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A SECOND ORDER LOGIC OF VARIABLE — BINDING OPERATORS *

It is well-known that Frege distinguished between first- and second-level concepts or functions. First-level concepts he associated with properties and relations between objects. These concepts Frege characterized as functions which assigned truth-values (the true or the false) to (n-tuples of) objects [1]. An (n-tuple of) object(s) was said to *fall under* such a concept if it was assigned the true by that concept. In his *Begriffsschrift* these concepts were indicated by predicate variables.

Second-level concepts or functions, on the other hand, correspond to variable-binding operators, e.g., the universal quantifier or, as in Frege's later development of the *Begriffsschrift*, the course-of-values abstraction operator. The latter assigns to a monadic concept the class which is its extension while the former assigns a truth-value. Second-level concepts, i.e., second-level functions corresponding to variable-binding operators of the quantifier type, accordingly, can be associated with properties or relations between properties and relations of objects in a sense analogous to (but also different from) that in which first-level concepts are associated with properties or relations between objects. In Frege's terminology, while an (n-tuple of) object(s) is said to fall under a first-level concept, the latter is said to *fall within*, not *under*, a second-level concept if it is assigned the true by that concept.

Third-level concepts corresponding to quantifiers binding predicate variables were also introduced into the *Begriffsschrift*, but Frege seems to have had some doubts regarding their ontological or objective significance. Indeed, Frege's attitude toward third-level concepts seems in general to resemble the nominalists' attitude toward second-level concepts, viz., that they are merely formal or

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[1] Frege apparently allowed only for binary relations. We extend his framework to include n-ary relations for arbitrary finite $n \geqslant 2$. In addition, we refer to all these relations as (n-ary) concepts. (Frege referred only to properties as concepts.)

§ 1. Syntax

In the system to be proposed here we distinguish between four disjoint types of *categorial constants*: (1) *individual constants* standing for objects, (2) n-place *predicate constants* standing for n-ary first-level concepts, (3) k-ary n-place (individual variable binding) *quantifier constants* standing for k-ary n-place second-level concepts [7], and (4) k-ary *truth-functional operators* standing for k-ary truth-functions, where k, n are positive integers. We understand a *language* to be a set of such constants.

In regard to truth-functions it is clear that Frege understood these to be first-level functions. This was because he considered the true and the false to be objects and comparable as such to objects in general. We, however, consider them not to be objects, or at least not objects of a type that can be indicated by individual variables, and similarly understand truth-functions to be incomparable with properties and relations of objects. Signs for truth-functions, accordingly, are not to be confused with predicates; nor are truth-functions themselves ever indicated by predicate variables. In what follows we shall assume that all the languages to be considered contain two special truth-functional operators, viz., $\sim$, the negation sign, and $\rightarrow$, the material conditional sign.

We take distinctive types of variables to be the logical proto-types of our categorial constants and assume that there are an enumerable infinity of variables of each type available. For brevity of presentation, we omit (quantifiable) variables for truth-functions, however, since the negation and material conditional signs suffice for the representation of all truth-functions [8]. We use 'α' and 'β' in our metalanguage to refer to individual variables and 'π^n', 'σ^n' to refer to n-place predicate variables and constants. We will usually drop the superscript for convenience, however. Where k, n are positive integers and

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[7] A k-ary n-place second-level concept takes k n-place first-level concepts as arguments in order to become saturated. That is, a k-ary n-place quantifier binds n individual variables and when applied to k sentence forms results in a sentence form. In general we shall use metalinguistic expressions of the form '$Q^k\alpha_1 \ldots \alpha_n$', where $\alpha_1, \ldots, \alpha_n$ are distinct individual variables, to refer to k-ary n-place quantifiers. It should be noted, however, that although names of the variables which a quantifier binds are part of the syntactical name of the quantifier, the quantifier itself need contain no occurrences of the variables named. It is only required that some syntactic means are available to determine which variables, and their order, a quantifier is understood to bind.

[8] This of course would no longer be true were we to extend the framework to include (n-ary) *modal* operators as well. In that case, besides quantifiable (n-ary) operator variables, we would also include quantifiable propositional variables having modally conditioned states of affairs as their values instead of Frege's two truth-values. We do not turn to this extension of the framework here partly for brevity of presentation and proof and partly to remain faithful to Frege's extensionalist position. Besides, the construction of the extension is fairly obvious by rather familiar techniques.

$\alpha_1, \ldots, \alpha_n$ are distinct individual variables, we use '$Q^k\alpha_1 \ldots \alpha_n$' to refer to k-ary n-place quantifier-variables or constants that bind the individual-variables $\alpha_1, \ldots, \alpha_n$.

Besides parentheses and brackets as auxiliary signs, the only logical or syncategorematic constant we shall use is $\forall$, the third-level universal quantifier for first-level predicate variables and second-level quantifier variables. For convenience, however, we shall also use this same sign as a categorial quantifier constant for individual variables, i.e., as a unary 1-place individual variable binding constant. We assume that every language to be considered here includes this sign among its categorial constants.

Where L is a language, we take the *terms* of L, TM_L, to be the set of individual variables and the individual constants in L. We use 'ζ', 'η' to refer to terms. By an *atomic formula* of L we understand an expression of the form $\pi(\zeta_1, \ldots, \zeta_n)$, where π is either an n-place predicate variable or an n-place predicate constant in L, and $\zeta_1, \ldots, \zeta_n$ are terms of L. The set of *formulas* of L, FM_L, is the smallest set K containing every atomic formula of L and such that (1) $\Delta[\varphi_1, \ldots, \varphi_n] \in K$ whenever $\varphi_1, \ldots, \varphi_n \in K$ and Δ is an n-ary truth-functional operator in L, (2) $(Q^k\alpha_1 \ldots \alpha_n)[\varphi_1, \ldots, \varphi_k] \in K$ whenever $\varphi_1, \ldots, \varphi_k \in K$, $\alpha_1, \ldots, \alpha_n$ are distinct individual variables and $Q^k\alpha_1 \ldots \alpha_n$ is either a k-ary n-place quantifier variable or a k-ary n-place quantifier constant in L, and (3) $(\forall\pi)\varphi$, $(\forall Q^k\alpha_1 \ldots \alpha_n)\varphi \in K$ whenever $\varphi \in K$, π is a predicate variable and $Q^k\alpha_1 \ldots \alpha_n$ is a k-ary n-place quantifier variable.

We note that in accordance with our assumptions regarding all the languages to be considered here: if α is an individual variable and φ, ψ are formulas of L, then so are $(\forall\alpha)\varphi$, $\sim\varphi$ and $(\varphi \to \psi)$. (We use the latter instead of $\to[\varphi, \psi]$.) In general, we use 'φ', 'ψ', 'χ' in our metalanguage to refer to formulas. We understand other truth-functional connectives which occur in this paper to be defined (in the metalanguage) in terms of $\sim$ and $\to$. Also, we take $(\exists\xi)\varphi$ to be $\sim(\forall\xi)\sim\varphi$, where ξ is any variable. Finally, we say that two variables or constants are of *the same type* if either both are terms or both are n-place predicate constants or variables, for some positive integer n, or for some positive integers k and n, both are k-ary n-place quantifier constants or variables.

§ 2. Substitution

We assume bondage and freedom of occurrences of variables in formulas to have been defined in the obvious way. Predicate and quantifier variables are of course bound only by the third-level syncategorematic universal quantifier whereas an individual variable is bound now not only by the second-level categorial universal quantifier but also by any quantifier variable or constant having that individual variable as one of its indices.

In regard to the *simultaneous* (*proper*) *substitution* of n terms $\zeta_1, \ldots, \zeta_n$ for n distinct individual variables $\alpha_1, \ldots, \alpha_n$ in a formula φ, we take

$$\varphi\begin{bmatrix}\alpha_1 \ldots \alpha_n \\ \zeta_1 \ldots \zeta_n\end{bmatrix}$$

to be the result of simultaneously replacing each free occurrence of $\alpha_1, \ldots, \alpha_n$ in φ by free occurrences of $\zeta_1, \ldots, \zeta_n$, respectively, if such a formula exists; otherwise it is just φ itself. We assume Church's characterization of the proper substitution of a formula φ with respect to $\alpha_1, \ldots, \alpha_n$ (as argument indicators) for the free occurrences of an n-place predicate variable π in a formula ψ:

$$\check{\mathrm{S}}_{\varphi}^{\pi(\alpha_1, \ldots, \alpha_n)}\psi\Big|$$

In the special case of substituting an n-place predicate-variable or constant for an n-place predicate-variable, we write $\psi[{}^{\pi}_{\sigma}]$ instead of

$$\check{\mathrm{S}}_{\sigma(\alpha_1, \ldots, \alpha_n)}^{\pi(\alpha_1, \ldots, \alpha_n)}\psi\Big|$$

We also assume Church's definition to be extended so as to allow for the simultaneous substitution of k formulas, all with respect to $\alpha_1, \ldots, \alpha_n$, for k n-place predicate variables $\pi_1, \ldots, \pi_k$ in a formula ψ:

$$\check{\mathrm{S}}_{\varphi_1 \qquad\qquad , \ldots, \varphi_k}^{\pi_1(\alpha_1, \ldots, \alpha_n), \ldots, \pi_k(\alpha_1, \ldots, \alpha_n)}\psi\Big|$$

We shall use the above form of simultaneous substitution in our characterization of what it means to properly substitute a formula χ with respect to k n-place predicate variables $\pi_1, \ldots, \pi_k$ (as argument indicators) for the free occurrences of a k-ary n-place quantifier-variable $Q^k\alpha_1 \ldots \alpha_n$ in a formula ψ:

$$\tilde{\mathrm{S}}_{\chi}^{Q^k\alpha_1 \ldots \alpha_n[\pi_1(\alpha_1, \ldots, \alpha_n), \ldots, \pi_k(\alpha_1, \ldots, \alpha_n)]}\psi\Big|$$

This expression shall stand for ψ unless (1) for each i, where $1 \leqslant i \leqslant n$, every subformula φ of ψ which occurs as the ith argument-formula of a free occurrence of $Q^k\alpha_1 \ldots \alpha_n$ in ψ can be properly substituted, with respect to $\alpha_1, \ldots, \alpha_n$, for π_i in χ (as defined by Church) and (2) no free occurrence of $Q^k\alpha_1 \ldots \alpha_n$ in ψ occurs in ψ within the scope of a quantifier expression binding a variable other than $\pi_1, \ldots, \pi_k$ which occurs free in φ; if these conditions are satisfied, then the above expression is the result of replacing each occurrence in ψ of any formula of the form $Q^k\alpha_1 \ldots \alpha_n[\varphi_1, \ldots, \varphi_k]$, where the initial occurrence of $Q^k\alpha_1 \ldots \alpha_n$ is a free occurrence in ψ, by:

$$\check{\mathrm{S}}_{\varphi_1 \qquad\qquad , \ldots, \varphi_k}^{\pi_1(\alpha_1, \ldots \alpha_n), \ldots, \pi_k(\alpha_1, \ldots, \alpha_n)}\chi\Big|$$

In the special case of substituting a quantifier variable or constant for a quantifier variable of the same type we write:

$$\psi\begin{bmatrix}Q^k\alpha_1 \ldots \alpha_n \\ P^k\beta_1 \ldots \beta_n\end{bmatrix}$$

instead of:

$$\tilde{\mathrm{S}}^{Q^k a_1 \ldots a_n [\pi_1(a_1, \ldots, a_n), \ldots, \pi_k(a_1, \ldots, a_n)]}_{P^k \beta_1 \ldots \beta_n [\pi_1(\beta_1, \ldots, \beta_n), \ldots, \pi_k(\beta_1, \ldots, \beta_n)]} \psi \Big|$$

where $\beta_1, \ldots, \beta_n$ are pairwise distinct individual variables. (Note that the variables $a_1, \ldots, a_n$ serve as argument indicators only for the subformulas of ψ that are argument-formulas of $Q^k a_1 \ldots a_n$ in ψ. The only relevent argument indicators in the substituend are the distinct n-place predicate variables $\pi_1, \ldots, \pi_k$.)

§ 3. Semantics

In our characterization of the set-theoretical structures which we shall use in our definition of validity, we extend the notion of a general model for standard second order logic. Each such model will be relativized to a language, i.e., a set of categorial constants, and will be said to be a model for that language. We define the latter notion first.

Where L is a language (containing $\sim$, $\rightarrow$, $\forall$), we understand $\mathfrak{A} = \langle D, R, K_n, M_{k,n} \rangle_{k,n \in \omega - \{0\}}$ to be a *model for L* if (1) D is a non-empty set, (2) $K_n \subseteq \{0, 1\}^{D^n}$, (3) $M_{k,n} \subseteq \{0, 1\}^{(K_n)^k}$, and (4) R is a function with L as domain and such that: (a) $R(\zeta) \in D$, for each individual constant ζ in L; (b) $R(\pi) \in K_n$, for each n-place predicate constant in L; (c) $R(Q^k a_1 \ldots a_n) \in M_{k,n}$, for each k-ary n-place quantifier constant in L; and (d) $R(\Delta) \in \{0, 1\}^{\{0,1\}^n}$, for each n-ary truth-functional operator in L. We also require that for each $A \in K_1$ and each individual variable a, $R(\forall a)(A) = 1$ iff A is the characteristic function of D^1 and that $R(\sim)$ and $R(\rightarrow)$ are the negation and material conditional truth-functions, respectively [9].

Where $\mathfrak{A}$ is a model defined as above, we say that $\mathfrak{a}$ is an *assignment* (of values to variables) *in* $\mathfrak{A}$ if $\mathfrak{a}$ is a function with the set of variables (of all types) as its domain and such that (1) $\mathfrak{a}(a) \in D$, where a is an individual variable, (2) $\mathfrak{a}(\pi) \in K_n$, where π is an n-place predicate variable, and (3) $\mathfrak{a}(Q^k a_1 \ldots a_n) \in M_{k,n}$, where $Q^k a_1 \ldots a_n$ is a k-ary n-place quantifier-variable. Where $\xi_1, \ldots, \xi_m$ are variables (of whatever types), we take $\mathfrak{a}\binom{\xi_1 \ldots \xi_m}{u_1 \ldots u_m}$ to be that assignment which is exactly like $\mathfrak{a}$ except (at most) in its assigning, for each i, $1 \leqslant i \leqslant m$, u_i to ξ_i.

Relative to a model $\mathfrak{A}$ and an assignment $\mathfrak{a}$ in $\mathfrak{A}$, we define $ext_{\mathfrak{A},\mathfrak{a}}(\xi)$ as the *extension in $\mathfrak{A}$ of* a categorial constant or variable ξ of the language for which $\mathfrak{A}$ is a model: where ξ is a constant in L, $ext_{\mathfrak{A},\mathfrak{a}}(\xi) = R(\xi)$; and where ξ is a variable, $ext_{\mathfrak{A},\mathfrak{a}}(\xi) = \mathfrak{a}(\xi)$.

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[9] We understand each natural number to be the set of natural numbers less than it and take ω to be the set of natural numbers. By A^B we understand the set of functions from B into A.

In regard to the extension of a formula relative to an assignment of values to its free variables, we follow Frege in taking this to be one of the truth-values, which we here associate with 0 and 1 (for the false and the true). If $\mathfrak{A}$ is a model for a language L and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, then *the extension in* $\mathfrak{A}$ *relative to* $\mathfrak{a}$ *of* a formula φ of L is defined recursively as follows: for atomic formulas $\pi(\zeta_1, \ldots, \zeta_n)$ of L, $ext_{\mathfrak{A},\mathfrak{a}}(\pi(\zeta_1, \ldots, \zeta_n)) = ext_{\mathfrak{A},\mathfrak{a}}(\pi)(ext_{\mathfrak{A},\mathfrak{a}}(\zeta_1), \ldots, ext_{\mathfrak{A},\mathfrak{a}}(\zeta_n))$; for a complex formula $\xi[\lambda_1, \ldots, \lambda_n]$ of L, where ξ is a truth-functional operator in L, $ext_{\mathfrak{A},\mathfrak{a}}(\xi[\lambda_1, \ldots, \lambda_n]) = ext_{\mathfrak{A},\mathfrak{a}}(\xi)(ext_{\mathfrak{A},\mathfrak{a}}(\lambda_1), \ldots, ext_{\mathfrak{A},\mathfrak{a}}(\lambda_n))$; for a complex formula $Q^k a_1 \ldots a_n[\varphi_1, \ldots, \varphi_k]$ of L, where $Q^k a_1 \ldots a_n$ is a quantifier variable or a quantifier constant in L, $ext_{\mathfrak{A},\mathfrak{a}}(Q^k a_1 \ldots a_n[\varphi_1, \ldots, \varphi_k]) = ext_{\mathfrak{A},\mathfrak{a}}(Q^k a_1 \ldots a_n)$ (the characteristic function of $\{x \in D^n : ext_{\mathfrak{A},\mathfrak{a}\binom{a_1 \ldots a_n}{x_1 \ldots x_n}}(\varphi_1) = 1\}, \ldots,$ the characteristic function of $\{x \in D^n : ext_{\mathfrak{A},\mathfrak{a}\binom{a_1 \ldots a_n}{x_1 \ldots x_n}}(\varphi_k) = 1\}$); for a formula $(\forall\pi)\psi$ of L, where π is an n-place predicate variable, $ext_{\mathfrak{A},\mathfrak{a}}((\forall\pi)\psi) = 1$ if $ext_{\mathfrak{A},\mathfrak{a}\binom{\pi}{F}}(\psi) = 1$ for all $F \in K_n$, and otherwise $ext_{\mathfrak{A},\mathfrak{a}}((\forall\pi)\psi) = 0$; for a formula $(\forall Q^k a_1 \ldots a_n)\psi$ of L, $ext_{\mathfrak{A},\mathfrak{a}}((\forall Q^k a_1 \ldots a_n)\psi) = 1$ if for all $A \in M_{k,n}$, $ext_{\mathfrak{A},\mathfrak{a}}(Q^k_A a_1 \ldots a_n)$ $(\psi) = 1$, and otherwise $ext_{\mathfrak{A},\mathfrak{a}}((\forall Q^k a_1 \ldots a_n)\psi) = 0$.

We say of a formula φ of a language L for which $\mathfrak{A}$ is a model that φ *holds in* $\mathfrak{A}$ if $ext_{\mathfrak{A},\mathfrak{a}}(\varphi) = 1$ for all assignments $\mathfrak{a}$ in $\mathfrak{A}$. It is easily seen that if φ is a sentence, i.e., a formula in which no variable has a free occurrence, then φ holds in $\mathfrak{A}$ iff $ext_{\mathfrak{A},\mathfrak{a}}(\varphi) = 1$, for some assignment $\mathfrak{a}$ in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a *general model* for L if every instance of (CP-1) and (CP-2) below which is a formula of L holds in $\mathfrak{A}$:

(CP-1) $(\exists\pi)(\forall a_1) \ldots (\forall a_n)[\pi(a_1, \ldots, a_n) \leftrightarrow \varphi]$

where $a_1, \ldots, a_n$ are distinct individual variables occurring free in φ and π is an n-place predicate-variable which does not occur in φ.

(CP-2) $(\exists Q^k a_1 \ldots a_n)(\forall\pi_1) \ldots (\forall\pi_k)[Q^k a_1 \ldots a_n[\pi_1(a_1, \ldots, a_n), \ldots, \pi_k(a_1, \ldots, a_n)] \leftrightarrow \varphi]$

where $\pi_1, \ldots, \pi_k$ are distinct n-place predicate variables occurring free in φ and $Q^k a_1 \ldots a_n$ is a k-ary n-place quantifier-variable which does not occur in φ.[10]

Finally, we say that a formula φ is *valid* if φ holds in every general model for a language of which φ is a formula.

The following semantic lemma will be found useful in our proof of completeness. We omit its proof here as it is a straightforward (but notationally messy) inductive argument on the structure of φ.

[10] For the purposes of the proof of completeness given in § 5 below, we can restrict (CP-2) further so that φ is to contain no bound occurrences of any quantifier variable and in that sense be "predicative" with respect to second-level concepts. This move seems ineffectual, however, in as much as the impredicativity of (CP-1) is essential to our proof of completeness and yields impredicative first-level concepts which must be allowed to fall within the second-level concepts specified by (CP-2).

Semantic Lemma. *If φ is a formula of a language for which $\mathfrak{A}$ is a model, $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, ζ is an individual, predicate or quantifier variable, λ is a constant or variable of the same type as ξ, and λ can be properly substituted for ξ in φ, then:*

$$ext_{\mathfrak{A},\mathfrak{a}\left(\frac{\xi}{ext_{\mathfrak{A},\mathfrak{a}}(\lambda)}\right)}(\varphi) = ext_{\mathfrak{A},\mathfrak{a}}(\varphi[{}^{\xi}_{\lambda}]) \,.$$

§ 4. Axioms and Theorems

By a (universal) generalization of a formula φ we understand a formula of the form $(\forall\xi_0) \dots (\forall\xi_{n-1})\varphi$, where $\xi_0, \dots, \xi_{n-1}$ are individual, predicate or quantifier variables. (When $n = 0$, we understand the quantifier prefix to be null.)

As *axioms* we take (CP-1), (CP-2) and all (universal) generalizations of formulas of any of the following forms (where ξ is an individual, predicate or quantifier variable, a is an individual variable, ζ, η are terms with ζ distinct from a, π is a 1-place predicate variable, $a_1, \dots, a_n$ are pairwise distinct individual variable, for $n > 0$, $Q^k a_1 \dots a_n$ is a k-ary n-place quantifier variable or constant, Δ is an n-ary truth-functional operator, and $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n$ are formulas):

(A1) $\varphi \to (\psi \to \varphi)$

(A2) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$

(A3) $(\sim \psi \to \sim \varphi) \to (\varphi \to \psi)$

(A4) $(\forall\xi)[\varphi \to \psi] \to [(\forall\xi)\varphi \to (\forall\xi)\psi]$

(A5) $\varphi \to (\forall\xi)\varphi$, where ξ does not occur free in φ,

(A6) $(\exists a)(\forall\pi)[\pi(a) \to \pi(\zeta)]$

(A7) $(\forall\pi)[\pi(\zeta) \to \pi(\eta)] \to [\varphi \to \psi]$, where φ is an atomic formula and ψ is obtained from φ by replacing an occurrence of η by an occurrence of ζ,

(A8) $(\forall a_1) \dots (\forall a_n)[\varphi_1 \leftrightarrow \psi_1] \wedge \dots \wedge (\forall a_1) \dots (\forall a_n)[\varphi_n \leftrightarrow \psi_n]$
$\to (Q^k a_1 \dots a_n[\varphi_1, \dots, \varphi_n] \leftrightarrow Q^k a_1 \dots a_n[\psi_1, \dots, \psi_n])$

(A9) $[\varphi_1 \leftrightarrow \psi_1] \wedge \dots \wedge [\varphi_n \leftrightarrow \psi_n] \to (\Delta[\varphi_1, \dots, \varphi_n] \leftrightarrow \Delta[\psi_1, \dots, \psi_n])$ [11]

Proofs are understood in the usual way as finite sequences every constituent of which is either an axiom or obtained from preceding constituents by *modus ponens*, the only primitive inference rule of the system. A *theorem* is any formula

[11] If $\sim$, $\to$ are the only truth-functional constants available, then (A9) is derivable from (A1)–(A3) and modus ponens. With truth-functional variables, however, (A9) must be retained (where Δ is then understood as an n-ary truth-functional variable), and for that reason we include it here now. Of course it is known that with quantified truth-functional variables, the material biconditional sign alone suffices for the representation of every truth-function.

which terminates a proof. A set Γ of formulas is said to *yield* a formula φ, in symbols $\Gamma \vdash \varphi$, if there is a natural number n and n formulas $\psi_0, \ldots, \psi_{n-1}$ in Γ such that $[\psi_0 \wedge \ldots \wedge \psi_{n-1} \rightarrow \varphi]$ is a theorem. (When $n = 0$ we understand this "conditional" to be φ itself.) When Γ is empty we write '$\vdash \varphi$' for '$\Gamma \vdash \varphi$'. Obviously, φ is a theorem iff $\vdash \varphi$. Also obvious is the derivation of the deduction theorem from this definition.

Because of (A1)–(A3), every tautologous formula is a theorem. Moreover, because every (universal) generalization of an axiom is an axiom and, by (A4), generalization is preserved by modus ponens, the rule of generalization (for individual, predicate and quantifier variables) is derivable. Then, by generalization, (A4)–(A5) and (A8)–(A9) and a simple inductive argument on the structure of formulas, the interchange law for provably equivalent formulas is derivable. Leibniz' law, in the form:

(LL-1) $\quad \vdash (\forall \pi)[\pi(\zeta) \rightarrow \pi(\eta)] \rightarrow [\varphi \leftrightarrow \psi]$

where ψ comes from φ by replacing one or more free occurrences of η by a free occurrence of ζ, is now provable by an induction on the subformulas of φ. In the atomic case, (LL-1) is obtained from (A7) and truth-functional logic; otherwise, (LL-1) is obtained by an inductive argument utilizing truth-functional logic, (A4)–(A5), (A8)–(A9) and generalization.

Given (LL-1), the principle of universal instantiation of a term for a generalized individual variable,

(UI-1) $\quad \vdash (\forall a)\varphi \rightarrow \varphi[{}^{a}_{\zeta}]$

is derivable. For by repeated application of (LL-1), $(\forall \pi)[\pi(\zeta) \rightarrow \pi(a)] \rightarrow (\varphi \leftrightarrow \varphi[{}^{a}_{\xi}])$ is provable, and therefore by generalization on a, truth-functional logic and axioms (A4)–(A6), (UI-1) follows.

The analogue of (LL-1) for first-level concepts (in our extensional system) is:

(LL-2) $\quad \vdash (\forall a_1) \ldots (\forall a_n)[\pi(a_1, \ldots, a_n) \leftrightarrow \varphi] \rightarrow |\psi \leftrightarrow \check{S}_{\varphi}^{\pi(a_1,\ldots,a_n)}\psi|]$

which is proved by an inductive argument on the subformulas of ψ (and where $a_1, \ldots, a_n$ are assumed to be pairwise distinct). If ψ is atomic, (LL-2) follows by n applications of (UI-1); otherwise, (LL-2) follows by the inductive hypothesis, truth-functional logic, generalization, (A4)–(A5), (A8)–(A9) and the fact that the substitution is voided (in which case (LL-2) is tautologous) if ψ begins with a quantifier expression binding a variable other than $a_1, \ldots, a_n$ which occurs free in φ.

Just as (UI-1) is derivable from (LL-1), we obtain from (LL-2) the principle of universal instantiation of a formula relative to certain distinct individual variables $a_1, \ldots, a_n$ (as argument indicators), for a generalized n-place predicate variable:

(UI-2) $\vdash (\forall \pi)\psi \rightarrow S_{\varphi}^{\pi(a_1,\ldots,\varphi_n)}\psi|$

If π does not occur in φ and $\alpha_1, \ldots, \alpha_n$ occur free in φ, then by generalization on (LL-2), truth-functional logic, (A4)–(A5) and (CP-1), (UI-2) follows. This qualified form of (UI-2), i.e., where π does not occur in φ and $\alpha_1, \ldots, \alpha_n$ occur free in φ, suffices to establish the rewrite rule for bound predicate variables, from which the unqualified form of (UI-2) follows by a rewrite of the occurrences of π in $(\forall\pi)\psi$ to an n-place predicate variable new to φ together with the interchange rule regarding the provable equivalence of φ with $(\varphi\wedge[\pi(\alpha_1, \ldots, \alpha_n) \leftrightarrow \pi(\alpha_1, \ldots, \alpha_n)]$ (so as to guarantee free occurrences of $\alpha_1, \ldots, \alpha_n$). To prove the rewrite rule use $\sigma(\alpha_1, \ldots, \alpha_n)$ in place of φ in the qualified form of (UI-2), where σ is an n-place predicate variable distinct from π and new to ψ. Then by generalization, (A4)–(A5), $\vdash (\forall\pi)\psi \rightarrow (\forall\sigma)\psi[^{\pi}_{\sigma}]$, from which, together with the interchange rule for provably equivalent formulas, the rewrite rule, and thereby the unqualified (UI-2), follows.

With the rewrite rule for bound predicate variables we can define identity as indiscernibility without bothering to specify which monadic predicate variable occurs in the definiens:

$$\zeta = \eta =_{\mathrm{df}} (\forall\pi)[\pi(\zeta) \rightarrow \pi(\eta)]$$

The analogue of (LL-1) and (LL-2) for second-level concepts is:

(LL-3) $$\vdash (\forall\pi_1) \ldots (\forall\pi_k)(Q^k\alpha_1 \ldots \alpha_n[\pi_1(\alpha_1, \ldots, \alpha_n), \ldots, \pi_k(\alpha_1, \ldots, \alpha_n)] \leftrightarrow \varphi)$$
$$\rightarrow (\psi \leftrightarrow \tilde{\mathrm{S}}_{\varphi}^{Q^k\alpha_1\ldots\alpha_n[\pi_1(\alpha_1,\ldots,\alpha_n),\ldots,\pi_k(\alpha_1,\ldots,\alpha_n)]}\psi|)$$

which is also proved by an inductive argument on the structure of ψ. If ψ is atomic or the substitution is not proper, then (LL-3) is tautologous. Assume then that the substitution is proper and that ψ is complex. If ψ is of the form $\Delta[\chi_1, \ldots, \chi_m]$, where Δ is an m-ary truth-functional operator, then (LL-3) follows from the inductive hypothesis, (A9) and truth-functional logic. If ψ is of the form $(\forall\beta)\chi$, $(\forall\sigma)\chi$ or $(\forall P^j\beta_1 \ldots \beta_m)\chi$, then (LL-3) follows from the inductive hypothesis, generalization, (A4)–(A5) and truth-functional logic. (We note that (A5) is applicable since the substitution is proper, and therefore neither β, σ nor $P^j\beta_1 \ldots \beta_m$ occur free in φ.) Suppose finally that ψ is of the form $P^j\beta_1 \ldots \beta_m[\chi_1, \ldots, \chi_j]$, where $P^j\beta_1 \ldots \beta_m$ is not $Q^k\alpha_1 \ldots \alpha_n$. We shall add the temporary qualification to (LL-3) that $\alpha_1, \ldots, \alpha_n$ do not occur free in φ. Later once we prove the general rewrite law for bound individual variables, we can drop this qualification. With this qualification here, however, (LL-3) follows from the inductive hypothesis by generalization, (A4)–(A5) and (A8). Suppose finally that $P^j\beta_1 \ldots \beta_m$ is $Q^k\alpha_1 \ldots \alpha_n$, i. e., suppose ψ is $Q^k\alpha_1 \ldots \alpha_n[\chi_1, \ldots, \chi_k]$. Note that where

$$\chi_i' = \tilde{\mathrm{S}}_{\varphi}^{Q^k\alpha_1\ldots\alpha_n[\pi_1(\alpha_1,\ldots,\alpha_n),\ldots,\pi_k(\alpha_1,\ldots,\alpha_n)]}\chi_i|$$

for $1 \leqslant i \leqslant k$, then in this special case:

$$\tilde{\mathrm{S}}_{\varphi}^{Q^k\alpha_1\ldots\alpha_n[\pi_1(\alpha_1,\ldots,\alpha_n),\ldots,\pi_k(\alpha_1,\ldots,\alpha_n)]}\psi| = \tilde{\mathrm{S}}_{\chi_1',\ldots,\chi_k'}^{\pi_1(\alpha_1,\ldots,\alpha_n),\ldots,\pi_k(\alpha_1,\ldots,\alpha_n)}\varphi|\,.$$

Now since by assumption the substitution is proper, then by (UI-2) (applied k times):

$$\vdash (\forall \pi_1) \dots (\forall \pi_k)(Q^k \alpha_1 \dots \alpha_n[\pi_1(\alpha_1, \dots, \alpha_n), \dots, \pi_k(\alpha_1, \dots, \alpha_n)] \leftrightarrow \varphi)$$
$$\to \big(Q^k \alpha_1 \dots \alpha_n[\chi_1', \dots, \chi_k'] \leftrightarrow \check{\mathrm{S}}^{\pi_1(\alpha_1, \dots, \alpha_n), \dots, \pi_k(\alpha_1, \dots, \alpha_n)}_{\chi_1', \dots, \chi_k'} \varphi|\big) .$$

But since by the inductive hypothesis, generalization, (A4)–(A5) and (A8),

$$\vdash (\forall \pi_1) \dots (\forall \pi_k)(Q^k \alpha_1 \dots \alpha_n[\pi_1(\alpha_1, \dots, \alpha_n), \dots, \pi_k(\alpha_1, \dots, \alpha_n)] \leftrightarrow \varphi)$$
$$\to (Q^k \alpha_1 \dots \alpha_n)[\chi_1, \dots, \chi_k] \leftrightarrow Q^k \alpha_1 \dots \alpha_n[\chi_1', \dots, \chi_k'])$$

then (LL-3) follows for this case from the above two theorems.

Also analogous to the way (UI-2) is derivable from (LL-2), we obtain from (LL-3) the following principle of universal instantiation:

$$\text{(UI-3)} \qquad \vdash (\forall Q^k \alpha_1 \dots \alpha_n)\psi \to \tilde{\mathrm{S}}^{Q^k \alpha_1 \dots \alpha_n[\pi_1(\alpha_1, \dots, \alpha_n), \dots, \pi_k(\alpha_1, \dots, \alpha_n)]}_{\varphi} \psi| .$$

If $Q^k \alpha_1 \dots \alpha_n$ does not occur in φ and $\pi_1, \dots, \pi_k$ are distinct n-place predicate variables occurring free in φ, then (UI-3) follows from (CP-2) and (LL-3) by generalization, truth-functional logic and (A4)–(A5). This qualified form of (UI-3) suffices to establish the rewrite law for bound quantifier variables, from which the unqualified form follows by rewriting $Q^k \alpha_1 \dots \alpha_n$ to a quantifier variable new to φ together with the interchange of φ with $\big(\varphi \wedge (\forall \alpha_1) \dots (\forall \alpha_n)[\pi_1(\alpha_1, \dots, \alpha_n) \to \pi_1(\alpha_1, \dots, \alpha_n)] \wedge \dots \wedge (\forall \alpha_1) \dots (\forall \alpha_n)[\pi_k(\alpha_1, \dots, \alpha_n) \to \pi_k(\alpha_1, \dots, \alpha_n)]\big)$ so as to guarantee free occurrences of $\pi_1, \dots, \pi_k$ in the substituend. To prove the rewrite law itself use $P^k \beta_1 \dots \beta_n[\pi_1(\beta_1, \dots, \beta_n), \dots, \pi_k(\beta_1, \dots, \beta_n)]$ in place of φ in the qualified form of (UI-3), where $P^k \beta_1 \dots \beta_n$ is a k-ary n-place quantifier variable distinct from $Q^k \alpha_1 \dots \alpha_n$ and where $\beta_1, \dots, \beta_n$ are new to ψ. (Note that this substitution satisfies the temporary qualification for (LL-3) that $\alpha_1, \dots, \alpha_n$ do not occur free in the substituend.) Then by generalization, (A4)–(A5),

$$\vdash (\forall Q^k \alpha_1 \dots \alpha_n)\psi \to (\forall P^k \beta_1 \dots \beta_n)\psi \begin{bmatrix} Q^k \alpha_1 \dots \alpha_n \\ P^k \beta_1 \dots \beta_n \end{bmatrix}$$

which together with the interchange rule for provably equivalent formulas yields the rewrite law for bound quantifier variables. Finally, we observe that with the rewrite law for bound quantifier variables the general rewrite law for bound individual variables follows.

We close this section with a lemma the proof for which we omit here as it is completely analogous to the proof of the same lemma in standard systems.

Lemma. *If $\Gamma \vdash \varphi[\frac{\xi}{\delta}]$, where ξ is a variable, δ is a constant of the same type as ξ, and δ does not occur in any member of Γ, then $\Gamma \vdash (\forall \xi)\varphi$.*

§ 5. Soundness and Completeness

As usual, we understand the *consistency* of a set of formulas of a language L to consist in the failure of that set to yield every formula of L. Such a set is also said to be *maximally consistent* if the addition to it of any formula of L not already in it results in an inconsistent set. The following lemma is a well-known and obvious consequence of maximal consistency. We omit the proof here.

Lemma. *If Γ is a maximally consistent set of formulas of a language L and Δ is a k-ary truth-functional constant (or variable) in L, then there is exactly one k-ary truth-function $f \in \{0, 1\}^{\{0,1\}^k}$ such that for all $x \in \{0, 1\}^k$ and all formulas $\varphi_1, \dots, \varphi_k$ of L, if for $1 \leqslant i \leqslant k$, $x_i = 1$ iff $\varphi_i \in \Gamma$, then $f(x_1, \dots, x_k) = 1$ iff $\Delta[\varphi_1, \dots, \varphi_k] \in \Gamma$.*

Where Γ is a set of formulas, we say that Γ is *simultaneously satisfiable* in a model $\mathfrak{A}$ if there is an assignment $\mathfrak{a}$ in $\mathfrak{A}$ such that for all $\psi \in \Gamma$, $ext_{\mathfrak{A},\mathfrak{a}}(\psi) = 1$. In terms of this notion, our present system is seen to be sound and complete as follows.

Soundness. *If Γ is simultaneously satisfiable by an assignment $\mathfrak{a}$ in a general model $\mathfrak{A}$, and $\Gamma \vdash \varphi$, then $\Gamma \cup \{\varphi\}$ is simultaneously satisfiable in $\mathfrak{A}$ by $\mathfrak{a}$.*

Proof. Suppose for all $\psi \in \Gamma$, $ext_{\mathfrak{A},\mathfrak{a}}(\psi) = 1$, for some assignment $\mathfrak{a}$ in $\mathfrak{A}$, and that $\vdash (\psi_1 \wedge \dots \wedge \psi_k \to \varphi)$, for some $\psi_1, \dots, \psi_k \in \Gamma$. Since, by definition, $\to$ must be interpreted as the material conditional truth-function, to show $ext_{\mathfrak{A},\mathfrak{a}}(\varphi) = 1$ it suffices to show $ext_{\mathfrak{A},\mathfrak{a}}(\chi) = 1$, whenever χ is a theorem formulable in the language for which $\mathfrak{A}$ is a model. To show this, however, since modus ponens preserves truth, it suffices to show $ext_{\mathfrak{A},\mathfrak{a}}(\chi) = 1$ whenever χ is an axiom. The latter is easily seen to be the case whenever χ is an axiom of the form (A1)–(A6) or (A8)–(A9) since these axioms hold in all models, general or otherwise. Also, since $\mathfrak{A}$ is a general model, then χ holds in $\mathfrak{A}$ trivially if it is an instance of (CP-1) or (CP-2). Finally, if χ is of the form (A7), then it is seen to hold in $\mathfrak{A}$ because of (CP-1). (Q.E.D)

Completeness. *Every consistent set of formulas is simultaneously satisfiable in some general model.*

Proof. Suppose Γ is a consistent set of formulas of some language L, and let λ be the least infinite ordinal equinumerous with or greater than L. We add to L new individual constants $\zeta_0, \dots, \zeta_\mu, \dots (\mu \in \lambda)$, new n-place predicate constants, for each $n \in \omega - \{0\}$, $\pi_0^n, \dots, \pi_\mu^n, \dots, (\mu \in \lambda)$, and new k-ary n-place quantifier constants, for $k, n \in \omega - \{0\}$, $P^k\beta_1 \ldots \beta_n, \ldots, P^k_\mu \beta_1 \ldots \beta_n, \ldots$ $(\mu \in \lambda)$, where $\beta_1, \ldots, \beta_n$ are certain (e.g. the first n) distinct individual variables. We call the resulting language L^*.

Let $\Sigma_1, \ldots, \Sigma_\mu, \ldots$ $(\mu \in \lambda)$ be an ordering of the set of formulas of L^* of the form $(\exists \alpha)\varphi$, $(\exists \pi)\varphi$ or $(\exists Q^k \alpha_1 \ldots \alpha_n)\varphi$. We define the chain $A_0, \ldots, A_\mu, \ldots$ $(\mu \in \lambda)$ by ordinal recursion as follows: (1) $A_0 = \Gamma$; (2) if $\bigcup_{\mu \in \nu} A_\mu \cup \{\Sigma_\nu\}$ is not consistent, then $A_\nu = \bigcup_{\mu \in \nu} A_\mu$; (3) if it is consistent, however, then where $\Sigma_\nu = (\exists \xi)\varphi$,

$$A_\nu = \bigcup_{\mu \in \nu} A_\nu \cup \{\varphi[\tfrac{\xi}{\delta_\iota}]\}$$

where δ_ι is a constant in $L^* - L$ of the same type as ξ and where ι is the least ordinal such that δ_ι does not occur in any member of $\bigcup_{\mu \in \nu} A_\mu \cup \{\Sigma_\nu\}$.

By ordinal induction, we show first that each A_ν is consistent, for $\nu \in \lambda$. If $\bigcup_{\mu \in \nu} A_\mu \cup \{\Sigma_\nu\}$ is not consistent, then A_ν is consistent by definition and the inductive hypothesis. Suppose then that $\bigcup_{\mu \in \nu} A_\nu \cup \{\Sigma_\nu\}$ is consistent but that A_ν is not consistent. Then, where $\Sigma_\nu = (\exists \xi)\varphi$, we have by the deduction theorem

$$\bigcup_{\mu \in \nu} A_\nu \vdash \sim \varphi[\tfrac{\xi}{\delta_\iota}]$$

and therefore by generalization on a constant not occurring in the premise set,

$$\bigcup_{\mu \in \nu} A_\nu \vdash (\forall \xi) \sim \varphi$$

which is impossible since then $\bigcup_{\mu \in \nu} A_\nu \cup \{\Sigma_\nu\}$ is not consistent. Accordingly, not only is each A_ν consistent but so is $\bigcup_{\mu \in \lambda} A_\mu$. Hence by Lindenbaum's lemma we conclude that there is a maximally consistent set W of formulas of L^* such that $\bigcup_{\mu \in \lambda} A_\mu \subseteq W$.

We now proceed to the construction of the general model based on W. Where η is a term of L^*, let $[\eta] = \{\delta\colon \delta$ is a term of L^* and $\eta \equiv \delta \in W\}$. If $\eta \equiv \delta \in W$, then since identity is an equivalence relation $[\eta] = [\delta]$. Also, by (A6) and the way W was constructed, $[\eta] = [\zeta_\iota]$, for some $\iota \in \lambda$. Accordingly, we can set the domain of discourse $D = \{[\zeta_\iota]\colon \iota \in \lambda\}$. Now let R be that function with L^* as domain and such that (1) $R(\eta) = [\eta]$, for each individual constant η of L^*; (2) for each n-place predicate constant σ in L^*, $R(\sigma)$ is that function in $\{0, 1\}^{D^n}$ such that for all terms $\eta_1, \ldots, \eta_n$ of L^*, $R(\sigma)([\eta_1], \ldots, [\eta_n]) = 1$ iff $\sigma(\eta_1, \ldots, \eta_n) \in W$; (3) for each k-ary n-place quantifier constant $Q^k\alpha_1 \ldots \alpha_n$ in L^*, $R(Q^k\alpha_1 \ldots \alpha_n)$ is that function in $\{0, 1\}^{B^k}$, where $B = \{R(\sigma)\colon \sigma$ is an n-place predicate constant in $L^*\}$, such that for all n-place predicate constants $\sigma_1, \ldots, \sigma_k$ in L^*, $R(Q^k\alpha_1 \ldots \alpha_n)\big(R(\sigma_1), \ldots, R(\sigma_k)\big) = 1$ iff $Q^k\alpha_1 \ldots \alpha_n[\sigma_1(\alpha_1, \ldots, \alpha_n), \ldots, \sigma_k(\alpha_1, \ldots, \alpha_n)] \in W$; and (4) for each k-ary truth-functional operator Δ in L^*, $R(\Delta)$ is that function in $\{0, 1\}^{\{0,1\}^k}$ as determined by Δ and W in accordance with the above lemma.

$R(\sim)$ and $R(\rightarrow)$, because every instance of (A1)–(A3) is in W, are easily seen to be negation and conditional truth-functions. Also, because of the way quantifiers are interpreted above, $R(\forall a)(R(\sigma)) = 1$, where σ is any monadic predicate constant in L^*, iff $(\forall a)\sigma(a) \in W$, i.e., by (UI-1) and the way W was constructed, iff $R(\sigma)([\eta]) = 1$ for all terms η of L^*, and therefore, iff $R(\sigma)$ is the characteristic function of D^1. Finally, from the way W was constructed it follows that for each n-place predicate constant σ in L^*, there is, by (CP-1) and (UI-1), an $\iota \in \lambda$ such that $R(\sigma) = R(\pi^n_\iota)$; moreover, for each k-ary n-place quantifier constant $Q^k a_1 \ldots a_n$ in L^*, there is, by (CP-2) and (UI-2), a $\mu \in \lambda$ such that $R(Q^k a_1 \ldots a_n) = R(P^k_\mu \beta_1 \ldots \beta_n)$. Accordingly, for $k, n \in \omega - \{0\}$, we can set $K_n = \{R(\pi^n_\iota)\colon \iota \in \lambda\}$ and $M_{k,n} = \{R(P^k_\mu \beta_1 \ldots \beta_n)\colon \mu \in \lambda\}$. Then setting $\mathfrak{A} = \langle D, R, K_n, M_{k,n}\rangle_{k,n\in\omega-\{0\}}$, we note that $\mathfrak{A}$ is a model for L^*.

We now take $\mathfrak{a}$ to be that function with the set of variables as domain and which is such that (1) for each individual variable a, $\mathfrak{a}(a) = [a]$, (2) for each n-place predicate variable σ, $\mathfrak{a}(\sigma) = R(\pi^n_\iota)$, where ι is the least ordinal such that $(\forall a_1) \ldots (\forall a_n)[\sigma(a_1, \ldots, a_n) \leftrightarrow \pi^n_\iota(a_1, \ldots, a_n)] \in W$, where $a_1, \ldots, a_n$ are distinct individual variables, and (3) for each k-ary n-place quantifier variable $Q^k a_1 \ldots a_n$, $\mathfrak{a}(Q^k a_1 \ldots a_n) = R(P^k_\mu \beta_1 \ldots \beta_n)$, where μ is the least ordinal such that

$$(\forall \sigma_1) \ldots (\forall \sigma_k)\{Q^k a_1 \ldots a_n[\sigma_1(a_1, \ldots, a_n), \ldots, \sigma_k(a_1, \ldots, a_n)] \leftrightarrow \\ \leftrightarrow P^k_\mu \beta_1 \ldots \beta_n[\sigma_1(\beta_1, \ldots, \beta_n), \ldots, \sigma_k(\beta_1, \ldots, \beta_n)]]$$

is in W, where $\sigma_1, \ldots, \sigma_k$ are distinct n-place predicate variables. It is clear that $\mathfrak{a}$ is an assignment in $\mathfrak{A}$.

We now show that relative to $\mathfrak{a}$ the extension in $\mathfrak{A}$ of a formula φ of L^* is 1 iff $\varphi \in W$. The proof is by induction on the number of truth-functional operators and quantifier constants or variables occurring in φ. When that number is zero, then φ is an atomic formula $\sigma(\eta_1, \ldots, \eta_n)$. By definition $ext_{\mathfrak{A},\mathfrak{a}}(\sigma(\eta_1, \ldots, \eta_n)) = ext_{\mathfrak{A},\mathfrak{a}}(\sigma)(ext_{\mathfrak{A},\mathfrak{a}}(\eta_1), \ldots, ext_{\mathfrak{A},\mathfrak{a}}(\eta_n)) = R(\pi^n_\iota)([\eta_1], \ldots, [\eta_n])$ 1 iff $\pi^n_\iota(\eta_1, \ldots, \eta_n) \in W$ iff $\sigma(\eta_1, \ldots, \eta_n) \in W$, since π^n_ι is either σ or coextensive with σ in W.

If φ is of the form $\Delta[\psi_1, \ldots, \psi_k]$, where Δ is a k-ary truth-functional operator in L^*, then since $ext_{\mathfrak{A},\mathfrak{a}}(\Delta)$ is the k-ary truth-function determined by the above lemma, the desired conclusion follows trivially from the inductive hypothesis and the conditions specified in that lemma.

Suppose now that φ is of the form $Q^k a_1 \ldots a_k[\psi_1, \ldots, \psi_k]$. By (CP-1) and the way W was constructed, for $1 \leqslant i \leqslant k$, there is an $\iota_i \in \lambda$ such that

$$(\forall a_1) \ldots (\forall a_n)[\pi^n_{\iota_i}(a_1, \ldots, a_n) \leftrightarrow \psi]$$

is in W[12]. Therefore, by (A8) and the fact that W is maximally consistent, $Q^k a_1 \ldots a_n[\psi_1, \ldots, \psi_k] \in W$ iff $Q^k a_1 \ldots a_n[\pi_{\iota_1}(a_1, \ldots, a_n), \ldots, \pi_{\iota_k}(a_1 \ldots, a_n)] \in W$. Also, by (UI-1), $\psi_i[^{a_1 \ldots a_n}_{\eta_1 \ldots \eta_n}] \in W$ iff $\pi_{\iota_i}(\eta_1, \ldots, \eta_n) \in W$, where $\eta_1, \ldots, \eta_n$ are arbitrary

[12] If $a_1, \ldots, a_n$ do not occur free in ψ_i, use (CP-1) on $(\psi_i \wedge a_1 \equiv a_1 \wedge \ldots \wedge a_n \equiv a_n)$ first and then use interchange.

individual constants in L^*. Therefore, by the inductive hypothesis, $ext_{\mathfrak{A},\mathfrak{a}}(\psi_i[{}^{a_1 \dots a_n}_{\eta_1 \dots \eta_n}]) = 1$ iff $\pi_{\iota_i}(\eta_1, \dots, \eta_n) \in W$; and therefore, since $D = \{[\zeta_\iota]: \iota \in \lambda\}$, $\{x \in D^n: ext_{\mathfrak{A},\mathfrak{a}\left({}^{a_1 \dots a_n}_{x_1 \dots x_n}\right)}(\psi_i) = 1\} = R(\pi_{\iota_i})$. Accordingly, $ext_{\mathfrak{A},\mathfrak{a}}(Q^k a_1 \dots a_n[\psi_1, \dots, \psi_k])$ $= ext_{\mathfrak{A},\mathfrak{a}}(Q^k a_1 \dots a_n)\big(R(\pi_{\iota_1}), \dots, R(\pi_{\iota_k})\big) = 1$ iff

$$P^k_\mu \beta_1 \dots \beta_n[\pi_{\iota_1}(\beta_1, \dots, \beta_n), \dots, \pi_{\iota_k}(\beta_1, \dots, \beta_n)] \in W\,,$$

where $P^k_\mu \beta_1 \dots \beta_n$ is $Q^k a_1 \dots a_n$ (if the latter is a constant) or "co-extensive" with $Q^k a_1 \dots a_n$ in W, and therefore, iff

$$Q^k a_1 \dots a_n[\pi_{\iota_1}(a_1, \dots, a_n), \dots, \pi_{\iota_k}(a_1, \dots, a_n)] \in W$$

iff $Q^k a_1 \dots a_n[\psi_1, \dots, \psi_k] \in W$.

Finally, suppose φ is either of the form $(\forall \pi)\psi$ or $(\forall Q^k a_1 \dots a_n)\psi$. Since the argument is similar for each, we restrict ourselves to the latter case and note that $ext_{\mathfrak{A},\mathfrak{a}}\big((\forall Q^k a_1 \dots a_n)\psi\big) = 1$ iff, by the definition of $M_{k,n}$, for all $\mu \in \lambda$, $ext_{\mathfrak{A},\mathfrak{a}\left({}^{Q^k a_1 \dots a_n}_{R(P^k_\mu \beta_1 \dots \beta_n)}\right)}(\psi) = 1$ iff, by the semantic lemma of § 3, for all $\mu \in \lambda$, $ext_{\mathfrak{A},\mathfrak{a}}\left(\psi\left[{}^{Q^k a_1 \dots a_n}_{P^k_\mu \beta_1 \dots \beta_n}\right]\right) = 1$ iff, by the inductive hypothesis, for all $\mu \in \lambda$, $\psi\left[{}^{Q^k a_1 \dots a_n}_{P^k_\mu \beta_1 \dots \beta_n}\right] \in W$ iff, from the way W was constructed, $(\forall Q^k a_1 \dots a_n)\psi \in W$.

We conclude then that for each formula φ of L^*, $ext_{\mathfrak{A},\mathfrak{a}}(\varphi) = 1$ iff $\varphi \in W$. Since W is maximally consistent, all universally closed instances of (CP-1) and (CP-2) are in W and therefore hold in $\mathfrak{A}$, from which it follows that $\mathfrak{A}$ is a general model. Finally, since $\Gamma \subseteq W$, Γ is simultaneously satisfiable in $\mathfrak{A}$. (Q.E.D.)

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