KOSZUL ALGEBRAS OF TWO GENERATORS AND A N_p PROPERTY OVER A RULED SURFACE

Eunkyung You

Submitted to the faculty of the University Graduate School
in partial fulfillment of the requirements
for the degree
Doctor of Philosophy
in the Department of Mathematics
Indiana University
August 2006

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Valery Lunts, Ph.D.

Jee Heub Koh, Ph.D.

Michael Larsen, Ph.D.

Shouhong Wang, Ph.D.

April 4, 2006

Acknowledgements

It is my pleasure to thanks my adviser, Professor Valery Lunts, for his guidance and support. Without his teaching and patient contribution to my mathematical education, I could not have accomplished my degree.

I am deeply grateful to Professor Michael Larsen for his advise, encouragement and support. Also I thanks Professor Jee Heub Koh and Shouhong Wang for their active participation in my PhD committee.

Finally I would like to thank all faculty members of the IU department of Mathematics for their help and dedication.

Abstract

This dissertation will discuss Koszul algebras and Koszul type N_p property. It is a basic problem of homological algebra to compute cohomology algebras of various augmented algebras. One of main purpose is to find the conditions that make algebra Koszul and to find some conditions on Hilbert series of the Koszul algebras. The other purpose is to find conditions which make a line bundle satisfy N_p on a projective variety over an algebraically closed field.

First consider quadratic algebras with two generator. We classify the quadratic algebras with two generator and investigate conditions under which these quadratic algebras are Koszul algebras. It turns out that one can formulate these conditions in terms of the dimensions of homogeneous degree two and three parts of the algebras.Whether an algebra is a Poincaré-Birkhoff-Witt algebra or not depends not only on the presentation of the algebra, but also on the field.

Second consider the Koszul-type concepts of the syzygies of varieties. Mark L. Green invented the N_p property of line bundles on a projective variety. We show that line bundles L and L^{k+p} satisfy the property N_p if L is a globally generated ample line bundle on a projective variety such that $H^1(X, L^s) = 0$ for all $s \geq 1$. We also sharpen a result of Park's work on a smooth complex projective surface.

Contents

1. INTRODUCTION

This dissertation will discuss Koszul algebras and Koszul type N_p property. It is a basic problem of homological algebra to compute cohomology algebras of various augmented algebras. One of main purpose is to find the conditions that make algebra Koszul and to find some conditions on Hilbert series of the Koszul algebras. The other purpose is to find conditions which make a line bundle satisfy N_p on a projective variety over an algebraically closed field.

Chapter 1 introduces basic definitions and theorems about Koszul algebras. We follow the notations from the published studies of A. Polishcuck, L. Positselski and S. B. Priddy. S. B. Priddy constructed a general resolution for a large class of augmented algebras by using bar resolution. He defines Koszul algebras using this resolution of an associated quadratic algebra. We also review the properties of Koszul algebras. In the work of A. Polishcuck, L. Positselski we find the following conjecture that any Koszul algebra A of finite global homological dimension d has the number of generators $\dim A_1$ greater or equal to d. However in general it is not true. We give an example that $d > \dim A_1$.

Chapter 2 introduces the definition of Poincaré-Birkhoff-Witt algebras and the connection between Poincaré-Birkhoff-Witt algebras and Koszul algebras. According to Priddy's work, we know that every Poincaré-Birkhoff-Witt algebra is a Koszul algebra, but the reverse is not true in general. We can find an example that is a Poincaré-Birkhoff-Witt algebra but not a Koszul algebra.

Chapter 3 investigates quadratic algebras which have two generators. We assume that the ground field is an arbitrary field. We investigate conditions under which these quadratic algebras are Koszul algebras. It turns out that one can formulate these conditions in terms of the dimensions of homogeneous degree two and three parts of the algebras. Whether an algebra is a Poincaré-Birkhoff-Witt algebra or not depends not only on the presentation of the algebra, but also on the field. Finally we list all possible Hilbert series of quadratic algebras which has two generators.

In Chapter 4, we assume that the ground field is algebraically closed, uncountable, and has characteristic zero. We establish connections between the set of quadratic algebras and the Grassmann variety. These connections have been researched and reviewed, but our chapter explains them in more detail.

Chapter 5 introduces the Koszul-type concepts of the syzygies of varieties. Mark L. Green invented the N_p property of line bundles on a projective variety. We show that line bundles L and L^{k+p} satisfy the property N_p if L is a globally generated ample line bundle on a projective variety such that $H^1(X, L^s) = 0$ for all $s \geq 1$. We also sharpen a result of Park's work on a smooth complex projective surface.

CHAPTER 1

GENERAL DISCUSSION OF KOSZUL ALGEBRAS

Consider a quadratic algebra $A = \bigoplus_{i=0}^{\infty} A_i = T(V)/J$ where $A_1 = V$ is a vector space over a field $\Bbbk$ and J is a two-sided ideal generated by quadratic relations. We want to find the conditions which make A Koszul algebra if $\dim V = 2$. Before that, we must know some basic definitions and theorems. We follow the notations from [1]. Throughout this paper, algebra A means an associative algebra with a fixed unit 1_A over a fixed ground field $\Bbbk$. Unless otherwise specified, we only consider graded algebras $A = \bigoplus_{i\geq 0} A_i$ such that $\dim_{\Bbbk} A_i < \infty$, for all $i \geq 0$ and $A_0 = \Bbbk$.

Definition 1. *For a graded algebra A and graded A-modules M, N, we will denoted by $Ext_A^i(M,N) = \bigoplus_{j\in Z} Ext_A^{ij}(M,N)$ the derived functor of the graded homomorphisms functor $Ext_A^0(M,N) = Hom_A(M,N) = \bigoplus_{j\in Z} Hom_A^j(M,N)$, where $Hom_A^j(M,N)$ is the space of all homomorphisms mapping M_t to M_{t-j}. The first grading i is called the homological grading and the second j is called the internal one.*

Definition 2. *A graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is called one-generated if the natural map $p : T(A_1) = \bigoplus_{i=0}^{\infty} T^i(A_1) \to A$ from the tensor algebra generated by A_1 is surjective where $T^i(A_1) = A_1^{\otimes i}$ for all i. A one-generated algebra A is called quadratic if $\ker p$ is generated by its subspace $I_A = \ker p \cap T^2(A_1) \subset A_1^{\otimes 2}$. We denote the quadratic algebra A by $A = \{A_1, I_A\}$. For one generated graded algebra A there is a uniquely defined quadratic algebra $\mathrm{q}A^!$ together with an algebra homomorphism $\mathrm{q}A^! \to A$ which is an isomorphism in degree 1 and a monomorphism in degree 2. Then $\mathrm{q}A^! = \{A_1, I_A\}$ is called the quadratic part of A. A quadratic dual algebra $A^!$*

of a quadratic algebra $A = \{V, I\}$ *is defined as* $A^! = \{V^*, I^\perp\}$ *where* $V = A_1$, V^* *is the dual vector space to* V *and* $I^\perp \subset V^* \otimes V^*$ *is the orthogonal complement to* I *with respect to the natural pairing* $\langle v_1 \otimes v_2, v_1^* \otimes v_2^* \rangle = \langle v_1 \otimes v_1^* \rangle \langle v_2 \otimes v_2^* \rangle$ *between* $V \otimes V$ *and* $V^* \otimes V^*$.

Let $A = \{V, I\}$ *be a quadratic algebra and* M *be a left* A*-module.* M *is called quadratic if* $M_i = 0$ *for* $i < 0$*, the natural map* $A \otimes M_0 \to M$ *is surjective and its kernel* J_M *is generated by the subspace* $K_M = J_M \cap A_1 \otimes M_0$*. Then we denote this as* $M = \langle M_0, K_M \rangle$*. For any graded algebra* A *and graded* A*-module* M *such that* $M_i = 0$ *for* $i > 0$*, there is a uniquely defined quadratic module* $\mathrm{q}_A M$ *over* $\mathrm{q}A$ *together with a morphism* $\mathrm{q}_A M \to M$ *of modules over* $\mathrm{q}A$ *which is an isomorphism in degree* 0 *and a monomorphism in degree* 1*. For a quadratic module* $M = \langle M_0, K \rangle$ *over a quadratic algebra* $A = \{V, I\}$*, we define the quadratic dual module* $M^!$ *over* $A^!$ *as* $M^! = \langle M_0^*, K^\perp \rangle$ *where* $K^\perp \subset V^* \otimes M_0^*$ *is orthogonal complement to* $K \subset V \otimes M_0$.

Definition 3. *A graded quadratic algebra* $A = \bigoplus_{i=0}^{\infty} A_i$ *over a field* $\Bbbk$ *is called* n*-Koszul if* $Ext_A^{ij}(\Bbbk, \Bbbk) = 0$ *for* $i < j \leq n$*.* A *is called Koszul if* $Ext_A^{ij}(\Bbbk, \Bbbk) = 0$ *for all* $i \neq j$.

Note 1.

(1) By above definition, a graded quadratic algebra A is Koszul if and only if A is n-Koszul for all n.
(2) It is proved in [1] that a quadratic algbra A is Koszul if and only if the dual algebra $A^!$ of A is Koszul.

Let $V = \oplus_{t \in \mathbb{Z}} V_t$ be a graded vector space; then its Hilbert series is a formal power series defined as

$$h_V(x) = \textstyle\sum_{t \in \mathbb{Z}} (\dim V_t) x^t$$

For a graded module M over a graded algebra A, consider the homological double *Poincaré series*

$$P_{A,M}(x,y) = \sum_{i,j\in\mathbb{Z}}(\dim Ext^{i,j}(M,\Bbbk))x^i y^j$$

We get the following facts in [1]; for a graded module M over a graded algebra A,

(1) $h_A(x)P_{A,M}(-1,x) = h_M(x)$.

(2) $h_A(x)h_{A^!}(-x) = 1$ if A is a Koszul algebra.

(3) A graded algebra A is Koszul if and only if $P_{A,\Bbbk}(x,y) = h_{A^!}(xy)$.

The inverse of (2) is not true. We can find a counterexample of (2) in [4]. For an augmented grade algebra $A = \Bbbk \oplus A_+$ and a left $A-$module M, consider the following bar-resolution $\widetilde{\mathcal{B}_\bullet}(A,M)$;

$$\cdots \to A\otimes_\Bbbk A_+ \otimes_\Bbbk A_+ \otimes_\Bbbk M \to A\otimes_\Bbbk A_+ \otimes_\Bbbk M \to A\otimes_\Bbbk M \to 0$$

where $\widetilde{\mathcal{B}}_i(A,M) = A\otimes_\Bbbk {A_+}^i \otimes_\Bbbk M$, the differential is

$$\partial(a_0\otimes\cdots\otimes a_i\otimes m) = \sum_{s=1}^{i}(-1)^s a_0\otimes\cdots\otimes a_{s-1}a_s\otimes\cdots a_i\otimes m$$
$$+(-1)^{i+1}a_0\otimes\cdots\otimes a_i m,$$

and A acts by the left multiplications.

Note 2. Consider the bar-resolution $\widetilde{\mathcal{B}_\bullet}(A,M)$ where A is a quadratic algebra and M is a A-module. We easily check the following fact;

$$H_i(\widetilde{\mathcal{B}}_i(A,M)) = 0 \text{ for } i>0.$$

Then $\widetilde{\mathcal{B}_\bullet}(A,M)$ is a free resolution of the A-module M.

We have by definition

$$Ext^i_A(M,N) = H^i(Hom_A(\widetilde{\mathcal{B}}_i(A,M),N))$$

Let A be a graded algebra and $B = \mathrm{q}A^!$ be the algebra dual to its quadratic part. The image of the element $e_{\mathrm{q}A}\in A_1\otimes A^!_1$ under the natural homomorphism $\mathrm{q}A^!\to A$ defines an element $e_A\in A_1\otimes B_1$ with a property $e_A^2 = 0$ in $A\otimes B$ (see Lemma 9.1 in [3]). Let R be a graded right A-module and L be a graded left B-module. We define the Koszul complex $K^A(R,L)$ of the modules L and R as follows:

$$K^A_{ij}(R,L) = R_q \otimes L^*_p \text{ where } i = p \text{ and } j = p+q$$

then the differential maps are mappings from K_{ij} to $K_{i-1,j}$ and are induced by e_A. Let M be a left graded A-module with $M_i = 0$ for $i < 0$ and $N = \mathrm{q}_A M^!$ be the quadratic dual left B-module. Set

$$'K_\bullet(A,M) = K^A_\bullet(A,N);$$

then $'K_\bullet(A,M)$ is a complex of free graded left A-modules

$$\cdots \to A \otimes_{\Bbbk} N^*_3 \to A \otimes_{\Bbbk} N^*_2 \to A \otimes_{\Bbbk} N^*_1 \to A \otimes_{\Bbbk} N^*_0 \to 0.$$

Specifically for a quadratic algebra $A = \bigoplus_{i=0}^{\infty} A_i = \{V,R\}$ where $A_1 = V$, consider $K^A_\bullet(A, \Bbbk^!_A)$ Koszul complex of A;

$$K_\bullet : \cdots \to A \otimes_{\Bbbk} K_3 \to A \otimes_{\Bbbk} K_2 \to A \otimes_{\Bbbk} K_1 \to A \otimes_{\Bbbk} K_0 \to 0,$$

where $K_j = \bigcap_{i=0}^{i=j-2} V^{\otimes i} \otimes R \otimes V^{\otimes j-i-2}$ for all $j \geq 2$, $K_0 = \Bbbk$, and $K_1 = V$. We know that the differential $\varphi_i : A \otimes K_i \to A \otimes K_{i-1}$ is the restriction of $A \otimes V^{\otimes i} \to A \otimes V^{\otimes (i-1)}$ such that $v \otimes (w_1 \otimes w_2 \otimes ... \otimes w_i) \mapsto vw_1 \otimes (w_2 \otimes w_3 \otimes ... \otimes w_i)$. Then $\varphi_i \varphi_{i+1} = 0$, i.e., $\ker \varphi_i \supset Im\varphi_{i+1}$. From now on, this is our Koszul complex. In $K^A_\bullet(A, \Bbbk^!_A)$ Koszul complex of a quadratic algebra A, we get the following result.

Claim 1. *Let $A = \{V,R\}$ be a a quadratic algebra. In Koszul complex $K_\bullet$,*

$$[K_r \otimes V] \bigcap [V \otimes K_r] = K_{r+1} \textit{ for all } r \geq 2$$

Proof. For all $r \geq 2$,

$$[K_r \otimes V] \bigcap [V \otimes K_r]$$

$$= ([\bigcap_{i=0}^{i=r-2} V^{\otimes i} \otimes R \otimes V^{\otimes r-i-2}] \otimes V) \bigcap (V \otimes [\bigcap_{i=0}^{i=r-2} V^{\otimes i} \otimes R \otimes V^{\otimes r-i-2}])$$

$$= (\bigcap_{i=0}^{i=r-2} V^{\otimes i} \otimes R \otimes V^{\otimes (r+1)-i-2}) \bigcap (\bigcap_{i=0}^{i=r-2} V^{\otimes i+1} \otimes R \otimes V^{\otimes r-i-2})$$

$$= (R \otimes V^{\otimes r-1}) \bigcap (\bigcap_{i=1}^{i=r-2} V^{\otimes i} \otimes R \otimes V^{\otimes r-i-1}) \bigcap (\bigcap_{i=0}^{i=r-3} V^{\otimes i+1} \otimes R \otimes V^{\otimes r-i-2})$$
$$\bigcap (V^{\otimes r-1} \otimes R)$$

$$= (R \otimes V^{\otimes r-1}) \bigcap (\bigcap_{i=1}^{i=r-2} V^{\otimes i} \otimes R \otimes V^{\otimes r-i-1}) \bigcap (V^{\otimes r-1} \otimes R)$$

$= \bigcap_{i=0}^{i=r-1} V^{\otimes i} \otimes R \otimes V^{\otimes r-i-2}$

$= K_{r+1}.$ $\square$

PROPOSITION 1. (*Proposition* 0.1 *in* [1]) *For any graded algebra A and graded A-module M with $M_i = 0$ for $i < 0$, we have $Ext_A^{ij}(M, \Bbbk) = 0$ for $i \neq j$ if and only if the complex $'K_\bullet(A, M)$ is a resolution of M.*

Until now we introduced the definitions and the facts about Koszul algebras. Now we discuss three conjectures in [1];

(1) The Hilbert series of a Koszul algebra A is a rational function. In particular, the growth of the sequence $\dim A_i$ is either polynomial, or exponential.
(2) Any Koszul algebra A of finite global homological dimension d has the number of generators $\dim A_1$ greater or equal to d.
(3) If the dimension of the components of both quadratic dual Koszul algebras A and $A^!$ has a polynomial growth, then the Hilbert series of A, $h_A(x)$ coincides with that of the tensor product of a symmetric algebra and an exterior algebra. i.e.,

$$h_A(x) = \tfrac{(1+x)^a}{(1-x)^b} \text{ for some } a, b \in \mathbb{Z} \geq 0.$$

In [1] they proved that the first and third conjectures are true if a quadratic algebra A is a PBW-algebra (We can find the definition of PBW-algebras in Chapter 3). We know that the global dimension of a Koszul algebra A, gl.dim A is the projective dimension of $\Bbbk$ (as A-module) over A and $Ext^{ii}(\Bbbk, \Bbbk) = A^!$. The second conjecture means that gl.dim A is $min\{t \mid \dim A_t^! \neq 0 \text{ and } \dim A_{t+1}^! = 0\}$ and it claims that gl.dim $A \leq \dim A_1$. In [5] they showed that if a quadratic algebra A is a I-type algebra, then gl.dim $A = \dim A_1$ but in general it is not true. We show a case such that gl.dim $A > \dim A_1$.

EXAMPLE 1. *Consider a quadratic algebra* $A = \{V, R\}$ *such that* V *has a basis* $\{x, y, z\}$. *and* R *is generated by* $\{x^2 + xz, y^2 + yz, yx, zx\}$. *We show that* A *satisfies* $gl.\dim A = 4 > 3 = \dim V$.

PROOF. The dual quadratic algebra $A^! = \{V^*, R^\perp\}$ has two-sided ideal $R^\perp$ which has a basis $\{y^{*2} - z^*y^*, x^{*2} - x^*z^*, x^*y^*, z^{*2}, z^*y^*\}$. Take a order $y^* > z^* > x^*$.

(1) $A^!_2$ has a basis $\{x^{*2}, y^*x^*, y^*z^*, z^*x^*\}$

(2) $A^!_3$ is generated by $\{x^{*3}, y^*x^{*2}, y^*z^*x^*, z^*x^{*2}\}$. We get the following relations in $A^!_3$; $x^{*3} = x^*z^*x^* = x^{*2}z^* = x^*z^{*2} = 0$, $y^*x^{*2} = y^*x^*z^*$, $y^*z^*x^* = y^{*2}x^*$, and $z^*x^{*2} = z^*x^*z^*$. So $A^!_3$ has a basis $\{y^*x^{*2}, y^*z^*x^*, z^*x^{*2}\}$.

(3) $A^!_4$ is generated by $\{y^*x^{*3}, y^*z^*x^{*2}, z^*x^{*3}\}$. We get the following relations in $A^!_4$; $y^*x^{*3} = z^*x^{*3} = 0$ and $y^*z^*x^{*2} = y^*z^*y^*z^* = y^{*2}x^{*2} = y^{*2}x^*z^* \neq 0$. So $A^!_4$ has a basis $\{y^*z^*x^{*2}\}$.

(4) $A^!_5$ is generated by $\{y^*z^*x^{*3}\}$ and $y^*z^*x^{*3} = 0$ in $A^!_5$. Then $A^!_5 = 0$.

$\square$

CHAPTER 2

PBW PROPERTIES OF KOSZUL ALGEBRAS

In this chapter we review the concept of Poincaré-Birkhoff-Witt(PBW) algebra and the properties of PBW-algebra.

Suppose that $A = \{V, R\}$ is a quadratic algebra where $\dim V = n$ and V has a fixed basis $\{x_i : i \in I\}$. Let B be a $\Bbbk$-module basis for A consisting of $1, x_i$ and certain monomials $x_{i_1} x_{i_2} \cdots x_{i_m}$, $i_j \in I$. A set

$$S \subset \cup_{n=1}^{\infty} I_1 \times I_2 \cdots \times I_n, \qquad I_j = I$$

is called a labeling set for B if for each $\{X = x_{i_1} x_{i_2} \cdots x_{i_n}\}$ there is a unique $\alpha = (i_1, i_2, ..., i_n) \in S$. The pair (B, S) is called a labeled basis for A. The set $\cup_{n=1}^{\infty} I_1 \times I_2 \cdots \times I_n$ is ordered first by length and then by the lexicographical ordering derived from the order of I. For a multiindex $\alpha = (i_1, i_2, ..., i_n)$, $i_k \in \{1, 2, .., m\}$, denoted by X^α the monomial $x_{i_1} x_{i_2} \cdots x_{i_n} \in T(V)$.

Lemma 1. (*Lemma* 2.1 *in* [1]) *Let V be a vector space with a basis w_α numbered by a linearly ordered set of indices Q which has a smallest value and $K \subset V$ be a vector subspace. Consider the subset $S \subset Q$ consisting of all α for which w_α cannot be presented as a linear combination of w_β with $\beta < \alpha$ modulo K. Then the images of w_α with $\alpha \in S$ from a basis of V/K. The subset S is a unique one for which there is a basis of K of the form*

$$u_\beta = w_\beta - \textstyle\sum_{\alpha < \beta} c_{\alpha\beta} w_\alpha, \qquad \beta \in \overline{S} = Q \backslash S$$

Consider a quadratic algebra $A = \{V, R\}$. Let S^2 be a subset of S as Lemma 1 such that $S^2 = \{\alpha \in S | X^\alpha \in A_2\}$. For $n \geq 3$, let

$$S^n = \{(i_1, i_2, ..., i_n) | (i_1, i_2) \in S^2, (i_2, i_3) \in S^2, ..., (i_{n-1}, i_n) \in S^2\}.$$

We call (B, S) a PBW-basis if it satisfies the following properties;

(1) $(i_1, ..., i_k)$ and $(j_1, ..., j_t) \in S$ implies that $(i_1, ..., i_k, j_1, ..., j_t) \in S$.

(2) for $k \geq 2$, $(i_1, i_2, ..., i_k) \in S$ if and only if $(i_j, i_{j+1}) \in S$ for all $1 \leq j < k$.

In other words, (B, S) is a PBW-basis if $S = \cup_{n=1}^{\infty} S^n$. If A has a PBW-basis, A is called a PBW-algebra.

THEOREM 1. (*Theorem* 2.2 *in* [1]) *Let* $A = \{V, R\}$ *be a quadratic graded algebra. If the cubic monomials* $x_{i_1} x_{i_2} x_{i_3}$, $(i_1, i_2, i_3) \in S^3$ *are linearly independent in* A_3*, then the same is true in any degree* n*; therefore this* A *is a PBW-algebra.*

NOTE 3. By the definition of PBW-algebras, we know that PBW-property depends on bases of quadratic algebras. For example, consider a quadratic algebra $A = T(\{x, y\})/\langle y^2 - x^2 \rangle$ over $\mathbb{C}$. Then $A = T(\{x, y\})/\langle y^2 - x^2 \rangle$ is isomorphic to $B = T(\{z, w\})/\langle zw - wz \rangle$.

(1) Given any order of $\{x, y\}$, $\{\alpha | X^\alpha \in A_2\} = \{(1, 2), (2, 1), (2, 2)\} \Rightarrow$ $S^3 = \{(1, 2, 1), (2, 1, 2), (1, 2, 2), (2, 2, 1), (2, 2, 2)\}$. $\{x, y\}$ are not a PBW-basis of A since $x^3 = xy^2 = y^2 x$ and $y^3 = x^2 y = yx^2$.

(2) Given any order of $\{z, w\}$, $\{\alpha \in S^2 | X^\alpha \in B_2\} = \{(1, 1), (2, 1), (2, 2)\} \Rightarrow$ $S^3 = \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2)\}$. But $\{z, w\}$ is a PBW-basis of B since B_3 has a basis $\{z^3, wz^2, w^2 z, w^3\}$ or $\{z^3, zw^2, z^2 w, w^3\}$.

THEOREM 2. (*Theorem* 5.3 *in* [2]) *Any quadratic PBW-algebra is Koszul.*

THEOREM 3. (*Theorem* 2.4 *in* [1]) *A quadratic algebra* A *is a PBW-algebra if and only if the dual algebra* $A^!$ *of* A *is also a PBW-algebra.*

THEOREM 4. (*Theorem* 2.6 *in* [2]) *Let* $A = \{V, R\}$ *be a PBW-algebra where* $\dim V = n$ *and* V *has a basis* $\{x_i : i \in I\}$*. Let* $\overline{S^2}$ *be the complement of* S^2 *in* $I \times I$*. The Hilbert series of* A*,* $h_A(x)$ *is*

$$h_A(x) = \frac{det(1+xM_{\overline{S^2}})}{det(1-xM_{S^2})}$$

where M_{S^2} *is the* $n \times n$ *matrix of zeros and ones corresponding to* S*.*

DEFINITION 4. *Let* $A = \{V, R\}$ *be a quadratic algebra.* A *is called a monomial algebra if* R *is generated by quadratic relations which are monomial.*

We easily get that all monomial algebras are PBW-algebras. By Theorem 2, we know that all PBW-algebras are Koszul but the opposite is not true. We get an example which are Koszul but not a PBW-algebra.

EXAMPLE 2. *Consider a quadratic algebra* $A = T(\{x, y\})/\langle x^2 + y^2\rangle$ *over rational numbers* $\mathbb{Q}$*. We will see that the* A *is a Koszul algebra in Chapter 3.*
By any linear transformations, $x^2 + y^2$ *is of the form* $ax^2 + bxy + byx + cy^2$ *where* $a, b, c \in \Bbbk$ *and* $a, c \neq 0$*. Given any basis of* A *and any order of the basis, we get that* $S^2 = \{(1, 2), (2, 1), (2, 2)\}$ *and* $\dim A_3 = 4$*. By Theorem 1, every basis of* A *can not be a PBW-basis of* A*. So* A *is not a PBW-basis.*

CHAPTER 3

KOSZUL ALGEBRAS ON TWO GENERATORS

Now, let $\Bbbk$ be our ground field (not necessarily an algebraically closed field). We consider a quadratic algebra $A = \bigoplus_{i=0}^{\infty} A_i = \{V, R\}$ where $V = A_1$ is a 2-dimensional vector space over $\Bbbk$ and R is a set of all quadratic relations of A. Since $\dim V = 2$, R can have at most 4 linearly independent elements. Denote by $|R|$ the maximum number of linearly independent elements of R. Denote by $\varphi_i : A \otimes K_i \to A \otimes K_{i-1}$ differential i-th map on the Koszul complex $K_\bullet$. Trivially we know that $\ker \varphi_i \supset Im\varphi_{i+1}$ for all i. We will show that

THEOREM 5. *We assume that* $A = \bigoplus_{i=0}^{\infty} A_i = \{V, R\}$ *is a quadratic algebra, where* $V = A_1$, $\dim V = 2$, *and* R *is is a set of all quadratic relations of* A.

(1) *When* $|R| = 0, 1, 3, 4$, *this* A *is Koszul*
(2) *When* $|R| = 2$, *only* A *which has* $dimA_3 = 2$ *is Koszul*

We will prove this on a case-by-case basis. By duality of Koszul algebras, it suffices to consider cases when $|R| = 4, 3, 2$. Unless specified, we usually omit $\otimes$ within A or K_i. If $|R| = j$ and $\{l_1, ..., l_j\}$ is a basis of R, we denote $R = \langle l_1, .., l_j\rangle$. Each l_i can be expressed in the form $a_{i1}x^2 + a_{i2}y^2 + a_{i3}xy + a_{i4}yx$ where all $a_{it} \in \Bbbk$. Then

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \\ yx \end{pmatrix} = G \begin{pmatrix} x^2 \\ y^2 \\ xy \\ yx \end{pmatrix}$$

By the elementary row operations, we can get row-reduced echelon matrix $G' = (b_{ik})_{1\leq i,k\leq 4}$ which is equivalent to G, so we can assume that each $l_j = b_{j1}x^2 + b_{j2}y^2 + b_{j3}xy + b_{j4}yx$.

0.1. $|R| = 4$**.** Trivially we know that this is a monomial algebra. So this is Koszul.

0.2. $|R| = 3$**.** We get 4 cases such that R has a basis as follows:

(1) $R = \langle x^2, xy, yx\rangle$,
(2) $R = \langle y^2 + ax^2, xy, yx\rangle$,
(3) $R = \langle y^2 + axy, x^2 + bxy, yx\rangle$,
(4) $R = \langle y^2 + ayx, x^2 + byx, xy + cyx\rangle$,

Lemma 2. *When $R = \langle x^2, xy, yx\rangle$, the A is Koszul.*

Proof. This is Koszul since A is a monomial algebra. □

Lemma 3. *When $R = \langle y^2 + ax^2, xy, yx\rangle$, the A is Koszul.*

Proof. When $a = 0$, it is Koszul since A is a monomial algebra. Assume that $a \neq 0$. Using dual algebra, we want to prove that $A = \{V, \langle ay^2 - x^2\rangle\}$ is Koszul. First we show that $K_3 = 0$.
If $v \in K_3$, there exists $\alpha, \beta, \gamma, \delta \in \Bbbk$ such that

$$v = (\alpha x + \beta y) \otimes (ay^2 - x^2)$$
$$= (ay^2 - x^2) \otimes (\gamma x + \delta y)$$

Then $\alpha = \beta = \gamma = \delta = 0 \Rightarrow v = 0$. Also we can easily find that $E_n = \{y^n, y^{n-1}x, y^{n-2}xy, \cdots, yxyx\cdots, xyxy\cdots\}$ is linear independent of B_n, for each n. We know that E_n is a basis of $A_n$by using the following facts;

$$y^t xyxy\cdots xyy = y^{t+2}xyxy\cdots x,$$

$$y^t xyxy\cdots yxx = ay^{t+2}xyxy\cdots y,$$
$$xy^{2t}xyxy\cdots = ay^{2t+2}yxy\cdots, \text{ and}$$
$$xy^{2t-1}xyxy\cdots = y^{2t-2}xyxy\cdots, \text{ for all } t\geq 1.$$

Finally if $v = \omega\otimes(ay^2 - x^2)\in\ker\varphi_2$ for some $\omega\in B$, $\omega x = \omega y = 0$ in B. For any n, it suffice to show that $\omega = 0$ in B_n. We can assume that $\omega\in B_n$. Using a basis E_n, we can express ω as $\omega = a_1y^n + a_2y^{n-1}x + \cdots + a_{n+1}xyxy\cdots$, for some $a_j\in\Bbbk$.

$$\begin{aligned}\omega x &= a_1y^nx + a_2y^{n-1}x^2 + \cdots + a_{n+1}xyxy\cdots x\\ &= aa_2y^{n+1} + a_1y^nx + aa_4y^{n-2}xy + a_4y^{n-3}xyx + \cdots.\end{aligned}$$

By linearly independence of E_n, each a_j is zero and $\omega = 0$. Then $\ker\varphi_2 = 0$. It is a Koszul algebra. □

Lemma 4. *When $R = \langle y^2 + axy, x^2 + bxy, yx\rangle$, the A is Koszul.*

Proof. (1) When $a = 0$ or $b = 0$, take the order $x > y$. Then $S^2 = \{(1,2)\}$ and $S^3 = \varnothing$. This is a PBW-algebra.

(2) When $a, b\neq 0$, $S^3 = \{(2,2,2)\}$ by taking any order of $\{x,y\}$ but $x^3 = y^3 = 0$ in A. This is not a PBW-algebra.

By a linear mapping $f : x\mapsto x$ and $y\mapsto y - ax$, then

$y^2 + ayx\mapsto y^2 - axy$, $x^2 + byx\mapsto(1-ab)x^2 + bxy$, and $yx\mapsto -ax^2 + yx$.

Then we can replace R as $\langle -ax^2 + yx, y^2 - axy, (1-ab)yx + bxy\rangle$.

When $1-ab = 0$, consider the dual algebra which is $T(\{x^*, y^*\})/\langle x^{*2} + ay^*x^*\rangle$. By a linear transformation $g : x^*\mapsto x^*$ and $y^*\mapsto y^* - \frac{1}{a}x^*$, it is isomorphic to a monomial algebra $T(\{x^*, y^*\})/\langle y^*x^*\rangle$. This is Koszul.

When $1 - ab\neq 0$, we can replace $\langle -ax^2 + yx, y^2 - axy, (1-ab)yx + bxy\rangle$ as $\langle xy, x^2, yx\rangle$ by the a linear mapping $f : x\mapsto x + \frac{1}{a}y$ and $y\mapsto y$. So this is Koszul.

□

LEMMA 5. *When $R = \langle y^2 + ayx, x^2 + byx, xy + cyx\rangle$, the A is Koszul.*

PROOF. (1) When $a = 0$, take a order $x > y$. Then $S^2 = \{(1, 2)\}$ and $S^3 = \varnothing$. This is a PBW-algebra.

(2) When $b = 0$, take a order $y > x$. Then $S^2 = \{(1, 2)\}$ and $S^3 = \varnothing$. This is a PBW-algebra.

(3) When $a, b \neq 0$, consider the dual algebra $A^! = \{V^*, R^\perp\}$ where $R^\perp$ has a basis $\{T = y^*x^* - ay^{*2} - bx^{*2} - cx^*y^*\}$.

We get that

$$-cx^*T + aTy^* - ay^*T - Tx^*$$
$$= (b + bc)x^{*3} + (c^2 - ab)x^{*2}y^* + a(c + 1)y^*x^*y^* + (ab - 1)y^*x^{*2}.$$

Given any order of $\{x^*, y^*\}$, $S^3 = \{(1, 2, 2), (2, 2, 1), (2, 2, 2), (1, 2, 1), (2, 1, 2)\}$.

If $c = -1$ and $ab = 1$, the A is a PBW-algebra.

If $c \neq -1$ or $ab \neq 1$, $R^\perp$ can be replaced by $\langle x^{*2} + \frac{c+1}{b}x^*y^* + \frac{ab+c}{b^2}y^{*2}\rangle$ by the linear mapping $f : x^* \mapsto x^* + \frac{1}{b}y^*$. It is Koszul by Lemma 4.

□

0.3. $|R| = 2$. We get 6 cases such that R has a basis as follows: for $a, b, c, d \in \Bbbk$

(1) $R = \langle xy, yx\rangle$
(2) $R = \langle y^2 + axy, yx\rangle$
(3) $R = \langle x^2 + ay^2 + bxy, yx\rangle$
(4) $R = \langle y^2 + ayx, xy + byx\rangle$
(5) $R = \langle x^2 + by^2 + cyx, xy + dyx\rangle$
(6) $R = \langle x^2 + axy + byx, y^2 + cxy + dyx\rangle$

We show that A is Koszul if R has a basis as follows:

(1) $R = \langle xy, yx\rangle$,
(2) $R = \langle x^2, yx\rangle$,

(3) $R = \langle y^2, yx\rangle$,

(4) $R = \langle y^2, xy + ayx\rangle$,

(5) $R = \langle x^2, xy + ayx\rangle$,

(6) $R = \langle y^2 + ayx, xy - yx\rangle$,

(7) $R = \langle x^2 + cy^2 + dyx, xy - yx\rangle$,

(8) $R = \langle x^2 + ay^2, xy \pm yx\rangle$,

(9) $R = \langle x^2 + axy + ayx, y^2\rangle$,

(10) $R = \langle x^2, y^2 + axy + ayx\rangle$,

(11) $R = \langle x^2 + ayx, y^2 + byx\rangle$ and $1 + ab = 0$,

(12) $R = \langle x^2 + axy, y^2 + bxy\rangle$ and $1 + ab = 0$,

(13) $R = \langle x^2 + axy + ayx, y^2 + bxy + byx\rangle$.

where $a, b \in \Bbbk - \{0\}$ and $c, d \in \Bbbk$.

LEMMA 6. *If $R = \langle xy, yx\rangle$ then this A is Koszul.*

PROOF. This is Koszul since A is a monomial algebra. $\square$

LEMMA 7. *When $R = \langle y^2 + axy, yx\rangle$, this A is Koszul if and only if $a = 0$.*

PROOF. If $a = 0$, it is Koszul since A is a monomial algebra. Assume that $a \neq 0$. First we show that K_3 has a basis $\{y^2x + axyx\}$. If $v \in K_3$, there exists $\alpha_i, \beta i, \gamma i, \delta i \in \Bbbk, i = 1, 2$ such that

$$v = (\alpha_1 x + \beta_1 y) \otimes (y^2 + axy) + (\gamma_1 x + \delta_1 y) \otimes yx$$

$$= (y^2 + axy) \otimes (\alpha_2 x + \beta_2 y) + yx \otimes (\gamma_2 x + \delta_2 y)$$

$$\alpha_1(xy^2 + ax^2y) + \beta_1(y^3 + ayxy) + \gamma_1 xyx + \delta_1 y^2x$$

$$= \alpha_2(y^2x + axyx) + \beta_2(y^3 + axy^2) + \gamma_2 yx^2 + \delta_2 yxy$$

By the term x^2y and yx^2, $\alpha_1 = \gamma_2 = 0 \Rightarrow \beta_2 = \beta_1 = 0 \Rightarrow \delta_2 = 0$ and $\gamma_1 = a\delta_1$. Then v is of the form $\delta_1(y^2x + axyx)$ for all $\delta_1 \in \Bbbk$.

Now we show that $\ker\varphi_2 \supsetneq Im\varphi_3$. In A, $y^3 = -ayxy = 0$ and $\varphi_2[(-axy) \otimes (y^2 + axy)] = (-axy^2 - a^2xyx) \otimes y = y^3 \otimes y = 0$. Then $0 \neq (-axy) \otimes (y^2 + axy) \in \ker\varphi_2$. By K_3, all elements in $Im\varphi_3$ are of the form $(vy + avx) \otimes yx$ for some $v \in A$. $(-axy) \otimes (y^2 + axy)$ is not in $Im\varphi_3$. This is not Koszul. □

LEMMA 8. *When $R = \langle x^2 + ay^2 + bxy, yx\rangle$, this is Koszul if and only if $a = b = 0$*

.

PROOF. If $v \in K_3$, there exists $\alpha_i, \beta i, \gamma i, \delta i \in \Bbbk, i = 1, 2$ such that

$$v = (\alpha_1 x + \beta_1 y) \otimes (x^2 + ay^2 + bxy) + (\gamma_1 x + \delta_1 y) \otimes yx$$

$$= (x^2 + ay^2 + bxy) \otimes (\alpha_2 x + \beta_2 y) + yx \otimes (\gamma_2 x + \delta_2 y).$$

$$\alpha_1 = \alpha_2, a\alpha_1 = b\beta_2, b\alpha_1 = \beta_2, \beta_1 = \gamma_2$$

$$a\beta_1 = a\beta_2, b\beta_1 = \delta_2, \gamma_1 = b\alpha_2, \delta_1 = a\alpha_2.$$

Then $a\alpha_1 = b^2\alpha_1$, $a\beta_1 = ab\alpha_1$, $\gamma_1 = b\alpha_1$, and $\delta_1 = a\alpha_1$.

$$\begin{pmatrix} a - b^2 & 0 & 0 & 0 \\ b & 0 & -1 & 0 \\ ab & -a & 0 & 0 \\ a & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(1) When $a = b = 0$, it is Koszul since this is a monomial algebra.

(2) When $a = 0$ and $b \neq 0$, we can replace R as $\langle \frac{-1}{b}x^2 + yx, xy\rangle$ by a linear map $y \mapsto (\frac{-1}{b}x + y)$ and $x \mapsto x$. By a linear map $y \mapsto x$ and $x \mapsto y$, we also replace R as $\langle y^2 - bxy, yx\rangle$. By Lemma 7, this is not Koszul.

(3) When $a \neq 0$ and $b = 0$, $K_3 = 0$ and $x^3 = -ay^2x = 0$ in A. $\varphi_2(x^2 \otimes (x^2 + bxy)) = x^3 \otimes x + bx^3 \otimes y = 0$. Since $0 \neq x^2 \otimes (x^2 + bxy) \in \ker\varphi_2$, this is not Koszul.

(4) When $a \neq b^2$ and $a \neq 0$, $K_3 = 0$ and $y^3 = \frac{-1}{a}y(x^2 + bxy) = 0$ in A. $\varphi_2(y^2 \otimes yx) = y^3 \otimes x = 0$. Since $0 \neq y^2 \otimes yx \in \ker\varphi_2$, this is not Koszul.

(5) When $a = b^2$ and $a \neq 0$,

K_3 has a basis $\{x^3 + axy^2 + bx^2y + byx^2 + aby^3 + b^2yxy + bxyx + b^2y^2x\}$. In A, $y^3 = \frac{-1}{a}y(x^2 + bxy) = 0$ and $\varphi_2(y^2 \otimes yx) = y^3 \otimes x = 0$. Then $0 \neq y^2 \otimes yx \in \ker\varphi_2$. All element in $Im\varphi_3$ is of the form $v(x + by) \otimes (x^2 + ay^2 + bxy) + bv(x + by) \otimes yx$ for some $v \in A$. $y^2 \otimes yx$ is not in $Im\varphi_3$. Then $\ker\varphi_2 \supsetneqq Im\varphi_3$. This A is not Koszul.

□

Lemma 9. *When $R = \langle y^2 + ayx, xy + byx\rangle$, we show that the following facts:*

(1) *When $a = 0$, A is Koszul.*
(2) *When $a \neq 0$, This A is Koszul if $b = -1$; this Koszul algebra is isomorphic to $A = \{\{x, y\}, \langle xy, yx\rangle\}$.*

Proof. Let $a, b \in \Bbbk$ be non zero.

(1) When $R = \langle y^2, xy\rangle$, it is Koszul since it is a monomial algebra.
(2) When $R = \langle y^2, xy + byx\rangle$, we show that it has a PBW-basis. Assume that $x > y$ in order. We know that $\{X^\alpha : \alpha \in S^3\} = \{x^3, yx^2\}$ since $S^2 = \{(1, 1), (2, 1)\}$ and $S^3 = \{(1, 1, 1), (2, 1, 1)\}$. In A, $x^2y = -bxyx = b^2yx^2 \neq 0$ and $y^2x = yxy = xy^2 = y^3 = 0$. Since A_3 has a basis $\{x^3, x^2y\}$ and by Theorem 1, A has a PBW-basis.
(3) When $R = \langle y^2 + ayx, xy\rangle$, we show that it is not Koszul.
   (a) We show that K_3 has a basis $\{xy^2 + axyx\}$.

   If $v \in K_3$, there are $a_i, b_i, c_i, d_i \in \Bbbk$ such that

$$v = (a_1x + b_1y) \otimes (y^2 + ayx) + (a_2x + b_2y) \otimes xy$$
$$= (y^2 + ayx) \otimes (c_1x + d_1y) + xy \otimes (c_2x + d_2y)$$
$$a_1(xy^2 + axyx) + a_2x^2y + b_1(y^3 + ay^2x) + b_2yxy$$
$$= c_1(y^2x + ayx^2) + c_2xyx + d_1(y^3 + ayxy) + d_2xy^2$$

By the terms x^2y and yx^2, $a_2 = c_1 = 0$

$\Rrightarrow$ by the term y^2x, $b_1 = 0$

$\Rrightarrow$ by the term y^3, $d_1 = 0$

$\Rrightarrow$ by the term yxy, $b_2 = 0$.

Then K_3 has a basis $\{xy^2 + axyx\}$.

(b) We show that $\ker\varphi_2 \supsetneqq Im\varphi_3$. In A, $y^2x = -\frac{1}{a}y^3 = yxy = 0$. $\varphi_2[y^2 \otimes xy] = y^2x \otimes y = 0$. All element of $Im\varphi_3$ is of the form $vx \otimes (y^2 + ayx)$ and $y^2 \otimes xy$ can not be in $Im\varphi_3$. This A is not Koszul.

(4) When $R = \langle y^2 + ayx, xy + byx\rangle$, we show that it is not Koszul.

By a linear map: $x \mapsto x - \frac{1}{a}y$ and $y \mapsto y$.

$xy + byx \mapsto -\frac{1}{a}(b + 1)y^2 + byx + xy$,

$y^2 + ayx \mapsto yx$.

R can be replaced by $\langle -\frac{1}{a}(b + 1)y^2 + xy, yx\rangle$.

If $b = -1$, it is Koszul since this is a monomial algebra.

If $b \neq -1$, we can replace R as $\langle\frac{b+1}{a}y^2 + yx, xy\rangle = \langle y^2 + \frac{a}{b+1}yx, xy\rangle$ by the linear map $x \mapsto x + \frac{b+1}{a}y$. By above (3), it is not Koszul.

$\square$

Lemma 10. *When $R = \langle x^2 + by^2 + cyx, xy + dyx\rangle$, this A is Koszul if*

(1) $b = c = 0$,

(2) $d = -1$,

(3) $c = 0$ *and* $d = 1$

Proof. If $v \in K_3$, there exists $\alpha_i, \beta i, \gamma i, \delta i \in \Bbbk$, $i = 1, 2$ such that

$$v = (\alpha_1x + \beta_1y) \otimes (x^2 + by^2 + cyx) + (\gamma_1x + \delta_1y) \otimes (xy + dyx)$$

$$= (x^2 + by^2 + cyx) \otimes (\alpha_2x + \beta_2y) + (xy + dyx) \otimes (\gamma_2x + \delta_2y)$$

$\alpha_1 = \alpha_2$, $b\alpha_1 = \delta_2$, $c\alpha_1 + d\gamma_1 = \gamma_2$, $\gamma_1 = \beta_2$

$$\beta_1 = c\alpha_2 + d\gamma_2,\ b\beta_1 = b\beta_2,\ c\beta_1 + d\delta_1 = b\alpha_2,\ \delta_1 = c\beta_2 + d\delta_2.$$

Then

$$(dc+c)\alpha_1 + d^2\gamma_1 - \beta_1 = 0, \quad -b\alpha_1 + c\beta_1 + d\delta_1 = 0,$$

$$-b\gamma_1 + b\beta_1 = 0, \qquad bd\alpha_1 + c\gamma_1 - \delta_1 = 0.$$

We get the following matrix-($*$);

$$\begin{pmatrix} c(d+1) & d^2 & -1 & 0 \\ -b & 0 & c & d \\ 0 & -b & b & 0 \\ bd & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} c(d+1) & d^2 & -1 & 0 \\ bd^2-b & cd & c & 0 \\ 0 & -b & b & 0 \\ bd & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $a, b, c, d \in \Bbbk$ be non-zeros.

(1) When $R = \langle x^2, xy\rangle$, it is Koszul since A is a monomial algebra.

(2) When $R = \langle x^2 + by^2, xy\rangle$, we show that it is not Koszul. By ($*$), we get

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ -b & 0 & 0 & 0 \\ 0 & b & -b & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then $K_3 = 0$. In A, $yx^2 = -by^3 = x^2y = 0$. $\varphi_2[yx \otimes xy] = yx^2 \otimes y = 0$. But $0 \neq yx \otimes xy \in \ker \varphi_2$. This A is not Koszul.

(3) When $R = \langle x^2 + cyx, xy\rangle$, we show that this is not Koszul. By ($*$), we get

$$\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then K_3 has a basis $\{x^2y+cy^2x\}$. $y^2x = \frac{1}{c^2}x^3 = \frac{-1}{c}xyx = 0$ in A and $\varphi_2[y^2 \otimes (xy)] = y^2x \otimes y = 0$. All element in $Im\varphi_3$ is of the form $(vx + cvy) \otimes (xy)$ for some $v \in A$. Since $0 \neq yx \otimes xy \in \ker \varphi_2$, this A is not Koszul.

(4) When $R = \langle x^2, xy + dyx \rangle$, we show that A has a PBW-basis.

Assume that $x > y$ in order. We can find that $\{X^\alpha : \alpha \in S^3\} = \{y^2x, y^3\}$ since $S^2 = \{(2,1),(2,2)\}$. In A_3, $xy^2 = -dxyx = d^2yx^2 \neq 0$. Since A_3 is generated by $\{y^2x, y^3\}$ and by Theorem 1, A has a PBW-basis.

(5) When $R = \langle x^2 + by^2 + cyx, xy \rangle$, we show that this is not Koszul. By $(*)$, we get

$$\begin{pmatrix} c & 0 & -1 & 0 \\ -b & 0 & c & 0 \\ 0 & b & -b & 0 \\ 0 & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} c^2 - b & 0 & 0 & 0 \\ -b & c & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In A, $x^3 = -(bxy^2 + cxyx) = 0$ and $\varphi_2[x^2 \otimes xy] = x^3 \otimes y = 0$.

Then $0 \neq x^2 \otimes xy \in \ker \varphi_2$.

If $c^2 \neq b$, $K_3 = 0$. It is not Koszul since $\ker \varphi_2 \neq 0$.

If $c^2 = b$, K_3 has a basis

$$\{cx^2y + (x^3 + bxy^2 + cxyx) + c(yx^2 + by^3 + cy^2x) + c^2yxy\}.$$

If there exists $w \in A \otimes K_3$ such that $\varphi_3(w) = x^2 \otimes xy$, we can express w as the following; for some $\alpha, \beta \in \Bbbk$

$w = (\alpha x + \beta y) \otimes [cx^2y + (x^3 + bxy^2 + cxyx) + c(yx^2 + by^3 + cy^2x) + c^2yxy]$.

In A,

$$c\alpha x^2 + c\beta yx + c^2(\alpha xy + \beta y^2) = x^2,$$

$$\alpha x^2 + \beta yx + c\alpha xy + c\beta y^2 = 0,$$

$$\Rightarrow x^2 = 0 \text{ in } A.$$

It is impossible that $w \in A \otimes K_3$ such that $\varphi_3(w) = x^2 \otimes xy$. This is not Koszul since $\ker \varphi_2 \supsetneqq Im\varphi_3$.

(6) When $R = \langle x^2 + by^2, xy + dyx \rangle$, we show that this is Koszul if $d^2 = 1$.

In A_3, $x^3 = -bxy^2 = -bd^2y^2x = d^2x^3$, $d^2yx^2 = x^2y = -by^3 = yx^2$

When $d^2 = 1$, assume that $x < y$ in order. $\{X^\alpha : \alpha \in S^3\} = \{x^3, x^2y\}$ since $S^2 = \{(2,1),(2,2)\}$. Since $\{x^3, x^2y\}$ is linearly independent in A_3 and by Theorem 1, A is a PBW-algebra.

When $d^2 \neq 1$, we show that it is not Koszul. By $(*)$, we get

$$\begin{pmatrix} 0 & d^2-1 & 0 & 0 \\ bd^2-b & 0 & 0 & 0 \\ 0 & b & -b & 0 \\ bd & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$K_3 = 0$ and $x^3 = x^2y = 0$. $\varphi_2(x^2 \otimes (x^2 + by^2)) = x^3 \otimes x + bx^2y \otimes y = 0$. $0 \neq x^2 \otimes (x^2 + by^2) \in \ker \varphi_2$. A is not Koszul.

(7) When $R = \langle x^2 + cyx, xy + dyx \rangle$, we can replace R as $\langle yx, -\frac{1+d}{c}y^2 + xy \rangle$ by a linear map : $y \mapsto y - \frac{1}{c}x$, $x \mapsto x$.

If $d + 1 = 0$, it is Koszul since this is a monomial algebra.

If $d + 1 \neq 0$, it is not Koszul by Lemma 8.

(8) When $R = \langle x^2 + by^2 + cyx, xy + dyx\rangle$, in A,

$$-by^2x - cyx^2 = x^3 = -bxy^2 - cxyx = -bd^2y^2x + cdyx^2,$$

$$\Rightarrow b(d^2-1)y^2x - c(d+1)yx^2 = 0,$$

$$d^2yx^2 = x^2y = -by^3 - cycy = yx^2 + cy^2x + cdy^2x.$$

$$\Rightarrow (d^2-1)yx^2 - c(d+1)y^2x = 0,$$

$$x^2y + cyxy = -by^3 = yx^2 + cy^2x = d^2x^2y - dcyxy,$$

$$x^2y = -by^3 + cdy^2x = -by^3 - d(yx^2 + by^3) = -b(d+1)y^3 - dyx^2$$
$$= (1+d)(yx^2 + cy^2x) - dyx^2 = yx^2 + c(1+d)y^2x,$$

$$\Rightarrow (d-1)x^2y = cyxy \text{ and } (d-1)yx^2 = cy^2x \text{ if } d \neq \pm 1,$$

$$\Rightarrow (d^2-1)yx^2 = 0 \text{ if } d \neq \pm 1.$$

(a) If $d = 1$ and $char\mathbb{k} \neq 2$, $K_3 = 0$ and $yx^2 = y^2x = yxy = 0$ in A. $\varphi_3(yx \otimes (xy + yx)) = (yx^2 \otimes x + yxy \otimes y) = 0$ and $0 \neq yx \otimes (xy + yx) \in \ker\varphi_2$. This is not Koszul.

(b) If $d = -1$, we have that $\{X^\alpha | \alpha \in S^3\}$ is $\{x^3, x^2y\}$ or $\{y^3, y^2x\}$. These two sets are linearly independent in A_3. It is Koszul by Theorem 1.

(c) We assume that $d \neq \pm 1$ and we get

$$\begin{pmatrix} c^2(d+1) - b(d-1)^2(d+1) & 0 & 0 & 0 \\ bd^2 - b & c(d+1) & 0 & 0 \\ 0 & 1 & -1 & 0 \\ bd & c & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In A, $yx^2 = x^2y = 0 \Rrightarrow A_3 = 0$.

When $[c^2 - b(d-1)^2] \neq 0$, $K_3 = 0$. $\varphi_2(y^2 \otimes (xy + dyx)) = y^2x \otimes y + dy^3 \otimes x = 0$. Since $0 \neq y^2 \otimes (xy + dyx) \in \ker\varphi_2$, it is not Koszul.

When $[c^2 - b(d-1)^2] = 0$, K_3 has a basis $\{c(x^3 + bxy^2 + cxyx) - b(d -$

$1)(yx^2+by^3+cy^2x)-b(d-1)(x^2y+dxyx)+bc(yxy+dy^2x)\}$. Then $\varphi_2(x^2\otimes(x^2+by^2+cyx))=0$ and $0\neq x^2\otimes(x^2+by^2+cyx)$ can not be in $Im\varphi_3$. This is not Koszul.

□

LEMMA 11. *In $R=\langle x^2+axy+byx, y^2+cxy+dyx\rangle$, this A is Koszul if*

(1) $R=\langle x^2,y^2\rangle$
(2) $R=\langle x^2+axy+ayx,y^2\rangle$ *where* $a\neq 0$*; it is isomorphic to* $A=\{\{x,y\},\langle x^2,y^2\rangle\}$
(3) $R=\langle x^2,y^2+cxy+cyx\rangle$ *where* $c\neq 0$ *; it is isomorphic to* $A=\{\{x,y\},\langle x^2,y^2\rangle\}$
(4) $R=\langle x^2+byx,y^2+cxy\rangle$ *where* $1+bc=0$ *; it is isomorphic to* $A=\{\{x,y\},\langle x^2,xy\rangle\}$
(5) $R=\langle x^2+axy,y^2+dyx\rangle$ *where* $1+ad=0$ *; it is isomorphic to* $A=\{\{x,y\},\langle x^2,xy\rangle\}$
(6) $R=\langle x^2+axy+ayx,y^2+cxy+cyx\rangle$ *where* $a\neq 0$ *and* $c\neq 0$*; it is isomorphic to* $A=\{\{x,y\},\langle x^2+ay^2,xy+yx\rangle\}$.

PROOF. Let $a,b,c,d\in\Bbbk$ be non zeros.

(1) When $R=\langle x^2,y^2\rangle$, this is a monomial algebra. It is Koszul.
(2) When $R=\langle x^2+axy,y^2\rangle$, let $f:A\to A$ be a linear map such that $x\mapsto y, y\mapsto x-\frac{1}{a}y$. Replace R as $\langle yx, x^2-\frac{1}{a}y^2-\frac{1}{a}xy\rangle$. By linear map: $x\mapsto y$ and $y\mapsto x$, we can replace R as $\langle xy,y^2-\frac{1}{a}x^2-\frac{1}{a}yx\rangle$ By Lemma 10, this is not Koszul.
(3) When $R=\langle x^2+byx,y^2\rangle$, let $f:V\to V$ be a linear map such that $x\mapsto x-by, y\mapsto y$. Then $x^2+byx\mapsto x^2-bxy-byx+b^2y^2+byx-b^2y^2$. Replace R as $\langle x^2-bxy,y^2\rangle$. It is not Koszul by above case (2).
(4) When $R=\langle x^2,y^2+cxy\rangle$ or $R=\langle x^2,y^2+dyx\rangle$, this is case like above (2) or (3) by a linear map : $x\mapsto y,y\mapsto x$. This is not Koszul.

(5) When $R = \langle x^2, y^2 + cxy + dyx \rangle$, let $f : V \to V$ be a linear map such that $x \mapsto x$ and $y \mapsto -cx - dy$. Then
$y^2 + cxy + dyx \mapsto c^2x^2 + cdxy + cdyx + d^2y^2 - c^2x^2 - cdxy - cdx^2 - d^2yx$
We can replace R as $\langle x^2, dy^2 + (c-d)yx \rangle$. It is Koszul when $c = d$.

(6) When $R = \langle x^2 + axy + byx, y^2 \rangle$, this is the case like above (5) by a linear map such that $x \mapsto y$ and $y \mapsto x$. It is Koszul if $a = b$.

(7) When $R = \langle x^2 + byx, y^2 + dyx \rangle$,
let $f : V \to V$ be a linear map such that $x \mapsto x$ and $y \mapsto y - \frac{1}{b}x$.
Then $x^2 + byx \mapsto yx$ and $y^2 + dyx \mapsto y^2 + \frac{1+bd}{b}x^2 - \frac{1}{b}xy - \frac{1+bd}{b}yx$
We can replace R as $\langle yx, y^2 + \frac{1+bd}{b}x^2 - \frac{1}{b}xy \rangle$. By linear map: $x \mapsto y$ and $y \mapsto x$, we can replace R as $\langle xy, x^2 + \frac{1+bd}{b}y^2 - \frac{1}{b}yx \rangle$. By the Lemma 10, this is not Koszul.

(8) When $R = \langle x^2 + axy, y^2 + dyx \rangle$, this is the case like above (7) by a linear map such that $x \mapsto y$ and $y \mapsto x$. It is not Koszul.

(9) When $R = \langle x^2 + byx, y^2 + cxy \rangle$,
let $f : V \to V$ be linear map such that $x \mapsto x$ and $y \mapsto y - \frac{1}{b}x$.
Then $x^2 + byx \mapsto yx$.

$$y^2 + cxy \mapsto y^2 + \frac{1+bc}{b^2}x^2 - \frac{1+bc}{b}xy - \frac{1}{b}yx$$

We can replace R as $\langle yx, y^2 + \frac{1+bc}{b^2}x^2 - \frac{1+bc}{b}xy \rangle$. By a linear map such that $x \mapsto y$ and $y \mapsto x$, we can replace R as $\langle xy, x^2 + \frac{1+bc}{b^2}y^2 - \frac{1+bc}{b}yx \rangle$. By Lemma 10, this is Koszul if $1 + bc = 0$.

(10) When $R = \langle x^2 + axy, y^2 + dyx \rangle$, this is the case like above (9) by a linear map such that $x \mapsto y$ and $y \mapsto x$. It is Koszul if $1 + ad = 0$.

(11) When $R = \langle x^2 + byx, y^2 + cxy + dyx \rangle$,
let $f : V \to V$ be a linear map such that $y \mapsto -\frac{1}{b}x + y, x \mapsto x$
Then $y^2 + cxy + dyx \mapsto (\frac{1}{b^2} - \frac{c+d}{b})x^2 + (-\frac{1}{b} + c)xy + y^2 + (d - \frac{1}{b})yx$

$x^2 + byx \mapsto byx$

we can replace R as $\langle (\frac{1}{b^2} - \frac{c+d}{b})x^2 + (\frac{1}{b} - c)xy + y^2, yx\rangle$

By Lemma 7 and Lemma 8, it is not Koszul.

(12) When $R = \langle x^2 + axy, y^2 + cxy + dyx\rangle$,

let $f : V \to V$ be a linear map such that $y \mapsto -\frac{1}{a}y + x, x \mapsto y$

Then $y^2 + cxy + dyx \mapsto (\frac{1}{a^2} - \frac{c+d}{a})y^2 + (-\frac{1}{a} + c)yx + x^2 + (d - \frac{1}{a})xy$

$x^2 + axy \mapsto ayx$

we can replace R as $\langle (\frac{1}{a^2} - \frac{c+d}{a})x^2 + (-\frac{1}{a} + c)yx + y^2, xy\rangle$. By linear map: $x \mapsto y$ and $y \mapsto x$, we can replace R as $\langle (\frac{1}{a^2} - \frac{c+d}{a})y^2 + (-\frac{1}{a} + c)xy + x^2, yx\rangle$. By the Lemma 8, this is not Koszul.

(13) When $R = \langle x^2 + axy + byx, y^2 + dyx\rangle$ or $R = \langle x^2 + axy + byx, y^2 + cxy\rangle$, let $f : V \to V$ be a linear transformation such that $x \mapsto y, y \mapsto x$. This is the case like above (11) or (12). It is not Koszul.

(14) When $R = \langle x^2 + axy + byx, y^2 + cxy + dyx\rangle$ where $c + d \neq 0$ or $a + c \neq 0$, if $c + d \neq 0$, let $t = -\frac{1}{c+d} \neq 0$ and let $f : x \mapsto x + ty, y \mapsto x$ be a linear map of V. Then

$$x^2 + axy + byx \mapsto x^2 + txy + tyx + t^2y^2 + axy + aty^2 + byx + tby^2$$
$$\mapsto x^2 + (a + t)xy + (b + t)yx + (t^2 + at + bt)y^2$$

$y^2 + cxy + dyx \mapsto (1 + t(c + d))y^2 + cxy + dyx \mapsto cxy + dyx$

Then R can be replaced as

$\langle x^2 + (b + t - \frac{da+dt}{c})yx + (t^2 + at + bt)y^2, xy + \frac{d}{c}yx\rangle$.

By Lemma 10, this is Koszul if $a = b$ and $c = d$

If $a + c \neq 0$, this is same case of $c + d \neq 0$.

(15) When $R = \langle x^2 + axy - ayx, y^2 + cxy - cyx\rangle$ and $char\mathbb{k} \neq 2$, we can assume that $a = 1$ by linear transformation $f : y \mapsto \frac{1}{a}y, x \mapsto x$. Let $f : A \to A$ be a linear transformation such that $x \mapsto x + ay, y \mapsto y$.

Then $x^2 + axy - ayx \mapsto x^2 + a^2y^2 + 2axy$,

$y^2 + cxy - cyx \mapsto y^2 + cxy - cyx$

we can replace R by $\langle x^2 + (2a - ca^2)xy + ca^2yx, y^2 + cxy - cyx\rangle$

But if $2a - ca^2 = -ca^2 \mapsto a = 0$, this is a contradiction because of $a \neq 0$. It is the case like case (14). This is not Koszul.

(16) When $R = \langle x^2 + axy - ayx, y^2 + cxy - cyx\rangle$ and $char\mathbb{k} = 2$, we can assume that $a = 1$. $y^3 = cx^2y = cyx^2$, $cx^3 = xy^2 = y^2x$, $cyxy = y^3 + c^2x^3$, and $xyx = x^3 + y^3$ in A. Since $\{X^\alpha | \alpha \in S^3\}$ can be $\{x^3, x^2y\}$ or $\{y^3, y^2x\}$. These two sets are linearly independent in A_3. By Theorem 1, this is Koszul.

$\square$

By above result about a quadratic algebra $A = \{V, R\}$ we get the followings:

(1) When $|R| = 0$, $A = T(\{x, y\})$ is a PBW-algebra. The Hilbert series of A is
$h_A(x) = \sum_{n=1}^{n=\infty} 2^{n-1}x^n$.

(2) When $|R| = 1$, A is a Koszul. The Hilbert series of A is
$h_A(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$ or
$h_A(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + \cdots$.

Then we get the following table.

Table 1:

For $a, b \in \mathbb{k} - 0$ and $c, d, e \in \mathbb{k}$

Let a quadratic algebra $B = \{V, W\}$ be isomorphic to $A^!$.

a basis of $R^\perp$	a basis of W	$\dim A_3$
$\langle x^2\rangle$ or $\langle y^2\rangle$	$\langle x^2\rangle$	1
$\langle x^2+ay^2\rangle$	$\langle x^2+ay^2\rangle$	0
$\langle xy+eyx\rangle$,$\langle exy+yx\rangle$	$\langle xy+ayx\rangle$	0
$\langle xy-bx^2\rangle$, $\langle xy-bx^2\rangle$,	$\langle xy\rangle$	0
$\langle xy-bx^2-ay^2\rangle, ab=1$	$\langle xy\rangle$	0
$\langle xy-bx^2-ay^2\rangle, ab\neq 1$	$\langle x^2\rangle$	1
$\langle yx-cy^2-dxy\rangle$,$\langle yx-cx^2-dxy\rangle, d\neq 1$	$\langle xy+eyx\rangle$	0
$\langle yx-ay^2-xy\rangle$,$\langle yx-cx^2-xy\rangle$,	$\langle x^2+xy-yx\rangle$	1
$\langle yx+ay^2+bx^2-xy\rangle$, if $ab=1$	$\langle x^2\rangle$	0
$\langle yx+ay^2+bx^2-xy\rangle$, if $ab\neq 1$	$\langle x^2+ay^2\rangle$	0

When $\dim A_3=1$, A is isomorphic to $T(\{x,y\})/\langle x^2\rangle$.

When $\dim A_3=0$, A is isomorphic to $T(\{x,y\})/\langle xy-ayx\rangle$, $T(\{x,y\})/\langle x^2+xy-yx\rangle$, $T(\{x,y\})/\langle x^2+by^2\rangle$ where $a,b\in\Bbbk$ and $b\neq 0$.

If $\Bbbk$ is an algebraically closed field, $T(\{x,y\})/\langle x^2+by^2\rangle$, $b\neq 0$ is isomorphic to $T(\{x,y\})/\langle xy-ayx\rangle$ for some $a,b\neq 0$. And if A is isomorphic to $T(\{x,y\})/\langle x^2\rangle$, $T(\{x,y\})/\langle xy-ayx\rangle$, or $T(\{x,y\})/\langle x^2+by^2\rangle$, then A is a PBW-algebra.

(3) When $|R|=3$, A is a Koszul algebra. The Hilbert series of A is

$h_A(x)=1+2x+x^2$ or

$h_A(x)=1+2x+x^2+x^3+x^4+\cdots$.

(4) When $|R|=4$, A is a PBW-algebra. The Hilbert series of A is

$h_A(x)=1+2x$.

(5) When $|R|=2$, A is a Kosszul algebra if only if $\dim A_3=2$. i.e., every Koszul algebras such that $|R|=2$ are PBW-algebras, and we get the Table 2. The Hilbert series of A is of the form that follows:

(a) If $\dim A_2 = 0$, $\quad h_A(x) = 1 + 2x + 2x^2$.

(b) If $\dim A_2 = 1$, $\quad h_A(x) = 1 + 2x + 2x^2 + x^3 + x^4 + x^5 + \cdots$ or

$$h_A(x) = 1 + 2x + 2x^2 + x^3.$$

(c) If $\dim A_2 = 2$, $h_A(x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \cdots$.

When $\dim A_3 = 0$, $\quad A$ is isomorphic to $T(\{x,y\})/\langle xy, x^2 - ayx + y^2\rangle$, $a \in \mathbb{k}$.

When $\dim A_3 = 1$, $\quad A$ is isomorphic to $T(\{x,y\})/\langle x^2, yx + y^2\rangle$ or

$$T(\{x,y\})/\langle xy, yx + y^2\rangle.$$

When $\dim A_3 = 2$, $\quad A$ is isomorphic to $T(\{x,y\})/\langle x^2, y^2\rangle$, $T(\{x,y\})/\langle x^2 + xy + yx, y^2 + bxy + byx\rangle$, $T(\{x,y\})/\langle x^2, xy - ayx\rangle$, or $T(\{x,y\})/\langle xy, yx\rangle$ where $a \in \mathbb{k}$ and $b \in \mathbb{k}^*$. If $\mathbb{k}$ is an algebraically closed field, then $T(\{x,y\})/\langle x^2 + xy + yx, y^2 + bxy + byx\rangle$ is isomorphic to $T(\{x,y\})/\langle x^2, y^2\rangle$.

We get the following table.

Table 2

Assume that a,b,c,d $\in \mathbb{k}$ are not zero and $\lambda \in \mathbb{k} - \{1, -1\}$.

Let a quadratic algebra $B = \{V, W\}$ be isomorphic to A.

a basis of R	a basis of W	$\dim A_3$
$\langle xy, yx\rangle$	$\langle xy, yx\rangle$	2
$\langle x^2, yx\rangle$,$\langle x^2, xy\rangle$	$\langle x^2, yx\rangle$	2
$\langle x^2, y^2\rangle$	$\langle x^2, y^2\rangle$	2
$\langle x^2 + ayx, xy\rangle$,$\langle x^2 + bxy, yx\rangle$	$\langle y^2 + yx, xy\rangle$	1
$\langle x^2 + ay^2, yx\rangle$,$\langle x^2 + ay^2, xy\rangle$	$\langle x^2 + y^2, xy\rangle$	0
$\langle x^2 + ay^2 + byx, xy\rangle$ $b^2 = a$	$\langle x^2, yx + y^2\rangle$	1
$\langle x^2 + ay^2 + byx, xy\rangle, b^2 \neq a$	$\langle x^2 + y^2 + \lambda yx, xy\rangle$	0
$\langle x^2, xy + byx\rangle$,$\langle y^2, xy + byx\rangle$	$\langle x^2, xy + byx\rangle$	2

a basis of R	a basis of W	$\dim A_3$
$\langle x^2+axy, xy-yx\rangle$ $\langle y^2+ayx, xy-yx\rangle$	$\langle xy, yx\rangle$	2
$\langle x^2+axy, yx+bxy\rangle, b\neq -1$	$\langle y^2+yx, xy\rangle$	1
$\langle x^2+by^2, xy\pm yx\rangle$	$\langle x^2, y^2\rangle$,$\langle x^2, xy+byx\rangle$	2
$\langle x^2+by^2+cyx, xy-yx\rangle$	$\langle xy, yx\rangle$	2
$\langle x^2+by^2+cyx, xy+dyx\rangle, d\neq -1$	$\langle x^2, y^2+yx\rangle$	1
$\langle x^2+axy, y^2\rangle$,$\langle x^2+byx, y^2\rangle$	$\langle x^2+y^2+\lambda yx, xy\rangle$	0
$\langle x^2, y^2+axy+ayx\rangle$,$\langle y^2, x^2+axy+ayx\rangle$	$\langle x^2, y^2\rangle$	2
$\langle x^2, y^2+axy+byx\rangle, a\neq b$	$\langle x^2, yx+y^2\rangle$	1
$\langle x^2+axy, y^2+byx\rangle$ $\langle x^2+byx, y^2+axy\rangle, ab=-1$	$\langle x^2, xy\rangle$	2
$\langle x^2+axy, y^2+byx\rangle$ $\langle x^2+byx, y^2+axy\rangle, ab\neq -1$	$\langle x^2+y^2+\lambda yx, xy\rangle$	0
$\langle x^2+byx, y^2+cxy+dyx\rangle$ if $1=b(c+d)$ $\langle x^2+axy, y^2+cxy+dyx\rangle$ if $1=a(c+d)$	$\langle y^2+yx, xy\rangle$	1
$\langle x^2+byx, y^2+cxy+dyx\rangle$,if $1\neq b(c+d)$ $\langle x^2+axy, y^2+cxy+dyx\rangle$, if $1\neq a(c+d)$	$\langle x^2+y^2+\lambda yx, xy\rangle$	0
$\langle x^2+axy+ayx, y^2+cxy+cyx\rangle$	$\langle y^2+yx, xy\rangle$, itself	2
$\langle x^2+axy+byx, y^2+cxy+dyx\rangle$ if this is not above two cases	$\langle x^2+y^2+\lambda yx, xy\rangle$	0

CHAPTER 4

GEOMETRIC EXPLANATION

From now, we assume that the ground field $\Bbbk$ is algebraically closed, uncountable. Let $A = \{V, R\}$ be a quadratic algebra where $\dim V = m$ and R be a set of quadratic relations. Let $Q_{m,s} = \{A | A$ is quadratic algebra such that $\dim A_1 = m$,$\dim A_2 = s\}$ be the Grassmann variety $G(m^2 - s, m^2)$. Let $Q_{m,s,u}$ be the set of all elements of $Q_{m,s}$ such that $\dim A_3 = u$. Then $Q_{m,s,u}$'s define a stratification of $Q_{m,s}$ by locally closed subvarieties.

THEOREM 6. ($Corollary$ 4.1 in [1]) *The set of all n-Koszul algebra in $Q_{m,s,u}$ is an open subset in Zariski's topology. Therefore the set of all Koszul algebra in $Q_{m,s,u}$ is a countable intersection of open subsets.*

Now, we consider $m = 2$ case.

REMARK 1. *By Chapter 3 we know that $Q_{2,1}$ and $Q_{2,0}$ consists of Koszul algebras. Trivially every a quadratic algebra in $Q_{2,0}$ are PBW-algebras. Since the ground field $\Bbbk$ is algebraically closed, a quadratic algebra $T(\{x, y\})/\langle x^2 + by^2\rangle$, $b \neq 0$ is isomorphic to $T(\{x, y\})/\langle xy - ayx\rangle$ or $T(\{x, y\})/\langle x^2 + xy - yx\rangle$. Every quadratic algebras in $Q_{2,1}$ are PBW-algebras. Let $A = T(V)/R \in Q_{2,1}$ where $\{x, y\}$ is a basis of $V = A_1$. By Table 1, we get that every a quadratic algebra in $Q_{2,1,1}$ is isomorphic to algebra dual to $T(\{x, y\})/\langle x^2\rangle$ and $Q_{2,1,0}$ is isomorphic to algebra dual to $T(\{x, y\})/\langle xy - \lambda yx\rangle$, $\lambda \in \Bbbk$ or $T(\{x, y\})/\langle x^2 + xy - yx\rangle$.*

REMARK 2. *Consider $Q_{2,2}$. Since V has two dimensional vector space and R has two basis elements, we know that* $\dim A_3 \leq 3$ *and $Q_{2,2} = Q_{2,2,0} \cup Q_{2,2,1} \cup Q_{2,2,2}$. By Table 2, every quadratic algebras in $Q_{2,2}$ are PBW-algebras.*

(1) *The $Q_{2,2,0}$ consists of algebras isomorphic to $T(\{x,y\})/\langle xy, x^2 - \lambda yx + y^2\rangle$, where $\lambda \in \mathbb{P}^1_{\Bbbk}$.*

(2) *The $Q_{2,2,1}$ consists of two components which consist of the algebras isomorphic to $T(\{x,y\})/\langle x^2, yx + y^2\rangle$ or $T(\{x,y\})/\langle xy, yx + y^2\rangle$. Let E_1 be the closed variety of the component which consists of algebras isomorphic to $T(\{x,y\})/\langle x^2, yx + y^2\rangle$ and E_2 be the closed variety of the component which consists of algebras isomorphic to $T(\{x,y\})/\langle xy, yx+y^2\rangle$. Then E_1 contains the components which consist of algebras isomorphic to $T(\{x,y\})/\langle x^2, y^2\rangle$, $T(\{x,y\})/\langle x^2, xy - \lambda yx\rangle$. E_2 contains the components which consist of algebras isomorphic to $T(\{x,y\})/\langle xy, yx\rangle$, $T(\{x,y\})/\langle x^2, xy - \lambda yx\rangle$ in $Q_{2,2,2}$.*

(3) *The $Q_{2,2,2}$ consists of three irreducible components. These have the following generic points respectively: $T(\{x,y\})/\langle x^2, y^2\rangle$, $T(\{x,y\})/\langle xy, yx\rangle$, $T(\{x,y\})/\langle x^2, xy - \lambda yx\rangle$. Let $f : x \mapsto x + \alpha y$ and $y \mapsto x + \beta y$, $g : x \mapsto x+y$ and $y \mapsto y$ be linear maps of V where $\alpha+\beta = 2$ and $\alpha \neq \beta$. Then $T(\{x,y\})/\langle x^2, y^2\rangle$ is isomorphic to $T(\{x,y\})/\langle 2y^2 + xy + yx, (x+y)^2\rangle$ by f. $T(\{x,y\})/\langle x^2, xy + yx\rangle$ is isomorphic to $T(\{x,y\})/\langle 2y^2 + xy + yx, (x+y)^2\rangle$ by g. Let $h : x \mapsto x+y$ and $y \mapsto y$ and $q : x \mapsto \alpha x + y$ and $y \mapsto x + \beta y$, be linear maps of V where $\alpha\beta \neq 1, \alpha\beta + 1 = \beta$. $T(\{x,y\})/\langle x^2, xy - yx\rangle$ is isomorphic to $T(\{x,y\})/\langle x^2 + 2xy + y^2, xy - yx\rangle$ by h. $T(\{x,y\})/\langle xy, yx\rangle$ is isomorphic to $T(\{x,y\})/\langle x^2 + 2xy + y^2, xy - yx\rangle$ by q, so the first component and the third components have intersection with the second one at $\lambda = -1$,$\lambda = 1$ respectively.*

In $\Bbbk^4 = G(2,4) = Q_{(2,2)}$, let (x^2, xy, yx, y^2) be the coordinate of $\Bbbk^4$.

Points in $G(2,4)$	*line* L_i *which contain* P_i
$P_1 = \langle x^2, y^2 \rangle$	$(0,\alpha,\beta,0)$,*where* $\alpha,\beta \in \Bbbk$ *can not be zero simultaneously*
$P_2 = \langle x^2, xy + yx \rangle$	$(0,\gamma,-\gamma,\delta)$, *where* $\gamma,\delta \in \Bbbk$ *can not be zero simultaneously.*
$P_3 = \langle x^2, xy - yx \rangle$	$(0,\eta,\eta,\theta)$, *where* $\eta,\theta \in \Bbbk$ *can not be zero simultaneously.*
$P_4 = \langle xy, yx \rangle$	$(\pi,0,0,\tau)$,*where* $\pi,\tau \in \Bbbk$ *can not be zero simultaneously.*

space S_{ij} *which contains* L_i, L_j	*Expression of* $S_{i,j}$ *using Plucker coordinate in* $\mathbb{P}^6_{\Bbbk}$
S_{12}	$\{[0,0,0,-\alpha\gamma-\beta\gamma,\alpha\delta,\beta\delta]\}$
S_{34}	$\{[\pi\theta,\pi\eta,\pi\theta,0,\tau\theta,\tau\eta]\}$
S_{23}	$\{[0,0,0,2\gamma\eta,\gamma\theta-\delta\eta,-\gamma\theta-\delta\eta]\}$

Each S_{ij} *is isomorphic to* $\mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$. *and the lines* $\mathbb{P}^1_{\Bbbk} \times \{\pm 1\}$ *in* S_{23} *are identified with the diagonals in* S_{12}, S_{34}, *but* S_{12}, S_{34} *has no intersection. Until now, we assume that* $\Bbbk$ *is an uncountable algebraically closed field. If not, we have more generic points. For example, there does not exist* $B^* = T(\{x,y\})/\langle x^2 + xy - yx \rangle$ *which is isomorphic to* $A^* = T(\{x,y\})/\langle x^2 + by^2 + cxy + dyx \rangle$, $b \neq 0$ *since* $\Bbbk$ *may not have an element* α *such that* $\alpha^2 + b = 0$ *in general. Even if* $\Bbbk$ *is not a algebraically closed field, algebras are Koszul in* $Q_{2,2}$ *when algebras is in* $Q_{2,2,2}$.

CHAPTER 5

SYZYGIES ON PROJECTIVE VARIETIES

In this chapter we show one relation between the line bundle L and the power of L. Also we show one sharper result on Theorem 1.7 in [10]. Before we mention some facts, we introduce Green-Lazarsfeld's definition of property N_p. Let $\Bbbk$ be an algebraically closed field. Let X be a projective variety of dimension n over $\Bbbk$ and L be a globally generated, ample line bundle on X, defining an embedding

$$\varphi_L : X \longrightarrow \mathbb{P}^r = \mathbb{P}(H^0(X, L)^*), \quad \text{by } x \mapsto \{\sigma \in H^0(X, L) | \sigma(x) = 0\}.$$

Let $S = \oplus_h Sym^h H^0(X, L)$ be the homogeneous coordinate ring of the projective space $\mathbb{P}^r$ and consider the graded S-module $G = \oplus_d H^0(X, L^d)$. Let $E_\bullet$ be a minimal graded free resolution of G:

$$E_\bullet : \quad \cdots \to E_{r+1} \to E_r \to \cdots \to E_1 \to E_0 \to G \to 0$$

where $E_i = \oplus_j S(-a_{ij})$ and $S(-a_{ij})$ is S with a shift of degree by $-a_{ij}$.

Then the line bundle L satisfies property $N_p (p \in \mathbb{N})$ if and only if

$E_0 = S$, and

$E_i = \oplus S(-i-1)$ (i.e., all $a_{ij} = i+1$) for $1 \leq i \leq p$.

Thus the definition can be summarized very concretely as follows:

L satisfies property $N_0 \Leftrightarrow X$ embeds in $\mathbb{P}H^0(L)$ as a projectively normal variety;

L satisfies property $N_1 \Leftrightarrow N_0$ holds for L and the homogeneous ideal I is generated by quadrics;

L satisfies property $N_2 \Leftrightarrow L$ satisfies Property N_1 and the S-module of

syzygies among quadratic generators $Q_i \in I$
is generated by the relations of the form
$\sum A_i Q_i = 0$, where the A_i are linear polynomials,

and so on.

Properties N_0 and N_1 are called normal generation and normal presentation, respectively, by Mumford.

DEFINITION 5. ([15]) *Let X be a projective variety and L be a base point free line bundle on X. Let M_L be the kernel of the evaluation map*

$$H^0(X, L) \otimes \mathcal{O}_X \to L.$$

Thus the following exact sequence on X holds:

$$0 \to M_L \to H^0(X, L) \otimes_{\Bbbk} \mathcal{O}_X \to L \to 0.$$

LEMMA 12. ([7]) *Let L be an ample line bundle that is globally generated on a projective variety X. Assume that $H^1(X, L^t) = 0$ for all $t \geq 1$. Then L satisfies N_p if and only if*

$$H^1(X, \textstyle\bigwedge^t M_L \otimes L^j) = 0 \text{ for } 0 \leq t \leq p+1 \text{ and } j \geq 1.$$

THEOREM 7. ([11] *Lemma* 4.1) *Let L be an ample line bundle that is globally generated on a projective variety X over an algebraically closed field $\Bbbk$ of characteristic zero. Then L satisfies property N_p if*

$$H^1(X, M_L^{\otimes t} \otimes L^j) = 0 \text{ for } 0 \leq t \leq p+1 \text{ and } j \geq 1.$$

Let L be an ample line bundle that is globally generated on a projective variety X. Let N be a line bundle on X and set $V = H^0(X, L)$. For integers $a, q \geq 0$ one can form the Koszul-type complex

$$K_\bullet : \cdots \to \textstyle\bigwedge^{a+1} V \otimes H^0(L^{b-1} \otimes N) \to \bigwedge^{a} V \otimes H^0(L^{b} \otimes N) \to$$
$$\textstyle\bigwedge^{a-1} V \otimes H^0(L^{b+1} \otimes N) \to \cdots$$

LEMMA 13. ([12] *Lemma* 1.4) *Let* L *be an ample line bundle that is globally generated on a irreducible projective variety* X*. Let* N *be a line bundle on* X*. Assume that* $H^1(X, L^{b-1} \otimes N) = 0$*. Then* $K_\bullet$ *is exact in the term* $\bigwedge^{a+1} V \otimes H^0(L^{b-1} \otimes N) \to \bigwedge^a V \otimes H^0(L^b \otimes N) \to \bigwedge^{a-1} V \otimes H^0(L^{b+1} \otimes N)$ *if and only if* $H^1(X, \bigwedge^{a+1} M_L \otimes L^{b-1} \otimes N) = 0$*.*

In [15], Rubei shows that if L is an ample line bundle on a complex torus X such that L satisfies N_0 or N_1, $L^{\otimes s}$ satisfies N_{s-1}, N_s for all $s \geq 1$,respectively. In general this is not true; we can find a counterexample in [16]. But we want to show one relation between a line bundle and the power of the line bundle. This is one of the main results.

THEOREM 8. *Let* X *be a irreducible projective variety over an algebraically closed field* $\Bbbk$ *with characteristic zero. Let* L *be a globally generated ample line bundle on* X *such that* $H^1(X, L^s) = 0$ *for all* $s \geq 1$*. If* L *satisfies* N_p*, then* L^{k+p} *satisfies* N_p *for all* $k \geq 1$*.*

PROOF. For convenience, set $M^{\otimes d}_{L^{\otimes t}} \otimes L^{\otimes q} = M^d_{t,q}$ for all t, d, q and set $H^i(X, T) = H^i(T)$ for any vector bundle T on X. Let $W = H^0(X, L)$. When L satisfies N_0, we know that the mapping $H^0(L^s) \otimes H^0(L^t) \to H^0(L^{s+t})$ is surjective for all $s, t \geq 1$. Then L^k satisfies N_0 for all $k \geq 1$. Assume that $p \geq 1$ and fix $k \geq 2$. For $1 \leq a \leq p$, we want to show the following: for $q \geq a + 1$ and $a \leq t \leq p$,

(1) $H^1(X, \bigwedge^{t+1-a} M_L \otimes M^a_{L^k} \otimes L^q) = 0$,
(2) $H^1(X, M^{a+1}_{L^k} \otimes L^q) = 0$.

We prove this by induction on a. We get the following commutative diagram (⋇) where each vertical line is exact; for $1 \leq a \leq t$ and $q \geq a + 1$,

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
\xrightarrow[\alpha_{1,a}]{} & \bigwedge^{t-a+1} W \otimes H^0(M^a_{k,q}) & \xrightarrow[\alpha_a]{} & \bigwedge^{t-a} W \otimes H^0(M^a_{k,q+1}) & \xrightarrow[\alpha_{2,a}]{} \cdots \\
 & \downarrow & & \downarrow & \\
\xrightarrow[\beta_{1,a}]{} & \bigwedge^{t-a+1} W \otimes H^0(L^k) \otimes H^0(M^{a-1}_{k,q}) & \xrightarrow[\beta_a]{} & \bigwedge^{t-a} W \otimes H^0(L^k) \otimes H^0(M^{a-1}_{k,q+1}) & \xrightarrow[\beta_{2,a}]{} \cdots \\
 & \downarrow & & \downarrow & \\
\xrightarrow[\gamma_{1,a}]{} & \bigwedge^{t-a+1} W \otimes H^0(M^{a-1}_{k,k+q}) & \xrightarrow[\gamma_a]{} & \bigwedge^{t-a} W \otimes H^0(M^{a-1}_{k,k+q+1}) & \xrightarrow[\gamma_{2,a}]{} \cdots \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

First we prove that (1) and (2) is true when $a = 1$. Assume $a = 1$. $\ker\beta_1 = \mathrm{Im}\beta_{1,1}$, $\mathrm{Im}\beta_1 = \ker\beta_{2,1}$, $\ker\gamma_1 = \mathrm{Im}\gamma_{1,1}$, and $\mathrm{Im}\gamma_1 = \ker\gamma_{2,1}$ since L satisfies N_p. We can easily check that $\mathrm{Im}\alpha_1 = \ker\alpha_{2,1}$. By Lemma 13, $H^1(X, \bigwedge^t M_L \otimes M_{L^k} \otimes L^q) = 0$ for all $q \geq 2$. So (1) is true when $a = 1$. Since $H^1(X, M_L \otimes M_{L^k} \otimes L^q) = 0$ for all $q \geq 2$, we get the following commutative diagram where each vertical line is exact;

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^0(L^r) \otimes M_1 \otimes M_k \otimes L^q & \xrightarrow[\lambda_p]{} & M_{L^{r+1}} \otimes M_k \otimes L^q & & \\
\downarrow & & \downarrow & & \\
H^0(L^r) \otimes H^0(L) \otimes M_k \otimes L^q & \xrightarrow[\theta_p]{} & H^0(L^{r+1}) \otimes M_k \otimes L^q & & \\
\downarrow & & \downarrow & & \\
H^0(L^r) \otimes M_k \otimes L^{q+1} & \xrightarrow[\eta_p]{} & M_k \otimes L^{q+r+1} & \xrightarrow[\delta_p]{} & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

We get the following commutative diagram $(*)$ where each vertical line is exact;

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(L^r) \otimes H^0(M_1 \otimes M_k \otimes L^q) & \underset{\lambda_p'}{\longrightarrow} & H^0(M_{r+1} \otimes M_k \otimes L^q) \\
\downarrow & & \downarrow \\
H^0(L^r) \otimes H^0(L) \otimes H^0(M_k \otimes L^q) & \underset{\theta_p'}{\longrightarrow} & H^0(L^{r+1}) \otimes H^0(M_k \otimes L^q) \\
\downarrow & & \downarrow \\
H^0(L^r) \otimes H^0(M_k \otimes L^{q+1}) & \underset{\eta_p'}{\longrightarrow} & H^0(M_k^p \otimes L^{q+r+1}) \\
\downarrow & & \downarrow \varphi_p \\
0 \to & & H^1(M_{r+1} \otimes M_k \otimes L^q) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Consider the following exact sequence $(**)$;

$$0 \to M_r \otimes M_k \otimes L^{q+1} \to H^0(L^r) \otimes M_k \otimes L^{q+1} \underset{\eta_p}{\longrightarrow} M_k \otimes L^{q+r+1} \to 0$$

We prove (2) by induction on r when $a = 1$. When $r = 1$, we get that $\eta_p^{'}$ is surjective by induction hypothesis and the exact sequence $(**)$. By the diagram $(*)$,

$$H^1(X, M_2 \otimes M_k \otimes L^q) = 0, \text{ for all } q \geq 2.$$

Assume that there exists $r \geq 1$ such that for all $r_0 \leq r < k$;

$$H^1(X, M_r \otimes M_k \otimes L^q) = 0 \text{ for all } q \geq 2.$$

Since $(*)$ is a commutative diagram and since $\eta_p^{'}$ and φ_p are surjective, we get the following: $H^1(X, M_{r+1} \otimes M_k \otimes L^q) = 0$, for all $q \geq 2$. So we get $H^1(X, M_{L^{k_1}} \otimes M_{L^k} \otimes L^q) = 0$ for all $q \geq 2$ and $1 \leq k_1 \leq k$.

Second, suppose that there exists a such that

- $1 \leq a \leq p$.
- $H^1(X, \bigwedge^{t+1-a} M_L \otimes M_{L^k}^a \otimes L^q) = 0$, for $q \geq a+1$ and $a \leq t \leq p$.

- $H^1(X, M_{L^k}^{a+1} \otimes L^q) = 0$ for $q \geq a+1$ and $a \leq t \leq p$.

We show that (1) and (2) are true for $a+1$. If $a = p$, we are done. Assume that $a < p$. By induction and by Lemma 13 $\ker\beta_{a+1}$ $=\mathrm{Im}\beta_{1,a+1}$, $\mathrm{Im}\beta_{a+1} = \ker\beta_{2,a+1}$, $\ker\gamma_{a+1}$ $=\mathrm{Im}\gamma_{1,a+1}$, and $\mathrm{Im}\gamma_{a+1} = \ker\gamma_{2,a+1}$ in the diagram (⋇). Then we get that $\mathrm{Im}\alpha_{a+1} = \ker\alpha_{2,a+1}$. By Lemma 13, $H^1(X, \bigwedge^{t-a} M_L \otimes M_{L^k}^{a+1} \otimes L^q) = 0$ for all $q \geq a+2$ and $a+1 \leq t \leq p$. $H^1(X, M_L \otimes M_{L^k}^{a+1} \otimes L^q) = 0$ for all $q \geq a+2$.
We get the following commutative diagram where each vertical line is exact;

$$\begin{array}{ccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & H^0(L^r) \otimes M_1 \otimes M_k^{a+1} \otimes L^q & \to & M_{L^{r+1}} \otimes M_k^{a+1} \otimes L^q \\
 & \downarrow & & \downarrow \\
 & H^0(L^r) \otimes H^0(L) \otimes M_k^{a+1} \otimes L^q & \to & H^0(L^{r+1}) \otimes M_k^{a+1} \otimes L^q \\
 & \downarrow & & \downarrow \\
 & H^0(L^r) \otimes M_k^{a+1} \otimes L^{q+1} & \to & M_k^p \otimes L^{q+r+1} \to 0 \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}$$

Using the same way in the case $a = 1$ and by above diagram, we get

$$H^1(X, M_{L^{k_1}} \otimes M_{L^k}^{a+1} \otimes L^q) = 0 \text{ for all } q \geq a+2 \text{ and } 1 \leq k_1 \leq k.$$

Then we prove that (1) and (2) are true. So $H^1(X, M_{L^k}^{p+1} \otimes L^q) = 0$ for all $k \geq 1$ and $q \geq p+1$. Since $\Bbbk$ is an algebraically closed field with characteristic zero and $H^1(X, L^k) = 0$ for all $k \geq 1$, L^{k+p} satisfies N_p for all $k \geq 1$. $\square$

Definition 6. *For an ample and globally generated line bundle L on a projective variety X, a sheaf $\mathcal{F}$ is called $m-$regular with respect to L in the sense of Castelnuovo-Mumford, if*

$$H^i(X, \mathcal{F} \otimes L^{m-i}) = 0 \text{ for all } i \geq 1.$$

and L is called $m-$autoregular, if it is $m-$regular with respect to itself.

Lemma 14. ([13]) *Let L be an ample line bundle which is globally generated on a projective variety X and let*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Suppose that $\mathcal{F}_2$ is m-regular with respect to L, that $\mathcal{F}_3$ is $(m-1)$-regular with respect to L, and that the map of global sections is surjective, i.e.,

$$H^0(X, \mathcal{F}_2 \otimes L^{m-1}) \to H^0(X, \mathcal{F}_3 \otimes L^{m-1}) \text{ is surjective.}$$

Then $\mathcal{F}_1$ is m-regular with respect to L.

Theorem 9. ([13]) *Let L be an ample line bundle which is globally generated on a projective variety X. Let $\mathcal{F}$ be a sheaf on X that is m-regular with respect to L. Then for $t \geq 0$,*

(1) *$\mathcal{F} \otimes L^{m+t}$ is generated by its global sections.*
(2) *The natural map $H^0(X, \mathcal{F} \otimes L^m) \otimes H^0(X, L^t) \to H^0(X, \mathcal{F} \otimes L^{m+t})$ is surjective.*
(3) *$\mathcal{F}$ is $(m+t)-$regular with respect to L.*

We cite some definitions and facts from [9].

Definition 7. ([9]) *Let C be a smooth projective curve of genus g. For a vector bundle $\mathcal{F}$ on C, the slope is defined as $\mu(\mathcal{F}) = deg(\mathcal{F})/rank(\mathcal{F})$ and $\mu^-(\mathcal{F}) = min\{\mu(\mathcal{Q}) | \mathcal{F} \to \mathcal{Q} \to 0 \text{ is exact}\}$ where $\mathcal{Q}$ is a vector bundle on C.*

LEMMA 15. ([9] *Lemma 1.12*) *For vector bundles $\mathcal{F}, \mathcal{G}$ and $\mathcal{E}$ on a smooth projective curve C of genus g.*

(1) $\mu^-(\mathcal{F} \otimes \mathcal{G}) = \mu^-(\mathcal{F}) + \mu^-(\mathcal{G})$.
(2) *If* $\mu^-(\mathcal{E}) > 2g-2$, *then* $\dim_{\Bbbk} H^1(C, \mathcal{E}) = h^1(C, \mathcal{E}) = 0$.
(3) *If* $\mu^-(\mathcal{E}) > 2g-1$, *then* $\mathcal{E}$ *is globally generated.*
(4) *If* $\mu^-(\mathcal{E}) > 2g$, *then* $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ *is very ample.*

DEFINITION 8. ([9]) *Let E be a vector bundle over a scheme Y, and $X = \mathbb{P}(E) \xrightarrow{\pi} Y$. If $\mathcal{F}$ is a coherent sheaf over X, we say $\mathcal{F}$ is j π-regular when*

$$R^i\pi_*(\mathcal{F}(j-i)) = 0 \text{ for all } i > 0.$$

LEMMA 16. ([9]*Lemma 3.1*) *Let Y be a projective variety and E a rank n vector bundle over Y. If $X = \mathbb{P}(E) \xrightarrow{\pi} Y$ and V is a vector bundle over X, then the following are equivalent.*

(1) *V is j π-regular.*
(2) *For all $y \in Y$, $\dim_{\Bbbk} H^i(X_y, v_y^*(V(j-i))) = h^i(X_y, v_y^*(V(j-i))) = 0$ for all $i > 0$ where $v_y^*(V(j-i))$ is the pull-back of $V(j-i)$ to X_y.*

LEMMA 17. ([9]*Lemma 3.1*) *Let E be a vector bundle over a projective variety Y and $X = \mathbb{P}(E) \xrightarrow{\pi} Y$. If V and W are k and l π-regular vector bundles over X respectively, then*

(1) *$V \otimes W$ is $(k+l)$ π-regular.*
(2) *If $k \leq 0$, $l \leq 0$ and Y is a curve, then*

$$\mu^-(\pi_*(V \otimes W)) \geq \mu^-(\pi_*V) + \mu^-(\pi_*W).$$

THEOREM 10. ([9] *4.1 Theorem*) *(char$(\Bbbk) = 0$ or $g \leq 1$). Let E be a rank n vector bundle over a smooth projective curve C of genus g, and $X = \mathbb{P}(E)$. If V and*

W are 0 π-regular vector bundles over X, and B_1 and B_2 are (-1) π-regular vector bundles over X such that

(1) $\mu^-(\pi_* V) + \mu^-(\pi_* B_1) \geq 2g$ *and*

(2) $\mu^-(\pi_* W) + \mu^-(\pi_* B_2) > 2g$,

then the map $H^0(X, V \otimes B_1) \otimes H^0(X, W \otimes B_2) \to H^0(X, V \otimes B_1 \otimes W \otimes B_2)$ *is surjective.*

From this point, we follow the notation and terminology from [8]. We mean that a smooth complex projective surface is an integral smooth projective scheme of dimension 2 over $\mathbb{C}$. Let C be a curve of genus g, and let E be a vector bundle of rank 2 on C which is normalized, i.e., $H^0(C, E) \neq 0$ but for all invertible sheaves $\mathcal{F}$ on C with $deg(\mathcal{F}) < 0$, we have $H^0(C, E \otimes \mathcal{F}) = 0$. Let $X = \mathbb{P}_C(E)$ be the associated ruled surface with projection morphism $\pi : X \to C$. Then there is a section $\sigma : C \to X$ with image C_0, such that $\mathcal{L}(C_0) = \mathcal{O}_X(1)$. Let ε be the divisor on C corresponding to the invertible sheaf $\wedge^2 E$, denote $e = -deg(\varepsilon)$. Let us fix a section C_0 of X with $\mathcal{L}(C_0) = \mathcal{O}_X(1)$. If $\mathsf{b} \in \mathrm{Pic} C$, the we denote the divisor $\pi^* \mathsf{b}$ on X by $\mathsf{b} f$, by abuse of notation. Thus any element of Pic$X$ can be written $aC_0 + \mathsf{b} f$ with $a \in \mathbb{Z}$ and $\mathsf{b} \in \mathrm{Pic} C$. Let Num$X$ be a group of divisors modulo numerical equivalence. Then any element of Num$X$ can be written $aC_0 + bf$ with $a, b \in \mathbb{Z}$. We denote the fibre of π at $y \in C$ as X_y.

Remark 3. ([14]) *Under the situation just stated, we mention some basic facts about X:*

(1) *Let $L \in PicX$ be a line bundle in the numerical class of $aC_0 + bf$. If $\mu^-(\pi_* L) = b - ae \geq 2g+1$ and $a \geq 1$, then L is very ample and $H^1(X, L^t) = H^1(C, \pi_* L^t) = 0$ for all $t \geq 1$ by Lemma 15.*

(2) $R^i \pi_*(\mathcal{O}_X(l)) = 0$ *for all $l \geq -1$ and $i \geq 1$.*

(3) *Projection formula; let $g : Y \to T$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_T$-module of finite rank. Then*

$$R^i g_*(\mathcal{F} \otimes g^*\mathcal{E}) \cong R^i g_*(\mathcal{F}) \otimes \mathcal{E}.$$

The following is our second principal result.

THEOREM 11. *Let $\pi : X \to C$ be a ruled surface. Let L be a line bundle in the numerical class of $aC_0 + bf$ with the numerical invariant $e \geq 0$. Then L satisfies N_p if $\mu^-(\pi_*L) = b - ae \geq 2g + 1$ and $a \geq p + 1$.*

PROOF. Assume that $\mu^-(\pi_*L) = b - ae \geq 2g + 1$ and $a \geq p + 1 \geq 1$. By Remark 3, we know that L is very ample and $H^1(X, L^h) = 0$ for all $h \geq 1$. Let $v_y : X_y \to X$ be the natural map for all $y \in C$. Note that if T and M are sheaves on X, then for all $n \in \mathbb{Z}$,

$$v_y^*(T \otimes_{\mathcal{O}_X} M) = v_y^*(T) \otimes_{\mathcal{O}_{X_y}} v_y^*(M), \text{ and } v_y^*\mathcal{O}_X(n) = (v_y^*\mathcal{O}_X(1))^{\otimes n}.$$

By Remark 3, $R^i\pi_*(\mathcal{O}_X(l)) = 0$ for all $l \geq -1$ and $i \geq 1$. By Lemma 16, for all $l \geq -1, i \geq 1$ and all $y \in C$,

$$H^i(X_y, v_y^*\mathcal{O}_X(l)) = H^i(X_y, v_y^*\mathcal{O}_X(a) \otimes_{\mathcal{O}_{X_y}} (v_y^*\mathcal{O}_X(1))^{\otimes l-a}) = 0,$$

i.e., $v_y^*\mathcal{O}_X(a)$ is $(l - a + 1)$-regular with respect to $v_y^*\mathcal{O}_X(1)$ for all $l \geq -1$ and all $y \in C$. Since $a \geq 1$ and $l \geq -1$, $v_y^*\mathcal{O}_X(a)$ is (-1)-regular with respect to $v_y^*\mathcal{O}_X(1)$ for all $y \in C$. By Theorem 9, we get that $v_y^*\mathcal{O}_X(a)$ is r-regular with respect to $v_y^*\mathcal{O}_X(1)$ for all $y \in C$ and $r \geq -1$. Then $\mathcal{O}_X(a)$ is r π-regular for $r \geq -1$. By projective formula, L is r π-regular for $r \geq -1$, i.e., for all $y \in C$, $i \geq 1$ and $r \geq -1$,

$$H^i(X_y, v_y^*(L \otimes \mathcal{O}_X(r - i))) = H^i(X_y, v_y^*(L) \otimes_{\mathcal{O}_{X_y}} (v_y^*\mathcal{O}_X(1))^{r-i}) = 0.$$

So $v_y^*(L)$ is r-regular with respect to $v_y^*(\mathcal{O}_X(1))$ for all $y \in C$ and $r \geq -1$. Since $\mu^-(\pi_*L) \geq 2g + 1$ and by Lemma 15, π_*L is globally generated. It means that α :

$H^0(X, L) \to H^0(X_y, v_y^*(L))$ is surjective for all $y \in C$, since $H^0(X, L) \cong H^0(C, \pi_* L)$. Consider the exact sequence; for all $y \in C$,

$$0 \to v_y^*(M_L) \to H^0(L) \otimes v_y^*(\mathcal{O}_X) \to v_y^*(L) \to 0.$$

Using the fact that cohomology commutes with tensoring with a vector space, $H^0(X, L) \otimes v_y^*(\mathcal{O}_X)$ is s-regular with respect to $v_y^*(\mathcal{O}_X(1))$ for all $y \in C$ and all $s \geq 0$. Since α is surjective and by Lemma 16, $v_y^*(M_L)$ is 1-regular with respect to $v_y^*(\mathcal{O}_X(1))$ for all $y \in C$ and $v_y^*(M_L \otimes \mathcal{O}_X(1))$ is 0-regular with respect to $v_y^*(\mathcal{O}_X(1))$ for all $y \in C$. So M_L is 1 π-regular and $M_L \otimes \mathcal{O}_X(1)$ is 0 π-regular. Also we can check that $L^h \otimes \mathcal{O}_X(-t)$ is (-1) π-regular for all $h \geq 1$ and $0 \leq t \leq p$. We have the following facts; for all $h \geq 1$ and $0 \leq t \leq p$,

(1) $(M_L \otimes \mathcal{O}_X(1))^{\otimes t}$ is 0 π-regular by Lemma 17.
(2) $L^h \otimes \mathcal{O}_X(-t)$ is (-1) π-regular.
(3) $\mu^-(\pi_* L^h \otimes \mathcal{O}_X(-t)) = hb - (ha - t)e \geq h(b - ae) = h\mu^-(\pi_* L) \geq 2g + 1$.
(4) $\mu^-(\pi_* \mathcal{O}_X) + \mu^-(\pi_* L) \geq 2g + 1$.

By Theorem 10, we know that for all $h \geq 1$ and $0 \leq t \leq p$,

(1) $\quad H^0(X, M_L^t \otimes L^h) \otimes H^0(X, L) \to H^0(X, M_L^t \otimes L^{h+1})$ is surjective.

Consider the following exact sequences; for all $h \geq 1$ and $0 \leq t \leq p$,

(2) $\quad 0 \to M_L^{t+1} \otimes L^h \to H^0(X, L) \otimes M_L^t \otimes L^h \to M_L^t \otimes L^{h+1} \to 0.$

Taking cohomology of (2), we get the following exact sequences; for all $h \geq 1$ and $0 \leq t \leq p$,

$$\cdots \to H^0(X, L) \otimes H^0(X, M_L^t \otimes L^h) \to H^0(X, M_L^t \otimes L^{h+1}) \to H^1(X, M_L^{t+1} \otimes L^h) \to$$
$$H^0(X, L) \otimes H^1(X, M_L^t \otimes L^h) \to \cdots .$$

Fix t such that $0 \leq t \leq p$. If $H^1(X, M_L^t \otimes L^h) = 0$, then $H^1(X, M_L^{t+1} \otimes L^h) = 0$ by (1). It suffices to show that $H^1(X, L^h) = 0$ for all $h \geq 1$, but it is true by Remark 3.

So for $0 \leq t \leq p+1$ and all $h \geq 1$, $H^1(X, M_L^t \otimes L^h) = 0$ if $\mu^-(\pi_* L) = b - ae \geq 2g+1$ and $a \geq p+1$. By Theorem 7, we get our result. $\square$

THEOREM 12. *Let $\pi : X \to C$ be a ruled surface where C has genus g. Let L be a line bundle in the numerical class of $aC_0 + bf$ with the numerical invariant $e \geq 0$. Then $K_X \otimes L^{\otimes(p+1)}$ satisfies N_p if $a \geq 3$ and $\mu^-(\pi_* L) = b - ae \geq 3$.*

PROOF. We know that $K_X \otimes L^{\otimes(p+1)}$ is the line bundle in the numerical class of $(-2+(p+1)a)C_0 + (2g-2-e+(p+1)b)f$. Then

$$(2g-2-e+(p+1)b) - e(-2+(p+1)a) - 2g - 1 = (p+1)(b-ea) - 3 + e \geq 0$$
$$-2+(p+1)a \geq p+1$$

By Theorem 11, $K_X \otimes L^{\otimes(p+1)}$ satisfies N_p. Actually $K_X \otimes L^{\otimes(p+1)}$ satisfies $N_{-2+(p+1)a}$. $\square$

REMARK 4. *Let $\pi : X \to C$ be a ruled surface. Let L be an ample line bundle in the numerical class of $aC_0 + bf$ with the numerical invariant $e \geq 0$. Then L is very ample and nef. By Theorem in* [10]*, we know that $K_X \otimes L^{\otimes(p+4)}$ satisfies N_p.*

Bibliography

[1] Polishcuck, A. and Positselski, L. (2000). *Quadratic algebras*. Preprint.

[2] Priddy, S. B. (1970). *Koszul resolutions*. Transactions of the American Mathematical Society. 152, 39-60.

[3] Manin, Yu.I. (1988). *Quantum Groups and Non-Commutative Geometry*, Centre de Recherches Mathematiques.

[4] Positselski, L. (1995). *Relation between the Hilbert series of dual quadratic algebras does not imply Koszulity*. Functional Analysis and Its Applications, 29(3), 213-217.

[5] Tate, John and van den Bergh, Michel. (1996). *Homological properties of Sklyanin algebras*. Inventiones Math. 124, 1-3, 619-649.

[6] Green, M. (1984). *Koszul cohomology and the geometry of projective varieties*, Journal of Differential Geometry. 19, 125-171.

[7] Green, M. and Lazarsfeld, Robert. (1985) *On the projective normality of complete linear series on an algebraic curve*, Inventiones Mathematicae. 83, no. 1, 73-90.

[8] Cox, David and Dickenstein, Alicia (2003) *Vanishing and codimension theorems for complete toric varieties*, Math.AG/0310108 v1.

[9] Butler, David C. (1004) *Normal generation of vector bundles over a curve*, Journal of Differential Geometry. 39, p.1-34.

[10] Park, Euisung. (2004) *On higher syzygies of ruled surface*, arXiv.math.AG/0401100 v3 19 Jan 2004.

[11] Parschi, Giuseppe. (2000) *Syzygies of abelian varieties*, Journal of the American Mathematical society, v13, n3, 651-664.

[12] Ein, L and Lazarsfeld, R. (1993) *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Inventiones Mathematicae. 111(1), 51-67.

[13] Hering, Milena (2004) *Syzyies,regularity and toric varieties*, arXiv:math. AG/0402328 v1.

[14] Hartshorne, Robin. (1997) *Algebraic Geometry*, no.52, Springer-Velag New York.

[15] Rubei, Elena. (2001) *On Koszul rings,syzygies,and abelian varieties*. Communications in algebra. 29(12), 5631-5740.

[16] Rubei, Elena. (2001) *A strange example concering property of* N_p arXiv:math. AG/0111168.

CURRICULUM VITAE

1971 born in Seoul, Korea.

1994 B.S in Mathematics, Seoul Women's University, Korea.

1995 Studying in graduate school, Mathematics,
Seoul Women's University, Korea.

2006 Ph.D in Mathematics, Indiana University.