

# NONCOMPUTABLE SPECTRAL SETS

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For my Mama, whose \*-minimal index is computable (because it's finite).

Love the Ma.

## Acknowledgements

It seems nothing short of a miracle that I now find myself in a position to graduate with a doctorate in mathematics. This thesis is the product of contributions from a few outstanding individuals that I've known (in the biblical sense) along my academic path. Without the personal connections that I am about to mention, there would indeed be no work here to acknowledge.

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## Abstract

It is a basic fact that, given a computer language and a computable integer function, there exists a shortest program in that language which computes the desired function. Once a programmer establishes the correctness of a program, she then need only verify that the program is the shortest possible in order to declare complete victory. Unfortunately, she can't. A creature that could identify minimal programs would not only be able to decide the halting problem, but she could even decide the halting problem for machines with halting set oracles. Such a creature exceeds the powers of ordinary machine cognition, and must therefore be a divine jument-numen.

Suppose, however, the programmer would be satisfied to know just whether or not her program is minimal up to finitely many errors. In this case, even the jument-numen is helpless: something much stronger is needed. Just as we can associate equality with the ordinary notion of "minimal," we can associate an equivalence relation,  $=^*$ , with the principle of "minimal up to finitely many errors." This thesis is the first to explore the extensive realm of minimal indices beyond the  $=^*$  relation. Every equivalence relation gives rise to a notion of minimality, modulo that relation. We call the resulting collection of minimal indices a *spectral set*, because it selects exactly one function from each equivalence class. Spectral sets are rare, natural examples of non-index sets which are neither  $\Sigma_n$  nor  $\Pi_n$ -complete.

In this thesis, we classify spectral sets according to their thinness and information content. We give optimal immunity results for the spectral sets (that is, we identify types of sets which are not contained in spectral sets), and we place these sets in the arithmetic hierarchy (which quantitatively measures their information contents). Some lower bounds in the arithmetic hierarchy follow from immunity properties alone, but we further refine



these immunity bounds using direct techniques. We also measure information content with Turing equivalences. In fact, the spectral sets become canonical iterations of the halting set when we list our computer programs in the right order. Regardless of the particular numbering, a reasonable amount of information is always present in such sets.

We now informally describe the contents of some spectral sets. The  $\Pi_3$ -Separation Theorem says that the spectral sets pertaining to  $\equiv_1$ ,  $=^*$ , and  $\equiv_m$  each have the same complexity with respect to the arithmetic hierarchy, yet each of these sets is immune against a different level of the arithmetic hierarchy. We can thus quantitatively compare the strength of equivalence relations by way of immunity. We also prove a result which we call the Forcing Lowness Lemma. Using this lemma, we show that  $\emptyset''''$  is decidable in  $\text{MIN}^T$  (the spectral set for  $\equiv_T$ ) together with  $\emptyset''$ . This result is probably optimal, and we apply the Forcing Lowness Lemma again to show that, in some formal sense, this fact will be difficult to prove.

Armed with this new machinery, we highlight its utility with some new applications. First, we prove the Peak Hierarchy Theorem: there exists a set which neither contains nor is disjoint from any infinite arithmetic set, yet the set is majorized by a computable function. Furthermore, the set that we construct is natural in the sense that it contains a spectral set. Along the way, we construct a computable sequence of c.e. sets in which no set can be computed from the join of the others, for any iteration of the jump operator.

We use the machinery of spectral sets to quantitatively compare the power of nondeterminism with the power of the jump operator. We show that in the world of computably enumerable sets, nondeterminism contributes nothing to immunity. In this respect, the jump operator outshines nondeterminism. Nonetheless, we can ascend naturally from the lowest level of the spectral hierarchy using nondeterminism.

Finally, we present connections to the Arslanov Completeness Criterion which stand as immediate consequences of immunity properties for spectral sets.

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## CHAPTER 1

# Introduction

### 1.1. Episode IV: A New Hierarchy

It all begins with Occam's razor. Since the fourteenth century, and certainly not before then, mankind has established a universal preference for simplicity. For centuries, people have enjoyed the pleasures of short descriptions and the joys of simple explanations. In the twentieth century, this predisposition abundantly manifests itself in computer science: humans love short computer programs.

We now consider two types of people who are especially keen (resp. not keen) on short programs. Note that the length of the shortest program describing an algorithm is a measure of its complexity. An output which follows a simple, constructive pattern can not be seen as random. In particular, outputs with short descriptions are not *Kolmogorov random*. Therefore, we expect that fans of Kolmogorov random strings will not like shortest programs (unless the programs are really long). In the tradition of minimal indices, on the other hand, machine learnists are generally dissatisfied with machines that merely learn the index of a target function [1]. They prefer indices which are close to minimal. Machine learning thus provides a practical application for the theory of shortest programs.

Minimal indices, or shortest programs, are generalizations of Kolmogorov random strings. According to Kolmogorov complexity, strings which lack short descriptions contain more information than those that have. This point-of-view matches our intuition: a string which is truly random does not follow a simple pattern and requires many bits to describe it. The index of a shortest program is always a Kolmogorov random string because if it were not, then some smaller (a.k.a shorter) index would compute that program. The set MIN (Definition 1.2.16) thus contains the shortest descriptions, or "random strings" for c.e. sets.

Later we shall consider further generalizations, such as  $\text{MIN}^T$ , the shortest descriptions for c.e. Turing degrees.

If simplicity constitutes our objective, then we have no better place to begin our study of computability theory than with the following eloquent observation:

*The set of shortest programs is not computable.*

By the “set of shortest programs,” we mean the set succinctly characterized by

$$\text{f-MIN} := \{e : (\forall j < e) [\varphi_j \neq \varphi_e]\}.$$

Despite its outwardly congenial appearance, f-MIN exhibits some barbaric properties. In 1972, Meyer demonstrated that f-MIN admits a neat Turing characterization, namely  $\text{f-MIN} \equiv_T \emptyset''$  [31]. Yet it is difficult, if not impossible, to pin down the degree of f-MIN for stronger reductions (see Section 1.4) [20], [39]. Along these lines, Schaefer showed that, unlike the familiar index sets, a strong reduction will never reduce the halting set to f-MIN (Section 3.4.1) [39].

At this point, the reader is likely wondering, “In the definition of f-MIN, what happens if we replace the functions with sets, and also try to replace equality with other equivalence relations?” We trace the scant history of this problem. In Spring 1990 (according to the best recollection of the author), John Case issued a homework assignment with the following definition [5]:

$$\text{f-MIN}^* := \{e : (\forall j < e) [\varphi_j \neq^* \varphi_e]\},$$

where  $=^*$  means equal except for a finite set. Case notes that f-MIN\* is  $\Sigma_2$ -immune, although his assignment exclusively refers to the  $\Sigma_2$  sets as “lim-r.e.” sets:

DEFINITION 1.1.1. *A is lim-r.e. iff there exists a computable function  $g$  satisfying*

$$(\forall x) [\chi_A(x) = \liminf_{t \rightarrow \infty} g(x, t)].$$

On October 1, 1996, six years after the initial homework assignment, Case introduced the set f-MIN\* to Marcus Schaefer in an email.

The following year, Schaefer published a Master’s Thesis on minimal indices [39], which became the first public account of f-MIN\*. In his survey thesis, Schaefer proved that

$f\text{-MIN}^* \oplus \emptyset' \equiv_T \emptyset'''$ , leaving open the tantalizing question of whether or not  $f\text{-MIN} \equiv_T \emptyset'''$ . All that would be required to answer this question affirmatively is to show that  $f\text{-MIN}^* \geq_T \emptyset'$ , care of Schaefer's result. The reader is encouraged to attempt this reduction before proceeding. This concludes our historical remarks. The remaining scholarship on this subject is probably contained in this thesis.

In attempt to comprehend the world of minimal indices, we characterize the spectral sets from Section 1.2.3 in three ways. First, we describe the sets in terms of the arithmetic hierarchy. The arithmetic hierarchy gives us an idea of the complexity of sets by describing how many quantifiers are needed to determine membership. The arithmetic hierarchy does not, however, tell us exactly which sets are computable from a set in question. For this reason, we devote Chapter 2 to a discussion of Turing degrees for minimal indices. We discuss several reduction techniques, and ultimately discover that spectral sets contain as much information as any set with the same complexity (modulo some nontrivial iteration of the halting set). Even without an oracle, this can still be true. In particular, we show that it possible to effectively enumerate the partial computable functions in such a way that the spectral sets are Turing equivalent to the familiar sets  $\emptyset'$ ,  $\emptyset''$ ,  $\emptyset'''$ ,  $\dots$  (see Chapter 5). Both Chapter 2 and Chapter 5 make use of the Forcing Lowness Lemma (Lemma 2.3.2), an interesting result in its own right.

In Chapter 3, we classify spectral sets in terms of immunity. Immunity is a measure of the “thinness” of a set, and it is an especially appropriate benchmark for spectral sets. All of the sets we consider are  $\omega$ -immune and not hyperimmune (see Section 3.4.1 and Section 6.2), but another immunity notion is useful for comparisons. In particular, we examine immunity with respect to the arithmetic hierarchy. Weak equivalence relations give rise to “thin” spectral sets which are immune against high levels of the arithmetic hierarchy. This may be the first time that a class of sets has been characterized in this manner. Of note in Chapter 3 is the  $\Pi_3$ -Separation Theorem (Theorem 3.1.3), which gives a neat, if not surprising, way for distinguishing between spectral sets in  $\Pi_3$ .

Chapter 4 crushes false generalizations that one might surmise after reading just the first three chapters. At the same time, it provides additional direction by introducing an

operator on equivalence relations. We learn that spectral sets occupy every level of the arithmetic hierarchy, including  $\Sigma_n - \Pi_n$ , and we gain intuition for why a simple converse to the immunity-completeness theorems does not appear in Section 3.3. Furthermore, we observe that the location of an equivalence relation within the arithmetic hierarchy tells us little about its immunity. The operator in this chapter offers a way to compare the power of nondeterminism against the jump operator within the realm of the c.e. sets. If we accept the notion that weaker relations indicate greater computational power, then the jump operator comes out on top.

Finally, what is the sparsest set you can imagine? We follow this line of thought into the last chapter. In particular, there exists a spectral set so sparse that it doesn't contain any infinite arithmetic sets. Most of the work in Chapter 6 is devoted to showing that this remarkable spectral set is not hyperimmune. Consequently, we are able to show that there exists a  $\mathbf{0}$ -majorized set which takes a bite out of every arithmetic set but never eats the whole thing.

## 1.2. Preliminaries

The computability background required for this paper is completely covered in Soare's book [44], and we use the standard notation from his book throughout this thesis. The other important reference for this thesis is Schaefer's survey on minimal indices [39]. Schaefer's work is approachable and comprehensive. We will not cover all his results here.

**1.2.1. Basic computability theory.** p.c. stands for partial computable, and c.e. stands for computably enumerable [44]. "c.e. in  $A$ ", or equivalently, " $A$ -c.e." means enumerable with an  $A$  oracle.  $A$ -computable means computable with an  $A$  oracle. We say  $A$  is *co-c.e.* if  $A$  is c.e. Unless otherwise specified, we assume a fixed enumeration of the partial computable functions,  $\varphi_0, \varphi_1, \dots$ .  $W_0, W_1, \dots$  is an enumeration of their domains.  $\text{dom } f$  denotes the domain for a partial function  $f$ , and  $\text{range } f$  is the range of  $f$ .  $\downarrow$  is for converge, and  $\uparrow$  is for diverge.  $\varphi_{e,t}(x) \downarrow$  means that  $\varphi_e(x)$  converges in  $t$  steps.  $(\mu x)$  means "the smallest  $x$  such that  $\dots$ ".  $K := \{x : \varphi_x(x) \downarrow\}$  denotes the halting set,  $'$  denotes the

jump operator,  $(^n)$  is the  $n^{\text{th}}$  iteration of the jump operator, and  $\emptyset^{(\omega)} := \{\langle x, n \rangle : x \in \emptyset^{(n)}\}$ . “lim” means limit.

For any set  $A$ ,  $\overline{A}$  denotes the complement of  $A$ .  $|A|$  denotes the cardinality of  $A$ .  $\chi_A$  is the characteristic function for  $A$ , which we sometimes write as just  $A$ .  $A(n)$  is the  $n^{\text{th}}$  element of  $A$  under the usual ordering.  $\omega$  denotes the set of natural numbers.  $\langle \cdot, \cdot \rangle : \omega \times \omega \rightarrow \omega$  is a bijective pairing function.

For any equivalence class  $\equiv_\alpha$ , the  $\equiv_\alpha$ -degree of a set  $A$  is the class of all sets equivalent to  $A$  under  $\equiv_\alpha$ . If no equivalence relation is specified, we mean Turing equivalence. The degree containing  $\emptyset$  is  $\mathbf{0}$ , the degree containing  $\emptyset'$  is  $\mathbf{0}'$ , and the degree containing  $\emptyset^{(n)}$  is  $\mathbf{0}^{(n)}$ . An acquaintance with the statements of the  $s$ - $m$ - $n$  Theorem, the Recursion Theorem, and the Jump Theorem [44, Theorem III.2.3] are recommended.

DEFINITION 1.2.1.  $\Psi_e^A(x)$  denotes the output of oracle Turing machine  $e$  with oracle  $A$  on input  $x$ .  $\psi_e^A(x)$  is the corresponding *use function*, the maximum query made to the oracle during computation (if it converges).  $W_e^A$  denotes the domain of  $\Psi_e^A(x)$ .

Occasionally, we will also use  $\psi$  to denote a partial computable function. In this case  $\psi$  will not receive an oracle superscript, so as not to be confused with the use function.

DEFINITION 1.2.2. For any set  $A$ ,

$$A \upharpoonright n := A \cap \{0, \dots, n\}$$

is the  $n^{\text{th}}$  *initial segment* of  $A$ .

The join operator allows us to use two or more sets as oracles simultaneously.

DEFINITION 1.2.3. Let  $A$  and  $B$  be sets. We define the *join* of  $A$  and  $B$ , denoted  $A \oplus B$ , to be the set

$$A \oplus B := \{2x : x \in A\} \cup \{2x + 1 : x \in B\}.$$

For a sequence of sets  $\{A_i\}$ , define the *infinite join* to be

$$\bigoplus_{i \in \omega} A_i := \{\langle x, i \rangle : x \in A_i\}.$$

Sometimes we only care about the first number in an ordered pair:



NOTATION (projections). Let  $\pi_1 : \omega \rightarrow \omega$  denote the function which maps pairs to their first coordinates, i.e.

$$\pi_1(\langle x, y \rangle) := x.$$

Similarly,

$$\pi_2(\langle x, y \rangle) := y.$$

DEFINITION 1.2.4. An integer  $n$  is an  $i^{\text{th}}$  prime power if  $n = p_i^k$  for some  $k \geq 1$ , where  $p_i$  is the  $i^{\text{th}}$  prime number. If  $n$  is an  $i^{\text{th}}$  prime power for some  $i$ , then we may simply say  $n$  is a prime power.

DEFINITION 1.2.5. Let  $A$  and  $B$  be thing. Wacka wacka.

DEFINITION 1.2.6.  $A$  is called an *index set* if

$$[x \in A \quad \& \quad \varphi_y = \varphi_x] \implies y \in A.$$

DEFINITION 1.2.7. A few familiar index sets will come into play. For  $n \geq 0$ :

$$\text{INF} := \{e : |W_e| = \infty\},$$

$$\text{TOT} := \{e : W_e = \omega\},$$

$$\text{COF} := \{e : W_e =^* \omega\},$$

$$\text{mCOMP} := \{e : W_e \equiv_{\text{m}} K\},$$

$$\text{LOW}^n := \{e : (W_e)^{(n)} \equiv_{\text{T}} \emptyset^{(n)}\},$$

$$\text{HIGH}^n := \{e : (W_e)^{(n)} \equiv_{\text{T}} \emptyset^{(n+1)}\}.$$

Note that

$$\text{LOW}^0 = \{e : W_e \equiv_{\text{T}} \emptyset\},$$

and

$$\text{HIGH}^0 = \{e : W_e \equiv_{\text{T}} K\}.$$

We sometimes view a set of natural numbers as a matrix, in which case the rows may have special meanings:

DEFINITION 1.2.8. For any set  $A$ ,

$$A^{[y]} := \{\langle x, y \rangle : \langle x, y \rangle \in A\},$$

$$A^{[\bar{x}]} := \{\langle x, y \rangle : \langle x, y \rangle \in A\}.$$

Here  $A^{[y]}$  is the “ $y^{\text{th}}$  row of  $A$ ,” and  $A^{[\bar{x}]}$  is the “ $x^{\text{th}}$  column of  $A$ ,”

$$A^{[\leq y]} := \bigcup_{z \leq y} A^{[z]}, \quad \text{and}$$

$$A^{[> y]} := \bigcup_{z > y} A^{[z]}.$$

DEFINITION 1.2.9. (I) A subset  $A \subseteq B$  is a *thick* subset if  $A^{[y]} =^* B^{[y]}$  for all  $y$ .

(II)  $B$  is *piecewise computable* if  $B^{[y]}$  is computable for all  $y$ .

DEFINITION 1.2.10. Let  $(D_e)_{e \in \omega}$  be the canonical numbering of the finite sets.

(I) A set is *immune* if it is infinite and contains no infinite c.e. sets.

(II) A set  $A$  is *hyperimmune* if it is infinite and there is no computable function  $f$  such that:

(a)  $(D_{f(i)})_{i \in \omega}$  is a family of pairwise disjoint sets, and

(b)  $D_{f(i)} \cap A \neq \emptyset$ .

**1.2.2. Reductions and arithmetic hierarchy.** We list the main reductions and equivalence relations.

DEFINITION 1.2.11 (reductions). Let  $A$  and  $B$  be sets.

(I) Let  $f$  and  $g$  be functions.

$$f =^* g \iff (\exists N)(\forall x > N) [f(x) = g(x)].$$

If  $f$  and  $g$  are the characteristic functions for  $A$  and  $B$  respectively, then

$$A =^* B \iff f =^* g.$$

Furthermore,  $A \subseteq^* B$  when  $A =^* C$  for some  $C \subseteq B$ .

(II)  $A \leq_m B$  if there exists a computable function  $f$  such that

$$x \in A \iff f(x) \in B.$$

- (III)  $A \leq_1 B$  if  $A \leq_m B$  via an *injective* function  $f$ .
- (IV) Let  $\{\sigma_n\}$  be an enumeration of all propositional truth tables with predicates of the form “ $k \in S$ ,” where  $k \in \omega$ . We say that a set  $X$  *satisfies* a truth table  $\sigma_n$ , or  $X \models \sigma_n$  if the proposition  $\sigma_n$  is true when “ $S$ ” is interpreted as  $X$ .

$A \leq_{tt} B$  just in case there exists a computable function  $f$  such that

$$x \in A \iff B \models \sigma_{f(x)}.$$

- (V)  $A \leq_{btt} B$  means that  $A \leq_{tt} B$  via an  $f$  which requires only constant many queries to  $B$ .
- (VI)  $A \leq_T B$  if there exists an index  $e$  such that  $\chi_A = \Psi_e^B$ . In general, we write  $f \leq_T B$  for a function  $f$  if  $f = \Psi_e^B$  for some  $e$ .
- (VII)  $A \leq_{bT} B$  if  $A \leq_T B$  and the largest query to the  $B$  oracle is computably bounded. That is,  $A \leq_{bT} B$  if there exists a computable function  $f$  and an index  $e$  such that for all  $x$ ,

$$\chi_A(x) = \Psi_e^{B \upharpoonright f(x)}(x).$$

Alternatively,  $A \leq_{bT} B$  if there exists a computable function  $f$  such that

$$x \in A \iff B \models \xi_{f(x)},$$

where  $\{\xi_n\}$  is an enumeration of p.c. truth tables which converge upon satisfaction and diverge otherwise. For this reason, *bT*-reductions are also called “weak truth-table” reductions.

A function  $f \leq_{bT} B$  if  $f \leq_T B$  and we can computably bound the largest query to  $B$ .

- (VIII) For every  $n < \omega$ , let

$$A \leq_{T^{(n)}} B \iff A^{(n)} \leq_T B^{(n)},$$

and

$$A \leq_{T^{(\omega)}} B \iff (\exists n) [A^{(n)} \leq_T B^{(n)}].$$

If  $A \leq_\alpha B$  and  $B \leq_\alpha A$  for some partial ordering  $\leq_\alpha$ , then we write  $A \equiv_\alpha B$ . If  $A \leq_\alpha B$  and  $B \not\leq_\alpha A$ , then  $A <_\alpha B$ . If  $A \not\leq_\alpha B$  and  $B \not\leq_\alpha A$ , then  $A \parallel_\alpha B$ . This notation applies to all of the reductions in this definition. Finally, two sets are equal if they are equal.

We define the member classes of the arithmetic hierarchy:  $\Delta_n$ ,  $\Sigma_n$ , and  $\Pi_n$  for  $n \geq 0$ .

DEFINITION 1.2.12 (arithmetic hierarchy). Let  $n \geq 1$ .

- (I)  $\Delta_0 = \Sigma_0 = \Pi_0$  is the class of computable sets.
- (II)  $A \in \Sigma_n$  if there exists a computable relation  $R$  such that

$$x \in A \iff (\exists y_1)(\forall y_2)(\exists y_3) \dots (Qy_n) R(x, y_1, \dots, y_n),$$

where  $Q$  is  $\forall$  if  $n$  is even, and  $\exists$  if  $n$  is odd. Similarly,

- (III)  $A \in \Pi_n$  if there exists a computable relation  $R$  such that

$$x \in A \iff (\forall y_1)(\exists y_2)(\forall y_3) \dots (Qy_n) R(x, y_1, \dots, y_n),$$

where  $Q$  is  $\exists$  if  $n$  is even, and  $\forall$  if  $n$  is odd.

- (IV)  $\Delta_n := \Sigma_n \cap \Pi_n$ .

We also relativize the arithmetic hierarchy in the following way:

DEFINITION 1.2.13 (relativized arithmetic hierarchy). Let  $S$  be a set, and let  $n \geq 1$ .

- (I)  $\Delta_0^S = \Sigma_0^S = \Pi_0^S$  is the class of  $S$ -computable sets.
- (II)  $A \in \Sigma_n^S$  is just as in Definition 1.2.12(ii), except that “a computable relation  $R$ ” is replaced with “an  $S$ -computable relation  $R$ .”
- (III)  $A \in \Pi_n^S$  is just as in Definition 1.2.12(iii), except that “a computable relation  $R$ ” is replaced with “an  $S$ -computable relation  $R$ .”
- (IV)  $\Delta_n^S := \Sigma_n^S \cap \Pi_n^S$ .

DEFINITION 1.2.14. A set  $A$  is:

- (I)  $\Sigma_n$ -complete if  $A \in \Sigma_n$  and for every  $B \in \Sigma_n$ ,  $B \leq_m A$ .
- (II)  $\Pi_n$ -complete if  $A \in \Pi_n$  and for every  $B \in \Pi_n$ ,  $B \leq_m A$ .

The reader who is reading about the arithmetic hierarchy for the first time should familiarize herself with (Relativized) Post's Theorem, the Hierarchy Theorem, and the Limit Lemma [44].

**1.2.3. Minimal indices.** We formally define our objects of study.

DEFINITION 1.2.15. Let  $\equiv_\alpha$  be an equivalence relation on sets. Then

$$\text{MIN}^{\equiv_\alpha} := \{e : (\forall j < e) [W_j \not\equiv_\alpha W_e]\}.$$

Similarly, for an equivalence relation  $\equiv_\beta$  on functions we define,

$$\text{f-MIN}^{\equiv_\beta} := \{e : (\forall j < e) [\varphi_j \not\equiv_\beta \varphi_e]\}.$$

A set of either form is called a *spectral set*, or, equivalently, a *MIN-set*.

We will refer to certain spectral sets often, and we use the following abbreviations for these sets. We employ equivalence relations from Definition 1.2.11.

DEFINITION 1.2.16. For notational clarity, we sometimes abbreviate the relations  $=^*$ ,  $\equiv_m$ ,  $\equiv_T$ , and  $\equiv_{T^{(n)}}$  as  $*$ ,  $m$ ,  $T$  and  $T^{(n)}$ , respectively. The following are in effect, for  $n \geq 0$ :

$$\begin{aligned} \text{MIN} &:= \{e : (\forall j < e) [W_j \neq W_e]\}, \\ \text{MIN}^* &:= \{e : (\forall j < e) [W_j \neq^* W_e]\}, \\ \text{MIN}^m &:= \{e : (\forall j < e) [W_j \not\equiv_m W_e]\}, \\ \text{MIN}^T &:= \{e : (\forall j < e) [W_j \not\equiv_T W_e]\}, \\ \text{MIN}^{T^{(n)}} &:= \{e : (\forall j < e) [W_j \not\equiv_{T^{(n)}} W_e]\}, \end{aligned}$$

and

$$\begin{aligned} \text{MIN}^{T^{(\omega)}} &:= \bigcap_{n \in \omega} \text{MIN}^{T^{(n)}} \\ &= \{e : (\forall j < e)(\forall n) [(W_j)^{(n)} \not\equiv_T (W_e)^{(n)}]\}. \end{aligned}$$

In the case of  $\text{MIN}^m$ , we modify the usual definition of  $\equiv_m$  so that all recursive sets, including  $\emptyset$  and  $\omega$ , have the same  $m$ -degree. This makes Theorem 3.2.1 true without modification.

A similar set of notations applies for indices of minimal functions, but only for a few specific equivalence relations.

NOTATION. We shall consider the following “function” spectral sets.

$$\begin{aligned} \text{fR} &:= \{e : (\forall j < e) [\varphi_j(0) \neq \varphi_e(0)]\}, \\ \text{f-MIN} &:= \{e : (\forall j < e) [\varphi_j \neq \varphi_e]\}, \\ \text{f-MIN}^* &:= \{e : (\forall j < e) [\varphi_j \neq^* \varphi_e]\}. \end{aligned}$$

Occasionally, we will want to compute the minimal index for a c.e. set:

DEFINITION 1.2.17. For every equivalence relation  $\equiv_\alpha$ , we define a function  $\min^{\equiv_\alpha}$  by

$$\min^{\equiv_\alpha}(e) := (\mu x) [W_x \equiv_\alpha W_e].$$

If  $\equiv_\alpha$  is not specified in the notation, we mean equality.

The following proposition is easily verified.

PROPOSITION 1.2.18. *Let  $\equiv_\alpha$  and  $\equiv_\beta$  be equivalence relations. Assume that for all  $X, Y \subseteq \omega$ ,*

$$X \equiv_\alpha Y \implies X \equiv_\beta Y.$$

*Then  $\text{MIN}^{\equiv_\alpha} \supseteq \text{MIN}^{\equiv_\beta}$ .*

- COROLLARY 1.2.19. (I)  $\text{f-MIN} \supseteq \text{MIN} \supseteq \text{MIN}^*$ ,  
 (II)  $\text{f-MIN} \supseteq \text{f-MIN}^* \supseteq \text{MIN}^*$ ,  
 (III)  $\text{MIN} \supseteq \text{MIN}^* \supseteq \text{MIN}^m \supseteq \text{MIN}^T \supseteq \text{MIN}^{T'} \dots$ .

In the following proposition,  $\equiv_\alpha$  can be taken to be any familiar intermediate reduction, such as  $\equiv_{\text{btt}}$ ,  $\equiv_{\text{tt}}$ , or  $\equiv_{\text{bT}}$ . It might appear, in light of Proposition 1.2.20, that the spectral sets form a simple, linear ordering under reverse inclusion. However, in Chapter 4 and Section A.8 we explore a natural class of equivalence relations which do not fit between  $\text{MIN}^{T^{(n)}}$  and  $\text{MIN}^{T^{(n+1)}}$  for any  $n$ .

PROPOSITION 1.2.20. (I) *For every  $n \geq 0$ ,  $\text{MIN}^{m^{(n+1)}} = \text{MIN}^{T^{(n)}}$ .*

(II) Let  $\equiv_\alpha$  be any equivalence relation which is weaker than  $\equiv_1$  and stronger than  $\equiv_T$ .

For any sets  $A$  and  $B$ , let

$$A \equiv_{\alpha^{(n)}} B \iff A^{(n)} \equiv_\alpha B^{(n)},$$

and define

$$\text{MIN}^{\alpha^{(n)}} := \{e : (\forall j < e) [W_j \not\equiv_{\alpha^{(n)}} W_e]\}.$$

Then for all  $n$ ,

$$\text{MIN}^{T^{(n)}} \supseteq \text{MIN}^{\alpha^{(n+1)}} \supseteq \text{MIN}^{T^{(n+1)}}.$$

PROOF. (I). It suffices to show that for any sets  $A$  and  $B$ ,

$$(1.1) \quad A^{(n+1)} \equiv_m B^{(n+1)} \iff A^{(n)} \equiv_T B^{(n)}.$$

We show (1.1) by proving the Jump Theorem [44, Theorem III.2.3], namely:

$$A' \leq_m B' \iff A \leq_T B.$$

Assume  $A' \leq_m B'$  via a computable function  $h$ . Then

$$A \leq_m A' \leq_m B',$$

$$\bar{A} \leq_m A' \leq_m B'.$$

Indeed,  $A \leq_m A'$  via the function  $f$  defined by

$$\Psi_{f(x)}^A(n) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise,} \end{cases}$$

because

$$x \in A \iff \Psi_{f(x)}^A[f(x)] \downarrow \iff f(x) \in A'.$$

An analogous function  $g$  yields  $\bar{A} \leq_m A'$ .

It follows that  $A$  and  $\bar{A}$  are c.e. in  $B$  via the enumerations  $x \in A_s \iff h \circ f(x) \in B'_s$  and  $x \in \bar{A}_s \iff h \circ g(x) \in B'_s$ , where  $B'_s$  is a  $B$ -enumeration of  $B'$ . Therefore  $A \leq_T B$ .

Conversely, assume  $A \leq_T B$ . Since  $A'$  is c.e. in  $A$ ,  $A'$  must be c.e. in  $B$ . This means that  $A' \leq_m B'$ , since  $B'$  is  $m$ -complete relative to  $B$ .  $\square$

(II). According to the Jump Theorem [44, Theorem III.2.3],

$$A \equiv_T B \iff A' \equiv_1 B'.$$

Therefore,

$$A^{(n)} \equiv_T B^{(n)} \implies A^{(n+1)} \equiv_1 B^{(n+1)} \implies A^{(n+1)} \equiv_\alpha B^{(n+1)},$$

and more obviously,

$$A^{(n+1)} \equiv_\alpha B^{(n+1)} \implies A^{(n+1)} \equiv_T B^{(n+1)}. \quad \square$$

$\square$

### 1.3. Complexity of spectral sets

We place spectral sets in the arithmetic hierarchy. Our lower bounds immediately show that MIN-sets are not computable, although our laconic proofs do not involve the familiar technique of reduction to the halting set. Spectral sets can be found in every level of the arithmetic hierarchy. Unlike index sets, which are always  $\geq_m K$  (Rice's Theorem [44]), spectral sets never have this property. In fact,  $K$  doesn't even *btt*-reduce to MIN-sets (Corollary 3.4.3).

Based on Corollary 1.2.19, it would be reasonable to extrapolate that  $A \supseteq B$  implies that  $B$  lies in a higher arithmetic level than  $A$ . This turns out not to be the case when we considered minimal indices of functions. Indeed, there is a notable exception:

DEFINITION 1.3.1 (Schaefer [39]). We call

$$\text{f-MIN} = \{e : (\forall j < e) [\varphi_j \neq \varphi_e]\}$$

the *set of minimal indices for functions*, and

$$\text{fR} = \{e : (\forall j < e) [\varphi_j(0) \neq \varphi_e(0)]\}$$

denotes the *set of shortest descriptions for nonnegative integers*.

$\text{f-MIN} \supseteq \text{fR}$ , yet  $\text{f-MIN} \in \Sigma_2 - \Pi_2$  and  $\text{fR} \in \Delta_2$ .  $\text{fR}$ , which does not appear to have a spectral analogue for sets, highlights a potential difference between minimal indices for sets and minimal indices for functions. We shall exhibit an infinite  $\Delta_2$  subset of MIN in



Section 3.4.3, however, it is not a spectral set. We remark that the results in Sections 1.3.1 and Sections 1.3.2 are by-and-large subsumed by Corollary 2.4.1.

**1.3.1. Upper bounds.** We reveal upper bounds for a number of sets, including the following rare example.

DEFINITION 1.3.2.

$$\text{MIN}^{\text{time}} := \{e : (\forall j < e) (\exists \langle x, t \rangle) [W_{j,t}(x) \neq W_{e,t}(x)]\}.$$

The  $\equiv_{\text{time}}$  identifies indices which are not only equal, but their respective computations converge in exactly the same amount of time.  $\equiv_{\text{time}}$  is the only MIN-set in this thesis which contains MIN; the rest are subsets of either MIN or f-MIN.

In Theorem 1.3.4, we will show that the following upper bounds are optimal (except for part (i), which follows from Theorem 2.2.1). In light of Proposition 1.2.20, Proposition 1.3.3 also shows that  $\text{MIN}^{\text{m}(n)} \in \Pi_{n+3}$ .

PROPOSITION 1.3.3. *Let  $n \geq 0$ . Then*

- (I)  $\text{MIN}^{\text{time}} \in \Sigma_1$ .
- (II)  $\text{fR} \in \Delta_2$ .
- (III)  $\text{MIN}, \text{f-MIN} \in \Sigma_2$ .
- (IV)  $\text{MIN}^*, \text{f-MIN}^* \in \Pi_3$ .
- (V)  $\text{MIN}^{\text{m}} \in \Pi_3$ .
- (VI)  $\text{MIN}^{\overline{=1}} \in \Pi_3$ .
- (VII)  $\text{MIN}^{\text{T}(n)} \in \Pi_{n+4}$ .

PROOF. (i). Immediate from the definition. □

(ii).  $\varphi_j(0) = \varphi_e(0)$  can be decided with a  $\emptyset'$  oracle. So  $\text{fR} \in \Delta_2$  by the Limit Lemma [44]. □

(iii).  $\{\langle j, e \rangle : W_j = W_e\} \in \Pi_2$  [44]. □

(iv).  $\{\langle j, e \rangle : W_j =^* W_e\} \in \Sigma_3$  [44]. □

(v). For any c.e. sets  $A$  and  $B$ ,

$$A \leq_m B \iff (\exists e)(\forall x) [\varphi_e(x) \downarrow \ \& \ (x \in A \iff \varphi_e(x) \in B)],$$

which shows that  $A \leq_m B$  is a  $\Sigma_2^{\emptyset'}$  relation. It follows that  $A \equiv_m B$  is also a  $\Sigma_2^{\emptyset'}$  relation.

In particular, for

$$C := \{\langle j, e \rangle : W_j \equiv_m W_e\},$$

we have

$$C \in \Sigma_2^{\emptyset'} = \Sigma_3.$$

Hence

$$e \in \text{MIN}^m \iff (\forall j < e) [\langle j, e \rangle \notin C],$$

which places  $\text{MIN}^m \in \Pi_3$ . □

(VI). The same idea from (v) works because injectivity can be tested with a  $\emptyset'$  oracle. □

(VII). For any sets  $A$  and  $B$ ,

$$\begin{aligned} A \leq_T B &\iff (\exists e) [A = \Psi_e^B] \\ &\iff (\exists e)(\forall x) [\Psi_e^B(x) \downarrow \ \& \ (x \in A \iff \Psi_e^B(x) = 1)], \end{aligned}$$

which shows that  $A \leq_T B$  is a

$$\Sigma_2^{B' \oplus (A \oplus B)} = \Sigma_2^{A \oplus B'}$$

relation, and it follows that  $A \equiv_T B$  is a  $\Sigma_2^{A' \oplus B'}$  relation. In particular, for

$$C_n := \left\{ \langle j, e \rangle : (W_j)^{(n)} \equiv_T (W_e)^{(n)} \right\},$$

we have

$$C_n \in \Sigma_2^{((W_j)^{(n)})' \oplus ((W_e)^{(n)})'} \subseteq \Sigma_2^{\emptyset^{(n+2)}} = \Sigma_{n+4}.$$

It follows that

$$e \in \text{MIN}^{T^{(n)}} \iff (\forall j < e) [\langle j, e \rangle \notin C_n],$$

which makes  $\text{MIN}^{T^{(n)}} \in \Pi_{n+4}$ . □

□

**1.3.2. Lower bounds.** It's not too hard to show that MIN-sets are noncomputable (modulo a few well-known theorems), however, the more familiar method of  $m$ -reduction to the halting set doesn't work. Theorem 1.3.4(ii) was known to Meyer [31], and I attribute Theorem 1.3.4(i) to Lance Fortnow.

**THEOREM 1.3.4.** *Let  $n \geq 0$ .*

- (I)  $\text{fR} \notin \Sigma_1 \cup \Pi_1$ .
- (II)  $\text{MIN}, \text{f-MIN} \notin \Pi_2$ .
- (III)  $\text{MIN}^*, \text{f-MIN}^* \notin \Sigma_3$ .
- (IV)  $\text{MIN}^m \notin \Sigma_3$ .
- (V)  $\text{MIN}^{\equiv 1} \notin \Sigma_3$ .
- (VI)  $\text{MIN}^{\text{T}^{(n)}} \notin \Sigma_{n+4}$ .

**PROOF.** (I).  $\text{fR} \notin \Sigma_1$  follows immediately from the fact that  $\text{fR}$  is immune [39]. Suppose  $\text{fR} \in \Pi_1$ . Let  $a$  be the smallest index such that  $\varphi_a(0) \uparrow$ . Define a computable function  $f$  by way of the  $s$ - $m$ - $n$  Theorem [44] and the following constant function:

$$\varphi_{f(x)}(y) := \begin{cases} (\mu t) [\varphi_{x,t}(0) \downarrow] & \text{if } \varphi_x(0) \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Let

$$K_0 := \{e : \varphi_e(0) \downarrow\}.$$

$K_0$  is  $\Sigma_1$ -complete. Note that

$$\begin{aligned} e \in K_0 &\iff \varphi_{f(e)}(0) \downarrow \\ &\iff (\exists j \in [\{0, \dots, f(e)\} \cap \text{fR}] - \{a\}) \left[ \varphi_j(0) \downarrow \ \& \ \varphi_{e, \varphi_j(0) \downarrow}(0) \downarrow \right], \\ &\iff (\exists j \leq f(e)) \left[ j \in \text{fR} - \{a\} \ \& \ \varphi_j(0) \downarrow \ \& \ \varphi_{e, \varphi_j(0) \downarrow}(0) \downarrow \right]. \end{aligned}$$

This means that  $\overline{K_0} \in \Sigma_1$ , since  $j \in \text{fR} - \{a\} \implies \varphi_j(0) \downarrow$ . But that's a contradiction, because now  $K_0$  is computable.  $\square$

(II). Suppose that  $\text{MIN} \in \Pi_2$ , let  $a$  be the minimal index for  $\omega$ , and recall that

$$\text{TOT} = \{e : W_e = \omega\}$$

is  $\Pi_2$ -complete [44]. Then

$$\begin{aligned} \text{TOT} &= (\text{MIN} \cap \text{TOT}) \cup (\overline{\text{MIN}} \cap \text{TOT}) \\ &= \{a\} \cup \{e : (\forall j < e) [j \in \text{MIN} - \{a\} \implies W_j \neq W_e]\}. \end{aligned}$$

Now  $\text{TOT} \in \Sigma_2$ , since  $W_j = W_e$  can be decided in  $\Pi_2$ , and because  $\text{MIN} - \{a\} \in \Pi_2$  by assumption. This contradicts the fact that  $\text{TOT}$  is  $\Pi_2$ -complete.  $\square$

(III). We reuse the argument from part (ii). Suppose  $\text{MIN}^* \in \Sigma_3$ , let  $a$  be the  $*$ -minimal index for  $\omega$ , and recall that the set of cofinite indices

$$\text{COF} := \{e : W_e =^* \omega\}$$

is  $\Sigma_3$ -complete [44]. Then

$$\begin{aligned} \text{COF} &= (\text{MIN}^* \cap \text{COF}) \cup (\overline{\text{MIN}^*} \cap \text{COF}) \\ &= \{a\} \cup \{e : (\forall j < e) [j \in \text{MIN}^* - \{a\} \implies W_j \neq^* W_e]\} \end{aligned}$$

Now  $\text{COF} \in \Pi_3$ , since  $W_j =^* W_e$  can be decided in  $\Sigma_3$ , and because  $\text{MIN}^* - \{a\} \in \Sigma_3$  by assumption. This contradicts the fact that  $\text{COF}$  is  $\Sigma_3$ -complete.  $\square$

(IV).  $\{e : W_e \equiv_m C\}$  is  $\Sigma_3$ -complete whenever  $C \neq \emptyset$ ,  $C \neq \omega$ , and  $C$  is c.e. [48].  $\square$

(V).  $\{e : W_e \equiv_1 C\}$  is  $\Sigma_3$ -complete whenever  $C$  is c.e., infinite, and coinfinite [17]. Since  $W_j \equiv_1 W_e$  is decidable in  $\Sigma_3$ , the same argument again applies.  $\square$

(VI). Combining the Yates Index Set Theorem with the Sacks Jump Theorem yields

$$\text{HIGH}^n = \{e : W_e \equiv_{T^{(n)}} \emptyset'\}$$

is  $\Sigma_{n+4}$ -complete, which is exactly what is needed to prove the theorem. This fact seems to have been first observed by Schwarz in his PhD thesis [41, Theorem 3.3.1], [44, Theorem XII.4.4]. He writes simply,

“We discovered the unexpectedly short argument [that  $\text{HIGH}_n$  is  $\Sigma_{n+4}$ -complete] quite by accident, after having given up on finding any more direct line of proof.”  $\square$

$\square$

### 1.4. Noneffective orderings and other disasters

MIN is sensitive to the order in which we list the partial computable functions. This is exacerbated by the fact that some c.e. classes can be enumerated without repetition [45], [9].

DEFINITION 1.4.1. A *numbering* of a set  $S$  is a surjective mapping of  $\omega$  onto  $S$ . If a numbering  $\varphi$  is computable, we say  $\varphi$  is a *computable numbering*. If  $S$  is not specified, we mean the set of partial computable functions. A p.c. function  $\varphi$  is a *p.c. numbering* if

$$e \mapsto \varphi(\langle e, \cdot \rangle)$$

maps onto the partial computable functions. For any p.c. numbering  $\varphi$ , we denote the function  $\varphi(\langle e, \cdot \rangle)$  by  $\varphi_e$ .

DEFINITION 1.4.2. A *Gödel numbering*  $\varphi$  is a p.c. numbering such that if  $\psi$  is a p.c. function, then there exists a computable function  $f$  satisfying

$$\varphi_{f(e)}(x) = \psi(\langle e, x \rangle).$$

If in addition  $f$  is linearly bounded, we say that  $\varphi$  is a *Kolmogorov numbering*.

Since the  $\psi$  in Definition 1.4.2 might itself be a numbering, we can effectively find a  $\varphi$ -index for any algorithm when  $\varphi$  is a Gödel numbering. Furthermore, any reasonably encoded universal Turing machine is a Kolmogorov numbering [40]. We use a subscript to indicate the numbering for a MIN-set, as in  $\text{MIN}_\varphi$ . If the subscript is omitted, then we mean an arbitrary Gödel numbering.

**1.4.1. Gödel numberings.** The degrees for spectral sets are not always invariant with respect to Gödel numberings. For example, while we do not yet know the truth table degree of  $\text{fR}$  [39], we do have Theorem 1.4.4(i). Theorem 1.4.4(ii) is due to Schaefer [39], and we shall revisit this result again in Theorem 5.2.3. Most surprising, however, are Martin Kummer's results on the truth-table degree of  $\text{RAND}$ , the set of *Kolmogorov random strings*:

DEFINITION 1.4.3. For any finite string  $x$ , the Kolmogorov complexity of  $x$  is

$$C(x) := \min\{|e| : \varphi_e(0) = x\},$$

where  $|\cdot|$  denotes the length of an integer encoded in binary. The set of *Kolmogorov random strings* is

$$\text{RAND} := \{x : C(x) \geq |x|\}.$$

THEOREM 1.4.4 (Schaefer [39], Kummer [23]).

- (I) For any Gödel numbering  $\varphi$ ,  $\text{RAND}_\varphi \equiv_{\text{bT}} \emptyset' \equiv_{\text{bT}} \text{fR}_\varphi$ .
- (II) There exists a Kolmogorov numbering  $\varphi$  such that  $\text{fR}_\varphi \equiv_{\text{tt}} \emptyset'$ .
- (III) For any Kolmogorov numbering  $\varphi$ ,  $\text{RAND}_\varphi \equiv_{\text{tt}} \emptyset'$ .
- (IV) There exists a Gödel numbering  $\varphi$  such that  $\text{RAND}_\varphi \not\equiv_{\text{tt}} \emptyset'$ .

We turn to numberings for f-MIN. Using Proposition 3.1.2, Young gave a short proof of the following fact.

THEOREM 1.4.5 (Meyer [31]). For any Gödel numbering  $\varphi$ , there exists a Gödel numbering  $\psi$  such that  $\text{f-MIN}_\psi <_1 \text{f-MIN}_\varphi$ .

A more complicated argument reveals even more sensitivity. Kinber was the first to prove the following two results (both for Gödel numberings), however Schaefer's proof of (ii) is decidedly simpler.

THEOREM 1.4.6 (Kinber [20], Schaefer [39]). (I) There exist Gödel numberings  $\varphi$  and  $\psi$  such that  $\text{f-MIN}_\varphi \not\equiv_{\text{btt}} \text{f-MIN}_\psi$ .

- (II) There exists a Kolmogorov numbering  $\varphi$  such that  $\text{f-MIN}_\varphi \equiv_{\text{tt}} \emptyset''$ .

The “closer” for f-MIN numberings, however, is still at-large. Namely, Meyer's question from 1972 of whether  $\text{f-MIN}_\varphi \equiv_{\text{tt}} \emptyset''$  for all Gödel numberings  $\varphi$  remains open [31].

#### 1.4.2. Enumeration without repetition.

DEFINITION 1.4.7 ([48]). For any equivalence relation  $\equiv_\alpha$ , we define

$$G_{\equiv_\alpha}(C) := \{e : W_e \equiv_\alpha C\}.$$

Yates proved the following theorem only for c.e. sequences in the case  $\equiv_\alpha$  equal to  $\equiv_T$  [49], but as we demonstrate here, his argument easily generalizes to other relations. In the following theorem, when we say the  $\equiv_\alpha$  degrees can not be enumerated, we mean that it is impossible to make a list consisting of exactly one index from each  $\equiv_\alpha$  equivalence class.

**THEOREM 1.4.8.** *Let  $\equiv_\alpha$  be an equivalence relation satisfying*

$$\{\langle i, j \rangle : W_i \equiv_\alpha W_j\} \in \Sigma_n.$$

*Assume there is some c.e. set  $C$  such that  $G_{\equiv_\alpha}(C)$  is  $\Sigma_n$ -complete. Then no  $\Sigma_n$  sequence of c.e. sets enumerates the  $\equiv_\alpha$ -degrees without repetition.*

**PROOF.** Let  $\equiv_\alpha$  be a relation satisfying

$$\{\langle i, j \rangle : W_i \equiv_\alpha W_j\} \in \Sigma_n.$$

Suppose there is some  $A \in \Sigma_n$  which contains exactly one index from each  $\equiv_\alpha$  class. Let  $c \in A$  be the index such that  $W_c \equiv_\alpha C$ . Then

$$G_{\equiv_\alpha}(C) = \{e : (\forall k) [k \in A - \{c\} \implies W_e \not\equiv_\alpha W_k]\} \in \Pi_n,$$

which implies that  $G_{\equiv_\alpha}(C)$  is not  $\Sigma_n$ -complete. □

**COROLLARY 1.4.9.** *Let  $n \geq 0$ . Then*

- (I) *the  $=^*$  degrees can not be enumerated without repetition,*
- (II) *the  $\equiv_m$  degrees can not be enumerated without repetition, and*
- (III) *the  $\equiv_{T^{(n)}}$  degrees can not be enumerated without repetition.*

**PROOF.** (I).  $=^*$  is a  $\Sigma_3$  relation on c.e. sets, and  $G_*(\omega) = \text{COF}$  is  $\Sigma_3$ -complete [44]. □

(II).  $\equiv_m$  is a  $\Sigma_3$  relation on c.e. sets, and  $G_m(K)$  is  $\Sigma_3$ -complete (see Theorem 1.3.4 (iv)). □

(III).  $\equiv_T$  is a  $\Sigma_{n+4}$  relation on c.e. sets, and  $G_{T^{(n)}}(K)$  is  $\Sigma_{n+4}$ -complete (see Theorem 1.3.4(vi)). □

□

Theorem 1.4.8 also eliminates the possibility that the  $=^*$  sets might be enumerable using a  $\emptyset''$  oracle, that the  $\equiv_m$  might be enumerable using a  $\emptyset''$  oracle, and that the  $\equiv_{\Gamma(n)}$  sets might be enumerable using a  $\emptyset^{(n+3)}$  oracle.

Theorem 1.4.8 does not apply when  $\equiv_\alpha$  is the equality relation, since  $G_=(\omega) = \text{TOT} \in \Pi_2$ . Consequently, Friedberg was able to prove the following theorem [13] [15], but we cite Kummer for the elegance of his later proof. Kummer's construction of a Friedberg ordering is an application of the set MIN.

**THEOREM 1.4.10** (Kummer [22]). *The c.e. sets can be enumerated without repetition.*

The noneffective Friedberg ordering  $\psi$  makes  $\text{MIN}_\psi = \omega$ . If we are willing to entertain arbitrary numberings, then MIN can be any set we like (or don't like). The partial computable functions admit analogous pathological numberings for f-MIN, thus threatening to turn our study of minimal indices into a triviality. For this reason, we hereby restrict our attention to Gödel numberings. The remaining results in this thesis do not depend on the particular choice of Gödel numbering. So from this point forth, we simply fix an enumeration of the partial computable functions (with one exception, Chapter 5).



## CHAPTER 2

### Turing characterizations

When squeezed gently, a fair amount of information can be extracted from spectral sets. To show that  $\emptyset^{(n)}$  reduces to a spectral set, one first tries to achieve this (difficult) reduction with the aid of some oracle. By repeatedly substituting with successively weaker oracles, eventually one eliminates the oracle entirely (hopefully). Each time that a weaker oracle is introduced, a new reduction technique is required. This chapter is organized according to technique. Each section describes one or more reduction methods which pertain to oracles of particular strength.

#### 2.1. Generic reductions

Lemma 2.1.1 shows how to “drop” a MIN-set “down one level.” We demonstrate an especially short proof which is peculiar to  $\text{MIN}^m$ , however there is a canonical strategy which works for MIN-sets in general. The canonical strategy is presented in the proofs of (i) and (iv). In each case, we give the reduction in only one direction because the opposite directions are immediate from our arithmetic upper bounds (Proposition 1.3.3). (i) and (ii) first appeared in [39] and [31] for f-MIN and f-MIN\*, respectively.

LEMMA 2.1.1. *For  $n \geq 0$ ,*

- (I)  $\text{MIN} \oplus \emptyset' \equiv_{\text{T}} \emptyset''$ ,
- (II)  $\text{MIN}^* \oplus \emptyset'' \equiv_{\text{T}} \emptyset'''$ ,
- (III)  $\text{MIN}^m \oplus \emptyset'' \equiv_{\text{T}} \emptyset'''$ ,
- (IV)  $\text{MIN}^{\text{T}^{(n)}} \oplus \emptyset^{(n+3)} \equiv_{\text{T}} \emptyset^{(n+4)}$ .

PROOF. (I). Let  $a$  be the minimal index for TOT, and let  $e$  be any index. Note that  $W_e = W_x$  for exactly one  $x$  in

$$B := \{0, \dots, e\} \cap \text{MIN}.$$

Since

$$\{\langle j, e \rangle : W_j \neq W_e\} \in \Sigma_2,$$

we can enumerate all the indices  $y \in B$  such that  $W_y \neq W_e$  using a  $\emptyset'$  oracle. Eventually, we enumerate all of the indices except for one. If the leftover index is  $a$ , then  $W_e = W_a$ , so  $e \in \text{TOT}$ . Otherwise,  $e \notin \text{TOT}$ . Thus, we can decide membership for a  $\Pi_2$ -complete set using only a  $\text{MIN} \oplus \emptyset'$  oracle.  $\square$

(II). Schaefer's proof of  $\text{f-MIN} \oplus \emptyset'' \equiv_{\text{T}} \emptyset'''$  uses the fact that there is an ordering  $\varphi$  such that  $\text{f-MIN}_{\varphi}^* \equiv_{\text{T}} \emptyset'''$  [39]. The argument in (iv) with COF substituted for  $\text{HIGH}^n$  yields an analogous result, without taking into consideration other Gödel numberings.  $\square$

(III). Define a  $\overline{\text{MIN}}^m$ -computable function  $f$  by

$$f(e) := (\mu i) [i \in \text{MIN}^m \quad \& \quad i > e].$$

Then

$$(\forall e) [W_e \not\equiv_m W_{f(e)}].$$

Since  $\overline{\text{MIN}}^m \in \Sigma_3$ , it follows from the  $\equiv_m$ -Completeness Criterion (Theorem 3.3.4(i), [18]) that

$$\text{MIN}^m \oplus \emptyset'' \equiv_{\text{T}} \overline{\text{MIN}}^m \oplus \emptyset'' \equiv_{\text{T}} \emptyset'''. \quad \square$$

(IV). Recall that  $\min^{\text{T}^{(n)}}(e)$  denotes the function which computes the  $\equiv_{\text{T}^{(n)}}$ -minimal index of  $e$ . We claim that

$$\min^{\text{T}^{(n)}} \leq_{\text{T}} \text{MIN}^{\text{T}^{(n)}} \oplus \emptyset^{(n+3)}.$$

Let  $a$  denote the  $\text{T}^{(n)}$ -minimal index for  $\emptyset^{(n+1)}$ . In Theorem 1.3.3(vii), we showed

$$\{\langle j, e \rangle : W_j \equiv_{\text{T}^{(n)}} W_e\} \in \Sigma_{n+4},$$

so we can enumerate the pairs of  $\equiv_{\text{T}^{(n)}}$ -equivalent c.e. sets using a  $\emptyset^{(n+3)}$  oracle.

For any index  $e$ ,  $W_e \equiv_{T^{(n)}} W_x$  for exactly one  $x$  in

$$\{0, \dots, e\} \cap \text{MIN}^{T^{(n)}}.$$

Since a unique  $x$  is guaranteed to exist, we have that  $x = \min^{T^{(n)}}(e)$  can be computed from a  $\text{MIN}^{T^{(n)}} \oplus \emptyset^{(n+3)}$  oracle. This proves the claim.

Now since

$$\text{HIGH}^n = \{e : W_e \equiv_{T^{(n)}} \emptyset'\}$$

is  $\Sigma_{n+4}$ -complete (see Theorem 1.3.4(vi)), it suffices to determine, using a  $\text{MIN}^{T^{(n)}} \oplus \emptyset^{(n+3)}$  oracle, whether a given index  $e$  is in  $\text{HIGH}^n$ . To do this, just compute  $\min^{T^{(n)}}(e)$ , and check whether it is equal to  $a$ . □

□

Note that Lemma 2.1.1(iii) gives us another way of showing that  $\text{MIN}^{\text{im}} \in \Pi_3 - \Sigma_3$ . We obtain similar arithmetic results from the other parts of Lemma 2.1.1.

## 2.2. (Old)-timers

Prior to this work, the only technique which was successful in reducing a MIN-set by a second “level” was to use MIN queries to build a “timer” for the convergence of some function, thereby turning an enumerable object into something computable. Unlike the technique of Lemma 2.1.1, however, the “timer” method appears to be peculiar to the equivalence relation under consideration. We demonstrate this method in Lemma 2.2.2.

The following theorem isolates the main idea of Lemma 2.2.2. Reading this proof may help to remember the proof of  $\text{MIN} \geq_{\text{bT}} \emptyset'$ . We defined  $\text{MIN}^{\text{time}}$  in Definition 1.3.2.

**THEOREM 2.2.1.**  $\text{MIN}^{\text{time}} \geq_{\text{bT}} \emptyset'$ .

**PROOF.** It suffices to determine whether  $W_e = \emptyset$  using a  $\text{MIN}^{\text{time}}$  oracle. Let  $a$  be the minimal index of the function which diverges everywhere. For  $j \in \text{MIN}^{\text{time}} - \{a\}$ , define a function  $s$  by

$$s(j) := (\mu\langle t, x \rangle) [\varphi_{j,t}(x) \downarrow],$$

and let

$$S(i) := \max_{\substack{j \leq i \\ (j \in \text{MIN}^{\text{time}} - \{a\})}} s(j).$$

$S(e)$  is computable from  $\text{MIN}^{\text{time}}$ , as  $s$  converges everywhere on its domain. Now either  $e \in \text{MIN}^{\text{time}}$ , or else  $\varphi_e$  duplicates the computation of some  $\varphi_j$ ,  $j < e$ . That is, either  $\varphi_{e,t}(x)$  converges for some  $\langle t, x \rangle \leq S(e)$ , or else  $W_e = \emptyset$ .  $\square$

The reverse inequality for Theorem 2.2.1 is immediate, as  $\text{MIN}^{\text{time}} \in \Sigma_1$  and hence  $\text{MIN}^{\text{time}} \leq_m \emptyset'$ .

Schaefer proved Lemma 2.2.2 for f-MIN and f-MIN\*, but a similar proof works for both sets and functions.

LEMMA 2.2.2 (Schaefer [39]).

- (I)  $\text{MIN} \geq_{\text{bT}} \emptyset'$ ,
- (II)  $\text{MIN}^* \oplus \emptyset' \geq_{\text{T}} \emptyset''$ .

PROOF. (I). Let  $e$  be an index. We show how to decide whether  $\varphi_e(e) \downarrow$  with a MIN oracle. Using the  $s$ - $m$ - $n$  Theorem, define a computable function  $f$  by

$$(2.1) \quad \varphi_{f(i)}(x) := \begin{cases} 1 & \text{if } \varphi_{i,x}(i) \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Now  $e \in K$  iff  $W_{f(e)} \neq \emptyset$ . Since  $\varphi_{f(i)}(x)$  effectively counts the steps in computation  $\varphi_i(x)$ , we can now proceed as in Lemma 2.2.1.

Let  $a$  be the minimal index of the function which diverges everywhere. Define a function  $s : \text{MIN} - \{a\} \rightarrow \omega$  by

$$s(j) := (\mu x) [\varphi_j(x) \downarrow],$$

and let

$$S(i) := \max_{\substack{j \leq i \\ (j \in \text{MIN} - \{a\})}} s(j).$$

Since  $\varphi_{f(e)}$  agrees with some index in  $\text{MIN} \cap \{0, \dots, f(e)\}$ , it must be the case that

$$\begin{aligned} W_{f(e)} \neq \emptyset &\iff W_{f(e)} \cap \{0, \dots, S[f(e)]\} \neq \emptyset \\ &\iff \varphi_{e, S[f(e)]}(e) \downarrow. \end{aligned}$$

Since  $S$  is computable in  $\text{MIN}$ , we can decide  $W_{f(e)} \neq \emptyset$ .  $\square$

(II). Recall that  $\text{TOT} \equiv_{\text{T}} \emptyset''$ . Since  $\overline{\text{TOT}}$  is c.e. in  $\emptyset'$ , it suffices to enumerate  $\text{TOT}$  using a  $\text{MIN}^* \oplus \emptyset'$  oracle. Define computable functions  $f$  and  $g$  by

$$\begin{aligned} \varphi_{f(i)}(x) &:= \begin{cases} \langle x, (\mu s) (\forall y \leq x) [\varphi_{i,s}(y) \downarrow] \rangle & \text{if such an } s \text{ exists,} \\ \uparrow & \text{otherwise.} \end{cases} \\ \varphi_{g(i)}(x) &:= \begin{cases} \pi_2[\varphi_i(y)] & \text{if } (\exists y) [y \geq x \ \& \ \varphi_i(y) \downarrow] \\ \uparrow & \text{otherwise.} \end{cases} \end{aligned}$$

Let  $a$  be the  $=^*$ -minimal index for the function which diverges everywhere. Define

(2.2)

$$A := \left\{ e : (\exists j, N) \left[ j \in [\text{MIN}^* - \{a\}] \cap \{0, \dots, f(e)\} \ \& \ (\forall x) [\varphi_{e, \max\{N, \varphi_{g(j)}(x)\}}(x) \downarrow] \right] \right\}.$$

We claim:

- (1)  $A$  is enumerable with a  $\text{MIN}^* \oplus \emptyset'$  oracle, and
- (2)  $A = \text{TOT}$ .

Note that  $W_j$  is infinite when  $j \in \text{MIN}^* - \{a\}$ , which makes  $\varphi_{g(j)}$  a total function. The bracketed clause in (2.2) is therefore computable in  $\text{MIN}^* \oplus \emptyset'$ , which proves (1).

If  $e \in A$  then the universal clause in (2.2) is satisfied, so  $e \in \text{TOT}$ . Conversely, assume  $e \in \text{TOT}$ . Then  $f(e) \in \text{INF}$ , so  $f(e)$ 's  $=^*$ -minimal index is not  $a$ . Let  $j$  be the  $=^*$ -minimal index for  $f(e)$ , choose  $n$  large enough so that

$$(\forall x > n) [W_j(x) = W_{f(e)}(x)],$$

and choose  $N$  large enough so that

$$(\forall x \leq n) [\varphi_{e,N}(x) \downarrow].$$

Then for all  $x$ ,

$$\max\{N, \varphi_{g(j)}(x)\} \geq \pi_2[\varphi_{f(e)}(x)],$$

because  $\pi_2[\varphi_{f(e)}]$  is a nondecreasing function. Hence

$$(\forall x) [\varphi_{e, \max\{N, \varphi_{g(j)}(x)\}}(x) \downarrow],$$

so our selected pair  $\langle j, N \rangle$  exhibits that  $e \in A$ . □

□

### 2.3. The Forcing Lowness Lemma

We show how to “drop”  $\text{MIN}^{\text{T}^{(n)}}$  by a second “level.” Lemma 2.3.2 is easiest to digest when we recall that  $\text{LOW}^0$  is the set of indices with computable domains. The lemma gives slightly more than we need to prove the main theorem of this section, which is Theorem 2.3.7. The argument in Theorem 2.3.7 only depends on knowing the index  $a_{\langle k, n \rangle}(0)$ , however the entire countable sequence  $a_{\langle k, n \rangle}(0), a_{\langle k, n \rangle}(1), \dots$ , as well as uniformity in  $n$ , will be required for Theorem 5.1.2.

We state a simple version of [37, Theorem 6.3] by Sacks for use in the next lemma. Sacks does not explicitly mention uniformity in his original proof, however Soare does [44, Theorem VIII.3.1].

**THEOREM 2.3.1** (Sacks Jump Theorem [36]). *Let  $B$  be any set, and let  $S$  be c.e. in  $B'$  with  $B' \leq_{\text{T}} S$ . Then there exists a  $B$ -c.e. set  $A$  with  $A' \equiv_{\text{T}} S$ . Furthermore, an index for  $A$  can be found uniformly from an index for  $S$ .*

**LEMMA 2.3.2** (forcing lowness). *There exists a ternary computable function  $a_{\langle k, n \rangle}(i)$  such that for every index  $k$  and any number  $i$ ,  $W_{a_{\langle k, n \rangle}(i)} \leq_{\text{T}^{(n)}} W_k$ . In particular, and furthermore:*

- (I)  $k \in \text{LOW}^n \implies (\forall i) [a_{\langle k, n \rangle}(i) \in \text{LOW}^n]$ ,
- (II)  $k \notin \text{LOW}^n \implies (\forall i \neq j) [W_{a_{\langle k, n \rangle}(i)} \upharpoonright_{\text{T}^{(n)}} W_{a_{\langle k, n \rangle}(j)}]$ .

*In either case,  $a_{\langle k, n \rangle}(i) \in \text{LOW}^{n+1}$  for all  $k, n$ , and  $i$ .*

PROOF. This lemma is secretly [44, Exercises VII.2.7 and VII.2.3], in mild disguise. Indeed, we shall combine finite injury ([12], [32]) with standard permitting ([6], [46]) by playing the Friedberg-Muchnik strategy ([32], [12]) under  $(W_k)^{(n)}$ . Our construction follows [42].

Given inputs  $n$  and  $k$ , we show how to effectively find  $\emptyset^{(n)}$ -c.e. sets  $A_0, A_1, \dots$  so that  $A_0 = (W_{a_{(k,n)}(0)})^{(n)}$ ,  $A_1 = (W_{a_{(k,n)}(1)})^{(n)}$ ,  $\dots$  etc. satisfy the conclusions of the theorem. If  $n$  is nonzero, then we can subsequently (and uniformly) find appropriate indices for c.e. sets by iteratively applying the Sacks Jump Theorem (Theorem 2.3.1). For clarity purposes, we adopt the following abbreviations:

$$B_i := \bigoplus_{j \neq i} A_j,$$

$$(B_i)_s := \bigoplus_{j \neq i} (A_j)_s,$$

where  $(A_j)_0 \subseteq (A_j)_1 \subseteq \dots$  is a  $\emptyset^{(n)}$ -enumeration for  $A_j$ .

If  $k \in \text{LOW}^n$ , our construction will satisfy for all  $i$ ,

$$Q_i : A_i \equiv_{\Gamma^{(n)}} \emptyset,$$

and if  $k \notin \text{LOW}^n$ , our construction will meet the requirements, for all  $i$  and  $e$ :

$$N_i : A_i \leq_{\Gamma^{(n)}} W_k,$$

$$R_{\langle e, i \rangle} : A_i \neq \Psi_e^{B_i}.$$

In the following construction, we imagine  $Y$  to be the set  $\emptyset^{(n)}$ . We write  $Y$  in place of  $\emptyset^{(n)}$  simply to emphasize that our algorithm is independent of the choice of oracle. Furthermore, our construction will be uniform in  $k$ . Let

$$C_k := (W_k)^{(n)} \oplus \omega.$$

Now  $C_k$  is c.e. in  $\emptyset^{(n)}$ , and an index for  $C_k$  (with  $\emptyset^{(n)}$  oracle) can be found uniformly from  $k$ . The “ $\omega$ ” is added into the definition of  $C_k$  just to ensure that the set is infinite. Since our construction will no longer refer to the value  $k$ , we abbreviate with  $C := C_k$ . Using the  $\emptyset^{(n)}$ -index for  $C$ , we can effectively find a 1:1 function  $c \leq_{\Gamma} \emptyset^{(n)}$  such that  $c(0), c(1), c(2), \dots$  is an enumeration of  $C$ .

*Construction.*

*Stage  $s = 0$ .* Define  $r(\langle e, i \rangle, 0) = -1$  for all  $\langle e, i \rangle$ . Set  $(A_i)_0 = \emptyset \oplus Y$  for all  $i$ .

*Stage  $s + 1$  ( $s + 1$  is an  $i^{\text{th}}$  prime power).* Choose the least  $e$  such that

$$(2.3) \quad r(\langle e, i \rangle, s) = -1 \quad \& \quad (\exists \text{ even } x) \left[ x \in \omega^{[\langle e, i \rangle]} - (A_i)_s \quad \& \quad \Psi_{e,s}^{(B_i)}(x) \downarrow = 0 \right. \\ \left. \& \quad (\forall \langle z, j \rangle < \langle e, i \rangle) [r(\langle z, j \rangle, s) < x] \quad \& \quad c(s) \leq x \right].$$

If there is no such  $e$ , then do nothing and go to stage  $s + 2$ . If  $e$  exists, then we say  $R_{\langle e, i \rangle}$  acts at stage  $s + 1$ , Perform the following steps.

*Step 1.* Enumerate  $x$  in  $A_i$ .

*Step 2.* Define  $r(\langle e, i \rangle, s + 1) = s + 1$ .

*Step 3.* For all  $\langle z, j \rangle > \langle e, i \rangle$ , define  $r(\langle z, j \rangle, s + 1) = -1$ .

*Step 4.* For all  $\langle z, j \rangle < \langle e, i \rangle$ , define  $r(\langle z, j \rangle, s + 1) = r(\langle z, j \rangle, s)$ .

When  $r(\langle z, j \rangle, s + 1)$  is reset to  $-1$ , we say that requirement  $R_{\langle z, j \rangle}$  is *injured*.

*Stage  $s + 1$  ( $s + 1$  is not a prime power).* Do nothing. Get some coffee.

CLAIM 2.3.3. For all  $i$ ,  $A_i \leq_T C$ .

PROOF. To decide whether  $x \in A_i$ , wait for a stage  $s$  such that all the elements of  $C$  below  $x + 1$  have been enumerated into  $C$ , i.e.,

$$C \upharpoonright x \subseteq \{c(0), c(1), \dots, c(s)\}.$$

Such a stage  $s$  is guaranteed to exist, and the oracle  $C$  lets us identify when this occurs. The final clause of (2.3), “ $c(s) \leq x$ ,” ensures that no element  $\leq x$  get enumerated into  $A_i$  after stage  $s$ . Hence

$$x \in A_i \iff x \in (A_i)_{s+1}. \quad \square$$

If  $C \leq_T \emptyset^{(n)}$ , then by Claim 2.3.3,  $A_i$  is  $\emptyset^{(n)}$ -computable for every  $i$ . This proves case (i). It remains to consider case (ii).

CLAIM 2.3.4. If requirement  $R_{\langle e, i \rangle}$  acts at some stage  $s + 1$  and is never later injured, then requirement  $R_{\langle e, i \rangle}$  is met and  $r(\langle e, i \rangle, t) = s + 1$  for all  $t \geq s + 1$ .



PROOF. Suppose  $R_{\langle e, i \rangle}$  acts at stage  $s + 1$  and say  $e$  is an  $i^{\text{th}}$  prime power. Then

$$\Psi_e^{(B_i)_s}(x) \downarrow = 0$$

for some  $x \in (A_i)_{s+1}$ . Since no  $R_{\langle z, j \rangle}, \langle z, j \rangle < \langle e, i \rangle$  ever acts after stage  $s + 1$ , it follows by induction on  $t > s$  that  $R_{\langle e, i \rangle}$  never acts again and  $r(\langle e, i \rangle, t) = s + 1$  for all  $t > s$ . Hence no  $R_{\langle z, j \rangle}, \langle z, j \rangle > \langle e, i \rangle$ , enumerates any  $x \leq s$  into any  $A_j$  ( $j \neq i$ ) after stage  $s + 1$ . Therefore,

$$B_i \upharpoonright s = (B_i)_s \upharpoonright s$$

and

$$\Psi_e^{B_i}(x) \downarrow = 0 \neq A_i(x). \quad \square$$

CLAIM 2.3.5. *Assume  $C \succ_{\text{T}} \emptyset^{(n)}$ . Then for every  $\langle e, i \rangle$ , requirement  $R_{\langle e, i \rangle}$  is met, acts at most finitely often, and  $r(\langle e, i \rangle) := \lim_s r(\langle e, i \rangle, s)$  exists.*

PROOF. Fix  $\langle e, i \rangle$  and assume the statement holds for all  $R_{\langle z, j \rangle}, \langle z, j \rangle < \langle e, i \rangle$ . Let  $v$  be the greatest stage when some such  $R_{\langle z, j \rangle}$  acts, if ever, and  $v = 0$  if none exists. Then  $r(\langle e, i \rangle, v) = -1$ , and this persists until some stage  $s + 1 > v$  (if ever) when  $R_{\langle e, i \rangle}$  acts. If  $R_{\langle e, i \rangle}$  acts at some stage  $s + 1$ , then  $R_{\langle e, i \rangle}$  becomes satisfied and never acts again. It then follows from Claim 2.3.4 that  $r(\langle e, i \rangle, t) = s + 1$  for all  $t \geq s + 1$ .

Either way,  $r(\langle e, i \rangle)$  exists and  $R_{\langle e, i \rangle}$  acts at most finitely often. Now suppose that  $R_{\langle e, i \rangle}$  is not met. Then

$$A_i = \Psi_e^{B_i}.$$

By stage  $v$ , at most finitely many elements  $x \in \omega^{[\langle e, i \rangle]}$  have been enumerated in  $A_i$ . No further elements are enumerated from  $\omega^{[\langle e, i \rangle]}$  because only requirement  $R_{\langle e, i \rangle}$  can enumerate in this row. Let  $x \in \omega^{[\langle e, i \rangle]} - (A_i)_v$  be such that  $x > v$ . Eventually there will be a stage  $s$  such that

$$\Psi_{e,s}^{(B_i)_s}(x) \downarrow = 0,$$

because  $x \notin A_i$ . Since  $x$  never becomes a witness that  $R_{\langle e, i \rangle}$  is satisfied, it must be the permitting clause “ $c(s) \leq x$ ” in (2.3) which prevents this from happening. Therefore

$$C \upharpoonright x = \{c(0), \dots, c(s)\} \upharpoonright x.$$

Since  $x$  was chosen arbitrarily, we now have an algorithm to compute any finite initial segment of  $C$ . Our algorithm used only a  $\emptyset^{(n)}$  oracle to compute the function  $c$ . Therefore  $C \leq_T \emptyset^{(n)}$ , contrary to assumption. So requirement  $R_{\langle e, i \rangle}$  must be met.  $\square$

Case (ii) is now satisfied because the requirements  $R_{\langle e, i \rangle}$  are met. Finally,

CLAIM 2.3.6 (Soare [43]). *For every  $k, n$ , and  $i$ , we have  $a_{\langle k, n \rangle}(i) \in \text{LOW}^{n+1}$ .*

PROOF. We may assume  $C >_T \emptyset^{(n)}$  because otherwise the result follows immediately from Claim 2.3.3. Using the relativized  $s$ - $m$ - $n$  theorem, define a computable function  $f$  such that for all  $Y \subseteq \omega$ ,

$$\Psi_{f(e)}^Y(x) := \begin{cases} 0 & \text{if } \Psi_e^Y(e) \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

$\Psi_{f(e)}^Y$  is either the constant zero function or diverges everywhere, depending on  $Y$ . Define a computable “witness” function  $w$  by

$$w(\langle e, i \rangle, s) := \begin{cases} \text{most recent member of } A_i \cap \omega^{[\langle e, i \rangle]} \text{ after stage } s, \text{ or} \\ \langle 0, \langle e, i \rangle \rangle \text{ if none exists.} \end{cases}$$

Since each requirement acts only finitely often (Claim 2.3.5), the limit

$$\hat{w}(e, i) := \lim_s w(\langle e, i \rangle, s)$$

exists and witnesses  $\Psi_e^{B_i}[\hat{w}(e, i)] \neq A_i[\hat{w}(e, i)]$ . Finally, define a sequence of functions  $g_i \leq_{T(n)} \emptyset$  by

$$g_i(e, s) := \begin{cases} 1 & \text{if } \Psi_{f(e), s}^{(B_i)_s}(w[\langle f(e), i \rangle, s]) \downarrow = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We show that

$$(2.4) \quad \hat{g}_i(e) := \lim_s g_i(e, s)$$

is the characteristic function for  $(B_i)'$ , which implies that  $(B_i)' \leq_{T(n)} \emptyset'$  by the Limit Lemma.

Let  $t$  be a large enough stage so that  $R_{\langle f(e), i \rangle}$  never gets injured after stage  $t$ , and large enough so that  $w(\langle f(e), i \rangle, \cdot)$  has settled, i.e.

$$(\forall s > t) (w[\langle f(e), i \rangle, s] = w[\langle f(e), i \rangle, t] = \hat{w}[\langle f(e), i \rangle]).$$

For clarity, let  $\tilde{w}$  denote the value  $\hat{w}[f(e), i]$ , and let  $v_s$  denote the function

$$v_s(x) := \Psi_{f(e), t}^{(B_i)_s}(x).$$

Now for all  $s > t$ ,  $g_i(e, s) = g_i(e, t)$ , so the limit in (2.4) exists. Indeed, if  $v_t(\tilde{w}) \downarrow = 0$ , and at some later stage  $s$ ,  $\neg[v_s(\tilde{w}) \downarrow = 0]$ , this would force our construction to find a new witness for  $R_{\langle e, i \rangle}$ , contradicting the fact that  $\tilde{w}$  is the final witness. If, on the other hand,  $\neg[v_t(\tilde{w}) \downarrow = 0]$ , then this computation on  $\tilde{w}$  must be preserved forever, lest  $R_{\langle e, i \rangle}$  acts again.

Since  $\hat{g}_i(e) = g_i(e, t)$ , it follows that

$$\begin{aligned} \hat{g}_i(e) = 1 &\iff \Psi_{f(e), t}^{(B_i)_t}(\hat{w}[f(e), i]) \downarrow = 0 \\ &\iff \Psi_{f(e)}^{B_i}(\hat{w}[f(e), i]) \downarrow = 0 \\ &\iff \Psi_e^{B_i}(e) \downarrow. \end{aligned}$$

Therefore  $\hat{g}_i$  is the characteristic function for  $(B_i)'$ . This proves  $a_{\langle k, n \rangle}(j) \in \text{LOW}^{n+1}$  for all  $j \neq i$ , as  $a_{\langle k, n \rangle}(j)$  is the  $\emptyset^{(n)}$ -index for  $A_j \leq_T B_i$ . Since  $i$  was chosen arbitrarily, we conclude that, in fact,  $a_{\langle k, n \rangle}(i) \in \text{LOW}^{n+1}$  for all  $i \in \omega$ .  $\square$

$\square$

Our first application of Lemma 2.3.2 is the following theorem:

**THEOREM 2.3.7.**  $\text{MIN}^{\text{T}^{(n)}} \oplus \emptyset^{(n+2)} \geq_T \emptyset^{(n+3)}$ .

**PROOF.** Since  $\text{LOW}^n$  is  $\Sigma_{n+3}$ -complete, it suffices to determine membership in  $\text{LOW}^n$  using a  $\text{MIN}^{\text{T}^{(n)}} \oplus \emptyset''$  oracle. On input  $k$ , first compute  $a_{\langle k, n \rangle}(0)$ , where  $a_{\langle k, n \rangle}$  is the computable function defined in Lemma 2.3.2, and let  $c$  be the least index such that

$$W_c \equiv_{\text{T}^{(n)}} \emptyset,$$

(i.e.,  $c \in \text{LOW}^n$ ). We would like to know whether  $\min^{\text{T}^{(n)}}(k) = c$ .

Let

$$e := a_{\langle k, n \rangle}(0),$$

and

$$S_e := \{0, \dots, e\} \cap \text{MIN}^{\text{T}^{(n)}}.$$

There exists a unique  $x \in S_e$  satisfying  $W_x \equiv_{\mathsf{T}^{(n)}} W_e$ , however unlike in Theorem 2.1.1(iv), we can not discover which one it is by direct enumeration because we are now missing the  $\emptyset^{(n+3)}$  oracle. So we use “double enumeration” instead. Since  $e \in \text{LOW}^{n+1}$ , the set

$$Y_e := S_e \cap \{y : W_y \leq_{\mathsf{T}^{(n)}} W_e\}$$

is c.e. in  $\text{MIN}^{\mathsf{T}^{(n)}} \oplus \emptyset^{(n+2)}$  (Proposition 1.3.3(vii)). Let  $Y_{e,t}$  denote the elements which have been added into  $Y_e$  after  $t$  steps of this enumeration. We remark that  $Y_{e,t} \leq_{\mathsf{T}} \text{MIN}^{\mathsf{T}^{(n)}} \oplus \emptyset^{(n+2)}$ .

CLAIM 2.3.8. *Define a function  $Z$  from  $\text{range}[a_{\langle \cdot, n \rangle}(0)]$  to finite sets by*

$$Z(e) := Y_e \cap \{y : W_e \leq_{\mathsf{T}^{(n)}} W_y\}.$$

Then

- (I)  $Z \leq_{\mathsf{T}} \text{MIN}^{\mathsf{T}^{(n)}} \oplus \emptyset^{(n+2)}$ , and
- (II)  $Z(e) = \{\min^{\mathsf{T}^{(n)}}(e)\}$ .

PROOF. (ii) is immediate because  $z \in Z(e)$  implies  $W_z \equiv_{\mathsf{T}^{(n)}} W_e$ , and  $\min^{\mathsf{T}^{(n)}}(e)$  is the unique member of  $S_e$  with this property. It remains to compute  $Z(e)$  with a  $\text{MIN}^{\mathsf{T}^{(n)}} \oplus \emptyset^{(n+2)}$  oracle. Note that when  $y \in Y_{e,t}$ , the relation

$$(2.5) \quad (\exists i \leq t) (\forall x) \left[ \Psi_i^{(W_y)^{(n)}}(x) \downarrow \quad \& \quad \left( x \in (W_e)^{(n)} \iff \Psi_i^{(W_y)^{(n)}}(x) = 1 \right) \right]$$

is in  $\Pi_1^{\emptyset^{(n+1)}} = \Pi_{n+2}$  because  $y \in \text{LOW}^{n+1}$ . Therefore knowing *a priori* that we are considering only members of  $Y_{e,t}$ , we can decide membership in (2.5) using the  $\emptyset^{(n+2)}$  oracle.

The algorithm for  $Z$  is as follows. Assume that we have not yet converged by stage  $t$ . For each  $y \in Y_{e,t}$ , we check using  $\emptyset^{(n+2)}$  whether  $y$  satisfies (2.5). If we find a  $y \in Y_{e,t}$  satisfying (2.5), then we know  $W_e \leq_{\mathsf{T}^{(n)}} W_y$ , hence  $Z(e) = \{y\}$ , so the algorithm terminates. Otherwise we proceed similarly in stage  $t+1$ . Eventually we will discover a  $y \in Y_e$  satisfying (2.5), namely  $y = \min^{\mathsf{T}^{(n)}}(e)$ .

We have glossed over one important detail of our algorithm, namely whether or not we can check for membership in (2.5) *uniformly in  $e$* . In fact, we can. In order to make the algorithm uniform in  $e$ , we not only need to know that  $(W_y)' \leq_{\mathsf{T}^{(n)}} \emptyset'$ , but we also

need to know explicitly what the reduction is so that we can make the correct queries to  $\emptyset''$  (regarding (2.5)).

Here are the missing details. When we enumerate  $y$  into  $Y_e$ , we automatically obtain a witness for  $W_y \leq_{T^{(n)}} W_e$ , namely the index of this reduction. Using this witness, we can effectively find a second index witnessing  $(W_y)' \leq_{T^{(n)}} (W_e)'$ . Finally,  $e$  is a special set of the form  $a_{\langle \cdot, n \rangle}(0)$ , and so Claim 2.3.6 gives a recipe for deciding membership in  $(W_e)^{(n+1)}$  given  $\emptyset^{(n+1)}$ .  $\square$

By Lemma 2.3.2,

$$\begin{aligned} Z(e) = \{c\} &\iff \min^{T^{(n)}}(e) = c \\ &\iff a_{\langle k, n \rangle}(0) = e \in \text{LOW}_n \\ &\iff k \in \text{LOW}^n. \end{aligned}$$

Thus, membership in  $\text{LOW}^n$  is decidable in  $\emptyset^{(n+2)} \oplus \text{MIN}^{T^{(n)}}$ .  $\square$

## 2.4. Conclusion

We summarize the main results of this chapter.

- COROLLARY 2.4.1. (I)  $\text{fR} \equiv_{\text{bT}} \emptyset'$ .  
 (II)  $\text{MIN} \equiv_T \emptyset''$ ,  
 (III)  $\text{MIN}^* \oplus \emptyset' \equiv_T \emptyset'''$ .  
 (IV)  $\text{MIN}^{\text{m}} \oplus \emptyset'' \equiv_T \emptyset'''$ .  
 (V)  $\text{MIN}^{T^{(n)}} \oplus \emptyset^{(n+2)} \equiv_T \emptyset^{(n+4)}$ .

PROOF. (I). Use the proof from Lemma 2.2.2(i), but in (2.1) make  $f$  check for convergence on 0 rather than  $i$ .  $\square$

(II), (III). Combine Lemma 2.1.1 with Lemma 2.2.2.  $\square$

(IV). Lemma 2.1.1.  $\square$

(V). Combine Lemma 2.1.1 with Theorem 2.3.7.  $\square$

$\square$

It would be interesting to know whether or not the  $\emptyset'$ ,  $\emptyset''$ , or  $\emptyset^{(n+2)}$  oracle is necessary in any of the above reductions. Theorem 5.2.3 shows, in a formal sense, that a positive answer to this question will be difficult to prove.

## CHAPTER 3

### Immunity and fixed points

We discuss “thinness” of spectral sets. Spectral sets are naturally sparse, as weaker relations give rise to thinner spectral sets (for example,  $\text{MIN}^T \subsetneq \text{MIN}$ ). The notion of “thinness” is formally captured by immunity. Based on the examples of  $\text{MIN}$ ,  $\text{MIN}^*$ , and  $\text{MIN}^T$ , one might be tempted to extrapolate that  $\text{MIN}$ -sets which are higher in the arithmetic hierarchy are also more immune. In general, however, arithmetic level turns out to be a crude and inaccurate indicator of thinness. It is even possible to find a pair of  $\text{MIN}$ -sets where the arithmetic level is higher in one set and immunity is greater in the other. For example,  $\text{MIN}^m \in \Pi_3$  and  $\text{MIN}^{\text{Thick-}\equiv^*} \in \Sigma_4 - \Pi_4$  (see Section 4.2), but the first set is  $\Sigma_3$ -immune while the latter is only  $\Sigma_2$ -immune.

The theorems in this chapter provide an alternative method to Chapter 1.3 for showing that  $\text{MIN}$ -sets are noncomputable. We illustrate a connection between these methods and generalizations of the Arslanov completeness criterion.

**DEFINITION 3.0.2.** Let  $\mathcal{C}$  be a family of sets. A set is  $\mathcal{C}$ -immune if it is infinite and contains no infinite members of  $\mathcal{C}$ . If  $\mathcal{C}$  is the class of c.e. sets, then we write *immune* in place of  $\mathcal{C}$ -immune.

For example, the  $\Pi_1$  set of Kolmogorov random strings,  $\text{RAND}$  (Definition 1.4.3), is immune [26, Corollary 2.7.1]. In fact,  $\overline{\text{RAND}}$  is a natural example of a *simple* set, being infinite, c.e., and having a complement which is immune. Simple sets were first invented by Emil Post in attempt to exhibit a c.e. set  $A$  satisfying  $\emptyset < A < \emptyset'$  [35]. Post’s program ultimately failed, however Post’s problem (and consequently his notion of immunity) shaped the focus of computability theory in the 1940’s and 1950’s [44].

### 3.1. Lower spectral sets

Marcus Schaefer [39] made the following observations with regards to minimal functions, but the results translate easily into sets. He attributes the main idea of (ii) to Blum [4, Theorem 3] and (iii) to John Case:

- THEOREM 3.1.1 (Schaefer [39]). (I)  $\text{fR}$  is immune.  
 (II)  $\text{MIN}$  is immune.  
 (III)  $\text{MIN}^*$  is  $\Sigma_2$ -immune.

The ideas from Theorem 3.1.1 will come in handy when we prove the  $\Pi_3$ -Separation Theorem (Theorem 3.1.3).

First, we consider the problematic relation  $\equiv_1$ . One might be tempted to modify this relation by identifying finite sets, but we decline to do this here. Consequently,  $\text{MIN}^{\equiv_1}$  contains a representative of each finite size. The reason for doing this is not only that this finite property makes the  $\Pi_3$ -Separation Theorem sparkle, but also because “finiteness” is essentially an unavoidable aspect of 1:1 equivalence on computable sets:

PROPOSITION 3.1.2 (Dekker and Myhill [7]). *Let  $A$  be an immune set of nonnegative integers. Then*

$$A <_1 A \cup \{-1\} <_1 A \cup \{-1, -2\} <_1 A \cup \{-1, -2, -3\} <_1 \dots$$

PROOF. Let

$$\begin{aligned} A_0 &:= A, \\ A_1 &:= A \cup \{-1\}, \\ A_2 &:= A \cup \{-1, -2\}, \\ &\vdots \end{aligned}$$

Clearly,  $A_n \leq_1 A_{n+1}$  via the identity function. Suppose towards a contradiction that  $A_{n+1} \leq_1 A_n$  for some  $n$ , and let  $f$  be the computable function that witnesses this relation.



Let  $x \in A_{n+1} - A_n$ . Then the sequence

$$\begin{aligned} & f(x) \\ & f \circ f(x) \\ & f \circ f \circ f(x) \\ & \vdots \end{aligned}$$

has no repetitions, and is therefore an infinite c.e. subset of  $A_n$ . Indeed,  $x$  is not in the range of  $f$ , so a repetition of values would indicate that  $f$  is not injective. This means that  $A_n$  is not immune, a contradiction.  $\square$

The following theorem shows that immunity can be used to distinguish between certain MIN-sets, even when the arithmetic hierarchy can not.

**THEOREM 3.1.3** ( $\Pi_3$ -Separation).  $\text{MIN}^m$ ,  $\text{MIN}^*$ , and  $\text{MIN}^{\equiv 1}$  are all in  $\Pi_3 - \Sigma_3$ , but

- (I)  $\text{MIN}^m$  is  $\Sigma_3$ -immune, whereas
- (II)  $\text{MIN}^*$  contains an infinite  $\Sigma_3$  set, and
- (III)  $\text{MIN}^{\equiv 1}$  contains an infinite  $\Sigma_2$  set.

**PROOF.** We already showed  $\text{MIN}^m, \text{MIN}^*, \text{MIN}^{\equiv 1} \in \Pi_3 - \Sigma_3$  in Theorem 1.3.4.

(I).  $\text{MIN}^m$  is infinite because it's noncomputable (Theorem 1.3.4(iv)). Let  $A$  be an infinite,  $\Sigma_3$  set, and suppose  $A \subseteq \text{MIN}^m$ . Since  $A$  is infinite and c.e. in  $\emptyset''$ , we can define a  $\emptyset''$ -computable function  $g$  by

$$g(e) = \pi_1 \left( (\mu(i, t)) [i > e \ \& \ i \in A_t] \right),$$

where  $\{A_t\}$  is a  $\emptyset''$ -enumeration of  $A$ .

Now for all  $e$ ,  $g(e) > e$  and  $g(e) \in \text{MIN}^m$ . Therefore

$$(\forall e) [W_e \not\equiv_m W_{g(e)}],$$

contradicting a theorem of Jockusch et al. (Theorem 3.2.1): for every  $f \leq_T \emptyset''$ ,

$$(\exists e) [W_e \equiv_m W_{f(e)}]. \quad \square$$

(II). For every  $k$ , let

$$\begin{aligned} P_k &:= \{n : n \text{ is a } k^{\text{th}} \text{ prime power}\}, \\ A_k &:= \{e : W_e \subseteq^* P_k\} \cap \text{INF}, \\ A &:= \{e : (\exists k) (\forall j < e) [e \in A_k \ \& \ j \notin A_k]\}. \end{aligned}$$

Now  $A \subseteq \text{MIN}^*$ , as  $e \in A$  implies  $W_j \not\equiv^* W_e$  for all  $j < e$ . Since the  $A_k$ 's are disjoint, any infinite  $B$  satisfies  $B \subseteq^* A_k$  for at most one  $k$ . Moreover, each  $A_k$  contributes a distinct element to  $A$ , hence  $A$  is infinite. Finally,

$$\begin{aligned} W_e \subseteq^* P_k &\iff (\exists N) (\forall x \geq N) [x \in W_e \implies x \in P_k] \\ &\iff (\exists N) (\forall x \geq N) [x \notin W_e \vee x \in P_k] \\ &\iff (\exists N) (\forall x \geq N) (\forall t) [x \notin W_{e,t} \vee x \in P_k], \end{aligned}$$

which makes  $A_k \in \Delta_3$ , on account of  $\text{INF} \in \Pi_2$ . It follows that  $A \in \Sigma_3$ .  $\square$

(III). Define a sequence of finite sets by

$$A_k := \{x : 0 \leq x \leq k\}.$$

Furthermore, define

$$B_k := \{e : W_e \text{ has at least } k \text{ elements}\} \in \Sigma_1,$$

which means that

$$C_k := \{e : W_e \text{ has exactly } k \text{ elements}\} = B_k \cap \overline{B_{k+1}} \in \Delta_2.$$

It follows from the Pigeonhole Principle that

$$W_e \equiv_1 A_k \iff e \in C_k,$$

and therefore

$$\{\langle e, k \rangle : W_e \equiv_1 A_k\} \in \Delta_2.$$

Now

$$A := \{e : (\exists k) (\forall j < e) [W_j \not\equiv_1 A_k \ \& \ W_e \equiv_1 A_k]\}$$

is a  $\Sigma_2$  set. Moreover,  $A$  is infinite because each  $A_k$  represents a distinct  $\equiv_1$  class. Since  $A \subseteq \text{MIN}^{\equiv_1}$ , it follows that  $\text{MIN}^{\equiv_1}$  is not  $\Sigma_2$ -immune.  $\square$

□

REMARK. It is worth noting that  $\text{MIN}^{\equiv 1}$  is immune (simply because it is a subset of  $\text{MIN}$ ).

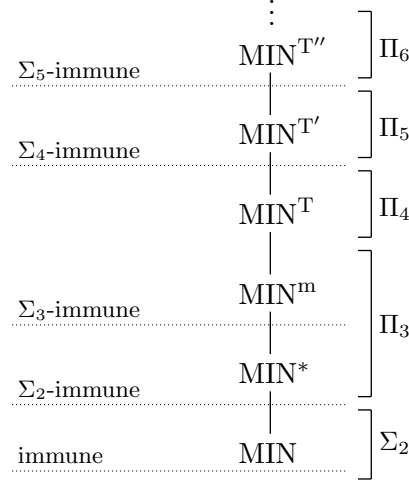


FIGURE 3.1. A naïve approach to spectral sets, by reverse inclusion.

The set inclusions and relations in Figure 3.1 becomes nonlinear when we add in spectral sets such as  $\text{fR}$ ,  $\text{f-MIN}$ ,  $\text{MIN}^{\equiv 1}$ ,  $\text{MIN}^{\text{Thick-*}}$  and  $\text{MIN}^{\text{a.e.-T}}$  (see Chapter 4 and Appendix A.8).  $\text{MIN}^{\text{Thick-*}}$ , in particular, does not seem to fit into this picture at all. Indeed,  $\text{MIN}^{\text{Thick-*}} \in \Sigma_4 - \Pi_4$  (Theorem 4.2.2) but is only  $\Sigma_2$ -immune (Theorem 4.3.2). A simple, general pattern connecting arithmetics and immunity does not seem to exist.

### 3.2. Upper spectral sets

The goal of this section is to determine the immunity of  $\text{MIN}^{\text{T}^{(n)}}$ . In the following theorem, the cases  $=^*$  and  $\equiv_{\text{T}}$  were first proved by Arslanov, and the remaining cases are due to Jockusch et al.

**THEOREM 3.2.1** (generalized fixed points, Arslanov [2], Jockusch et al. [18]). *For every  $n \leq \omega$ ,*

$$(I) \quad f \leq_{\text{T}} \emptyset' \implies (\exists e) [W_e =^* W_{f(e)}],$$

$$(II) \quad f \leq_{\text{T}} \emptyset'' \implies (\exists e) [W_e \equiv_{\text{m}} W_{f(e)}],$$

(III)  $f \leq_T \emptyset^{(n+2)} \implies (\exists e) [W_e \equiv_{T^{(n)}} W_{f(e)}]$ .

Furthermore,  $e$  can be found effectively from  $n$  and an index for  $f$ .

COROLLARY 3.2.2. For all  $n < \omega$ ,  $\text{MIN}^{T^{(n)}}$  is  $\Sigma_{n+3}$ -immune.

PROOF. We follow the proof of the  $\Pi_3$ -Separation Theorem (Theorem 3.1.3(i)), and as before,  $\text{MIN}^{T^{(n)}}$  is infinite (by Theorem 1.3.4(vi)).

Let  $n \geq 0$ , and let  $A$  be an infinite,  $\Sigma_{n+3}$  set. Suppose  $A \subseteq \text{MIN}^{T^{(n)}}$ . Since  $A$  is infinite and c.e. in  $\emptyset^{(n+2)}$ , we can define a  $\emptyset^{(n+2)}$ -computable function  $g$  by

$$g(e) = \pi_1 \left( (\mu \langle i, t \rangle) [i > e \ \& \ i \in A_t] \right),$$

where  $\{A_t\}$  is a  $\emptyset^{(n+2)}$ -enumeration of  $A$ .

Now for all  $e$ ,  $g(e) > e$  and  $g(e) \in \text{MIN}^{T^{(n)}}$ . Therefore

$$(\forall e) [W_e \not\equiv_{T^{(n)}} W_{g(e)}],$$

contradicting Theorem 3.2.1. □

We now show that Corollary 3.2.2 is optimal.

DEFINITION 3.2.3. A set  $A$  is called *low* if  $A' \equiv_T \emptyset'$ .

DEFINITION 3.2.4. Let  $A$  and  $B$  be c.e. sets.  $A$  and  $B$  are *pairwise minimal* if

- (I)  $A, B >_T \emptyset$ , and
- (II) For every c.e. set  $C$ ,

$$[C \leq_T A \ \& \ C \leq_T B] \implies C \text{ is computable.}$$

The original minimal pairs construction is due to Lachlan [24] and Yates [47]. We generalize their result as follows:

THEOREM 3.2.5. There exists a computable sequence of c.e. sets  $A_0, A_1, \dots$  which are low and pairwise minimal.

PROOF. We shall assume familiarity with Lachlan's tree construction for minimal pairs [25] as given in [8], since only a minor modification is needed to prove the theorem. Lachlan's tree construction meets the following requirements:

$$R_e : \overline{A} \neq W_e,$$

$$Q_e : \overline{B} \neq W_e,$$

$$N_{\langle i,j \rangle} : [\Psi_i^A = \Psi_j^B = f \ \& \ f \in \text{TOT}] \implies f \text{ is computable.}$$

Since we are constructing a sequence of noncomputable sets, we replace  $R_e$  and  $Q_e$  with an appropriate requirement  $R_{\langle k,e \rangle}$ . We also add the lowness requirement  $L_{\langle k,e \rangle}$  from [44, Theorem VII.1.1]:

$$R_{\langle k,e \rangle} : \overline{A_k} \neq W_e,$$

$$L_{\langle k,e \rangle} : (\exists^\infty s) \left[ \Psi_{e,s}^{(A_k)_s}(e) \downarrow \right] \implies \Psi_e^{A_k}(e) \downarrow,$$

$$N_{\langle i,j,m,n \rangle} : [\Psi_i^{A_m} = \Psi_j^{A_n} = f \ \& \ f \in \text{TOT}] \implies f \text{ is computable.}$$

Requirements  $R_{\langle k,e \rangle}$  and  $N_{\langle i,j,m,n \rangle}$  are satisfied in exactly the same way as the original construction, once we place these requirements on the analogous levels of the tree. The lowness requirement,  $L_{\langle k,e \rangle}$ , combines easily with the the  $N_{\langle i,j,m,n \rangle}$  requirement because both are negative requirements which only try to protect existing computations. We satisfy  $L_{\langle k,e \rangle}$  simply by adding an extra constraint on the witnesses chosen to satisfy  $R_{\langle k,e \rangle}$ : define a computable restraint function  $r$  by

$$r(k, e, s) := \psi_{e,s}^{(A_k)_s}(e),$$

where  $\psi$  denotes the use function. Then restrain, in stage  $s$  with priority  $\langle k, e \rangle$  (lower numbers having higher priority), any element less than  $r(k, e, s)$  from entering  $(A_k)_{s+1}$ . In some stage  $t$ , after finite many injuries,  $r$  eventually protects the computation on  $e$  with oracle  $(A_k)_t$  (whether or not the computation converges), thereby satisfying  $L_{\langle k,e \rangle}$ .  $\square$

Theorem 3.2.5 easily relativizes:

**THEOREM 3.2.6.** *For every  $n$ , there exists a computable sequence of c.e. sets  $A_0, A_1, \dots$  such that for all C c.e. in  $\emptyset^{(n)}$  and  $i \neq j$ ,*

- (I)  $\emptyset^{(n)} <_T (A_i)^{(n)}$ .
- (II)  $(A_i)^{(n+1)} \equiv_T \emptyset^{(n+1)}$ ,
- (III)  $[C \leq_T (A_i)^{(n)} \ \& \ C \leq_T (A_j)^{(n)}] \implies C \leq_T \emptyset^{(n)}$ .

PROOF. We perform the construction from Theorem 3.2.5 above  $\emptyset^{(n)}$ , but with each c.e. set  $A_k$  replaced with a respective set  $B_k$  which is c.e. in  $\emptyset^{(n)}$ .  $n$  applications of the Sacks Jump Theorem (Theorem 2.3.1) then gives the desired reduction:  $(A_k)^{(n)} \equiv_T B_k$ .  $\square$

Theorem 3.2.6 will be useful in the proof of Theorem 3.2.7.

THEOREM 3.2.7. *For all  $n \geq 0$ ,  $\text{MIN}^{\text{T}^{(n)}}$  is not  $\Sigma_{n+4}$ -immune.*

PROOF. Let  $n \geq 0$ , and let  $A_0, A_1, \dots$  be the corresponding sequence of sets obtained from Theorem 3.2.6. Recall that

$$\text{LOW}^n := \{e : (W_e)^{(n)} \equiv_T \emptyset^{(n)}\},$$

and define

$$B_k := \left[ \{x : W_x \leq_{\text{T}^{(n)}} A_k\} \cap \overline{\text{LOW}^n} \right],$$

$$B := \{e : (\exists k) (\forall j < e) [e \in B_k \ \& \ j \notin B_k]\}.$$

The proof of Proposition 1.3.3(vii) mentions that  $B \leq_T A$  is a  $\Sigma_2^{B \oplus A'}$  relation. Since, for any  $x$ , both  $(W_x)^{(n)} \leq_T \emptyset^{(n+1)}$  and  $(A_k)^{(n+1)} \leq_T \emptyset^{(n+1)}$ , it follows that

$$\left\{ x : (W_x)^{(n)} \leq_T (A_k)^{(n)} \right\} \in \Sigma_2^{\emptyset^{(n+1)}} = \Sigma_{n+3}.$$

This places  $B_k \in \Delta_{n+4}$ , on account of  $\overline{\text{LOW}^n} \in \Pi_{n+3}$ . Therefore  $B \in \Sigma_{n+4}$ .

It remains to show that  $B$  is an infinite subset of  $\text{MIN}^{\text{T}^{(n)}}$ . Note that  $B_i \cap B_j = \emptyset$  for  $i \neq j$ . Indeed, if  $e \in B_i \cap B_j$ , then

$$W_e \leq_{\text{T}^{(n)}} A_i \ \& \ W_e \leq_{\text{T}^{(n)}} A_j \ \& \ e \notin \text{LOW}^n,$$

contradicting Property (iii) of Theorem 3.2.6. Now since  $B_k \neq \emptyset$  and each  $B_k$  contributes exactly one element to  $B$ ,  $B$  must be infinite.

Finally, assume  $e \in B$ , and let  $k$  be such that  $e \in B_k$  and  $j \notin B_k$  for all  $j < e$ . Then for  $j < e$ ,

$$W_e \leq_{T^{(n)}} A_k \quad \& \quad W_j \not\leq_{T^{(n)}} A_k,$$

which implies  $W_e \not\equiv_{T^{(n)}} W_j$ . So  $e \in \text{MIN}^{T^{(n)}}$ . That is,  $B \subseteq \text{MIN}^{T^{(n)}}$ .  $\square$

### 3.3. Completeness criterion

As a application of our immunity results, we obtain generalizations of the Arslanov Completeness Criterion. The classical theorem and Arslanov's original generalization can be stated as follows:

**THEOREM 3.3.1** (= and =\*-Completeness Criterion, Arslanov [3]). (I) *Let  $A$  be c.e.*

*Then*

$$A \equiv_T \emptyset' \iff (\exists f \leq_T A) (\forall e) [W_e \neq W_{f(e)}].$$

(II) *Let  $A \in \Sigma_2$  and  $\emptyset' \leq_T A$ . Then*

$$A \equiv_T \emptyset'' \iff (\exists f \leq_T A) (\forall e) [W_e \neq^* W_{f(e)}].$$

The forward directions of Theorem 3.3.1 follow immediately from the fact that  $\text{MIN}$  is not  $\Sigma_2$ -immune and  $\text{MIN}^*$  is not  $\Sigma_3$ -immune (recall  $\text{MIN} \in \Sigma_2$ , and Theorem 3.1.3(ii)). Proofs are analogous to Corollary 3.3.2 below. According to [18], the hypothesis “ $A$  is c.e. (resp.  $\Sigma_2$ )” in Theorem 3.3.1 can be strengthened to “ $A$  is  $k$ -c.e. (resp.  $k$ -c.e. in  $\emptyset'$ ).” A  $k$ -c.e. set is a limit computable set in which the function witnessing this fact never changes its mind more than  $k$  times. By “ $A$  is  $k$ -c.e.,” we mean that the theorem holds for any  $k$  (in fact, we can assume only  $k$ -REA, see [19] for a definition).

Using immunity properties of  $\text{MIN}$ -sets, we are able to give a completeness criterion for 1:1 equivalence:

**COROLLARY 3.3.2** ( $\equiv_1$ -Completeness Criterion). *Let  $A$  be c.e. Then*

$$A \equiv_T \emptyset' \iff (\exists f \leq_T A) (\forall e) [W_e \equiv_1 W_{f(e)}].$$

PROOF. Let  $A \equiv_T \emptyset'$ , and suppose

$$(\forall f \leq_T A) (\exists e) [W_e \equiv_1 W_{f(e)}].$$

This contradicts the  $\Pi_3$ -Separation Theorem (Theorem 3.1.3): under this assumption, the immunity argument in Theorem 3.1.3(i) shows that  $\text{MIN}^{\equiv_1}$  is  $\Sigma_2$  immune, but part (iii) of that theorem says that it isn't.

The reverse direction is a direct application of the  $=$ -Completeness Criterion (Theorem 3.3.1). Let  $A$  be c.e., and assume

$$(\exists f \leq_T A) (\forall e) [W_e \not\equiv_1 W_{f(e)}].$$

Then clearly this same assertion holds for equality:

$$(\exists f \leq_T A) (\forall e) [W_e \neq W_{f(e)}],$$

which implies that  $A \equiv_T \emptyset'$ . □

The following result was first, along with its converse (modulo an appropriate assumption about  $A$ ), was first proved for  $n = 0$  by Arslanov [3, Corollary 2.3] and for  $n > 0$  by Jockush et al. [18, Corollary 5.17].

**COROLLARY 3.3.3** (necessary generalized completeness). *Let  $A$  be a set. Then for any  $n$ ,*

$$A \equiv_T \emptyset^{(n+3)} \implies (\exists f \leq_T A) (\forall e) [W_e \not\equiv_{T^{(n)}} W_{f(e)}].$$

PROOF. Let  $A \equiv_T \emptyset^{(n+3)}$ , and suppose

$$(\forall f \leq_T A) (\exists e) [W_e \equiv_{T^{(n)}} W_{f(e)}].$$

The argument in Corollary 3.2.2 shows that  $\text{MIN}^{T^{(n)}}$  is  $\Sigma_{n+4}$ -immune for all  $n$ , contradicting Theorem 3.2.7. □

Because  $\text{MIN}^{m^{(n+1)}} = \text{MIN}^{T^{(n)}}$  (Proposition 1.2.20), Corollary 3.3.3 shows that Theorem 3.2.1 is optimal in the sense that  $\equiv_{T^{(n+1)}}$  cannot be replaced with  $\equiv_{m^{(n+1)}}$ . In contrast, the authors of [18] note that  $\equiv_T$  can be substituted with  $\equiv_m$ . The converse for Corollary 3.3.3 is known to hold when  $A$  is  $k$ -c.e. in  $\emptyset^{(n+2)}$  (or  $k$ -REA in  $\emptyset^{(n+2)}$ ) for some  $k$ , and



$\emptyset^{(n+2)} \leq_T A$ . Furthermore, this additional condition is necessary [18]. For “completeness,” we state these theorems explicitly for the  $\Sigma_n$  sets:

**THEOREM 3.3.4** ( $\equiv_m$  and  $\equiv_{T(n)}$ -Completeness Criterion, Jockusch et al. [18]).

(I) *Let  $A \in \Sigma_3$  and  $\emptyset'' \leq_T A$ . Then*

$$A \equiv_T \emptyset''' \iff (\exists f \leq_T A) (\forall e) [W_e \not\equiv_m W_{f(e)}].$$

(II) *Let  $A \in \Sigma_{n+3}$  and  $\emptyset^{(n+2)} \leq_T A$ . Then*

$$A \equiv_{T(n)} \emptyset^{(n+3)} \iff (\exists f \leq_T A) (\forall e) [W_e \not\equiv_{T(n)} W_{f(e)}].$$

In summary, fixed points give us immunity:

**THEOREM 3.3.5.** *Let  $n \geq 0$ . Then*

$$\left( \forall f \leq_T \emptyset^{(n)} \right) (\exists e) [W_e \equiv_\alpha W_{f(e)}] \implies \text{MIN}^{\equiv_\alpha} \text{ is } \Sigma_{n+1}\text{-immune.}$$

A simple converse to Theorem 3.3.5, however, may not be forthcoming. In Lemma 4.3.1, we develop the notion of *semi-fixed points*, which are sufficient to ensure certain immunity properties (see Theorem 4.3.2(ii)). In fact, Theorem 4.3.2(ii) holds even if  $\nu$  were not computable, but merely  $\nu \leq_T \emptyset''$ . Thus it appears that fixed points are a strictly stronger notion than immunity for MIN-sets.

### 3.4. Refinements

**3.4.1.  $\omega$ -immunity.** Let  $D_0, D_1, \dots$  be a computable numbering of the finite sets.

**DEFINITION 3.4.1** (Fenner and Schaefer [10]). (I) A set  $A$  is called *k-immune* if it is

infinite and there is no computable function  $f$  such that

- (a)  $(D_{f(n)})_{n \in \omega}$  is a family of pairwise disjoint sets,
- (b)  $D_{f(n)} \cap A \neq \emptyset$ , and
- (c)  $|D_{f(n)}| \leq k$ .

(II) A set is called  $\omega$ -immune if is  $k$ -immune for every  $k$ .

THEOREM 3.4.2 (Fenner and Schaefer [10]). *Let  $A$  be a set.*

- (I) MIN is  $\omega$ -immune.
- (II) RAND is  $\omega$ -immune.
- (III) If  $A$  is  $\omega$ -immune, then  $\emptyset' \not\leq_{\text{btt}} A$ .

Schaefer actually proves that f-MIN is immune, but the same proof works for MIN [39].

COROLLARY 3.4.3. *Let  $A \subseteq \text{MIN}$ . Then  $\emptyset' \not\leq_{\text{btt}} A$ . This includes all the MIN-sets mentioned in this paper.*

PROOF. Any subset of MIN is also  $\omega$ -immune. □

**3.4.2.  $\Pi_n$ -immunity.** The  $\Pi_3$ -Separation Theorem 3.1.3 gives us an optimal immunity result for  $\text{MIN}^{\text{m}}$ , but the analogous theorems for MIN,  $\text{MIN}^*$ , and  $\text{MIN}^{\text{T}}$  leave room for improvement. We can say a bit about  $\Pi_n$  subsets in general, if they exist.

THEOREM 3.4.4. *Let  $n \geq 0$ .*

- (I) *Let  $A$  be an infinite  $\Pi_1$  subset of MIN. Then  $A \equiv_{\text{bT}} \emptyset'$ ,  $A$  is not hyperimmune, and  $A \not\equiv_{\text{btt}} \emptyset'$ .*
- (II) *Let  $A$  be an infinite  $\Pi_2$  subset of  $\text{MIN}^*$  such that  $A \geq_{\text{T}} \emptyset'$ . Then  $A \equiv_{\text{T}} \emptyset''$ , but  $A \not\equiv_{\text{btt}} \emptyset''$ .*
- (III) *Let  $A$  be an infinite  $\Pi_{n+3}$  subset of  $\text{MIN}^{\text{T}(n)}$  such that  $A \geq_{\text{T}} \emptyset^{(n+2)}$ . Then  $A \equiv_{\text{T}} \emptyset^{(n+3)}$ , but  $A \not\equiv_{\text{btt}} \emptyset^{(n+3)}$ .*

PROOF. (i). Suppose MIN has an infinite subset  $A \in \Pi_1$ . Since MIN is strongly effectively immune [34],  $A$  must also be strongly effectively immune. Thus  $\bar{A}$  is effectively simple, and it follows immediately that  $A \equiv_{\text{bT}} \emptyset'$  [29]. Furthermore, since a hypersimple set can never be bT-complete [14],  $A$  must not be hyperimmune. Finally,  $\bar{A} \not\equiv_{\text{btt}} \emptyset'$  follows from the fact that  $A$  is simple [33, Theorem III.8.8], [? ]. □

(II). The argument is quite the same as part (i), but now we turn to our  $\omega$ -immunity results. Suppose  $\text{MIN}^*$  has an infinite subset  $A \in \Pi_2$ . Define  $f \leq_T \bar{A}$  by

$$f(e) := (\mu x) [x \in A \ \& \ x > e].$$

Then

$$(\forall e) [W_{f(e)} \neq^* W_e].$$

Therefore  $\bar{A} \equiv_T \emptyset''$  by the  $=^*$  Completeness Criterion (Theorem 3.3.1).

In place of simplicity,  $A \subseteq \text{MIN}^*$  implies that  $A$  is  $\omega$ -immune, and therefore  $\bar{A} \not\leq_{\text{btt}} \emptyset'$  by Theorem 3.4.2. □

(III). same as part (ii). □

□

**3.4.3.  $\Delta_n$ -immunity.** We observed in the introduction to Chapter 1.3 that  $\text{fR} \subseteq \text{f-MIN}$ , which proves  $\text{f-MIN}$  is not  $\Delta_2$ -immune. We now show that  $\text{MIN}$  is not  $\Delta_2$ -immune either, although our witness to this fact will not be a spectral set.

DEFINITION 3.4.5. (I) Let  $A \subseteq \omega$ . We say that a set  $A$  is a *partial function* if

$$(\forall x) [\langle x, y_1 \rangle, \langle x, y_2 \rangle \in A \implies y_1 = y_2].$$

(II) Define the  $\Pi_1$  set

$$\text{FUN} := \{e : W_e \text{ is a partial function}\}.$$

PROPOSITION 3.4.6.  $\text{FUN}$  is  $\Pi_1$ -complete.

PROOF. Using the *s-m-n* Theorem, define a computable function  $f$  by

$$\varphi_{f(e)}(x) := \begin{cases} 1 & \text{if } x = e = \langle \pi_1(e), \pi_2(e) \rangle \text{ and } \varphi_e(e) \downarrow, \\ 1 & \text{if } x = \langle \pi_1(e), \pi_2(e) + 1 \rangle, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then

$$e \in K \iff f(e) \notin \text{FUN}. \quad \square$$

NOTATION. If  $e \in \text{FUN}$ , we use  $\hat{e}$  to denote the function represented by  $W_e$ . In more detail,  $\hat{e}(x)$  is the unique integer  $y$  satisfying  $\langle x, y \rangle \in W_e$  if such a  $y$  exists, and  $\hat{e}(x)$  diverges otherwise.

DEFINITION 3.4.7.

$$\text{FUN-fR} := \{e : e \in \text{FUN} \ \& \ (\forall j \in \text{FUN} \cap \{0, \dots, e\}) [\hat{j}(0) \neq \hat{e}(0)]\}.$$

In the following Corollary, the general techniques from parts (ii) and (iii) can be applied to part (i). We choose to have FUN, however, because FUN hypostatizes an essential connection between sets and functions.

COROLLARY 3.4.8. *Let  $n \geq 0$ .*

- (I) MIN is not  $\Delta_2$ -immune.
- (II)  $\text{MIN}^*$  is not  $\Delta_3$ -immune.
- (III)  $\text{MIN}^{\text{T}^{(n)}}$  is not  $\Delta_{n+4}$ -immune.

PROOF. (I). We show that  $\text{FUN-fR} \subseteq \text{MIN}$ . First, note that  $\text{FUN-fR} \in \Delta_2$ . Indeed,  $\text{FUN} \leq_{\text{T}} \emptyset'$ , and then convergence of  $\hat{j}(0)$  and  $\hat{e}(0)$  can be decided by asking  $\emptyset'$  whether the following sets are nonempty:

$$\begin{aligned} &\{y : \langle 0, y \rangle \in W_j\}, \\ &\{y : \langle 0, y \rangle \in W_e\}. \end{aligned}$$

Since there are infinitely many possible values for  $\hat{e}(0)$ , FUN-fR must be infinite.

Finally, it is clear that MIN contains FUN-fR:

$$\begin{aligned} e \in \text{FUN-fR} &\implies (\forall j \in \text{FUN} \cap \{0, \dots, e\}) [\hat{j}(0) \neq \hat{e}(0)] \\ &\implies (\forall j < e) [W_j \neq W_e]. \end{aligned} \quad \square$$

(II). By Theorem 3.1.3,  $\text{MIN}^*$  contains an infinite  $\Sigma_3$  set  $A$ . Since  $A$  is c.e. in  $\emptyset''$ ,  $A$  contains a set  $B \leq_{\text{T}} \emptyset''$ , namely  $B = \{b_0 < b_1 < b_2 < \dots\}$  where

$$\begin{aligned} b_0 &:= \text{any member of } A, \text{ and} \\ b_n &:= \pi_1((\mu\langle x, t \rangle) [x \in A_t \ \& \ x > b_{n-1}]). \end{aligned}$$

Thus  $B$  is an infinite,  $\Delta_3$  subset of  $\text{MIN}^*$ . □

(III). Similar to part (ii), Theorem 3.2.7 provides an infinite  $\Sigma_{n+4}$  subset of  $\text{MIN}^{\text{T}^{(n)}}$ ,  
which in turn contains a  $\Delta_{n+4}$  subset. □

□

## CHAPTER 4

### Thickville: nonuniformity vs. the jump operator

“... Oriental onlookers are dubbed with pungent comments such as ‘He’s roasting King Kong’ ...”

*–The New York Times, 6/27/1963*

We provide intuition for the fact that  $\text{MIN}$  is a  $\Sigma$ -set while  $\text{MIN}^*$ ,  $\text{MIN}^m$ , and  $\text{MIN}^T$  are all  $\Pi$ -sets.

#### 4.1. Intuition

DEFINITION 4.1.1. Let  $\equiv_\alpha$  be an equivalence relation, and let  $A, B \subseteq \omega$ . Define the relation

$$A \equiv_{\text{Thick-}\equiv_\alpha} B \iff (\forall n) [A^{[n]} \equiv_\alpha B^{[n]}].$$

Similarly,

$$A \leq_{\text{Thick-}\equiv_\alpha} B \iff (\forall n) [A^{[n]} \leq_\alpha B^{[n]}].$$

Note that for any equivalence relation  $\equiv_\alpha$  on  $\omega$ ,  $\equiv_{\text{Thick-}\equiv_\alpha}$  is also an equivalence relation. Informally,  $\text{Thick-}\equiv_\alpha$  requires agreement on every row.

DEFINITION 4.1.2. Let  $\equiv_\alpha$  be an equivalence relation. Then

$$\text{Thick-MIN}^{\equiv_\alpha} := \text{MIN}^{\equiv_\alpha} \cap \text{MIN}^{\text{Thick-}\equiv_\alpha}$$

This definition is justified by the fact that  $\text{Thick-}\equiv_\alpha$  is intuitively a stronger notion than  $\equiv_\alpha$ . Indeed, for any  $A, B \subseteq \omega$ , define  $m$ -equivalent sets  $X \equiv_m A$  and  $Y \equiv_m B$  by

$$\begin{aligned} X^{[0]} &:= A & Y^{[0]} &:= B \\ X^{[n+1]} &:= \emptyset & Y^{[n+1]} &:= \emptyset. \end{aligned}$$

Then

$$X \equiv_{\text{Thick-}\equiv_{\alpha}} Y \implies A \equiv_{\alpha} B.$$

Note also that  $\text{MIN}^* = \text{MIN}^* \cap \text{MIN}^{\text{Thick-}*} \supseteq \text{MIN}^{\text{Thick-}*}$ , eliminating the need for the  $\equiv_{\text{m}}$ -equivalent sets  $X$  and  $Y$  at the  $*$ -level:

PROPOSITION 4.1.3. *For any equivalence relations  $\equiv_{\alpha}$  and  $\equiv_{\beta}$ ,*

- (I)  $\text{MIN}^{\equiv_{\alpha}} \supseteq \text{Thick-MIN}^{\equiv_{\alpha}}$ , and
- (II) *If  $(\forall A, B \subseteq \omega) [A \equiv_{\alpha} B \implies A \equiv_{\beta} B]$  then  $\text{Thick-MIN}^{\equiv_{\alpha}} \supseteq \text{Thick-MIN}^{\equiv_{\beta}}$ .*

Proposition 4.1.3(i) formally insinuates that “Thick- $\equiv_{\alpha}$ ” is a stronger relation than  $\equiv_{\alpha}$ , simply because it is always possible to move all the information encoded in a set into a single row.

Theorem 4.1.4(iv) gives basic set-theoretic properties of the modified thick operator from Definition 4.1.2. If one wishes to deal strictly with MIN-sets, however, a brief inspection of Theorem 4.1.4(iv) also reveals that  $\text{MIN}^{\text{Thick-T}^{(n)}} \supseteq \text{MIN}^{\text{T}^{(n+1)}}$ . On the other hand,  $\text{MIN}^{\text{T}^{(n)}} \supseteq \text{MIN}^{\text{Thick-T}^{(n)}}$  is not true in general because two sets which are Turing equivalent need not contain their respective “information” in identical rows.

We illustrate this last point with an example. Define sets  $A$  and  $B$  by

$$\begin{aligned} A^{[0]} &:= K & B^{[0]} &:= \emptyset \\ A^{[i+1]} &:= \emptyset & B^{[1]} &:= K \\ & & B^{[i+2]} &:= \emptyset. \end{aligned}$$

Then clearly  $A \equiv_{\text{T}^{(n)}} B$ , but  $A \not\equiv_{\text{Thick-T}^{(n)}} B$ .

THEOREM 4.1.4. *Let  $n \geq 0$ . Then:*

- (I)  $\text{MIN}^* \supseteq \text{MIN}^{\text{Thick-}*}$ .
- (II)  $\text{Thick-fR} = \text{f-MIN}$ .
- (III)  $\text{MIN}^{\text{Thick-}=} = \text{MIN}$ .
- (IV)  $\text{MIN}^{\text{T}^{(n)}} \supseteq \text{Thick-MIN}^{\text{T}^{(n)}} \supseteq \text{MIN}^{\text{T}^{(n+1)}}$ .

PROOF. (i).  $A =^* B$  implies  $A \equiv_{\text{Thick-}*} B$ . □

(II). We interpret fR as a single row in f-MIN. Define

$$\begin{aligned} \text{Thick-fR} &:= \{e : (\forall j < e) (\forall n) [\varphi_e(n) \neq \varphi_j(n)]\} \\ &= \text{f-MIN}. \end{aligned} \quad \square$$

(III). Two sets are equal if they agree on all rows. □

(IV). Assume  $A^{(n)} \leq_{\text{Thick-T}} B^{(n)}$ . Then  $A^{(n+1)} \leq_{\text{Thick-m}} B^{(n+1)}$  by the Jump Theorem [44]. For every  $i$ , let  $f_i$  be the computable function that witnesses  $(A^{(n+1)})^{[i]} \leq_m (B^{(n+1)})^{[i]}$ .

We create a function  $h$  which captures the values for all the  $f_i$ 's.

Define a computable function  $h$  by

$$h(\langle i, x \rangle, s) := \begin{cases} f_i(x) & \text{if } [x \in \omega^{[i]} \ \& \ i \leq s], \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\hat{h}(\langle i, x \rangle) := \lim_s h(\langle i, x \rangle, s).$$

By the Limit Lemma,  $\hat{h} \leq_T \emptyset'$ . Since  $\hat{h}(\langle i, x \rangle) = f_i(x)$  for all  $i$ , we have that

$$A^{(n+1)} \leq_T B^{(n+1)} \oplus \emptyset' \equiv_T B^{(n+1)}.$$

Hence

$$A^{(n)} \equiv_{\text{Thick-T}} B^{(n)} \implies A^{(n+1)} \equiv_T B^{(n+1)},$$

which gives the second inclusion (by Proposition 1.2.18). The first inclusion follows immediately from Proposition 4.1.3(i). □

□

Theorem 4.1.4(ii) gives some intuition why  $\text{MIN} \in \Sigma_2$ .  $\text{MIN}$  trivially equals  $\text{MIN}^{\text{Thick}=\}$ , so unlike the other “natural”  $\text{MIN}$ -sets,  $\text{MIN}$  actually doubles as a “thick” set. In the case of functions, we note that f-MIN doubles as a nontrivial thick set.

REMARK. Theorem 4.1.4(iv) unambiguously shows that the jump operator defeats nondeterminacy on c.e. sets when we interpret rows as nondeterministic enumerations of c.e. sets.



## 4.2. Arithmetics

We now redeem the thick operator by showing that, just like the jump operator, thick “kicks” complete sets up one level in the arithmetic hierarchy. Still, jump and thick are juxtaposed here as antithetical: thick “kicks” relations into  $\Pi_n$  whenever the jump operator “kicks” them into  $\Sigma_n$ .

PROPOSITION 4.2.1. *Let  $n \geq 0$ . Then*

- (I)  $\text{MIN}^{\text{Thick-}*} \in \Sigma_4$ .
- (II)  $\text{MIN}^{\text{Thick-m}} \in \Sigma_4$ .
- (III)  $\text{MIN}^{\text{Thick-T}^{(n)}} \in \Sigma_{n+5}$ .

PROOF. (I).  $\{\langle j, e \rangle : W_j =^* W_e\} \in \Sigma_3$ , so  $\{\langle j, e \rangle : W_j \equiv_{\text{Thick-}*} W_e\} \in \Pi_4$ . □

(II).  $\{\langle j, e \rangle : W_j \equiv_{\text{Thick-m}} W_e\} \in \Pi_4$ . □

(III).  $\{\langle j, e \rangle : W_j \equiv_{\text{Thick-T}^{(n)}} W_e\} \in \Pi_{n+5}$ . □

□

Theorem 4.2.2 gives the lower bounds.

THEOREM 4.2.2. *Let  $n \geq 0$ . Then*

- (I)  $\text{MIN}^{\text{Thick-}*} \notin \Pi_4$ .
- (II)  $\text{MIN}^{\text{Thick-m}} \notin \Pi_4$ .
- (III)  $\text{MIN}^{\text{Thick-T}^{(n)}} \notin \Pi_{n+5}$ .

PROOF. (I). Let  $A \in \Pi_4$ . Then there exists a relation  $R \in \Sigma_3$  such that

$$x \in A \iff (\forall y) R(x, y).$$

Since COF is  $\Sigma_3$ -complete [44], there exists a computable function  $g$  such that  $R(x, y)$  iff  $W_{g(x,y)}$  is cofinite. Therefore

$$x \in A \iff (\forall y) [W_{g(x,y)} =^* \omega].$$

Define a computable function  $f$  by

$$\varphi_{f(x)}^{[y]} := \varphi_{g(x,y)}.$$

Then

$$\begin{aligned} W_{f(x)} \equiv_{\text{Thick-}^*} \omega &\iff (\forall y) [W_{g(x,y)} =^* \omega] \\ &\iff x \in A, \end{aligned}$$

which makes

$$\text{Thick-COF} := \{e : W_e \equiv_{\text{Thick-}^*} \omega\}$$

$\Pi_4$ -complete.

Now, as in Theorem 1.3.4, suppose towards a contradiction that  $\text{MIN}^{\text{Thick-}^*} \in \Pi_4$ , and let  $a$  be the  $\equiv_{\text{Thick-}^*}$ -minimal index for  $\omega$ . Then

$$\begin{aligned} \text{Thick-COF} &= \{e : W_e \equiv_{\text{Thick-}^*} \omega\} \\ &= \{a\} \cup \left\{ e : (\forall j < e) \left[ j \in \text{MIN}^{\text{Thick-}^*} - \{a\} \implies W_j \not\equiv_{\text{Thick-}^*} W_e \right] \right\}. \end{aligned}$$

Now  $\text{Thick-COF} \in \Sigma_4$ , since  $W_j \equiv_{\text{Thick-}^*} W_e$  can be decided in  $\Pi_4$ , and because

$$\text{MIN}^{\text{Thick-}^*} - \{a\} \in \Pi_4$$

by assumption. This contradicts the fact that  $\text{Thick-COF}$  is  $\Pi_4$ -complete.  $\square$

(II). Let  $K^\omega$  be the c.e. set in which each row is the halting set; for all  $k$ ,

$$(K^\omega)^{[k]} := K,$$

and recall that

$$\text{mCOMP} = \{e : W_e \equiv_{\text{m}} K\}$$

is  $\Sigma_3$ -complete (Theorem 1.3.4(iv)). By an argument analogous to part (i), we have that

$$\text{Thick-mCOMP} := \{e : W_e \equiv_{\text{Thick-m}} K^\omega\}$$

is  $\Pi_4$ -complete.

Suppose  $\text{MIN}^{\text{Thick-m}} \in \Pi_4$ . Following the same line of reasoning as before, and noting that  $W_j \equiv_{\text{Thick-m}} W_e$  can be decided in  $\Pi_4$ , we obtain a contradiction.  $\square$

(III). We use the same reasoning a third time. Let  $K^{(n)^\omega}$  be the c.e. set given by

$$\left(K^{(n)^\omega}\right)^{[i]} := K^{(n)},$$

for all  $i$ , and recall that

$$\text{HIGH}^n = \left\{ e : (W_e)^{(n)} \equiv_{\text{T}} K^{(n)} \right\}$$

is  $\Sigma_{n+4}$ -complete (see Theorem 1.3.4(vi)). By an argument analogous to part (i), we have that

$$\text{Thick-HIGH}^n := \left\{ e : (W_e)^{(n)} \equiv_{\text{Thick-T}} K^{(n)^\omega} \right\}$$

is  $\Pi_{n+5}$ -complete.

Suppose  $\text{MIN}^{\text{Thick-m}} \in \Pi_{n+5}$ . Following the same line of reasoning as before, and noting that  $W_j \equiv_{\text{Thick-T}(n)} W_e$  can be decided in  $\Pi_{n+5}$ , we obtain a contradiction.  $\square$

$\square$

### 4.3. Immunity

Thickness contributes nothing to immunity, as evidenced by Theorem 4.3.2.

LEMMA 4.3.1 (semi-fixed points). *There exists a computable function  $\nu$  such that*

- (I)  $f \leq_{\text{T}} \emptyset' \implies (\exists e) [W_{\nu(e)} \equiv_{\text{Thick-*}} W_{f(e)}]$ ,
- (II)  $f \leq_{\text{T}} \emptyset'' \implies (\exists e) [W_{\nu(e)} \equiv_{\text{Thick-m}} W_{f(e)}]$ ,
- (III)  $f \leq_{\text{T}} \emptyset^{(n+2)} \implies (\exists e) [W_{\nu(e)} \equiv_{\text{Thick-T}(n)} W_{f(e)}]$ .

PROOF. (I). The proof for part (ii) will work.  $\square$

(II). Using the  $s$ - $m$ - $n$  Theorem, define a computable function  $\nu$  by

$$\varphi_{\nu(x)}(\langle z, n \rangle) := \begin{cases} \varphi_{\varphi_x(n)}(z) & \text{if } \varphi_x(n) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

so that for any  $x \in \text{TOT}$ ,

$$W_{\nu(x)}^{[n]} = W_{\varphi_x(n)}.$$

Let  $f \leq_T \emptyset''$ , and define, again using the  $s$ - $m$ - $n$  Theorem, a computable sequence of  $\emptyset''$ -computable functions  $\{f_n\}$  by

$$\varphi_{f_n(x)}(z) := \varphi_{f(x)}(\langle z, n \rangle)$$

so that

$$W_{f_n(x)} = W_{f(x)}^{[n]}.$$

By the Generalized Fixed Point Theorem 3.2.1, we can uniformly find a computable sequence  $\{e_n\}$  such that for all  $n$ ,

$$W_{e_n} \equiv_m W_{f_n(e)}.$$

Let  $e$  be an index so that

$$\varphi_e(n) := e_n.$$

Then for all  $n$ ,

$$W_{\nu(e)}^{[n]} = W_{\varphi_e(n)} = W_{e_n} \equiv_m W_{f_n(e)} = W_{f(e)}^{[n]}.$$

This means that

$$(4.1) \quad (\forall f \leq_T \emptyset'') (\exists e) [W_{\nu(e)} \equiv_{\text{Thick-m}} W_{f(e)}],$$

which is what we intended to show. □

(III). Our proof of (ii) used no specific properties of  $\equiv_m$  except that this relation satisfies Generalized Fixed Point Theorem 3.2.2. Since  $\equiv_{T(n)}$  satisfies analogous fixed point properties, the same argument will work. □

□

Comparing Theorem 4.3.2 with the results from Chapter 3, we note that the thick operator does not at all affect the immunity of our main equivalence relations:

**THEOREM 4.3.2.** *Let  $n \geq 0$ . Then*

- (I)  $\text{MIN}^{\text{Thick-*}}$  is  $\Sigma_2$ -immune but not  $\Sigma_3$ -immune.
- (II)  $\text{MIN}^{\text{Thick-m}}$  is  $\Sigma_3$ -immune (and not  $\Sigma_4$ -immune).
- (III)  $\text{MIN}^{\text{Thick-T}^{(n)}}$  is  $\Sigma_{n+3}$ -immune but not  $\Sigma_{n+4}$ -immune.

PROOF. (I).  $\text{MIN}^{\text{Thick-*}}$  is  $\Sigma_2$ -immune follows immediately from Theorem 4.1.4(i) and Theorem 3.1.1(iii). We show  $\text{MIN}^{\text{Thick-*}}$  is not  $\Sigma_3$  immune by modifying the proof of Theorem 3.1.3(ii). All that is needed is to change the definition of  $A_k$  so that it only applies to the first row of each c.e. set:

$$A_k := \left\{ e : W_e^{[0]} \subseteq^* P_k \right\} \cap \text{INF}.$$

The rest of the proof is the same.  $\square$

(II). We tweak the proof of  $\Pi_3$ -Separation Theorem 3.1.3(i). Let  $A$  be an infinite,  $\Sigma_3$  set. Suppose  $A \subseteq \text{MIN}^{\text{Thick-m}}$ . Since  $A$  is infinite and c.e. in  $\emptyset''$ , we can define a  $\emptyset''$ -computable function  $f$  by

$$f(x) = \pi_1 \left( (\mu \langle i, t \rangle) [i > \nu(x) \ \& \ i \in A_t] \right),$$

where  $\{A_t\}$  is a  $\emptyset''$ -enumeration of  $A$  and  $\nu$  is the computable function from Lemma 4.3.1.

Now for all  $x$ ,  $f(x) > \nu(x)$  and  $f(x) \in \text{MIN}^{\text{Thick-m}}$ . Therefore

$$(\forall x) [W_{\nu(x)} \not\equiv_{\text{Thick-m}} W_{f(x)}],$$

contradicting Lemma 4.3.1.  $\square$

(III). A rehash of ideas from parts (i) and (ii). To show that  $\text{MIN}^{\text{Thick-T}^{(n)}}$  is not  $\Sigma_4$ -immune, we use the proof from Theorem 3.2.7 in place of Theorem 3.1.3(ii): just redefine

$$B_k := \left[ \left\{ x : \left( (W_x)^{(n)} \right)^{[0]} \leq_{\text{T}} (A_k)^{(n)} \right\} \cap \overline{\text{LOW}^n} \right],$$

and then the proof from Theorem 3.2.7 works.

The argument in (ii) shows that  $\text{MIN}^{\text{T}^{(n)}}$  is  $\Sigma_3$  immune.  $\square$

$\square$

## CHAPTER 5

### A Kolmogorov numbering

For certain Gödel numberings, we can exactly determine the truth-table degree of  $\text{MIN}$ ,  $\text{MIN}^*$ , and  $\text{MIN}^{\text{m}}$  as well as the Turing degrees of  $\text{MIN}^{\text{T}^{(n)}}$ , and  $\text{MIN}^{\text{Thick-}^*}$ . The main result of this chapter, Theorem 5.2.3, provides a Kolmogorov numbering in which  $\text{MIN}$ -sets exactly characterize the Turing degrees  $\mathbf{0}$ ,  $\mathbf{0}'$ ,  $\mathbf{0}''$ ,  $\dots$ .

#### 5.1. Numberings I & II

**5.1.1. Numbering I.** Theorem 5.1.1, restricted to  $\text{f-MIN}_\psi$  and  $\text{f-MIN}_\psi^*$ , was first proved by Schaefer [39]. He also mentions a Gödel ordering satisfying (i) (see Theorem 1.4.4). The majority of constructions here are inspired by [39, Theorem 2.17].

**THEOREM 5.1.1.** *There exists a Kolmogorov numbering  $\psi$  simultaneously satisfying:*

- (I)  $\text{fR}_\psi \geq_{\text{tt}} \emptyset'$ ,
- (II)  $\text{MIN}, \text{f-MIN}_\psi \geq_{\text{tt}} \emptyset''$ ,
- (III)  $\text{MIN}^*, \text{f-MIN}_\psi^* \geq_{\text{tt}} \emptyset'''$ ,
- (IV)  $\text{MIN}_\psi^{\text{Thick-}^*} \geq_{\text{tt}} \emptyset'''$ ,
- (V)  $\text{MIN}_\psi^{\text{Thick-m}} \geq_{\text{tt}} \emptyset'''$ , and
- (VI)  $\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \geq_{\text{tt}} \emptyset^{(n+4)}$ .

**PROOF.** We first construct a Gödel numbering  $\psi$  satisfying (vi). We later argue that our construction can be modified to produce a Kolmogorov numbering satisfying all six parts of the lemma.

Let  $\varphi$  be any Gödel numbering, and let  $n \geq 0$ . We define the numbering  $\psi$  as follows. Define an increasing, computable function  $f$  by

$$\begin{aligned} f(0) &:= 0, \\ f(k+1) &:= 4[f(k) + 1] + 1, \end{aligned}$$

Let  $i \geq 0$ . If  $i = f(k)$  for some  $k$ , then we define  $\psi_i := \varphi_k$ . This makes  $\psi$  an effective ordering. Otherwise, for some  $k$ ,  $f(k) < i < f(k+1)$ . In this case we define

$$(5.1) \quad \psi_i(\langle x, y \rangle) := \begin{cases} 1 & \text{if } [y - f(k) \text{ is odd} \ \& \ y = i \ \& \ \varphi_x(x) \downarrow], \\ 1 & \text{if } [y - f(k) \text{ is even} \ \& \ y = i - 1 \ \& \ \varphi_k(x) \downarrow], \\ \uparrow & \text{otherwise.} \end{cases}$$

The functions  $\psi_{f(k)+1}, \psi_{f(k)+3}, \dots, \psi_{4[f(k)+1]-1}$  code the halting set into distinct rows, and the remaining functions between  $f(k)$  and  $f(k+1)$  are used for comparisons.

It remains now only to show that

$$\text{HIGH}_\varphi^n \leq_{\text{tt}} \text{MIN}_\psi^{\text{Thick-T}^{(n)}},$$

because  $\text{HIGH}_\varphi^n$  is  $\Sigma_{n+4}$  complete (Theorem 1.3.4(vi)). Here we use the subscript “ $\varphi$ ” to emphasize that we are considering  $\text{HIGH}^n$  with respect to the numbering  $\varphi$ .

We claim that

$$(5.2) \quad e \in \text{HIGH}_\varphi^n \iff \left[ \text{MIN}_\psi^{\text{Thick-T}^{(n)}} \cap \{f(k) + 2, f(k) + 4, \dots, 4f(k) + 4\} \right] = \emptyset,$$

where  $k$  is such that  $f(k) \leq e < f(k+1)$ . The claim follows by inspecting pairs of functions  $\{\psi_i, \psi_{i+1}\}$ . Indeed, assume  $e \in \text{HIGH}_\varphi^n$ . Then for all  $y$ , including  $y = f(k) + 1$ ,

$$(\text{dom } \psi_{f(k)+1})^{[y]} \equiv_{\text{T}^{(n)}} (\text{dom } \psi_{f(k)+2})^{[y]}.$$

Therefore

$$\text{dom } \psi_{f(k)+1} \equiv_{\text{Thick-T}^{(n)}} \text{dom } \psi_{f(k)+2},$$

which means that

$$f(k) + 2 \notin \text{MIN}_\psi^{\text{Thick-T}^{(n)}}.$$

Similarly,

$$f(k) + 4, f(k) + 6, \dots, 4f(k) + 4 \notin \text{MIN}_\psi^{\text{Thick-T}^{(n)}},$$

which proves the first direction.

Conversely, assume that  $e \notin \text{HIGH}_\varphi^n$ . Then for all  $i \neq j$ , with

$$i, j \in \{f(k) + 1, f(k) + 2, \dots, 4f(k) + 4\},$$

we have

$$\psi_i \not\equiv_{\text{Thick-T}^{(n)}} \psi_j.$$

This means that for  $k \geq 1$ ,

$$[4f(k) + 4] - f(k) = 3f(k) + 4$$

distinct  $\equiv_{\text{Thick-T}^{(n)}}$ -equivalence classes are represented in

$$(5.3) \quad \{\psi_{f(k)+1}, \psi_{f(k)+2}, \dots, \psi_{4f(k)+4}\}.$$

It follows that at least

$$[3f(k) + 4] - (f(k) + 1) = 2f(k) + 3$$

of the indices from (5.3) are  $\equiv_{\text{Thick-T}^{(n)}}$ -minimal, since only those classes also represented in  $\{\psi_1, \dots, \psi_{f(k)}\}$  could be  $\equiv_{\text{Thick-T}^{(n)}}$ -nonminimal. Thus, any subset from

$$\{f(k) + 1, f(k) + 2, \dots, 4f(k) + 4\}$$

with cardinality at least  $f(k) + 2$  must contain a  $\equiv_{\text{Thick-T}^{(n)}}$ -minimal index. In particular,

$$\left[ \text{MIN}_\psi^{\text{Thick-T}^{(n)}} \cap \{f(k) + 2, f(k) + 4, \dots, 4f(k) + 4\} \right] \neq \emptyset.$$

Hence we conclude that

$$\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \geq_{\text{tt}} \emptyset^{(n+4)}.$$

We now describe separate orderings satisfying (i) – (v), and then we show that all six numberings can be combined together into a single Gödel numbering. Finally, we argue that this Gödel numbering can be made into an Kolmogorov numbering by ambiguously appealing to [39, Theorem 2.17].

The remaining, individual numberings are either identical or similar to the numbering  $\psi$  which we just constructed. For instance, the same  $\psi$  satisfies

$$\text{MIN}_\psi^{\text{Thick-m}} \geq_{\text{tt}} \emptyset'''.$$



In fact, we need only change  $\text{HIGH}_\varphi^n$  to  $\text{mCOMP}_\varphi$  in the verification (5.2), and then the same proof works. For  $\equiv_{\text{Thick-*}}$ ,  $=^*$ , and  $=$ , we use a different numbering, say  $\nu$ , which is exactly like  $\psi$  except the condition “ $\varphi_x(x) \downarrow$ ” is omitted from (5.1). To verify this numbering works, we swap either  $\text{COF}_\varphi$  or  $\text{TOT}_\varphi$  for  $\text{HIGH}_\varphi^n$  in (5.2). For  $\text{fR}$ , we substitute (5.1) with

$$\xi_i(x) := \begin{cases} \langle i, 1 \rangle & \text{if } i \text{ is odd,} \\ \langle i - 1, 1 \rangle & \text{if } [i \text{ is even} \ \& \ \varphi_k(k) \downarrow], \\ \uparrow & \text{otherwise.} \end{cases}$$

In the verification for  $\text{fR}$ , we replace  $\text{HIGH}_\varphi^n$  in (5.2) with the halting set complement,  $\overline{K_\varphi}$ .

We now merge the numberings  $\psi$ ,  $\nu$ , and  $\xi$  into a single Gödel numbering  $\rho$  satisfying (i) – (vi). All we do is change the p.c. functions filling the coding “gap” between  $f(k)$  and  $f(k+1)$ , so that  $\psi$  fills the first gap,  $\nu$  fills the second gap,  $\xi$  fills the third gap,  $\psi$  again fills the fourth, etc. Furthermore, we must repeat each  $\varphi_k$  function three times, so that each of numbering strategies may ask questions to it. For this reason, we let  $\varphi$  be a Kolmogorov numbering such that  $\varphi_k = \varphi_{k+1} = \varphi_{k+2}$  whenever  $k \equiv 0 \pmod{3}$ . We could settle for a Gödel numbering for the moment, but we’ll need  $\varphi$  to be a Kolmogorov numbering anyway after the next paragraph.

We define

$$\rho_i := \varphi_k \quad \text{when } i = f(k) \text{ for some } k.$$

Otherwise,  $f(k) < i < f(k+1)$  for some  $k$ . If  $k \equiv 0 \pmod{3}$  then we use the  $\psi$  strategy for  $i$ , if  $k \equiv 1 \pmod{3}$  we use the  $\nu$  strategy for  $i$ , and if  $k \equiv 2 \pmod{3}$  we use the  $\xi$  strategy for  $k$ . So, for example, if  $i = 3 \cdot 4567 + 1$ , then

$$\rho_i(\langle x, y \rangle) := \begin{cases} 1 & \text{if } [y - f(k) \text{ is odd} \ \& \ y = i], \\ 1 & \text{if } [y - f(k) \text{ is even} \ \& \ y = i - 1 \ \& \ \varphi_k(x) \downarrow], \\ \uparrow & \text{otherwise.} \end{cases}$$

We can now make truth-table queries to the appropriate spectral sets, just as before.

Finally, we transform  $\rho$  into a Kolmogorov numbering. The idea is to enumerate a large number of  $\varphi_k$ ’s between each coding “gap” instead of just the one  $k$  from  $f(k)$ . In the  $s^{\text{th}}$

gap, we code a crib for  $\varphi_s$  in the same manner as we did with  $\rho$ . More formally we define, by induction,

$$(5.4) \quad g(0) := 0,$$

$$(5.5) \quad h(0) := 0,$$

$$(5.6) \quad g(k+1) := g(k) + h(k) + 2[g(k) + 1],$$

$$(5.7) \quad h(k+1) := 2(h(k) + 2[g(k) + 1]).$$

Our new numbering is split into blocks  $h(k) \leq i < h(k+1)$  rather than  $f(k) \leq i < f(k+1)$  as before. For  $i$  with

$$h(k) \leq i < h(k) + 2[g(k) + 1],$$

we apply the familiar coding scheme from  $\rho$  (on  $\varphi_k$ ), and for  $i$  with

$$h(k) + 2[g(k) + 1] \leq i < h(k+1),$$

we simply enumerate  $\varphi_{g(k)}$  up to  $\varphi_{g(k+1)-1}$ . This construction is a Kolmogorov numbering by [39, Theorem 2.17], where this same induction appears.  $\square$

### 5.1.2. Numbering II.

**THEOREM 5.1.2.** *There exists a Kolmogorov numbering  $\psi$  such that for all  $n \geq 0$ :*

- (I)  $\text{MIN}_{\psi}^m \geq_{\text{tt}} \emptyset'''$ .
- (II)  $\text{MIN}_{\psi}^{\Gamma^{(n)}} \geq_{\text{tt}} \emptyset^{(n+3)}$ .

**PROOF.** As in Theorem 5.1.1, we shall first construct a Gödel numbering  $\psi$  satisfying (i) and (ii), and we later argue that the construction can be modified so as to achieve a single Kolmogorov numbering.

Let  $\varphi$  be an arbitrary Gödel numbering, and assume  $\langle \cdot, \cdot \rangle$  is a bijective pairing function satisfying  $\langle 0, 0 \rangle = 0$ . Let  $a$  be the computable function from Lemma 2.3.2, defined in terms of this ordering. Define a computable function  $f$  by

$$f(0) := 0,$$

$$f(k+1) := 2f(k) + 3.$$

The numbering  $\psi$  is defined as follows. Let  $C$  be an arbitrary computable set, and let  $\psi_0$  be such that

$$\text{dom } \psi_0 := C.$$

Let  $i \geq 1$ . If  $i = f(\langle k, n \rangle)$  for some pair  $\langle k, n \rangle$ , then  $\psi_i := \varphi_{\langle k, n \rangle}$ . Otherwise,  $f(\langle k, n \rangle) < i < f(\langle k, n \rangle + 1)$  for some  $\langle k, n \rangle$ . In this case,

$$\psi_i := \varphi_{a_{\langle k, n \rangle}(i)}.$$

Let  $\text{LOW}_\varphi^n$  and  $\text{LOW}_\psi^n$  denote the  $\text{LOW}^n$  indices in terms of  $\varphi$ -indices and  $\psi$ -indices, respectively.

We claim, for  $\langle k, n \rangle > 0$ ,

$$\text{MIN}_\psi^{\text{T}^{(n)}} \cap \{f(\langle k, n \rangle) + 1, f(\langle k, n \rangle) + 2, \dots, 2f(\langle k, n \rangle) + 2\} \neq \emptyset \iff k \in \overline{\text{LOW}_\varphi^n}.$$

Indeed, if  $k \in \text{LOW}_\varphi^n$ , then  $a_{\langle k, n \rangle}(i) \in \text{LOW}_\varphi^n$  for all  $i$ , hence

$$\{f(\langle k, n \rangle) + 1, \dots, 2f(\langle k, n \rangle) + 2\} \subseteq \text{LOW}_\psi^n,$$

and so

$$\text{MIN}_\psi^{\text{T}^{(n)}} \cap \{f(\langle k, n \rangle) + 1, \dots, 2f(\langle k, n \rangle) + 2\} = \emptyset.$$

Conversely, if  $k \in \overline{\text{LOW}_\varphi^n}$ , then by definition of  $a$ , each of the  $\psi$ -indices

$$(5.8) \quad f(\langle k, n \rangle) + 1, \dots, 2f(\langle k, n \rangle) + 2$$

represents a distinct  $\text{T}^{(n)}$ -degree. At most  $f(\langle k, n \rangle) + 1$  degrees are represented with smaller indices, so at least one of the  $f(\langle k, n \rangle) + 2$  degrees in (5.8) must be minimal. That is,

$$\text{MIN}_\psi^{\text{T}^{(n)}} \cap \{f(\langle k, n \rangle) + 1, \dots, 2f(\langle k, n \rangle) + 2\} \neq \emptyset.$$

Since  $\text{LOW}^n$  is  $\Sigma_{n+3}$ -complete, this proves that  $\psi$  satisfies (ii).

Similarly, for  $k > 0$ ,

$$\text{MIN}_\psi^{\text{m}} \cap \{f(\langle k, 0 \rangle) + 1, \dots, 2f(\langle k, 0 \rangle) + 2\} \neq \emptyset \iff k \in \overline{\text{LOW}_\varphi^0},$$

which shows that  $\psi$  satisfies (i). One can now transform  $\varphi$  into a Kolmogorov numbering by following the familiar procedure from Theorem 5.1.1, starting from (5.4).  $\square$

## 5.2. Truth-table apogee

We present a Kolmogorov numbering for which MIN-sets achieve maximal truth-table and Turing degrees.

LEMMA 5.2.1. *Let  $n \geq 0$ .*

- (I)  $\text{MIN}^{\text{Thick-}*} \oplus \emptyset''' \equiv_{\text{bT}} \emptyset''''$ ,
- (II)  $\text{MIN}^{\text{Thick-m}} \oplus \emptyset''' \equiv_{\text{bT}} \emptyset''''$ ,
- (III)  $\text{MIN}^{\text{Thick-T}^{(n)}} \oplus \emptyset^{(n+4)} \equiv_{\text{bT}} \emptyset^{(n+5)}$ .

PROOF. The same proof from Lemma 2.1.1(i) works here when we substitute the fact that either Thick-COF is  $\Pi_4$ -complete, Thick-mCOMP is  $\Pi_4$ -complete, or Thick-HIGH $^n$  is  $\Pi_{n+5}$ -complete for the fact that TOT is  $\Pi_2$ -complete. Definitions for Thick-COF and Thick-mCOMP appear in the proof of Theorem 4.2.2.  $\square$

Combining the orderings from Lemma 5.1.1 and Lemma 5.1.2 (using techniques from these lemmas), we obtain:

THEOREM 5.2.2. *There exists a Kolmogorov numbering  $\psi$  satisfying*

- (I)  $\text{fR}_\psi \geq_{\text{tt}} \emptyset'$ ,
- (II)  $\text{MIN}_\psi, \text{f-MIN}_\psi \geq_{\text{tt}} \emptyset''$ ,
- (III)  $\text{MIN}_\psi^*, \text{f-MIN}_\psi^* \geq_{\text{tt}} \emptyset'''$ ,
- (IV)  $\text{MIN}_\psi^{\text{m}} \geq_{\text{tt}} \emptyset'''$ ,
- (V)  $\text{MIN}_\psi^{\text{T}^{(n)}} \geq_{\text{tt}} \emptyset^{(n+3)}$ ,
- (VI)  $\text{MIN}_\psi^{\text{Thick-}*} \geq_{\text{tt}} \emptyset'''$ ,
- (VII)  $\text{MIN}_\psi^{\text{Thick-m}} \geq_{\text{tt}} \emptyset'''$ ,
- (VIII)  $\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \geq_{\text{tt}} \emptyset^{(n+4)}$ .

Using the numbering from Theorem 5.2.2, together with Lemma 5.2.1 and Lemma 2.1.1, we can conclude the following.

COROLLARY 5.2.3. *There exists a Kolmogorov numbering  $\psi$  simultaneously satisfying:*

- (I)  $\text{fR}_\psi \equiv_{\text{tt}} \emptyset'$ ,

- (II)  $\text{MIN}_\psi \equiv_{\text{tt}} \text{f-MIN}_\psi \equiv_{\text{tt}} \emptyset''$ ,
- (III)  $\text{MIN}_\psi^* \equiv_{\text{tt}} \text{f-MIN}_\psi^* \equiv_{\text{tt}} \emptyset'''$ ,
- (IV)  $\text{MIN}_\psi^{\text{m}} \equiv_{\text{tt}} \emptyset'''$ ,
- (V)  $\text{MIN}_\psi^{\text{T}^{(n)}} \equiv_{\text{T}} \emptyset^{(n+4)}$ ,
- (VI)  $\text{MIN}_\psi^{\text{Thick-*}} \equiv_{\text{T}} \emptyset''''$ ,
- (VII)  $\text{MIN}_\psi^{\text{Thick-m}} \equiv_{\text{T}} \emptyset''''$ , *and*
- (VIII)  $\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \equiv_{\text{T}} \emptyset^{(n+5)}$ .

## CHAPTER 6

### Hyperimmunity and the Peak Hierarchy Theorem

In Corollary 6.2.4, we exhibit an infinite sequence of indices which is common to all spectral sets herein. We conclude that spectral sets are not hyperimmune, and we use this fact to build a special “cutting set” in the last section.

#### 6.1. A computable sequence of intermediate degrees

The main goal of this section is to prove Theorem 6.1.1.

##### 6.1.1. Main theorem.

**THEOREM 6.1.1.** *There exists a computable sequence  $\{x_k\}$  such that for all  $n$  and  $i$ ,*

$$(W_{x_i})^{(n)} \not\leq_T \bigoplus_{j \neq i} (W_{x_j})^{(n)}.$$

In particular,  $(W_{x_i})^{(n)} \upharpoonright_T (W_{x_j})^{(n)}$  whenever  $i \neq j$ .

**PROOF.** The proof uses three lemmata. Lemma 6.1.2 creates a computable sequence “upstairs,” above  $\emptyset'$ . Lemma 6.1.5 brings that sequence “downstairs” using infinite injury from the Sacks Jump Theorem. Finally, we “take the elevator to the top” by way of Lemma 6.1.9. We now prove Theorem 6.1.1, assuming these lemmas.

Let  $\{a_k\}$  be the sequence of computable functions guaranteed by Lemma 6.1.5. Then by (6.3), for any  $s$  and  $i \neq j$ ,

$$(6.1) \quad W_{a_i(s)}^Y \oplus Y \not\equiv_T W_{a_j(s)}^Y \oplus Y.$$

By Lemma 6.1.9, let  $x$  be the fixed point for the sequence  $\{a_k\}$  satisfying

$$\Psi_{a_k(x)}^Y = \Psi_{x_k}^Y,$$

so that  $\{x_k\}$  is computable. Let  $i \neq j$ . Taking  $Y$  to be the empty set and applying (6.2) and (6.1) yields

$$\begin{aligned} (W_{x_i})' &\equiv_{\mathsf{T}} \left( W_{x_i}^\emptyset \oplus \emptyset \right)' = \left( W_{a_i(x)}^\emptyset \oplus \emptyset \right)' \equiv_{\mathsf{T}} W_{a_i(x)}^{\emptyset'} \oplus \emptyset' \\ &\not\equiv_{\mathsf{T}} W_{a_j(x)}^{\emptyset'} \oplus \emptyset' \equiv_{\mathsf{T}} \left( W_{a_j(x)}^\emptyset \oplus \emptyset \right)' = \left( W_{x_j}^\emptyset \oplus \emptyset \right)' \equiv_{\mathsf{T}} (W_{x_j})'. \end{aligned}$$

In general, applying (6.2) and (6.3) from Lemma 6.1.5 yields

$$\begin{aligned} (W_{x_i})^{(n)} &\equiv_{\mathsf{T}} \left( W_{x_i}^\emptyset \oplus \emptyset \right)^{(n)} = \left( W_{a_i(x)}^\emptyset \oplus \emptyset \right)^{(n)} \equiv_{\mathsf{T}} \left( W_{a_i(x)}^{\emptyset'} \oplus \emptyset' \right)^{(n-1)} \\ &\quad \vdots \\ &\equiv_{\mathsf{T}} W_{a_i(x)}^{\emptyset^{(n)}} \oplus \emptyset^{(n)} \not\leq_{\mathsf{T}} \bigoplus_{j \neq i} \left( W_{a_j(x)}^{\emptyset^{(n)}} \oplus \emptyset^{(n)} \right) \\ &\quad \equiv_{\mathsf{T}} \bigoplus_{j \neq i} \left( W_{a_j(x)}^{\emptyset^{(n-1)}} \oplus \emptyset^{(n-1)} \right)' \\ &\quad \quad \quad \vdots \\ &\quad \equiv_{\mathsf{T}} \bigoplus_{j \neq i} \left( W_{a_j(x)}^\emptyset \oplus \emptyset \right)^{(n)} \\ &\quad = \bigoplus_{j \neq i} \left( W_{x_j}^\emptyset \oplus \emptyset \right)^{(n)} \\ &\quad \equiv_{\mathsf{T}} \bigoplus_{j \neq i} (W_{x_j})^{(n)}. \quad \square \end{aligned}$$

**6.1.2. Three lemmas.** Lemma 6.1.2 was first proved by Kleene and Post in 1954 [21, Theorem 3.3.1]. A more recent, nonrelativized exposition appears in Odifreddi's book [33, Proposition V.2.7]. We isolate this proof in order to clarify intuition for Lemma 6.1.5.

LEMMA 6.1.2. *There exists a computable sequence  $\{a_i\}$  such that for any  $Y \subseteq \omega$  and  $i \in \omega$ ,*

$$W_{a_i}^Y \oplus Y \not\leq_{\mathsf{T}} \bigoplus_{j \neq i} \left( W_{a_j}^Y \oplus Y \right).$$

PROOF. We reuse the proof of Lemma 2.3.2. Our construction here is exactly the same as before, but without permitting (we omit “ $c(s) \leq x$ ” from (2.3)). The only other difference is that we use  $A_i$  to denote the relevant set  $W_{a_i}^Y \oplus Y$  rather than the irrelevant set,  $(W_{a_{\langle k,n \rangle}(i)})^{(n)}$ . The claims from before then follow verbatim:

CLAIM 6.1.3. *If requirement  $R_{\langle e, i \rangle}$  acts at some stage  $s + 1$  and is never later injured, then requirement  $R_{\langle e, i \rangle}$  is met and  $r(\langle e, i \rangle, t) = s + 1$  for all  $t \geq s + 1$ .*

CLAIM 6.1.4. *For every  $\langle e, i \rangle$ , requirement  $R_{\langle e, i \rangle}$  is met, acts at most finitely often, and  $r(\langle e, i \rangle) := \lim_s r(\langle e, i \rangle, s)$  exists.*

Recall that  $R_{\langle e, i \rangle}$  was the requirement

$$R_{\langle e, i \rangle} : A_i \neq \Psi_e^{\oplus_{j \neq i} A_j}.$$

Thus the lemma follows immediately from Claim 6.1.4.  $\square$

LEMMA 6.1.5. *There exists a computable sequence of computable functions  $\{a_i\}$  such that for any computable sequence  $\{s_i\}$  and  $Y \subseteq \omega$ ,*

$$(6.2) \quad (W_{a_i(s)}^Y \oplus Y)' \equiv_T W_{s_i}^{Y'} \oplus Y' \quad \text{and}$$

$$(6.3) \quad W_{a_i(s)}^Y \oplus Y \not\leq_T \bigoplus_{j \neq i} (W_{a_j(s)}^Y \oplus Y),$$

where  $s$  is such that  $\varphi_s(i) = s_i$ .

PROOF. We mix the “true stages” proof of the Sacks Jump Theorem [44] together with Lemma 6.1.2, however we omit the “avoid the cone” strategy. The main idea is as follows. In the proof of the Sacks Jump Theorem (Theorem 2.3.1), one constructs a single set  $A$  satisfying the thickness requirement

$$P_{\langle e, 0 \rangle} : A_0^{[e]} =^* B_0^{[e]},$$

where  $A_0$  and  $B_0$  are analogous to sets defined below. In this proof, we will simultaneously construct infinitely many sets  $\{A_i\}$  satisfying requirement  $R_{\langle e, i \rangle}$  from Lemma 6.1.2 by playing the strategy of Lemma 6.1.2 on the first column of each matrix,  $A_0^{[\tilde{0}]}, A_1^{[\tilde{0}]}, \dots$ . This strategy won't interfere with the corresponding  $P_{\langle e, i \rangle}$  requirements because we're changing just one point in each row.

We now prove the lemma. We construct  $\{a_i\}$  uniformly in  $s$  and independent of  $Y$ . For convenience, let  $S_i$  denote the set  $W_{s_i}^{Y'} \oplus Y'$ , and let  $\{A_i\}$  denote the sequence of sets we wish to construct, namely  $\{W_{a_i(s)}^Y \oplus Y\}$ . Thus  $\{a_i\}$  will be defined implicitly.



Each  $S_i \in \Sigma_2^Y$ , (by [44, Theorem IV.3.2], relativized to  $Y$ ) gives us a computable function  $h_i$  such that

$$\begin{aligned} z \in S_i &\implies |W_{h_i(z)}^Y| < \infty, \text{ and} \\ z \notin S_i &\implies W_{h_i(z)}^Y = \omega. \end{aligned}$$

Define a  $Y$ -c.e. set  $B_i$  by

$$B_i^{[z]} := \{ \langle x, z \rangle : x \in W_{h_i(z)}^Y \}$$

so that for all  $z$  and  $i$ ,

$$(6.4) \quad \begin{aligned} z \in S_i &\implies |B_i^{[z]}| < \infty, \text{ and} \\ z \notin S_i &\implies B_i^{[z]} = \omega. \end{aligned}$$

Given any computable enumeration  $\{C_s\}$  of a c.e. set  $C$ , define, for  $s > 0$ ,

$$\begin{aligned} \hat{u}(C_s) &:= \begin{cases} (\mu x) [x \in C_s - C_{s-1}] & \text{if } C_s - C_{s-1} \neq \emptyset, \\ \max(C_s \cup \{s\}) & \text{otherwise;} \end{cases} \\ \hat{\Psi}_{e,s}^{C_s}(x) &:= \begin{cases} \Psi_{e,s}^{C_s}(x) & \text{if defined and } \psi_{e,s}^{C_s}(x) < \hat{u}(C_s), \\ \uparrow & \text{otherwise;} \end{cases} \\ \hat{\psi}_{e,s}^{C_s}(x) &:= \begin{cases} \psi_{e,s}^{C_s}(x) & \text{if } \hat{\Psi}_{e,s}^{C_s}(x) \downarrow, \\ -1 & \text{otherwise;} \end{cases} \end{aligned}$$

$$T(C) := \{s : C_s \upharpoonright \hat{u}(C_s) = C \upharpoonright \hat{u}(C_s)\},$$

and according to Definition 1.2.8,

$$\omega^{[\widetilde{x}_k]} := \{ \langle x, \langle y, k \rangle \rangle : y \in \omega \}.$$

is the  $k^{\text{th}}$  row of the  $x^{\text{th}}$  column.

We now construct  $\{A_i\}$  uniformly in  $\{S_i\}$ . In order to ensure that  $A_i \not\leq_T \bigoplus_{j \neq i} A_j$  and  $(A_i)' \equiv_T S_i$ , we meet the following requirements:

$$\begin{aligned} R_{\langle e, i \rangle} &: A_i \neq \Psi_e^{\bigoplus_{j \neq i} A_j}, \\ P_{\langle e, i \rangle} &: A_i^{[e]} =^* B_i^{[e]}, \end{aligned}$$

and we attempt to meet the ‘‘pseudo-requirement’’

$$Q_{\langle e, i \rangle} : (\exists^\infty s) [\hat{\Psi}_{e,s}^{(A_i)_s}(e) \downarrow] \implies \Psi_e^{A_i}(e) \downarrow.$$

$R_{\langle e, i \rangle}$  will make  $\{A_i\}$  computably independent.  $P_{\langle e, i \rangle}$  guarantees that  $S_i \leq_T (A_i)'$  because

$$F_i(z) := \lim_x A_i(\langle x, z \rangle)$$

exists for all  $z$  and is the characteristic function of  $\overline{S_i}$  by (6.4), and because  $F_i \leq_T (A_i)'$  by the Limit Lemma. We don't actually meet  $Q_{\langle e, i \rangle}$ , since that would force  $(A_i)'$  to be limit computable in  $Y$  and hence  $(A_i)' \leq_T Y'$ , but we do meet  $Q_{\langle e, i \rangle}$  well enough to ensure  $(A_i)' \leq_T S_i$ .

*Construction.* Let

$$\hat{q}_i(e, s) := \hat{\psi}_{e,s}^{(A_i)_s}(e).$$

For each  $i$ , fix a computable sequence  $\{(B_i)_s\}_{s \in \omega}$  such that  $B_i = \bigcup_s (B_i)_s$ . In this construction,  $\hat{q}_i(e, \cdot)$  will be the restraint function for  $Q_{\langle e, i \rangle}$ , and  $r(\langle e, i \rangle, \cdot)$  will be the restraint function from Lemma 6.1.2. Define  $Y$ -computable functions  $\hat{Q}_i$  by

$$\hat{Q}_i(e, s) := \max_{j \leq e} \hat{q}_i(j, s).$$

*Stage  $s = 0$ .* Let  $r(\langle e, i \rangle, 0) = -1$  for all  $\langle e, i \rangle$ . Set  $(A_i)_0 = \emptyset \oplus Y$  for all  $i$ .

*Stage  $s + 1$ .* Do the following when  $s + 1$  is an  $i^{\text{th}}$  prime power. If  $s + 1$  is not a prime power, do nothing.

*Step 1.* For every even  $x \notin \omega^{\widehat{0}}$  and every  $e$  such that

$$x \in (B_i^{[e]})_{s+1} \quad \text{and} \quad x > \max \left\{ \hat{Q}_i(e, s), r(\langle e, i \rangle, s) \right\},$$

enumerate  $x$  into  $(A_i^{[e]})_{s+1}$ .

*Step 2.* We modify Stage  $s+1$  from Lemma 6.1.2 so that the Lemma 6.1.2 strategy happens on the first column of each  $A_i$ . Choose the least  $e$  such that:

$$\begin{aligned} r(\langle e, i \rangle, s) = -1 \quad & \& \quad (\exists \text{ even } x) \left[ x \in \omega^{[0_{\langle e, i \rangle}]} - (A_i)_s \right. \\ & \& \quad \left. \Psi_{e,s}^{\oplus_{j \neq i} (A_j)_s}(x) \downarrow = 0 \quad \& \quad (\forall \langle z, j \rangle < \langle e, i \rangle) [r(\langle z, j \rangle, s) < x] \right]. \end{aligned}$$

Here  $\omega^{[0_{\langle e, i \rangle}]}$  is the  $\langle e, i \rangle^{\text{th}}$  row of the 0 column. If there is no such  $e$ , then do nothing and go to stage  $s+2$ . If  $e$  exists, then  $R_{\langle e, i \rangle}$  acts at stage  $s+1$ .

Perform the following steps.

- (a) Enumerate  $x$  into  $(A_i^{[\tilde{0}]})_{s+1}$ .
- (b) Define  $r(\langle e, i \rangle, s+1) = s+1$ .
- (c) For all  $\langle z, j \rangle > \langle e, i \rangle$ , define  $r(\langle z, j \rangle, s+1) = -1$ .
- (d) For all  $\langle z, j \rangle < \langle e, i \rangle$ , define  $r(\langle z, j \rangle, s+1) = r(\langle z, j \rangle, s)$ .

Finally, we have  $A_i = \bigcup_s (A_i)_s$ .

**CLAIM 6.1.6.** *For every  $\langle e, i \rangle$ ,  $R_{\langle e, i \rangle}$  is satisfied.*

**PROOF.** Claims 6.1.3 and 6.1.4 from Lemma 6.1.2 each hold in this construction (with the same proofs) because the restraint function  $r$  protects the computations in the same way as before.  $\hat{q}_i(e, s)$  does not at all restrict enumeration into  $A_i^{[\tilde{0}]}$ . Since Step 2 only enumerates in the first column of each  $A_i$ , the construction in fact satisfies the stronger relation

$$A_i^{[\tilde{0}]} \neq \Psi_e^{\oplus_{j \neq i} A_j}. \quad \square$$

(6.3) follows immediately from Claim 6.1.6.

**CLAIM 6.1.7.** *For every  $\langle e, i \rangle$ ,  $P_{\langle e, i \rangle}$  is satisfied.*

**PROOF.** Since Step 2 of the construction affects only points in  $A_i^{[\tilde{0}]}$ , the points added in Step 2 have no bearing on the satisfaction of  $P_{\langle e, i \rangle}$ .

We would like to show that for every  $\langle e, i \rangle$ ,

$$\hat{L}(\langle e, i \rangle) := \lim_s \max \left\{ \hat{Q}_i(e, s), r(\langle e, i \rangle, s) \right\}$$

is finite, because if this is true, then for all  $x > \hat{L}(\langle e, i \rangle)$ ,  $x$  is enumerated into  $A_i^{[e]}$  iff  $x \in B_i^{[e]}$ . That is,  $A^{[e]} =^* B^{[e]}$ . Claim 6.1.4 shows that

$$r(\langle e, i \rangle) := \lim_s [r(\langle e, i \rangle, s)] < \infty,$$

so it suffices to prove

$$(6.5) \quad \lim_s \hat{Q}_i(e, s) < \infty.$$

We won't be able to show this, however, because the limit in (6.5) probably doesn't exist. Nevertheless, if we restrict ourselves to the set of "true stages," we can not only guarantee that the restricted limit exists, but also that it's finite. Let

$$T_i := T(A_i).$$

$T_i$  is the set of *true stages* in the enumeration  $\{(A_i)_s\}_{s \in \omega}$ . A true stage  $t \in T_i$  guarantees that all nonzero computations below  $\hat{u}[(A_i)_t]$  are correct. Thus any *apparent* computation  $\hat{\Psi}_{e,t}^{(A_i)_t}(x) = y$  is, in fact, a *true* computation  $\Psi_e^{A_i}(x) = y$ , for any  $x$ .

$T_i$  is infinite because  $\{(A_i)_s\}$  is an enumeration of  $A_i$ . Indeed, it must happen infinitely often that we enumerate an element  $x$  into  $(A_i)_t$  so that  $(A_i)_t \Vdash x = A_i \Vdash x$ . Furthermore,

$$(6.6) \quad \lim_{t \in T_i} \hat{Q}_i(e, t)$$

is finite because once  $\hat{q}_i(j, t)$  converges at some true stage  $t$  (which must happen),  $\hat{q}_i(j, \cdot)$  remains unchanged through all subsequent stages.

Finally, why is it sufficient for the limit (6.6) to be finite *only on the true stages*? Because  $A_i^{[e]}$  enumerates, in sufficiently late true stages, all elements in  $B_i^{[e]}$  which are greater than both (6.6) and  $r(\langle e, i \rangle)$ . This makes  $A_i^{[e]} =^* B_i^{[e]}$ , as desired.  $\square$

CLAIM 6.1.8. *For every  $i$ ,  $(A_i)' \leq_T S_i$ .*

PROOF. We determine membership in  $(A_i)'$  using an  $S_i \oplus Y' \equiv_T S_i$  oracle. Let

$$T_i^e := T\left(A_i^{\lfloor \leq e \rfloor}\right),$$

and observe that

$$(6.7) \quad e \in (A_i)' \iff \Psi_e^{A_i}(e) \downarrow \iff (\exists t) \left[ t \in T_i^e \quad \& \quad \hat{\Psi}_{e,t}^{(A_i)_t}(e) \downarrow \right].$$

Indeed, the second  $\iff$  in (6.7) follows from the fact that enumerations into  $A_i^{[>e]}$  cannot, by definition of  $\hat{q}_i$ , disturb settled computations on  $e$  with oracle  $A_i^{[\leq e]}$ .

It remains to show that the main predicate in (6.7) is  $Y$ -computable and can be constructed from  $S_i$ , because then membership in  $A_i$  is determined by a formula decidable in  $Y'$ . Note that  $A_i^{[\leq e]}$  is computable because  $A_i^{[\leq e]} =^* B_i^{[\leq e]}$ , and because  $B$  is piecewise computable. Hence  $T_i^e \equiv_T A_i^{[\leq e]}$  is computable. Moreover, the index for  $T_i^e$  can be found from the following series of *uniform* reductions:

$$T_i^e \leq_T A_i^{[\leq e]} \leq_T B_i^{[\leq e]} \oplus (Y' \oplus \{i\}) \leq_T S_i \oplus \{i\},$$

where  $Y' \oplus \{i\}$  is used to compute the finite,  $Y$ -c.e. set  $A_i^{[\hat{0}]} \cap \omega^{[\leq e]}$ . This means that the computable predicate  $t \in T_i^e$  can be constructed from  $S_i \oplus \{i\}$ . Furthermore,  $\hat{\Psi}_{e,t}^{(A_i)_t}(e) \downarrow$  is a  $Y$ -computable predicate, which makes

$$(\exists t) \left[ t \in T_i^e \quad \& \quad \hat{\Psi}_{e,t}^{(A_i)_t}(e) \downarrow \right]$$

from (6.7) decidable in  $Y'$ . Thus

$$(A_i)' \leq_T (S_i \oplus \{i\}) \oplus Y' \equiv_T S_i. \quad \square$$

Claim 6.1.7 and Claim 6.1.8 together now prove (6.2). □

LEMMA 6.1.9. *Let  $\{a_k\}$  be a computable sequence of partial computable functions. Then there exists a computable sequence  $\{x_k\}$  such that for all  $Y \subseteq \omega$ ,*

$$\begin{aligned} \Psi_{a_0(x)}^Y &= \Psi_{x_0}^Y \\ \Psi_{a_1(x)}^Y &= \Psi_{x_1}^Y \\ &\vdots \end{aligned}$$

where  $x$  is such that  $\varphi_x(k) = x_k$  for all  $k$ .

PROOF. Let  $\{a_k\}$  be a computable sequence of p.c. functions. Define a partial computable function  $f$  by

$$f(z, k) := a_k(z).$$

By the *s-m-n* Theorem, there is a computable function  $g$  such that

$$\varphi_{g(z)}(k) = f(z, k).$$

By the Recursion Theorem, there exists a fixed point  $x$  satisfying

$$\varphi_x(k) = \varphi_{g(x)}(k) = f(x, k).$$

Thus

$$a_k(x) = f(x, k) = \varphi_x(k) := x_k.$$

It follows that for all  $Y \subseteq \omega$ ,

$$\Psi_{a_k(x)}^Y = \Psi_{x_k}^Y. \quad \square$$

## 6.2. Properties of $\text{MIN}^{\text{T}(\omega)}$

DEFINITION 6.2.1. Let  $f$  and  $g$  be total functions, and let  $A = \{a_0 < a_1 < \dots\}$  be a set.

- (I)  $f$  majorizes  $g$  if  $(\forall n) [f(n) > g(n)]$ .
- (II)  $f$  dominates  $g$  if  $(\forall^\infty n) [f(n) > g(n)]$ , where  $\forall^\infty$  means “for all but finitely many.”
- (III) The function  $p_A(n) := a_n$  is called the *principal function* of  $A$ .
- (IV) A function  $f$  majorizes a set  $A$  if  $(\forall n) [f(n) > p_A(n)]$ .
- (V) Let  $\mathbf{a}$  be a Turing degree. A set  $A$  is  *$\mathbf{a}$ -dominated* (resp.  *$\mathbf{a}$ -majorized*) if there exists an  $\mathbf{a}$ -computable function  $f$  which dominates (resp. majorizes)  $A$ .

THEOREM 6.2.2 ([30], [42]). *An infinite set  $A$  is hyperimmune iff  $A$  is not  $\mathbf{0}$ -dominated.*

We obtain the following paradoxical result:

THEOREM 6.2.3 (peak hierarchy).  $\text{MIN}^{\text{T}(\omega)}$

- (I) *is infinite,*
- (II) *contains no infinite arithmetic sets, and*
- (III) *is not hyperimmune.*

PROOF. (I). Theorem 6.1.1 provides a denumerable list of distinct  $\equiv_{\text{T}(\omega)}$  classes.  $\square$

(II). Follows from Corollary 3.2.2, because  $\text{MIN}^{\text{T}(\omega)} \subseteq \text{MIN}^{\text{T}(n)}$  for every  $n$ .  $\square$

(III). We verify that  $\text{MIN}^{\text{T}(\omega)}$  gets majorized. Let  $\{x_k\}$  be as in Theorem 6.1.1. Then for all  $n$  and  $i \neq j$ ,

$$W_i \not\equiv_{\text{T}(n)} W_j.$$

Without loss of generality,  $x_0 < x_1 < \dots$  since  $\{x_k\}$  is computable. Define the computable function

$$f(0) := x_1$$

$$f(n+1) := x_{[2f(n)]},$$

and let  $p$  be the principal function of  $\text{MIN}^{\text{T}(\omega)}$ . Note that  $f(0) > 0 = p(0)$ , and assume for the purposes of induction that  $f(n) > p(n)$ . Note that

$$p(n) \leq x_{p(n)} < x_{f(n)} < x_{f(n)+1} < \dots < x_{2f(n)} = f(n+1),$$

so at least  $f(n)$   $x_k$ 's lie strictly between  $p(n)$  and  $f(n+1)$ , namely

$$\{x_{f(n)}, x_{f(n)+1}, \dots, x_{2f(n)-1}\}.$$

Hence, at least  $f(n)$  distinct  $\equiv_{\text{T}(\omega)}$ -equivalence classes are represented by indices strictly between  $p(n)$  and  $f(n+1)$ . Since less than  $f(n)$  classes are represented in indices up to  $p(n)$ , there necessarily must be a new  $\equiv_{\text{T}(\omega)}$ -class introduced strictly between  $p(n)$  and  $f(n+1)$ . This forces  $p(n+1) < f(n+1)$ . Hence  $f$  majorizes  $\text{MIN}^{\text{T}(\omega)}$ . The result now follows immediately from Theorem 6.2.2. □

□

Consequently, the other MIN-sets in this thesis share properties (i) and (iii):

**COROLLARY 6.2.4.** *Every set containing  $\text{MIN}^{\text{T}(\omega)}$  is infinite but not hyperimmune.*

**REMARK.** If we only wanted to prove that  $\text{MIN}^{\text{T}}$  is  $\mathbf{0}$ -dominated, we could have simplified the proof of Theorem 6.2.3 by omitting Lemmas 6.1.5 and 6.1.9, thereby avoiding infinite injury.

$\emptyset^{(\omega)}$  is another familiar set which is hyperarithmetical and  $\mathbf{0}$ -dominated. However, unlike  $\text{MIN}^{\text{T}(\omega)}$ ,  $\emptyset^{(\omega)}$  contains a copy of  $\emptyset'$ . This means that  $\emptyset^{(\omega)}$  is not at all immune.

### 6.3. A strange “cutting” set

Lusin once constructed a set of reals which neither contains nor is disjoint from any perfect set [27], [28, Theorem 2.25]. By modifying Lusin’s construction and gently expanding  $\text{MIN}^{\text{T}(\omega)}$ , we obtain an analogous construction for the arithmetic hierarchy which is remarkably well-behaved.

COROLLARY 6.3.1. *There exists a set  $X \supseteq \text{MIN}^{\text{T}(\omega)}$  such that  $X$ :*

- (I) *contains no infinite arithmetic sets,*
- (II) *is not disjoint from any infinite arithmetic set, and*
- (III) *is  $\mathbf{0}$ -majorized.*

PROOF. We simultaneously enumerate disjoint sets  $X$  and  $Y$  so that both  $X$  and  $Y$  intersect every infinite arithmetic set. Let  $A_0, A_1, A_2, \dots$  be an enumeration of the arithmetic sets. Let

$$X_0 := \text{MIN}^{\text{T}(\omega)},$$

$$Y_0 := \emptyset.$$

Now assume that  $X_n$  and  $Y_n$  have already been constructed, with

$$X_n =^* \text{MIN}^{\text{T}(\omega)},$$

$$Y_n =^* \emptyset,$$

$X_n \cap Y_n = \emptyset$ , and for all  $k < n$ ,

$$X_n \cap A_k \neq \emptyset,$$

$$Y_n \cap A_k \neq \emptyset.$$

Since  $\text{MIN}^{\text{T}(\omega)}$  does not contain any infinite arithmetic sets (Theorem 6.2.3), it follows that  $A_n \cap \overline{\text{MIN}^{\text{T}(\omega)}}$  is infinite, which means that

$$R_n := (A_n \cap \overline{X_n}) - Y_n$$



is infinite. Enumerate the least element in  $R_n$  into  $X_n$ , and call this expanded set  $X_{n+1}$ .

Now

$$S_n := (A_n \cap \overline{Y_n}) - X_{n+1}$$

is also infinite. Enumerate the least element in  $S_n$  into  $Y_n$ . Finally, let

$$X := \bigcup_{n \in \omega} X_n,$$

$$Y := \bigcup_{n \in \omega} Y_n.$$

This concludes the construction.

It is clear that  $X_{n+1}$  satisfies the same inductive hypotheses as  $X_n$ . Consequently  $X$  and  $Y$  are disjoint, because otherwise a common element would have been introduced after finitely many stages. Furthermore,

$$X_n \cap A_n \neq \emptyset,$$

$$Y_n \cap A_n \neq \emptyset,$$

for every  $n$ , which proves (ii) and (i). Finally, (iii) follows because  $\text{MIN}^{\text{T}(\omega)}$  is an infinite subset of  $X$ , by Corollary 6.2.4 and Theorem 6.2.2.  $\square$

**REMARK.** It is straightforward to construct a set which satisfies properties (i) and (ii) from Corollary 6.3.1 but not (iii). In the construction of Corollary 6.3.1, one can achieve this by first replacing  $X_0 = \text{MIN}^{\text{T}(\omega)}$  with  $X_0 = \emptyset$ , and then, during enumeration into  $X_n$ , by choosing an element so that  $X(n) > A_n(n)$  rather than just enumerating the least element in  $R_n$ .

## APPENDIX A

### Open problems

#### A.1. Truth table degrees

Meyer's original question from 1972 remains open: is  $\text{f-MIN} \equiv_{\text{tt}} \emptyset''$  [31]? A reduction  $\text{f-MIN} \geq_{\text{bT}} \emptyset''$  would suffice to show  $\text{f-MIN} \equiv_{\text{tt}} \emptyset''$ , if it were the case that  $\emptyset' \leq_{\text{tt}} \text{MIN}$  [39, Section 8]. Similarly, Schaefer asks, is  $\text{fR} \equiv_{\text{tt}} \emptyset'$  [39]? The fact that we know  $\text{RAND}_\varphi \equiv_{\text{tt}} \emptyset'$  for any Kolmogorov numbering  $\varphi$  (Theorem 1.4.4(iii)) but we don't know the truth-table degree of its cousin  $\text{fR}$  indicates that there is still much to learn about similarities between randomness and minimal indices.

#### A.2. Is $\text{MIN}^{\text{T}} \equiv_{\text{T}} \emptyset''''$ ?

We conjecture that Corollary 5.2.3 does not hold for arbitrary Gödel numberings. In particular, we conjecture that that Corollary 2.4.1 is optimal in the following sense:

CONJECTURE A.2.1. *Let  $n \geq 0$ .*

- (I) *There exists a Gödel numbering  $\varphi$  such that  $\text{MIN}_\varphi^* \not\equiv_{\text{T}} \emptyset'$ .*
- (II) *There exists a Gödel numbering  $\varphi$  such that  $\text{MIN}_\varphi^{\text{m}} \oplus \emptyset' \not\equiv_{\text{T}} \emptyset''$ .*
- (III) *There exists a Gödel numbering  $\varphi$  such that  $\text{MIN}_\varphi^{\text{T}^{(n)}} \oplus \emptyset^{(n+1)} \not\equiv_{\text{T}} \emptyset^{(n+2)}$ .*

Even showing  $\text{MIN}_\varphi^{\text{m}} \not\equiv_{\text{T}} \emptyset''$  or  $\text{MIN}_\varphi^{\text{T}} \not\equiv_{\text{T}} \emptyset''$  for some Gödel numbering  $\varphi$  would be enough to resolve the Turing degree of  $\text{MIN}^{\text{m}}$  or  $\text{MIN}^{\text{T}}$ .

All of the initial information in a  $=^*$  set can be faulty [39], so intuitively one needs a halting set oracle to extract useful information from  $\text{MIN}^*$ . Similarly,  $\text{MIN}^{\text{m}}$  and  $\text{MIN}^{\text{T}}$  presume knowledge of total functions, making  $\emptyset'' \equiv_{\text{T}} \text{TOT}$  undecidable relative to these sets. The difficulty in constructing the necessary numberings for Conjecture A.2.1 is revealed by

considering a simpler problem where we try to find *any*  $A \in \Sigma_4$  satisfying:

$$A \oplus \emptyset'' \equiv_{\text{T}} \emptyset'''' ,$$

$$A \not\equiv_{\text{T}} \emptyset'' .$$

A set  $A$  with these properties can be constructed using two iterations of the Sacks Jump Theorem (Theorem 2.3.1), however this is already a nontrivial, infinite injury construction. Making this construction work with  $A = \text{MIN}_{\varphi}^{\text{T}}$  for some *Gödel* numbering  $\varphi$  can only be more complicated.

If Conjecture A.2.1 holds, then spectral sets are (possibly the first) natural examples of sets which are not Turing equivalent to any of the canonical  $\Sigma_n$ -complete sets. If Conjecture A.2.1 fails, then spectral sets are a new and remarkable characterizations of the Turing degrees  $\mathbf{0}'$ ,  $\mathbf{0}''$ ,  $\mathbf{0}'''$ ,  $\dots$ .

One approach to solving the  $\text{MIN}^*$  problem is to look first at the related problem of  $\text{MIN}^{\text{m}}$ . This approach is promising because it has not received much attention. It is also promising for mathematical reasons. We now sing praises of  $\text{MIN}^{\text{m}}$ . If indeed  $\text{MIN}^{\text{m}} \oplus \emptyset'' \equiv_{\text{T}} \emptyset''''$  and  $\text{MIN}^* \oplus \emptyset' \equiv_{\text{T}} \emptyset''''$  are both optimal results (in the sense of Conjecture A.2.1), then it seems easy to find a numbering  $\varphi$  in which  $\text{MIN}_{\varphi}^{\text{m}}$  avoids (merely) the cone of degrees above  $\emptyset''$ , when compared to the (daunting) task of forcing  $\text{MIN}_{\varphi}^*$  to avoid the the cone above  $\emptyset'$ . The second reason to take up  $\text{MIN}^{\text{m}}$  is for the elegance and brevity of results given in this thesis which are unique to  $\text{MIN}^{\text{m}}$ . The Generalized Fixed Point Theorem 3.2.1(ii) immediately gives optimal immunity for  $\text{MIN}^{\text{m}}$  (see  $\Pi_3$ -Separation Theorem 3.1.3), and our purported optimal result for the Turing degree of  $\text{MIN}^{\text{m}}$ , Lemma 2.1.1(iii), follows directly from the  $\equiv_{\text{m}}$ -Completeness Criterion 3.3.4(i). Finally, we have a satisfying proof of the fact that  $\text{MIN}_{\psi}^{\text{m}} \equiv_{\text{tt}} \emptyset''''$  for some Kolmogorov numbering  $\psi$  (Theorem 5.2.3). This same argument finds only a Turing degree for  $\text{MIN}_{\psi}^{\text{T}^{(n)}}$ .

### A.3. MIN vs. f-MIN

We know that  $\text{MIN} \equiv_{\text{T}} \emptyset'' \equiv_{\text{T}} \text{f-MIN}$ . What can be said about stronger reductions? For example is it true, in general, that  $\text{MIN}_{\varphi} \equiv_{\text{btt}} \text{f-MIN}_{\varphi}$ ? We know that there exists a Kolmogorov numbering  $\psi$  such that  $\text{MIN}_{\psi} \equiv_{\text{tt}} \text{f-MIN}_{\psi}$  (Theorem 5.2.3(ii)), and for any numbering  $\varphi$ , there is a Gödel numbering  $\psi$  such that  $\text{MIN}_{\varphi} \not\equiv_{\text{btt}} \text{f-MIN}_{\psi}$  (by Theorem 1.4.6). But do there exist any Gödel numberings  $\varphi$  and  $\psi$  such that  $\text{MIN}_{\varphi} \equiv_{\text{btt}} \text{f-MIN}_{\psi}$ ? Given a Gödel numbering  $\varphi$ , does there always exist a Gödel numbering  $\psi$  such that  $\text{MIN}_{\varphi} \equiv_{\text{tt}} \text{f-MIN}_{\psi}$ ?

### A.4. $\Pi_n$ -immunity

In Chapter 3, we prove optimal results with respect to  $\Sigma_n$ -immunity. What about  $\Pi_n$ -immunity? In particular, is  $\text{MIN}$   $\Pi_1$ -immune? Is  $\text{MIN}^*$   $\Pi_2$ -immune? Is  $\text{MIN}^{\text{T}(n)}$   $\Pi_{n+3}$ -immune? We do have an optimal  $\Pi_n$ -immunity result for  $\text{MIN}^{\text{m}}$  ( $\Pi_3$ -Separation Theorem 3.1.3), however this argument does not generalize to other spectral sets.

Is there a weak form of the Arslanov Completeness Criterion which is equivalent to immunity for  $\text{MIN}$ -sets? Does  $\text{MIN}$  contain a  $\Delta_2$  spectral set, as  $\text{f-MIN}$  does? Does there exist a direct reduction from  $\text{f-MIN}$  to  $\text{fR}$  which does not go through the halting set?

### A.5. A question of Friedman

$\text{MIN}^*$  and  $\text{MIN}^{\text{T}}$  are not the only sets with short descriptions whose Turing degree remains elusive. Consider the set

$$A := \{n : (\exists j < n) [\max(W_j) = n]\}.$$

Friedman asks [38], what is the complexity of  $A$ ? A straightforward argument shows that  $A \leq_{\text{T}} \emptyset'$ , however it is surprising that we do not know whether or not  $A \in \Sigma_1$ ,  $A \in \Pi_1$ , or  $A' \equiv_{\text{T}} \emptyset'$ .

On a related note, we visit a set reminiscent of the Kolmogorov random strings. For any numbering  $\varphi$ , let

$$\text{sR}_{\varphi} := \{x : (\forall j < x) [\varphi_j(0) \neq x]\}.$$

What are the possible degrees for  $\text{sR}_\varphi$  when  $\varphi$  is a Kolmogorov numbering? Can  $\text{sR}_\varphi$  be finite?

### A.6. Intermediate degrees

PROPOSITION A.6.1. *Let  $A_0, A_1, \dots$  be a sequence of sets, and let  $I$  be a computable set. Then*

$$\bigoplus_{i \in I} (A_i)' \leq_T \left( \bigoplus_{i \in I} A_i \right)'$$

PROOF. For all  $k \in I$ ,

$$A_k \leq_T \bigoplus_{i \in I} A_i.$$

It then follows from the Jump Theorem [44] that

$$(A_k)' \leq_T \left( \bigoplus_{i \in I} A_i \right)',$$

so

$$\bigoplus_{i \in I} (A_i)' \leq_T \left( \bigoplus_{i \in I} A_i \right)' \oplus I \equiv_T \left( \bigoplus_{i \in I} A_i \right)'. \quad \square$$

In light of Theorem 6.1.1, where we found a computable sequence  $\{x_k\}$  satisfying for all  $n$  and  $i$

$$(W_{x_i})^{(n)} \not\leq_T \bigoplus_{j \neq i} (W_{x_j})^{(n)},$$

Proposition A.6.1 leaves us with the burning question of whether or not there is a computable sequence  $\{z_k\}$  satisfying for all  $n$  and  $i$ ,

$$(W_{z_i})^{(n)} \not\leq_T \left( \bigoplus_{j \neq i} W_{z_j} \right)^{(n)}.$$

It would be sufficient to show that for any computable  $I \subseteq \omega$ , the indices  $\{a_i\}$  from Lemma 6.1.5 satisfy the additional condition,

$$\left( \bigoplus_{i \in I} \left( W_{a_i(s)}^Y \oplus Y \right) \right)' \equiv_T \bigoplus_{i \in I} \left( W_{s_i}^{Y'} \oplus Y' \right).$$

because then we can pull the jump operator outside of the join, as would be needed in the proof of Theorem 6.1.1.

### A.7. Other complexity measures and variants

What can be said about approximability, autoreducibility, and size-minimal indices of spectral sets? These questions (and some answers to them) appear in Schaefer's paper [39].

DEFINITION A.7.1. A set  $A$  is *autoreducible* if, for all  $x$ , one can decide whether  $x \in A$  by querying only elements in  $A - \{x\}$ .

Schaefer showed that f-MIN is autoreducible [39]. Are there other spectral sets which are autoreducible?

DEFINITION A.7.2. (I) A set  $A$  is  $(1, k)$ -*computable* if there exists a computable function  $f$  such that for every set  $X \subseteq \omega$  with  $|X| = k$ , there is some  $x \in X$  such that  $f(x) = \chi_A(x)$ .

(II) A set  $A$  is *approximable* if  $A$  is  $(1, k)$ -computable for some  $k$ .

f-MIN is not  $(1, 2)$ -computable, and there exists a Gödel numbering  $\varphi$  such that  $\text{f-MIN}_\varphi$  is not approximable [39]. Does there exist a Gödel numbering  $\psi$  such that  $\text{MIN}_\psi$  is approximable?

DEFINITION A.7.3. For a Gödel numbering  $\varphi$  and a (total) size function  $s$ , define

$$\text{f-MIN}_{\varphi, s} = \{e : (\forall i) [s(i) < s(e) \implies \varphi_i \neq \varphi_e]\},$$

to be the set of *size-minimal indices* of  $\varphi$ .

In contrast to the MIN-sets in this paper, there is a computable size function  $s$  (independent of the Gödel numbering  $\varphi$ ) such that  $\text{f-MIN}_{\varphi, s}$  is hyperimmune [39]. It is an open problem to determine whether  $\text{f-MIN}_{\varphi, s} \equiv_{\text{T}} \emptyset''$  whenever  $\varphi$  is a Gödel numbering and  $s$  is computable.

### A.8. Almost thickness

Ken Harris made the following observation, generalizing a familiar representation for  $\Sigma_3$  sets:

THEOREM A.8.1 ( $\Sigma_n$ -Representation, Harris [16]). *If  $A \in \Sigma_n$ , then there is a computable function  $g$  such that*

$$\begin{aligned} x \in A &\iff (\forall^\infty y_1) (\forall y_2) (\forall^\infty y_3) \dots [W_{g(x,\bar{y})} = \omega], \\ x \in \bar{A} &\iff (\forall y_1) (\forall^\infty y_2) (\forall y_3) \dots [W_{g(x,\bar{y})} \text{ is finite}], \end{aligned}$$

where  $\dots$  denotes the remaining of the  $n - 3$  quantifiers.

Similarly, if  $A \in \Pi_n$ , then there is a computable function  $g$  such that

$$\begin{aligned} x \in A &\iff (\forall y_1) (\forall^\infty y_2) (\forall y_3) \dots [W_{g(x,\bar{y})} = \omega], \\ x \in \bar{A} &\iff (\forall^\infty y_1) (\forall y_2) (\forall^\infty y_3) \dots [W_{g(x,\bar{y})} \text{ is finite}]. \end{aligned}$$

Here  $\forall^\infty x$  means “for all but finitely many  $x$ .” This makes a.e.- a natural complement to the Thick- operator:

DEFINITION A.8.2. Let  $\equiv_\alpha$  be an equivalence relation on sets. Then

$$A \equiv_{\text{a.e.-}\equiv_\alpha} B \iff (\forall^\infty n) [A^{[n]} \equiv_\alpha B^{[n]}].$$

Using the Representation Theorem A.8.1 and the methods from Section 4.2, we arrive at the following observations:.

- PROPOSITION A.8.3. (I)  $\text{MIN}^{\text{a.e.-}=} \in \Pi_3 - \Sigma_3$ ,  
 (II)  $\text{MIN}^{\text{a.e.-}*} \in \Pi_5 - \Sigma_5$ ,  
 (III)  $\text{MIN}^{\text{a.e.-m}} \in \Pi_5 - \Sigma_5$ ,  
 (IV)  $\text{MIN}^{\text{a.e.-T}^{(n)}} \in \Pi_{n+6} - \Sigma_{n+6}$ .

We remark that a.e.-\* is an example of an equivalence relation for which

$$\text{MIN}^{\text{T}^{(n)}} \not\subseteq \text{MIN}^{\text{a.e.-}*} \not\subseteq \text{MIN}^{\text{T}^{(n+1)}}$$

for all  $n$ , however  $\text{MIN}^{\text{a.e.-}}$  sets are not hyperimmune. What can be said about the immunity of  $\text{MIN}^{\text{a.e.-}}$ -sets?

PROPOSITION A.8.4.  $\text{MIN}^{\text{a.e.-}=}$  is not  $\Sigma_3$ -immune.

PROOF. Let

$$\omega\text{-INF} := \{e : (\forall N) (\exists y \geq N) (\exists x) [x \in W_e^{[y]}}\}.$$

Similar to the proof of Theorem 3.1.3(ii), let

$$P_k := \{\langle x, y \rangle : y \text{ is a } k^{\text{th}} \text{ prime power}\},$$

and set

$$A_k := \left\{ e : (\exists N) (\forall y \geq N) \left[ W_e^{[y]} \subseteq P_k^{[y]} \right] \right\} \cap \omega\text{-INF},$$

$$A := \{e : (\exists k) (\forall j < e) [e \in A_k \ \& \ j \notin A_k]\}.$$

Now  $A_k \in \Delta_3$ , because  $\omega\text{-INF} \in \Pi_2$  and

$$\left\{ e : W_e^{[y]} \subseteq P_k^{[y]} \right\}$$

is in  $\Pi_1$ . Hence  $A \in \Sigma_3$ .  $A$  is infinite because  $A$  contains a member from each  $A_k$ , and the  $A_k$ 's are pairwise disjoint. Finally,  $A \subseteq \text{MIN}^{\text{a.e.-}}$  because  $e \in A$  implies  $e \in A_k$  for some  $k$ , and any set which is equal to  $W_e$  must eventually be contained in  $A_k$  (for sufficiently high rows). So  $\text{MIN}^{\text{a.e.-}}$  contains an infinite,  $\Sigma_3$  subset.  $\square$

CONJECTURE A.8.5. For  $n \geq 0$ ,

- (I)  $\text{MIN}^{\text{a.e.-}^*}$  is not  $\Sigma_4$ -immune.
- (II)  $\text{MIN}^{\text{a.e.-}^{\text{m}}}$  is not  $\Sigma_5$ -immune.
- (III)  $\text{MIN}^{\text{a.e.-}^{\text{T}^{(n)}}}$  is not  $\Sigma_{n+5}$ -immune.

While it is clear that Thick- and a.e.- can be combined iteratively to obtain more equivalence relations (and hence more open questions), the author currently considers this direction somewhat esoteric.

### A.9. Ershov hierarchy

DEFINITION A.9.1. A set  $X$  is *d.c.e.* (*difference of c.e. sets*) if there exists c.e. sets  $A$  and  $B$  such that  $X = A - B$ . More generally,  $X$  is *n-c.e.* if there exists a sequence of c.e. sets  $A_1, A_2, \dots, A_n$  such that

$$X = A_1 - A_2 \cup A_3 - \dots \cup A_n,$$



where order of operations in left to right.

The d.c.e. sets can be enumerated as pairs of indices for c.e. sets, so it is natural to consider minimal indices for d.c.e. sets [16]. Let  $V_0, V_1, \dots$  be an enumeration of the d.c.e. sets, and let

$$2\text{-MIN} := \{e : (\forall j < e) [V_j \neq V_e]\}.$$

What is the Turing degree of 2-MIN? A similar question can be asked about the  $n$ -c.e. sets, whose indices are also enumerable.

### A.10. Polynomial-time

Computational complexity intersects nontrivially with Kolmogorov complexity [11], so it is natural to ask what applications computational complexity has in the generalized world of minimal indices (and vice-versa). We examine, for example, a familiar notion from resource-bounded complexity within the context of minimal indices.

DEFINITION A.10.1. Let  $A$  and  $B$  be sets. We write  $A \leq_p B$  if there exists a computable algorithm  $f$ , running in time polynomial in input length, which satisfies

$$x \in A \iff f(x) \in B$$

for all  $x$ . If  $A \leq_p B$  and  $B \leq_p A$ , we write  $A \equiv_p B$ . Now

$$\text{MIN}^p := \{e : (\forall j < e) [W_j \not\equiv_p W_e]\}.$$

It is immediate that  $\text{MIN}^p$  is  $\Sigma_2$ -immune, as  $\text{MIN}^* \supseteq \text{MIN}^p$  (Theorem 3.1.1(iii)), however the Turing degree of  $\text{MIN}^p$  is not known.

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## Education

### *Graduate:*

- Jan. 2007 • PhD Mathematics, *Indiana University*.
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- 2006 • Certificate in Logic, Language, and Computation, *Indiana University*.
- 2004 • Doctoral Minor composition, *Indiana University School of Music*.
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### *Undergraduate:*

- 2000 • BA, *Pomona College*.
  - Mathematics and Classical Guitar Performance.
- Fall 1998 • Attended *Budapest Semesters in Mathematics*.
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### *Early years:*

- 1978–1996 • Born & raised in Atlanta, Georgia (USA).

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- Hungarian, Hebrew, Spanish.

## Grants and Awards

- NSF summer grant in quantum computing:
  - PCP and quantum computing with Zhenghan Wang at IU. (2003)
- US Fulbright Fellowship to Hungary:
  - To study descriptive set theory at Eötvös Loránd University with Péter Komjáth. (2000-2001)
- William Blanchard Prize in composition:
  - from Pomona College. (2000)
- Pomona College Scholar:
  - (1997,1999)
- Scholarships from the American Recorder Society and Canto Antiguo:
  - To attend Canto Antiguo workshop. (1998)
- Summer Undergraduate Research Experience (SURP) grant:
  - Summer research grant for “archaeometrics” project, a study of statistical models of archaeology, at Pomona College and in Israel. Joint work with Adolfo Rumbos and Donald Bentley. (1998)

## Employment and Other Activities

- 2005-2006
  - Teaching Assistant at University of Chicago.  
*Courses taught:*  
Honors Introduction to Computer Science I–Scheme (assisting). Introduction to Computer Science II–C (assisting).
- 2005
  - Attended theory of computing seminars at Princeton University, and at Institute for Advanced Study.
- 2001–2004
  - Associate Instructor at Indiana University.  
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Basic Algebra, Business Calculus, Calculus I & II (recitations), Finite Mathematics (assisting), and WeB-WorK (web-based homework system for undergraduates).
- 2000–2001
  - US Fulbright Fellowship to Hungary.
- Summer 1999
  - Joint Program in Survey Methodology (JPSM) Junior Fellow at US Census Bureau.
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  - SURP research at Pomona College in Archaeometrics.
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  - Product distributor for Coca-Cola at 1996 Olympic Games.

## Research

- *Research interests:*  
computability theory, theory of computing, logic and foundations of mathematics.



## Research (continued)

### *Publications*

- 2007 • *PhD thesis:*  
*Noncomputable Spectral Sets* (draft). PhD thesis, Indiana University, 2007.
- 2005 • P.O. Vontobel, R. Smarandache, N. Kiyavash, J. Teutsch, and D. Vukobratovic. *On the minimal pseudo-codewords of codes from finite geometries*. *Proc. IEEE Intern. Symp on Inform Theory 2005*, Adelaide, Australia, p. 980–984, Sept. 4–9, 2005. [arxiv]
- 2000 • *Undergraduate thesis:*  
*Borel Determinacy and Other Wonders of Infinite Games* (not published).

### *Referee work:*

- 2005 • IEEE Transactions on Information Theory.

### *Conferences & Presentations:*

- 2006 • University of Chicago Logic Seminar (Oct. 6).  
*Presentation: Shortest programs*
- 2005 • University of Chicago Theory Seminar (Nov. 16 & Nov. 30).  
*Presentation:  $O\sqrt{\log n}$  approximation for Sparsest Cut [ARV]*
- Proof Theory Course (June 6–June 17),  
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- 2004 • Indiana University Logic Seminar (Nov. 10).  
*Presentation: A bit of PCP*
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## Research (continued)

- 2003 • International Conference on Computability and Complexity in Analysis,  
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## Solo bicycle tours

- July 2006 • Sister Bay–Postville–Nauvoo, 1200 miles.
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  - July 2003 • The Southeastern US, 3000 miles.
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  - 1993 • Northern California, 900 miles.
  - 1992 • Blue Ridge Parkway, 469 miles.