STUDENTS' MEANINGS FOR EXTENSIVE QUANTITATIVE UNKNOWNS

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A series of three design experiments was conducted with middle school students to investigate relationships between students' rational number knowledge and algebraic reasoning. After the first experiment a change was made in the investigation of students' construction of extensive quantitative unknowns. Students were asked to represent in pictures and equations the values for an unknown height measured in two different, multiplicatively-related measurement units. The work of 13 students operating at two levels of multiplicative reasoning was analyzed to identify differences and similarities. Students operating at the lower level of reasoning required substantial support to construct unknowns with implicit quantitative relationships, while students operating at the higher level of reasoning constructed unknowns with explicitly embedded units.

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It is well known that secondary students develop concepts for unknowns and variables that are often quite different from what teachers and curriculum developers intend. Common and persistent issues include that students see letters as standing for labels of objects rather than quantities and for known rather than indeterminate values (e.g., Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Küchemann, 1981). In addition, an increasing number of secondary students take algebra courses (Stein, Kaufman, Sherman, & Hillen, 2011), and so algebra teachers are tasked with working with a greater diversity of students. This phenomenon has been managed in several ways, from tracking lower-skilled students into double periods of algebra (Nomi & Allensworth, 2013) to teaching all students in heterogeneous groups with supports such as a 2-year algebra course (Boaler & Staples, 2008). In whatever ways diversity is managed, more needs to be known about the algebraic thinking and learning of a wide range of secondary students in order to inform the kind of supports that both students and teachers need.

To address these issues, we conducted three iterative design experiments with small groups of cognitively diverse middle school students in which we studied relationships between their rational number knowledge and algebraic reasoning. Analysis of the first experiment led us to revise our approach to developing concepts of unknowns. The purpose of this paper is to describe and account for how the students in the second and third experiments conceived of what we call *extensive quantitative unknowns* (EQUs). For example, consider the relationship between two values of the unknown height of a school measured in feet and inches: The number of inches that can fit into the height is 12 times the number of feet that can fit into the height because each foot in the height is equivalent to 12 inches. We call these problems Single Unknown Problems, and we posed them with non-standard measurement units to promote rethinking of relationships that students might take for granted when working with standard ones. Our research questions are: (1) How did the students conceive of EQUs? (2) How can we account for their conceptions?

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Approach to Algebraic Reasoning and an Analytical Tool

A Quantitative Approach

In our experiments we took a quantitative approach to algebraic reasoning, in which a *quantity* is a property of one's concept of an object or phenomenon "that can be subjected to comparison" (Steffe & Olive, 2010, p. 49). Quantities can be extensive or intensive (Schwartz, 1988): Extensive quantities can be directly counted or measured (e.g., a distance), while intensive quantities are often created out of multiplicative comparisons of extensive quantities (e.g., amount of distance covered per unit of time). In our experiments we focused first on extensive quantities because, generally speaking, they are more basic than intensive quantities.

To conceive of an extensive quantity requires conceiving of a measurement unit, of the property as subdivided into these units, and of a way to enumerate these units to find a value (Thompson, 2011). However, extensive quantities can be thought about absent values, which makes them useful algebraically (Smith & Thompson, 2008). We conceptualized an EQU as an extensive quantity for which a value is not known, but for which a value could be determined. In our experiment we focused primarily on distances. So, an EQU could be thought of as a distance for which we have a measurement unit; we can imagine subdividing the distance into those units; but we don't know how many of those units will be needed to span the distance.

Students' Multiplicative Concepts

In our work we use students' multiplicative concepts as a key analytical tool. We conceive of students' multiplicative concepts as the interiorized results of students' units-coordinating schemes (Steffe, 1994). *Interiorization* refers to re-processing the result of a scheme so that students can anticipate it prior to activity. A *units coordination* involves two composite units (units of units), and it means to distribute the units of one composite unit across the units of another composite unit. For example, a units coordination of 5 and 7 involves distributing 7 units of 1 across each of the units of the 5 to get a unit of 35 that students structure in various ways.

Students who have interiorized two levels of units (MC2 students) can treat a length as a unit of units, or composite unit, prior to activity (Hackenberg & Tillema, 2009; Steffe & Olive, 2010). For example, MC2 students can imagine taking a 1-meter length and partitioning it into 5 equal parts without having to actually make the partitions. In other words, they can treat a length that represents 1 meter as a unit containing 5 units, a two-levels-of-units structure.

Furthermore, MC2 students can make three levels of units in activity: They can insert units into each unit in solving a problem. For example, they can insert 7 parts into each of the 5 parts in the 5/5-meter and determine that they have made 35 parts in all. However, in further activity the 35/35-meter becomes only a unit of 35 units; these students do not continue to view the 35/35-meter as a unit of 5 units each containing 7 units. So, MC2 students can consistently take two levels of units as given and create three-levels-of-units structures, but they don't maintain these structures in further operating. In contrast, students who have interiorized three levels of units (MC3 students) can maintain three-levels-of-units structures in further operating.

MC2 and MC3 students are the focus of this paper, so we don't discuss students who have interiorized only one level of units (MC1 students) here. We note that operating with a multiplicative concept is relatively stable: Progressing from one concept to another requires a significant reorganization of schemes that can take two years (Steffe & Cobb, 1988). Current estimates are that about one-third of incoming sixth-grade students are MC1 students, with MC2 and MC3 students making up the rest of the population (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015).

Method and Data Analysis

To launch each of the three 18-episode design experiments, we implemented a selection process with 21-24 7th- and 8th-grade students: a 30-minute interview and a 12-item worksheet. The interview questions and worksheet were designed to assess students' multiplicative concepts and fractions knowledge. Our aim was to use this process to invite three MC1, three MC2, and three MC3 students to participate in each experiment; however, MC1 students declined to participate. So, we invited six MC2 and three MC3 students to participate in each experiment. In the second and third experiments there was some attrition, resulting in 7 students in the second experiment (4 MC2, 3 MC3) and 6 students in the third experiment (3 MC2, 3 MC3).

In each experiment the 18 1-hour episodes ran twice per week and were video-recorded with one stationary and two roaming cameras. During episodes students often worked in groups of two or three using a software program called JavaBars (Biddlecomb & Olive, 2000), and student work was recorded with Screenflow (Telestream LLC, 2013). Sometimes students worked in groups that were cognitively more homogeneous (e.g., all MC2 students) and sometimes more heterogeneous (e.g., both MC2 and MC3 students).

One researcher (the first author) served as the teacher. Other team members operated roaming cameras, took notes, and interacted with students. Between episodes the team processed data, kept an Episode Index, watched video, took notes, and discussed conjectures to prepare for the next episode. Following each experiment each student participated in a 45-minute interview to assess the student's understanding of topics from the experiment and experience of the class.

For this paper we engaged in two phases of analysis. First we formulated a second-order model (Steffe, von Glasersfeld, Richards, & Cobb, 1983) of each of the 13 students' rational number and algebraic reasoning as it was addressed in the experiments, including students' concepts of unknowns. Second-order models are generated out of researchers' theoretical constructs, models from prior research, and a commitment to use constructs in an orienting but not deterministic way (Clement, 2000). To accomplish this analysis, we repeatedly viewed video files, transcribed major portions of video, and took detailed notes (Cobb & Gravemeijer, 2008). We also wrote and discussed memos (Corbin & Strauss, 2008): interpretations of and conjectures about students' ways of operating and about interactions in which those ways of operating occurred. Then, in the second phase we looked across the students to articulate differences in how students operated with unknowns and to account for these differences based on our models.

Analysis

We found qualitative differences in how MC2 and MC3 students represented and thought about EQUs: MC2 students constructed *implicit* quantitative unknowns, while the MC3 students constructed unknowns with explicitly embedded units. We support these claims by demonstrating how students worked on Single Unknown Problems. Often MC2 students did not represent the two values for the unknown height accurately (from our perspective) in pictures or equations; those who did represent the two values accurately in pictures and equations still demonstrated conflations that to us indicate their equations did not reflect embedded measurement units. Sometimes MC3 students also developed incorrect equations (from our perspective) but revised their work upon questioning; furthermore, their comments about their revised equations demonstrated that the equations did reflect embedded units. So, while both kinds of thinkers showed evidence of the construction of EQUs, their concepts of them differed significantly.

Due to space limitations, we cannot demonstrate student work here; we will do so in the presentation of this paper and in a manuscript that we will develop out of this work.

Discussion and Conclusions

This study suggests that MC2 students require substantial teacher support to create equations representing quantitative unknowns, and they may develop only an implicit understanding of the quantitative structure that we see. In contrast, the construction of EQUs, complete with an awareness of embedded units, is in the province of MC3 students. However, in our study, both MC2 and MC3 students benefitted from explicit discussion about the meanings they attributed to letters. We see developing both common and different supports for algebraic learners like these students to be at the heart of what is needed for access to algebra.

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