}_i$ that contains all but finitely many vertices of $\mathcal{G}_i$, for each $i \in \{1, \ldots, k\}$. Next we pick a subtree from each of the collections in $\{\mathcal{G}_i\}_1^k$ and we compute the μ_p probability that each of the components F_i is open and associated to nuclei in those trees respectively. We show that this probability is obtained as a finite sum of telescoping sums and as a result the existence of the limit in p is easy to establish. Finally, by repeating the above computation for all the finitely many possible choices of subtrees of $\mathcal{G}_i$ and taking their sum we obtain the required cylinder probability.

For $v \in F_1$, let $t_v := \max_{u \in \bigcup_1^k F_i} d(u,v)$ and define

$$\mathrm{Rt}_1 := \partial_1\Big(\bigcup_{v \in F_1} B(v, t_v) \Big). \tag{2.1}$$

For any $x_1 \in \mathrm{Rt}_1$, let T_{x_1} denote the $(d-1)$-ary subtree rooted at x_1 and obtained by deleting the unique neighbor of x_1 closest to F_1. The collection of such subtrees with roots in Rt_1 is the collection in $\mathcal{G}_1$ associated to F_1 mentioned above. The collections of subtrees associated to the other components depend on a choice of a subtree in $\mathcal{G}_1$, or equivalently the choice of a vertex in Rt_1. For $x_1 \in \mathrm{Rt}_1$ and $2 \le j \le k$, define

$$\mathrm{Rt}_{j,x_1} := \partial_j\Big(\bigcup_{v \in F_1} B\big(v, d(v, x_1) - 1\big) \Big). \tag{2.2}$$

Now, for a given vertex $x_1 \in \mathrm{Rt}_1$, the collection of subtrees from $\mathcal{G}_j$ associated to F_j, is given by the collection of subtrees $T_{x_j}$with $x_j \in \mathrm{Rt}_{j,x_1}$. The notation Rt is used to indicate that these sets of vertices serve as roots of subtrees of interest. The motivation for these definitions is to obtain a method of organization for the calculation of the cylinder probability by controlling the positions of the nuclei $\{c_i\}_i$.

Choose $x_1 \in \mathrm{Rt}_1$ and $x_i \in \mathrm{Rt}_{i,x_1}$ for $i = 2, \ldots, k$. We compute the probability that the nuclei c_i are in T_{x_i}, and the components F_i are all open. The condition that all edges in K are closed is an automatic consequence of this construction since the end points of any edge in K lie in different components and hence are associated to different nuclei. By taking a sum, over all choices of $(x_1, \ldots, x_k)$ of the above probabilities we obtain an approximation of the cylinder probability when $p > 0$, which converges to the total probability in the limit as $p \to 0$. Hence, we have

$$\lim_{p \to 0} \mu_p\Big(\omega(F_i) = 1,\ c_1 \in T_{x_1},\ 1 \le i \le k\Big) = \lim_{p \to 0} \sum \mu_p\Big(\omega(F_i) = 1,\ c_i \in T_{x_i},\ 1 \le i \le k\Big),$$

where the sum is over the set $(x_2,\dots,x_k)\in\prod_2^k \mathrm{Rt}_{j,x_1}$ and

$$\{\omega(F_i)=1\} \coloneqq \{\omega(f)=1,\ \forall f\in E(F_i)\}.$$

Taking the sum over $x_1\in\mathrm{Rt}_1$ in the above equation, we obtain

$$\lim_{p\to 0}\ \mu_p\big(C(F,K)\big)=\lim_{p\to 0}\sum_{x_1\in\mathrm{Rt}_1}\sum \mu_p\Big(\omega(F_i)=1,\ c_i\in T_{x_i},\ 1\le i\le k\Big), \tag{2.3}$$

where the second sum is over is $(x_2,\dots,x_k)\in\prod_2^k \mathrm{Rt}_{j,x_1}$. Hence, in order to prove Theorem 2.1, it suffices to prove the following proposition.

Proposition 2.1. *Let $\{F_i\}_1^k$ be a collection of finite connected subgraphs satisfying the condition that for any $i\in\{1,\dots,k\}$ there exists $j\in\{1,\dots,k\}$ such that $d(F_i,F_j)=1$. For $x_1\in Rt_1$ and $x_i\in Rt_{i,x_1}$, for $2\le i\le k$, the following limit exists:*

$$\lim_{p\to 0}\ \mu_p\Big(\omega(F_i)=1,\ c_i\in T_{x_i},\ 1\le i\le k\Big). \tag{2.4}$$

Before proving Proposition 2.1 in full generality, we prove two special cases of it: when F consists of a single connected component F_1, and next when F consists of two connected components F_1 and F_2. Calculations in these cases are illustrative of the general case, and in addition we obtain explicit formulas for computing probabilities under μ_0 for simple events which are of interest in themselves.

We introduce some notation and preliminary lemmas. Since vertices are associated to nuclei by the comparison of labels of vertices we write, $M(A)\coloneqq\min\{X_a\colon a\in A\}$ for any finite set $A\subset V$. We make use of the following simple lemma for comparing the minimal labels in two disjoint sets.

Lemma 2.1. *Let A,B be two disjoint sets with $|A|=a$ and $|B|=b$. We have*

$$\mathbb{P}_p\big(M(A)\le M(B)\big)=\frac{a}{a+b}.$$

Now, we show that the asymptotic probability that a finite connected subgraph is open exists.

Proposition 2.2. *For a connected finite subgraph F_1, $\lim_{p\to 0} \mu_p\big(\omega(F_1) = 1\big)$ exists.*

Proof. For all the edges of F_1 to be open, we need all the vertices of F_1 to be associated to the same nucleus. For the event in the proposition to hold, we fix the position of the nucleus c_1 associated to F_1, which in turn determines two sets of vertices: one which must not contain any nuclei, and another which must contain no nuclei with a label smaller than that of c_1. By summing the probability of this event over all the possible positions of c_1, we obtain the probability that F_1 is open. We formalize this argument below.

We organize the calculation by controlling the position of the nucleus c_1 as indicated previously. In this case, $\mathcal{G}_1 = \mathbb{T}_d$ and $\mathrm{Rt}_1 = \partial\Big(\bigcup_{v\in F_1} B(v,t_v)\Big)$. Fix a vertex x in Rt_1. Associated to this we define $N_{1,x} := \bigcup_{v\in F_1} B\big(v, d(v,x)-1\big)$ and $N_{1,x}(n) := \bigcup_{v\in F_1} B\big(v, d(v,x)-1+n\big)$ for $n \geq 0$. In particular, $N_{1,x}(0) = N_{1,x.}$ Let $T_x(n)$ denote the vertices at depth n in the tree T_x, and define $\mathcal{T}_{1,x}(n) := \partial_1\big(N_{1,x}(n)\big)$ for all $n \geq 0$. When the dependence on x is clear, we will write $\mathcal{T}_1$ instead of $\mathcal{T}_{1,x}$. Consequently, we have $T_x(n) \subset \mathcal{T}_1(n)$. In Figure 2.1 below, we illustrate the various sets appearing in the computation.

If the nucleus c_1 were to be one of the vertices of $T_x(n)$, then we must have $M\big(T_x(n)\big) < p$, $M\big(T_x(n)\big) < M\big(\mathcal{T}_1(n)\setminus T_x(n)\big)$ and $M\big(N_{1,x}(n)\big) > p$. Here $N_{1,x}(n)$ indicates the region which must be forced to have no nuclei if c_1 lies in $T_x(n)$. Hence, we can write the following equation:

$$
\begin{aligned}
&\mu_p\big(\omega(F_1) = 1,\ c_1 \in T_x\big) \\
&\quad = \sum_{n=0}^{\infty} \mathbb{P}\Big(M\big(N_{1,x}(n)\big) > p,\ M\big(T_x(n)\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_1(n)\setminus T_x(n)\big)\Big) \\
&\quad = \sum_{n=0}^{\infty} \mathbb{P}\Big(M\big(N_{1,x}(n)\big) > p\Big)\ \mathbb{P}\Big(M\big(T_x(n)\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_1(n)\setminus T_x(n)\big)\Big) \qquad (2.5)\\
&\quad = \sum_{n=0}^{\infty} q^{|N_{1,x}(n)|}\ \mathbb{P}\Big(M\big(T_x \cup (\mathcal{T}_x(n)\setminus T_x(n))\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_1(n)\setminus T_x(n)\big)\Big) \\
&\quad = \sum_{n=0}^{\infty} q^{|N_{1,x}(n)|}\ \mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < p\Big)\mathbb{P}\Big(M\big(T_x(n)\big) < M\big(\mathcal{T}_1(n)\setminus T_x(n)\big)\Big)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} q^{|N_{1,x}(n)|} \left(1 - q^{|\mathcal{T}_1(n)|}\right) \frac{|T_x(n)|}{|\mathcal{T}_1(n)|}, \tag{2.6}$$

where we used the fact that the minima of labels over disjoint sets are independent to write the product in (2.5) and used Lemma 2.1 to simplify the second term in that equation.

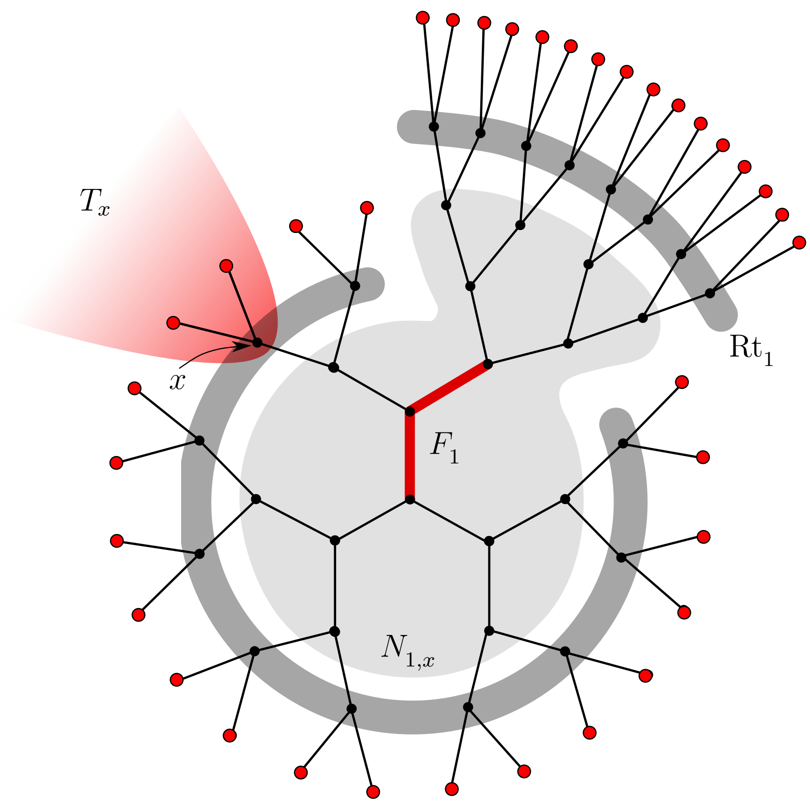


Figure 2.1: The component F_1 is denoted by the thick red edges. The binary tree T_x with its root at x is enclosed in the red shaded region. The vertices within the light grey region form the set $N_{1,x}$, and those within the dark grey region constitute Rt_1. The vertices in red form the set $\mathcal{T}_{1,x}(1)$, and here we assume one of the vertices in $T_x(1)$ is the nucleus c_1.

Since we are working with the tree $\mathbb{T}_d$, we have that $\{|T_x(n)|/|\mathcal{T}_1(n)|\}_n$ is a constant sequence in n, as both the numerator and denominator grow by a factor of $d-1$ as n increases by one. Furthermore, we have $|N_{1,x}(n) \cup \mathcal{T}_1(n)| = |N_{1,x}(n+1)|$. Using these we can simplify (2.6) to obtain

$$\mu_p\big(\omega(F) = 1,\ c_1 \in T_x\big) = \frac{|T_x(0)|}{|\mathcal{T}_1(0)|} \sum_{n=0}^{\infty} \Big(q^{|N_{1,x}(n)|} - q^{|N_{1,x}(n+1)|}\Big) = \frac{1}{|\partial\big(N_{1,x}(0)\big)|}\ q^{|N_{1,x}(0)|}.$$

Taking the sum over all $x \in \mathrm{Rt}_1$, and taking the limit as $p \to 0$, we get

$$\lim_{p\to 0} \mu_p\big(\omega(F_1) = 1\big) = \sum_{x\in \mathrm{Rt}_1} \frac{1}{|\partial N_{1,x}|}, \tag{2.7}$$

which shows that the limit exists. □

As an application of this, consider the case when F is a single edge (u, v) in $\mathbb{T}_d$. We have $\mathrm{Rt}_1 = \partial\big(B(u, 1) \cup B(v, 1)\big)$, and as a result $|\mathrm{Rt}_1| = 2(d-1)^2$. Additionally, for $x \in \mathrm{Rt}_1 \cap \partial B(u, 1)$ we have $N_{1,x}(0) = \partial\big(B(u, 1) \cup B(v, 2)\big)$ and $|\partial N_{1,x}(0)| = d(d-1)^2$. By symmetry we conclude that $\partial N_{1,x}$ is the same for every $x \in \mathrm{Rt}_1$. Hence, using (2.7), we obtain the edge-marginal probability

$$\lim_{p\to 0} \mu_p\big(\omega(u, v) = 1\big) = \frac{2}{d}.$$

We remark that the expression in (2.7) was simple for two reasons. First, we had a natural partition of the vertex set $V(\mathbb{T}_d)$ which allowed us to position the nucleus c_1 conveniently. Next, we had that $\{|T_x(n)|/|\partial\mathcal{T}_1(n)|\}_n$ is a constant sequence, that modified the sum to a telescoping sum. Neither the partitioning nor the simplification to a telescoping sum is available on other graphs, and this restricts our method from being carried out more generally.

Next, we consider another special case, where the cylinder set is described by two adjacent finite connected open components, separated by a closed edge. This case will illustrate some more aspects of why the general proof works. The strategy of the proof is similar to that in previous case. As mentioned earlier, we decompose the event according to the positions of the nuclei c_i associated to the components F_i and show that we obtain telescoping sums.

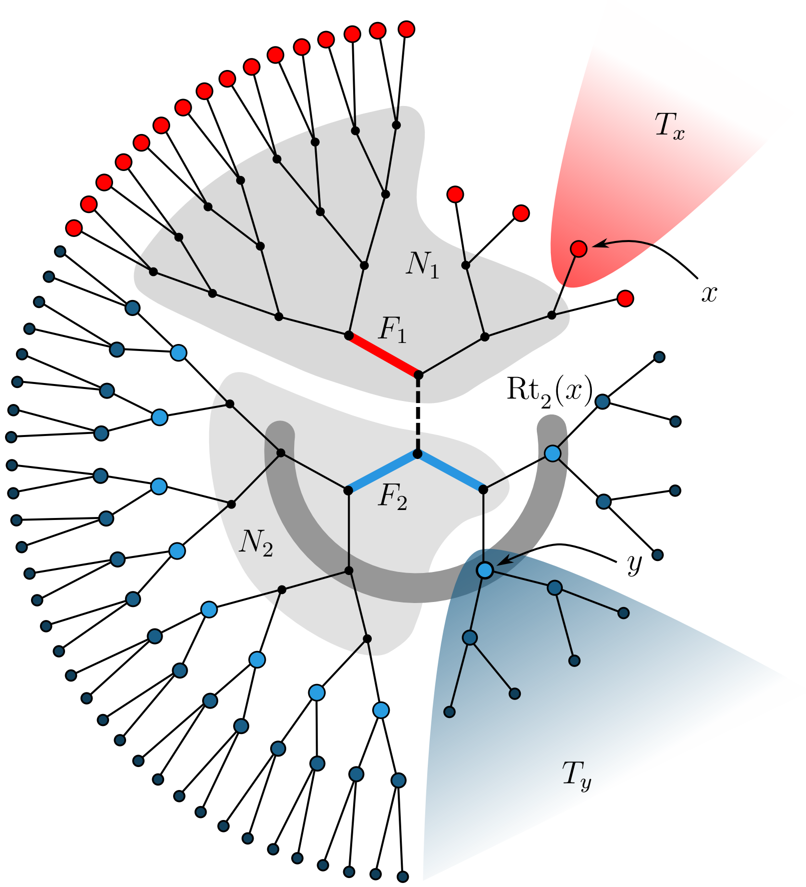


Figure 2.2: The components F_1 and F_2 are colored red and blue respectively. The rooted binary trees T_x and T_y are enclosed in shaded regions. The dashed edge represents the closed edge between the components F_1 and F_2. The red vertices constitute the set $\mathcal{T}_x(0)$ and the vertices depicted with the same size and shade of blue form the set $\mathcal{T}_y(n)$: the lightest shade of blue and largest size represents $n = 0$, the one with the darkest shade and smallest size represents $n = 2$, and the other is $n = 1$. Finally, the vertices within the light grey regions are $N_1(0)$ and $N_2(0)$, and those within the dark grey region are $\mathrm{Rt}_2(x)$.

Since we are interested in the limit as $p \to 0$, the probability that the nuclei c_i associated to F_i lie in any given finite region of the graph can be ignored. To avoid multiple subscripts, let $x := x_1$ and $y := x_2$, with $x \in \mathrm{Rt}_1$ and $y \in \mathrm{Rt}_2(x)$. We will also write $\mathcal{T}_x := \mathcal{T}_{1,x}$ and $\mathcal{T}_y := \mathcal{T}_{2,y}$. To keep track of the region forced to have no nuclei due to the positions of

c_1 and c_2 we define $N_2(y) \coloneqq \bigcup_{v \in F_2} B(v, d(v,y) - 1) \cap \mathcal{G}_2$. Since we fixed y we will write N_2 instead of $N_{2,y}$ when the context is clear. Furthermore, analogous to $N_{1,x}(n)$ we define $N_2(n) \coloneqq \bigcup_{v \in F_2} B\big(v, d(v,y) - 1 + n\big) \cap \mathcal{G}_2$ and $\mathcal{T}_i(n) \coloneqq \partial_i\big(N_i(n)\big)$ for $i = 1,2$. We illustrate these sets in a particular case in Figure 2.2.

We state the two component case of Proposition 2.1 below.

Proposition 2.3. *Let F_1 and F_2 be connected finite subgraphs of $\mathbb{T}_d$ with $d(F_1, F_2) = 1$, and T_x, T_y as described above. Then the following limit exists:*

$$\lim_{p \to 0} \mu_p\big(\omega(F_1) = 1,\, \omega(F_2) = 1,\, c_1 \in T_x,\, c_2 \in T_y\big).$$

Proof. Suppose $c_1 \in T_x(n)$ for some $n \geq 0$. The possible positions for c_2 in T_y are restricted by the position of c_1 and we must have $c_2 \in T_y(n) \cup T_y(n+1) \cup T_y(n+2)$. This follows because otherwise a vertex of F_1 would be closer to c_2, or a vertex of F_2 closer to c_1. The condition $c_1 \in T_x(n)$ forces the set $N_1(n)$ to contain no nuclei. Similarly, the condition $c_2 \in T_y(k)$ forces $N_2(k)$ to contain no nuclei for $k \in \{n, n+1, n+2\}$. Furthermore, we observe that when $c_1 \in T_x(n)$ we must have $M\big(T_x(n)\big) < M\big(\mathcal{T}_x(n) \setminus T_x(n)\big)$. Similarly when $c_2 \in T_y(n+j)$ we must have $M\big(T_y(n+j)\big) < M\big(\mathcal{T}_y(n+j) \setminus T_y(n+j)\big)$, for $j \in \{0,1,2\}$. We can now write the probability that both components F_1 and F_2 are open, with c_1 in T_x and c_2 in T_y as follows:

$$\begin{aligned}
\mu_p\Big(\omega(F_1) = 1,\, \omega(F_2) = 1,\, c_1 \in T_x,\, c_2 \in T_y\Big) &= \sum_{n=0}^{\infty} \mathbb{P}\Big(M\big(N_1(n) \cup N_2(n)\big) > p\Big) \times \\
\times \Bigg[\mathbb{P}\Bigg(M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n)\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_x(n) \setminus T_x(n)\big),& \\
M\big(T_y(n)\big) < M\big(\mathcal{T}_y(n) \setminus T_y(n)\big)\Bigg) +& \\
+ \mathbb{P}\Bigg(M\big(\mathcal{T}_x(n)\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_x(n) \setminus T_x(n)\big), M\big(\mathcal{T}_y(n)\big) > p,\ M\big(\mathcal{T}_y(n+1)\big) < p,& \\
M\big(T_y(n+1)\big) < M\big(\mathcal{T}_y(n+1) \setminus T_y(n+1)\big)\Bigg) +&
\end{aligned}$$

$$+\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big) < p,\ M\big(T_x(n)\big) < M\big(\mathcal{T}_x(n)\setminus T_x(n)\big),$$
$$M\big(\mathcal{T}_y(n)\cup\mathcal{T}_y(n+1)\big) > p,\ M\big(T_y(n+2)\big) < M\big(\mathcal{T}_y(n+2)\setminus T_y(n+2)\big)\Big)\Big].$$

We can simplify this probability by noting that when comparing the minimal labels of two disjoint sets, the comparison is independent of where the minimal labels occur within those sets. Using this, we get

$$\mu_p\big(\omega(F_1)=1,\,\omega(F_2)=1,\,c_1\in T_x,\,c_2\in T_y\big)=\sum_n \frac{|T_x(n)|}{|\mathcal{T}_x(n)|}\,\frac{|T_y(n)|}{|\mathcal{T}_y(n)|}\mathbb{P}\big(M\big(N_1(n)\cup N_2(n)\big)>p\big)\times$$
$$\times\Big[\mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n)\big) < p\Big) + \mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < p, M\big(\mathcal{T}_y(n+1)\big) < p, M\big(\mathcal{T}_y(n)\big) > p\Big)$$
$$+\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big) < p,\ M\big(\mathcal{T}_y(n)\cup\mathcal{T}_y(n+1)\big) > p\Big)\Big].$$

We can now reduce the sum of three terms within the square brackets above into a difference of two. Ultimately, this is the source of the telescoping which we exploit repeatedly. Consider the following decomposition of the events appearing above:

$$\mathbb{1}\Big(M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n)\big) < p\Big) = \mathbb{1}\Big(M\big(\mathcal{T}_x(n)\big) < p\Big)\times$$
$$\times\Big(\mathbb{1}\Big(M\big(\mathcal{T}_y(n)\big) > M\big(\mathcal{T}_x(n)\big)\Big) - \mathbb{1}\Big(M\big(\mathcal{T}_y(n)\big) > p\Big)\Big),$$
$$\mathbb{1}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big) < p\Big) = \mathbb{1}\Big(M\big(\mathcal{T}_x(n)\big) < p\Big)\ \mathbb{1}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big)\Big).$$

Along with Lemma 2.1 and the decomposition above, we can simplify the expression further as

$$\mu_p\big(\omega(F_1)=1,\,\omega(F_2),\,c_1\in T_x,\,c_2\in T_y\big) = \sum_n \frac{|T_x(n)|}{|\mathcal{T}_x(n)|}\,\frac{|T_y(n)|}{|\mathcal{T}_y(n)|}\ q^{|N_1(n)\cup N_2(n)|}\times \tag{2.8}$$
$$\times\,\mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < p\Big)\Big[\mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n)\big)\Big) - \mathbb{P}\Big(M\big(\mathcal{T}_y(n)\big) > p\Big)+$$

$$+\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n)\big) > p\Big)\,\Big(1-\mathbb{P}\Big(M\big(\mathcal{T}_y(n+1)\big) > p\Big)\Big)+$$

$$+\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n)\big) > p\Big)\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n+1)\big) > p\Big)\,\mathbb{P}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big)\Big)\Bigg]$$

Rewriting $\mathbb{P}\Big(M\big(\mathcal{T}_y(n+2)\big) < M\big(\mathcal{T}_x(n)\big)\Big)$ as $1-\mathbb{P}\Big(M\big(\mathcal{T}_y(n+2)\big) > M\big(\mathcal{T}_x(n)\big)\Big)$ and simplifying the expression, we get the following telescoping sum:

$$\mu_p\big(\omega(F_1)=1, \omega(F_2)=1,\, c_1\in T_x,\, c_2\in T_y\big) = \sum_n \frac{|T_x(n)|}{|\mathcal{T}_x(n)|}\frac{|T_y(n)|}{|\mathcal{T}_y(n)|}\; q^{|N_1(n)\cup N_2(n)|}\times$$

$$\times\Bigg[\mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < p,\; M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n)\big)\Big)-$$

$$-\,\mathbb{P}\Big(M\big(\mathcal{T}_x(n)\big) < p,\; M\big(\mathcal{T}_y(n)\cup\mathcal{T}_y(n+1)\big) > p,\; M\big(\mathcal{T}_x(n)\big) < M\big(\mathcal{T}_y(n+2)\big)\Big)\Bigg]$$

$$= \sum_n \frac{|T_x(n)|}{|\mathcal{T}_x(n)|}\frac{|T_y(n)|}{|\mathcal{T}_y(n)|}\; q^{|N_1(n)\cup N_2(n)|}\times\Bigg[\frac{|\mathcal{T}_x(n)|}{|\mathcal{T}_x(n)|+|\mathcal{T}_y(n)|}\;\Big(1-q^{|\mathcal{T}_x(n)|+|\mathcal{T}_y(n)|}\Big)$$

$$-\,\frac{|\mathcal{T}_x(n)|}{|\mathcal{T}_x(n)|+|\mathcal{T}_y(n+2)|}\;q^{|\mathcal{T}_y(n)\cup\mathcal{T}_y(n+1)|}\Big(1-q^{|\mathcal{T}_x(n)|+|\mathcal{T}_y(n+2)|}\Big)\Bigg]. \qquad (2.9)$$

Again, as in the single-component case, the sequences of fractions appearing in the above expression are constant sequences since $|T_x(n)|, |\mathcal{T}_x(n)|, |T_y(n)|$ and $|\mathcal{T}_y(n)|$ grow by a factor of $d-1$ when n is increased to $n+1$. Note that $|N_2(n)|+|\mathcal{T}_y(n)\cup\mathcal{T}_y(n+1)| = |N_2(n+2)|$. Furthermore, the boundary of the region forced to have no nuclei can be decomposed as $\partial\big(N_1(n)\cup N_2(n)\big) = \mathcal{T}_x(n)\cup\mathcal{T}_y(n)$ and $\partial\big(N_1(n)\cup N_2(n+2)\big) = \mathcal{T}_x(n)\cup\mathcal{T}_y(n+2)$. Consequently, the sum in equation 2.9 can be reduced to a difference of two telescoping series. Hence, in the limit as $p\to 0$ we obtain

$$\lim_{p\to 0}\mu_p\big(\omega(F_1)=1,\,\omega(F_2)=1,\,c_1\in T_x,\,c_2\in T_y\big)$$

$$= \frac{1}{|\mathcal{T}_x(0)|\,|\mathcal{T}_y(0)|}\left(\frac{|\mathcal{T}_x(0)|}{|\mathcal{T}_x(0)|+|\mathcal{T}_y(0)|}-\frac{|\mathcal{T}_x(0)|}{|\mathcal{T}_x(0)|+|\mathcal{T}_y(2)|}\right).$$

Summing over the set $\big\{(x,y)\mid x\in \mathrm{Rt}_1,\; y\in\mathrm{Rt}_2(x)\big\}$ we get an explicit expression for the cylinder probability $\lim_{p\to 0}\ \mu_p\big(\omega(F_1)=1,\, \omega(F_2)=1,\, c_1\neq c_2\big)$. $\square$

We now prove the general case of Proposition 2.1.

Proof. Suppose the cylinder set is of the form discussed previously with k connected components given by $F_1,\ldots,F_k$. Let Rt_1 be as before and choose $x_1\in\mathrm{Rt}_1$. This in turn defines $\{\mathrm{Rt}_{i,x_1}\}_2^k$ as given in (2.2). Now, for a choice of $\{x_i\in\mathrm{Rt}_{i,x_1}\}_2^k$, analogous to the definition in the previous cases but more generally, we define

$$N_{1,x_1} \coloneqq \mathcal{G}_1\cap\Big(\bigcup_{v\in F_1} B\big(v,d(v,x_1)-1\big)\Big),\quad N_{i,x_i} \coloneqq \mathcal{G}_i\cap\Big(\bigcup_{v\in F_i} B\big(v,d(v,x_i)-1\big)\Big),\tag{2.10}$$

which help keep track of regions in $\mathcal{G}_i$ devoid of nuclei when c_i are in T_{x_i}. Furthermore, let

$$N_{i,x_i}(n) \coloneqq \mathcal{G}_i\cap\Big(\bigcup_{v\in F_i} B\big(v,d(v,x_i)-1+n\big)\Big).$$

Again, to simplify notation, we write N_i for N_{i,x_i}, N_i for $N_i(0)$, and let $N(n)\coloneqq\bigcup_1^k N_i(n)$, for $n\geq 0$. Finally, let

$$\mathcal{T}_{x_i}(n) \coloneqq \partial_i\big(N_i(n)\big) = \partial_i\Big(\bigcup_{v\in F_i} B\big(v,d(v,x_i)-1+n\big)\Big).$$

If we specify the position of the nucleus c_1 in T_{x_1}, this imposes conditions on the positions of c_i in T_{x_i}, for $2\leq i\leq k$. In particular, if $c_1\in T_{x_1}(n)$, then for a component F_i with $d(F_1,F_i)=m$, we must have $c_i\in\displaystyle\bigcup_{j=n}^{n+2m} T_{x_i}(j)$, else we would have a vertex in F_1 closer to c_i than c_1, or a vertex of F_i closer to c_1 than c_i. Furthermore, if $c_i\in T_{x_i}(n)$ or $c_i\in T_{x_i}(n+2m)$ we would need to compare the labels of c_1 and c_i to obtain the Voronoi decomposition. The position of c_i is also influenced by the position of the nuclei of components F_j which lie on the path from F_i to F_1 or for which F_i lies on the path from F_j to F_1.

The conditions on the labels of the vertices in $\mathcal{G}_i$ when the nucleus associated to F_i lies in $T_{x_i}(n)$, can be summarized by the following three conditions:

- $M\big(T_{x_i}(n)\big) < p$, since c_i lies in $T_{x_i}(n)$,
- $M\big(N_i(n)\big) > p$, since for every vertex $v \in N_i(n)$, we have $d\big(v, F_i\big) < d\big(T_{x_i}(n), F_i\big)$,
- $M\big(T_{x_i}(n)\big) < M\big(\mathcal{T}_{x_i}(n) \setminus T_{x_i}(n)\big)$, since vertices in $\mathcal{T}_{x_i}(n)$ could possibly be nuclei, but with smaller labels than c_i, and c_i lies in $T_{x_i}(n)$.

In addition to these conditions we need to keep track of the configuration of the depths of nuclei c_i in T_{x_i}.

To aid our calculation, we associate to the collection of components $\{F_i\}_1^k$ a finite rooted directed graph (G, o), which we call the *directed graph associated to the components* $\{F_i\}_i$, or more simply the *associated graph.* Let the vertices of G be in one-to-one correspondence with components F_i, with the root $o \in G$ corresponding to F_1. Two vertices are defined to be adjacent in G if the corresponding components are adjacent in $\mathbb{T}_d$. Finally, we direct the edges away from the root to obtain (G, o). Note that the associated graph will always be a finite tree. The associated graph of a collection of seven components is depicted in figure 2.4 below.

For any directed rooted graph (H, o) we define a collection of functions

$$\mathcal{F}_H^n \coloneqq \Big\{ f \colon V(H) \to \mathbb{R} \,\Big|\, f(o) = n,\ f(v) \in \{f(u), f(u)+1, f(u)+2\},\ \forall (u,v) \in \vec{E}(H) \Big\},$$

which we call *admissible functions* on H. We view $f \in \mathcal{F}_G^n$ as prescribing depths in T_{x_i}. Note that when $c_i \in T_{x_i}\big(f(F_i)\big)$ there is a positive probability that all components are open, i.e., $\omega(F_i) = 1$ for all $1 \le i \le k$. Conversely, if $c_1 \in T_{x_1}(n)$, then the depths of c_i within T_{x_i} must be described by some function in $\mathcal{F}_G^n$ in order to ensure that all components F_i are open with positive probability. Anticipating the notational requirements in the proof to follow, we will write $G \coloneqq G_0$, and $\mathcal{F}_G^n = \mathcal{F}_{G_0}^n =: \mathcal{F}_0^n$. In the course of the proof, we will prune G and view the resulting graph as a subgraph of G with the same root. This process is illustrated in Figure 2.3 below. The graphs obtained in this process, which will be defined later, will be denoted by G_j and the associated collection of functions will be denoted by $\mathcal{F}_j^n$.

As a final ingredient in describing the event in Proposition 2.1 we introduce some more notation. The conditions here refine those mentioned in the bullet points above. Let

$$\Phi_{u,v}(n,m) := \begin{cases} \mathbb{1}\big(M(\mathcal{T}_u(n)) < M(\mathcal{T}_v(m)) < p\big) & \text{if } m = n, \\ \mathbb{1}\big(M\big(\mathcal{T}_u(n)\big) < p,\ M\big(\mathcal{T}_v(m)\big) < p\big) & \text{if } m = n+1, \\ \mathbb{1}\big(M\big(\mathcal{T}_v(m)\big) < M\big(\mathcal{T}_u(n)\big) < p\big) & \text{if } m = n+2\ , \end{cases} \tag{2.11}$$

where $n \geq 0$, and $(u,v) \in \vec{E}(G)$. For $0 \leq n \leq m$ and $u \in V(G)$, define

$$I_u[n,m) := \begin{cases} \mathbb{1}\Bigg(M\Big(\bigcup_{j=n}^{m-1} \mathcal{T}_u(j)\Big) > p\Bigg) & \text{if } n < m, \\ 1 & \text{if } n = m, \end{cases} \tag{2.12}$$

and we write $I_u(n) := I_u\big[0,n)$. Finally, for $u \in V(G)$ and $n \geq 0$, let

$$\rho_u(n) := \mathbb{1}\Big(M\big(T_u(n)\big) < M\big(\mathcal{T}_u(n) \setminus T_u(n)\big)\Big).$$

Now, the required probability in Proposition 2.1 can be expressed in this new notation as follows. Taking the sum over the depth n of c_1 in T_{x_1} and varying the depths of c_i within T_{x_i} according to the prescription $f \in \mathcal{F}_0^n$, we write

$$\begin{aligned} \mu_p\Big(\omega(F_i) = 1,\, c_i \in T_{x_i},\, 1 \leq i \leq k\Big) = \sum_{n=0}^{\infty} \sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Bigg[\, \mathbb{1}\Big(M\big(N(0)\big) > p\Big) \times \\ \times \prod_{(u,v) \in \vec{E}_0} \Phi_{u,v}(f(u), f(v)) \prod_{u \in V(G_0)} I_u(f(u))\, \rho_u(f(u))\Bigg]. \end{aligned} \tag{2.13}$$

Here, the term involving $N(0)$ ensures there are no nuclei closer to F_i than c_i, and Φ describes the necessary comparisons of the minimal labels corresponding to the components represented by u and v, while I describes the region in $\mathcal{G}_i$ which is forced to have no nuclei by virtue of the positions of c_i in T_{x_i}.

Although $\prod I_u\big(f(u)\big)$ is independent of $\prod_{(u,v)} \Phi_{u,v}\big(f(u), f(v)\big)$ and $\prod_u \rho(f(u))$ we won't express the expectation above as a product since this suffices for our argument. However, the comparison of labels implicit in $\prod_{(u,v)} \Phi_{u,v}\big(f(u), f(v)\big)$ is independent of where the minimum is achieved, which in turn is expressed in $\prod_u \rho_u(f(u))$. Hence, using this observation and Lemma 2.1, we can simplify (2.13) to obtain

$$\sum_{n=0}^{\infty} \sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Big(\mathbb{1}\Big(M\big(N(0)\big) > p\Big)\Big)\mathbb{E}\Big[\prod_{(u,v)\in\vec{E}_0} \Phi_{u,v}(f(u), f(v)) \prod_{u\in V(G_0)} I_u(f(u))\Big] \prod_{u\in V(G_0)} \mathbb{E}\Big(\rho_u\big(f(u)\big)\Big)$$

$$= \sum_{n=0}^{\infty} \sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Big(\mathbb{1}\big(M\big(N(0)\big) > p\big)\Big)\mathbb{E}\Big[\prod_{(u,v)\in\vec{E}_0} \Phi_{u,v}(f(u), f(v)) \prod_{u\in V(G_0)} I_u(f(u))\Big] \prod_{u\in V(G_0)} \frac{|T_u(f(u))|}{|\mathcal{T}_u(f(u))|}.$$

In addition, for any fixed vertex $u \in V(G_0)$, we have

$$\frac{|T_u\big(f(u)\big)|}{|\mathcal{T}_u\big(f(u)\big)|} = \frac{|T_u(0)|}{|\mathcal{T}_u(0)|} = \frac{1}{|\mathcal{T}_u(0)|},$$

since $|T_u(m+1)| = (d-1)|T_u(m)|$ and $|\mathcal{T}_u(m+1)| = (d-1)|\mathcal{T}_u(m)|$, for all $m \geq 0$. This in turn shows that the ratio above is independent of f. Hence,

$$\mu_p\big(\omega(F_i) = 1,\, c_i \in T_{x_i},\, 1 \leq i \leq k\big) = \Big(\prod_{u\in V(G)} \frac{1}{|\mathcal{T}_u(0)|}\Big)\, \mathbb{E}\Big(\mathbb{1}\Big(M\big(N(0)\big) > p\Big)\Big) \times \tag{2.14}$$

$$\times \sum_{n=0}^{\infty} \sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Big[\prod_{(u,v)\in\vec{E}} \Phi_{u,v}\big(f(u), f(v)\big) \prod_{u\in V(G)} I_u\big(f(u)\big)\Big].$$

Therefore, to prove Proposition 2.1 it suffices to prove that the following limit exists:

$$\lim_{p\to 0} \sum_{n=0}^{\infty} \sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Big[\prod_{(u,v)\in\vec{E}_0} \Phi_{u,v}\big(f(u), f(v)\big) \prod_{u\in V(G_0)} I_u\big(f(u)\big)\Big]. \tag{2.15}$$

We simplify this expression in a manner similar to the two-component case. Call a vertex v an *offspring* of u if $(u,v) \in \vec{E}_0$, and call w a *descendant* of u if there is a directed path from u to w. Let $O(u)$ denote the set of offspring of u and $D(u)$ the set of descendants of u.

Note that by our definition $u \notin D(u)$.

Select a vertex $v_1 \in V(G_0)$ such that all the offspring of v_1 are leaves in G_0 and let $G_1 \coloneqq G_0 \setminus D(v_1)$. We can express $\mathcal{F}_0^n$ as a disjoint union, indexed by functions in $\mathcal{F}_1^n$. First, for any $h \in \mathcal{F}_1^n$, define $\mathcal{A}_0^n(h) \coloneqq \Big\{ f \in \mathcal{F}_0^n \colon f\big|_{G_1} = h \Big\}$. Using this we can write $\mathcal{F}_0^n = \bigcup_{h \in \mathcal{F}_1^n} \mathcal{A}_0^n(h)$. With this decomposition of $\mathcal{F}_0^n$, and rewriting the product over all vertices in $V(G_0)$ as product of the product over vertices in $V(G_1)$ and the product over the offspring of v_1, we can write (2.15) as

$$
\begin{aligned}
&\sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_0} \Phi_{u,v}\big(f(u), f(v)\big) \prod_{u \in V(G_0)} I_u\big(f(u)\big) \Bigg] \\
&\quad = \sum_{h \in \mathcal{F}_1^n} \sum_{f \in \mathcal{A}_0^n(h)} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_1} \Phi_{u,v}\big(h(u), h(v)\big) \prod_{u \in V(G_1)} I_u\big(h(u)\big) \prod_{w \in O(v_1)} I_w\big(h(v_1)\big) \times \\
&\qquad\qquad \times \prod_{w \in O(v_1)} \Phi_{v_1,w}(f(v_1), f(w))\ I_w\big[f(v_1), f(w)\big) \Bigg] \\
&\quad = \sum_{h \in \mathcal{F}_1^n} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_1} \Phi_{u,v}\big(h(u), h(v)\big) \prod_{u \in V(G_1)} I_u\big(h(u)\big) \prod_{w \in O(v_1)} I_w\big(h(v_1)\big) \times \\
&\qquad\qquad \times \Bigg(\sum_{f \in \mathcal{A}_0^n(h)} \prod_{w \in O(v_1)} \Phi_{v_1,w}\big(f(v_1), f(w)\big)\ I_w\big[f(v_1), f(w)\big) \Bigg) \Bigg].
\end{aligned}
\tag{2.16}
$$

Now, we evaluate the sum over $\mathcal{A}_0^n(h)$ below. We claim

$$
\begin{aligned}
&\sum_{f \in \mathcal{A}_0^n(h)} \prod_{w \in O(v_1)} \Phi_{v_1,w}\big(f(v_1), f(w)\big)\, I_w\big[f(v_1), f(w)\,\big) \\
&\quad = \mathbb{1}\Big(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < p \Big) \prod_{w \in O(v_1)} \Bigg[\mathbb{1}\Big(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < M\Big(\mathcal{T}_w(h(v_1))\Big) \Big) - \\
&\qquad\qquad - \mathbb{1}\Big(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < M\Big(\mathcal{T}_w(h(v_1)+2)\Big) \Big)\, I_w\big[h(v_1), h(v_1)+2\,\big) \Bigg] \\
&\quad = \mathbb{1}\Big(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < p \Big) \sum_{i_1 \in \{0,2\}^{O(v_1)}} (-1)^{\frac{1}{2}|i_1|} \Bigg(\prod_{w \in O(v_1)} I_w\big[h(v_1), h(v_1) + i_1(w)\,\big) \Bigg) \times
\end{aligned}
$$

$$\times \mathbb{1}\Bigg(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < M\Big(\bigcup_{w\in O(v_1)} \mathcal{T}_w(h(v_1)+i_1(w))\Big)\Bigg), \quad (2.17)$$

where we define $|i| \coloneqq \sum_{a\in A} |i(a)|$, for any $i \in \{0,2\}^A$. Since we will need this form of the computation again, we state it as a lemma and prove it in detail. By a slight absue of notation we use $\mathcal{A}^n_G(h)$ and $\mathcal{A}^n_0(h)$ interchangeably and write $\mathcal{A}^n_{H_r}(h)$ for $\mathcal{A}^n_r(h)$ to emphasize the underying graph.

Lemma 2.2. *Let (G, o) be the rooted directed graph as before and let v be a vertex such that all its offspring $\{w_j\}_1^r$ are leaves in G. Let $H_0 \coloneqq G \setminus O(v)$, and $H_m \coloneqq H_{m-1} \cup \{w_m\}$ for $1 \le m \le r$. If $h \in \mathcal{F}^n_{H_0}$ and $\mathcal{A}^n_G(h)$ is as before, then*

$$\sum_{f\in\mathcal{A}^n_G(h)} \prod_{j=1}^r \Phi_{v,w_j}\big(f(v), f(w_j)\big)\ I_{w_j}\big[f(v), f(w_j)\big) =$$

$$= \mathbb{1}\Bigg(M\Big(\mathcal{T}_v(h(v))\Big) < p\Bigg) \sum_{i\in\{0,2\}^{O(v)}} (-1)^{\frac{1}{2}|i|}\Bigg(\prod_{j=1}^r I_{w_j}\big[h(v), h(v)+i(w_j)\,\big)\Bigg)\times$$

$$\times \mathbb{1}\Bigg(M\Big(\mathcal{T}_v(h(v))\Big) < M\Big(\bigcup_{j=1}^r \mathcal{T}_{w_j}\big(h(v)+i(w_j)\big)\Big)\Bigg).$$

Proof. By $\mathcal{A}^n_G(h)$ we mean the collection of functions in $\mathcal{F}^n_G$ which coincide with h on the domain of h. We can further partition $\mathcal{A}^n_{H_r}(h)$ as

$$\mathcal{A}^n_{H_r}(h) = \bigcup_{g\in\mathcal{A}^n_{H_{r-1}}(h)} \Big\{ f\in\mathcal{A}^n_{H_r}(h) \ \Big|\ f|_{H_{r-1}} = g\Big\} = \bigcup_{g\in\mathcal{A}^n_{H_{r-1}}(h)} \mathcal{A}^n_{H_r}(g)$$

So, we can write the expression in the statement of the lemma as

$$\sum_{f\in\mathcal{A}^n_{H_r}(h)} \prod_{j=1}^r \Phi_{v,w_j}\big(f(v), f(w_j)\big)\ I_{w_j}\big[f(v), f(w_j)\,\big)$$

$$= \sum_{g\in\mathcal{A}^n_{H_{r-1}}(h)} \ \sum_{f\in\mathcal{A}^n_{H_r}(g)} \prod_{j=1}^r \Phi_{v,w_j}\big(f(v), f(w_j)\big)\ I_{w_j}\big[f(v), f(w_j)\,\big)$$

$$= \sum_{g\in\mathcal{A}^n_{H_{r-1}}(h)} \prod_{j=1}^{r-1} \Phi_{v,w_j}\big(g(v),g(w_j)\big)\ I_{w_j}\big[g(v),g(w_j)\big)\times \tag{2.18}$$

$$\times \sum_{f\in\mathcal{A}^n_{H_r}(g)} \Phi_{v,w_r}\big(f(v),f(w_r)\big)\ I_{w_r}\big[f(v),f(w_r)\big).$$

The sum over $\mathcal{A}_{H_r}(g)$ can be simplified exactly as done in (2.8) in the two-component case. Note that $\mathcal{A}^n_{H_r}(g)$ consists of exactly three elements when $g\in\mathcal{A}^n_{H_{r-1}}(h)$. So, expanding the sum, we get

$$\sum_{f\in\mathcal{A}^n_{H_r}(g)} \Phi_{v,w_r}\big(f(v),f(w_r)\big)\ I_{w_r}\big[f(v),f(w_r)\big) = \mathbb{1}\Big(M\Big(\mathcal{T}_v(f(v))\Big) < M\Big(\mathcal{T}_{w_r}(f(v))\Big) < p\Big)+$$
$$+\,\mathbb{1}\Big(M\Big(\mathcal{T}_v(f(v))\Big) < p,\ M\Big(\mathcal{T}_{w_r}(f(v)+1)\Big) < p\Big)\ I_{w_r}\big[f(v),f(v)+1\big)+$$
$$+\,\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v)+2)\Big) < M\Big(\mathcal{T}_v(f(v))\Big) < p\Big)\ I_{w_r}\big[f(v),f(v)+2\big)$$

$$= \mathbb{1}\Big(M\Big(\mathcal{T}_v(f(v))\Big) < p\Big)\Big[\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v))\Big) > M\Big(\mathcal{T}_v(f(v))\Big)\Big) - \mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v))\Big) > p\Big)+$$
$$+\Big(1-\mathbb{1}\Big(M\big(\mathcal{T}_{w_r}(f(v)+1)\big) > p\Big)\Big)\ I_{w_r}\big[f(v),f(v)+1\big)+$$
$$+\Big(1-\mathbb{1}\Big(M\big(\mathcal{T}_{w_r}(f(v)+2)\big) > M\big(\mathcal{T}_v(f(v))\big)\Big)\Big)\ I_{w_r}\big[f(v),f(v)+2\big)\Big]$$
$$= \mathbb{1}\Big(M\Big(\mathcal{T}_v(f(v))\Big) < p\Big)\Big[\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v))\Big) > M\Big(\mathcal{T}_v(f(v))\Big)\Big)-$$
$$- I_{w_r}\big[f(v),f(v)+2\big)\,\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v)+2)\Big) > M\Big(\mathcal{T}_v(f(v))\Big)\Big)\Big].$$

This follows from noting that

$$\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(f(v)+1)\Big) > p\Big)\ I_{w_r}\big[f(v),f(v)+1\big) = I_{w_r}\big[f(v),f(v)+2\big),$$

and simplifying the resulting expression.

Moreover, note that for every $f\in\mathcal{A}^n_{H_r}(g)$ we have $f(v)=g(v)$ and in addition to this, for

$g \in \mathcal{A}^n_{H_{r-1}}(h)$ we have $g(v) = h(v)$. Hence, replacing $f(v)$ by $h(v)$ in the resulting equation above and substituting it back in (2.18) gives

$$\sum_{f \in \mathcal{A}^n_{H_r}(h)} \prod_{j=1}^{r} \Phi_{v,w_j}\big(f(v), f(w_i)\big)\ I_{w_i}\big[f(v), f(w_i)\,\big) = \mathbb{1}\Big(M\Big(\mathcal{T}_v(h(v))\Big) < p\Big)\times$$

$$\times \Big[\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(h(w_r))\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big) -$$

$$- I_{w_r}\big[h(v), h(v)+2\,\big)\,\mathbb{1}\Big(M\Big(\mathcal{T}_{w_r}(h(w_r)+2)\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big)\Big]\times$$

$$\times \sum_{g \in \mathcal{A}^n_{H_{r-1}}(h)} \prod_{j=1}^{r-1} \Phi_{v,w_j}\big(g(v), g(w_j)\big)\ I_{w_j}\big[g(v), g(w_j)\,\big).$$

We see that in this equation the sum at the end is of the same form as the term that was just simplified. Hence, repeating the same calculation $r-1$ times we obtain the following:

$$\sum_{f \in \mathcal{A}^n_{H_0}(h)} \prod_{j=1}^{r} \Phi_{v,w_j}\big(f(v), f(w_j)\big)\ I_{w_j}\big[f(v), f(w_j)\,\big)$$

$$= \mathbb{1}\Big(M\Big(\mathcal{T}_v(h(v))\Big) < p\Big) \prod_{j=1}^{r} \Big[\mathbb{1}\Big(M\Big(\mathcal{T}_{w_j}(h(v))\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big) -$$

$$- I_{w_j}\big[h(v), h(v)+2\,\big)\,\mathbb{1}\Big(M\Big(\mathcal{T}_{w_j}(h(v)+2)\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big)\Big].$$

Expanding the product of differences above and writing it as a sum over all the $\{0,2\}$ labellings of the offspring $\{w_j\}_1^r$, we obtain

$$\mathbb{1}\Big(M\Big(\mathcal{T}_v(h(v))\Big) < p\Big) \sum_{i \in \{0,2\}^{O(v)}} (-1)^{\frac{1}{2}|i|} \prod_{j=1}^{r} \Big[I_{w_j}\big[h(v), h(v)+i(w_j)\,\big)\times$$

$$\times\,\mathbb{1}\Big(M\Big(\mathcal{T}_{w_j}(h(v)+i(w_j))\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big)\Big].$$

The lemma now follows from fact that

$$\prod_{j=1}^{r} \mathbb{1}\Big(M\Big(\mathcal{T}_{w_j}(h(v)+i(w_j))\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big) =$$

$$\mathbb{1}\Big(M\Big(\bigcup_{j=1}^{r} \mathcal{T}_{w_j}(h(v)+i(w_j))\Big) > M\Big(\mathcal{T}_v(h(v))\Big)\Big). \qquad \square$$

Returning to the proof of Proposition 2.1, we now substitute (2.17) back in to (2.16) obtain

$$\begin{aligned} &\sum_{f\in\mathcal{F}_0^n} \mathbb{E}\Bigg[\prod_{(u,v)\in\vec{E}_0} \Phi_{u,v}\big(f(u),f(v)\big) \prod_{u\in V(G_0)} I_u\big(f(u)\big)\Bigg] \qquad (2.19)\\ &= \sum_{h\in\mathcal{F}_1^n} \mathbb{E}\Bigg[\prod_{(u,v)\in\vec{E}_1} \Phi_{u,v}\big(h(u),h(v)\big) \prod_{u\in V(G_1)} I_u\big(h(u)\big) \prod_{w\in O(v_1)} I_w\big(h(v_1)\big)\mathbb{1}\Big(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < p\Big)\times\\ &\times \sum_{i_1\in\{0,2\}^{O(v_1)}} (-1)^{\frac{1}{2}|i_1|}\Bigg(\prod_{w\in O(v_1)} I_w\big[h(v_1), h(v_1)+i_1(w)\,\big)\Bigg)\times\\ &\qquad\times \mathbb{1}\Bigg(M\Big(\mathcal{T}_{v_1}(h(v_1))\Big) < M\Big(\bigcup_{w\in O(v_1)} \mathcal{T}_w(h(v_1)+i_1(w))\Big)\Bigg)\Bigg]. \end{aligned}$$

Before attempting to simplify the above equation any further, we introduce the pruning procedure that we alluded to earlier. This will help in the inductive argument to follow.

Select a sequence of vertices in $V(G)$ as follows. First, select as v_1 any vertex in G_0 such that all its offspring are leaves in G_0. Let $G_1 = G\setminus O(v_1)$ and select a vertex v_2 from G_1, such that all its vertices are leaves in G_2. In general, after selecting $v_1,\ldots,v_l$, set $S_l \coloneqq \bigcup_{j=1}^{l} O(v_j)$, $G_l \coloneqq G_0 \setminus S_l$, and choose a vertex v_{l+1} in G_l all of whose offspring are leaves in G_l. When this process is completed, we have a sequence $v_1,\ldots,v_{m-1}$, and we are left with a star with the root as its internal vertex. At this stage choose the root o as the vertex v_m. Hence, after removing the leaves of o in $V(G_{m-1})$ we are left with the graph G_m composed of a single vertex and no edges. An illustration of this selection procedure is shown in Figure 2.3 below.

We refer to these chosen vertices v_j as *reducible vertices*, and let $\mathcal{R}_j \coloneqq \{v_1,\ldots,v_j\}$. We call a set of reducible vertices a *complete reducible set* if it contains the root, i.e., the procedure described above is completed.

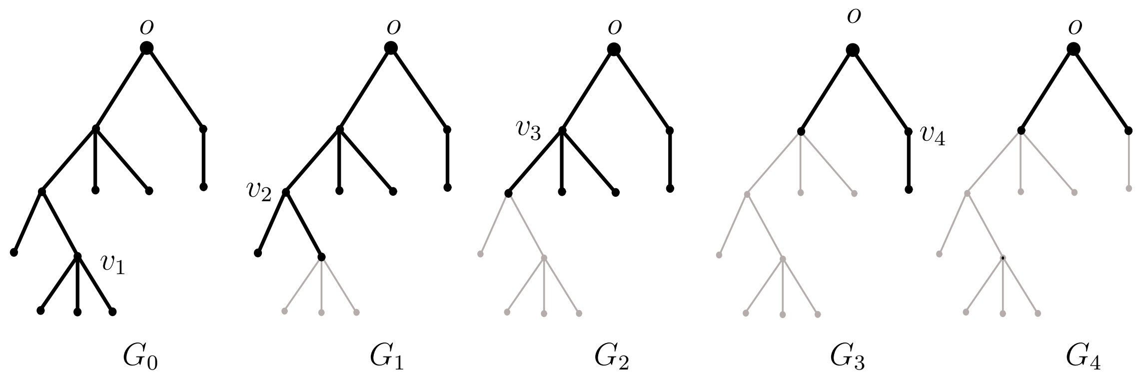


Figure 2.3: Starting with the graph G_0 on the left, we obtain a complete reducible set $\{v_1, \ldots, v_4, o\}$. In each graph we choose a vertex v_i in G_{i-1} such that all its offspring are leaves. In each graph, the root o is shown on top, G_i is depicted in black, and the vertices of S_i are in grey.

Now, given any $\{0, 2\}$-labeling i_j of vertices in S_j, we define the following functions analogous to Φ and I defined previously. First, for $m \geq 0$, define

$$\mathcal{T}_u^{i_j}(m) := \begin{cases} \mathcal{T}_u(m) \cup \bigcup_{w \in D(u)} \mathcal{T}_w\Big(m + \sum_{v \in \mathcal{P}(u,w) \setminus u} i_j(v)\Big) & \text{if } u \in v_j \cup S_j \\ \mathcal{T}_u(m) & \text{if } u \notin v_j \cup S_j, \end{cases}$$

where $\mathcal{P}(u, w)$ denotes the path from u to w in G. Using this, define $\Phi_{u,v}^{i_j}$ on the set of directed edges $\vec{E}_j$ of G_j, for $n \geq 0$ and $m \in \{n, n+1, n+2\}$, as

$$\Phi_{u,v}^{i_j}(n, m) := \begin{cases} \mathbb{1}\big(M\big(\mathcal{T}_u^{i_j}(n)\big) < M\big(\mathcal{T}_v^{i_j}(m)\big) < p\big) & \text{if } m = n, \\ \mathbb{1}\big(M\big(\mathcal{T}_u^{i_j}(n)\big) < p,\ M\big(\mathcal{T}_v^{i_j}(m)\big) < p\big) & \text{if } m = n+1, \\ \mathbb{1}\big(M(\mathcal{T}_v^{i_j}(m)) < M\big(\mathcal{T}_u^{i_j}(n)\big) < p\big) & \text{if } m = n+2. \end{cases}$$

Finally, for $0 \le n < m$, define

$$I_u^{i_j}\big[n,m\big) := \begin{cases} I_u\big[n,m\big) \times \displaystyle\prod_{w \in D(u)} I_w\Big[n,\ m + \sum_{v \in \mathcal{P}(u,w)\setminus u} i_j(v)\Big) & \text{if } u \in v_j \cup S_j, \\ I_u\big[n,m\big) & \text{if } u \notin v_j \cup S_j, \end{cases}$$

and if $n = m$, then $I_u^{i_j}\big[n,m\big) := 1$. As in the previous case let $I_u^{i_j}(m) := I_u^{i_j}\big[0,m\big)$.

We now rewrite (2.19) with this new notation. Note that for any edge (u,v) of G, with $u, v \notin \mathcal{R}_j \cup S_j$, we have $\Phi_{u,v}^{i_j}(n,m) = \Phi_{u,v}(n,m)$. Similarly, for any $u \notin \mathcal{R}_j \cup S_j$ we have $I_u^{i_j}(n) = I_u(n)$. Again, from the definition of I, it follows that

$$I_v\big(h(v_1)\big) \prod_{w \in O(v_1)} I_w\big(h(v_1)\big) I_w\big[h(v_1), h(v_1) + i_1(w)\,\big) = I_{v_1}^{i_1}\big(h(v_1)\big).$$

Integrating over the minimal label in $\mathcal{T}_{v_1}(h(v_1))$, (2.19) reduces to

$$\begin{aligned}
&\sum_{i_1 \in \{0,2\}^{S_1}} (-1)^{\frac{1}{2}|i_1|} \sum_{h \in \mathcal{F}_1^n} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_1} \Phi_{u,v}^{i_1}\big(h(u), h(v)\big) \prod_{u \in V(G_1)} I_u^{i_1}\big(h(u)\big)\, \mathbb{1}\Big(M\Big(\mathcal{T}_v(h(v))\Big) < p \Big) \\
&\qquad\qquad \times \mathbb{1}\Big(M\big(\mathcal{T}_{v_1}\big(h(v_1)\big)\big) < M\big(\mathcal{T}_{v_1}^{i_1}\big(h(v_1)\big) \setminus \mathcal{T}_{v_1}\big(h(v_1)\big)\big) \Big) \Bigg] \\
&= \sum_{i_1 \in \{0,2\}^{S_1}} (-1)^{\frac{1}{2}|i_1|} \sum_{h \in \mathcal{F}_1^n} \frac{|\mathcal{T}_{v_1}\big(h(v_1)\big)|}{|\mathcal{T}_{v_1}^{i_1}\big(h(v_1)\big)|}\, \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_1} \Phi_{u,v}^{i_1}\big(h(u), h(v)\big) \prod_{u \in V(G_1)} I_u^{i_1}\big(h(u)\big) \Bigg].
\end{aligned}$$

The coefficient $|\mathcal{T}_{v_1}\big(h(v_1)\big)|/|\mathcal{T}_{v_1}^{i_1}\big(h(v_1)\big)|$ is not dependent on h since

$$\frac{|\mathcal{T}_{v_1}\big(h(v_1)\big)|}{|\mathcal{T}_{v_1}^{i_1}\big(h(v_1)\big)|} = \frac{|\mathcal{T}_{v_1}(0)|}{|\mathcal{T}_{v_1}^{i_1}(0)|}.$$

Summarizing the calculation up to this point, the probability expressed in Proposition 2.1 is given by

$$\mu_p\Big(\omega(F_i) = 1,\, c_i \in T_{x_i},\, 1 \le i \le k\Big) = \Bigg(\prod_{u \in V(G)} \frac{1}{|\mathcal{T}_u(0)|} \Bigg)\, \mathbb{E}\Big[M\big(N(0) > p\big) \Big] \times$$

$$\times \sum_{i_1 \in \{0,2\}^{O(v_1)}} (-1)^{\frac{1}{2}|i_1|} \frac{|\mathcal{T}_{v_1}(0)|}{|\mathcal{T}_{v_1}^{i_1}(0)|} \sum_{n=0}^{\infty} \sum_{h \in \mathcal{F}_1^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_1} \Phi_{u,v}^{i_1}\big(h(u),h(v)\big) \prod_{u \in V(G_1)} I_u^{i_1}\big(h(u)\big)\Bigg].$$

The double sum above is similar in form to that occurring in (2.15) with the additional modification by i_1 and a different index set; now indexing over admissible functions on G_1 instead of G_0.

In the computation that follows, the idea is to carry out this calculation over the remaining reducible vertices to ultimately obtain a linear combination of telescoping series. For simplicity, let *reduction at* v refer to the procedure of selecting a reducible vertex v and simplifying in the manner above, i.e., splitting the sum over the set $\mathcal{F}$ into a sum over the set $\mathcal{A}$. So, the specific calculation carried out here was the reduction at v_1. Let $\mathcal{R} = \{v_1, \dots, v_{m-1}, v_m\}$, where $v_m = o$, be a fixed complete reducible set for G. For $1 \le r < m-1$, assume that the reduction over the vertex v_r results in

$$\begin{aligned} &\sum_{f \in \mathcal{F}_0^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_0} \Phi_{u,v}\big(f(u),f(v)\big) \prod_{u \in V(G_0)} I_u\big(f(u)\big)\Bigg] \\ &\qquad = \sum_{\{0,2\}^{S_r}} (-1)^{-\frac{1}{2}|i_r|} C_{i_r} \sum_{h \in \mathcal{F}_r^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_r} \Phi_{u,v}^{i_r}\big(h(u),h(v)\big) \prod_{u \in V(G_r)} I_u^{i_r}\big(h(u)\big)\Bigg], \end{aligned} \tag{2.20}$$

where $C_{i_r} := \prod_{j=1}^{r} \frac{|\mathcal{T}_{v_j}(0)|}{|\mathcal{T}_{v_j}^{i_j}(0)|}$, and $i_j := i_r\big|_{S_j}$.

We claim that the reduction over the next vertex v_{r+1} preserves the form above, with r replaced by $r+1$. The following calculation essentially mimics that in the reduction over v_1, with the appropriate attention to the superscripts.

Split the sum over the admissible functions on G_r into smaller sums over $\mathcal{A}_r^n(h)$, indexed by admissible functions h on G_{r+1} to get

$$\sum_{i_r \in \{0,2\}^{S_r}} (-1)^{\frac{1}{2}|i_r|} C_{i_r} \sum_{f \in \mathcal{F}_r^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_r} \Phi_{u,v}^{i_r}\big(f(u),f(v)\big) \prod_{u \in V(G_r)} I_u^{i_r}\big(f(u)\big)\Bigg]$$

$$= \sum_{i_r \in \{0,2\}^{S_r}} (-1)^{\frac{1}{2}|i_r|} C_{i_r} \sum_{h \in \mathcal{F}^n_{r+1}} \sum_{f \in \mathcal{A}^n_r(h)} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_{r+1}} \Phi^{i_r}_{u,v}\big(h(u), h(v)\big) \prod_{u \in V(G_{r+1})} I^{i_r}_u\big(h(u)\big) \times$$
$$\times \prod_{w \in O(v_{r+1})} I^{i_r}_w\big(h(v_{r+1})\big) \prod_{w \in O(v_{r+1})} \Phi^{i_r}_{v_{r+1},w}\big(f(v_{r+1}), f(w)\big)\ I^{i_r}_w\big[f(v_{r+1}), f(w)\,\big)\Bigg]. \tag{2.21}$$

Using the linearity of expectation, we first evaluate the sum over $\mathcal{A}^n_r(h)$. Following the proof of Lemma 2.2, with the modification of ornamenting Φ and I by i_r, we obtain the following equation.

$$\sum_{f \in \mathcal{A}^n_r(h)} \prod_{w \in O(v_{r+1})} \Phi^{i_r}_{v_{r+1},w}\big(f(v_{r+1}), f(w)\big)\ I^{i_r}_w\big[f(v_{r+1}), f(w)\,\big)$$
$$= \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < p\Big) \prod_{w \in O(v_{r+1})} \Bigg[\mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}} h(v_{r+1})\Big) < M\Big(\mathcal{T}^{i_r}_w\big(h(v_{r+1})\big)\Big)\Big) -$$
$$- I^{i_r}_w\big[h(v_{r+1}), h(v_{r+1}) + 2\,\big)\, \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < M\Big(\mathcal{T}^{i_r}_w\big(h(v_{r+1}) + 2\big)\Big)\Big)\Bigg]$$
$$= \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < p\Big) \sum_{j \in \{0,2\}^{O(v_{r+1})}} (-1)^{\frac{1}{2}|j|} \prod_{w \in O(v_{r+1})} I^{i_r}_w\big[h(v_{r+1}), h(v_{r+1}) + j(w)\big) \times$$
$$\times \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < M\Big(\mathcal{T}^{i_r}_w\big(h(v_{r+1}) + j(w)\big)\Big)\Big)$$
$$= \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < p\Big) \sum_{j \in \{0,2\}^{O(v_{r+1})}} (-1)^{\frac{1}{2}|j|} \Bigg(\prod_{w \in O(v_{r+1})} I^{i_r}_w\big[h(v_{r+1}), h(v_{r+1}) + j(w)\big) \Bigg) \times$$
$$\times \mathbb{1}\Big(M\Big(\mathcal{T}^{i_r}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < M\Big(\bigcup_{w \in O(v_{r+1})} \mathcal{T}^{i_r}_w\big(h(v_{r+1}) + j(w)\big)\Big)\Big). \tag{2.22}$$

The above equation can be simplified notationally. Note that $v_{r+1} \notin S_r$ and as a consequence $O(v_{r+1}) \notin S_r$. Since i_r gives a $\{0,2\}$-labeling of S_r and j gives one for $O(v_{r+1})$, let $i_r \bullet j$ denote the labeling of $S_{r+1} = S_r \cup O(v_{r+1})$ that coincides with j on $O(v_{r+1})$ and i_r on S_r. The union of sets appearing in the last term above can be rewritten using $i_r \bullet j$ as follows.

Recall that $D(w)$ represents the set of descendants of w, and let $m \geq 0$. We have

$$\begin{aligned}
\bigcup_{w \in O(v_{r+1})} \mathcal{T}_w^{i_r}\big(m + j(w)\big) &= \bigcup_{w \in O(v_{r+1})} \left[\mathcal{T}_w\big(m + j(w)\big) \cup \bigcup_{u \in D(w)} \mathcal{T}_u\Big(m + j(w) + \sum_{v \in \mathcal{P}(w,u) \setminus w} i_r(v)\Big)\right] \\
&= \bigcup_{w \in O(v_{r+1})} \left[\mathcal{T}_w\big(m + j(w)\big) \cup \bigcup_{u \in D(w)} \mathcal{T}_u\Big(m + j(w) + \sum_{v \in \mathcal{P}(w,u) \setminus w} (i_r \bullet j)(v)\Big)\right] \\
&= \bigcup_{w \in O(v_{r+1})} \left[\mathcal{T}_w\big(m + (i_r \bullet j)(w)\big) \cup \bigcup_{u \in D(w)} \mathcal{T}_u\Big(m + \sum_{v \in \mathcal{P}(w,u)} (i_r \bullet j)(v)\Big)\right].
\end{aligned}$$

Since $\mathcal{P}(w,u) = \mathcal{P}(v_{r+1}, u) \setminus v_{r+1}$, and w is the only vertex on $\mathcal{P}(v_{r+1}, w) \setminus v_{r+1}$, we get

$$\bigcup_{w \in O(v_{r+1})} \mathcal{T}_w^{i_r}\big(m + j(w)\big) = \bigcup_{u \in D(v_{r+1})} \mathcal{T}_u\Big(m + \sum_{v \in \mathcal{P}(v_{r+1},u) \setminus v_{r+1}} (i_r \bullet j)(v)\Big) = \mathcal{T}_{v_{r+1}}^{i_r \bullet j}(m) \setminus \mathcal{T}_{v_{r+1}}(m).$$

Therefore, we obtain

$$\begin{aligned}
&\mathbb{1}\bigg(M\Big(\mathcal{T}_{v_{r+1}}^{i_r}\big(h(v_{r+1})\big)\Big) < M\Big(\bigcup_{w \in O(v_{r+1})} \mathcal{T}_w^{i_r}\big(h(v_{r+1}) + j(w)\big)\Big)\bigg) \\
&\qquad = \mathbb{1}\bigg(M\Big(\mathcal{T}_{v_{r+1}}^{i_r \bullet j}\big(h(v_{r+1})\big)\Big) < M\Big(\mathcal{T}_{v_{r+1}}^{i_r \bullet j}\big(h(v_{r+1})\big) \setminus \mathcal{T}_{v_{r+1}}^{i_r \bullet j}\big(h(v_{r+1})\big)\Big)\bigg).
\end{aligned}$$

A similar decomposition shows that

$$I_{v_{r+1}}^{i_r}(m) \times \prod_{w \in O(v_{r+1})} I_w^{i_r}(m)\, I_w^{i_r}[m, m + j(w)) = I_{v_{r+1}}^{i_r \bullet j}(m).$$

Note that $I_w^{i_r}(m) I_w^{i_r}\big[m, m + j(w)\big) = I_w^{i_r}\big(m + j(w)\big)$, and $I_{v_{r+1}}^{i_r}(m) = I_{v_{r+1}}(m)$. So, expanding using the definitions, and grouping the descendants of v_{r+1} as a union of the offspring of v_{r+1} and their respective descendants, we get

$$\begin{aligned}
&I_{v_{r+1}}^{i_r}(m) \times \prod_{w \in O(v_{r+1})} I_{v_w}(m)\, I_w\big[m, m + j(w)\big) \\
&= I_{v_{r+1}}(m) \times \prod_{w \in O(v_{r+1})} \bigg(I_w\big(m + j(w)\big) \times \prod_{u \in D(w)} I_u\Big(m + j(w) + \sum_{v \in \mathcal{P}(w,u) \setminus w} i_r(v)\Big)\bigg)
\end{aligned}$$

$$= I_{v_{r+1}}(m) \times \prod_{w \in O(v_{r+1})} \left(I_w\big(m + j(w)\big) \times \prod_{u \in D(w)} I_u\Big(m + \sum_{v \in \mathcal{P}(v_{r+1},u) \setminus v_{r+1}} i_r(v) + j(v)\Big)\right)$$

$$= I_{v_{r+1}}(m) \times \prod_{u \in D(v_{r+1})} I_u\Big(m + \sum_{v \in \mathcal{P}(v_{r+1},u) \setminus v_{r+1}} i_r \bullet j(v)\Big)$$

$$= I^{i_r \bullet j}_{v_{r+1}}(m),$$

where the penultimate step again used the fact that w is the only vertex on $\mathcal{P}(v_{r+1}, w) \setminus v_{r+1}$, and $\big(i_r \bullet j(w)\big) = j(w)$, for $w \in O(v_{r+1})$.

Using the two simplifications above and substituting the modified version of (2.22) back into (2.21) we obtain

$$\sum_{i_r \in \{0,2\}^{S_r}} (-1)^{\frac{1}{2}|i_r|} C_{i_r} \sum_{f \in \mathcal{F}^n_r} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_r} \Phi^{i_r}_{u,v}\big(f(u), f(v)\big) \prod_{u \in V(G_r)} I^{i_r}_u\big(f(u)\big)\Bigg] \tag{2.23}$$

$$= \sum_{i_r \in \{0,2\}^{S_r}} (-1)^{\frac{1}{2}|i_r|}\, C_{i_r} \sum_{j \in \{0,2\}^{O(v_{r+1})}} (-1)^{\frac{1}{2}|j|} \sum_{h \in \mathcal{F}^n_{r+1}} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_{r+1}} \Phi^{i_r \bullet j}_{u,v}\big(h(u), h(v)\big) \prod_{u \in V(G_{r+1})} I^{i_r \bullet j}_u\big(h(u)\big) \times$$

$$\times\, \mathbb{1}\Big(M\Big(\mathcal{T}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < p \Big) \times$$

$$\times\, \mathbb{1}\Big(M\Big(\mathcal{T}_{v_{r+1}}\big(h(v_{r+1})\big)\Big) < M\Big(\mathcal{T}^{i_r \bullet j}_{v_{r+1}}\big(h(v_{r+1})\big) \setminus \mathcal{T}_{v_{r+1}}\big(h(v_{r+1})\big)\Big)\Big)\Bigg].$$

Conditioning on the minimal label in $\mathcal{T}_{v_{r+1}}\big(h(v_{r+1})\big)$ and using Lemma 2.1, as done earlier, the expression on the right in the above equation reduces to

$$\sum_{i_r \bullet j \in \{0,2\}^{S_{r+1}}} (-1)^{\frac{1}{2}|i_r| + |j|}\, C_{i_r} \frac{|\mathcal{T}_{v_{r+1}}(0)|}{|\mathcal{T}^{i_r \bullet j}_{v_{r+1}}(0)|} \sum_{h \in \mathcal{F}^n_{r+1}} \mathbb{E}\Bigg[\prod_{(u,v) \in \vec{E}_{r+1}} \Phi^{i_r \bullet j}_{u,v}\big(h(u), h(v)\big) \prod_{u \in V(G_{r+1})} I^{i_r \bullet j}_u\big(h(u)\big)\Bigg].$$

Since $i_r \bullet j$ varies over all possible labels of S_{r+1}, w rewrite it as i_{r+1}. Additionally, using the definition of C_{i_r}, we have

$$C_{i_r} \frac{|\mathcal{T}_{v_{r+1}}(0)|}{|\mathcal{T}^{i_r \bullet j}_{v_{r+1}}(0)|} = \left(\prod_{l=1}^{r} \frac{|\mathcal{T}_{v_l}(0)|}{|\mathcal{T}^{i_l}_{v_l}(0)|} \right) \frac{|\mathcal{T}_{v_{r+1}}(0)|}{|\mathcal{T}^{i_{r+1}}_{v_{r+1}}(0)|} = \prod_{l=1}^{r+1} \frac{|\mathcal{T}_{v_l}(0)|}{|\mathcal{T}^{i_l}_{v_l}(0)|} = C_{i_{r+1}}.$$

Hence,

$$\sum_{i_r \in \{0,2\}^{S_r}} (-1)^{\frac{1}{2}|i_r|}\, C_{i_r} \sum_{h \in \mathcal{F}_r^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_r} \Phi_{u,v}^{i_r}\big(h(u),h(v)\big) \prod_{u \in V(G_r)} I_u^{i_r}\big(h(u)\big)\Bigg]$$
$$= \sum_{i_{r+1} \in \{0,2\}^{S_{r+1}}} (-1)^{\frac{1}{2}|i_{r+1}|}\, C_{i_{r+1}} \sum_{h \in \mathcal{F}_{r+1}^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_{r+1}} \Phi_{u,v}^{i_{r+1}}\big(h(u),h(v)\big) \prod_{u \in V(G_{r+1})} I_u^{i_{r+1}}\big(h(u)\big)\Bigg],$$

and this completes the proof of the induction step.

Restating the above equation, the reduction at the penultimate vertex $v_{m-1} \in \mathcal{R}_m$ gives

$$\sum_{h \in \mathcal{F}_0^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_0} \Phi_{u,v}\big(f(u),f(v)\big) \prod_{u \in V(G_0)} I_u\big(f(u)\big)\Bigg] \tag{2.24}$$
$$= \sum_{i_{m-1} \in \{0,2\}^{S_{m-1}}} (-1)^{\frac{1}{2}|i_{m-1}|}\, C_{i_{m-1}} \sum_{h \in \mathcal{F}_{m-1}^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_{m-1}} \Phi_{u,v}^{i_{m-1}}\big(h(u),h(v)\big) \prod_{u \in V(G_{m-1})} I_u^{i_{m-1}}\big(h(u)\big)\Bigg].$$

In fact, we can again carry out the reduction procedure at o. Note that G_{m-1} is a star with internal vertex o, and consequently $G_m = \{o\}$ and $\mathcal{F}_m^n$ consists of just one function h satisfying $h(o) = n$. Writing $\mathcal{F}_{m-1}^n = \bigcup_{h \in \mathcal{F}_m^n} \mathcal{A}_{m-1}^n(h) = \mathcal{A}_{m-1}^n(h)$, we have

$$\sum_{i_{m-1} \in \{0,2\}^{S_{m-1}}} (-1)^{\frac{1}{2}|i_{m-1}|}\, C_{i_{m-1}} \sum_{h \in \mathcal{F}_{m-1}^n} \mathbb{E}\Bigg[\prod_{(u,v)\in \vec{E}_{m-1}} \Phi_{u,v}^{i_{m-1}}\big(h(u),h(v)\big) \prod_{u \in V(G_{m-1})} I_u^{i_{m-1}}\big(h(u)\big)\Bigg]$$
$$= \sum_{i_{m-1} \in \{0,2\}^{S_{m-1}}} (-1)^{\frac{1}{2}|i_{m-1}|}\, C_{i_{m-1}} \sum_{h \in \mathcal{A}_{m-1}^n} \mathbb{E}\Bigg[I_o(n) \prod_{w \in O(o)} \Phi_{o,w}^{i_{m-1}}\big(n,h(w)\big) I_w^{i_{m-1}}\big(h(w)\big)\Bigg]$$
$$= \sum_{i_{m-1} \in \{0,2\}^{S_{m-1}}} (-1)^{\frac{1}{2}|i_{m-1}|}\, C_{i_{m-1}} \mathbb{E}\Bigg[I_o(n) \prod_{w \in O(o)} I_w^{i_{m-1}}(n) \times$$
$$\times \sum_{h \in \mathcal{A}_{m-1}^n} \prod_{w \in O(o)} \Phi_{o,w}^{i_{m-1}}\big(n,h(w)\big) I_w^{i_{m-1}}\big(h(w)\big)\Bigg].$$

Using (2.22) and simplifying the resulting product, with j prescribing $\{0,2\}$ labels to vertices in $O(o)$, we obtain

$$\sum_{i_{m-1}\bullet j\in\{0,2\}^{S_m}} (-1)^{\frac{1}{2}|i_{m-1}\bullet j|}\, C_{i_{m-1}}\, \mathbb{E}\Big[\, I_o^{i_{m-1}\bullet j}(n)\Big]\, \mathbb{E}\Bigg[\mathbb{1}\Big(M\big(\mathcal{T}_o(n)\big) < p\Big)\times \\ \times\, \mathbb{1}\bigg(M\Big(\mathcal{T}_o(n)\Big) < M\Big(\mathcal{T}_o^{i_{m-1}\bullet j}(n)\setminus \mathcal{T}_o(n)\Big)\bigg)\Bigg].$$

Integrating over the density of $M\big(\mathcal{T}_o(n)\big)$, we get

$$\sum_{i_{m-1}\bullet j\in\{0,2\}^{S_m}} (-1)^{\frac{1}{2}|i_{m-1}\bullet j|}\, C_{i_{m-1}}\, \frac{|\mathcal{T}_o(0)|}{|\mathcal{T}_o^{i_{m-1}\bullet j}(0)|}\, \mathbb{E}\Big[\, I_o^{i_{m-1}\bullet j}(n)\Big]\mathbb{E}\bigg[1-\mathbb{1}\bigg(M\Big(\mathcal{T}_o^{i_{m-1}\bullet j}(n)\Big) > p\bigg)\bigg].$$

Replace $i_{m-1}\bullet j$ by i_m. By definition, we have $I_o^{i_m}(n)\,\mathbb{1}\Big(M\big(\mathcal{T}_o^{i_m}(n)\big) > p\Big) = I_o^{i_m}(n+1)$. Therefore, we can use independence to rewrite the product of the expectations to get a difference, which will subsequently give us a telescoping series. Hence, in summary, reducing the vertex o results in

$$\sum_{h\in\mathcal{F}_0^n} \mathbb{E}\bigg[\prod_{(u,v)\in\vec{E}_0} \Phi_{u,v}\big(f(u),f(v)\big) \prod_{u\in V(G_0)} I_u\big(f(u)\big)\bigg] \\ = \sum_{i_m\in\{0,2\}^{S_m}} (-1)^{\frac{1}{2}|i_m|}\, C_{i_m}\, \bigg(\mathbb{E}\Big[I_o^{i_m}(n)\Big] - \mathbb{E}\Big[I_o^{i_m}(n+1)\Big]\bigg). \tag{2.25}$$

The probability of the cylinder event in Proposition 2.1 can be evaluated, using (2.14), as follows.

$$\mu_p\big(\omega(F_i)=1,\, c_i\in T_{x_i},\, 1\le i\le k\big) \\ = \bigg(\prod_{u\in V(G)} \frac{1}{|\mathcal{T}_u(0)|}\bigg)\, \mathbb{E}\Big(\mathbb{1}\Big(M\big(N(0)\big) > p\Big)\Big)\times \\ \times \sum_{n=0}^{\infty}\sum_{f\in\mathcal{F}_0^n} \mathbb{E}\bigg[\prod_{(u,v)\in\vec{E}} \Phi_{u,v}\big(f(u),f(v)\big) \prod_{u\in V(G)} I_u\big(f(u)\big)\bigg] \\ = \bigg(\prod_{u\in V(G)} \frac{1}{|\mathcal{T}_u(0)|}\bigg)\, \mathbb{E}\Big(\mathbb{1}\Big(M\big(N(0)\big) > p\Big)\Big)\times$$

$$\times \sum_{i_m \in \{0,2\}^{S_m}} (-1)^{\frac{1}{2}|i_m|} C_{i_m} \sum_{n=0}^{\infty} \Big(\mathbb{E}\Big[I_o^{i_m}(n) \Big] - \mathbb{E}\Big[I_o^{i_m}(n+1) \Big] \Big)$$
$$= \Big(\prod_{u \in V(G)} \frac{1}{|\mathcal{T}_u(0)|} \Big) \mathbb{E}\Big(\mathbb{1}\Big(M\big(N(0)\big) > p \Big) \Big) \sum_{i_m \in \{0,2\}^{S_m}} (-1)^{\frac{1}{2}|i_m|} C_{i_m} \mathbb{E}\Big(I_o^{i_m}(0) \Big).$$

Recall that $N(0) = \bigcup_1^k N_i(0)$, where $N_i(0)$ is a finite set of vertices in $\mathcal{G}_i$. The term involving $N(0)$ above expresses the condition that there be no nuclei there; taking the limit in p this term tends to 1.

In conclusion, we have

$$\lim_{p \to 0} \mu_p\big(\omega(F_i) = 1,\, c_i \in T_{x_i},\, 1 \le i \le k\big)$$
$$= \Big(\prod_{u \in V(G)} \frac{1}{|\mathcal{T}_u(0)|} \Big) \sum_{i_m \in \{0,2\}^{G \setminus \{o\}}} (-1)^{\frac{1}{2}|i_m|} C_{i_m}, \qquad (2.26)$$

and this completes the proof of the proposition. □

As remarked earlier, taking the sum over all the choices of $x_1 \in \mathrm{Rt}_1$ and $(x_2, \dots, x_k) \in \prod_{i=2}^k \mathrm{Rt}_{i,x_i}$, we get the limit probability that the components $\{F_i\}_1^k$ are all open and the edges in between the components are closed. This completes the proof of existence of μ_0 on regular trees $\mathbb{T}_d$.

The following pseudocode summarizes the calculation of the cylinder probability of components $\{F_i\}_1^k$, of the type described in proposition 2.1. For a choice of a complete reducible sequence of vertices $\mathcal{R}$ in the associated graph G, the output of the pseudocode *CylProb* is the limit probability of the given subgraphs being open and in distinct components.

Algorithm 1 CylProb

Require: G and $\mathcal{R}$

Find Rt_1

for $x_1 \in \mathrm{Rt}_1$ **do**

3: Compute $|\mathcal{T}_{1,x_1}(0)|$

4: **for** $i \in \{2, \ldots, k\}$ **do**

5: Find $\mathrm{Rt}_{i,x_1}, |\mathcal{T}_{i,x_1}(0)|$

6: **end for**

7: **for** $(x_2, \ldots, x_k) \in \prod_{i=2}^{k} \mathrm{Rt}_{i,x_1}$ **do**

8: $\Pi_{(x_1,\ldots,x_k)} \leftarrow \prod_{i=1}^{k} \dfrac{1}{|\mathcal{T}_{i,x_1}(0)|}$

9: **for** $\alpha \in \{0,2\}^{G \setminus F_1}$ **do**

10: $C_{\alpha,(x_1,\ldots,x_k)} \leftarrow (-1)^{\frac{1}{2}|\alpha|} \prod_{j \in \mathcal{R}} \dfrac{|\mathcal{T}_{j,x_1}(0)|}{|\mathcal{T}^{\alpha}_{j,x_1}(0)|}$

11: **end for**

12: **end for**

13: **return** $\Pi_{(x_1,\ldots,x_k)} \sum_{\alpha} C_{\alpha,(x_1,\ldots,x_k)}$

14: **end for**

15: **return** $\mathit{CylProb} \leftarrow \sum_{(x_1,\ldots,x_k)} \Pi_{(x_1,\ldots,x_k)} \sum_{\alpha} C_{\alpha,(x_1,\ldots,x_k)}$

We illustrate a part of the calculation with a specific example. Consider the collection of subgraphs shown in Figure 2.4 below, with the sets of vertices as indicated. For this choice of nuclei $x_1, \ldots, x_k$, and a specific $\alpha \in \{0,2\}^{G \setminus F_1}$ we find the output in line 10 and line 8 of the pseudocode above.

Recall that by definition, we have $\mathcal{T}_{i,x_i}(0) = \partial_i\big(\bigcup_{v \in F_i} B\big(v, d(v,x_i) - 1\big)\big)$, and these are the colored vertices in Figure 2.4, colored the same as the subgraphs they are related to. The associated adjacency graph of the components F_i is also shown in the figure and we have a complete sequence of reducible vertices in $\mathcal{R} = \{F_3, F_2, F_1\}$. In this case, we have the following:

$$\mathcal{T}_1(0) = 5 \cdot 2^4 \qquad \mathcal{T}_3(0) = 2^5 \qquad \mathcal{T}_5(0) = 2^5 \qquad \mathcal{T}_7(0) = 2^3$$

$$\mathcal{T}_2(0) = 5 \cdot 2^2 \qquad \mathcal{T}_4(0) = 5 \cdot 2^3 \qquad \mathcal{T}_6(0) = 2^4$$

Using these values, we have

$$\Pi_{(x_1\dots,x_k)} = \prod_1^k \frac{1}{|\mathcal{T}_i(0)|} = \frac{1}{5^3\ 2^{26}}.$$

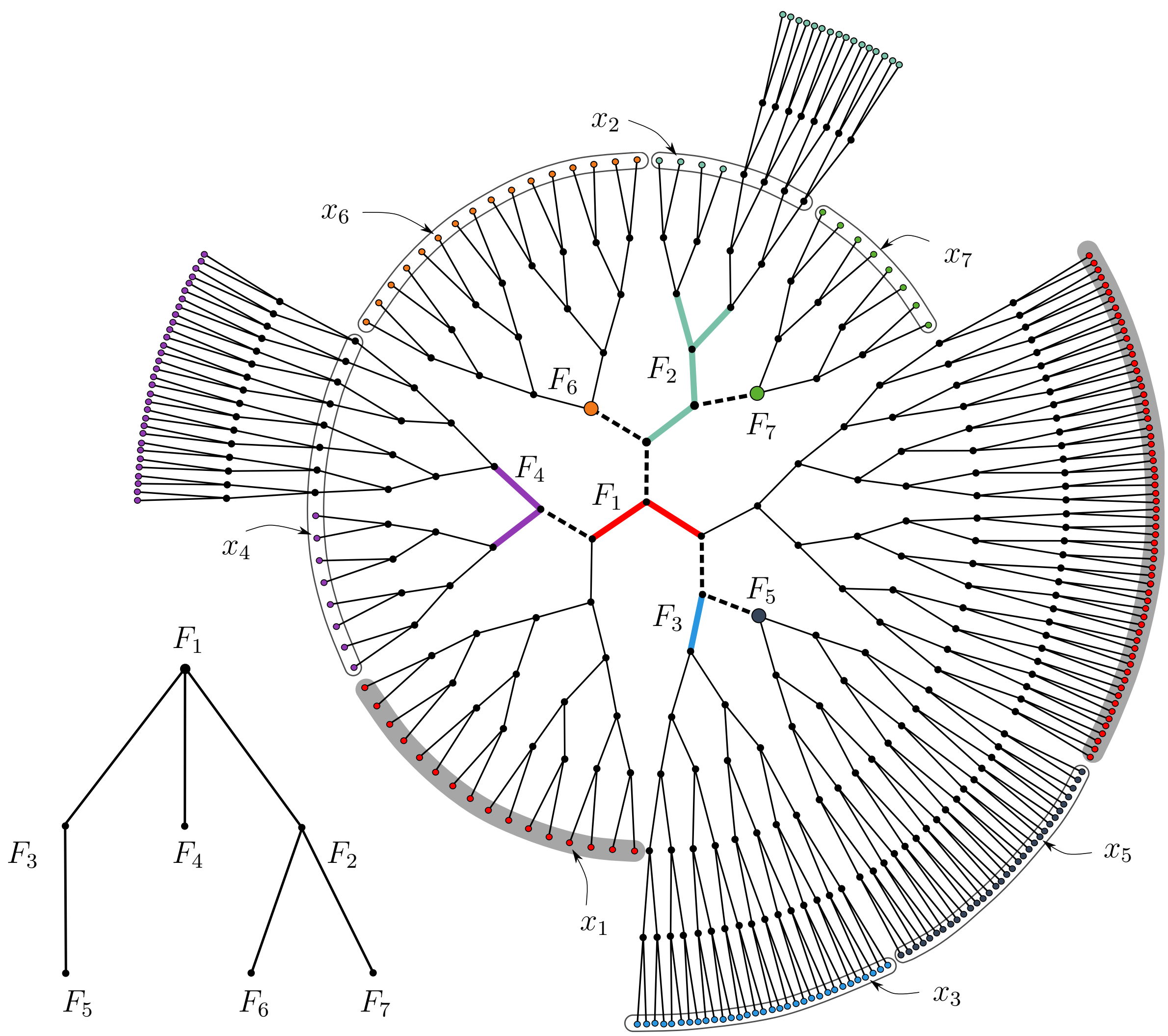


Figure 2.4: The associated graph G of the collection of subgraphs $\{F_i\}_1^7$ is shown on the left. The components F_i are depicted with thick colored edges, and in the case of F_5, F_6 and F_7 thick colored vertices. The vertices within the dark grey region constitute the set Rt_1. For the depicted choice of x_1, vertices within the grey contiguous regions constitute R_{i,x_1}, for $2 \le i \le k$. Finally, corresponding to the choice of $\{x_i\}_1^k$, vertices colored the same as the components constitute the sets $\mathcal{T}_{i,x_i}(0)$, for $1 \le i \le k$.

Now, consider $\alpha \in \{0,2\}^{G\setminus F_1}$ given by $\big(\alpha(F_1),\ldots,\alpha(F_k)\big) = \big(0,2,0,2,0,0\big)$. To compute $C_{\alpha,(x_1,\ldots,x_k)}$ for this configuration, we need $\mathcal{T}_i^\alpha(0)$, for $i \in \{3,2,1\}$. Using the weights provided by α on the path from any descendant to the reducible vertex, we have

$$|\mathcal{T}_3^\alpha(0)| = |\mathcal{T}_3(0) \cup \mathcal{T}_5(0))| = 5 \cdot 2^5, \qquad |\mathcal{T}_2^\alpha(0)| = |\mathcal{T}_2(0) \cup \mathcal{T}_6(0) \cup \mathcal{T}_7(0)| = 11 \cdot 2^2,$$
$$|\mathcal{T}_1^\alpha(0)| = |\mathcal{T}_1(0) \cup \mathcal{T}_2(0) \cup \mathcal{T}_3(2) \cup \mathcal{T}_4(0) \cup \mathcal{T}_5(4) \cup \mathcal{T}_6(0) \cup \mathcal{T}_7(0)| = 93 \cdot 2^5.$$

Furthermore, we have that $|\mathcal{T}_i(m)| = 2^m |\mathcal{T}_i(0)|$. Finally using this, we get

$$C_{\alpha,(x_1,\ldots,x_k)} = \frac{|\mathcal{T}_1(0)|}{|\mathcal{T}_1^\alpha(0)|} \cdot \frac{|\mathcal{T}_2(0)|}{|\mathcal{T}_2^\alpha(0)|} \cdot \frac{|\mathcal{T}_3(0)|}{|\mathcal{T}_3^\alpha(0)|} = \frac{1}{5^2\ 2^{31}}.$$

Properties of Bernoulli–Voronoi clusters

In this section, we use the mass-transport principle (see [LP16]) to obtain some results about the measure μ_0 on $\mathbb{T}_d$. Although some of these can be obtained using the results of the previous section, the use of the mass-transport principle allows us to circumvent some of the tedious calculations which would otherwise arise.

First, we state the principle. Let G be a Cayley graph of a group, and let $F(x,y,\omega)$ be a function of three variables, $x,y \in V(G)$ and $\omega \in \{0,1\}^{E(G)}$, such that it is invariant under the diagonal action of the automorphism group $\mathrm{Aut}(G)$ of the graph G, i.e., $F(x,y,\omega) = F(\gamma x, \gamma y, \gamma\omega)$, for $\gamma \in \mathrm{Aut}(G)$. Under these conditions the mass-transport principle states that

$$\sum_{x \in V(G)} \mathbb{E}\, F(o,x,\omega) = \sum_{x \in V(G)} \mathbb{E}\, F(x,o,\omega). \tag{2.27}$$

On any configuration $\omega \in \Omega$, we think of $F(x,y,\omega)$ as the amount of *mass* sent from x

to y. The mass-transport principle can then be viewed as stating that the average amount of *mass sent out* from any arbitrary distinguished vertex o is equal to the average amount of *mass received* at o.

Using (2.27), we can obtain the single-edge marginal on $\mathbb{T}_d$, which was evaluated in the previous section.

Lemma 2.3. *For any fixed edge $e \in E(\mathbb{T}_d)$ and vertex $o \in V(\mathbb{T}_d)$, we have $\mathbb{E}_0\big(\deg_\omega(o)\big) = 2$ and $\mu_0\big(\omega(e) = 1\big) = \dfrac{2}{d}$.*

Proof. For any vertex $x \in V$, we denote the connected percolation component of x by $K(x)$ and the nucleus associated to it by c_x. Consider the transport function which sends mass $\deg_\omega(x)$ from every vertex x to its nucleus c_x, i.e.,

$$F(x, y, \omega) := \begin{cases} \deg_\omega(x) & \text{if } y = c_x, \\ 0 & \text{if } y \neq c_x, \end{cases}$$

where $\deg_\omega(x)$ denotes the induced degree of x in $K_\omega(x)$. From the definition of the model it is clear that this function is diagonally invariant. Since $F(o, x, \omega)$ is non-zero only when $x = c_o$ we have $\sum_x F(o, x, \omega) = F(o, c_o, \omega) = \deg_\omega(o)$, and it follows that

$$\mathbb{E}_p\Big[\sum_{x \in V(\mathbb{T}_d)} F(o, x, \omega)\Big] = \mathbb{E}_p\big(\deg_\omega(o)\big).$$

By the definition of F, mass is received at o only if o is a nucleus, and in that case it receives $\deg_\omega(x)$ from every vertex x in $K(o)$. Hence,

$$\mathbb{E}_p\Big[\sum_{x \in V(\mathbb{T}_d)} F(x, o, \omega)\Big] = p\,\mathbb{E}_p\Big[\sum_{x \in K(o)} \deg_\omega(x) \mid o = c_o\Big] = p\,\mathbb{E}_p\Big[2\big(|K(o)| - 1\big) \mid o = c_o\Big],$$

where the last equality follows from the observation that on finite graphs the sum over the degrees of all vertices counts every edge twice, and the number of edges in a finite tree is one less than the number of vertices. As a result, we obtain

$$\mathbb{E}_p\big[\deg_\omega(o)\big] = 2p\Big(\mathbb{E}_p\big[|K(o)| \mid o = c_o\big] - 1\Big) \tag{2.28}$$

To evaluate this expression, we use another transport function to find $\mathbb{E}_p\big[|K(o)| \mid o = c_o\big]$. Consider the function G which sends mass 1 from every vertex to its associated nucleus, i.e.,

$$G(x,y,\omega) := \begin{cases} 1 & \text{if } y = c_x, \\ 0 & \text{if } y \neq c_x. \end{cases}$$

The average mass sent from o is exactly 1, and mass is received at o only if o is itself a nucleus. Using the mass-transport principle, we have

$$1 = \mathbb{E}_p\Big[\sum_{x\in V} G(o,x,\omega)\Big] = E_p\Big[\sum_{x\in V} G(x,o,\omega)\Big] = p\,\mathbb{E}_p\Big[|K(o)| \mid o = c_o\Big]. \tag{2.29}$$

Hence, the expected size of the component of o, conditioned on o being a nucleus, is exactly $1/p$. Substituting this in (2.28), we get

$$\mathbb{E}_p\big[\deg_\omega(o)\big] = 2(1-p).$$

Taking the limit gives us the first statement in the lemma. Furthermore, using the fact that $\mathbb{T}_d$ is edge transitive, we have $\mathbb{E}[\deg(o)] = d\,\mu_p(\omega(e) = 1)$, for an edge e adjacent to o. This completes the proof of the lemma. □

Note that from results in [Hä97], we know that the maximal marginal edge density for automorphism invariant edge percolation measures on $\mathbb{T}_d$ with only finite clusters is at most $\frac{2}{d}$. Hence the collection of measures $\{\mu_p\}_{p>0}$ give us examples of percolation measures which get arbitrarily close to this upper bound.

Using a similar argument as in the lemma above, we can relate the degree distribution of a vertex x to the expected proportion of vertices in its component having that degree. Let $D_x(k)$ denote the number of vertices in $K(x)$ having induced degree k.

Lemma 2.4. *For any* $1 \leq k \leq d$ *and* $p \geq 0$, $\mathbb{E}_p\left[\dfrac{D_o(k)}{|K(o)|}\right] = \mu_p(\deg_\omega(o) = k)$.

Proof. Fix $1 \leq k \leq d$ and define the following transport function

$$F_k(x, y, \omega) := \begin{cases} \dfrac{1}{|K(x)|} & \text{if } y \in K(x) \text{ and } \deg_\omega(y) = k, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, for an arbitrary fixed vertex o, the total mass sent out from o is $\sum_x F_k(o, x, \omega) = D_o(k)/|K(o)|$. Mass is received at o only of $\deg_\omega(o) = k$, and on that event it receives a total of mass 1. Conditioning on this event, we have

$$\mathbb{E}_p\Big[\sum_x F_k(x, o, \omega)\Big] = \mu_p\big(\deg_\omega(o) = k\big)\, \mathbb{E}_p\Big[\sum_x F_k(x, o, \omega) \;\big|\; \deg_\omega(o) = k\Big] = \mu_p\big(\deg_\omega(o) = k\big).$$

Now, using the mass-transport principle, we have the statement of the lemma,

$$\mathbb{E}_p\Big[\frac{D_o(k)}{|K(o)|}\Big] = \mathbb{E}_p\Big[\sum_{x \in V} F_k(o, x, \omega)\Big] = \mathbb{E}_p\Big[\sum_{x \in V} F_k(x, o, \omega)\Big] = \mu_p\big(\deg_\omega(o)\big). \qquad \square$$

Using this, we can easily obtain the degree distribution of a fixed vertex in $\mathbb{T}_3$, the limit of which turns out to be the uniform distribution. For any finite subgraph of $\mathbb{T}_3$, the number of degree 3 vertices is two less than the number of degree 2 vertices. In particular, for the component $K(o)$, we have $D_o(3) = D_o(2) - 2$. Using this relation between $D_o(3)$ and $D_o(2)$, we get the following relation

$$1 = \frac{D_o(1) + D_o(2) + D_o(3)}{|K(o)|} = \frac{D_o(1)}{|K(o)|} + 2\,\frac{D_o(2)}{|K(o)|} - 2\frac{1}{|K(o)|}.$$

Taking the expectation and using the previous lemma, we get

$$1 + 2p = \mu_p(\deg_\omega(o) = 1) + 2\mu_p(\deg_\omega(o) = 2). \tag{2.30}$$

In the above equation we used the relation $\mathbb{E}_p[1/|K(o)|] = p$, which is obtained by using a mass-transport argument with the transport function which sends mass $1/|K(x)|$ to c_x from each x.

Note that from Lemma 2.3 we know that the expected degree of o is $2(1-p)$. Solving these two equations simultaneously, we obtain the relations

$$\begin{aligned}
\mu_p\big(\deg_\omega(o) = 1\big) &= 2p + \mu_p\big(\deg_\omega(o) = 3\big), \\
\mu_p\big(\deg_\omega(o) = 2\big) &= 1 - 2p - 2\mu_p\big(\deg_\omega(o) = 3\big).
\end{aligned}$$

Using (2.30) along with the above equations, we obtain

$$\mu_p\big(\deg_\omega(o) = 1\big) = \frac{1+2p}{3}, \quad \mu_p\big(\deg_\omega(o) = 2\big) = \frac{1+2p}{3}, \quad \mu_p(\deg_\omega(o) = 3) = \frac{1-4p}{3}.$$

Taking the limit as p tends to zero, we get

$$\mu_0\big(\deg_\omega(o) = 1\big) = \mu_0\big(\deg_\omega(o) = 2\big) = \mu_0\big(\deg_\omega(o) = 3\big) = \frac{1}{3}.$$

In fact we can obtain the degree distribution more generally on $\mathbb{T}_d$ by using the marginal probability formula (2.26) obtained in the previous section.

Lemma 2.5. *For any $1 \leq k \leq d$, and $o \in \mathbb{T}_d$ and $d \geq 3$, we have*

$$\mu_0\big(\deg(o) = k\big) = \frac{1}{d-2} \cdot \frac{(d-1)!}{(k-1)!} \cdot \frac{\Gamma\Big(k - 1 + \frac{1}{d-2}\Big)}{\Gamma\Big(d + \frac{1}{d-2}\Big)} \tag{2.31}$$

As a consequence, $\lim_{d\to\infty} \mu_0\big(deg(o) = 1\big) = 1$.

Proof. Since $\mathbb{T}_d$ is edge-transitive and the measure μ_p is automorphism invariant, for any $1 \le k \le d$, we have

$$\mu_0\big(\deg_\omega(o) = k\big) = \binom{d}{k}\,\mu_0\Big(\omega(e_i) = 1,\ \omega(f_j) = 0,\ 1 \le i \le k,\ k < j \le d\Big). \tag{2.32}$$

Note that e_i is an open edge and f_j is a closed edge adjacent to o. Using (2.26) we can compute the probability of the cylinder event appearing on the right in the above equation. Note that the cylinder set, described in the above event, is composed of $d-k+1$ components: F_1 is a star formed by k edges, with internal vertex o, and the remaining $d-k$ components are formed by singleton vertices. This implies, in particular, that the associated graph of this cylinder event is a star with internal vertex F_1.

Using the notation of the previous section, we get $\mathrm{Rt}_1 = S(o,4) \cap \mathcal{G}_1$, and $|\mathrm{Rt}_1| = k(d-1)^3$, where $S(o,4)$ is the sphere of radius 4 centered at o. By symmetry, for any $x_1 \in \mathrm{Rt}_1$, we have $\mathrm{Rt}_{i,x_1} = S(o,4) \cap \mathcal{G}_i$ and $|\mathrm{Rt}_{i,x_1}| = (d-1)^3$, for $2 \le d-k+1$. For any $x_i \in \mathrm{Rt}_{i,x_1}$, since the components $\{F_i\}_2^{d-k+1}$ are singleton vertices in $\mathbb{T}_d$, we have $|\mathcal{T}_{i,x_1}(0)| = |\mathcal{T}_i(0)| = (d-1)^3$. Furthermore, for any $x_1 \in \mathrm{Rt}_1$, we have $|\mathcal{T}_1(0)| = (d-1)^3 + (k-1)(d-1)^5$. We can write

$$\begin{aligned}\mu_0\Big(\omega(e_i) = 1,\ \omega(f_j) = 0,\ 1 \le i \le k,\ k < j \le d\Big)&\\ = \sum_{x_1 \in \mathrm{Rt}_1} \sum_{\substack{(x_2,\dots,x_{d-k+1})\\ \in \prod_{i=2}^{d-k+1} \mathrm{Rt}_{i,x_1}}} \prod_{i=1}^{d-k+1} \frac{1}{|\mathcal{T}_i(0)|} \sum_{\alpha \in \{0,2\}^{G\setminus F_1}} (-1)^{\frac{1}{2}|\alpha|} C_\alpha.&\end{aligned} \tag{2.33}$$

First, by symmetry, the summand is the same for every tuple $(x_1,\dots,x_{d-k+1})$, and there are $k(d-1)^{3(d-k+1)}$ such tuples. Moreover, the only reducible vertex in the associated graph is F_1, and as a result, we have a simple form of C_α, given by $C_\alpha = |\mathcal{T}_1(0)|/|\mathcal{T}_1^\alpha(0)|$. In addition, $|\mathcal{T}_1^\alpha(0)|$ is determined by the number of components amongst $\{F_i\}_2^{d-k+1}$ which receive the label 2. If j components are labeled 2, then we have

$$\mathcal{T}_1^\alpha(0) = (d-1)^3 + (k-1)(d-1)^5 + \big(j(d-1)^5 + (d-k-j)(d-1)^3\big)$$

$$= d(d-1)^3\big((k+j-1)(d-2)+1\big).$$

Using the above equations to simplify (2.33), we obtain

$$\begin{aligned}
&\mu_0\Big(\omega(e_i)=1,\ \omega(f_j)=0,\ 1\le i\le k,\ k<j\le d\Big)\\
&\qquad = \frac{k(d-1)^{3(d-k+1)}}{(d-1)^{3(d-k)}}\sum_{j=0}^{d-k}(-1)^j\binom{d-k}{j}\frac{1}{d(d-1)^3\big((k+j-1)(d-2)+1\big)}\\
&\qquad = \frac{k}{d}\cdot\frac{(d-k)!}{d-2}\cdot\frac{\Gamma\Big(k-1+\frac{1}{d-2}\Big)}{\Gamma\Big(d+\frac{1}{d-2}\Big)}.
\end{aligned}$$

Multiplying the expression on the right by the binomial factor in (2.32) and simplifying, we obtain the degree distribution

$$\mu_0\big(\deg(o)=k\big)=\frac{1}{d-2}\cdot\frac{(d-1)!}{(k-1)!}\cdot\frac{\Gamma\Big(k-1+\frac{1}{d-2}\Big)}{\Gamma\Big(d+\frac{1}{d-2}\Big)}.$$

As a special case, for $k=d$, we have

$$\mu_0\big(\deg(o)=d\big)=\frac{1}{d-2}\cdot\frac{\Gamma\Big(d-1+\frac{1}{d-2}\Big)}{\Gamma\Big(d+\frac{1}{d-2}\Big)}=\frac{1}{d^2-3d+3}.$$

Furthermore, for $k=1$, we get

$$\mu_0\big(\deg(o)=1\big)=\frac{(d-1)!}{d-2}\cdot\frac{\Gamma\Big(\frac{1}{d-2}\Big)}{\Gamma\Big(d+\frac{1}{d-2}\Big)}. \tag{2.34}$$

Note that the gamma function satisfies the recurrence relation $\Gamma(1+x)=x\Gamma(x)$ using which we can simplify the denominator in equation 2.34. Simiplifying the expression we get

$$\mu_0\big(\deg(o)=1\big) = \frac{(d-1)!}{\prod_{i=1}^{d-1}\left((d-i)+\frac{1}{d-2}\right)} = \frac{1}{\prod_{i=1}^{d-1}\left(1+\frac{1}{i(d-2)}\right)}.$$

Taking the log of the denominator in the final expression above, simplifying it and comparing comparing its growth to the growth of the harmonic series we can conclude that it tends to zero as d tends to infinity. As a consequence, we have that the limit in equation 2.34 tends to 1 as the degree d tends to infinity.

A more careful analysis of the asymptotics in d shows that the term on the right in equation 2.34 behaves as follows:

$$\frac{(d-1)!}{d-2}\cdot\frac{\Gamma\left(\frac{1}{d-2}\right)}{\Gamma\left(d+\frac{1}{d-2}\right)} \sim 1-\frac{\log(d)+\gamma}{d}+O\left(\frac{1}{d^2}\right),$$

where γ is the Euler–Mascheroni constant. This constant makes its appearance via a comparision with the growth of the harmonic series. In particular, for large d, the degree of any fixed vertex is 1 with high probability. □

Paths

Consider two vertices x and y such that $d(x,y)=m$ and let P_m denote the path between x and y. We compute the probability that x and y lie in the same component. Using (2.7) we can find an exact expression for this probability. The trees T_i, $i=1,\dots,n$ obtained from $G\backslash P$ can be classified into two types. We refer to trees whose roots are adjacent to the end vertices of P as type A. The remaining trees are of type B. We further classify a tree of type B into one of the classes $\{B_j\}_{j=1}^{m-1}$ if the root of that tree is adjacent to the j^{th} vertex on the path P between x and y. Here, on a $d-$regular tree, the number of trees n is

$2(d-1)+(m-1)(d-2)$. For a tree T_i of type A, we have

$$\begin{aligned} a_0^i &= (d-1)^m + (d-1) + \sum_{k=1}^{m-1}(d-2)(d-1)^k \\ &= (d-1)^{m+1} + (d-1)^m. \end{aligned}$$

A similar calculation for a tree T_i of type B_j gives

$$\begin{aligned} a_0^i &= (d-1)^{j+1} + (d-1)^{m-j+1} + \sum_{k=1}^{m-1}(d-2)(d-1)^{|k-j|} \\ &= d(d-1)^j + d(d-1)^{m-j} - d. \end{aligned}$$

Hence, we have

$$\mu(x \sim y) = \sum_{i=1}^{n} \frac{1}{a_0^i} = \frac{2}{d(d-1)^{m-1}} + \sum_{j=1}^{m-1} \frac{(d-2)}{d(d-1)^j + d(d-1)^{m-j} - d},$$

where $x \sim y$ denotes the event that the path between x and y is open. Examining the sum obtained above we have, for constants c_1 and c_2 independent of m,

$$c_1 \frac{1}{(d-1)^{m/2}} \leq \sum_{j=1}^{m-1} \frac{(d-2)}{d(d-1)^j + d(d-1)^{m-j} - d} \leq c_2 \frac{1}{(d-1)^{m/2}}$$

as $m \to \infty$. So, the probability that two vertices at distance m from satisfies

$$\mu(x \sim y) \asymp (d-1)^{-m/2}$$

Distribution of the normalized label of the nucleus

Since, by definition, the label of the nucleus c_o of $K(o)$ is smaller than p, consider the normalized label $\frac{1}{p}X_{c_o}$. For any $\theta \in [0,1]$, its distribution is given by

$$\mu_p(X_{c_o} \leq p\theta) = p\theta + \sum_{n=1}^{\infty} q^{b(n-1)}(1-(1-p\theta)^{s(n)}).$$

This is obtained by decomposing the event depending on the distance of the nucleus c_o from o. When the nucleus is at distance n from o we have no nuclei in the ball of radius $n-1$ about o. Additionally, the minimal label of the vertices in $S(n)$, the sphere of radius n centered at o, is smaller than $p\theta$. For small values of p we have $1-p\theta \leq (1-p)^{\theta} \leq (1-p)^{\theta+\delta}$, where $\delta = \delta(p)$. Hence, using this lower bound we have

$$\mu_p(X_{c_o} \leq p\theta) \geq p\theta + \sum_{n=1}^{\infty} q^{b(n-1)} - q^{b(n-1)+\theta s(n)})) \geq p\theta + \sum_{n=1}^{\infty} \theta s(n) q^{c_n} \ln(1/q),$$

where $b(n)$ and $s(n)$ denotes the volume of a ball and sphere, of radius n, respectively and $b(n-1) < c_n < b(n-1) + \theta s(n)$. The last inequality is a consequence of applying the mean value theorem to the function q^x. Let $c'_n \in (b(n-1), b(n))$ satisfy the equation $q^{b(n-1)} - q^{b(n)} = (b(n)-b(n-1))q^{c'_n} \ln(1/q) = s(n)q^{c'_n} \ln(1/q)$. Then, we have $c'_n \geq c_n$. Hence,

$$\mu_p(X_{c_o} \leq p\theta) \geq p\theta + \sum_{n=1}^{\infty} \theta s(n) q^{c'_n} \ln(1/q) = \theta,$$

where the last equality follows from the observation that $\sum s(n) q^{c'_n} \ln(1/q) = \sum (q^{b(n-1)} - q^{b(n)}) = q^{b(0)}$. We note that this lower bound holds for any sequence $b(n)$, and under some further restrictions on growth, we can determine the distribution of this normalized label in the low-intensity limit.

Suppose $\lim s(n)/b(n-1) = 0$, then with c_n, c'_n as before it follows that for $n \geq N$ we have $c_n/c'_n > b(n-1)/b(n) > 1/(1+\epsilon)$. For p small and some $\delta = \delta(p) < 1$, we have

$$\begin{aligned}\mu_p(X_{c_o} < p\theta) &\leq \epsilon' + (\theta+\delta)\sum_N^\infty s(n) q^{c_n}\ln(1/q)\\ &\leq \epsilon' + (\theta+\delta)\sum_N^\infty s(n) q^{c'_n/(1+\epsilon)}\ln\Big(\frac{1}{q^{1/(1+\epsilon)}}\Big)\frac{1}{1+\epsilon} \leq \epsilon' + \frac{\theta+\delta}{1+\epsilon}.\end{aligned}$$

Hence, choosing p small and N large, we have an upper bound on θ, which we state in the following lemma.

Lemma 2.6. *For any graph G, we have $\liminf \mu_p(X_{c_o} < p\theta) \geq \theta$, for $\theta \in [0,1]$. Furthermore, if the growth sequence of the graph G satisfies $\lim_{n\to\infty} \frac{s(n)}{b(n-1)} = 0$, then we have that $\lim_{p\to 0} \mu_p(X_{c_o} < \theta) = \theta$.*

The limit measure μ_0 on other vertex-transitive graphs

In this section, we obtain some conditions on a vertex-transitive graph G such that the limit Bernoulli–Voronoi measure μ_0 exists and is trivial. Here, triviality means the measure is concentrated on the graph G.

Sufficient conditions for triviality

Proposition 2.4. *Let G be a vertex-transitive graph of degree d, with the growth of the volume of balls given by the sequence $\{b_n\}_n$. If $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = 1$ then μ_0 exists and is trivial.*

Proof. It suffices to show that for any fixed vertex o, we have $\lim_{p\to 0} \mu_p\big(\deg_\omega(o) = d\big) = 1$. We obtain a lower bound on this probability as follows. Let R be the distance from o to its nucleus c_o. If c_o is the unique nucleus in $S(o,R)$, and there are no nuclei in $S(o,R+1)\cup S(o,R+2)$, then c_o is the unique closest nucleus to all the neighbors of o. Consequently, on this event o

has degree d. Hence, we have

$$\mu_p\big(\deg(o)=d\big) \geq \sum_{n=1}^{\infty} q^{b_{n-1}} \binom{s_n}{1} p\, q^{s_n-1}\, q^{s_{n+1}+s_{n+2}}$$
$$= \frac{p}{q} \sum_{n=1}^{\infty} s_n\, q^{b_{n+2}}, \tag{2.35}$$

where we write s_n instead of $s(o,n)$.

By assumption we have $\lim \frac{b_{n+1}}{b_n} = 1$. Therefore, for any given ϵ we can choose $N = N(\epsilon)$ such that for all $n \geq N$, we have $|\frac{b_n}{b_{n-1}} - 1| < \epsilon$. It follows that for $n \geq N$ we have $b_{n+2} < (1+\epsilon)^3 b_{n-1}$, and $q^{b_{n+2}} > q^{(1+\epsilon)^3 b_{n-1}}$. Hence, substituting this in (2.35) we obtain

$$\mu_p\big(\deg(o)=d\big) \geq \frac{p}{q} \sum_{n=N}^{\infty} \big(b_n - b_{n-1}\big) q^{(1-\epsilon)^3 b_{n-1}} \geq \frac{p}{q} \int_{b_N}^{\infty} q^{(1+\epsilon)^3 x}\, dx$$
$$= \frac{q^{(1+\epsilon)^3 b_N}}{q} \cdot \frac{1}{(1+\epsilon)^3} \cdot \frac{p}{\ln(1/q)}.$$

Taking the low-intensity limit we obtain $\lim_{p\to 0} \mu_p\big(\deg(o)=d\big) = 1$. Hence, it follows that the limit measure μ_0 exists and is trivial. □

The hypothesis in the proposition above is satisfied by graphs whose volume growth is polynomial. Interestingly, the subexponential growth condition $\lim_n \sqrt[n]{b_n} = 1$ doesn't easily imply that the ratio of volumes of consecutive balls tends to 1. In fact, the existence of this limit for Cayley graphs of groups of sub-exponential growth remains open, and is stated as Question 34.B (page 206) in [dlH00].

An alternate way of obtaining a sufficient condition for triviality is to express the probability that a fixed edge is open, and obtain a condition which ensures that we get a telescoping sum. Consider the following formulation. For a neighbor v of o we define $A_v^{n,m} := \{u\colon d(o,u) = n,\, d(v,u) = m\}$ and $a_v(n,m) := |A_v^{n,m}|$ where $|n-m| \geq 0$. Using this notation we obtain a decomposition of the sphere of radius r, centered at o, given by $S(o,r) = A_v^{r,r-1} \cup A_v^{r,r} \cup A_v^{r,r+1}$. Using this notation we obtain another sufficient condition for the triviality of μ_0.

Proposition 2.5. *Let G be a infinite vertex-transitive graph, and o an arbitrary distinguished vertex, with $A_v^{n,m}$ and $a_v(n,m)$ as defined above. If, for all neighbors v of o, we have*

$$\lim_{n\to\infty} \frac{a_v(n,n)}{a_v(n,n)+a_v(n,n+1)+a_v(n+1,n)} + 2\lim_{n\to\infty}\frac{a_v(n,n+1)}{s_{n+1}} = 1, \tag{2.36}$$

and both limits exist, then the measure μ_0 exists and is trivial.

If G is a vertex-transitive bipartite graph, and for all neighbors v of o we have the limit $\lim_{n\to\infty}\frac{a_v(n,n-1)}{s_n} = \frac{1}{2}$, then μ_0 exists and is trivial.

Before we begin the proof we illustrate a case where condition 2.36 is satisfied. Consider the triangular lattice which is the dual of the hexagonal lattice. Here, the evaluation of $s(n)$ and $a_v(n,m)$ is relatively simple and can be obtained by viewing the sphere of radus n as a hexagon of side length $n-1$. As a result, for a fixed neighbour v of o, we can see that $s(n) = 6+6(n-1)$, $a_v(n,n+1) = 3+2(n-1)$ and $a_v(n,n-1) = 1+2(n-1)$. Consequently, the limits of the ratios in equation 2.36 can be easily evaluated and we see that the triangular lattice satisfies the condition in the above proposition.

Proof. Let R be the random distance at which o is connected to its nucleus c_o. For a fixed neighbor v of o, we can express the probability that the edge (o,v) is open as

$$\begin{aligned}
&\mu_p\big(\omega(o,v)=1\big) \\
&\quad = \mu_p\Big(c_o \in A_v^{R,\,R-1}\Big) + \mu_p\Big(c_o \in A_v^{R,\,R},\ M\big(A_v^{R,\,R-1}\big) > p\Big) + \\
&\quad + \mu_p\Big(c_o \in A_v^{R,\,R+1},\ M\big(A_v^{R,\,R} \cup A_v^{R,\,R-1}\big) > p,\ M\big(A_v^{R,\,R+1}\big) < M\big(A_v^{R+1,\,R+1} \cup A_v^{R+2,\,R+1}\big)\Big).
\end{aligned}$$

Conditioning on R, we can write the above probability as

$$\mu_p\big(\omega(o,v)=1\big) = \mathbb{P}\Big(M\big(A_v^{0,1}\big) < p,\ M\big(A_v^{0,1}\big) < M\big(A_v^{1,1} \cup A_v^{2,1}\big)\Big) +$$

$$+\sum_{n=1}^{\infty} q^{b_{n-1}}\mathbb{P}\Big(M\big(A_v^{n,n-1}\big) < p,\ M\big(A_v^{n,n-1}\big) < M\big(A_n^{n,n} \cup A_n^{n,n+1}\big)\Big)+$$

$$+\sum_{n=1}^{\infty} q^{b_{n-1}}\mathbb{P}\Big(M\big(A_v^{n,n}\big) < p,\ M\big(A_v^{n,n-1}\big) > p,\ M\big(A_v^{n,n}\big) < M\big(A_v^{n,n+1} \cup A_v^{n+1,n}\big)\Big)+$$

$$+\sum_{n=1}^{\infty} q^{b_{n-1}}\mathbb{P}\Big(M\big(A_v^{n,n+1}\big) < p,\ M\big(A_v^{n,n} \cup A_v^{n,n-1} \cup A_v^{n+1,n}\big) > p,$$

$$M\big(A_v^{n,n+1}\big) < M\big(A_v^{n+1,n+1} \cup A_v^{n+2,n+1}\big)\Big).$$

For notational simplicity we write $a(n,m)$ for $a_v(n,m)$, suppressing the dependence on v. Using Lemma 2.1, we can simplify this further as

$$\begin{aligned}
&\mu_p\big(\omega(o,v)=1\big) = \\
&\quad \frac{a(0,1)}{s(v,1)}\big(1-q^{s(v,1)}\big) + \sum_{n=1}^{\infty}\frac{a(n,n-1)}{s(o,n)}\big(q^{b_{n-1}}-q^{b_n}\big) + \sum_{n=1}^{\infty}\frac{a(n,n+1)}{s(v,n+1)}\big(q^{b_n}-q^{b_{n+1}}\big)+ \\
&\quad + \sum_{n=1}^{\infty}\frac{a(n,n)}{a(n,n)+a(n,n+1)+a(n+1,n)}\big(q^{b_{n-1}+a(n,n-1)}-q^{b_n+a(n+1,n)}\big) \\
&\quad = \frac{1}{s_1}(1+q)(1-q^{s_1}) + \ 2\sum_{n=1}^{\infty}\frac{a(n,n+1)}{s_{n+1}}(q^{b_n}-q^{b_{n+1}})+ \\
&\quad + \sum_{n=1}^{\infty}\frac{a(n,n)}{a(n,n)+a(n,n+1)+a(n+1,n)}\big(q^{b_{n-1}+a(n,n-1)}-q^{b_n+a(n+1,n)}\big),
\end{aligned} \tag{2.37}$$

where we used the observation that $a(n,n+1)=a(n+1,n)$ to obtain the second equality above.

If both the limits $\frac{a(n,n+1)}{s_{n+1}}$ and $\frac{a(n,n)}{a(n,n)+a(n,n+1)+a(n+1,n)}$ exist, then we can approximate the two infinite sums in (2.37) by telescoping sums, to obtain a limit as $q\to 1$. Imposing condition 2.36 we can ensure that the edge probability tends to 1 as $q\to 1$. If the triviality of the edge-marginals holds for all edges incident to o, then we would have that the measure μ_0 is itself trivial.

Note that when G is bipartite, we have that $a(n,n)=0$, for every edge (o,v) and $n>0$.

Hence, we can ignore the second sum in (2.37). In that case, we get

$$\lim_{p\to 0} \mu_p\big(\omega(o,v)=1\big) = 2 \lim_{p\to 0} \sum_{n=1}^{\infty} \frac{a(n,n+1)}{s_{n+1}} \left(q^{b_n} - q^{b_{n+1}}\right). \tag{2.38}$$

Therefore, if $\lim_{n\to\infty} \frac{a(n,n-1)}{s_n} = \frac{1}{2}$, by the same reasoning as in the general case, the above sum can be approximated by a telescoping series. Taking the limit as p tends to zero, we would obtain $\lim\limits_{p\to 0} \mu_p\big(\omega(o,v)=1\big) = 1$. □

When the limit in equation 2.36 exists and is not 1, this calculation doesn't prove the existence of the limit measure. However, assuming that the limit measure exists, it gives us a method of computing some elementary cylinder probabilities. Moreover, this method of showing triviality of the limit measure doesn't work on graphs of exponential growth. In particular, if the graph is bipartite, the condition $\lim_n(a(n,n-1)/s_n) = 1/2$ will not be met on such graphs, as we show in the following remark.

Remark 2.1. *Let G be an infinite bipartite, vertex-transitive graph. If $\lim\limits_{n\to\infty} \dfrac{a_v(n,n-1)}{s_n} = \dfrac{1}{2}$ for every $v \sim o$, then G has subexponential growth.*

Proof. For a fixed neighbour v of o we claim that $a_v(m,m+1) = a_v(m+1,m)$ $\forall m \geq 0$. Note that this is true for $m=0$ since $a_v(0,1) = a_v(1,0) = 1$. Assume that $a_v(n-1,n) = a_v(n,n-1)$ holds. We can write the cardinality of the sphere of radius n in two ways as follows:

$$|S(o,n)| = a_v(n,n-1) + a_v(n,n+1)$$
$$|S(v,n)| = a_v(n-1,n) + a_v(n+1,n).$$

Using the assumption above, and the vertex-transitivity of G, we conclude that $a_v(n,n+1) = a_v(n+1,n)$. This induction argument proves the claim. To prove the statement in the remark consider the following equation:

$$1 = \frac{s_n}{s_n} = \frac{a_v(n,n-1) + a_v(n,n+1)}{s_n} = \frac{a_v(n,n-1)}{s_n} + \frac{a_v(n+1,n)}{s_{n+1}} \frac{s_{n+1}}{s_n}.$$

Now, using the hypothesis that $\lim \frac{a_v(n,n-1)}{s_n} = \frac{1}{2}$ it follow from the equation above that $\lim \frac{s_{n+1}}{s_n} = 1$. This proves that G has sub-exponential growth.

□

Note that both Proposition 2.4 and Proposition 2.5 rely on the computation of some ratio of statistics of balls in G. In general, even when possible, computing these statistics can be exceptionally tedious. These ratios do capture more information than just the growth rate, as remarked earlier. Just to illustrate some of the difficulties of dealing with these ratios, we note that the existence of the limit of b_{n+1}/b_n is not necessarily independent of the choice of the generating set in a Cayley graph. As noted in [GdlH97], the limit of the ratio of the sizes of consecutive balls doesn't exist for $\mathbb{Z}_2 * \mathbb{Z}_3$, with the standard generators $\{a, b\}$, but it does exist for the generating set $\{a, ab\}$.

In the rest of this section, we compute the single-edge marginals in three cases: a Cayley graph of the lamplighter group on $\mathbb{Z}$, the Trofimov graph on $\mathbb{T}_3$, and a Cayley graph of the modular group $\mathbb{Z}_2 * \mathbb{Z}_3$. In the first two cases we show that the asymptotic probability exists and is strictly less than 1, but in the last case the limit doesn't exist.

Lamplighter group

For a particular Cayley graph of the lamplighter group, we compute $\lim_n a(n, n-1)/s_n$ and show that this limit is less than $\frac{1}{2}$. This would imply, by Proposition 2.5, that if μ_0 exists it must be non-trivial.

The calculation of this limit is simpler when we use the generators which make it the Diestel-Leader graph $D\text{-}L(2,2)$. In this case, the resulting graph is a subgraph of the Cartesian product of two 3-regular trees $\mathbb{T}$ and $\mathbb{T}'$, and it can be described as follows. Fix two ends ω and ω', and two vertices o and o' in the two trees $\mathbb{T}$ and $\mathbb{T}'$ respectively. This determines horodistance functions h and h' on the trees, using which we can describe the vertex set of $D\text{-}L(2,2)$ by $V(D\text{-}L(2,2)) = \{(x, x') \in \mathbb{T} \times \mathbb{T}' : h(x) + h'(x') = 0\}$.

The vertices (x, x') and (y, y') are adjacent if and only if $x \sim y$ in $\mathbb{T}$ and $x' \sim y'$ in $\mathbb{T}'$. We can visualize this graph by placing $\mathbb{T}$ and $\mathbb{T}'$ next to each other and in 'opposite directions', with o and o' on the same horizontal, such that the horocycles $H(-k)$ on $\mathbb{T}$ and $H'(k)$ lie on the same level, as depicted in the Figure 2.5 below.

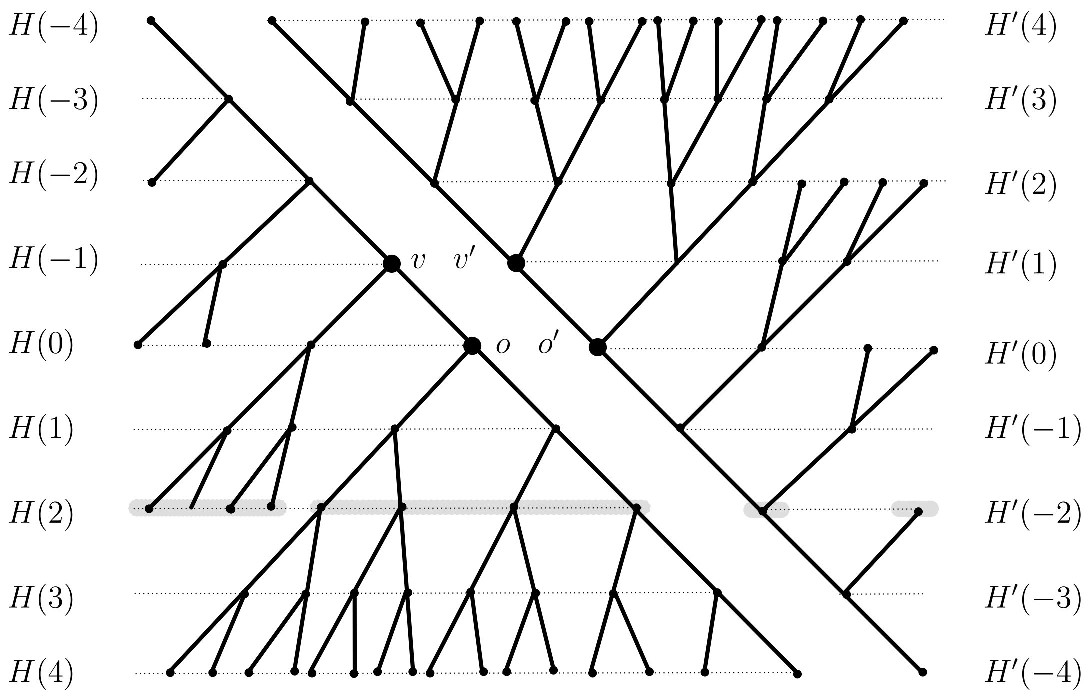


Figure 2.5: Vertices within a ball of radius 4 in D-$L(2,2)$. The set $H(2)$ is the union of the sets $L_0(2)$ and $L_1(2)$ which are indicated by contiguous shaded regions. Similarly $H'(-2)$ is the union of $R_0(-2)$ and $R_1(-2)$ and these are shown by contiguous shaded regions on the right

The graph distances can be represented in terms of the distances on the component trees and the horodistance function as $d\big((x,x'),(y,y')\big) = d(x,y) + d(x',y') - |h(x) - h(y)|$. Now, with these preliminaries we can determine the volume of the sphere of radius n. To keep track of the counting procedure, we introduce some notation. We use a decomposition of the horocycles into equivalence classes determined by the distances from o in $\mathbb{T}$ and o' in $\mathbb{T}'$ as follows

$$H(k) = \bigcup_{j=0}^{\infty} L_j(k) \text{ and } H'(k) = \bigcup_{j=0}^{\infty} R_j(k),$$

where $L_j(k) \coloneqq \{x \in H(k)\colon d(x,o) = |k| + 2j\}$ and $R_j(k) \coloneqq \{x \in H(k)\colon d(x',o') = |k| + 2j\}$. Furthermore, for $k > 0$, we have $|L_0(k)| = 2^k$, $|L_0(-k)| = 1$, $|L_j(k)| = 2^{k+j-1}$ and $|L_j(-k)| = 2^{j-1}$, for $j > 0$. As a consequence of symmetry, we have $|R_j(k)| = |L_j(-k)|$, for all k.

If $(x,x') \in S(n)$, then $x \in H(k)$ and $x' \in H(-k)$ for some k. Note that we use $S(n)$ for $S((o,o'),n)$ the sphere of radius n centered at (o,o) in D-$L(2,2)$. Using the above formulation, we have $x \in L_j(k)$ and $x' \in R_i(-k)$, for some i and j. Using the graph distance, we have the relation $n = d\big((o,o'),(x,x')\big) = (|k| + 2j) + (|k| + 2i) - |k|$, which yields $i = ((n - |k|)/2) - j$. So, if $(x,x') \in S(n)$, then for some k with the same parity as n, and $|k| \le n$ we have

$$(x,x') \in \bigcup_{j=0}^{\frac{n-|k|}{2}} \Big(L_j(k) \times R_{\frac{n-|k|}{2}-j}(-k)\Big).$$

Due to the parity condition, we compute volume of $S(n)$ separately for n even and odd, beginning with n even.

Using symmetry, we can restrict our attention to (x,x'), with $x \in H(-k)$, for $0 \le k \le n$. We have,

$$|S(n)| = 2 \sum_{\substack{2 \le k \le n,\\ k \text{ even}}} \sum_{j=0}^{\frac{n-k}{2}} \Big(|L_j(-k)| \cdot |R_{\frac{n-k}{2}-j}(k)|\Big) + \sum_{j=0}^{n/2} |L_j(0)| \cdot |R_{\frac{n}{2}-j}(0)|$$

where the terms in the second summation correspond to the vertices in $H(0)$. Expanding the above expression further, we get

$$|S(n)| = 2\big(|L_0(-n)| \cdot |R_0(n)|\big) + 2 \sum_{\substack{k=2,\\ k \text{ even}}}^{n-2} \Big(|L_0(-k)| \cdot |R_{\frac{n-k}{2}}(k)| + |L_{\frac{n-k}{2}}(-k)| \cdot |R_0(k)|\Big)$$

$$+\,2\sum_{\substack{k=2,\\ k\text{ even}}}^{n-2}\sum_{j=1}^{\frac{n-k}{2}-1}\Big(|L_j(-k)|\cdot|R_{\frac{n-k}{2}-j}(k)|\Big)+\Big(|L_0(0)|\cdot|R_{\frac{n}{2}}(0)|\Big)+\Big(|L_{\frac{n}{2}}(0)|\cdot|R_0(0)|\Big)$$

$$+\sum_{j=1}^{\frac{n}{2}-1}|L_j(0)|\cdot|R_{\frac{n}{2}-j}(0)|.$$

Taking $n=2r$ and $k=2m$, and substituting the size of each of the sets appearing above, we have the simplification,

$$|S(2r)|=2(2^{2r})+2\sum_{m=1}^{r-1}(2^{r+m}+2^{r-m-1}2^{2m})+2\sum_{m=1}^{r-1}\sum_{j=1}^{r-m-1}2^{j-1}2^{2m+r-m-j-1}$$
$$+\,2^{r-1}+2^{r-1}+\sum_{j=1}^{r-1}2^{j-1}2^{r-j-1}.$$

Evaluating this expression, we obtain the volume of a sphere of even radius

$$S(2r)=9(2^{2r-1})-2^{r-2}(3r+13) \tag{2.39}$$

Using the same approach, we find $|S(2r+1)|$. In this case, vertices $x\in\mathbb{T}$ with $x\in H(0)$ needn't be considered by the parity condition. Hence, we have

$$|S(n)|=2\sum_{\substack{k=1,\\ k\text{ odd}}}^{n}\sum_{j=0}^{\frac{n-k}{2}}\Big(|L_j(-k)|\cdot|R_{\frac{n-k}{2}-j}(k)|\Big)$$
$$=2|L_0(-n)|\cdot|R_0(n)|+2\sum_{\substack{k=1,\\ k\text{ odd}}}^{n-2}\Big(|L_0(-k)|\cdot|R_{\frac{n-k}{2}}(k)|+|L_{\frac{n-k}{2}}(-k)|\cdot|R_0(k)|\Big)$$
$$+\,2\sum_{\substack{k=1,\\ k\text{ odd}}}^{n-2}\sum_{j=1}^{\frac{n-k}{2}-1}\Big(|L_j(-k)|\cdot|R_{\frac{n-k}{2}-j}(k)|\Big).$$

Taking $n = 2r + 1$ and $k = 2m - 1$ and simplifying, we have

$$|S(2r+1)| = 2(2^{2r+1}) + 2\sum_{m=1}^{r}(2^{m+r-1} + 2^{m+r-1}) + 2\sum_{m=1}^{r}\sum_{j=1}^{r-m}(2^{j-1}2^{2m-1+r-m+1-j-1})$$
$$= 2^{2r+2} + 2^r(5(2^r) - r - 5).$$

Hence, we have the volume of a sphere of odd radius given by

$$|S(2r+1)| = 9(2^{2r}) - 2^r(r+5). \tag{2.40}$$

Now, consider a neighbor (v, v') of (o, o'), such that $v \in H(-1)$ and $v' \in H'(-1)$, with v and v' on the spine of the trees $\mathbb{T}$ and $\mathbb{T}'$, as depicted in Figure 2.5. Since D-$L(2,2)$ is a bipartite graph, we have $A^{n,n} = \varnothing$. Consider $A^{n,n-1} = S\big((o,o'), n\big) \cap S\big((v,v'), n-1\big)$.

Let n and k be odd, with $k > 0$. For $(x, x') \in S\big((o,o'), n\big)$, we have $d\big((o,o'),(v,v')\big) = d(o,v) + d(o',v') + |h(v) - h(x)|$, where $h(v) = -1$. If $(x,x') \in L_j(-k) \times R_{\frac{n-k}{2}-j}(k)$, we have $|h(x) - h(v)| = |-k+1| = k-1$. Since $k > 0$, the path from o to x must contain v, and as a result $d(o,x) = d(v,x) + 1$. If $d\big((v,v'),(x,x')\big) = n-1$, we must have $d(o',x') = 1 + d(v',x')$, i.e., v' lies on the path from o' to x'. This property is satisfied by only half the vertices in the set $R_0(k)$. Hence, the number of possible vertices in $\bigcup_{k>0}\Big(H(-k) \times H'(k)\Big) \bigcap A^{n,n-1}$ is given by

$$\frac{1}{2}\sum_{\substack{k=1,\\ k \text{ odd}}}^{n} |L_{\frac{n-k}{2}}(-k)| \cdot |R_0(k)| = \frac{1}{2}\Big(|L_0(-n)| \cdot |R_0(n)| + \sum_{\substack{k=1,\\ k \text{ odd}}}^{n-2} |L_{\frac{n-k}{2}}(-k)| \cdot |R_0(k)|\Big)$$
$$= \frac{1}{2}(2^{2r+1} + \sum_{m=1}^{r} 2^{r+m-1}) = 3(2^{2r-1}) - 2^{r-1}.$$

Next, we consider the case when $(x,x') \in H(k) \times H'(-k)$, for $k > 0$. For such vertices we have $|h(x) - h(v)| = k + 1$. Since $x' \in H'(-k)$ and $k > 0$, it follows that the path from o' to x' doesn't include v', and hence $d(o',x') + 1 = d(v',x')$. If $d\Big((v,v'),(x,x')\Big) = n-1$, then it follows that $d(o,x) - 1 = d(v,x)$, i.e., v lies on the path from x to o. This implies $x \notin L_0(k)$.

In addition to this, if $(x,x') \in H(n) \times H'(-n)$, then $d\big((v,v'),(x,x')\big) \geq n+1$.

Hence, the number of vertices in $\bigcup_{k>0}\big(H(k)\times H'(-k)\big)\cap A^{n,n-1}$ is given by the expression

$$\sum_{\substack{k=1,\\ k \text{ odd}}}^{n-2} \sum_{j=1}^{\frac{n-k}{2}} |L_j(k)| \cdot |R_{\frac{n-k}{2}-j}(-k)| = \sum_{\substack{k=1,\\ k \text{ odd}}}^{n-2} \Big(|L_{\frac{n-k}{2}}(k)| \cdot |R_0(-k)| + \sum_{j=1}^{\frac{n-k}{2}-1} |L_j(k)| \cdot |R_{\frac{n-k}{2}-j}(-k)| \Big)$$
$$= \sum_{m=1}^{r} (2^{r+m-1} + \sum_{j=1}^{r+m-1} 2^{r+m-2}) = 3(2^{2r-1}) - 2^{r-1}(r+3).$$

Using the above two expressions, we have

$$a(2r+1,2r) = 6(2^{2r-1}) - 2^{r-1}(r+4).$$

Since $a(n,n-1) = a(n-1,n)$ we also have

$$a(2r+2,2r+1) = a(2r+1,2r+2) = s(2r+1) - a(2r+1,2r)$$
$$= 6(2^{2r}-2)^{r-1}(r+6).$$

Hence,

$$\lim_{r\to\infty} \frac{a(2r+1,2r)}{s(2r+1)} = \lim_{r\to\infty} \frac{3(2^r) + 2^{r-1}(r+4)}{9(2^{2r}) - 2^r(r+5)} = \frac{1}{3}$$

and

$$\lim_{r\to\infty} \frac{a(2r+2,2r+1)}{s(2r+2)} = \lim_{r\to\infty} \frac{3(2^{2r+1}) - 2^{r-1}(r+6)}{9(2^{2r+1}) - 2^{r-1}(3r+16)} = \frac{1}{3}.$$

It now follows from Proposition 2.5 that if the limit Bernoulli–Voronoi measure exists, then it must be non-trivial on D-$L(2,2)$.

Trofimov Graph

The Trofimov graph, or the grandfather graph, is constructed from a three-regular tree $\mathbb{T}_3$ (or more generally $\mathbb{T}_d$) as follows. The vertex set of the Trofimov graph is the same as the vertex set $V(\mathbb{T}_3)$. Fix an arbitrary distinguished vertex o. The ends of $\mathbb{T}_3$ can be identified with the set of infinite simple paths from o. Fix an end. Each vertex u has a unique edge in the direction of the chosen end. Letting (u, v) be this edge, we say v is the father of u. Extending this further, we say that the unique father of v is the grandfather of u. Hence, for each vertex we can assign a unique grandfather vertex. The Trofimov graph is obtained by adding to the set of edges of $\mathbb{T}_3$ new edges between every vertex and its grandfather vertex.

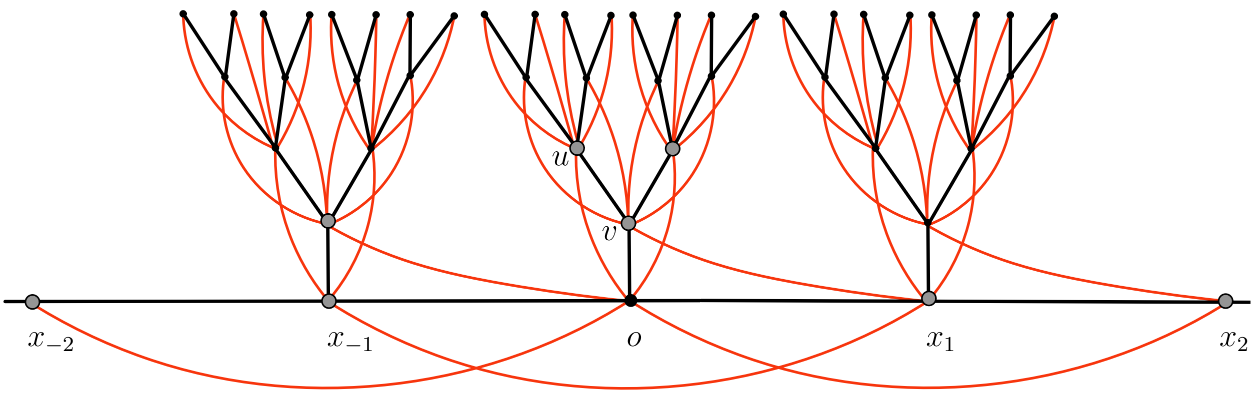


Figure 2.6: The edges of $\mathbb{T}_3$ are drawn black, and the edges between a vertex and its grandfather vertex are drawn red. The distinguished end is represented by the infinite path from o to the right. The grey vertices represent the sphere of radius 1 centered at o.

As in the figure above, fix a bi-infinite path and label by $\{x_n\}_{n>0}$ the vertices in the direction of the end in that order. Similarly label the vertices on the other side of o by $\{x_{-n}\}_{n>0}$. We refer to the subtrees of $\mathbb{T}_3$ rooted at x_k as T_{x_k} and the subtree rooted at o as T_o. Furthermore we refer to the vertices at depth m in this tree as $T_{x_k}(m)$. The trees T_{x_k} are isomorphic to binary trees with an edge appended to the root, hence $|T_{x_k}(m)| = 2^{m-1}$ for $m \geq 1$.

To compute s_n, we first note that $S(n) \cap T_{x_k} = \varnothing$ if $|k| > 2n$. Within T_o we have

$S(n) \cap T_o = T_o(2n) \cup T_o(2n-1)$. All paths from o to vertices in T_{x_k}, $k > 0$, pass through x_k and $d(o, x_{2j}) = d(o, x_{2j-1}) = j$ for $0 < j < n-1$. Hence, for $0 < k < 2n-1$

$$S(n) \cap T_{x_k} = T_{x_k}(2(n-j)) \cup T_{x_k}(2(n-j)-1),$$

where $j = \lceil k/2 \rceil$, and

$$|S(n) \cap T_{x_k}| = \frac{3}{4}\, 4^{n-j}.$$

For $j \in \{2n-1, 2n\}$, we have $S(n) \cap T_{x_j} = \{x_j\}$.

The trees on either side of o are not symmetric with respect to the distance function. For the subtrees indexed by even integers $-2j$, there are geodesic paths from o to vertices in these trees passing through x_{-2j}. As a result we have

$$S(n) \cap T_{x_{-2j}} = T_{x_{-2j}}(2(n-j)) \cup T_{x_{-2j}}(2(n-j)-1),$$

and

$$|S(n) \cap T_{x_{-2j}}| = \frac{3}{4}\, 4^{n-j},$$

for $0 < j < n$. For the odd integers $-2j+1$, we have

$$S(n) \cap T_{n_{-2j+1}} = T_{x_{-2j+1}}(2(n-j)) \cup T_{x_{-2j+1}}(2(n-j)+1),$$

and

$$|S(n) \cap T_{n_{-2j+1}}| = \frac{3}{2}\, 4^{n-j},$$

for $0 < j < n$. Furthermore, we have $S(n) \cap T_{-2n} = \{x_{-2n}\}$, and $S(n) \cap T_{x_{-2n+1}} = \{x_{-2n+1}\} \cup T_{x_{-2n+1}}(1)$. Hence, summing up sizes of these fragments of $S(n)$, we have

$$s_n = 2\sum_{j=1}^{n-1} \frac{3}{4}\, 4^{n-j} + 2 + 3(4^{n-1}) + \sum_{j=1}^{n-1} \frac{3}{4}\, 4^{n-j} + \sum_{j=1}^{n-1} \frac{3}{2}\, 4^{n-j} + 3$$

$$= 8(4^{n-1}).$$

Note that G is not edge transitive. There is no automorphism of G which takes a red edge onto a black edge as depicted in the figure above. Hence, we consider the two cases separately and being with the edge (o, v). We compute $a_v(n, n), a_v(n, n+1)$ and $a_v(n, n-1)$, which gives us a decomposition of s_n. We have

$$A_v^{n,n} = T_o(2n) \cup \bigcup_{j=1}^{n} \big(S(n) \cap T_{x_{2j-1}}\big) \implies a_v(n, n) = 3(4^{n-1}).$$

Similarly, all vertices in $S(n)$ lying in the subtrees indexed by the negative integers and positive even integers are in $A_v^{n,n+1}$,

$$A_v^{n,n+1} = \bigcup_{k=-1}^{-2n} \big(S(n) \cap T_{x_k}\big) \cup \bigcup_{j=1}^{n} \big(S(n) \cap T_{x_{2j}}\big) \implies a_v(n, n+1) = 4^n.$$

Finally we have that the only vertices closer to v than to o in $S(n)$ are those in $T_o(2n-1)$, hence

$$A_v^{n,n-1} = T_o(2n-1) \implies a_v(n, n-1) = 4^{n-1}.$$

Using (2.37), we can show the following limit

$$\lim_{p \to 0} \mu_p\big(\omega((o, v)) = 1\big) = \lim_{n \to \infty} 2\,\frac{4^n}{8(4^n)} + \lim_{n \to \infty} \frac{3(4^{n-1})}{3(4^{n-1}) + 4^n + 4^n} = \frac{23}{44}.$$

Now, considering the other edge (o, u), we see that all the vertices in $S(n)$ which are not in the subtree T_o are strictly further from u than o. In addition to this, half of the vertices in $T_o(2n)$, i.e., those vertices for which geodesic paths from u must include v, are also strictly further from u than from o. Hence

$$a_u(n, n+1) = \Big| \bigcup_{k=1}^{n} \big(S(n) \cap T_{x_k}\big) \cup \bigcup_{k=-1}^{-2n} \big(S(n) \cap T_k\big) \Big| + \frac{1}{2} \, |T_o(2n)|$$
$$= \frac{3}{2} \, 4^n.$$

All vertices in $S(n) \cap T_o$ such that the geodesic paths from u to these vertices don't include v are strictly closer to u than to o. Hence

$$a_u(n, n-1) = \frac{1}{2} \, \big|T_o(2n) \cup T_o(2n-1)\big| = \frac{3}{2} \, 4^{n-1}.$$

Finally, all the vertices not accounted for lie in $A_u^{n,n}$,

$$a_u(n, n) = \frac{1}{2} \, \big|T_o(2n-1)\big| = \frac{1}{2} \, 4^{n-1}.$$

Once again using (2.37), we obtain

$$\lim_{p \to 0} \, \mu_p(\omega(o, u)) = 1) = \lim_{n \to \infty} 2 \, \frac{\frac{3}{2} 4^n}{8(4^n)} + \lim_{n \to \infty} \frac{\frac{1}{2} 4^{n-1}}{\frac{1}{2} 4^{n-1} + \frac{3}{2} 4^n + \frac{3}{2} 4^n}$$
$$= \frac{83}{200}.$$

The Trofimov graph is yet another example of a vertex-transitive graph on which, if μ_0 were to exist, it would be non-trivial.

Non-Existence of the limit measure

In this section, we prove that the low-intensity of the Bernoulli-Voronoi measure doesn't always exist. In particuar, we show that it doesn't exist on the standard Cayley graph of the Modular group $\mathbb{Z}_2 * \mathbb{Z}_3$. The graph is shown in figure 2.7 below.

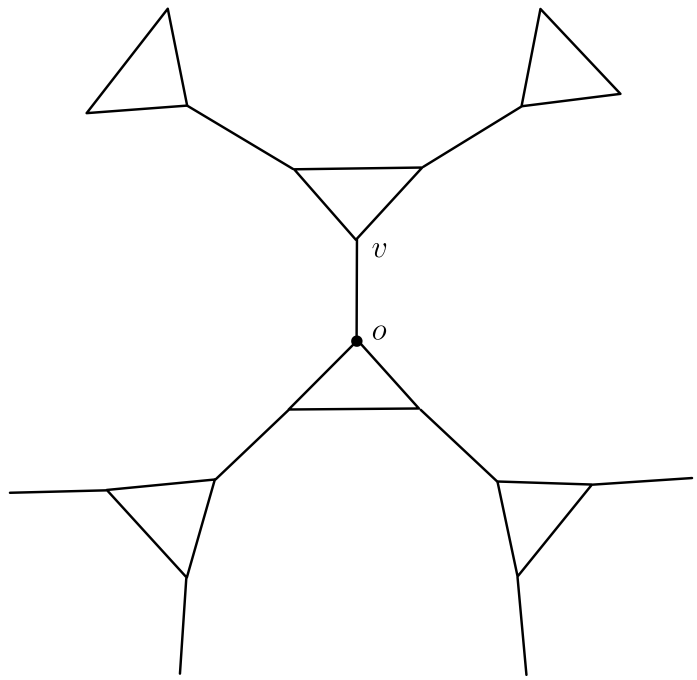


Figure 2.7: The ball of radius 4 of the standard Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$.

This is a regular graph of degree 3 containing cycles of length 3 and no other cycles. We can obtain this graph from the 3−regular tree by replacing every vertex there by a cycle of length 3. Furthermore, it is straightforward to compute the volume of balls and spheres. Since we require these expressions in the computation of the limit measure, we state these below:

$$\begin{aligned} s(2n) &= 2 \cdot 2^n, \qquad s(2n+1) = 3 \cdot 2^n, \\ b(2n) &= 7 \cdot 2^n - 6, \quad b(2n+1) = 10 \cdot 2^n - 6. \end{aligned} \tag{2.41}$$

Note that this is a bi-partite graph and as a result $a(n,n) = 0, \forall n$. The volumes of the subsets of the sphere are also straight forward to calculate and are given below:

$$a(2n, 2n-1) = a(2n, 2n+1) = a(2n+1, 2n) = a(2n-1, 2n) = 2^n. \tag{2.42}$$

Theorem 2.2. *The low-intensity limit of the Bernoulli-Voronoi measure on the standard Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ does not exist.*

Proof. It suffices to show that for some fixed edge the limit of the single-edge marginal probability doesn't exist. We will show that the single-edge marginal of the edge marked (o, v) in figure 2.7 deosn't exist. Consider the expression for the limit edge marginal from the proof of proposition 2.5

$$\lim_{p\to 0} \mu_p\big(\omega(o, v) = 1\big) = 2 \lim_{q\to 1} \sum_{n=1}^{\infty} \frac{a(n, n+1)}{s_{n+1}} \left(q^{b_n} - q^{b_{n+1}}\right). \tag{2.43}$$

Using the expressions in equation 2.42 and equation 2.41, we note that $a(n, n+1)/s_{n+1}$ is $1/3$ when n is even, and $1/2$ otherwise. We can rewrite the sum on the right in 2.43 as follows:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a(n, n+1)}{s_{n+1}} \left(q^{b_n} - q^{b_{n+1}}\right) &= \frac{1}{3} q^{b_1} + \frac{1}{6} \sum_{n=1}^{\infty} \left(q^{b_{2n-1}} - q^{b_{2n}}\right) \\ &= \frac{1}{3} q^{b_1} + \frac{1}{6q^6} \sum_{n=1}^{\infty} \left(q^{5\cdot 2^n} - q^{7\cdot 2^n}\right). \end{aligned} \tag{2.44}$$

To show that the edge marginal limit does not exist it suffices to show that the limit of the sum on the right in the equation above does not exist, as q tends to 1. For simplicity, abusing the notation to rewrite q^5 as q we claim that the limit below does not exist:

$$\lim_{q\to 1} \sum_{n=1}^{\infty} \left(q^{2^n} - q^{2^{n+\alpha}}\right), \tag{2.45}$$

where $\alpha = \log_2(7/5)$. Note that an equivalent formulation of the edge marginal in $\mathbb{T}_3$ would have required that the above limit exist for $\alpha = 1$, in which case we have a telescoping series.

Taking the limit in equation 2.45 along the subsequence $\{q_0^{2^{-m}}\}_m$, for some $q_0 \in (0, 1)$, we note that, if the limit were to exist, it must be given a doubly infinite series, i.e.,

$$\lim_{q\to 1}\sum_{n=1}^{\infty}\left(q^{2^n}-q^{2^{n+\alpha}}\right)=\sum_{n=-\infty}^{\infty}\left(q_0^{2^n}-q_0^{2^{n+\alpha}}\right).$$

Writing $f(q_0)$ for the doubly infinite series on the right above, in order to prove that the limit in 2.45 does not exist, it suffices to show that $f(q')\neq f(q'')$ for some $q',q''\in(0,1)$. We show this using a numerical estimate

Note that the doubly infinite series is a series of positive terms. We first obtain an upper bound on the series $f(q)$ and show that the sum of sufficiently many terms in the series $f(q')$ is strictly larger than the upper bound on $f(q'')$ for some $q',q''\in(0,1)$.

To obtain an upper bound on $f(q)$ we obtain upper bounds on the right and left tails. Consider the tail on the right in $f(q)$. For $m>0$, we have

$$\sum_{n=m+1}^{\infty}\left(q^{2^n}-q^{2^{n+\alpha}}\right)=q^{2^{m+1}}+\sum_{n=m+1}^{\infty}\left(q^{2^{n+1}}-q^{2^{n+\alpha}}\right)<q^{2^{m+1}}, \tag{2.46}$$

where the inequality follows from the observation that the series in the middle is a series of negative terms.

To get a bound on the left tail we compare the term $q^{2^{n+\alpha}}$ to $q^{2^{n+1}}$ as follows:

$$\begin{aligned}\sum_{n=-\infty}^{-m-1}\left(q^{2^n}-q^{2^{n+\alpha}}\right)<\sum_{n=-\infty}^{-m-1}\left(q^{2^n}-q^{2^{n+1}}\right)&=\lim_{k\to-\infty}\sum_{n=k}^{-m-1}\left(q^{2^n}-q^{2^{n+1}}\right)\\&=1-q^{2^{-m}}.\end{aligned} \tag{2.47}$$

Using both the tail bounds, for $m>0$, we have

$$f(q)<1-q^{2^{-m}}+\sum_{n=-m}^{m}\left(q^{2^n}-q^{2^{n+\alpha}}\right)+q^{2^{m+1}}. \tag{2.48}$$

Taking $q' = 0.2$ and $q'' = 0.1$, we have

$$f(0.2) > \sum_{n=-20}^{20} \left(0.2^{2^n} - 0.2^{2^{n+\alpha}}\right) = 0.4854290723337563$$

$$f(0.1) < 1 - 0.1^{2^{-20}} + \sum_{n=-20}^{20} \left(0.1^{2^n} - 0.1^{2^{n+\alpha}}\right) + 0.1^{2^{21}} = 0.485425438025593.$$

This shows that $f(q)$ is not constant on the interval $(0, 1)$. In particular, the limit in equation 2.45 does not exist thereby proving the theorem. □

Chapter 3

Poisson–Voronoi construction in Euclidean and Hyperbolic spaces

Consider the Poisson process with intensity λ in the Euclidean space $\mathbb{R}^d$. Motivated by the constructions in the previous chapter, we consider the Poisson points as nuclei and obtain a partition of $\mathbb{R}^d$ into Voronoi cells, a sample of which is shown in Figure 3.1. Here, the object of study is the set of boundaries of the Voronoi cells, which we view as a random closed set, ∂. Analogous to the discrete case, a question of interest is whether there is a natural limit as the intensity of the Poisson process is reduced to zero. In this case, we are interested in the number of components produced by the limit, as well as the structure of the limiting random closed set. Comparing this to the limit Bernoulli–Voronoi measure on $\mathbb{Z}^d$ we might expect the limit to be trivial and we show that this is indeed the case. Here, by triviality we mean there is a unique Voronoi cell.

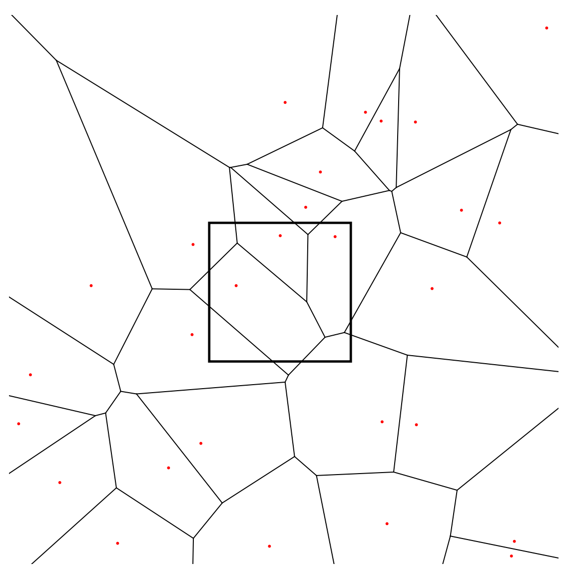


Figure 3.1: The picture within the bold square is a sample of the Poisson–Voronoi bounadaries on $\mathbb{R}^2$ generated by Poisson nuclei of intensity $\lambda = 2$.

Let $\mathcal{P}_\lambda$ denote the collection of Poisson points of intensity λ, and for any $x \in \mathbb{R}^d$ let $c_x \in \mathcal{P}_\lambda$ be the closest nucleus to x. Imitating the notation of the previous chapter, for a fixed point $o \in \mathbb{R}^d$, let $K_\lambda(o)$ denote the Voronoi cell of o, i.e.,

$$K_\lambda(o) \coloneqq \{x \in \mathbb{R}^d \mid d(x, c_o) < d(x, y), \forall y \in \mathcal{P}_\lambda\}.$$

To show that in the limit we obtain the trivial partition of $\mathbb{R}^d$, it suffices to show that for any $m > 0$, we have $B(o, m) \subset K_\lambda(o)$ with probability close to one as λ tends to zero. For the remainder of the chapter by $B(o, r)$ we mean the closed ball of radius r centered at o.

Proposition 3.1. *Let o be an arbitrary fixed point in $\mathbb{R}^d$, for $d \geq 1$. For any $m > 0$, we have*

$$\lim_{\lambda \to 0} \mu_\lambda\big(B(o, m) \subset K_\lambda(o)\big) = 1. \tag{3.1}$$

This implies $\mu_0\big(K(o) = \mathbb{R}^d\big) = 1$.

Proof. Suppose o is associated to a nucleus at a distance $r > m$, and assume $B(o, m) \subsetneq K_\lambda(o)$. It follows, by definition, that $\bigcup_{x \in B(o,m)} B(x, d(x, c_o))$ must contains no nuclei. To obtain a lower bound on the probability that $B(o, m) \subsetneq K_\lambda(o)$, we use the fact that $\bigcup_{x \in B(o,m)} B(x, d(x, c_o)) \subset B(o, r+2m)$ and estimate the probability that $B(o, r+2m) \setminus B(o, r)$ contains no nuclei other than c_o. Hence, representing the point c_o in polar co-ordinates as (r, θ), we have

$$\begin{aligned}
\mu_\lambda\big(B(o, m) \subsetneq K_\lambda(o)\big) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \exp\Big(- \lambda \Big| \bigcup_{x \in B(o,m)} B(x, d(x, (r, \theta))) \setminus B(o, r) \Big| \Big) \times \\
&\qquad \times \lambda \exp\big(- \lambda |B(o, r)|\big) \frac{d}{dr} |B(o, r)| \; dr \, d\theta \\
&\geq \int_m^{\infty} \lambda \exp\big(- \lambda |B(o, r + 2m)|\big) \frac{d}{dr} |B(o, r)| \; dr \, d\theta.
\end{aligned} \tag{3.2}$$

We note that $|B(o, r)| = Cr^d$, for some dimension dependent constant $C = C(d)$. Since

m is fixed, given any $\delta > 0$, we can choose $R = R(\delta)$ such that for all $r \geq R$ we have $\dfrac{r+2m}{r} \leq 1+\delta$. Given $\delta > 0$, we use such a choice of R to simplify the lower bound in (3.2) to obtain

$$\begin{aligned}\mu_\lambda\big(B(o,m) \subsetneq K_\lambda(o)\big) &\geq \int_R^\infty \lambda \exp\big(-\lambda\,(1+\delta)^d |B(o,r)|\big)\,\frac{d}{dr}|B(o,r)|\,dr \\ &= \frac{1}{(1+\delta)^d}\exp\big(-\lambda\,(1+\delta)^d\,|B(o,r)|\big).\end{aligned}$$

Taking the limit in λ, and noting that δ was an arbitrary positive number, we obtain

$$\lim_{\lambda\to 0} \mu_\lambda\big(B(o,m) \subset k_\lambda(o)\big) = 1. \qquad \square$$

The triviality of the limit here will be seen to be in contrast to the situation in $\mathbb{H}^2$, again resembling the contrast seen in the discrete case.

For the remainder of this chapter we consider the the Poincaré disk $\mathbb{H}^2$, represented by the interior of the unit disk $\mathbb{D}$ in $\mathbb{R}^2$, with the metric $d_H(x,y) := \operatorname{arccosh}\big(1+\delta(x,y)\big)$, where $\delta(x,y) := 2\,d(x,y)^2/\big((1-d(o,x)^2)(1-d(o,y)^2)\big)$ and $d(\cdot,\cdot)$ is the Euclidean metric on $\mathbb{D}$. More conveniently, distances from the origin have the form

$$d_H(o,x) = \log\frac{1+d(o,x)}{1-d(o,x)}.$$

Geodesics in this model are either straight lines through o or arcs of circles which are orthogonal to $\partial\,\mathbb{D}$, and this characterization is useful in what follows. Lastly, to help keep track of which metric is being used, we write $B_H(o,t)$ to represent the ball about o with hyperbolic radius t, and $B(o,t)$ to represent the ball with Euclidean radius t. When the context is clear and the radius mentioned is the hyperbolic radius we wil drop the subscript H in $B_H(o,\rho)$. In all cases to follow, $|A|$ denotes the hyperbolic area of the set A; we avoid $|\cdot|_H$ since we will only deal with hyperbolic areas. For more details and basic results refer [CFK⁺97].

Proposition 3.2. *Let $\mathcal{P}_\lambda$ be a Poisson process with intensity λ in $\mathbb{H}^2$. For an arbitrary fixed* *point $o \in \mathbb{H}^2$ and fixed $m > 0$, we have*

$$\lim_{\lambda \to 0} \mu_\lambda\big(B(o,m) \subset K_\lambda(o)\big) < 1.$$

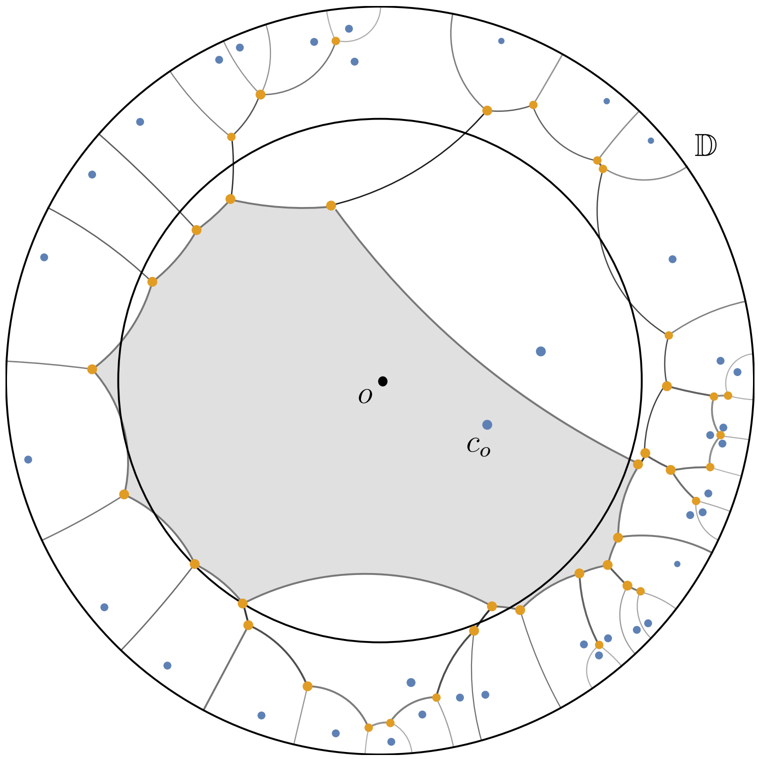


Figure 3.2: Sample of the Poisson–Voronoi boundaries with intensity $\lambda = 0.05$ in $B(o, 0.7)$. The shaded region is the Voronoi cell of o, with the blue points denoting nuclei and the orange points Voronoi vertices. Poisson points are generated in a large ball, whose radius is dependent on the position of c_o, such that the nucleus of every point within $B(o, 0.7)$ is determined.

Proof. Following the same method as in the previous proof, we suppose that o is associated to the nucleus c_o at distance r. If every point in $B(o,m)$ is associated to c_o, this would imply that there are no nuclei in the region $\bigcup_{x \in B(o,m)} B(x, d(x, c_o))$. Hence, we can write the required probability as

$$\mu_\lambda\big(B(o,m) \subset K_\lambda(o)\big) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \exp\Big(-\lambda\Big|\bigcup_{x\in B(o,m)} B(x,d(x,(r,\theta)))\setminus B(o,r)\Big|\Big)\times$$

$$\times\,\lambda\,\exp\big(-\lambda|B(o,r)|\big)\,\frac{d}{dr}|B(o,r)|\;dr\,d\theta. \tag{3.3}$$

To obtain an upper bound on this probability we use the fact that

$$B\Big((m,\pi+\theta), d\big((m,\pi+\theta),c_o\big)\Big) \subset \bigcup_{x\in B(o,m)} B\Big(x, d\big(x,(r,\theta)\big)\Big).$$

Hence,

$$\begin{aligned}\mu_\lambda\big(B(o,m)\subset K_\lambda(o)\big) &\geq \int_0^\infty \lambda\,\exp\big(-\lambda\,\big|B(o,r+m)\big|\big)\frac{d}{dr}|B(o,r)|\\ &= \int_0^\infty 2\pi\lambda\sinh(r)\exp\big(-2\pi\lambda(\cosh(r+m)-1)\big),\end{aligned}$$

where we used $|B(o,r)| = 2\pi(\cosh(r)-1)$. To simplify this further, we note that

$$\frac{\sinh(r)}{\sinh(m+r)} = \frac{\sinh(r)}{\sinh(m)\cosh(r)+\cosh(m)\sinh(r)} \leq \frac{1}{\sinh(m)+\cosh(m)}.$$

Using this, we obtain

$$\begin{aligned}&\mu_\lambda\big(B(o,m)\subset K_\lambda(o)\big)\\ &\qquad\leq \frac{1}{\sinh(m)+\cosh(m)}\int_0^\infty 2\pi\lambda\sinh(r+m)\exp\big(-2\pi\lambda(\cosh(r+m)-1)\big)\,dr\\ &\qquad= \frac{1}{\sinh(m)+\cosh(m)}\,\exp\big(-2\pi\lambda(\cosh(m)-1)\big).\end{aligned}$$

Hence, taking the limit as λ tends to zero, we have

$$\lim_{\lambda\to 0}\mu_\lambda\big(B(o,m)\subset K_\lambda(o)\big) \leq \frac{1}{\sinh(m)+\cosh(m)} = e^{-m} < 1. \qquad \square$$

Asymptotic probabilities

In this section, we turn our attention to the boundaries, denoted by ∂, of the Poisson–Voronoi cells, viewed as random closed sets in $\mathbb{H}^d$. Analogous to the question of existence of the measure μ_0 on T_d in the previous chapter, we investigate the existence of μ_0, the weak limit of μ_λ, viewed as a family of measures supported on closed sets of $\mathbb{H}^d$.

Let $\mathcal{F}$ be the space of closed sets of $\mathbb{H}^d$. We equip this space with the σ-algebra $\mathcal{B}(\mathcal{F})$ generated by the collection of sets $\{F \in \mathcal{F} : F \cap K \neq \varnothing\}$, for $K \in \mathcal{K}$, where $\mathcal{K}$ is the collection of compact sets of $\mathbb{H}^d$. A basic object of study associated to a random closed set X is the associated *capacity functional*, $T_X : \mathcal{K} \to [0,1]$, defined by $T_X(K) = \mathbb{P}\big(X \cap K \neq \varnothing\big)$ for $K \in \mathcal{K}$. Given a random closed set we can find its associated capacity functional, but more importantly, there is a characterization of those functionals $T : \mathcal{K} \to [0,1]$ which are capacity functionals of random closed sets. Before we give this characterization, due to Choquet, we need some preliminary definitions.

Let Δ denote the operation of *successive differences*; for a functional T and compact sets K and K' we define $\Delta_{K'}T(K) \coloneqq T(K) - T(K \cup K')$. If we begin with a capacity functional, we have a simple monotonicity property: if $K' \subset K$ then $T(K') \leq T(K)$. So, it follows from this observation that if T is a capacity functional then $\Delta_{K'}T(K) \leq 0$. We call a functional *completely alternating* if this property holds more generally which we state in the definition below.

Definition 3.1. *Let $\mathcal{K}$ denote the family of compact sets on $\mathbb{H}^d$. A real valued functional T is called completely alternating if*

$$\Delta_{K_n} \dots \Delta_{K_1} T(K) \leq 0, \ \forall n \geq 1 \text{ and } \forall \ K_1, \dots, K_n, \ K \in \mathcal{K}.$$

We can now state the characterization of capacity functionals, which holds more generally in locally compact Hausdorff spaces. For a detailed proof refer chapter 1 in [MM05].

Theorem 3.1. *Let Ω be a locally compact Hausdorff space. A functional $T\colon \mathcal{K} \to [0,1]$ satisfying $T(\varnothing) = 0$ is the capacity functional of a (necessarily unique) random closed set in Ω if and only if T is upper semicontinuous and completely alternating.*

Asymptotic intersection probabilities of connected compact sets.

Here, we view μ_λ as the law of the random closed set ∂, and let T_λ denote the corresponding capacity functional. A natural candidate for T_0 is the pointwise limit $\lim_{\lambda\to 0} T_\lambda(K)$ for $K \in \mathcal{K}$. In this pursuit, we prove the following result.

Theorem 3.2. *Let $K \in \mathcal{K}$ be a connected compact set in $\mathbb{H}^2$. The asymptotic non-intersection probability $\lim\limits_{\lambda\to 0} \mu_\lambda\big(\partial \cap K = \varnothing\big)$ exists.*

Without loss of generality, since μ_λ is invariant under automorphisms of $\mathbb{H}^d$, we assume that K contains the origin o. The hyperbolic metric on the Poincaré disk is given by d_H and the Euclidean metric by d.

None of the boundaries intersect K if and only if c_o is strictly the closest Poisson point to every point in K, or equivalently if $\bigcup_{x\in K} B\big(x, d_H(x, c_o)\big)$ contains no Poisson points other than c_o. This region, which is forced to be devoid of any other Poisson points, is denoted by $N_K(r,\phi)$, where $c_o = (r,\phi)$ in polar co-ordinates, and when the context is clear, we simply write $N_K(r)$. So, we have the following expression for the non-intersection probability.

$$\begin{aligned}\mu_\lambda(\partial \cap K = \varnothing) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \lambda\bigg(\frac{d}{dr}|B(o,r)|\bigg) e^{-\lambda|B(o,r)|}\mathbb{P}\bigg(\mathcal{P}_\lambda\big(N_K(r,\phi)\setminus B(o,r)\big) = \varnothing\bigg)\, dr\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \lambda\bigg(\frac{d}{dr}|B(o,r)|\bigg) e^{-\lambda|N_K(r,\phi)|}\, dr\, d\phi\end{aligned}$$

where $\mathbb{P}\big(\mathcal{P}_\lambda(A) = \varnothing\big)$, represents the void probability, i.e., the probability that there are no Poisson points in the Borel set A.

Changing the order of integration in the above integral, and assuming that the limit $\lim\limits_{r\to 1}\Big(|N_K(r,\phi)|/|B(o,r)|\Big)$ exists for every fixed ϕ, we have a simple expression for this probability.

Lemma 3.1. *If the following limit exists*

$$\lim_{r\to 1} \frac{|N_K(r,\phi)|}{|B(o,r)|} =: \alpha(\phi),$$

then

$$\lim_{\lambda\to 0} \mu_\lambda\big(\partial \cap K = \varnothing\big) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\alpha(\phi)}\, d\phi. \tag{3.4}$$

This follows from the observation that as we take the intensity of the Poisson process to zero, the probability of finding the Poisson point c_o in any given fixed ball about o decays to zero.

To prove Theorem 3.2 it suffices then to show that the limit of the ratio in the above lemma exists for all connected compact sets K. We state the first part of this requirement in the following proposition.

Proposition 3.3. *Let K be a compact connected set containing o, and let $c_o = (r,\phi)$, then the following limit of the ratio of areas exists.*

$$\lim_{r\to 1} \frac{|N_K(r,\phi)|}{|B(o,r)|} := \alpha(\phi). \tag{3.5}$$

Proof. For $\lambda > 0$, the boundary of the Poisson–Voronoi decomposition is connected since there are no infinite cells. Furthermore, the boundary is composed of geodesic segments. Hence, we can restrict our attention to convex compact connected sets K since the events $\{\partial \cap K = \varnothing\}$ is equivalent to $\{\partial \cap \mathrm{co}(K) = \varnothing\}$, where $\mathrm{co}(A)$ is the convex hull of the set A.

The exact evaluation of the area $|N_K(r,\phi)|$ is difficult even for simple sets K, but for large r and fixed ϕ, we can write this area as an integral of a polar function. Writing the hyperbolic area form in terms of the Euclidean distance r, the area enclosed by a curve $f(\theta)$,

$\theta \in [0, 2\pi)$, can be written as

$$\int_0^{2\pi} \int_0^{f(\theta)} \frac{4r}{(1-r^2)^2}\, dr\, d\theta = 2 \int_0^{2\pi} \frac{f(\theta)^2}{1-f(\theta)^2}\, d\theta.$$

Note that using the above equation, we get

$$|B\big(o, R\big)| = 4\pi \frac{R^2}{1-R^2}.$$

For a fixed ϕ, let $f(r, \phi, \theta)$ be the polar curve describing the boundary of $N_K(r, \phi)$, for $\theta \in [0, 2\pi)$. Returning to the ratio of areas in the proposition, we can write it as

$$\lim_{r\to 1} \frac{|N_K(r, \phi)|}{|B(o, r)|} = \lim_{r\to 1} \frac{1-r^2}{4\pi r^2} \int_0^{2\pi} \frac{2f(r, \phi, \theta)^2}{1-f(r, \phi, \theta)^2}\, d\theta = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-f(r, \phi, \theta)^2}\, d\theta, \quad (3.6)$$

where the last equality follows from the fact that $\lim_{r\to 1} f(r, \phi, \theta) = 1$, uniformly in θ.

To show that the limit in (3.6) exists, we show that the pointwise limit of the integrand exists and is integrable. To prove this, we prove a more general result about such limits, which we write as our next lemma.

Lemma 3.2. *Let $g : [0, 1) \to [0, 1)$ be any function, then $\lim_{x\to 1} \frac{1-x}{1-g(x)}$ exists if and only if $\lim_{x\to 1} d_H\big(o, x\big) - d_H\big(o, g(x)\big)$ exists.*

Proof. The relation between Euclidean and hyperbolic distances is made through hyperbolic trigonometric identities. For any $x \in [0, 1)$, viewed as a point on the x-axis unit disk, we have $x = \tanh\big(d_H(o, x)/2\big)$. To prove the *if* statement in the lemma, assume that the difference in hyperbolic distances has a limit, $\lim_{x\to 1} d_H(o, x) - d_H(o, g(x)) = C$, for some constant C. Given $\epsilon > 0$, there exists $x_0 > 0$ such that for all $x > x_0$, we have

$$1 - \tanh\Big(\big(d_H(o, x) - (C - \epsilon)\big)/2\Big) \le 1 - g(x) \le 1 - \tanh\Big(\big(d_H(o, x) - (C + \epsilon)\big)/2\Big).$$

Hence, we have

$$\frac{1-\tanh\big(d_H(o,x)/2\big)}{1-\tanh\Big(\big(d_H(o,x)-(C+\epsilon)\big)/2\Big)} \leq \frac{1-x}{1-g(x)} \leq \frac{1-d_H\big(o,x\big)/2}{1-\tanh\Big(\big(d_H(o,x)-(C-\epsilon)\big)/2\Big)}.$$

Using the definition of tanh, we have an expression in terms of exponentials given by

$$e^{-(C+\epsilon)} \leq \lim_{x\to 1} \frac{1-x}{1-g(x)} \leq e^{-(C-\epsilon)}.$$

Since ϵ was arbitrary we have the required limit.

To prove the other direction, assume that $\lim_{x\to 1}(1-x)/\big(1-g(x)\big) = c$, for some constant c. As before, given $\epsilon > 0$, there exists x_0 such that for all $x > x_0$ we have

$$1 - \frac{1-x}{c-\epsilon} \leq g(x) \leq 1 - \frac{1-x}{c+\epsilon}.$$

Since $d_H(o,x) - d_H(o,g(x)) = 2\left(\tanh^{-1}(x) - \tanh^{-1}\big(g(x)\big)\right)$, we can use the above inequality and rewrite the inverse hyperbolic tangent in terms of logarithms to obtain the following inequality:

$$\ln\frac{1+r}{2(c+\epsilon)-(1-r)} \leq d_H\big(o,x\big) - d_H\big(o,g(x)\big) \leq \ln\frac{1+r}{2(c-\epsilon)-(1-r)}.$$

Hence, in this case we have $\lim_{x\to 1} d_H\big(o,x\big) - d_H\big(o,g(x)\big) = \ln(1/c)$. □

Returning to the proof of Proposition 3.3, we apply the above lemma to a specific function. Let $R_\theta \coloneqq \{z : \arg(z) = \theta\}$ and fix any point $y \neq o$ in $\mathbb{H}^2$. Consider a point c_r with polar co-ordinates $(r,0)$ and let $g_\theta(y,c_r)$ be the Euclidean distance to 0 of the point given by the intersection of the hyperbolic circle, with center y and radius $d_H(y,c_r)$, with R_θ.

Lemma 3.3. *For a fixed point $y \neq o$ and $g_\theta(y,c_r)$ as above, the following limit exists.*

$$\lim_{r\to 1} \frac{1-r}{1-g_\theta(y,c_r)}.$$

Proof. By Lemma 3.2, it suffices to show that $d_H\big(o, r\big) - d_H\big(o, g_\theta(y, c_r)\big)$ has a limit as r approaches 1. This is equivalent to showing that $\cosh\big(d_H(o, r)\big)/\cosh\big(d_H(o, g_\theta(y, c_r))\big)$ has a limit in r. This follows easily from the hyperbolic cosine law. Applying it to the two triangles formed by $\Delta oy\, g_{_\theta}(y, c_r)$ and Δoyc_r, we get the following relations

$$\begin{aligned}\cosh\big(d_H(o, g_\theta(y, c_r))\big) &= \cosh\big(d_H(o, y)\big)\cosh(t_r) - \sinh\big(d_H(o, y)\big)\sinh(t_r)\cos\big(\angle o\, y\, g_\theta\big)\\ \cosh\big(d_H(o, r)\big) &= \cosh\big(d_H(o, y)\big)\cosh(t_r) - \sinh\big(d_H(o, y)\big)\sinh(t_r)\cos\big(\angle o\, y\, c_r\big),\end{aligned}$$

where we write $t_r = d_H(y, c_r)$. As r tends to 1, we have that both $\angle o\, y\, g_\theta$ and $\angle o\, y\, c_r$ have limits determined by the ideal boundary points $(1, \theta)$ and $(1, 0)$ respectively. The lemma follows from taking the limit of the ratio of the quantities above and noting that $\tanh(t_r) \to 1$ as $r \to 1$. □

Since K is compact and $o \in K$ we have $K \subset B(o, R)$ for some $0 < R < 1$ and hence, for $d_H(o, c_o)$ large enough, we have $N_K(c_o) \subset B\big(2R + d(o, c_o)\big)$ and as a result, for some constant M dependent on R, we get

$$1 \leq \frac{1 - r}{1 - f(r, \phi, \theta)} \leq M.$$

Recall that c_o is a point with polar co-ordinates $(r, 0)$ and $f(r, \phi, \theta)$ is the polar curve describing the boundary of the set $N_K(r, \phi)$. Suppose there are two sequences $\{r_n\}$ and $\{s_m\}$ along which we have the limits given by

$$\lim_{n\to\infty} \frac{1 - r_n}{1 - f(r_n, \phi, \theta)} = \alpha, \quad \text{and} \quad \lim_{m\to\infty} \frac{1 - s_m}{1 - f(s_m, \phi, \theta)} = \beta.$$

For notational simplicity, we suppress the dependence on ϕ and θ and write $f(\cdot)$ for $f(\cdot, \phi, \theta)$. Since f represents the boundary of $N_k(r, \phi)$, by definition, we have a point $y_n \in K$ satisfying the property that $d_H\Big(y_n, \big(f(r_n), \theta\big)\Big) = d_H\Big(y_n, \big(r_n, \phi\big)\Big)$, and similarly let $z_m \in K$ be the corresponding point for s_m.

Since both sequences $\{y_n\}_n$ and $\{z_m\}_m$ lie in a compact set, there are subsequences y_{n_i}, z_{m_j} converging to points $y, z \in K$ respectively. With a slight abuse of notation we refer to these subsequences themselves as $\{y_n\}$ and $\{z_m\}$. For simplicity, write $g_\theta(x, r)$ for $g_\theta(x, c_r)$. As in Lemma 3.3, let $g_\theta(x, r)$ be hyperbolic distance from o of the point of intersection of the hyperbolic sphere $S_H\Big(x, d_H\big(x, (r, \phi)\big)\Big)$ and the ray R_θ. We claim

$$\lim_{n\to\infty} \frac{1 - r_n}{1 - g_\theta(y, r_n)} = \alpha \quad \text{and} \quad \lim_{m\to\infty} \frac{1 - s_m}{1 - g_\theta(z, s_m)} = \beta. \tag{3.7}$$

This claim would follow directly, if we had the limits,

$$\lim_{n\to\infty} \frac{1 - f(r_n)}{1 - g_\theta(y, r_n)} = 1 \quad \text{and} \quad \lim_{m\to\infty} \frac{1 - f(s_m)}{1 - g_\theta(z, s_m)} = 1.$$

However, by Lemma 3.2 this is equivalent to showing

$$\begin{aligned} &\lim_{n\to\infty} d_H\big(o, (f(r_n), 0)\big) - d_H\big(o, (g_\theta(y, r_n), 0)\big) = 0 \quad \text{and,} \\ &\lim_{m\to\infty} d_H\big(o, (f(s_m), 0)\big) - d_H\big(o, (g_\theta(z, s_m), 0)\big) = 0. \end{aligned}$$

We observe that the above equations follow from

$$\lim_{n\to\infty} d_H\big(y, (f(r_n), \theta)\big) - d_H\big(y, (g_\theta(y, r_n), \theta)\big) = 0,$$

and the same equation with r_n replaced by s_m.

$$\begin{aligned} &d_H\Big(y, \big(f(r_n), \theta\big)\Big) - d_H\Big(y, \big(g_\theta(y, r_n), \theta\big)\Big) \\ &= \Big(d_H\big(y, (f(r_n), \theta)\big) - d_H\big(y_n, (f(r_n), \theta)\big)\Big) - \Big(d_H\big(y_n, (f(r_n), \theta)\big) - d_H\Big(y, \big(g_\theta(y, r_n), \theta\big)\Big)\Big) \\ &\qquad\qquad \leq 2C\, d_H\big(y_n, y\big), \end{aligned}$$

for some constant C. Since $y_n \to y$ this completes the proof of the in equation 3.7. Finally, from the definition of g_θ and f, we have that $g_\theta(z, r_n) \le f(r_n)$, and this implies

$$\alpha = \lim_{n\to\infty} \frac{1 - r_n}{1 - f(r_n)} \ge \lim_{n\to\infty} \frac{1 - r_n}{1 - g_\theta(z, r_n)}.$$

Using Lemma 3.3 and (3.7), we obtain

$$\lim_{n\to\infty} \frac{1 - r_n}{1 - g_\theta(z, r_n)} = \lim_{r\to 1} \frac{1 - r}{1 - g_\theta(z, r)} = \lim_{m\to\infty} \frac{1 - s_m}{1 - g_\theta(z, s_m)} = \beta.$$

Hence, $\alpha \ge \beta$ and the same argument with appropriate change in the notation, we have the opposite inequality as well, thus demonstrating the existence of the limit. Therefore this proves the existence of the limit of the integrand, as $r \to 1$ in (3.6). □

Although Theorem 3.2 guarantees the existence of the limit of the non-intersection probability of compact convex sets, it is not in general easy to compute this limit explicitly as this could involve the evaluation of complicated integrals. In what follows, we compute the asymptotic probabilities in two cases: a ball, and a line segment.

Before computing these probabilities, we obtain an explicit form of the numerator in the ratio of areas in (3.5), which could potentially simplify the calculations for other compact convex sets $K \subset \mathbb{D}$ as well. The main difficulty in obtaining the limit of the ratio of areas in (3.5) is in obtaining a convenient parametrization of the boundary of $N_K(c_o)$ in order to find its area. We can describe the boundary set $N_K(c_o)$ as follows: a point $z \in \mathbb{D}$ lies on the boundary of $N_K(c_o)$ if the perpendicular bisector, intersecting z, of the geodesic segment joining z and c_o is tangent to ∂K and doesn't intersect the geodesic segment joining o and c_o. We omit those bisectors which intersect the geodesic segment between o and c_o to obtain a simple closed curve since those points $z \ne c_o$ arising from this procedure would lie in the interior of $B(o, d_H(o, c_o))$, and by definition would be in the interior of $N_K(c_o)$. This is simply a restatement of the definition of $N_K(c_o)$, since this condition ensures the existence of a point $x \in K$ equidistant to both c_o and z, while ensuring that for all $y \in K$ we have $d_H(y, z) \ge d_H(y, c_o)$. In other words, the set of points forming the boundary $\partial N_K(c_o)$ are obtained by reflecting the point c_o (in the hyperbolic sense) in the collection of geodesics that are tangent to K and not separating o and c_o.

Let $c_o = (r, 0)$, in polar co-ordinates, which can be arranged after a rotation about o if necessary. Now, let $F = F_r$ be a Möbius transformation taking c_o to o and vice versa. The unique orientation-preserving isometry satisfying this condition is $F(z) := (r - z)/(1 - rz)$. After applying this transformation, the area we intend to evaluate is bounded by the curve obtained by reflecting the origin in the family of geodesics, which are tangent to $F(K)$ and do not intersect the geodesic segment joining o and c_o. To parametrize this curve, fix $\phi \in [0, 2\pi)$ and let $p(\phi)$ be the point on the curve in the direction ϕ. Now, we need an expression for $d(o, p(\phi))$.

Let the tangent geodesic γ, in which the reflection of o is $p(\phi)$, be given by the Euclidean circle with center $o'(\phi)$ and radius $\mathrm{rad}(\phi)$, as shown in Figure 3.3.

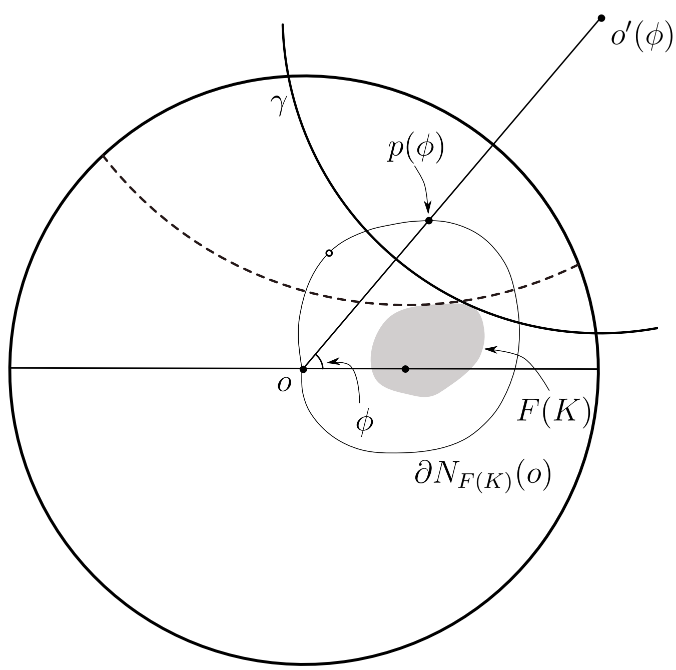


Figure 3.3: The circle is the unit disk $\mathbb{D}$, with the region $F(K)$ shown in gray and the point $F(o)$ marked inside it. The dotted line is another geodesic corresponding to the unmarked point on $\partial N_{F(K)}(o)$. The solid black circular is the geodesic γ corresponding to the point $p(o)$.

Since γ intersects $\partial\mathbb{D}$ orthogonally, we have the relation

$$d\big(o'(\phi),\ p(\phi)\big)\, d\big(o'(\phi), o\big) = \mathrm{rad}(\phi)^2.$$

Furthermore, from the Pythagoras theorem, we have $1 + \mathrm{rad}(\phi)^2 = d(o, o'(\phi))^2$, and in addition $d(o, p(\phi)) = d(o, o'(\phi)) - d(o'(\phi), p(\phi))$. From these relations, we derive

$$d\big(o, p(\phi)\big)\, d\big(o, o'(\phi)\big) = 1, \tag{3.8}$$

which shows that the curve $\partial N_{F(k)}(o)$ is the inversion of the curve formed by the Euclidean centers of the circles describing geodesics which are tangent to $F(k)$, and do not intersect the segment joining o and c_o.

Using the expression for the hyperbolic area enclosed by the curve $p(\phi)$, we have

$$|N_k(c_o)| = |F(N_k(c_o))| = 2\int \frac{d(o, p(\phi))^2}{1 - d(o, p(\phi))^2}\, d\phi = 2\int \frac{1}{d(o, o'(\phi))^2 - 1}\, d\phi. \tag{3.9}$$

Although this is an interesting and neat expression of the area, the difficulty of parametrizing the curve is not overcome but is instead hidden in the evaluation of $o'(\phi)$, for any given ϕ. However, in the two instances which follow, this formulation proves fruitful.

Asymptotic non-intersection probability of a ball

Here, we compute the asymptotic non-intersection probability of a ball.

Theorem 3.3. *For $0 \le R < 1$, let $B(o, R)$ denote the ball of Euclidean radius R in $\mathbb{D}$. Then*

$$\lim_{\lambda \to 0}\ \mu_\lambda\big(\partial \cap B(o, R) = \varnothing\big) = \frac{\pi}{4}\, \frac{(1 - R^2)^2}{R\big(1 - R^2\big) + \big(1 + R^2\big)^2 \arctan\left(\frac{1+R}{1-R}\right)}.$$

If ρ is the hyperbolic radius of the ball, we have

$$\lim_{\lambda\to 0}\ \mu_\lambda\big(\partial\cap B_H(o,\rho)=\varnothing\big)=\frac{\pi}{4\arctan(e^\rho)\cosh^2\rho+2\sinh\rho}.$$

Proof. Using the spherical symmetry of $B(o,R)$ we can simplify the expression for the probability in (3.4) to obtain

$$\lim_{\lambda\to 0}\mu_\lambda(\partial\cap B(o,R)=\varnothing)=\frac{1}{2\pi}\int_0^{2\pi}\lim_{r\to 1}\frac{|B(o,R)|}{|N_{B(o,R)}(r,\phi)|}\,d\phi=\lim_{r\to 1}\frac{|B(o,R)|}{|N_{B(o,R)}(r)|}. \tag{3.10}$$

Here, we are again restricting the nucleus c_o to the x-axis, and we write it in polar co-ordinates as $(r,0)$. We can now evaluate the hyperbolic area of $N_{B(o,R)}(r,0)$ using (3.8), but first we introduce some notation. Given a point $z\neq o$, the collection of geodesics passing through z is given by a collection of circles orthogonal to $\partial\mathbb{D}$ with their Euclidean centers on a Euclidean straight line l_z called the *polar line* of z. For more details refer [CG67]. Since the isometry $F_r(z)$ maps circles to circles (in the extended sense), we have that $F_r(S(o,R))$ is a circle. Euclidean circles contained strictly within $\mathbb{D}$ are also hyperbolic circles, but with possibly different centers. By symmetry we can conclude that both the hyperbolic and Euclidean centers of $F_r(B(o,R))$ lie on the x-axis. Let $(u,0)$, with $u=u(r)$, be its Euclidean center and $s=s(r)$ its Euclidean radius. In particular, u and s can be expressed in terms of r and R as

$$u=\frac{r\big(1-R^2\big)}{1-(rR)^2},\qquad s=\frac{R\big(1-r^2\big)}{1-(rR)^2}. \tag{3.11}$$

For any point $z_0=(x_0,y_0)$ on the boundary of $B(u,s)$ consider the unique geodesic γ described by the Euclidean circle with its center at the point of intersection of the polar line l_{z_0} and the straight line through the points $(u,0)$ and z_0, as shown in Figure 3.4. In this case, this geodesic γ is also tangent to $B(u,s)$ with z_0 as the point of tangency. Parametrize the Cartesian co-ordinates of the point z_0 as $x_0=u+s\cos(\theta)$ and $y_0=\sin(\theta)$, with θ in a subset of $[0, 2\pi)$. Consequently, denoting the center of the circle describing the geodesic by $(x(\theta), y(\theta))$, we have a parametrization of the collection of geodesics which are tangent to $B(u, s)$. The condition that the tangent geodesic does not intersect the line joining o and $F_r(o)$ is translated into a condition on the parameter θ. When $\theta = \pm \arccos(-s/u)$ the geodesic which is tangent to $B(u, s)$ passes through o, and is a Euclidean line through o. Hence, when $\theta \in [-\arccos(-s/u), \arccos(-s/u))$ the geodesic doesn't separate o and $(r, 0)$.

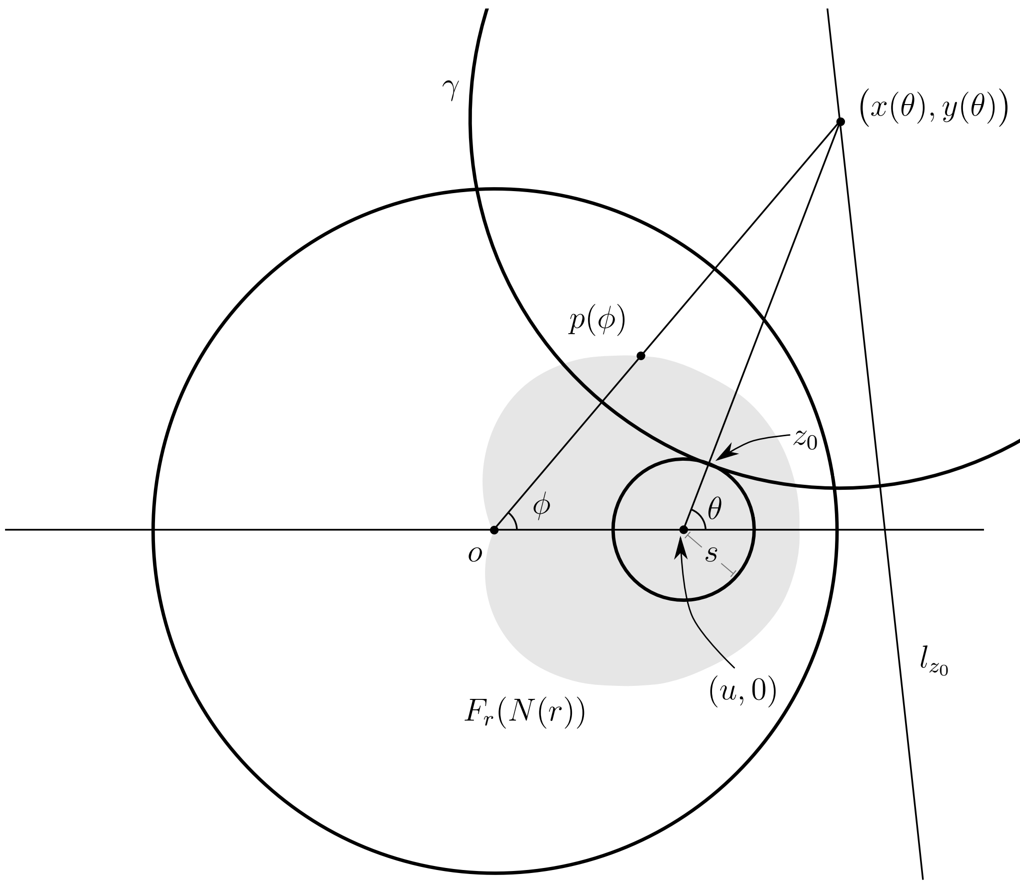


Figure 3.4: The gray region represents the set $F_r(N(r))$ and it contains the marked point with polar co-ordinates $(u, 0)$. The polar line l_{z_0} is the Euclidean line segment on the right.

For the point z_0 the polar line l_{z_0} has the equation

$$y = -\frac{x_0}{y_0}\, x + \frac{1 + x_0^2 + y_0^2}{2y_0}.$$

The co-ordinates of the intersection of l_{z_0} with the line through $(u, 0)$ and z_0 are given by

$$x = \frac{\left(1 + x_0^2 + y_0^2\right)\left(x_0 - u\right) + 2uy_0^2}{2y_0^2 + 2x_0\left(x_0 - u\right)}, \quad y = \frac{y_0}{x_0 - u}\frac{\left(1 + x_0^2 + y_0^2\right)\left(x_0 - u\right) + 2uy_0^2}{2y_0^2 + 2x_0\left(x_0 - u\right)}.$$

Rewriting this in terms of the parameter θ, we have,

$$x(\theta) = u + \frac{\left(1 - (u^2 - s^2)\right)\cos\theta}{2(s + u\cos\theta)}, \quad y(\theta) = \frac{\left(1 - (u^2 - s^2)\right)\sin\theta}{2\left(s + u\cos\theta\right)}.$$

Parametrizing the boundary of $F_r(N(r))$ by ϕ, for any $\phi \in [0, 2\pi)$, we can relate ϕ and θ by $\phi := h(\theta) = \arctan(y(\theta)/x(\theta))$. Hence, by using the change-of-variables formula in expression (3.9), we obtain

$$\begin{aligned} |N_{B(o,R)}(r, 0)| &= 2\int \frac{1}{d\left(o, o'(\phi)\right)^2 - 1}\, d\phi = 2\int \frac{1}{x(\theta)^2 + y(\theta)^2 - 1}\, h'(\theta)\, d\theta \\ &= 2\int \left(\frac{y'(\theta)x(\theta) - x'(\theta)y(\theta)}{x(\theta)^2 + y(\theta)^2 - 1} - \frac{y'(\theta)x(\theta) - x'(\theta)y(\theta)}{x(\theta)^2 + y(\theta)^2}\right) d\theta. \end{aligned} \tag{3.12}$$

The second term in the integrand above, when integrated over interval

$$\theta \in \Big[\arccos(-s/u),\, \arccos(-s/u)\Big)$$

yields a term which converges to 2π, as $r \to 1$. Hence, the majority of the contribution to the area is from the first term of this integrand. Simplifying this term, we have

$$y'(\theta)x(\theta) - x'(\theta)y(\theta) = \frac{\beta u\left(s\cos\theta + u\right) + \beta^2}{\left(s + u\cos\theta\right)^2},$$

where $\beta = \left(1 - (u^2 - s^2)\right)/2$. The denominator simplifies as follows:

$$x(\theta)^2 + y(\theta)^2 - 1 = \frac{\cos^2\theta\left(u^4 - u^2 + 2\beta u^2\right) + \cos\theta\left(2us(u^2 - 1) + 2\beta us\right) + \beta^2 + s^2(u^2 - 1)}{(s + u\cos(\theta))^2}$$

$$= \frac{(u^2s^2)\cos^2\theta + us\big(1-(u^2+s^2)\big)\cos\theta + \frac{1}{4}\big(1-(u^2+s^2)\big)^2}{\big(s+u\cos\theta\big)^2}$$

Factoring the numerator in this expression as a quadratic in $\cos\theta$, we find that it has a double root at $\big(1-(u^2+s^2)\big)/(2su)$, thereby simplifying the expression. Now, returning to the first term of the integrand in (3.12), the simplified expression is given by

$$\frac{y'(\theta)x(\theta)-x'(\theta)y(\theta)}{x(\theta)^2+y(\theta)^2-1} = \frac{4\beta\big(su\cos\theta+(\beta+u^2)\big)}{\big(2su\cos\theta-(1-(u^2+s^2))\big)^2}$$

$$= (1-(u^2-s^2))\frac{2su\cos\theta+\big(1+(u^2+s^2)\big)}{\big(2su\cos\theta-(1-(u^2+s^2))\big)^2}.$$

Since this is an even function of θ, the integral we need to evaluate is

$$|N_{B(o,R)}(r)| = 4\int_0^{\arccos(-s/u)}(1-(u^2-s^2))\frac{2su\cos\theta+\big(1+(u^2+s^2)\big)}{\big(2su\cos\theta-(1-(u^2+s^2))\big)^2}\,d\theta.$$

This integral can be simplified using the trigonometric substitution $t=\tan\frac{\theta}{2}$, using which we obtain

$$|N_{B(o,R)}(r)| = 8\int_0^{\sqrt{\frac{u+s}{u-s}}}(1-(u-s)(u+s))\;\frac{t^2\big(1+(u-s)^2\big)+\big(1+(u+s)^2\big)}{\Big(t^2(1-(u-s)^2)+(1-(u+s)^2)\Big)^2}\,dt.$$

The integrand here is a rational function in t and can be solved by elementary techniques. Completing the integration and substituting u and s from (3.11) we get the following result:

$$|N_{B(o,R)(r)}| = \frac{R^2(1-r^2)(1-R^2)}{1-(rR)^2}\frac{(1-(rR)^2)}{(1-r^2)^2(1-R^2)^2(1+R^2)}\times \left(\frac{2R(r+R)(1-rR)}{\sqrt{\frac{(r+R)(1-rR)}{(r-R)(1+rR)}}} + (1+r^2)(1+R^2)^2\;\arctan\frac{1+rR}{1-rR}\sqrt{\frac{(r+R)(1-rR)}{(r-R)(1+rR)}}\right). \tag{3.13}$$

The final ingredient in evaluating the probability in (3.10), is the hyperbolic volume of the ball, centered at o, of Euclidean radius r given by $|B(o,r)| = \dfrac{4\pi r^2}{1-r^2}$. We observe that there is a factor of $(1-r^2)$ in (3.13), and the rest of the coefficients have finite limits. Using this, we have

$$\lim_{\lambda\to 0} \mu_\lambda(\partial \cap B(o,R) = \varnothing) = \lim_{r\to 1} \frac{|B(o,R)|}{|N_{B(o,R)}(r)|} = \frac{\pi}{4} \frac{(1-R^2)^2}{R\big(1-R^2\big) + \big(1+R^2\big)^2 \arctan\left(\frac{1+R}{1-R}\right)}.$$

To convert the Euclidean radius into the hyperbolic radius, we use $\rho = \log \dfrac{1+r}{1-r}$ in the above expression which, after a straight forward algebraic manipulation yields

$$\lim_{\lambda\to 0} \mu_\lambda\big(\partial \cup B_H(o,\rho) = \varnothing\big) = \frac{\pi}{4\arctan(e^\rho)\cosh^2\rho + 2\sinh\rho}. \tag{3.14}$$

□

For large ρ, we have $4\arctan(e^\rho)\cosh^2\rho \sim \frac{\pi}{2} e^{2\rho}$ and $2\sinh\rho \sim e^\rho$. Hence, the probability in (3.14) is asymptotically $2e^{-2\rho}$. The graph in ρ of the non-intersection probability is plotted in Figure 3.5 below.

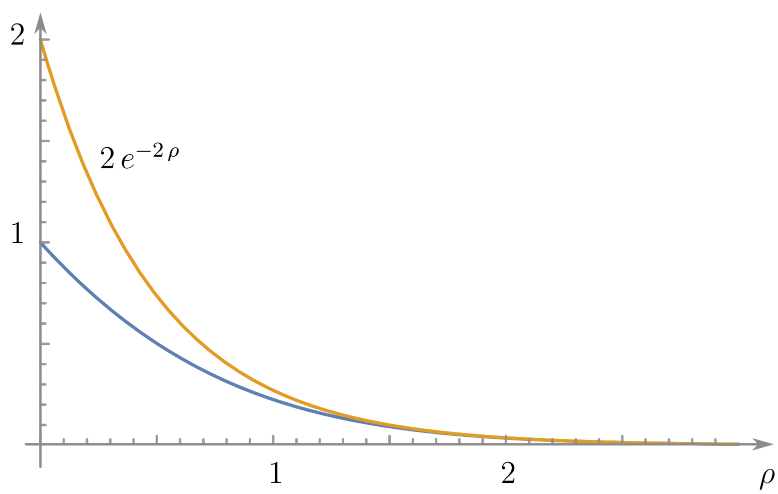


Figure 3.5: The non-intersection probability is plotted in blue and the asymptotic approximation is plotted in orange

Asymptotic non-intersection probability of a line segment

Another asymptotic probability which we can obtain an expression for, is the probability that two given points lie in different Voronoi cells. We have the following theorem.

Theorem 3.4. *Let L be a geodesic segment of hyperbolic length 2ρ. We have*

$$\lim_{\lambda\to 0} \mu_\lambda\big(\partial \cap L = \varnothing\big) =$$
$$\int_0^{\pi/2} \frac{2\, d\theta}{\cosh(\rho)\big(\pi + 2\arctan(\sinh(\rho)\,\tan\theta\,)\big) + \cos\theta\,\sinh(\rho)\big(\pi - 2\arctan(\cosh(\rho)\,\sin\theta\,)\big)}.$$

For ρ large, the above integral is asymptotic to $\exp(-\rho)$.

Proof. Using Lemma 3.1 again, we would have an expression for the non-intersection probability of the line segment L, if we can evaluate the limit in (3.5). Assume that the line segment is centered at o, with end points u and v.

Unlike in the previous case of the ball, the limit in (3.5) depends on both the distance of c_o from o as well as its position on the sphere $S(o, d(o, c_o))$. Let $c_o = (r, \theta)$ be the polar co-ordinates of the nucleus associated to o, where r is the Euclidean distance of o to c_o in $\mathbb{H}^2$, and let the geodesic segment L be centered at o. By definition, we have

$$|N_L(c_o)| = \Big|\bigcup_{x\in L} B\big(x, d_H(x, c_o)\big)\Big| = \big|B\big(u, d_H(u, c_o)\big) \cup B\big(v, d_H(v, c_o)\big)\big|,$$

where $|A|$ represents the hyperbolic area of the set A. Note that a circle in $\mathbb{H}^2$ is also a circle in the Euclidean metric, albeit possibly with a different center. For a hyperbolic circle with center p not at o its Euclidean center is on the Eulidean line-segment connecting p and o. The second equality in the equation above follows from this observation and the corresponding result in the Euclidean setting. Instead of varying the position of c_o on the sphere $S(o, d_H(o, c_o))$, we restrict c_o to the x-axis in the Poincaré disk and vary the angle L makes with the x-axis. This corresponds to a change of variable, and the limit in (3.5) remains unchanged.

Let $t := \log((1+r)/(1-r))$, be the hyperbolic distance between the points o and c_o. The limit as r approaches 1 is equivalent to t approaching infinity.

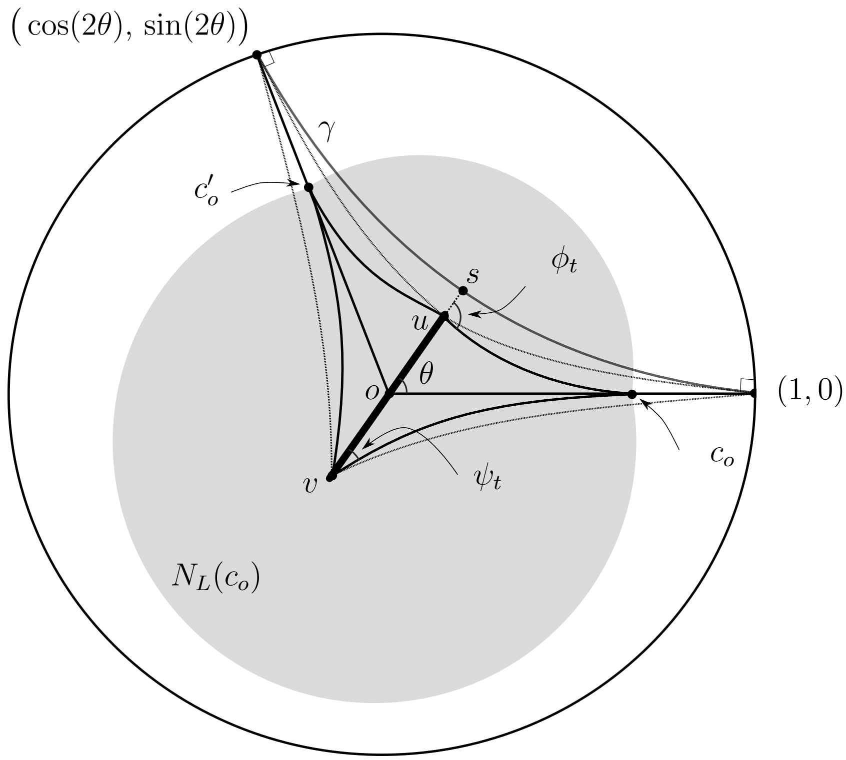


Figure 3.6: The gray region denotes $N_L(c_o)$, where the geodesic segment L is denoted by the thich line joining u to v. γ is the geodesic joining the ideal points with Cartesian co-ordinates $(\cos(2\theta), \sin(2\theta)) and (1, 0)$. The point s is the orthogonal projection of o onto γ. The other black lines are the geodesic segments joining the given pair of points.

$N_L(c_o)$ is the union of two balls and can be decomposed into a disjoint union of a quadrilateral and two sectors of disks. Quadrilaterals in $\mathbb{D}$ have area at most 2π, since triangles have area at most π. As a consequence, the quadrilateral region does not contribute any positive value to the limit in 3.5. The balls $B(u, d_H(u, c_o))$ and $B(v, d_H(v, c_o))$ intersect at two points c_o and c_o' as shown in Figure 3.6. Here c_o' is the reflection of the nucleus c_o in the line through o at angle θ. It suffices for our purpose to restrict θ to the range $[0, \pi/2)$.

The sectors of the balls that we are interested in are determined by the angle ϕ_t in $B(u, d_H(u, c_o))$, given by the angle $\angle suc_o$, and the angle ψ_t in $B(v, d_H(v, c_o))$ given by $\angle ovc_o$.

Note that The subscript t indicates the dependence of the angles on the position of c_o, which is described by t.

Using this notation, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{|N_L(c_o)|}{|B(o,t)|} &= \lim_{t\to\infty}\left(\frac{\phi_t}{\pi}\frac{\left|B\big(u,d_H(u,c_o)\big)\right|}{|B(o,t)|}+\left(1-\frac{\psi_t}{\pi}\right)\frac{\left|B\big(v,d_H(v,c_o)\big)\right|}{|B(o,t)|}\right)\\ &= \lim_{t\to\infty}\left(\frac{\phi_t}{\pi}\frac{\cosh(d_H(u,c_o))-1}{\cosh(t)-1}+\left(1-\frac{\psi_t}{\pi}\right)\frac{\cosh(d_H(v,c_o))-1}{\cosh(t)-1}\right), \end{aligned} \tag{3.15}$$

where we used the formula for the hyperbolic area of a ball, which is given by $|B(o,t)| = 2\pi(\cosh(t)-1)$.

To simplify the expression in (3.15) further, we use the hyperbolic cosine law to relate $\cosh\big(d_H(u,c_o)\big)$ and $\cosh\big(d_H(v,c_o)\big)$ to the angle at o and the length of the adjacent sides in triangles Δuoc_o and Δvoc_o respectively. This results in the following equations:

$$\begin{aligned}\cosh(d_H(u,c_o)) &= \cosh(\rho)\cosh(t)-\sinh(\rho)\sinh(t)\cos\theta\\ \cosh(d_H(v,c_o)) &= \cosh(\rho)\cosh(t)-\sinh(\rho)\sinh(t)\cos(\pi-\theta).\end{aligned}$$

Using the fact that $\lim_{t\to\infty}\tanh(t)=1$, we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{\left|B\big(u,d_H(u,c_o)\big)\right|}{|B(o,t)|} &= \cosh(\rho)-\sinh(\rho)\cos\theta,\\ \lim_{t\to\infty}\frac{\left|B\big(v,d_H(v,c_o)\big)\right|}{|B(o,t)|} &= \cosh(\rho)+\sinh(\rho)\cos\theta.\end{aligned} \tag{3.16}$$

Along with these equations, we need an expression for $\lim_{t\to\infty}\phi_t$ and $\lim_{t\to\infty}\psi_t$. The functions ϕ_t and ψ_t are bounded and eventually monotone, and hence have a limit which we represent by ϕ and ψ respectively. As t diverges to infinity c_o converges to the point $p := (1,0)$ and c_o' converges to the ideal point $p' := (1, 2\theta)$, represented in polar co-ordinates. In Figure 3.6, the limiting angle ϕ is given by $\angle sup$, and ψ by $\angle svp$, where s is the orthogonal projection of o onto the geodesic joining the ideal points p and p'. To find an expression for

ϕ and ψ in terms of ρ and θ we use the following lemma.

Lemma 3.4. *Let o be the origin in $\mathbb{D}$, and γ be a geodesic with end points given by the ideal points p and q. Let s be the orthogonal projection of o onto γ, and θ the angle $\angle sop = \angle soq$. We have the following relation*

$$\exp\big(-d_H(o,s)\big) = \tan\frac{\theta}{2}.$$

Proof of the lemma. Viewing $\mathbb{D}$ as the unit disk in $\mathbb{R}^2$, we note that the geodesics in $\mathbb{D}$ are given by arcs of Euclidean circles, which intersect the unit disk orthogonally. Using this description, without loss of generality, we assume the geodesic γ passes through the ideal point $(1,0)$ and is described by a Euclidean circle C with Euclidean center given by Cartesian co-ordinates $(1,y)$, for some $-\infty < y < \infty$. By symmetry, we can further assume $y > 0$. Let $\theta \in (0,\pi/2)$, be the angle made by the line joining o to $(1,y)$. Rewriting y in terms of θ, we have $y = \tan\theta$, and the orthogonal projection of o onto γ is the intersection of the line joining o and $(1,\tan\theta)$ with C. Call this point s.

The Euclidean distance between o and s is given by

$$d(o,s) = d\big(o,(1,\tan\theta)\big) - d\big(s,(1,\tan\theta)\big) = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}.$$

As a result, the hyperbolic distance between o and s is

$$d_H(o,s) = \log\frac{1+d(o,s)}{1-d(o,s)} = \log\frac{\cos\theta+1}{\sin\theta} = \log\frac{1}{\tan\frac{\theta}{2}}.$$

Hence, we have the statement of the lemma $\exp(-d_H(o,s)) = \tan(\theta/2)$. □

Note that this result is true for any point $x \in \mathbb{D}$, i.e., if $\gamma(x)$ denotes the orthogonal projection of x onto γ, then $\exp(-d_H(x,\gamma(x))) = \tan(\theta/2)$, where θ is the angle at the vertex x in the triangle formed by x, $\gamma(x)$ and one of the ideal end points of γ.

Let γ be the geodesic with end points $(1,0)$ and $(1,2\theta)$, in polar co-ordinates, as indicated in Figure 3.6. Note that the orthogonal projection of any point on the line L onto γ is the point s. Applying the result of the lemma thrice, with γ and the three points o, u and v, we get

$$\exp(-d_H(o,s)) = \tan\frac{\theta}{2}, \quad \exp(-d_H(u,s)) = \tan\frac{\phi}{2}, \quad \exp -d_H(v,s) = \tan\frac{\psi}{2}.$$

Using the fact that $d_H(o,s) = d_H(o,u) + d_H(u,s)$ and $d_H(v,s) = d_H(v,u) + d_H(u,s)$, and simplifying we obtain

$$\phi = 2\arctan\left(e^{\rho}\tan\frac{\theta}{2}\right), \text{ and } \psi = 2\arctan\left(e^{-\rho}\tan\frac{\theta}{2}\right).$$

Furthermore, using the sum and difference formulas for arctan, we get

$$\phi+\psi = 2\arctan\left(\frac{(e^{\rho}+e^{-\rho})\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right), \text{ and } \phi-\psi = 2\arctan\left(\frac{(e^{\rho}-e^{-\rho})\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right).$$

Returning to the expression (3.15), and substituting the limit of areas obtained in (3.16) along with the limits of ϕ_t and ψ_t, we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{|N_L(c_o)|}{|B(o,t)|} &= \cosh(\rho)\Big(\frac{\pi+(\phi-\psi)}{\pi}\Big) + \sinh(\rho)\cos\theta\Big(\frac{\pi-(\phi+\psi)}{\pi}\Big)\\ &= \frac{1}{\pi}\Big(\cosh(\rho)\big(\pi+2\arctan(\cosh(\rho)\sin\theta)\big) + \sinh(\rho)\cos\theta\big(\pi-2\arctan(\sinh(\rho)\tan\theta)\big)\Big).\end{aligned}$$

Substituting this expression in (3.4), we get the statement in the lemma

$$\lim_{\lambda\to 0}\mu_\lambda\big(\partial\cap L=\varnothing\big) = \int_0^{\pi/2}\frac{2\,d\theta}{\cosh(\rho)\big(\pi+2\arctan(\sinh(\rho)\ \tan\theta)\big)+\cos\theta\ \sinh(\rho)\big(\pi-2\arctan(\cosh(\rho)\ \sin\theta)\big)}.$$

Furthermore, to obtain the asymptotics in ρ, we note that

$$\lim_{\rho\to\infty} \cos\theta \sinh(\rho)\big(\pi - 2\arctan(\cosh(\rho)\sin\theta)\big) = 2\cot\theta, \text{ and}$$
$$\lim_{\rho\to\infty} \arctan\big(\sinh(\rho)\tan\theta\big) = \frac{\pi}{2}.$$

Therefore, we have

$$\int_0^{\pi/2} \frac{2\,d\theta}{\cosh(\rho)\big(\pi + 2\arctan(\sinh(\rho)\,\tan\theta)\big) + \cos\theta\,\sinh(\rho)\big(\pi - 2\arctan(\cosh(\rho)\,\sin\theta)\big)}$$
$$\sim \int_0^{\pi/2} \frac{2\,d\theta}{\pi e^\rho + 2\cot\theta} = \frac{\pi^2 e^\rho - 4r + \ln(16) - 4\ln(\pi)}{4 + \pi^2 e^{2\rho}} \sim e^{-\rho}. \qquad \square$$

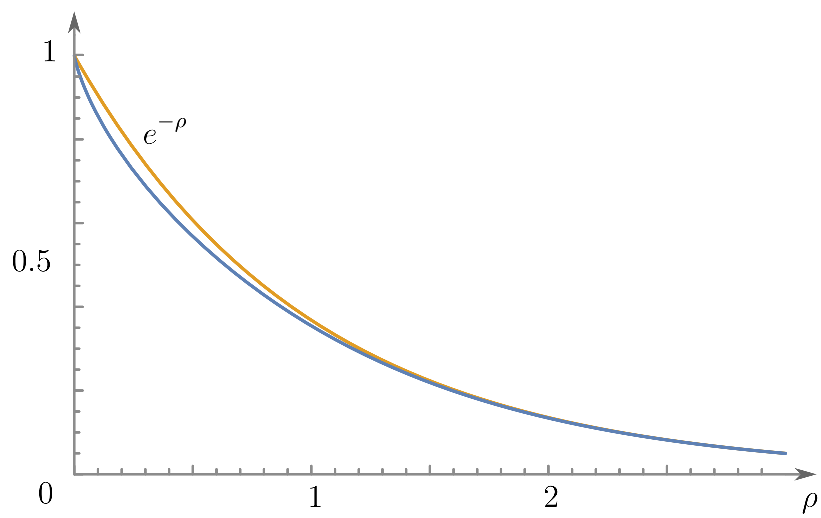


Figure 3.7: The probability of non-intersection with the line segment is plotted in blue and the asymptotic approximation function is plotted in orange.

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