LINEAR ALGEBRAIC PROPERTIES OF C_0-OPERATORS

Yun-Su Kim

Submitted to the faculty of the University Graduate School
in partial fulfillment of the requirements
for the degree
Doctor of Philosophy
in the Department of Mathematics
Indiana University
July, 2006

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Hari Bercovici, Ph.D.

Eric Bedford, Ph.D.

Scott Brown, Ph.D.

Bill Ziemer, Ph.D.

May 8, 2006

To my parents.

Acknowledgements

It is a great pleasure to acknowledge my thesis advisor, Professor Hari Bercovici, for the many hours of his guidance and support throughout the 6 years he has been my advisor. His responsible advising made me study hard and helped me learn how to research. I also thank the other members of my committee: Professor Eric Bedford for helping me as an editor, Professor Scott Brown for encouragement and support, and Professor Bill Ziemer for introducing me to Professor Hari Bercovici and inspiring self-confidence in me.

I would like to thank other faculties and good friends. Especially, Professor David Hoff, Professor Jee Heub Koh, Professor Ayelet Lindenstrauss, for their kindness, and Graduate Secretary Misty Cummings who was one of my best friends in Bloomington. I am indebted to my good friends in the Mathematics department, in particular Cristina Tone, Florentina Tone, Maiying Kong, Yesim Kurt, Jiexiang Li, Onur Yavuz, Qayum Khan, and Frederic Picard.

I would like to appreciate Andrea Becker who is both my English tutor and a good friend, and Erica Isaacson who included me in multiple holiday celebrations.

I would like to express my heartfelt thanks to Professor Byung Gai Kang at Sungshin Women's University for 16 years of support, Professor Byung Sik In at Sungshin Women's University for his kindness, and Dr. Hyunwoo Lee, a minister in Chicago, who was a professor in the Mathematics department when I was an undergraduate student and now is a wonderful advisor of life.

I was very lucky to meet wonderful friends Hyangmi Bang, Junghwa Son, Hyekyung Kim, Jungwon Kim, Hyunjoo Lee, and Jungsook Lee.

Finally, and most of all, I would like to thank my family who have given me everlasting support and encouragement with love: My father and mother for trusting me, my brother for supporting my parents and my sister for her warmth.

Abstract

Yun-Su Kim

Linear Algebraic Properties of C_0-operators

The theory of Jordan models for contractions is due to B. Sz.-Nagy - C. Foias, B. Moore - E.A. Nordgren, and H. Bercovici - D. Voiculescu. J.A. Ball introduced the class of C_0-operators relative to a multiply connected domain Ω. A. Zucchi provided a classification of C_0-operators relative to Ω. Since no analogue of the characteristic function of a contraction is available in that context, that study does not yield some of the results available for the unit disk. In this thesis we use a substitute for the characteristic function, suggested by an analogue of Beurling's theorem provided by M.A. Abrahamse and R.G. Douglas. This allows us to prove a relationship between the Jordan models of a C_0-operator relative to Ω, of its restriction to an invariant subspace, and of its compression to the orthocomplement of that subspace.

This thesis is organized as follows. In Chapter 2, by defining a *quasi-inner function*, we provide a generalized Beurling's Theorem. In Chapter 3, we primarily deal with C_0-operators relative to Ω. Finally, in Chapter 4, we study the modular lattice for C_0-Operators relative to the open unit disc.

Contents

CHAPTER 1

Introduction

An operator T in the set of bounded linear operators from a Hilbert space H to H, denoted by $L(H)$, is said to be *completely nonunitary* if there is no invariant subspace M for T such that the restriction $T|M$ of T to the space M is a unitary operator. B. Sz.-Nagy and C. Foias introduced a weak*-continuous functional calculus [**29**]

$$\Phi_1 : H^\infty \to L(H)$$

where H^∞ is the Banach algebra of bounded and analytic functions on $\mathbf{D}$ with the supremum norm. A completely nonunitary contraction T is an operator of class C_0 relative to the open unit disk $\mathbf{D}$ if the associated functional calculus has a non-trivial kernel. In fact, the kernel of a C_0-operator T relative to $\mathbf{D}$ is a principle ideal generated by an inner function called a *minimal function* of T. The C_0-operators relative to the open unit disk $\mathbf{D}$ was introduced by B. Sz.-Nagy [**28**]. One of important results is a classification theorem using Jordan model which is similar to Jordan's classical result. The theory of Jordan models of C_0-operators relative to the open unit disk $\mathbf{D}$ was studied by Sz.-Nagy and Foias [**29**], and B. Moore-E.A. Nordgren [**22**].

The study of C_0-operators relative to the open unit disk $\mathbf{D}$ was continued by H. Bercovici-D. Voiculescu [**6**], [**9**], [**8**] who gave other intrinsic characterizations of C_0-operators relative to the open unit disk $\mathbf{D}$, and introduced C_0-Fredholm operators.

As a fundamental example of C_0-operators relative to the open unit disk $\mathbf{D}$, for a given inner function $\theta \in H^\infty$, the *Jordan block* $S(\theta)$ is a C_0-operator acting on $H(\theta)$, the orthogonal complement of θH^2 in the Hardy space H^2 defined by :

$$S(\theta) = P_{H(\theta)} S | H(\theta) \tag{0.1}$$

where $S \in L(H^2(\Omega))$ is the shift operator defined by

$$(Sf)(z) = zf(z).$$

When we study Hardy space, a useful result is Beurling's theorem. First of all, it could be used when we find a minimal function of C_0-operators relative to $\mathbf{D}$. The paper [**9**] of Beurling is the immediate starting point for invariant subspaces. Beurling described every closed S-invariant subspaces in [**9**]. His characterization was obtained by means of function-theoretic analysis of individual analytic functions.

By giving an analogue (in the scalar case) of Beurling's theorem on invariant subspaces of the Hardy spaces of the unit disc, Sarason [**26**], Voivhick [**30**], and Hasumi [**18**] started operator theory related to function theory on multiply connected domains. It was continued in the work of Abrahamse−Douglas [1, 2], and of Ball [4, 5]. J. Agler [3] showed that the existence of normal doundary dilations − an analogue of Sz.-Nagy dilation theorem − still holds for annuli and it may fail for domains of connectivity greater than two (Dritschel−McCullough [**14**]). However it holds up to similarity (Douglas−Paulsen [**13**]); this allowed A. Zucchi [**31**] to provide a classification of C_0-operators relative to Ω. Instead of completely nonunitary operator, they consider an operator satisfying some requirements, called *hypothesis (h)*. By the same way as C_0-operators relative to $\mathbf{D}$ in [**29**], they provided a weak*-continuous functional calculus

$$\Phi_2 : H^\infty(\Omega) \to L(H)$$

where $H^\infty(\Omega)$ is the Banach algebra of bounded and analytic functions on Ω with the supremum norm. An operator T satisfying hypothesis (h) is said to be of *class* C_0-*operators relative to* Ω if the associated functional calculus has a non-trivial kernel. For each C_0-operator T relative to Ω, A. Zucchi found a model function

$$\Theta = \{\theta_\alpha \in H^\infty(\Omega) : \alpha < \gamma\}$$

a family of inner functions such that T is quasisimilar to

$$S(\Theta) = \bigoplus_{\alpha<\gamma'} S(\theta_\alpha)$$

called *Jordan model* of T.

In this paper, we study some linear algebraic properties for Jordan models of C_0-operators relative to multiply connected domains Ω in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves which is the same domain used by Ball [**4**] and Zucchi [**31**].

Sz.-Nagy and Foias introduced a *functional model* $S(\Theta_1)$ (which is also defined in the same way as equation 0.1) associated with an operator-valued inner function

$$\Theta_1 : \mathbf{D} \to L(H)$$

and proved that it is also a C_0-operators relative to $\mathbf{D}$. In order to define the similar function $S(\varphi)$ for an operator-valued function φ defined on Ω and to consider whether it is also a C_0-operator relative to Ω or not, in Chapter 2, we start extending Beurling's theorem.

In [**24**], H.L. Royden proved that for every fully invariant subspace M_1 of $H^p(\Omega)(1 \leq p \leq \infty)$, there is an inner function θ in $H^\infty(\Omega)$ such that

$$M_1 = \theta H^p(\Omega).$$

M.A. Abrahamse and R.G. Douglas also provided a generalization of Beurling's theorem. In their study of rationally invariant subspaces, they produced a bundle shift operator and inner bundle map in [**2**]. They proved that a closed subspace M_2 of $H^2(\Omega, E)$ is $R(\Omega)$-invariant for the bundle shift operator if and only if

$$M_2 = \Theta_2 H^2(\Omega, F),$$

where F is a flat unitary vector bundle over Ω and Θ_2 is an inner bundle map from F to E. To use this result, we provide a definition of a *quasi-inner function* in Definition 2.11. The proper setting here is that of maps of flat unitary vector bundles, i.e., of multiplicative multivalued operator-valued functions. We will convert these to usual single valued analytic functions by composing them with some bundle isomorphism. This has been done quite often in the scalar case where one can be somewhat more explicit and choose a multiplicative outer function to have constant absolute value on each boundary component, see, e.g., Royden [**24**]. Using the *quasi-inner function* instead of inner bundle map, we provide another generalization of Beurling's theorem in Theorem 2.14 without using vector bundles :

Let K be a Hilbert space. Then a closed subspace M of $H^2(\Omega, K)$ is $R(\Omega)$-invariant for the shift operator S_K if and only if there is a Hilbert space K' and quasi-inner function $\varphi : \Omega \to L(K', K)$ such that

$$M = \varphi H^2(\Omega, K').$$

To use this result, we need to study quasi-inner functions. As one of the characterizations of quasi-inner functions, in Corollary 2.17, we prove that for a function $\varphi \in H^\infty(\Omega, L(K_1, K_2))$ (K_i is a Hilbert space for $i = 1, 2$), following statements are equivalent :

(a) *φ is quasi-inner.*

(b) *The multiplication operator M_φ defined on Hardy space $H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is one-to-one and has closed range where M_φ is defined by $(M_\varphi f)(z) = \varphi(z) f(z)$ for $f \in H^2(\Omega, K_1)$.*

When we compare two subspaces $\varphi_1 H^2(\Omega, K_1)$ and $\varphi_2 H^2(\Omega, K_2)$ of $H^2(\Omega, K)$ for quasi-inner functions $\varphi_1 : \Omega \to L(K_1, K)$ and $\varphi_2 : \Omega \to L(K_2, K)$, we find some relationships between $\varphi_1 : \Omega \to L(K_1, K)$ and $\varphi_2 : \Omega \to L(K_2, K)$ in Corollary 2.16, that is,

$\varphi_1 H^2(\Omega, K_1)$ and $\varphi_2 H^2(\Omega, K_2)$ of $H^2(\Omega, K)$ are equal if and only if there exist a function $\varphi \in H^\infty(\Omega, L(K_1, K_2))$ such that $\varphi(z)$ is invertible for any $z \in \Omega$, $\sup \|\varphi(z)^{-1}\| < \infty$ *and*

$$\varphi_1(z) = \varphi_2(z)\varphi(z)$$

for any $z \in \Omega$.

From this relationship, we begin discussing the greatest common quasi-inner divisor in section 4 in Chapter 2 and we prove some characterizations of a quasi-inner divisor which are similar to results of an inner divisor in H^∞ ([**6**]) :

For any quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, the following assertions are equivalent:

(a) $\theta | \varphi$.
(b) $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$.
(c) $\varphi H^2(\Omega, K) \subset \theta H^2(\Omega, K)$.
(d) *There is a $\lambda > 0$ such that $\varphi(z)\varphi(z)^* \leq \lambda^2 |\theta(z)|^2 I_{\mathbb{C}^n}$ for any $z \in \Omega$.*

In Chapter 3, we deal primarily with C_0-operators relative to Ω. As desired before, we provide a C_0-operator $S(\varphi)$ for a quasi-inner function φ defined on Ω. To determine whether $S(\varphi)$ is a C_0-operator or not for a quasi-inner function φ defined on Ω, we use the concept of a "*scalar multiple*" the same way as on **D** and in Theorem 3.16 we describe which functions in $H(\Omega)$ belong to the kernel of the associated functional calculus. It follows that the least scalar multiple of φ is the minimal function of $S(\varphi)$.

For a quasi-inner function φ on Ω, $S(\varphi)$ is not always a C_0-operator relative to Ω. However, in Proposition 3.17, we show that for a quasi-inner function φ on Ω whose range is an operator on a finite dimensional Hilbert space, $S(\varphi)$ is a C_0-operator relative to Ω. Thus we primarily study with quasi-inner functions on Ω whose range is $L(\mathbb{C}^n)$. Further results in Proposition 3.17 provide another important property of a quasi-inner function φ on Ω whose range is an operator on a finite dimensional Hilbert space, that is, $\varphi(z)$ is invertible almost everywhere on $\partial\Omega$.

In section 3 of Chapter 3, we study some relationships between quasi-equivalence and quasi-similarity. First, Proposition 3.29 shows the following fundamental fact :

Let φ_1 and φ_2 be quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. If φ_1 and φ_2 be quasi-equivalent, then $S(\varphi_1)$ and $S(\varphi_2)$ are quasi-similar.

Naturally we have a question whether the converse of Proposition 3.29 is true or not and so in Corollary 3.31 we also derive :

Let φ_1 and φ_2 be quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. If $S(\varphi_1)$ is a quasi-affine transform of $S(\varphi_2)$, then φ_1 and φ_2 are quasi-equivalent.

From number theory, we know some useful facts for relatively prime numbers. When we consider two relatively prime quasi-inner functions on Ω, we obtain an interesting result which is used when we prove Proposition 3.29 :

Let $\varphi_1, \varphi_2 \in H^\infty(\Omega)$ be such that $\varphi_1 \wedge \varphi_2 \equiv 1$. If $f \in L^2(\partial\Omega, \mathbb{C}^n)$, $\varphi_1 f \in H^2(\partial\Omega, \mathbb{C}^n)$ and, $\varphi_2 f \in H^2(\partial\Omega, \mathbb{C}^n)$, then $f \in H^2(\partial\Omega, \mathbb{C}^n)$.

From the characterization of $R(\Omega)$-invariant subspaces for the shift operator in Theorem 2.14, we characterize a rationally invariant subspace for $S(\varphi)$ in Theorem 3.33 :

Let F and F' be two separable Hilbert spaces and φ be a quasi-inner function in $H^\infty(\Omega, L(F, F'))$.

(i) If $M \subset H(\varphi)$ is a rationally invariant subspace (i.e. $R(\Omega)$-invariant) for $S(\varphi)$, then there is a Hilbert space K and quasi-inner functions $\varphi_1 \in H^\infty(\Omega, L(F, K))$ and $\varphi_2 \in H^\infty(\Omega, L(K, F'))$ such that

$$\varphi(z) = \varphi_2(z)\varphi_1(z)$$

for $z \in \Omega$ and

$$M = \varphi_2 H^2(\Omega, K) \ominus \varphi H^2(\Omega, F) \tag{0.2}$$

(ii) Conversely, if K, φ_1 and φ_2 are as above , then (0.2) defines a rationally invariant subspace of $H(\varphi)$. Moreover, if

$$S(\varphi) = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is the triangularization of $S(\varphi)$ with respect to the decomposition $H(\varphi) = M \oplus (H(\varphi) \ominus M)$, then $T_2 = S(\varphi_2)$ and $S(\varphi_1)$ is similar to T_1.

In [**8**], H. Bercovici and D. Voiculescu proved following fact :

Let H be a separable Hilbert space. Assume that $T \in L(H)$ is an operator of class C_0 relative to $\boldsymbol{D}$, H' is an invariant subspace for T, and $T = \begin{pmatrix} T' & Y \\ 0 & T'' \end{pmatrix}$ is the triangularization of T with respect to the decomposition $H = H' \oplus (H \ominus H')$.

If $\oplus_{j<\omega} S(\theta_j)$, $\oplus_{j<\omega} S(\theta'_j)$, and $\oplus_{j<\omega} S(\theta''_j)$ are the Jordan models of T, T', T'', respectively, then

$$\theta_0 \cdots \theta_{k-1} | \theta'_0 \cdots \theta'_{k-1} \theta''_0 \cdots \theta''_{k-1}, \tag{0.3}$$

for every $k = 1, 2, 3, \ldots.$

To obtain a result similar to 0.3 for a C_0-operator T relative to Ω, we define a $R(\Omega)$-*generating set* and a *multiplicity* of T. In Proposition 3.36, we derive a similar result for a C_0-operator relative to Ω with finite multiplicity. Furthermore, we finish Chapter 3 by proving the same result without the assumption of finite multiplicity in Theorem 3.38.

We know that the collection of all subspaces of a Hilbert space is a *lattice*. A lattice is called *modular* if

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee \mathbf{N} \tag{0.4}$$

for any element $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ in the lattice such that $\mathbf{N} \subset \mathbf{L}$. In Chapter 4, we study the modular lattice for C_0-Operators relative to $\mathbf{D}$. An operator $T \in L(H)$ is said to have *Property (P)* if every injective operator in the commutant of T has dense range. Thus if $T \in L(H)$ has Property (P), then every injective operator in the commutant of T is a quasiaffinity. Property (P) of an operator was introduced by H. Bercovici in [**9**] and he characterized C_0-operators relative to $\mathbf{D}$ with Property (P).

Whenever $X \in \{A \in L(H) : AT_1 = T_2A\}$ for $T_i \in L(H)(i = 1, 2)$, we have a function

$$X_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$$

where $\mathtt{Lat}(T_i)$ denotes the collection of all closed invariant subspaces for T_i defined by

$$X_*(M) = (XM)^-. \tag{0.5}$$

If T_1 and T_2 are similar, and $\mathtt{Lat}(T_1)$ is modular, then clearly, $\mathtt{Lat}(T_2)$ is also modular. In Section 1.3 of Chapter 4, we consider when T_1 and T_2 are quasi-similar instead of similar. When $T_1 \in L(H_1)$ is a quasiaffine transform of $T_2 \in L(H_2)$ and

$$Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$$

is a quasiaffinity, $\mathtt{Lat}(T_2)$ might not be modular, even though $\mathtt{Lat}(T_1)$ is modular. However in Theorem 4.14, we prove :

If $Y_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$ is onto and $\mathtt{Lat}(T_1)$ is modular, then $\mathtt{Lat}(T_2)$ is also modular.

As an example of C_0-operator T such that $\mathtt{Lat}(T)$ is modular, we have the Jordan block, that is, $S(\theta)$ for an inner function θ in H^∞. For $\Theta = \{\theta_\alpha : \alpha < \gamma\}$ (γ is a cardinal number) and the Jordan operator $S(\Theta) = \bigoplus_{\alpha<\gamma} S(\theta_\alpha)$, the lattice of hyperinvariant subspaces for $S(\Theta)$ is also modular. However it is not proven yet whether the lattice of hyperinvariant subspaces for any C_0-operator relative to $\mathbf{D}$ is modular or not. For a lattice of all closed invariant subspaces for a C_0-operator relative to $\mathbf{D}$, we obtain the following result in Theorem 4.19 :

Let $T \in L(H)$ be an operator of class C_0 with property (P). Then $\mathtt{Lat}(T)$ is a modular lattice.

CHAPTER 2

Shift Operators

In this chapter,we summarize basic results about Hardy spaces relative to the open unit disk and also provide fundamental results about Hardy spaces over multiply-connected domains. References for this material are [**15**], [**25**], [**2**], [**19**] and [**31**].

We also present the standard definitions of analytic vector bundle and flat unitary vector bundle. References for this material are Steenrod [**27**], Husemoller [**21**], Abrahamse and Douglas [**2**]. In section 1, we present a shift operator S_K on a vector valued Hardy space over a multiply-connected domain and in section 3, we provide a generalized Beurling's theorem for the operator S_K.

1. Hardy spaces

Let Ω be a bounded finitely connected region in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves and let z_0 be a point of Ω. For each real-valued continuous function u on $\partial\Omega$, we associate a real number $\breve{u}(z_0)$, where $\breve{u}$ is the harmonic extension of u to Ω. The map $u \to \breve{u}(z_0)$ is linear and, because of the maximum principle, it is bounded. Thus, the Riesz representation theorem implies that there is a unique Borel probability measure m, called the *harmonic measure* on $\partial\Omega$ for z_0, such that

$$\breve{u}(z_0) = \int_{\partial\Omega} u \; dm$$

for every real-valued continuous function u on $\partial\Omega$.

For $1 \leq p < \infty$, the space $H^p(\Omega)$ is defined to be the space of analytic functions f on Ω such that the subharmonic function $|f|^p$ has a harmonic majorant on Ω. For a fixed $z_0 \in \Omega$, there is a norm on $H^p(\Omega)$ defined as follows :

$$\|f\|_p = \inf\{u(z_0)^{1/p} : u \text{ is a harmonic majorant of } |f|^p\}.$$

Let m be harmonic measure for the point z_0 and let $L^p(\partial\Omega)$ be the L^p-space of complex valued functions on the boundary of Ω with respect to m. If f is in $H^p(\Omega)$, then there is a function f^* in $L^p(\partial\Omega)$ such that $f(z)$ approaches $f^*(\lambda_0)$ as z approaches λ_0 nontangentially for almost every λ_0 relative to m.

Proposition 2.1. *The map $f \to f^*$ is an isometry from $H^p(\Omega)$ onto a closed subspace, denoted $H^p(\partial\Omega)$, of $L^p(\partial\Omega)$.*

In this way, $H^p(\Omega)$ can be viewed as a closed subspace of $L^p(\partial\Omega)$.

A function f defined on Ω is in $H^\infty(\Omega)$ if it is holomorphic and bounded. $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Omega)$ and it is a Banach algebra if endowed with the supremum norm. Finally, the mapping $f \to f^*$ is an isometry of $H^\infty(\Omega)$ onto a week*-closed subalgebra of $L^\infty(\partial\Omega)$. If Ω is the open unit disk $\mathbf{D}$, we denote $H^p(\mathbf{D})$ simply by $H^p(1 \leq p \leq \infty)$.

We recall that a sequence $\{f_n\}_{n=1}^\infty$ of elements in $H^\infty(\Omega)$ is weak*-convergent if and only if it is boundedly pointwise convergent, i.e., it converges pointwise and $\sup_{n\in\mathbf{N}} \|f_n\|_\infty$ is finite.

Proposition 2.2. [**16**] *Let $R(\Omega)$ be the algebra of rational functions with poles off $\overline{\Omega}$. Then $R(\Omega)$ is sequentially boundedly weak*-dense in $H^\infty(\Omega)$, more precisely, for any $f \in H^\infty(\Omega)$ there exists a sequence $\{r_n\}_{n=1}^\infty$ in $R(\Omega)$ weak*-convergent to f, and such that* $\max\{|r_n(z)| : z \in \overline{\Omega}\} \leq \|f\|_\infty$ *for all n.*

As is frequently the case in the unit disk, it is necessary to consider vector-valued Hardy spaces. If K is a Hilbert space, then $H^p(\Omega, K)$ is defined to be the space of analytic functions $f : \Omega \to K$ such that the subharmonic function $\|f\|^p$ is majorized by a harmonic function u. The norm on $H^p(\Omega,K)$ is defined as in the scalar case :

$$\|f\|_p = \inf \{u(z_0)^{1/p} : u \text{ is a harmonic majorant of } \|f\|^p\}.$$

Let m denote harmonic measure on $\partial\Omega$ relative to z_0 and let $L^p(\partial\Omega,K)$ be the L^p-space of K-valued functions on the boundary of Ω defined with respect to m. Let $H^p(\partial\Omega,K)$ be the set of functions f in $L^p(\partial\Omega,K)$ such that $\int_{\partial\Omega} f(z)g(z)dz = 0$ for every function g analytic in a neighborhood of the closure of Ω. As in the scalar case, one can show that a function f in $H^p(\Omega, K)$ has nontangential limits a.e. relative to m, that the boundary function f^* is in $H^p(\partial\Omega, K)$, and the map $f \to f^*$ is an isometry from $H^p(\Omega, K)$ onto $H^p(\partial\Omega, K)$. As in the scalar case, $H^p(\Omega, K)$ can be identified with a closed subspace of the space $L^p(\partial\Omega, K)$.

Define a shift operator $S_K : H^2(\Omega, K) \to H^2(\Omega, K)$ by

$$(S_K f)(z) = zf(z). \tag{1.1}$$

Let $\mathbb{C}$ and $L(K)$ be the set of complex numbers and the algebra of bounded linear operators from K to K respectively. Let $H^\infty(\Omega, L(K))$ denote the algebra of bounded $L(K)$-valued analytic functions on Ω, which we also consider as consisting of multiplication operators on $H^2(\Omega, K)$, that is, for each $g \in H^\infty(\Omega, L(K))$ we define a *multiplication operator* $M_g : H^2(\Omega, K) \to H^2(\Omega, K)$ by

$$(M_g f)(z) = g(z)f(z). \tag{1.2}$$

Evidently, for each $g \in H^\infty(\Omega, L(K))$, M_g is a bounded linear operator with norm $\|M_g\| = \|g\|_\infty$. Our work is based on the following result due to M.B. Abrahamse and R.G. Douglas.([**2**], Proposition 1.9)

Proposition 2.3. *The commutant* $(S_K)' = \{X \in L(H^2(\Omega, K)) : S_K X = X S_K\}$ *of* S_K *is* $H^\infty(\Omega, L(K))$.

2. Vector Bundles

2.1. Analytic Vector Bundles. We now present in this section and next section, the standard definitions of an analytic vector bundle and a flat unitary vector bundle. A family of Hilbert spaces over Ω is a topological space E together with:

(1) A continuous map p from E onto Ω.

(2) A complex Hilbert space structure on each fiber $E_z = p^{-1}(z)$, z in Ω, such that the Hilbert space topology on E_z is the same as the topology inherited from E.

If K is a Hilbert space and if U is an open subset of Ω, then a *coordinate mapping* from $U \times K$ into E is a homeomorphism $\varphi : U \times K \to p^{-1}(U)$ such that:

(1) For each (z, k) in $U \times K$, the point $\varphi(z, k)$ is in E_z.

(2) For each z in U, the map $\varphi^z(k) : K \to E_z$ defined by $\varphi^z(k) = \varphi(z, k)$ is a continuous linear isomorphism.

If each point z in Ω is contained in an open set U of Ω for which there is a Hilbert space K and a coordinate mapping from $U \times K$ into E, then the family of Hilbert spaces is said to be *locally trivial.* A locally trivial family of Hilbert spaces over Ω is called a *vector bundle* over Ω. Notice that local triviality implies that the function $z \to \dim E_z$ is locally constant and since Ω is connected, this function must be constant throughout Ω. The *dimension of* E is defined to be the common dimension of E_z.

An *isomorphism* between two bundles E and E' over Ω is a homeomorphism φ from E onto E' such that for each z in Ω, the restriction of φ to E_z is a continuous linear isomorphism of E_z onto E'_z. A vector bundle over Ω is said to be *trivial* if it is isomorphic to the product bundle $\Omega \times K$.

To define analytic vector bundles, let E be a vector bundle over Ω and let K be a Hilbert space with dimension equal to that of E. A *coordinate covering* $\{U_s\ ,\ \varphi_s\}$ *with fiber* K consists of an open cover $\{U_s\}$ of Ω and coordinate mapping φ_s from $U_s \times K$ into E. Let $GL(K)$ be the set of all invertible continuous linear operators on K. Then an *analytic vector bundle* is a vector bundle equipped with a coordinate covering $\{U_s\ ,\ \varphi_s\}$ such that:

If U_s and U_t are two open sets in $\{U_s\}$ with nonempty intersection , then there is an analytic *coordinate transition function* $f_{st} : U_s \cap U_t \longrightarrow GL(K)$ defined by $f_{st}(z) = (\varphi_s^z)^{-1}\varphi_t^z$.

To define *equivalence of analytic vector bundles*, let E_1 and E_2 be vector bundles over Ω, let Λ be a bundle isomorphism from E_1 onto E_2, let $\{U_s,\ \varphi_s\}$ be an analytic coordinate covering for E_1 with fiber K_1, and let $\{U_t,\ \varphi_t\}$ be an analytic coordinate covering for E_2 with fiber K_2. Since E_1 and E_2 are isomorphic, dim $K_1 =$ dim K_2 , so that there is an isometry V from K_2 onto K_1. It follows that for each coordinate mapping φ_s, the function $(z,\ h) \rightarrow \Lambda(\varphi_s(z,\ V(h))$ defines a coordinate mapping from $U_S \times K_2$ into E_2. We denote this coordinate mapping by $\Lambda \circ \varphi_s$. If all coordinate transition functions of the coordinate covering $\{U_t,\ \varphi_t\} \cup \{U_s,\ \Lambda \circ \varphi_s\}$ are analytic, then the bundle isomorphism Λ is said to be an *analytic isomorphism.*

To define an analytic coordinate covering for the product bundle $\Omega \times K$, let $\{U_s\}$ be the set of all open subsets of Ω and define φ_s: $U_S \times K \rightarrow \Omega \times K$ by setting $\varphi_s(z,\ k) = (z,\ k)$. The transition functions are then identically equal to the identity on Ω and hence, analytic. An analytic vector bundle over Ω which is analytically isomorphic to this bundle is said to be *analytically trivial.* Bungart [**10**] proved the following fundamental fact.

THEOREM 2.4. *Every analytic vector bundle over* Ω *is trivial.*

2.2. Flat Unitary Vector Bundles. Let E be a vector bundle over Ω.

DEFINITION 2.5. A *unitary coordinate covering* for E is a coordinate covering $\{U_s,\, \varphi_s\}$ such that for each s and for each z in U_s, the fiber map $\varphi_s^z : K \to E_z$ is unitary. The unitary coordinate cover $\{U_s,\, \varphi_s\}$ is said to be *flat* if the transition functions f_{st} on $U_s \cap U_t$ are constant. A *flat unitary vector bundle* is a vector bundle equipped with a flat unitary coordinate covering.

If Λ is a bundle isomorphism from E_1 onto E_2, let $\{U_s,\, \varphi_s\}$ be a flat unitary coordinate covering for E_1 , and let $\{U_t,\, \varphi_t\}$ be a flat unitary coordinate covering for E_2, then Λ is said to be *isomorphism* of flat unitary vector bundles if $\{U_t,\, \varphi_t\} \cup \{U_s,$ $\Lambda \circ \varphi_s\}$ is a flat unitary coordinate covering for E_2.

PROPOSITION 2.6. [**2**] *Every flat unitary vector bundle over* Ω *is equivalent to a flat unitary vector bundle that can be extended to an open set* Ω' *containing the closure of* Ω.

2.3. Bundle Shift. Let E be a vector bundle. A *cross section* of E over Ω is a continuous function f from Ω into E such that $p(f(z)) = z$ for all z in Ω. If E is an analytic vector bundle with fiber K. If E is an analytic vector bundle with fiber K and if $\{U_s,\, \varphi_s\}$ is a coordinate covering for E, then a cross section f of E is said to be *analytic* if for each s the function $z \to (\varphi_s^z)^{-1}(f(z))$ is an analytic function from U_s into K.

If E is a flat unitary vector bundle over Ω with fiber K and coordinate covering $\{U_s,\, \varphi_s\}$ and if f is a cross section of E, then for z in $U_s \cap U_t$, the operator $(\varphi_t^z)^{-1}\varphi_s^z$

is unitary so that $\|(\varphi_s^z)^{-1}(f(z))\| = \|(\varphi_t^z)^{-1}(f(z))\|$. This implies that there is a function h_f on Ω defined by $h_f^E(z) = \|(\varphi_s^z)^{-1}(f(z))\|$, where z is in U_s.

DEFINITION 2.7. We define $H^2(\Omega, E)$ to be the space of analytic cross sections f of E such that $(h_f^E)^2$ is majorized by a harmonic function.

Then we proceed as outlined in Section 1 to show that $H^2(\Omega, E)$ is a Hilbert space ([**19, 25**]) with the following norm:

$$\|f\|_2 = \inf\,\{u(z_0)^{1/2} : u \text{ is a harmonic majorant of } (h_f^E)^2\}.$$

We define the *bundle shift* $T_E : H^2(\Omega, E) \to H^2(\Omega, E)$ by

$$(T_E f)(z) = zf(z) \tag{2.1}$$

for $z \in \Omega$. The operator T_E admits a functional calculus defined on the algebra $R(\Omega)$ of rational functions with poles off $\overline{\Omega}$. More precisely, if $u \in R(\Omega)$, $(u(T_E)f)(z) = u(z)f(z)$ for $z \in \Omega$ and $f \in H^2(\Omega, E)$. A closed subspace M of $H^2(\Omega, E)$ is said to be $R(\Omega)$-*invariant* for T_E if $u(T_E)M \subset M$ for $u \in R(\Omega)$.

THEOREM 2.8. [**2**] *If E is a flat unitary vector bundle over Ω with fiber K, then the operator T_E is similar to S_K.*

2.4. Inner Bundle Map. Let E and F be flat unitary bundles over Ω that extend to Ω' which is an open set containing the closure of Ω, and Θ be a bounded holomorphic bundle map from E to F. Since the restriction of Θ to E_z is a bounded linear function from E_z into F_z, we consider Θ as an element of $H^\infty(\Omega, L(E_z, F_z))$. Then Θ can be shown to have nontangential limits a.e. relative to m on $\partial\Omega$. The limit at a point z of $\partial\Omega$ can be regarded as an operator from the fiber of E at z to the fiber of F at z.

Definition 2.9. A bounded holomorphic bundle map Θ is *inner* if the nontangential limits are isometric operators a.e. relative to m.

Recall that an inner function is a function $\theta \in H^\infty$ for which $|\theta^*| = 1$ a.e. on $\partial\mathbf{D}$. We know that Beurling's theorem characterizes every invariant subspace for the unilateral shift operator S on H^2 defined by $(Sf)(z) = zf(z)$ for any f in H^2, that is, every closed S-invariant subspace M_1 of H^2, other than $\{0\}$, contains an inner function θ such that

$$M_1 = \theta H^2.$$

M.A. Abrahamse and R.G. Douglas [**2**] proved the following generalization of Beurling's theorem.

Theorem 2.10. *Let T_E be a bundle shift on $H^2(\Omega, E)$. Then a closed subspace M_2 of $H^2(\Omega, E)$ is $R(\Omega)$-invariant for T_E if and only if*

$$M_2 = \Theta H^2(\Omega, F),$$

where F is a flat unitary vector bundle over Ω and Θ is an inner bundle map from F to E.

In the next section, we will use this result to provide another generalization of Beurling's theorem.

3. Generalization Of Beurling's Theorem

In the case of unit disk, B.Sz.- Nagy and C. Foias used the Beurling's Theorem illustrating invariant subspaces of the unilateral shift S when they thought about a minimal function of a C_0-operator T. In this section we will think about $R(\Omega)$-invariant subspace for the shift operator

$$S_K : H^2(\Omega, K) \to H^2(\Omega, K)$$

defined in the section 1. We will provide an extended Beurling's theorem by using the result provided by M.A. Abrahamse and R.G. Douglas [**2**].

To use the result provided by M.A. Abrahamse and R.G. Douglas [1], we need to extend the notion of (vector-valued) inner function ([**6**]) to bounded finitely connected regions.

Definition 2.11. Let K and K' be Hilbert spaces and let $\varphi \in H^\infty(\Omega, L(K, K'))$. We will say that φ is *quasi-inner* if there exists a constant $c > 0$ such that for every $k \in K$ and almost every $z \in \partial\Omega$ we have $c\,\|k\| \leq \|\varphi(z)k\|$.

That is, if $\varphi \in H^\infty(\Omega, L(K, K'))$ is quasi-inner, then operator $\varphi(z)$ is bounded below almost every $z \in \partial\Omega$. This definition allows us to reformulate Theorem 2.10 without using vector bundles.

By the Riesz representation theorem, we identify $L(\mathbb{C})$ with $\mathbb{C}$. Thus we also provide a definition of (scalar-valued) quasi-inner function.

Definition 2.12. $\theta \in H^\infty(\Omega)$ is said to be *quasi-inner* if there exists a constant $m > 0$ such that for almost every $z \in \partial\Omega$, $m \leq |\theta^*(z)|$.

Proposition 2.13. *Let $\varphi \in H^\infty(\Omega, L(K, K'))$. If φ is a quasi-inner function, then $\varphi H^2(\Omega, K)$ is closed subspace of $H^2(\Omega, K')$.*

Proof. Clearly $\varphi H^2(\Omega, K) \subset H^2(\Omega, K')$. Let $\{\varphi f_n\}$ be a Cauchy sequence in $\varphi H^2(\Omega, K)$. If f_n^* is the nontangential limits of f_n, then since φ is a quasi-inner function, there exists a constant $c > 0$ such that

$$\|\varphi(z)f_n^*(z) - \varphi(z)f_m^*(z)\| \geq c\,\|f_n^*(z) - f_m^*(z)\| \tag{3.1}$$

for a.e. $z \in \partial\Omega$.

Since $\|\varphi f_n^*\|_{H^2(\partial\Omega,K)} = \|\varphi f_n\|_{H^2(\Omega,K)}$, $\{\varphi f_n^*\}$ is a Cauchy sequence in $\varphi H^2(\partial\Omega, K)$, from the equation (3.1) we conclude that $\{f_n^*\}$ is also a Cauchy sequence in $H^2(\partial\Omega, K)$. We know that $H^2(\partial\Omega, K)$ is a Banach space. Thus $\lim f_n^*$ exists.

If $\lim f_n^* = f^* \in H^2(\partial\Omega, K)$, then by the Proposition 2.1, there is unique $f \in H^2(\Omega, K)$ such that $\lim f_n = f$ and so $\lim \varphi f_n = \varphi f \in \varphi H^2(\Omega, K')$ which proves this Proposition. □

THEOREM 2.14. *Let K be a Hilbert space. Then a closed subspace M of $H^2(\Omega, K)$ is $R(\Omega)$-invariant for the shift operator S_K if and only if there is a Hilbert space K' and quasi-inner function $\varphi : \Omega \to L(K', K)$ such that*

$$M = \varphi H^2(\Omega, K').$$

PROOF. ($\Leftarrow$) Let $M = \varphi H^2(\Omega, K')$ for some quasi-inner function $\varphi : \Omega \to L(K', K)$. Then by Proposition 2.13, M is closed in $H^2(\Omega, K)$ and for any $g \in M$, there is $f \in H^2(\Omega, K')$ such that $g(z) = \varphi(z)f(z)$. Since for any $r \in R(\Omega)$, $(r(S_K)g)(z) = r(z)g(z) = r(z)\varphi(z)f(z) = \varphi(z)[r(z)f(z)]$ and $rf \in H^2(\Omega, K')$, $rg \in M$. Thus M is invariant for $R(\Omega)$.

($\Rightarrow$) Consider a closed subspace $M \subset H^2(\Omega, K)$ which is $R(\Omega)$-invariant. Let

$$M' = \{G \in H^2(\Omega, \Omega \times K) \colon \forall z \in \Omega,\ G(z) = (z, g(z)) \text{ for some } g \in M\}.$$

Then M' is a closed subspace of $H^2(\Omega, \Omega \times K)$ which is invariant for $R(T_{\Omega\times K})$ and so by Theorem 2.10, there is a flat unitary bundle F over Ω with fiber K' and an inner bundle map $\Theta : F \to \Omega \times K$ such that

$$M' = \Theta H^2(\Omega, F).$$

By Proposition 2.6, there is a flat unitary vector bundle F' over Ω', an open set containing the closure of Ω, with fiber K' such that F is unitary equivalent to the bundle $F'|\Omega$. By Theorem 2.4, there is an analytic isomorphism

$$\Lambda : \Omega' \times K' \to F'.$$

Define an invertible operator $W : H^2(\Omega, K') \to H^2(\Omega'|\Omega, F'|\Omega)$ by

$$(Wf)(z) = \Lambda(z, f(z)) = \Lambda_z(f(z))$$

for $f \in H^2(\Omega, K')$. Then

$$M' = \Theta U W H^2(\Omega, K')$$

where $U : H^2(\Omega, F'|\Omega) \to H^2(\Omega, F)$ is a unitary operator. For each $z \in \Omega$, we define a bounded operator $W_z : K' \to F_z$ by

$$W_z a = U((Wh_a))(z)$$

for $a \in K'$ where $h_a \in H^2(\Omega, K')$ defined by $h_a(z) = a$.

By the definition of W, we obtain $W_z a = U\Lambda_z(a)$ for $a \in K'$. Since $a \in K'$ is arbitrary, for each $z \in \Omega$,

$$W_z = U\Lambda_z. \tag{3.2}$$

For a fixed $z \in \Omega$, define $\Theta_z : F_z \to K$ by $\Theta_z = \Theta|F_z$, i.e. for $f_z \in F_z$, $\Theta(f_z) = (z, \Theta_z(f_z))$.

For $z \in \Omega$, define

$$\varphi(z) = \Theta_z W_z, \tag{3.3}$$

then $\varphi(z) : K' \to K$. We will prove $\varphi \in H^\infty(\Omega, L(K', K))$.

Since Λ is an analytic isomorphism, if $\{U_t, \varphi_t\}$ is a unitary coordinate covering for F' and $\{U_s, \varphi_s\}$ is a unitary coordinate covering for $\Omega' \times K'$ (and so φ_s is identity), then $f_{ts} : U_t \cap U_s \to GL(K')$ is holomorphic. Since

$$f_{ts}(z) = (\varphi_t^z)^{-1}((\Lambda \circ \varphi_s)^z) = (\varphi_t^z)^{-1}(\Lambda_z)$$

for $z \in U_t \cap U_s$, $k(z) = \Lambda_z$ is holomorphic and so if $h : \Omega' \to L(K', F_z)$ defined by

$$h(z) = W_z,$$

then h is also holomorphic. From the fact that Θ is holomorphic, we conclude that φ is holomorphic.

Since $\Theta : F \to \Omega \times K$ is a bounded holomorphic bundle map, Θ_z can be shown to have nontangential limits Θ_z^* for all $z \in A(\subset \partial\Omega)$ with $m(A^c) = 0$. Since $\|\Theta_z\| = \|\Theta_z^*\|$ and Θ is an inner function, for $z \in \Omega$,

$$\|\Theta_z\|_\infty = 1.$$

Since $h : \Omega' \to L(K', F_z)$ is analytic and the norm function is also continuous, there is $M'' > 0$ such that $\|h(z)\| \leq M''$ for any $z \in \overline{\Omega}$. It follows that $\|\varphi(z)\| \leq \|\Theta_z\| \, \|W_z\| \leq M''$ and so

$$\|\varphi\|_\infty \leq M''.$$

To conclude our proof, we must verify that φ is quasi-inner.

By the same way as above, since $f_{st} : U_t \cap U_s \to GL(K')$ is holomorphic defined by $f_{st}(z) = ((\Lambda \circ \varphi_s)^z)^{-1}(\varphi_t^z) = (\Lambda_z)^{-1}(\varphi_t^z)$ for $z \in U_t \cap U_s$, and so $k_1(z) = (\Lambda_z)^{-1}$ is holomorphic on Ω'. Thus $h_1(z) = W_z^{-1}$ is also holomorphic on Ω' and so

$$\left\|{W_z}^{-1}\right\| \leq m_0$$

for some $m_0 > 0$ for any $z \in \Omega$.

Since $\|W_z(a)\| \geq \frac{\left\|{W_z}^{-1} W_z a\right\|}{\left\|W_z^{-1}\right\|} = \frac{\|a\|}{\left\|W_z^{-1}\right\|} \geq \frac{\|a\|}{m_0}$, by the equation (3.3),

$$\|\varphi(z)(a)\| \geq \frac{\|a\|}{m_0}$$

for all $z \in A$ and so φ is an quasi-inner function. By the definition of M', we conclude that $M = \varphi H^2(\Omega, K')$. □

Let K_1 and K_2 be separable Hilbert spaces and $\psi \in H^\infty(\Omega, L(K_1, K_2))$. In the same way as section 1, we define a *multiplication operator* $M_\psi : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ by

$$M_\psi(g)(z) = \psi(z)g(z)$$

for all g in $H^2(\Omega, K_1)$. We can check easily that $\|M_\psi\| = \|\psi\|_\infty$.

LEMMA 2.15. *Let K_1 and K_2 be separable Hilbert spaces. If $T : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is a bounded linear operator such that $TS_{K_1} = S_{K_2}T$, then there is a function $\psi \in H^\infty(\Omega, L(K_1, K_2))$ such that $T = M_\psi$.*

PROOF. Define $Y \in (S_{K_1 \bigoplus K_2})'$ by $Y = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$. Then by Proposition 2.3, $Y = M_\omega$ where $\omega \in H^\infty(\Omega, L(K_1 \oplus K_2))$. Let $\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$. Take $\psi = \omega_{21}$, then $T = M_\psi$. □

In Theorem 2.14, we see that if M is a $R(\Omega)$-invariant closed subspace for the shift operator S_K on $H^2(\Omega, K)$ then there are a quasi-inner function $\varphi : \Omega \to L(K', K)$ and a Hilbert space K' such that $M = \varphi H^2(\Omega, K')$. The following Corollary indicates when two $R(\Omega)$-invariant closed subspaces for the shift operator S_K on $H^2(\Omega, K)$ are equal.

COROLLARY 2.16. *Let $\varphi_1 : \Omega \to L(K_1, K)$ and $\varphi_2 : \Omega \to L(K_2, K)$ be quasi-inner functions (K,K_1,and K_2 are Hilbert spaces). Then two subspaces $\varphi_1 H^2(\Omega, K_1)$ and $\varphi_2 H^2(\Omega, K_2)$ of $H^2(\Omega, K)$ are equal if and only if there exist a function $\varphi \in H^\infty(\Omega, L(K_1, K_2))$ such that $\varphi(z)$ is invertible for any $z \in \Omega$, $\sup \|\varphi(z)^{-1}\| < \infty$ and*

$$\varphi_1(z) = \varphi_2(z)\varphi(z)$$

for any $z \in \Omega$. In particular, K_1 and K_2 have the same dimension.

PROOF. Evidently, if $\varphi_1(z) = \varphi_2(z)\varphi(z)$ for any $z \in \Omega$, $\varphi_1 H^2(\Omega, K_1) \subset \varphi_2 H^2(\Omega, K_2)$. Since $\sup \|\varphi(z)^{-1}\| < \infty$ and $\varphi_2(z) = \varphi_1(z)\varphi(z)^{-1}$, $\varphi_2 H^2(\Omega, K_1) \subset \varphi_1 H^2(\Omega, K_2)$.

Conversely, assume that $\varphi_1 H^2(\Omega, K_1) = \varphi_2 H^2(\Omega, K_2)$. Define an operator $T : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ as follows. For $f \in H^2(\Omega, K_1)$,

$$Tf = g$$

such that

$$\varphi_1 f = \varphi_2 g. \tag{3.4}$$

Since $\varphi_i (i = 1, 2)$ is a quasi-inner function, T is well-defined and invertible. Since $S_{K_2} T = T S_{K_1}$, by the previous Lemma 2.15,

$$T = M_\varphi$$

for a function $\varphi \in H^\infty(\Omega, L(K_1, K_2))$. By the invertibility of T, $S_{K_1} T^{-1} = T^{-1} S_{K_2}$. From the previous Lemma 2.15, $T^{-1} = M_\psi$ for some $\psi \in H^\infty(\Omega, L(K_2, K_1))$. It follows that $M_\varphi^{-1} = M_\psi$.

Since for any $f \in H^2(\Omega, K_2)$, $f(z) = (M_\varphi^{-1} M_\varphi f)(z) = (M_\psi M_\varphi f)(z) = \psi(z)\varphi(z) f(z)$ for $z \in \Omega$,

$$\psi(z)\varphi(z) = I_{K_1}. \tag{3.5}$$

Similarly, we can get

$$\varphi(z)\psi(z) = I_{K_2}. \tag{3.6}$$

From equations (3.5) and (3.6), we can conclude that $\varphi(z)$ is invertible for any $z \in \Omega$ and $\varphi(z)^{-1} = \psi(z)$.

Since $\psi \in H^\infty(\Omega, L(K_2, K_1))$, $\sup \|\varphi(z)^{-1}\| < \infty$. By the equation (3.4),

$$\varphi_1 f = \varphi_2 \varphi f$$

for $f \in H^2(\Omega, K_1)$. Since $f \in H^2(\Omega, K_1)$ is arbitrary, we can conclude that $\varphi_1 = \varphi_2 \varphi$. Since $\varphi(z)$ is invertible for any $z \in \Omega$, K_1 and K_2 have the same dimension. □

COROLLARY 2.17. *Let $\varphi \in H^\infty(\Omega, L(K_1, K_2))$. Then the following statements are equivalent.*

(a) φ is quasi-inner.

(b) M_φ *is one-to-one and has closed range.*

PROOF. ((a) → (b)) By Proposition 2.13, $M_\varphi H^2(\Omega, K_1) = \varphi H^2(\Omega, K_1)$ is closed. Since φ is quasi-inner, for some $c > 0$, $\|\varphi(z)a\| \geq c\,\|a\|$ for a.e. $z \in \partial\Omega$ and for any $a \in K_1$. Thus condition (a) implies $\|M_\varphi f\| \geq c\,\|f\|$ for $f \in H^2(\Omega, K_1)$. It follows that M_φ is one-to-one.

((b) → (a)) Since $\varphi H^2(\Omega, K_1)$ is is $R(\Omega)$-invariant for S_{K_2}, by Theorem 2.14,

$$\varphi H^2(\Omega, K_1) = \varphi_1 H^2(\Omega, K_0)$$

where K_0 is a Hilbert space and $\varphi_1 : \Omega \to L(K_0, K_2)$ is a quasi-inner function. As in the proof of Corollary 2.16, we can get an invertible function $\varphi_2 \in H^\infty(\Omega, L(K_1, K_0))$ such that $\sup \|\varphi_2(z)^{-1}\| < \infty$ and

$$\varphi(z) = \varphi_1(z)\varphi_2(z)$$

for any $z \in \Omega$.

Let $c_1 = \sup \|\varphi_2(z)^{-1}\| > 0$. Since $1 = \left\|\varphi_2(z)\varphi_2^{-1}\right\| \leq \|\varphi_2(z)\|\,\|\varphi_2(z)^{-1}\|$,

$$\frac{1}{c_1} \leq \frac{1}{\left\|\varphi_2^{-1}\right\|} \leq \|\varphi_2(z)\|\,. \tag{3.7}$$

From the fact that φ_1 is quasi-inner, we can conclude that there is a $c_2 > 0$ such that for any $a \in K_1$,

$$\|\varphi(z)a\| = \|\varphi_1(z)\varphi_2(z)a\| \geq c_2\,\|\varphi_2(z)a\| \geq \frac{c_2}{c_1}\,\|a\|\,, \tag{3.8}$$

for a.e.$z \in \partial\Omega$. It follows that for any $a \in K_1$, there is a $c = \frac{c_2}{c_1} > 0$ such that

$$\|\varphi(z)a\| \geq c\,\|a\|$$

for a.e.$z \in \partial\Omega$. □

4. Greatest Common Quasi-Inner Divisor

Let θ and θ' be two inner functions in H^∞. We say that θ *divides* θ' (or $\theta|\theta'$) if θ' can be written as $\theta' = \theta{\cdot}\phi$ for some $\phi \in H^\infty$. It is clear that $\phi \in H^\infty$ is also inner. We will use the notation $\theta \equiv \theta'$ if $\theta|\theta'$ and $\theta'|\theta$. Generally, the following definition shows how to define a *divisor* of a function in H^∞.

Definition 2.18. Let f and f' be two functions in H^∞. We say f *divides* f' (denoted $f|f'$) if f' can be written as fg for some $g \in H^\infty(\Omega)$.

We will use the notation $f \equiv f'$ if $f|f'$ and $f'|f$.

Proposition 2.19. *([**6**]) For any inner functions θ and θ' in H^∞, the following assertions are equivalent:*

(a) $\theta \mid \theta'$.

(b) $\theta' H^\infty \subset \theta H^\infty$.

(c) $\theta^\infty H^2 \subset \theta H^2$.

(d) $|\theta'(z)| \leq c\,|\theta(z)|$ for some $c > 0$ and for all $z \in \boldsymbol{D}$.

Let K be a Hilbert space. The time has come to consider divisibilities between a function in $H^\infty(\Omega)$ and a function in $H^\infty(\Omega, L(K))$ or between functions in $H^\infty(\Omega, L(K))$.

Definition 2.20. If $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, then we say that θ *divides* φ (denoted $\theta|\varphi$) if φ can be written as

$$\varphi = \theta \cdot \phi'$$

for some $\phi' \in H^\infty(\Omega, L(K))$.

Definition 2.21. If $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, then we say that φ *divides* θ (denoted $\varphi|\theta$) if there exists $\psi \in H^\infty(\Omega, L(K))$ satisfying the relations

$$\varphi(z)\psi(z) = \theta(z)I_K$$

and

$$\psi(z)\varphi(z) = \theta(z)I_K$$

for $z \in \Omega$.

Definition 2.22. If φ and φ' are functions in $H^\infty(\Omega, L(K))$, then we say that φ is a *left divisor* of φ' if

$$\varphi'(z) = \varphi(z)\varphi''(z) \text{ (denoted } \varphi|_l\varphi')$$

for some $\varphi'' \in H^\infty(\Omega, L(K))$, and we say that φ is a *right divisor* of φ' if

$$\varphi'(z) = \varphi''(z)\varphi(z) \text{ (denoted } \varphi|_r\varphi')$$

for some $\varphi'' \in H^\infty(\Omega, L(K))$. We will say φ *divides* φ' if φ is not only a left divisor but also a right divisor of φ'.

Definition 2.23. (a) Let F be a family of functions in $H^\infty(\Omega)$. A (quasi-inner) function $\theta \in H^\infty(\Omega)$ is called the *greatest common (quasi-inner) divisor* of F if θ divides every element in F and if θ is a multiple of any other common (quasi-inner) divisor of F. *The greatest common quasi-inner divisor* of F is denoted by $\bigwedge F$(or $\bigwedge_{i\in I} f_i$ if $F = \{f_i : i \in I\}$, or $f_1 \wedge f_2$ if $F = \{f_1, f_2\}$). If $\bigwedge F = 1$, then F is said to be *relatively prime.*

(b) Let φ be a functions in $H^\infty(\Omega, L(K))$. A (quasi-inner) function $\theta \in H^\infty(\Omega)$ is called the *greatest common (quasi-inner) divisor* of φ if θ divides φ and every (quasi-inner) divisors of φ is a divisor of θ. *The greatest quasi-inner divisor* of φ is denoted by $D(\varphi)$.

Lemma 2.24. *Let A and B be subsets of $H^\infty(\Omega)$. Then $\bigwedge AB = \bigwedge\{fg : f \in A$ and $g \in B\} \equiv (\bigwedge A)(\bigwedge B)$.*

Proof. Let $\bigwedge A = \phi$ and $\bigwedge B = \psi$. Then for any $f \in A$ and $g \in B$, $f = \phi Q_f$ and $g = \psi Q_g$ for some Q_f, $Q_g \in H^\infty(\Omega)$. Thus $fg = \phi\psi Q_f Q_g$ and so

$$\bigwedge AB = \phi\psi(\bigwedge\{Q_f Q_g : f \in A \text{ and } g \in B\}).$$

If $\bigwedge\{Q_f : f \in A\} \neq 1$ $(\bigwedge\{Q_g : g \in B\} \neq 1)$, then $\bigwedge A \neq \phi(\bigwedge B \neq \psi)$ and so $\bigwedge\{Q_f : f \in A\} = 1(\bigwedge\{Q_g : g \in B\} = 1)$. Let

$$d = \bigwedge\{Q_f Q_g : f \in A \text{ and } g \in B\},$$

then $d|Q_f Q_g$ for any $f \in A$ and $g \in B$. We can conclude that $d|Q_g$ for any $g \in B$, because $\bigwedge\{Q_f : f_A\} = 1$. Since $\bigwedge\{Q_g : g \in B\} = 1$, $d \equiv 1$. It follows that $\bigwedge AB \equiv (\bigwedge A)(\bigwedge B)$. □

Using this Lemma, we can get the following conclusion.

Corollary 2.25. *Let C and D be subsets of $H^\infty(\Omega)$. If C is relatively prime, then so is $C^k(= \{h^k : h \in C\})$ for every positive integer k. If D is also relatively prime, then so is CD.*

By the definition of divisibility, we can get following results similar to Proposition 2.19.

Theorem 2.26. *For any quasi-inner functions $\theta \in H^\infty(\Omega)$ and $\varphi \in H^\infty(\Omega, L(K))$, the following assertions are equivalent:*

(a) $\theta|\varphi$.

(b) $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$.

(c) $\varphi H^2(\Omega, K) \subset \theta H^2(\Omega, K)$.

(d) There is a $\lambda > 0$ such that $\varphi(z)\varphi(z)^* \leq \lambda^2|\theta(z)|^2 I_{\mathbb{C}^n}$ for any $z \in \Omega$.

Proof. If $\theta|\varphi$, $\varphi = \theta\varphi_1$ for some $\varphi_1 \in H^\infty(\Omega, L(K))$. Then

$$\varphi H^\infty(\Omega, K) = \theta\varphi_1 H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K).$$

Thus (a) implies (b).

Conversely, suppose that $\varphi H^\infty(\Omega, K) \subset \theta H^\infty(\Omega, K)$. Then $\varphi^* H^\infty(\partial\Omega, K) \subset \theta^* H^\infty(\partial\Omega, K)$.

Let $\{b_i : i \in I\}$ be an orthonormal basis of K and $g_i \in H^\infty(\partial\Omega, K)$ defined by $g_i(z) = b_i (i \in I)$. Since $\varphi^* H^\infty(\partial\Omega, K) \subset \theta^* H^\infty(\partial\Omega, K)$, there is $f_i \in H^\infty(\partial\Omega, K)$ such that $\varphi^* g_i = \theta^* f_i$, i.e. for $i \in I$,

$$\varphi^*(z) b_i = \theta^*(z) f_i(z). \tag{4.1}$$

Define $\varphi_1 : \partial\Omega \to L(K)$ by for $i \in I$,

$$\varphi_1(z) b_i = f_i(z). \tag{4.2}$$

For $i \in I$, define $\varphi_i \in H^\infty(\partial\Omega, L(K))$ by $\varphi_i(z) b_j = \delta_{ij} f_i(z) (j \in I)$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ Then

$$\varphi_1 = \sum_{i \in I} \varphi_i. \tag{4.3}$$

By (4.1), for each $i \in I$, $\varphi^*(z) b_i = \theta^*(z) \varphi_1(z) b_i$. Since $\varphi^*(z)$ and $\theta^*(z)\varphi_1(z)$ have the same value on a basis of K,

$$\varphi^* = \theta^* \varphi_1.$$

To prove that (b) implies (a), we have to show that $\varphi_1 \in H^\infty(\partial\Omega, L(K))$. Since $\theta \in H^\infty$ is a quasi-inner function, there is $c > 0$ such that $|\theta(z)| \geq c$ for every $z \in A \subset \partial\Omega$ with $m(A) = 0$. For any $x \in K$ with $\|x\| = 1$ and $z \in A$,

$$\|\varphi_1(z) x\| = \frac{\|\varphi^*(z) x\|}{|\theta^*(z)|} \leq \frac{\|\varphi\|_\infty}{c}. \tag{4.4}$$

From (4.3) and (4.4), we conclude that

$$\varphi_1 \in H^\infty(\partial\Omega, L(K)).$$

Therefore there is $\psi \in H^\infty(\Omega, L(K))$ such that $\varphi = \theta\psi$ and $\psi^* = \varphi_1$ which proves $(b) \Rightarrow (a)$.

By the same way as above, $(a) \Leftrightarrow (c)$ is proven. We begin to prove $(a) \Leftrightarrow (d)$. If $\theta|\varphi$, $\varphi = \theta\varphi_1$ for some $\varphi_1 \in H^\infty(\Omega, L(\mathbb{C}^n))$. Then

$$\varphi(z)\varphi(z)^* = \theta(z)\varphi_1(z)\ \varphi_1(z)^*\overline{\theta(z)} \leq \|\varphi_1\|_\infty^2 |\theta(z)|^2 I_{\mathbb{C}^n}.$$

Let $\lambda = \|\varphi_1\|_\infty$. Since φ is quasi-inner, $\varphi \neq 0$ and so $\lambda > 0$. Thus (a) implies (d).

Conversely, suppose that for any $z \in \Omega$,

$$\varphi(z)\varphi(z)^* \leq \lambda^2|\theta(z)|^2 I_K \tag{4.5}$$

for some $\lambda > 0$. For each $z \in \Omega$, we will define a linear mapping $F_z \in L(K)$. Let

$$A = \{z \in \Omega : \theta(z) = 0\}$$

and

$$B = \{z \in \Omega : \theta(z) \neq 0\}.$$

If $z \in A$, then let $F_z = 0$. If $z \in B$, then range of $\overline{\theta(z)} I_K$ is K and so we can define a linear mapping F_z from K to range of $\varphi(z)^*$ by

$$F_z(\overline{\theta(z)} f) = \varphi(z)^* f$$

for $f \in K$.

Since $\left\|F_z(\overline{\theta(z)} f)\right\|^2 = \|\varphi(z)^* f\|^2 = (\varphi(z)\varphi(z)^* f, f) \leq \lambda^2(|\theta(z)|^2 f, f) = \lambda^2 \|\theta(z) f\|^2$, that is,

$$\left\|F_z(\overline{\theta(z)} f)\right\| \leq \lambda \|\theta(z) f\|, \tag{4.6}$$

F_z is well-defined for $z \in B$. By definition of F_z, if $z \in B$, then $F_z\overline{\theta(z)} = \varphi(z)^*$, and so

$$\theta(z) F_z^* = \varphi(z). \tag{4.7}$$

If $z \in A$, by 4.5 $\|\varphi(z)\| = 0$ and so $\varphi(z) = 0$. Thus $\theta(z)F_z = \varphi(z)$ for any $z \in \Omega$.

Define a function $F : \Omega \to K$ by

$$F(z) = F_z^*.$$

Then by equation (4.15),

$$\varphi(z) = \theta(z)F(z)$$

for $z \in \Omega$. To finish this proof, we have to prove that $F \in H^\infty(\Omega, L(K))$. From inequality (4.6), we can get

$$\|F\|_\infty \leq \lambda \tag{4.8}$$

and so $F = \frac{\varphi}{\theta}$ has only removable singularities in Ω. Thus F can be defined on $\{z \in \Omega : \theta(z) = 0\}$ so that F is analytic and

$$\varphi = \theta F.$$

From (4.8), we can conclude that $F \in H^\infty(\Omega, L(K))$ which proves $(d) \Rightarrow (a)$. □

Note that for $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$, we can think $\varphi(z)(z \in \Omega)$ as an $n \times n$ matrix, say $(\varphi_{ij}(z))_{n\times n}$.

Lemma 2.27. *If* $\varphi = [\varphi_{ij}]_{i,j=1}^n$ *is a function in* $H^\infty(\Omega, L(\mathbb{C}^n))$*, then* $D(\varphi) = \bigwedge \varphi_{ij}$.

Proof. Let $\bigwedge \varphi_{ij} = \theta_1$. If $\theta \in H^\infty(\Omega)$ ia a quasi-inner function such that $\theta|\varphi$, then $\varphi = \theta\phi$ for some $\phi \in H^\infty(\Omega, L(\mathbb{C}^n))$. Thus

$$(\varphi_{ij})_{n\times n} = (\theta\phi_{ij})_{n\times n}$$

for each $i, j = 1, 2, \cdots, n$. Thus $\varphi_{ij} = \theta\phi_{ij}$ for each $i, j = 1, 2, \cdots, n$. Since $\phi \in H^\infty(\Omega, L(\mathbb{C}^n))$, $\phi_{ij} \in H^\infty(\Omega)$ and so $\theta|\varphi_{ij}$ for each $i, j = 1, 2, \cdots, n$. By definition of greatest common divisor, we can conclude that $\theta|\theta_1$. Clearly $\theta_1|\varphi$. Thus $D(\varphi) = \theta_1$. □

THEOREM 2.28. *For any quasi-inner functions* $\theta \in H^\infty(\Omega)$ *and* $\varphi \in H^\infty(\Omega, L(K))$, *the following assertions are equivalent:*

(a) $\varphi|\theta$.

(b) $\theta H^\infty(\Omega, K) \subset \varphi H^\infty(\Omega, K)$.

(c) $\theta H^2(\Omega, K) \subset \varphi H^2(\Omega, K)$.

(d) There is a $\lambda > 0$ *such that* $|\theta(z)|^2 I_K \leq \lambda^2 \varphi(z)\varphi(z)^*$ *for any* $z \in \Omega$.

PROOF. This theorem is proven by the same way as Theorem 2.26. □

PROPOSITION 2.29. *For any quasi-inner functions* φ *and* φ' *in* $H^\infty(\Omega, L(\mathbb{C}^n))$ *and* $H^\infty(\Omega, L(C^n))$ *respectively, the following assrtions are equivalent:*

(a) $\varphi|_l\varphi'$.

(b) $\varphi' H^\infty(\Omega, \mathbb{C}^n) \subset \varphi H^\infty(\Omega, \mathbb{C}^n)$.

(c) $\varphi' H^2(\Omega, \mathbb{C}^n) \subset \varphi H^2(\Omega, \mathbb{C}^n)$.

PROOF. In the same way as the proofs of Theorem 2.26, it is proven that (a) and (b) are equivalent and so are (a) and (c). □

COROLLARY 2.30. *Let* φ *and* φ' *be quasi-inner functions in* $H^\infty(\Omega, L(\mathbb{C}^n))$. *Then* $\varphi|_l\varphi'$ *and* $\varphi'|_l\varphi$ *if and only if there exist a function* $\varphi \in H^\infty(\Omega, L(K_1, K_2))$ *such that* $\varphi(z)$ *is invertible for any* $z \in \Omega$ *with* $\sup \|\varphi(z)^{-1}\| < \infty$ *and* $\varphi_1(z) = \varphi_2(z)\varphi(z)$ *for any* $z \in \Omega$.

PROOF. By Proposition 2.29, we can conclude that $\varphi|_l\varphi'$ and $\varphi'|_l\varphi$ if and only if $\varphi' H^2(\Omega, \mathbb{C}^n) = \varphi H^2(\Omega, \mathbb{C}^n)$. Then by Corollary 2.16, this Corollary is proven. □

For quasi-inner functions φ and φ' in $H^\infty(\Omega, L(\mathbb{C}^n))$, $\varphi(z)$ is a bounded operator on the Hilbert space $\mathbb{C}^n$. When we consider some relationships between $\varphi|_l\varphi'$ ($\varphi|_r\varphi'$) and $\varphi(z)|_l\varphi'(z)$ ($\varphi(z)|_r\varphi'(z)$, respectively) for each $z \in \Omega$, first of all, we obtain easily if $\varphi|_l\varphi(\varphi|_r\varphi')$ then $\varphi(z)|_l\varphi'(z)(\varphi(z)|_r\varphi'(z)$, respectively) for each $z \in \Omega$.

For every $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$, we construct a new function $\varphi^\sim \in H^\infty(\Omega, L(\mathbb{C}^n))$ by

$$\varphi^\sim(z) = \varphi(\overline{z})^*$$

for $z \in \Omega$.

LEMMA 2.31. *Let φ_1 and φ_2 be functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. Suppose that $z \in \Omega$ if and only if $\overline{z} \in \Omega$. Then following statements are equivalent:*

(a) $\varphi_1 |_l \varphi_2$.

(b) $\varphi_1^\sim |_r \varphi_2^\sim$.

PROOF. ($\Rightarrow$) Since $\varphi_1 |_l \varphi_2$, there is a function $\varphi_3 \in H^\infty(\Omega, L(\mathbb{C}^n))$ such that

$$\varphi_2(z) = \varphi_1(z)\varphi_3(z)$$

for $z \in \Omega$. By our assumption, $\varphi_2(\overline{z}) = \varphi_1(\overline{z})\varphi_3(\overline{z})$ for $z \in \Omega$. Thus

$$\varphi_2(\overline{z})^* = \varphi_3(\overline{z})^*\varphi_1(\overline{z})^*$$

for $z \in \Omega$ and so $\varphi_1^\sim |_r \varphi_2^\sim$.

($\Leftarrow$) By the same way as the proof of ($\Rightarrow$), this implication can be proven. □

From Lemma 2.31, we can get a following result.

COROLLARY 2.32. *Let φ_1 and φ_2 be functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. Suppose that $z \in \Omega$ if and only if $\overline{z} \in \Omega$. Then following statements are equivalent:*

(a) $\varphi_1 | \varphi_2$.

(b) $\varphi_1 |_r \varphi_2$ *and* $\varphi_1^\sim |_r \varphi_2^\sim$.

CHAPTER 3

C_0-Operators

In this chapter we will summarize basic results about classical C_0-operators relative to the open unit disk **D**. References are H. Bercovici [**6**], and B. Sz.-Nagy and C. Foias [**27**]. Also fundamental results of C_0-operator relative to a bounded finitely connected region Ω in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves will be provided. References for C_0-operator relative to a bounded finitely connected region Ω are J.A. Ball [**4**] and Adele Zucchi [**31**].

1. C_0-Operators Relative to D

1.1. A Functional Calculus. It is well-known that for every linear operator A on a finite dimensional vector space V over the field F, there is a minimal polynomial for A which is the (unique) monic generator of the ideal of polynomials over F which annihilate A. If the dimension of F is not finite, then generally there is no such a polynomial. However, to provide a function similar to a minimal polynomial, B. Sz.-Nagy and C. Foias focused on a contraction $T \in L(H)$ which is called to be *completely nonunitary*, i.e. there is no invariant subspace M for T such that the restriction $T|M$ of T to the space M is a unitary operator.

Let H be a subspace of a Hilbert space K and P_H be the orthogonal projection from K onto H. We recall that if $A \in L(K)$, and $T \in L(H)$, then A is said to be a *dilation* of T provided that for $n = 1, 2, ...,$

$$T^n = P_H A^n | H. \tag{1.1}$$

If A is an isometry (unitary operator) then A will be called an *isometric* (*unitary*) *dilation* of T. An isometric (unitary) dilation A of T is said to be *minimal* if no restriction of A to an invariant subspace is an isometric (unitary) dilation of T. B. Sz.-Nagy proved the following interesting result:

PROPOSITION 3.1. *Every contraction has a unitary dilation.*

Let $T \in L(H)$ be a completely nonunitary contraction with minimal unitary dilation $U \in L(K)$. For every polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ we have

$$p(T) = P_H p(U) | H, \tag{1.2}$$

and so this formula suggests that the functional calculus $p \to p(T)$ might be extended to more general functions p. Since the mapping $p \to p(T)$ is a homomorphism from the algebra of polynomials to the algebra of operators, we will extend it to a mapping which is also a homomorphism from an algebra to the algebra of operators. By Spectral Theorem, since $U \in L(H)$ is a normal operator, there is a unique *spectral measure* E on the Borel subsets of the spectrum of U denoted as usual by $\sigma(U)$ such that

$$U = \int_{\sigma(U)} z dE(z). \tag{1.3}$$

Since the spectral measure E of U is absolutely continuous with respect to Lebesgue measure on $\partial \mathbf{D}$, for $g \in L^\infty(\sigma(U), E)$, $g(U)$ can be defined as follows:

$$g(U) = \int_{\sigma(U)} g(z) dE(z). \tag{1.4}$$

It is clear that if g is a polynomial, then this definition agrees with the preceding one. Since the spectral measure of U is absolutely continuous with respect to Lebesgue measure on $\partial\mathbf{D}$, the expression $g(U)$ makes sense for every $g \in L^\infty = L^\infty(\partial\mathbf{D})$. We generalize formula 1.2, and so for $g \in L^\infty$, define $g(T)$ by

$$g(T) = P_H g(U)|H. \tag{1.5}$$

While the mapping $g \to g(T)$ is obviously linear, it is not generally multiplicative, i.e. it is not a homomorphism. Evidently it is convenient to find a subalgebra in L^∞ on which the functional calculus is multiplicative. Recall that H^∞ is the Banach space of all (complex-valued) bounded analytic functions on the open unit disk $\mathbf{D}$ with supremum norm [**29**]. It turns out that H^∞ is the unique maximal algebra making the map a homomorphism between algebras. In section 1, we saw that H^∞ can be regarded as a subalgebra of $L^\infty(\partial\mathbf{D})$.

We note that the functional calculus with H^∞ functions can be defined in terms of independent of the minimal unitary dilation. Indeed, if $u(z) = \sum_{n=0}^{\infty} a_n z^n$ is in H^∞, then

$$u(T) = \lim_{r\to 1} u(rT) = \lim_{r\to 1} \sum_{n=0}^{\infty} a_n r^n T^n, \tag{1.6}$$

where the limit exists in the strong operator topology.

B. Sz.-Nagy and C. Foias introduced this important functional calculus for completely nonunitary contractions.

Proposition 3.2. *Let $T \in L(H)$ be a completely nonunitary contraction. Then there is a unique algebra representation Φ_T from H^∞ into $L(H)$ such that :*

(i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;

(ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in \boldsymbol{D}$;

(iii) Φ_T is continuous when H^∞ and $L(H)$ are given the weak-topology.*

(iv) Φ_T is contractive, i.e. $\|\Phi_T(u)\| \le \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B.Sz.- Nagy and C. Foias [**29**] defined the *class* C_0 relative to the open unit disk $\mathbf{D}$ consisting of completely nonunitary contractions T on H such that the kernel of Φ_T is not trivial. If $T \in L(H)$ is an operator of class C_0, then

$$\ker\, \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak*-closed ideal of H^∞, and hence there is an inner function generating ker Φ_T. The *minimal function* m_T of an operator of class C_0 is the generator of ker Φ_T, and it seems as a substitute for the minimal polynomial. Also, m_T is uniquely determined up to a constant scalar factor of absolute value one [**6**]. The theory of class C_0 relative to the open unit disk has been described by B.Sz.- Nagy, C. Foias ([**29**]) and H. Bercovici ([**6**]).

1.2. Jordan Operator. We know that every $n \times n$ matrix over an algebraically closed field F is similar to a unique Jordan canonical form. To extend that theory to the C_0 operator $T \in L(H)$, B.Sz.- Nagy and C. Foias [**29**] introduced a weaker notion of equivalence. They defined a *quasiaffine transform* of T which is bounded operator T' defined on a Hilbert space H' such that there exists an injective operator $X \in L(H, H')$ with dense range in H' satisfying $T'X = XT$. We write

$$T \prec T'$$

if T is a quasiaffine transform of T'. Instead of similarity, they introduced *quasisimilarity* of two operators, namely, T and T' are *quasisimilar*, denoted by

$$T \sim T',$$

if $T \prec T'$ and $T' \prec T$.

Given an inner function $\theta \in H^\infty$, the *Jordan block* $S(\theta)$ is the operator acting on $H(\theta) = H^2 \ominus \theta H^2$, which means the orthogonal complement of θH^2 in the Hardy space H^2, as follows :

$$S(\theta) = P_{H(\theta)} S | H(\theta) \tag{1.7}$$

where $S \in L(H^2)$ is the unilateral shift operator defined by

$$(Sf)(z) = zf(z)$$

and $P_{H(\theta)} \in L(H^2)$ denotes the orthogonal projection of H^2 onto $H(\theta)$.

PROPOSITION 3.3. [**6**] *For every inner function θ in H^∞, the operator $S(\theta)$ is of class C_0 and its minimal function is θ.*

PROPOSITION 3.4. [**6**] *Let $T_1 \in L(H)$ and $T_2 \in L(H)$ be two completely nonunitary contarctions of class C_0. If T_1 and T_2 are quasisimilar, then $m_{T_1} \equiv m_{T_2}$.*

From Proposition 3.3 and Proposition 3.4, we can easily see that for every inner functions θ_1 and θ_2 in H^∞, if $S(\theta_1)$ and $S(\theta_2)$ are quasisimilar, then $\theta_1 \equiv \theta_2$. Conversely,

PROPOSITION 3.5. [**6**] *Let θ_1 and θ_2 be inner functions in H^∞. If $\theta_1 \equiv \theta_2$, then $S(\theta_1)$ and $S(\theta_2)$ are quasisimilar.*

Let

$$\Theta = \{\theta_i \in H^\infty(\Omega) : i = 1, 2, 3, \cdots\}$$

be a family of inner functions. Then Θ is called a *model function* if $\theta_i \mid \theta_j$ whenever $j \leq i$. The *Jordan operator* $S(\Theta)$ determined by the model function Θ is the C_0-operator defined as

$$S(\Theta) = \bigoplus_{i<\gamma'} S(\theta_i),$$

where $\gamma' = \min\{k : \theta_k \equiv 1\}$. We will call $S(\Theta)$ the *Jordan model* of the operator T if

$$S(\Theta) \sim T,$$

and in the sequel $\bigoplus_{i<\gamma'} S(\theta_i)$ always means a *Jordan operator* determined by a model function.

By using Jordan blocks, C_0-operators relative to the open unit disk $\mathbf{D}$ can be classified ([**6**] Theorem 5.1) :

THEOREM 3.6. *Any C_0-operator T relative to the open unit disk $\boldsymbol{D}$ acting on a Hilbert space is quasisimilar to a unique Jordan operator.*

THEOREM 3.7. *If Θ and Θ' are two model functions and $S(\Theta) \prec S(\Theta')$, then $\Theta \equiv \Theta'$ and hence $S(\Theta) = S(\Theta')$.*

From Theorem 3.6 and Theorem 3.7, we can conclude that " $\prec$ " is an equivalence relation on the set of C_0-operators.

2. C_0-Operator Relative To Ω

J.A. Ball([**4**]) defined the class of C_0 relative to a bounded finitely connected region Ω in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves.

2.1. Functional Calculus. Let H be a Hilbert space and K be a compact subset of the complex plane.

DEFINITION 3.8. If $T \in L(H)$ and $\sigma(T) \subseteq K$, we say that K is a *spectral set* for the operator T if

$$\|r(T)\| \leq \max \{|r(z)| : z \in K\},$$

whenever r is a rational function with poles off K, i.e. $r \in R(K)$.

To define a C_0-operator relative to Ω, we need to provide a representation defined on $H^\infty(\Omega)$ and need to provide an operator with other properties instead of a completely nonunitary operator.

If $T \in L(H)$ is an operator with $\overline{\Omega}$ as a spectral set and with no normal summand with spectrum in $\partial\Omega$, that is, T has no reducing subspace $M \subseteq H$ such that $T|M$ is normal and $\sigma(T|M) \subseteq \partial\Omega$, then we say that T satisfies *hypothesis (h).*

THEOREM 3.9. *([**31**], Theorem 3.1.4) Let $T \in L(H)$ be an operator satisfying hypothesis (h). Then there is a unique algebra representation Ψ_T of $H^\infty(\Omega)$ into $L(H)$ such that :*

(i) $\Psi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator.

(ii) $\Psi_T(g) = T$, if $g(z) = z$ for all $z \in \Omega$.

(iii) Ψ_T is continuous when $H^\infty(\Omega)$ and $L(H)$ are given the weak*−topology.

(iv) Ψ_T is contractive, that is, $\|\Psi_T(f)\| \leq \|f\|$ for all $f \in H^\infty(\Omega)$.

From now on we will denote $\Psi_T(f)$ by $f(T)$ for all $f \in H^\infty(\Omega)$. With this functional calculus, we can provide the definition of a C_0-operator relative to Ω as in the previous section.

DEFINITION 3.10. An operator T satisfying hypothesis (h) is said to be of *class* C_0 if the kernel of Ψ_T is not trivial.

2.2. Jordan Model. A.Zucchi provided a classification of C_0 operators relative to Ω by a similar way to that of C_0 operators relative to the open unit disk([**31**]).

A nonzero function θ in $H^\infty(\Omega)$ is said to be *inner* if $|\theta|$ is constant a.e. on each component of $\partial\Omega$. Let θ and θ' be two inner functions in $H^\infty(\Omega)$. We say that θ *divides* θ' (or $\theta|\theta'$) if θ' can be written as $\theta' = \theta \cdot \phi$ for some $\phi \in H^\infty(\Omega)$. We will use the notation

$$\theta \equiv \theta'$$

if θ and θ' are two inner functions in $H^\infty(\Omega)$ that differ only by a scalar factor. Thus the relations $\theta|\theta'$ and $\theta'|\theta$ imply that $\theta \equiv \theta'$.

Clearly every inner function is also quasi-inner. To use Theorem 2.14, we define a Jordan block $S(\theta)$ for a quasi-inner function θ instead of an inner function.

PROPOSITION 3.11. [**24**] *If M is a closed $R(\Omega)$-invariant subspace of $H^p(\Omega)$ $(1 \leq p \leq \infty)$, then there is an inner function φ in $H^\infty(\Omega)$ such that $M = \varphi H^p(\Omega)$.*

DEFINITION 3.12. (i) Given a quasi-inner function $\theta \in H^\infty(\Omega)$, the *Jordan block* $S(\theta)$ is the operator acting on the space $H(\theta) = H^2(\Omega) \ominus \theta H^2(\Omega)$ as follows :

$$S(\theta) = P_{H(\theta)} S | H(\theta)$$

where $S \in L(H^2(\Omega))$ is defined by $(Sf)(z) = zf(z)$.

(ii) Let

$$\Theta = \{\theta_i \in H^\infty(\Omega) : i = 1, 2, 3, \cdots\}$$

be a family of quasi-inner functions. Then Θ is called a *model function* if $\theta_i \mid \theta_j$ whenever $j \leq i$. The *Jordan operator* $S(\Theta)$ determined by the model function Θ is the C_0-operator defined as

$$S(\Theta) = \bigoplus_{i<\gamma'} S(\theta_i),$$

where $\gamma' = \min\{k : \theta_k \equiv 1\}$.

LEMMA 3.13. *If θ is a quasi-inner function in $H^\infty(\Omega)$, then there is an inner function $\theta_1 \in H^\infty(\Omega)$ such that $S(\theta) = S(\theta_1)$.*

PROOF. By Theorem 2.14, $\theta H^2(\Omega)$ is a closed $R(\Omega)$-invariant subspace of $H^2(\Omega)$ and so by Proposition 3.11, there is an inner function $\theta_1 \in H^\infty(\Omega)$ such that $\theta H^2(\Omega) = \theta_1 H^2(\Omega)$. It follows that $H(\theta) = H(\theta_1)$. Thus $S(\theta) = S(\theta_1)$. □

We will call $S(\Theta)$ the *Jordan model* of the operator T if $S(\Theta) \sim T$, and in the sequel $\bigoplus_{i<\gamma'} S(\theta_i)$ always means a *Jordan operator* determined by a model function. A.Zucchi [**31**] provided the following classification of C_0 operators relative to Ω.

THEOREM 3.14. *Every operator T of class C_0 relative to Ω is quasisimilar to a Jordan operator $S(\Theta)$. Moreover $S(\Theta)$ is uniquely determined by either of the relations $S(\Theta) \prec T$ or $T \prec S(\Theta)$.*

2.3. Scalar Multiples. Let K and K' be Hilbert spaces and $\varphi \in H^\infty(\Omega, L(K, K'))$ be a quasi-inner function. By Proposition 2.13, $\varphi H^2(\Omega, K)$ is a closed subspace of $H^2(\Omega, K')$. Thus we define

$$H(\varphi) = H^2(\Omega, K') \ominus \varphi H^2(\Omega, K)$$

and denote by $S(\varphi)$ the compression of $S_{K'}$ to $H(\varphi)$, i.e.,

$$S(\varphi) = P_{H(\varphi)} S_{K'} | H(\varphi),$$

where $P_{H(\varphi)}$ denotes the orthogonal projection onto $H(\varphi)$. Until now, a lot of problems for the C_0-operator relative to $\mathbf{D}$ have been solved. In this paper, we will prove some similar results for the C_0-operator relative to Ω by using quasi-inner functions instead of inner functions in H^∞.

In Proposition 3.3, we have already mentioned that for any inner function $\theta \in H^\infty$, the Jordan block $S(\theta)$ is of class C_0 with minimal function. Even though $S(\varphi)$ is defined by the same way as $S(\theta)$, φ is an operator-valued function. Thus to study the minimal function of $S(\varphi)$, we provide a new concept.

DEFINITION 3.15. The function $\varphi \in H^\infty(\Omega, L(K, K'))$ is said to have a *scalar multiple* $u \in H^\infty(\Omega)$, $u \neq 0$, if there exists $\psi \in H^\infty(\Omega, L(K', K))$ satisfying the relations $\varphi(z)\psi(z) = u(z) I_{K'}$ and $\psi(z)\varphi(z) = u(z) I_K$ for $z \in \Omega$, that is $\varphi | u$.

In fact, by using scalar multiple of a quasi-inner function $\varphi \in H^\infty(\Omega, L(K, K'))$, we can determine whether $S(\varphi)$ is of class C_0 relative to Ω or not.

THEOREM 3.16. *Suppose that $\varphi \in H^\infty(\Omega, L(K, K'))$ is a quasi-inner function and $u \in H^\infty(\Omega)$. Then the following assertions are equivalent :*

(a) u is a scalar multiple of φ.

(b) $u(S(\varphi)) = 0$.

(c) $uH^2(\Omega, K') \subset \varphi H^2(\Omega, K)$.

PROOF. Assume (a), and let $\psi \in H^\infty(\Omega, L(K', K))$ satisfy the relation $\varphi(z)\psi(z) = u(z)I_{K'}$ for $z \in \Omega$. Then

$$u(S(\varphi))H(\varphi) = P_{H(\varphi)}u(S_{K'})H(\varphi) \subset P_{H(\varphi)}uH^2(\Omega, K') \subset P_{H(\varphi)}\varphi H^2(\Omega, K).$$

Thus $u(S(\varphi)) = 0$. Thus $(a) \to (b)$.

Next, assume (b). Then $u(S_{K'})H(\varphi) = uH(\varphi) \subset \varphi H^2(\Omega, K)$. It follows that

$$uH^2(\Omega, K') = uH(\varphi) + u\varphi H^2(\Omega, K) \subset \varphi H^2(\Omega, K).$$

Thus $(b) \to (c)$.

In order to prove $(c) \to (a)$, let

$$M = \{f \in H^2(\Omega, K) : ug = \varphi f \text{ for some } g \in H^2(\Omega, K')\}.$$

Then $\overline{M}$ is rationally invariant. By Theorem 2.14, there is a Hilbert space K_1 and a quasi-inner function $\varphi_1 \in H^\infty(\Omega, L(K_1, K))$ such that

$$\overline{M} = \varphi_1 H^2(\Omega, K_1).$$

From the Theorem 2.2.4 in [**31**], $u = \theta F$ where θ is a function such that $|\theta|$ is constant almost everywhere on each component of $\partial\Omega$, i.e. an inner function in $H^\infty(\Omega)$, and F is an outer function in $H^\infty(\Omega)$. By the definition of M,

$$\theta H^2(\Omega, K') = \overline{\theta F H^2(\Omega, K')} = \overline{uH^2(\Omega, K')} = \overline{\varphi M} = \varphi\overline{M} = \varphi\varphi_1 H^2(\Omega, K_1).$$

Since θ is inner, $\theta I_{K'} \in H^\infty(\Omega, L(K'))$ is quasi-inner. Then by Corollary 2.16, there exist $\varphi_2 \in H^\infty(\Omega, L(K', K_1))$ such that $\theta I_{K'} = \varphi\varphi_1\varphi_2$. Then $uI_{K'} = \varphi(F\varphi_1\varphi_2)$, that is,

$$u(z)I_{K'} = \varphi(z)(F(z)\varphi_1(z)\varphi_2(z)).$$

Since $F\varphi_1\varphi_2 \in H^\infty(\Omega, L(K', K))$, it is proven. $\square$

If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ is given, then $\varphi(z)$ can be regarded as an $n \times n$ matrix. Let det be the determinant function on $n \times n$ matrices over $\mathbb{C}$ and for a $n \times n$ matrix A, $\operatorname{adj} A$ be the algebraic classical adjoint of A, i.e. $(\texttt{adj}\ \ A)A = A(\texttt{adj}\ \ A) = (\texttt{det}\ \ A)I$. Then we define two functions det φ and adj φ on Ω by $(\det\ \varphi)(z) = \det\ \varphi(z)$ and $(\operatorname{adj}\ \varphi)(z) = \operatorname{adj}\ \varphi(z)$.

In the next Proposition, we will see that when $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ is a quasi-inner function, det φ and adj φ are also quasi-inner.

Proposition 3.17. *Let K and K' be Hilbert spaces with* $\dim K = \dim K' = n(< \infty)$.

(a) If $\varphi \in H^\infty(\Omega, L(K, K'))$ is a quasi-inner function, then $\varphi(z)$ is invertible a.e. on $\partial\Omega$.

(b) If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ is a quasi-inner function, then θ, defined by $\theta(z)=$ $\det \varphi(z)$, is quasi-inner.

(c) If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ is a quasi-inner function, then $\operatorname{adj} \varphi$ *is quasi-inner.*

(d) If $\varphi \in H^\infty(\Omega, L(K, K'))$ be a quasi-inner function, then $S(\varphi)$ on $H(\varphi)$ is of class C_0.

Proof. (a) Since $\varphi \in H^\infty(\Omega, L(K, K'))$, φ has nontangential limits for any $z \in \partial\Omega \backslash A$ with $m(A) = 0$. For a fixed $z_0 \in \partial\Omega \backslash A$, since for some $c > 0$ and $a \in K$,

$$c\,\|a\| \le \|\varphi(z_0)a\|,$$

range of $\varphi(z_0)$ is closed, and $\varphi(z_0)$ is one-to-one. By the first isomorphism theorem,

$$\varphi(z_0)K \cong \text{range of } \varphi(z_0).$$

Thus K and range of $\varphi(z_0)$ have the same dimension.

Since dim $K =$ dim K' and range of $\varphi(z_0)$ is a closed subspace of K', we conclude range of $\varphi(z_0) = K'$, that is, $\varphi(z_0)$ is one-to-one and onto. Thus $\varphi(z)$ is invertible for $z \in \partial\Omega \backslash A$.

(b) If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$, then $\theta = \det \varphi$ is a polynomial and so $\theta \in H^\infty$. Since φ is quasi-inner, there exist $m(> 0)$, and $M(> 0)$ such that for $h \in \mathbb{C}^n$,

$$m \|h\| \leq \|\varphi(z)h\| \leq M \|h\| \ \texttt{a.e.} \quad z \in \partial\Omega.$$

Then

$$m^n \leq |\det \varphi(z)| = |\theta(z)| \leq M^n \texttt{a.e.} z \in \partial\Omega.$$

Thus $\theta \in H^\infty$ is quasi-inner.

(c) By (b), there is $m_1 > 0$ such that $m_1 \leq |\det\ \varphi(z)|$ a.e. $z \in \partial\Omega$. Since φ is quasi-inner, there exist $c_1(> 0)$, and $c_2(> 0)$ such that for $h \in \mathbb{C}^n$, $c_1 \|h\| \leq \|\varphi(z)h\| \leq c_2 \|h\|$ a.e. $z \in \partial\Omega$. Since $(\varphi(z)(\text{adj}\ \varphi(z)) = \det\ \varphi(z) I_{C^n}$, for $h \in C^n$, $m_1 \|h\| \leq |\det\ \varphi(z)\ |\ \|h\| = \|\varphi(z)\ \text{adj}\ \varphi(z)h\| \leq c_2 \|\text{adj}\ \varphi(z)h\ \|$. Thus for $h \in C^n$,

$$\frac{m_1}{c_2} \|h\| \leq \|adj\ \varphi(z)h\| \tag{2.1}$$

a.e. $z \in \partial\Omega$.

Next, $c_1 \|\ \text{adj}\, \varphi(z)h\| \leq \|\varphi(z)\, \text{adj}\, \varphi(z)h\| = |\det \varphi(z)|\, \|h\| \leq \|\det \varphi(z)\|_\infty\ \|h\|$ a.e. $z \in \partial\Omega$. Since $\varphi \in H^\infty$, $\|\det \varphi(z)\|_\infty = M_0 < \infty$. Thus

$$\|adj\ \varphi(z)h\| \leq \frac{M_0}{c_1} \|h\| \tag{2.2}$$

a.e. $z \in \partial\Omega$. Clearly adj φ is holomorphic, i.e. adj $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$. From (2.1) and (2.2), we conclude that adj φ is also quasi-inner.

(d) By Theorem 3.16, it's enough to prove that φ has a scalar multiple $u \in H^\infty(\Omega)$. Let $\psi(z) =$ adj $\varphi(z)$ and $u(z) = \det\ \varphi(z)$. Then by (c), $\psi \in H^\infty(\Omega, L(K', K))$ and by (b), $u \in H^\infty(\Omega)$. Since $\varphi(z)(\text{adj}\ \varphi(z)) = (\det\ \varphi(z)) I_{K'}$, for $z \in \Omega$, it is proven. □

COROLLARY 3.18. *Let K and K' be Hilbert spaces with dim K = dim K' = $n(<\infty)$. If $\varphi \in H^\infty(\Omega, L(K, K'))$ is quasi-inner, and $\psi \in H^\infty(\Omega, L(K', K))$ such that $\varphi(z)\psi(z) = u(z)I_{K'}$ for $u \in H^\infty(\Omega)$, $u \neq 0$, then $\psi(z)\varphi(z) = u(z)I_K$.*

PROOF. By Proposition 3.17 (a), $\varphi(z)$ is invertible a.e. on $\partial\Omega$. From the equation $\varphi(z)\psi(z) = u(z)I_{K'}$, we obtain

$$\psi(z) = \varphi(z)^{-1}[u(z)I_{K'}] = u(z)\varphi(z)^{-1}I_{K'} = u(z)I_K\varphi(z)^{-1}$$

for a.e. $z \in \partial\Omega$. It follows that for a.e. $z \in \partial\Omega$, $\psi(z)\varphi(z) = u(z)I_K$ which proves this Corollary. □

From Corollary 3.18, we see that if K and K' are Hilbert spaces with finite dimension, a quasi-inner function $\varphi \in H^\infty(\Omega, L(K, K'))$ has a scalar multiple if and only if there exists $\psi \in H^\infty(\Omega, L(K', K))$ satisfying the relations

$$\varphi(z)\psi(z) = u(z)I_{K'}$$

for $z \in \Omega$

COROLLARY 3.19. *(a) If $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $u \in H^\infty(\Omega)$ are quasi-inner functions such that*

$$\varphi(z)\psi(z) = u(z)I_{\mathbb{C}^n}$$

where $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$, then ψ is also quasi-inner.

(b) Conversely, if $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$ are quasi-inner functions such that

$$\varphi(z)\psi(z) = u(z)I_{\mathbb{C}^n} \text{ or } \psi(z)\varphi(z) = u(z)I_{\mathbb{C}^n}$$

for some $u \in H^\infty(\Omega)(u \neq 0)$, then u is quasi-inner.

PROOF. (a) It could be proven in the same way as the proof of Proposition 3.17 (c).

(b) Since $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $\psi \in H^\infty(\Omega, L(\mathbb{C}^n))$ are quasi-inner functions, there exist $m_1(>0)$, and $m_2(>0)$ such that for $h \in \mathbb{C}^n$, $m_1 \|h\| \le \|\varphi(z)h\|$ and $m_2 \|h\| \le \|\varphi(z)h\|$ a.e. $z \in \partial\Omega$. Then

$$\|\varphi(z)\psi(z)h\| \ge m_1 \|\psi(z)h\| \ge m_1 m_2 \|h\|$$

and so $\|\varphi(z)\psi(z)\| \ge m_1 m_2$. Let $m' = m_1 m_2$. Since $|u(z)| = \|\varphi(z)\psi(z)\|$, it is proven. □

From Corollary 3.19(a), we also conclude that If $u_1 \in H^\infty(\Omega)$ and $u_2 \in H^\infty(\Omega)$ are quasi-inner functions such that

$$u_1(z)u_3(z) = u_2(z)I_{\mathbb{C}^n}$$

where $u_3 \in H^\infty(\Omega)$, then u_3 is also quasi-inner.

Proposition 3.20. *Let $\varphi \in H^\infty(\Omega, L(\mathbb{C}^n))$ be a quasi-inner function and $\zeta \in H^\infty(\Omega)$ be a function defined by*

$$\zeta(z) = \text{adj } (\varphi(z)).$$

If $\theta_0 \in H^\infty(\Omega)$ is the minimal function of $S(\varphi)$, then

$$\det \varphi(z) \equiv \theta_0(z)D(\zeta)(z).$$

Proof. From the fact $(\text{adj } A)A = A(\text{adj } A) = (\det A)I_{\mathbb{C}^n}$ for any $n \times n$ matrix A, we obtain that

$$\varphi(z)\zeta(z) = \varphi(z)\,\text{adj}(\varphi(z)) = (\det \varphi(z))I_{\mathbb{C}^n}.$$

Since the minimal function of $S(\varphi)$ is θ_0, by Theorem 3.16, θ_0 is the least scalar multiple of φ and from the equation **??**, if $u(z) = \det \varphi(z)$, then $\theta_0 | u$. Let

$$\det \varphi(z) = \theta_0(z)q(z) \tag{2.3}$$

for $z \in \Omega$ where $q \in H^\infty(\Omega)$. Since θ_0 is the least scalar multiple of φ, there exists $\psi \in H^\infty(\ \Omega,\ L(\mathbb{C}^n))$ satisfying the relation

$$\varphi(z)\psi(z) = \theta_0(z)I_{\mathbb{C}^n}$$

for $z \in \Omega$. Then

$$\varphi(z)q(z)\psi(z) = q(z)\theta_0(z)I_{\mathbb{C}^n} = \det\,\varphi(z)I_{\mathbb{C}^n} = \varphi(z)\,\mathrm{adj}\,(\varphi(z)) = \varphi(z)\zeta(z),$$

and so

$$\varphi(z)[q(z)\psi(z) - \zeta(z)] = 0. \tag{2.4}$$

Since φ is a quasi-inner function, equation (2.4) implies that $q(z)\psi(z) = \zeta(z)$. Thus

$$q|D(\zeta) \tag{2.5}$$

and $\zeta(z) = \mathrm{adj}\,(\varphi(z)) = D(\mathrm{adj}\,(\varphi(z)))\psi'(z)$ for some $\psi' \in H^\infty(\,\Omega,\, L(\mathbb{C}^n))$.

It follows that $D(\mathrm{adj}\,(\varphi(z)))\psi'(z)\varphi(z) = [\mathrm{adj}\,(\varphi(z))]\varphi(z) = \det\,\varphi(z)I_{\mathbb{C}^n}$ or $\psi'(z)\varphi(z) = \varphi(z)\psi'(z) = [\det\,\varphi(z)/D(\mathrm{adj}\,(\varphi(z)))]I_{\mathbb{C}^n} = \frac{\det\varphi(z)}{D(\zeta)(z)}I_{\mathbb{C}^n}$. By Corollary 3.18,

$$\psi'(z)\varphi(z) = \varphi(z)\psi'(z) = \tfrac{\det\varphi(z)}{D(\zeta)(z)}I_{\mathbb{C}^n},$$

and so by Theorem 3.16,

$$\theta_0|\frac{\det\varphi(z)}{D(\zeta)(z)}. \tag{2.6}$$

By (2.5), there is a function $q_1 \in H^\infty(\Omega)$ such that $D(\zeta)(z) = q(z)q_1(z)$. Then from equation (2.3), we obtain $[\det\,\varphi(z)]q_1(z) = \theta_0(z)q(z)q_1(z) = \theta_0(z)D(\zeta)(z)$ and so

$$\det\varphi(z)|\theta_0(z)D(\zeta)(z). \tag{2.7}$$

From (2.6) and (2.7), we can conclude that $\det\,\varphi(z) \equiv \theta_0(z)D(\zeta)(z)$.

□

COROLLARY 3.21. *Let φ and ψ be quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. If $\psi(z)\varphi(z) = \phi(z)I_{\mathbb{C}^n}(z \in \Omega)$ for some $\phi \in H^\infty(\Omega)$ and $S(\varphi) \sim \bigoplus_{i<\omega} S(\theta_i)$ ($\theta_i \in H^\infty(\Omega)$ is a quasi-inner function for $i = 0,\, 1,\, 2...$), then $D(\psi)\theta_0 \equiv \phi$.*

PROOF. This Corollary can be proven if we substitute $\phi(z)$ and $\psi(z)$ instead of det $\varphi(z)$ and adj $(\varphi(z))$ respectively in Proposition 3.20. □

3. Quasi-equivalence and Quasi-similarity

3.1. Normal Form. Let M be an $m \times n$ matrix with entries in the polynomial algebra $F[x]$ over a field F, $1 \leq k \leq \min(m, n)$, and $\delta_k(M)$ is the greatest common divisor of the determinant of all $k \times k$ submatrices of M. It is well known that M is equivalent to a matrix N which is in normal form [**20**], i.e., every entry off the main diagonal of N is 0 and on the main diagonal of N there appear (in order) polynomials f_1,..., f_l such that f_k divides f_{k+1}, $1 \leq k \leq l-1$. In fact,

$$f_k = \frac{\delta_k(M)}{\delta_{k-1}(M)}. \tag{3.1}$$

To get a similar conclusion for any $n \times n$ matrices over $H^\infty(\Omega, K)$, we need another equivalence relation on the set of all $n \times n$ matrices over $H^\infty(\Omega, K)$.

DEFINITION 3.22. A *quasi-unit* X of order n is a collection of $n \times n$ matrices over $H^\infty(\Omega)$ such that for det $X = \{\det A : A \in X\}$, det X is relatively prime, that is,

$$\bigwedge_{A \in X} \det A \equiv 1.$$

DEFINITION 3.23. If A and B are $m \times n$ matrices over $H^\infty(\Omega)$, then A will be called *quasi-equivalent* to B if there exist quasi-units X and Y of order m and n respectively such that $XA = BY$.

In [**15**], Theorem 4.7.3 says that each function $f \in H^p(\Omega)$ has a factorization

$$f = \theta F \tag{3.2}$$

where θ is inner and F is an outer function in $H^p(\Omega)$. In this paper, $(f)_i$ denotes θ, i.e. inner part θ when we have the factorization 3.2.

DEFINITION 3.24. If A and B are $n \times n$ matrices over $H^\infty(\Omega)$ and δ is an $H^\infty(\Omega)$ function, then A will be called *δ-equivalent* to B if there are square matrices X and Y of order n such that $XA = BY$ and $(\det X)_i$ and $(\det Y)_i$ are factors of δ.

It is immediate that if A is *δ-equivalent* to B for all δ in a relatively prime family, then A is *quasi-equivalent* to B. A matrix E over $H^\infty(\Omega)$ is in *normal form* or simply, *normal* provided

$$E = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \tag{3.3}$$

where D is a diagonal matrix of nonzero quasi-inner functions, each with a positive first nonvanishing Taylor coefficient, and each one except the first divisible by its predecessor.

DEFINITION 3.25. Let $D_k(A)$ be the greatest common quasi-inner divisor of all minors of rank k of A (k is no larger than $\min\{m, n\}$) and D_0=1. Then the *invariant factors* for a $m \times n$ matrix A over $H^\infty(\Omega)$ are defined by

$$\xi_k(A) = D_k(A)/D_{k-1}(A)$$

for $k \geq 1$ such that some minors of rank k are not 0.

With these definitions, we obtain the following conclusion.

PROPOSITION 3.26. *Let K be a finite dimensional Hilbert space. Every $n \times n$ matrices over $H^\infty(\Omega, K)$ is quasi-equivalent to a unique normal matrix. In fact, given any $n \times n$ matrix A over $H^\infty(\Omega, K)$ and any quasi-inner function ψ, A is δ-equivalent to the normal matrix formed from the invariant factors of A for some δ relatively prime to ψ.*

PROOF. The proof is analogous to that of Theorem 3.1 in [**23**] □

Corollary 3.27. *Let φ be a quasi-inner function in $H^\infty(\Omega, L(\mathbb{C}^n))$. If A is a matrix over $H^\infty(\Omega)$ representing φ, and A is quasi-equivalent to a normal matrix N whose diagonal entries are $\theta_0, ..., \theta_{n-1}$, then*

$$\det \varphi(z) \equiv \theta_0 \cdots \theta_{n-1}.$$

Proof. Let $\psi = \theta_0 \cdots \theta_{n-1}$. By Proposition 3.26, there is $\delta_1 (\in H^\infty(\Omega))$ relatively prime to ψ such that A and N are δ_1-equivalent. Thus there are square matrices X_1 and Y_1 of order n such that $X_1 A = N Y_1$, $(\det X_1)_i | \delta_1$, and $(\det Y_1)_i | \delta_1$. Since δ_1 is relatively prime to $\psi = \det N$, so are $(\det X_1)_i$ and $(\det Y_1)_i$. It follows that

$$\det N = \theta_0 \cdots \theta_{n-1} | (\det A)(\det X_1)$$

and so

$$(\det N)|(\det A). \tag{3.4}$$

From Proposition 3.17, if $\omega = \det A$, then $\omega(z) = \det \varphi(z)$ is a quasi-inner function. By Proposition 3.26, there is $\delta_2 \in H^\infty(\Omega)$ relatively prime to ω such that A and N are δ_2-equivalent. Thus there are square matrices X_2 and Y_2 of order n such that

$$X_2 A = N Y_2,\ (\det X_2)_i | \delta_2, \text{ and } (\det Y_2)_i | \delta_2.$$

Clearly $(\det X_2)_i$ and $(\det Y_2)_i$ are relatively prime to ω. Because $X_2 A = N Y_2$ and $(\det Y_2)_i$ is relatively prime to $\omega = \det A$,

$$(\det A)|(\det N).$$

Therefore 3.4 and 3.1 imply that $\det A = \det \varphi(z) \equiv \det N \equiv \theta_0 \cdots \theta_{n-1}$. □

Let f_1 and f_2 be in $H^\infty(\Omega)$. If M is the w^*-closure of $f_1 H^\infty(\Omega) + f_2 H^\infty(\Omega)$, then by the same way as Theorem 1 in [**24**], we obtain $M = (f_1 \wedge f_2) H^\infty(\Omega)$.

Theorem 3.28. *Let φ_1, $\varphi_2 \in H^\infty(\Omega)$ be such that $\varphi_1 \wedge \varphi_2 \equiv 1$. If $f \in L^2(\partial\Omega, \mathbb{C}^n)$, $\varphi_1 f \in H^2(\partial\Omega, \mathbb{C}^n)$ and, $\varphi_2 f \in H^2(\partial\Omega, \mathbb{C}^n)$, then $f \in H^2(\partial\Omega, \mathbb{C}^n)$.*

PROOF. Since $\varphi_1 \wedge \varphi_2 \equiv 1$, w^*-closure of $\varphi_1 H^\infty(\partial\Omega) + \varphi_2 H^\infty(\partial\Omega)$ is $H^\infty(\partial\Omega)$. Thus there are nets $\{f_\alpha\}$ and $\{g_\alpha\}$($\alpha \in A$ which is a directed set) in $H^\infty(\partial\Omega)$ such that $h_\alpha = \varphi_1 f_\alpha + \varphi_2 g_\alpha$ *weak**-converges to 1, that is,

$$\int_{\partial\Omega} (h_\alpha - 1) h dm \to 0 \tag{3.5}$$

for any $h \in L^1(\partial\Omega)$. We will prove that $h_\alpha f \to f$ weakly in $L^2(\partial\Omega, \mathbb{C}^n)$, that is,

$$((h_\alpha f - f), g) \to 0$$

for any $g \in L^2(\partial\Omega, \mathbb{C}^n)$. If $f = (f_1, \cdots, f_n)$ and $g = (g_1, \cdots, g_n)$, then $((h_\alpha f - f), g) = \sum_{i=1}^{i=n} \int_{\partial\Omega} (h_\alpha - 1) f_i \bar{g}_i dm$. Since $f,\ g \in L^2(\partial\Omega, \mathbb{C}^n)$, f_i and $\bar{g}_i \in L^2(\partial\Omega)$ for any $i = 1, \cdots, n$. By the *Hölder'* inequality, $f_i \cdot \bar{g}_i \in L^1(\partial\Omega)$. From (3.5), $\int_{\partial\Omega} (h_\alpha - 1) f_i \bar{g}_i dm \to 0$. It follows that $\sum_i \int_{\partial\Omega} (h_\alpha - 1) f_i \bar{g}_i dm \to 0$ and so

$$h_\alpha f \to f \tag{3.6}$$

weakly in $L^2(\partial\Omega, \mathbb{C}^n)$. Since a subspace of a Banach space is norm closed if and only if it is weakly closed [**11**] and $H^2(\partial\Omega, \mathbb{C}^n)$ is a closed subspace of $L^2(\partial\Omega, \mathbb{C}^n)$, $H^2(\partial\Omega, \mathbb{C}^n)$ is weakly closed. Since $\varphi_1 f,\ \varphi_2 f \in H^2(\partial\Omega, \mathbb{C}^n)$, $h_\alpha f \in H^2(\partial\Omega, \mathbb{C}^n)$. From (3.6), we conclude that $f \in H^2(\partial\Omega, C^n)$. □

3.2. Quasi-similarity. In this section, we will discuss about some relations of quasi-equivalence, quasi-similarity, and quasi-affinity.

PROPOSITION 3.29. *Let φ_1 and φ_2 are quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. If φ_1 and φ_2 are quasi-equivalent, then $S(\varphi_1)$ and $S(\varphi_2)$ are quasisimilar.*

PROOF. Since " $\prec$ " is an equivalence relation on the set of C_0-operators, it will suffice to prove $S(\varphi_1) \prec S(\varphi_2)$. In the same way as Corollary 3.27, there are functions Δ and Λ in $H^\infty(\Omega, L(\mathbb{C}^n))$ whose determinants are relatively prime to those of φ_1

and φ_2 and satisfying the equation

$$\Delta\varphi_1 = \varphi_2\Lambda. \tag{3.7}$$

Since φ_2 is a quasi-inner function in $H^\infty(\Omega, L(\mathbb{C}^n))$, $\varphi_2 H^2(\Omega, \mathbb{C}^n)$ is a closed subspace of $H^2(\Omega, \mathbb{C}^n)$.

Define

$$X = P_{H(\varphi_2)}\Delta|H(\varphi_1).$$

Since $\Delta\varphi_1 H^2(\Omega, \mathbb{C}^n) \subset \varphi_2 H^2(\Omega, \mathbb{C}^n)$, for any $f \in H(\varphi_1)$, $XS(\varphi_1)f = P_{H(\varphi_2)}\Delta P_{H(\varphi_1)}Sf = P_{H(\varphi_2)}\Delta Sf = P_{H(\varphi_2)}S\Delta f = P_{H(\varphi_2)}SP_{H(\varphi_2)}\Delta f = S(\varphi_2)Xf$. Thus

$$XS(\varphi_1) = S(\varphi_2)X.$$

Next, we will prove that X is quasi-affinity. Suppose $v \in H(\varphi_2)$ is orthogonal to the range of X, and let $u \in H^2(\Omega, \mathbb{C}^n)$. Then $u = u_1 + \varphi_1 u_2$ for some $u_1 \in H(\varphi_1)$ and $u_2 \in H^2(\Omega, \mathbb{C}^n)$. Since $v \in H(\varphi_2)$,

$$(v, \Delta u_1) = (P_{H(\varphi_2)}v, \Delta u_1) = (v, P_{H(\varphi_2)}\Delta u_1) = (v, Xu_1) = 0,$$

and equation (3.7) implies that

$$(v, \Delta\varphi_1 u_2) = (v, \varphi_2\Lambda u_1) = 0.$$

Thus v is orthogonal to $\Delta H^2(\Omega, \mathbb{C}^n)$, which includes $(\det\Delta)H^2(\Omega, \mathbb{C}^n)$, because of adj $\Delta \in H^\infty(\Omega, \mathbb{C}^n)$. However v is also orthogonal to $\varphi_2 H^2(\Omega, \mathbb{C}^n)$, which include $(\det\varphi_2)H^2(\Omega, \mathbb{C}^n)$.

Since $\det\Delta$ and $\det\varphi_2$ are relatively prime, $(\det\Delta)H^2(\Omega, \mathbb{C}^n)$ and $(\det\varphi_2)H^2(\Omega, \mathbb{C}^n)$ span $H^2(\Omega, \mathbb{C}^n)$, thus $v = 0$. Consequently X has dense range.

Let $\omega \in H(\varphi_1)$ such that $X\omega = 0$, i.e.

$$\Delta\omega \in \varphi_2 H^2(\Omega, \mathbb{C}^n). \tag{3.8}$$

By Proposition 3.17, there is a measure zero set B of $\partial\Omega$ such that $\varphi_1(z)$ is invertible on $\partial\Omega \setminus B$ and ω has nontangential limits on $\partial\Omega \setminus B$ and so we can define $g : \partial\Omega \to \mathbb{C}^n$ such that $g(z) = \varphi_1(z)^{-1}\omega(z)$ for $z \in \partial\Omega \setminus B$. Then $g \in L^2(\Omega, \mathbb{C}^n)$, because φ_1 is uniformly bounded below. Since $\varphi_1(z)g(z) = \omega(z)$,

$$\varphi_1 g \in H^2(\Omega, \mathbb{C}^n) \tag{3.9}$$

From the fact that $\varphi_2 \Lambda g = \Delta\varphi_1{\varphi_1}^{-1}\omega = \Delta\omega$ a.e. on $\partial\Omega$, by (3.8) we conclude that $\varphi_2\Lambda g \in \varphi_2 H^2(\Omega, C^n)$. Since φ_2 is quasi-inner, φ_2 is invertible a.e. on $\partial\Omega$. Thus

$$\Lambda g \in H^2(\partial\Omega, \mathbb{C}^n). \tag{3.10}$$

(3.9) and (3.10) imply that $(\det \varphi_1)g \in H^2(\partial\Omega, \mathbb{C}^n)$ and $(\det \Lambda)g \in H^2(\partial\Omega, \mathbb{C}^n)$.

Since $(\det \varphi_1) \wedge (\det \Lambda) \equiv 1$, by Theorem 3.28, $g \in H^2(\partial\Omega, \mathbb{C}^n)$. Then there is a function $F \in H^2(\Omega, \mathbb{C}^n)$ such that $F^* = g$ and so $\varphi_1 F = \omega$ a.e. on $\partial\Omega$. It follows that $\varphi_1 F = \omega$ on Ω. Thus

$$\omega \in \varphi_1 H^2(\Omega, C^n) \cap H(\varphi_1),$$

and so $\omega = 0$. Hence X is one-to-one and so $S(\varphi_1) \prec S(\varphi_2)$. □

Corollary 3.30. *Let φ be a quasi-inner function in $H^\infty(\Omega, L(\mathbb{C}^n))$. If A is a matrix over $H^\infty(\Omega)$ representing φ and A is quasi-equivalent to a normal matrix N whose diagonal entries are $\theta_0 \cdots \theta_{n-1}(\theta_{i+1}|\theta_i$ for $i = 0, 1, \cdots, n-1)$, then*

$$S(\varphi) \sim \oplus_{i=0}^{n-1} S(\theta_i).$$

Proof. Since $S(N) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, by Proposition 3.29, $S(\varphi) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, because " $\sim$ " is an equivalence relation. □

Corollary 3.31. *Let φ_1 and φ_2 be quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$. If $S(\varphi_1)$ is a quasi-affine transform of $S(\varphi_2)$, then φ_1 and φ_2 are quasi-equivalent.*

PROOF. Let ${\varphi_1}'$ and ${\varphi_2}'$ be the normal matrices that are quasi-equivalent to φ_1 and φ_2 respectively. Then by Proposition 3.29 and the hypothesis, $S({\varphi_1}') \sim S(\varphi_1) \prec S(\varphi_2) \sim S({\varphi_2}')$ and so

$$S({\varphi_1}') \prec S({\varphi_2}'). \tag{3.11}$$

By Corollary 3.30, if $\theta_0, \cdots, \theta_{n-1}$ and $g_0, \cdots, g_{n-1}$ are diagonal entries of ${\varphi_1}'$ and ${\varphi_2}'$ respectively, then $S({\varphi_1}') \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$ and $S({\varphi_2}') \sim \bigoplus_{i=0}^{n-1} S(g_i)$. Thus by (3.11),

$$\bigoplus_{i=0}^{n-1} S(\theta_i) \prec \bigoplus_{i=0}^{n-1} S(g_i). \tag{3.12}$$

It was shown that if one Jordan operator is a quasiaffine transform of another, then they are both determined by the same nonconstant quasi-inner functions (Theorem 4.3.9. in [**31**]). Consequently, ${\varphi_1}' = {\varphi_2}'$. It follows by transitivity that φ_1 and φ_2 are quasi-equivalent. □

COROLLARY 3.32. *Let φ be a quasi-inner function in $H^\infty(\Omega, L(\mathbb{C}^n))$. If $S(\varphi) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, then* $\det\, \varphi(z) \equiv \theta_0(z), \cdots, \theta_{n-1}(z)$ $(z \in \Omega)$.

PROOF. Let N be a normal matrix whose diagonal entries are $\theta_0, \cdots, \theta_{n-1}$. Since $S(N) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, $S(\varphi) \sim S(N)$. By Corollary 3.31, φ and N are quasi-equivalent. Then by Corollary 3.27, $\det\, \varphi(z) \equiv \theta_0(z), \cdots, \theta_{n-1}(z)$ □

4. Tensor operations

By using Beurling's theorem, every invariant subspace M of $S(\theta)$ for an inner function $\theta \in H^\infty$ is characterized in Proposition 1.10 [**6**], that is, M has the form $\phi H^2 \ominus \theta H^2$. We extend this result to a $R(\Omega)$-invariant for $S(\varphi)$ where φ is a quasi-inner function in $H^\infty(\Omega, L(F, F'))$ for some separable Hilbert spaces F and F'. When we prove this result, we will use our generalized Beurling's theorem (Theorem 2.14) instead of Beurling's theorem.

THEOREM 3.33. *Let F and F' be two separable Hilbert spaces and φ be a quasi-inner function in $H^\infty(\Omega, L(F,F'))$.*

(i) If $M \subset H(\varphi)$ is a rationally invariant subspace (i.e.$R(\Omega)$-invariant) for $S(\varphi)$, then there is a Hilbert space K and quasi-inner functions $\varphi_1 \in H^\infty(\Omega, L(F,K))$ and $\varphi_2 \in H^\infty(\Omega, L(K,F'))$ such that $\varphi(z) = \varphi_2(z)\varphi_1(z)$ for $z \in \Omega$ and

$$M = \varphi_2 H^2(\Omega, K) \ominus \varphi H^2(\Omega, F) \tag{4.1}$$

(ii) Conversely, if K, φ_1 and φ_2 are as above , then 4.1 defines a rationally invariant subspace of $H(\varphi)$. Moreover, if

$$S(\varphi) = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is the triangularization of $S(\varphi)$ with respect to the decomposition $H(\varphi) = M \oplus (H(\varphi) \ominus M)$, then $T_2 = S(\varphi_2)$ and $S(\varphi_1)$ is similar to T_1.

PROOF. (i). Since M is rationally invariant, the space $M \oplus \varphi H^2(\Omega, F)$ is also rationally invariant subspace of $H^2(\Omega, F')$ and so Theorem 2.14 implies the existence of a Hilbert space K and a quasi-inner function $\varphi_2 \in H^\infty(\Omega, L(K,F'))$ such that

$$M \oplus \varphi H^2(\Omega, F) = \varphi_2 H^2(\Omega, K). \tag{4.2}$$

Since $M \subset H(\varphi)$, by (4.2) we get (4.1) holds.

The inclusion $\varphi H^2(\Omega, F) \subset \varphi_2 H^2(\Omega, K)$ implies that for any $f \in H^2(\Omega, F)$ there is $\phi_f \in H^2(\Omega, K)$ such that $\varphi f = \varphi_2 \phi_f$. Let

$$M' = \{\phi_f \in H^2(\Omega, K) : \varphi f = \varphi_2 \phi_f \ \texttt{for some} \ f \in H^2(\Omega, F)\}. \tag{4.3}$$

Since $\varphi(rf) = \varphi_2(r\phi_f)$ for any $r \in R(\Omega)$ and $f \in H^2(\Omega, F)$, M' is also a (*closed*) rationally invariant subspace of $H^2(\Omega, K)$, and so

$$M' = \varphi_3 H^2(\Omega, K')$$

for some Hilbert space K' and a quasi-inner function $\varphi_3 \in H^\infty(\Omega, L(K', K))$ by Theorem 2.14. It follows that

$$\varphi H^2(\Omega, F) = \varphi_2\varphi_3 H^2(\Omega, K')$$

by the definition of M'. By Corollary 2.16, there is a function $\varphi_4 \in H^\infty(\Omega, L(F, K'))$ such that $\varphi = \varphi_2\varphi_3\varphi_4$.

Let $\varphi_1 = \varphi_3\varphi_4 \in H^\infty(\Omega, L(F, K))$. Since φ and φ_2 are quasi-inner functions, so is φ_1. Then φ_1 is a quasi-inner function satisfying $\varphi = \varphi_2\varphi_1$.

(ii). The rationally invariance of the subspace M described by 4.1 is obvious.

Since $H(\varphi) \ominus M = H^2(\Omega, F') \ominus \varphi_2 H^2(\Omega, K) = H(\varphi_2)$,

$$T_2{}^* = S(\varphi)^* | H(\varphi) \ominus M = S_{F'}^* | H(\varphi_2) = S(\varphi_2)^*.$$

Thus $T_2 = S(\varphi_2)$.

It remains to prove similarity of T_1 and $S(\varphi_1)$. Define $Y : H^2(\Omega, K) \to \varphi_2 H^2(\Omega, K)$ by

$$Yf = \varphi_2 f.$$

Clearly Y is onto. Since φ_2 is a quasi-inner function, Y is one-to-one. Since $Y(\varphi_1 H^2(\Omega, F)) = \varphi_2\varphi_1 H^2(\Omega, F) = \varphi H^2(\Omega, F)$, $\varphi_2 H^2(\Omega, K) = M \oplus \varphi H^2(\Omega, F)$ and $H^2(\Omega, K) = H(\varphi_1) \oplus \varphi_1 H^2(\Omega, F)$,

$$P_M Y(H(\varphi_1)) = M.$$

Thus we define a bounded linear function $F : H(\varphi_1) \to M$ by

$$Fg = P_M \varphi_2 g \texttt{ for } g \in H(\varphi_1),$$

then F is onto. Since φ_2 is a quasi-inner function, ker $F = \{g \in H(\varphi_1) : \varphi_2 g \in \varphi H^2(\Omega, F)\} = \{g \in H(\varphi_1) : g \in \varphi_1 H^2(\Omega, F)\} = \{0\}$. It follows that $F \in L(H(\varphi_1), M)$

is bijective. By the Open Mapping Theorem, F is invertible and we can easily check that $T_1F = FS(\varphi_1)$. $\square$

We recall that the Hilbertian tensor product of the Hilbert spaces $F_1, F_2, \cdots, F_n$, $n \geq 1$, is a Hilbert space denoted $F_1 \otimes F_2 \otimes \cdots \otimes F_n$ which is the completion of the algebraic tensor product of the given spaces. The inner product in this space is defined by

$$(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = (x_1, y_1) \cdots (x_n, y_n),$$

for $x_i, y_i \in F_i$, and $1 \leq i \leq n$. If $T_i \in L(F_i)$ then there is an unique operator $T_1 \otimes T_2 \otimes \cdots \otimes T_n \in L(F_1 \otimes F_2 \otimes \cdots \otimes F_n)$ such that

$$(T_1 \otimes T_2 \otimes \cdots \otimes T_n)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = T_1x_1 \otimes T_2x_2 \otimes \cdots T_nx_n,$$

for $x_j \in F_j$, and $1 \leq j \leq n$.

Fix $n \geq 1$, and consider the mapping $\Gamma_n : L(F) \to L(\otimes^n F)$ given by $\Gamma_n(T) = T \otimes T \otimes \cdots \otimes T$, where F is a Hilbert space and $T \in L(F)$.

Define a unitary representation $\pi_n : S_n \to L(\otimes^n F)$, where S_n denotes the group of permutations of $\{1, 2, \cdots, n\}$, defined by

$$\pi_n(\sigma)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)},$$

for $\sigma \in S_n$, $x_j \in F$, and $1 \leq j \leq n$.

Next, we recall that the C^*-algebra $C^*(G)$ of a finite group G consists of all formal sums $\sum_{g \in G} \alpha_g g$, where $\alpha_g \in \mathbb{C}$ for $g \in G$, and multiplication is defined such that G can be regarded as a multiplicative group in $C^*(G)$, and $g^* = g^{-1}$ for g in G.

The homomorphism $\pi_n : S_n \to L(\otimes^n F)$ can be extended to a $*$-homomorphism, still denoted π_n, from $C^*(S_n)$ to $L(\otimes^n F)$.

Definition 3.34. If A is a finite-dimensional C^*-algebra, a (self-adjoint) projection $p \in A$ ia said to be *minimal* if $p \neq 0$ and the only projection q such that

$0 \leq q \leq p$ are 0 and p. A projection p which is minimal in the center of A will be called a *minimal central projection.*

For a minimal central projection $p \in C^*(S_n)$, define

$$\phi_p(T) = \Gamma_n(T)|\pi_n(p)(\otimes^n F).$$

We will use the alternating projection $a_n \in C^*(S_n)$ defined by

$$a_n = \tfrac{1}{n!}\textstyle\sum_{\sigma \in S_n} \epsilon(\sigma)\sigma,$$

where $\epsilon(\sigma)$ is the sign of σ, i.e. $\epsilon(\sigma) = +1$ or -1 according to whether σ is an even or odd permutation.

Let $\Phi \in H^\infty(\Omega, L(F))$ and define a new function $\Gamma_n(\Phi) \in H^\infty(\Omega, L(\otimes^n F))$ by

$$[\Gamma_n(\Phi)](z) = \Gamma_n(\Phi(z))$$

for $z \in \Omega$. The analyticity of $\Gamma_n(\Phi)$ follows from the fact that Γ_n is a homogeneous polynomial, and the estimate $\| \Gamma_n(\Phi) \|_\infty \leq \| \Phi \|_\infty^n$ is obvious.

If $p \in C^*(S_n)$ is a minimal central projection, we can define a function $\phi_p(\Phi) \in H^\infty(\Omega, L(\pi_n(p)(\otimes^n F)))$ by

$$[\phi_p(\Phi)](z) = \phi_p(\Phi(z))$$

for $z \in \Omega$.

If $\{M_i\}_{i \in I}$ is a family of subsets of the Hilbert space H, we denote by $\bigvee_{i \in I} M_i$ the closed linear span generated by $\bigcup_{i \in I} M_i$.

Definition 3.35. Let $T \in L(H)$ be an operator with spectrum in $\overline{\Omega}$. A subset $G \subseteq H$ with the property that

$$\bigvee\{r(T)m \; ; \; r \in R(\Omega), \, m \in G\} = H,$$

is called an $R(\Omega)$-*generating set* for T. The *multiplicity* μ_T of T is the smallest cardinality of an $R(\Omega)$-generating set for T. The operator T is said to be *multiplicity-free* if $\mu_T = 1$. If $\mu_T = 1$, any vector $x \in H$ such that

$$\bigvee\{r(T)x\ ;\ r \in R(\Omega)\} = H$$

is said to be $R(\Omega)$-*cyclic* for T.

Thus μ_T is the smallest number of $R(\Omega)$-cyclic subspaces that are needed to generate H, where an $R(\Omega)$-cyclic subspace for T is a subspace of the form $\bigvee\{r(T)x\ ;\ r \in R(\Omega)\}$ for some $x \in H$.

PROPOSITION 3.36. *Let H be a separable Hilbert space. Assume that $T \in L(H)$ is an operator of class C_0 relative to Ω such that $\mu_T = n < \infty$, H' is a $R(\Omega)$-invariant subspace for T, and $T = \begin{pmatrix} T' & Y \\ 0 & T'' \end{pmatrix}$ is the triangularization of T with respect to the decomposition $H = H' \oplus (H \ominus H')$.*

If $\oplus_{j<\omega} S(\theta_j)$, $\oplus_{j<\omega} S(\theta'_j)$, and $\oplus_{j<\omega} S(\theta''_j)$ are the Jordan models of T, T', and T'', respectively, then

$$\theta_0 \cdots \theta_{k-1} | \theta'_0 \cdots \theta'_{k-1} \theta''_0 \cdots \theta''_{k-1}, \tag{4.4}$$

for every k such that $1 \leq k < n$, and

$$\theta_0 \cdots \theta_{k-1} \equiv \theta'_0 \cdots \theta'_{k-1} \theta''_0 \cdots \theta''_{k-1}, \tag{4.5}$$

for every $k \geq n$.

PROOF. Let $f \in H^\infty(\Omega, L(\mathbb{C}^n))$ be a quasi-inner function such that if B is a matrix over $H^\infty(\Omega)$ representing f, then B is a normal matrix whose diagonal entries are $\theta_0, \cdots, \theta_{n-1}$. Then by Corollary 3.30,

$$S(f) \sim \oplus_{j=0}^{n-1} S(\theta_j).$$

Since $\mu_T = n$, $T \sim \oplus_{j<\omega} S(\theta_j) \sim \oplus_{j=0}^{n-1} S(\theta_j)$ and so $T \sim S(f)$. Thus there is an injective operator $X \in L(H, H(f))$ with dense range such that

$$S(f)X = XT. \tag{4.6}$$

Let $M = XH'$. Since H' is a $R(\Omega)$- invariant subspace for T, so is M for $S(f)$. Then by Theorem 3.33, there are quasi-inner functions $f_1 \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $f_2 \in H^\infty(\Omega, L(\mathbb{C}^n))$ such that $f = f_2 f_1$ and

$$M = f_2 H^2(\Omega, \mathbb{C}^n) \ominus f H^2(\Omega, \mathbb{C}^n).$$

If $S(f) = \begin{pmatrix} T_1 & Z \\ 0 & T_2 \end{pmatrix}$ is the triangularization of $S(f)$ with respect to the decomposition $H(f) = M \oplus (H(f) \ominus M)$, then T_1 is similar to $S(f_1)$ and $T_2 = S(f_2)$ by Theorem 3.33.

Let $X' = X|H'$. Then by equation (4.6),

$$T_1 X' = S(f)X|H' = XT|H' = X'T'. \tag{4.7}$$

Since $\mu_T \leq n$, we get $\mu_{T'} \leq n$ and so equation (4.7) implies that $T_1 \sim T' \sim \oplus_{j=0}^{n-1} S(\theta_j')$. Since T_1 is similar to $S(f_1)$,

$$S(f_1) \sim \oplus_{j=0}^{n-1} S(\theta_j').$$

Define $X'' : H(f) \ominus M \to H \ominus H'$ by

$$X'' = X^*|H(f) \ominus M.$$

Then X'' is injective with dense range in $H \ominus H'$ and

$$X'' {T_2}^* = X^* S(f)^*|H(f) \ominus M = T^* X^*|H(f) \ominus M = (T'')^* X''.$$

Thus $\mu_{T''} \leq n$ and $T_2 \sim T'' \sim \oplus_{j=0}^{n-1} S(\theta_j'')$. It follows that

$$S(f_2) \sim \oplus_{j=0}^{n-1} S(\theta_j'').$$

Fix k such that $1 \leq k < n$ and note that

$$\phi_{a_k}(f) = \phi_{a_k}(f_2)\phi_{a_k}(f_1).$$

Since $\phi_{a_k}(f)$ is also a diagonal inner function with diagonal entries $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}$ where $i_p \neq i_q$ for $p \neq q$, the minimal function of $S(\phi_{a_k}(f))$ is $\theta_0\theta_1\cdots\theta_{k-1}$. Similarly, the minimal functions of $S(\phi_{a_k}(f_1))$ and $S(\phi_{a_k}(f_2))$ are $\theta'_0\theta'_1\cdots\theta'_{k-1}$ and $\theta''_0\theta''_1\cdots\theta''_{k-1}$, respectively. By Theorem 3.16, there are functions g', $g'' \in H^\infty(\Omega, L(\pi_k(a_k)(\otimes^k C^n)))$ such that

$$g'\phi_{a_k}(f_1) = \theta'_0\theta'_1\cdots\theta'_{k-1}I \tag{4.8}$$

and

$$g''\phi_{a_k}(f_2) = \theta''_0\theta''_1\cdots\theta''_{k-1}I, \tag{4.9}$$

where I is the identity function on $\pi_k(a_k)(\otimes^k C^n))$. Combining these relations (4.8) and (4.9), we get

$$g'g''\phi_{a_k}(f) = g'g''\phi_{a_k}(f_2)\phi_{a_k}(f_1) = \theta'_0\theta'_1\cdots\theta'_{k-1}\theta''_0\theta''_1\cdots\theta''_{k-1}I,$$

and this Corollary follows because $\theta_0\theta_1\cdots\theta_{k-1}$ is the least scalar multiple of $\phi_{a_k}(f)$ by Theorem 3.16.

Next, for $k = n$, since $S(f) \sim \oplus_{j=0}^{n-1} S(\theta_j)$, $S(f_1) \sim \oplus_{j=0}^{n-1} S(\theta'_j)$, and $S(f_2) \sim \oplus_{j=0}^{n-1} S(\theta''_j)$, by Corollary 3.32,

$$\det f(z) \equiv \theta_0(z)\theta_1(z)\cdots\theta_{n-1}(z),$$

$$\det f_1(z) \equiv \theta'_0(z)\theta'_1(z)\cdots\theta'_{n-1}(z),$$

and

$$\det f_2(z) \equiv \theta''_0(z)\theta''_1(z)\cdots\theta''_{n-1}(z),$$

for $z \in \Omega$. From the fact $f(z) = f_2(z)f_1(z)$ for any $z \in \Omega$, we obtain $\det f(z) = (\det f_2(z))(\det f_1(z))$ which proves for the case $k = n$.

For $k \geq n$, since $\theta_k \equiv \theta'_k \equiv \theta''_k \equiv 1$, by the case of $k = n$,

$$\theta_0\theta_1\cdots\theta_{k-1} \equiv \theta_0\theta_1\cdots\theta_{n-1} \equiv \theta'_0\theta'_1\cdots\theta'_{n-1}\theta''_0\theta''_1(z)\cdots\theta''_{n-1} \equiv \theta'_0\theta'_1\cdots\theta'_{k-1}\theta''_0\theta''_1\cdots\theta''_{k-1}.$$

□

In Proposition 3.36, we have proved some relations (4.4) and (4.5) when $T \in L(H)$ is an operator of class C_0 relative to Ω such that $\mu_T = n < \infty$. In next Theorem, we will have a same result with Proposition 3.36 without the assumption $\mu_T = n < \infty$. We have following Proposition from Theorem 4.3.10. in [**31**].

PROPOSITION 3.37. *Let $T \in L(H)$ be an operator of class C_0 relative to Ω. If $\oplus_{j<\omega} S(\theta_j)$ is the Jordan model of T, then for any $k = 1, 2, 3, ...,$ there are $R(\Omega)$-invariant subspaces M_{-1}, M_0,..., M_{k-2} and h_0, h_1,...,h_{k-1} in H such that*

$$h_i \in M_{i-1} \text{ and } m_{h_i} = m_{T|M_{i-1}} \tag{4.10}$$

for $i = 0, 1, ..., k-1$, and

$$K_i \vee M_i = M_{i-1} \text{ and } K_i \cap M_i = \{0\} \tag{4.11}$$

for $i = 0, 1, ..., k-1$, where $M_{-1} = H$ and $K_i = \bigvee\{r(T)h_i : r \in R(\Omega)\}$

THEOREM 3.38. *Let H be a separable Hilbert space. Assume that $T \in L(H)$ is an operator of class C_0 relative to Ω, H' is a $R(\Omega)$-invariant subspace for T, and $T = \begin{pmatrix} T' & Y \\ 0 & T'' \end{pmatrix}$ is the triangularization of T with respect to the decomposition $H = H' \oplus (H \ominus H')$.*

If $\oplus_{j<\omega} S(\theta_j)$, $\oplus_{j<\omega} S(\theta'_j)$, and $\oplus_{j<\omega} S(\theta''_j)$ are the Jordan models of T, T', T'', respectively, then

$$\theta_0 \cdots \theta_{k-1} | \theta'_0 \cdots \theta'_{k-1} \theta''_0 \cdots \theta''_{k-1},$$

for every $k = 1, 2, 3, ...$

PROOF. Fix $k \geq 1$. Since $T \sim \oplus_{j<\omega} S(\theta_j)$, by Proposition 3.37, there are $R(\Omega)$-invariant subspaces M_{-1}, M_0,..., M_{k-2} and h_0, h_1,...,h_{k-1} in H satisfying (4.10) and (4.11).

Let $h_i = h_i{}' + h_i{}''$ where $h_i' \in H'$ and $h_i'' \in H \ominus H'$ $(i = 0, 1, ..., k-1)$, and $M (\subset H)$ be a $R(\Omega)$-invariant subspace for T generated by $h_0', ..., h_{k-1}'$ and $h_0'', ..., h_{k-1}''$.

If $T|M \sim \oplus_{j \leq 2k} S(\phi_j)$, clearly for $i = 0, ..., k-1$,

$$\phi_i | \theta_i.$$

Since $h_i \in M$ for $i = 0, ..., k-1$,

$$\theta_i | \phi_i$$

and so for $i = 0, ..., k-1$,

$$\theta_i \equiv \phi_i. \tag{4.12}$$

Let $T_1 = T|M$ and $H_1{}' = M \cap H'$. Then clearly $H_1{}'$ is a $R(\Omega)$-invariant subspace for T_1. If $T_1 = \begin{pmatrix} T_1{}' & Y_1 \\ 0 & T_1{}'' \end{pmatrix}$ is the triangularization of T_1 with respect to the decomposition $M = H_1' \oplus (M \ominus H')$ and $\oplus_j S(\phi_j)$, $\oplus_j S(\phi_j')$, and $\oplus_j S(\phi_j'')$ are the Jordan models of T_1, $T_1{}'$, and $T_1{}''$, respectively, then by Proposition 3.36 (note that $\mu_{T_1} \leq 2k$),

$$\phi_0 \cdots \phi_{k-1} \mid \phi_0' \cdots \phi_{k-1}' \phi_0'' \cdots \phi_{k-1}''.$$

By (4.12), we conclude that

$$\phi_0 \cdots \phi_{k-1} \equiv \theta_0 \cdots \theta_{k-1}, \tag{4.13}$$

and since $T'|H_1{}' = T_1{}'$,

$$\phi_i{}' | \theta_i{}', \tag{4.14}$$

for $i = 0, ..., k-1$.

Next, let $H_1{}'' = M \ominus H_1'$, $H'' = H \ominus H'$, and $P_{H''} : H_1{}'' \rightarrow H''$ be an orthogonal projection into H'', that is, $P_{H''} f = g$ if $f = g + h$ with $g \in H''$ and $h \in (H'')^{\perp}$. If

$a \in \ker P_{H''}$, then $a \in H' \cap (M \ominus H_1') \subset H' \cap M = H_1{}'$. Since $a \in H_1{}''(= M \ominus H_1')$, $a = 0$. Thus $P_{H''}$ is one-to-one. Moreover, H' is invariant for T, H'' is invariant for T^*. Thus

$$T^* P_{H''} = P_{H''} T^* P_{H''}$$

and so

$$P_{H''} T = (T^* P_{H''})^* = (P_{H''} T^* P_{H''})^* = P_{H''} T P_{H''} = T'' P_{H''}.$$

We have $P_{H''} T_1{}'' = P_{H''} P_{M \ominus H_1{}'} T | M \ominus H_1{}' = P_{H''} T | M \ominus H_1{}'$ and an easy calculation, we get

$$T'' P_{H''} = P_{H''} T_1{}''.$$

The fact that $P_{H''}$ is one-to-one implies

$$\phi_i{}'' | \theta_i{}'', \tag{4.15}$$

for $i = 0, ..., k-1$. Thus from (4.8), (4.13), (4.14), and (4.15), we can conclude that $\theta_0 \cdots \theta_{k-1} | \theta_0' \cdots \theta_{k-1}' \theta_0'' \cdots \theta_{k-1}''$. Since k is arbitrary, this theorem is proven. □

CHAPTER 4

Modular Lattice for C_0-Operators

1. Lattice of subspaces

1.1. Modular Lattice. Let H be a Hilbert space. If $F_i (i \in I)$ is a subset of H, then the closed linear span of $\bigcup_i F_i$ will be denoted by $\bigvee_i F_i$. The collection of all subspaces of a Hilbert space is a *lattice*. This means that the collection is partially ordered (by inclusion), and that any two elements $\mathbf{M}$ and $\mathbf{N}$ of it have a least upper bound or supremum (namely the span $\mathbf{M} \vee \mathbf{N}$) and a greatest lower bound or infimum (namely the intersection $\mathbf{M} \cap \mathbf{N}$). A lattice is called *distributive* if

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee (\mathbf{L} \cap \mathbf{N}) \tag{1.1}$$

for any element $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ in the lattice.

In the equation (1.1), if $\mathbf{N} \subset \mathbf{L}$, then $\mathbf{L} \cap \mathbf{N} = \mathbf{N}$, and so the identity becomes

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee \mathbf{N} \tag{1.2}$$

If the identity (1.2) is satisfied whenever $\mathbf{N} \subset \mathbf{L}$, then the lattice is called *modular*.

For an arbitrary operator $T \in L(H)$, $\mathtt{Lat}(T)$ denotes the collection of all closed invariant subspaces for T. The following fact is well-known([**17**]).

PROPOSITION 4.1. *The lattice of subspaces of a Hilbert space H is modular if and only if* dim *H is finite.*

We will think about $\mathtt{Lat}(T)$ for a C_0-operator T relative to $\mathbf{D}$.

DEFINITION 4.2. The *cyclic multiplicity* μ_T of an operator $T \in L(H)$ is the smallest cardinal of a subset $A \subset H$ with the property that $\bigvee_{n=0}^{\infty} T^n A = H$. The operator T is said to be *multiplicity-free* if $\mu_T = 1$.

Thus μ_T is the smallest number of cyclic subspaces for T that are needed to generate H, and T is multiplicity-free if and only if it has a cyclic vector.

1.2. Property (P)**.** In this section, every C_0-operator means C_0-operator relative to $\mathbf{D}$. Let H be a Hilbert space and for an operator $T \in L(H)$, T^* denote the adjoint of T. It is well known that H is finite-dimensional if and only if every operator $X \in L(H)$, with the property $\ker(X) = \{0\}$, also satisfies $\ker(X^*) = \{0\}$. The following definition is a natural extension of finite dimensionality.

DEFINITION 4.3. An operator $T \in L(H)$ is said to have *property (P)* if every operator $X \in \{T\}'$ with the property that $\ker(X) = \{0\}$ is a quasiaffinity, i.e., $\ker(X^*) = \ker(X) = \{0\}$.

From the fact that the commutant $\{0\}'$ of zero operator on H coincides with $L(H)$, we can see that H is finite-dimensional if and only if the zero operator on H has property (P).

Let T_1 and T_2 be operators in $L(H)$. Suppose that

$$X \in \{A \in L(H) : AT_1 = T_2A\}.$$

If M is in $\mathtt{Lat}(T_1)$, then $(XM)^-$ is in $\mathtt{Lat}(T_2)$. By using these facts, we define a function between $\mathtt{Lat}(T_1)$ and $\mathtt{Lat}(T_2)$ as following:

$$X_*(M) = (XM)^-. \tag{1.3}$$

The operator X is said to be a (T_1, T_2)-*lattice-isomorphism* if X_* is a bijection of $\mathtt{Lat}(T_1)$ onto $\mathtt{Lat}(T_2)$. We will use the name lattice-isomorphism instead of (T_1, T_2)-lattice-isomorphism if no confusion may arise.

If $X \in \{A \in L(H) : AT_1 = T_2A\}$, then $X^*T_2^* = T_1^*X^*$. Thus $(X^*)_* : \mathtt{Lat}(T_2^*) \to \mathtt{Lat}(T_1^*)$ is well-defined by

$$(X^*)_*(M') = (X^*M')^-.$$

PROPOSITION 4.4. [**6**] *(Theorem 7.1.9) Suppose that $T \in L(H)$ is an operator of class C_0 with Jordan model $\bigoplus_\alpha S(\theta_\alpha)$. Then T has property (P) if and only if*

$$\bigwedge_{j<\omega} \theta_j \equiv 1.$$

Thus, if T has property (P), then H is separable and T^* also has property (P).

PROPOSITION 4.5. [**6**] *An operator T of class C_0 fails to have property (P) if and only if T is quasisimilar to $T|N$, where N is a proper invariant subspace for T.*

PROPOSITION 4.6. [**6**]*(Lemma 7.1.20) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. If the mapping X_* is onto $\mathtt{Lat}(T_2)$ if and only if $(X^*)_*$ is one-to-one on $\mathtt{Lat}(T_2^*)$.*

COROLLARY 4.7. *Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. The mapping X_* is one-to-one on $\mathtt{Lat}(T_1)$ if and only if $(X^*)_*$ is onto $\mathtt{Lat}(T_1^*)$.*

PROOF. Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. By Proposition 4.6, $(X^*)_*$ is onto $\mathtt{Lat}(T_1^*)$ if and only if $(X^{**})_* = X_*$ is one-to-one on $\mathtt{Lat}(T_1)$. □

From Proposition 4.6 and Corollary 4.7, we obtain the following result.

COROLLARY 4.8. *If $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$, then X is a lattice-isomorphism if and only if X^* is a lattice-isomorphism.*

PROPOSITION 4.9. [**6**] *(Proposition 7.1.21) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class C_0, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ is an injection. If T_1 has property (P), then X is a lattice-isomorphism.*

Recall that if T is an operator on a Hilbert space, then ker T = (ran $T^*)^\perp$ and ker T^* = (ran $T)^\perp$.

COROLLARY 4.10. *Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class C_0, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ has dense range. If T_2 has property (P), then X is a lattice-isomorphism.*

PROOF. Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. Let $Y = X^*$ and so

$$YT_2^* = T_1^*Y. \tag{1.4}$$

From the fact that $\ker Y = \ker(X^*) = (\text{ ran } X)^\perp = \{0\}$, we conclude that Y is injective. Since T_2 has property (P), so does T_2^* by Proposition 4.4. By Proposition 4.9 and equation (1.4), $Y = X^*$ ia a lattice-isomorphism. From Corollary 4.8, it is proven that X is a lattice-isomorphism. □

COROLLARY 4.11. *Suppose that $T_i \in L(H_i)(i = 1, 2)$ is a C_0-operator and T_1 has property (P). If $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ and X is an injection, then X is a lattice-isomorphism.*

PROOF. Define $Y : H_1 \to (XH_1)^-$ by

$$Yh = Xh \text{ for any } h \in H_1.$$

Since X is an injection, so is Y. Clearly, Y has dense range. Note that $(XH_1)^-$ is invariant for T_2. By definition of Y,

$$YT_1 = (T_2|(XH_1)^-)Y. \tag{1.5}$$

It follows that $T_1 \prec (T_2|(XH_1)^-)$ and so $T_1 \sim (T_2|(XH_1)^-)$. By Proposition 4.9, it is proven. □

COROLLARY 4.12. *Suppose that $T_i \in L(H_i)(i = 1, 2)$ is a C_0-operator and T_2 has property (P). If $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ and X has a dense range, then X is a lattice-isomorphism.*

PROOF. By assumption, $X^*T_2^* = T_1^*X^*$. Since T_2 has property (P), by Proposition 4.4, so does T_2^*.

Because X has dense range, $X^* : H_2 \to H_1$ is an injection. By Corollary 4.11, X^* is a lattice isomorphism. From Corollary 4.8, X is also a lattice isomorphism. □

1.3. Quasi-Affinity and Modular Lattice. For operators $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$, then we define a function

$$Y_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$$

the same way as equation (1.3). For any $N \in \mathtt{Lat}(T_2)$, if $M = Y^{-1}(N)$, then $YT_1(M) = T_2Y(M) \subset T_2N \subset N$ and so $T_1(M) \subset M$. It follows that

$$M = Y^{-1}(N) \in \mathtt{Lat}(T_1)$$

for any $N \in \mathtt{Lat}(T_2)$. If Y is invertible, that is, T_1 and T_2 are similar, and $\mathtt{Lat}(T_1)$ is modular, then clearly, $\mathtt{Lat}(T_2)$ is also modular. In this section, we consider when T_1 and T_2 are quasi-similar instead of similar, and find an assumption in Theorem 4.14 such that $\mathtt{Lat}(T_2)$ is modular, whenever $\mathtt{Lat}(T_1)$ is modular.

PROPOSITION 4.13. *Let $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$. Suppose that $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$ and for any $N \in \mathtt{Lat}(T_2)$, the condition $M = Y^{-1}(N)$ implies that $Y_*(M) = N$.*

Then for any $M_i = Y^{-1}(N_i)$ with $N_i \in \mathtt{Lat}(T_2)$ $(i = 1, 2)$,

$$Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2).$$

PROOF. Assume that $N_i \in \texttt{Lat}(T_2)$ and $M_i = Y^{-1}(N_i)$ for $i = 1, 2$. Then by assumption, we obtain

$$Y_*(M_i) = N_i. \tag{1.6}$$

Since $Y^{-1}(N_1 \cap N_2) = Y^{-1}(N_1) \cap Y^{-1}(N_2) = M_1 \cap M_2$, by assumption,

$$Y_*(M_1 \cap M_2) = N_1 \cap N_2$$

which proves that $Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2)$ by equation (1.6).

□

THEOREM 4.14. *Let $T_1 \in L(H_1)$ be a quasiaffine transform of $T_2 \in L(H_2)$ and $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2 B\}$ be a quasiaffinity.*

If $Y_* : \texttt{Lat}(T_1) \to \texttt{Lat}(T_2)$ is onto and $\texttt{Lat}(T_1)$ is modular, then $\texttt{Lat}(T_2)$ is also modular.

PROOF. Suppose that $\texttt{Lat}(T_2)$ is not modular. Then there are invariant subspaces $N_i (i = 1, 2, 3)$ for T_2 such that

$$N_3 \subset N_1, \tag{1.7}$$

and

$$(N_1 \cap N_2) \vee N_3 \neq N_1 \cap (N_2 \vee N_3).$$

Let

$$M_i = Y^{-1}(N_i), \tag{1.8}$$

for $i = 1, 2, 3$. Since $YT_1 = T_2 Y$, definition (1.8) of M_i implies that for $i = 1, 2, 3$,

$$YT_1(M_i) = T_2 Y(M_i) \subset T_2 N_i \subset N_i.$$

It follows that $T_1 M_i \subset Y^{-1}(N_i) = M_i$ for $i = 1, 2, 3$. Thus M_i is a closed invariant subspace for T_1. Condition (1.7) implies that

$$M_3 \subset M_1.$$

Since $Y(M_i) \subset N_i$, for $i = 1, 2, 3$,

$$Y_*(M_i) = (Y(M_i))^- \subset N_i. \tag{1.9}$$

Since Y_* is onto, there is a function $\phi : \mathtt{Lat}(T_2) \to \mathtt{Lat}(T_1)$ such that $Y_* \circ \phi$ is the identity mapping on $\mathtt{Lat}(T_2)$. Hence for $i = 1, 2, 3$,

$$Y_*(\phi(N_i)) = Y(\phi(N_i))^- = N_i.$$

It follows that for $i = 1, 2, 3$,

$$\phi(N_i) \subset M_i. \tag{1.10}$$

Since $Y_* \circ \phi$ is the identity mapping on $\mathtt{Lat}(T_2)$, (1.10) implies that for $i = 1, 2, 3$,

$$N_i = Y_*(\phi(N_i)) \subset Y_*(M_i). \tag{1.11}$$

By (1.9) and (1.11), we get

$$Y_*(M_i) = N_i, \tag{1.12}$$

for $i = 1, 2, 3$. Hence we can easily see that function Y satisfies the assumptions of Proposition 4.13.

Thus by Proposition 4.13 and equation (1.12),

$$Y_*[M_1 \cap (M_2 \vee M_3)] = Y_*(M_1) \cap Y_*(M_2 \vee M_3) = N_1 \cap (N_2 \vee N_3). \tag{1.13}$$

Since $M_1 \cap M_2 = Y^{-1}(N_1) \cap Y^{-1}(N_2) = Y^{-1}(N_1 \cap N_2)$, by the same way as above, we obtain

$$Y_*(M_1 \cap M_2) = N_1 \cap N_2. \tag{1.14}$$

By equations (1.12) and (1.14), we obtain

$$Y_*[(M_1 \cap M_2) \vee M_3] = (N_1 \cap N_2) \vee N_3. \tag{1.15}$$

Since $(N_1 \cap N_2) \vee N_3 \neq N_1 \cap (N_2 \vee N_3)$, from equations (1.13) and (1.15), we can conclude that

$$(M_1 \cap M_2) \vee M_3 \neq M_1 \cap (M_2 \vee M_3).$$

Therefore $\mathtt{Lat}(T_1)$ is not modular.

□

2. Modular Lattice for C_0-Operators with Property (P)

We provide some operators, say T, of class C_0 relative to $\mathbf{D}$ such that $\mathtt{Lat}(T)$ is modular.

PROPOSITION 4.15. [**6**] *Let θ be a nonconstant inner function in H^∞. Then every invariant subspace M of $S(\theta)$ has the form*

$$\phi H^2 \ominus \theta H^2$$

for some inner devisor ϕ of θ.

We can easily check that if $\mathbf{M}_1 = \theta_1 H^2 \ominus \theta H^2$ and $\mathbf{M}_2 = \theta_2 H^2 \ominus \theta H^2$ where θ_i $(i = 1, 2)$ is an inner inner devisor of θ, then

$$\mathbf{M}_1 \cap \mathbf{M}_2 = (\theta_1 \vee \theta_2) H^2 \ominus \theta H^2 \tag{2.1}$$

and

$$\mathbf{M}_1 \vee \mathbf{M}_2 = (\theta_1 \wedge \theta_2) H^2 \ominus \theta H^2 \tag{2.2}$$

where $\theta_1 \vee \theta_2$ is the least inner multiple of θ_1 and θ_2. Note that if $\mathbf{M}_1 \subset \mathbf{M}_2$, then

$$\theta_2 | \theta_1. \tag{2.3}$$

LEMMA 4.16. *If θ is an inner function in H^∞, then $\mathtt{Lat}(S(\theta))$ is distributive.*

PROOF. Let $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ be invariant subspaces for $S(\theta)$. Then by Proposition 4.15, there are nonconstant inner functions θ_1, θ_2, and θ_3 in H^∞ such that

$$\mathbf{M}_i = \theta_i H^2 \ominus \theta H^2 \text{ for } i = 1, 2, 3.$$

From equations (2.1) and (2.2), we obtain that

$$\mathbf{M}_1 \cap (\mathbf{M}_2 \vee \mathbf{M}_3) = (\theta_1 \vee (\theta_2 \wedge \theta_3))H^2 \ominus \theta H^2, \tag{2.4}$$

and

$$(\mathbf{M}_1 \cap \mathbf{M}_2) \vee (\mathbf{M}_1 \cap \mathbf{M}_3) = ((\theta_1 \vee \theta_2) \wedge (\theta_1 \vee \theta_3)H^2 \ominus \theta H^2. \tag{2.5}$$

Since $\theta_1 \vee (\theta_2 \wedge \theta_3) = (\theta_1 \vee \theta_2) \wedge (\theta_1 \vee \theta_3)$, by equations (2.4) and (2.5), this lemma is proven.

□

In this section, we will consider a sufficient condition for $\mathtt{Lat}(T)$ of a C_0-operator T to be modular.

PROPOSITION 4.17. [**6**] *(*`Proposition 2.4.3`*) Let $T \in L(H)$ be a completely nonunitary contraction, and M be an invariant subspace for T. If*

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{2.6}$$

is the triangularization of T with respect to the decomposition $H = M \oplus (H \ominus M)$, then T is of class C_0 if and only if T_1 and T_2 are operators of class C_0.

PROPOSITION 4.18. [**6**] *(*`Corollary 7.1.17`*) Let $T \in L(H)$ is an operator of class C_0, M be an invariant subspace for T, and*

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{2.7}$$

be the triangularization of T with respect to the decomposition $H = M \oplus (H \ominus M)$. Then T has property (P) if and only if T_1 and T_2 have property (P).

Let H and K be Hilbert spaces and $H \oplus K$ denote the algebraic direct sum. Recall that $H \oplus K$ is also a Hilbert space with an inner product

$$(\langle h_1, k_1\rangle, \langle h_2, k_2\rangle) = (h_1, h_2) + (k_1, k_2)$$

THEOREM 4.19. *Let $T \in L(H)$ be an operator of class C_0 with property (P). Then $\mathtt{Lat}(T)$ is a modular lattice.*

PROOF. Suppose that T has property (P) and let M_i $(i = 1, 2, 3)$ be an invariant subspace for T such that $M_3 \subset M_1$. Then evidently,

$$(M_1 \cap M_2) \vee M_3 \subset M_1 \cap (M_2 \vee M_3). \tag{2.8}$$

Let $T_i = T|M_i$ $(i = 1, 2, 3)$. Define a linear transformation $X : M_2 \oplus M_3 \to M_2 \vee M_3$ by

$$X(a_2 \oplus a_3) = a_2 + a_3$$

for $a_2 \in M_2$ and $a_3 \in M_3$.

Then for $a_2 \oplus a_3 \in M_2 \oplus M_3$ with $\|a_2 \oplus a_3\| \leq 1$, $\|X(a_2 \oplus a_3)\| = \|a_2 + a_3\| \leq \|a_2\| + \|a_3\| \leq 2$. It follows that $\|X\| \leq 2$ and so X is bounded.

Since $M_2 \vee M_3$ is generated by $\{a_2 + a_3 : a_2 \in M_2 \text{ and } a_3 \in M_3\}$, X has dense range. By the definition of T_i $(i = 1, 2, 3)$,

$$X(T_2 \oplus T_3)(a_2 \oplus a_3) = Ta_2 + Ta_3$$

and

$$(T|M_2 \vee M_3)X(a_2 \oplus a_3) = Ta_2 + Ta_3.$$

Thus

$$X(T_2 \oplus T_3) = (T|M_2 \vee M_3)X.$$

By Proposition 4.17, since T is of class C_0, so do $T_2 \oplus T_3$ and $T|M_2 \vee M_3$ and by Proposition 4.18, since T has property (P), so do $T_2 \oplus T_3$ and $T|M_2 \vee M_3$. Thus, Proposition 4.4 implies that $T|M_2 \vee M_3$ has Property (P). By Corollary 4.12, X is a lattice-isomorphism.

Thus $X_* : \mathtt{Lat}(T_2 \oplus T_3) \to \mathtt{Lat}(T|M_2 \vee M_3)$ is onto. Let

$$M = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}. \tag{2.9}$$

Since $M = X^{-1}(M_1)$, M is a closed subspace of $M_2 \oplus M_3$. Evidently, M is invariant for $T_2 \oplus T_3$. From the equation (2.9), we conclude that

$$M = (M_1 \cap M_2) \oplus M_3. \tag{2.10}$$

Since $X^{-1}(M_1 \cap (M_2 \vee M_3)) = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1 \cap (M_2 \vee M_3)\} = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}$,

$$X^{-1}(M_1 \cap (M_2 \vee M_3)) = M$$

Since X is a lattice-isomorphism,

$$X_*M = (XM)^- = M_1 \cap (M_2 \vee M_3). \tag{2.11}$$

By equation (2.10) and definition of X,

$$X_*M = (XM)^- \subset (M_1 \cap M_2) \vee M_3. \tag{2.12}$$

From (2.11) and (2.12), we conclude that

$$M_1 \cap (M_2 \vee M_3) \subset (M_1 \cap M_2) \vee M_3. \tag{2.13}$$

Thus if T has property (P), then by (2.8) and (2.13), we obtain that

$$M_1 \cap (M_2 \vee M_3) = (M_1 \cap M_2) \vee M_3.$$

□

Bibliography

[1] M.B. Abrahamse and R.G. Douglas, *Operators on multiply- connected domains*, Proc. Roy. Irish Acad. Sect. A **74** (1974), 135-141.

[2] M.B. Abrahamse and R.G. Douglas, *A class of subnormal operators relatrd to multiply-connected domains*, Advances in Math. **19**(1976), 106-148.

[3] J. Agler, *Rational dilation on an annulus*, Ann. of Math. (2) **121** (1985), 537-563.

[4] J.A. Ball, *Operators of Calss C_{00} over Multiply-Connected Domains*, Mich. Math. J. **25** (1978), 183-196.

[5] J.A. Ball, *A lifting theorem for operator models of finite rank on multiply-connected domains*, J. Operator Theory **1** (1979), 3-25.

[6] H. Bercovici, *Operator theory and arithmetic in H^∞*, Amer. Math. Soc., Providence, Rhode island (1988).

[7] H. Bercovici, *C_0-Fredholm operators*, II, Acta Sci. Math. (Szeged) **42**(1980), 3-42.

[8] H. Bercovici and D. Voiculescu, *Tensor operations on characteristic functions of C_0 contractions*, Acta Sci. Math. **39**(1977), 205-231.

[9] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. **81** (1949), 239-255.

[10] L. Bungart, *On analytic fiber bundles. 1*, Topology **7**(1968), 55-68.

[11] R.G. Douglas, *Banach algebra techniques in operator theory*, N.Y., Springer(1998).

[12] R.G. Douglas, On Majorization, *Factorization, and Range Inclusion of Operators on Hilbert Space*, Proceedings of American Mathematical Society, Vol. 17, No. 2(Apr., 1996).

[13] R.G. Douglas and V. Paulsen, *Completely bounded maps and hypo-Dirichlet algebras*, Acta. Sci. Math. (Szeged) **50** (1986), 143-157.

[14] M.A. Dritschel and S. McCullough, *The failure of rational dilation on a triply connected domain*, J. Amer. Math. Soc. **18** (2005), 873-918.

[15] S. Fisher, *Function Theory on Planar Domains*, a second course in Complex Analysis, Wiley, New York (1983)

[16] T. Gamelin and G. Garnett, *Pointwise bounded approximation and Drichlet Algebras*, J. Finctional Analysis **8** (1971), 360-404

[17] P.R. Halmos, *A Hilbert Space Problem Book*, D. Van Nostrand Company, Princeton, N.Y., 1967

[18] M. Hasumi, *Invariant subspace theorems for finite Riemann surface*, Canad. J. Math. **18** (1966), 240-255.

[19] M.Heins, *Hardy classes on Riemann surfaces*, Lecture Notes in Math. vol.98, Springer-verlag, Berlin, 1969

[20] Kanneth Hoffman and Ray Kunze, *Linear Algebra*

[21] D.Husemoller, *Fiber bundles*, McGraw-Hill,New York: Springer-Verlag (1994)

[22] B. Moore, III and E.A. Nordgren, *On quasi-equvalence and quasi-similarity*, Acta Sci. Math. (Szeged) **34**(1973), 311-316.

[23] E.A. Nordgren, *On quasiequivalence of matrices over* H^∞, Acta. Sci. Math. (Szeged) **34**(1973), 301-310.

[24] H.L. Royden, *Invariant subspaces of* H^p *for multiply connected regions*, Pacific J. Math. **134**(1988), 151-172.

[25] W. Rudin, *Analytic functions of class* H_p, Trans. Amer. Math. Soc. **78**(1955), 44-66.

[26] D.E. Sarason, *The* H^p *spaces of an annulus*, Mem. Amer. Math. Soc. **56**(1965).

[27] N.E. Steenrod, *The topology of fibre bundles*, Princeton, N.J. : Princeton University Press, (1951).

[28] B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Sci Math. (Szeged) **15** (1953), 87-92

[29] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam(1970).

[30] M. Voichick, *Ideals and invariant subspaces of analytic functions*, Trans. Amer. Math. Soc. 111 (1964), 493-512.

[31] Adele Zucchi, *Operators of class* C_0 *with spectra in multiply connected regions*, Mem. Amer. Math. Soc. **127**(1997).

Curriculum Vita

Yun-Su Kim

Education :

Ph.D., Mathematics, Indiana University, Bloomington, Indiana, 2006
Advisor : Hari Bercovici
Thesis : Linear Algebraic Properties of C_0-operators.

M.S., Mathematics, Seoul National University, Seoul, Korea, 1998
Advisor : Suck-Jin Kang
Thesis : The Memorial Bases of Free Lie Superalgebras

B.S., Mathematics, Sungshin Women's University, Seoul, Korea, 1993
Advisor : Kyung Bae Kim

Areas of Interest :

C_0-Operators
Operator Spaces
Hilbert Modules
Positive Operators