BIFURCATION AND STABILITY PROBLEMS IN FLUID DYNAMICS

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To my family and my teachers

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Abstract

My thesis is to study the bifurcation, stability and phase transitions of incompressible fluid flows. Bifurcation is a versatile methodology to trace solutions of physical problems along with the system parameter and to investigate their structure. The study is oriented toward a nonlinear dynamic theory for the underlying physical problems consisting of 1)complete bifurcation when the system parameter crosses some critical values, 2)asymptotic stability of bifurcated solutions and 3)the structure/pattern of the bifurcated solutions and phase transitions in the physical spaces. The study in the first two directions is related to application of a new bifurcation theory, called *attractor bifurcation*, which was developed by T. Ma and S. Wang. The third direction of the study is related to geometric study of fluid flows and includes structural stability theory.

In Chapter 2 and 3, we present bifurcation and stability analysis on complex Ginzburg-Landau equation which is followed by the infinite Prandtl number convection in Chapter 4 and 5. It is proved that there are two different types of bifurcations, super-critical and sub-critical which depend on the degree of nonlinear dissipative terms. In the last Chapter we present the structure of bifurcated solutions of 2D infinite Prandtl number convection. Structural stability in a physical space is also explored and the roll structure of the solutions is obtained as many experiments have suggested.

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Introduction

My graduate thesis focuses on the bifurcation, stability and phase transitions of incompressible fluid flows. Bifurcation is a versatile methodology to determine how qualitatively different flow regimes appear and disappear as a control parameter varies, and to explore the theoretical limits of predicting these flow regimes. The study is oriented toward a nonlinear dynamic theory for the underlying physical problems consisting of

1)complete bifurcation when the system parameter crosses some critical values,

2)asymptotic stability of bifurcated solutions and

3)the structure/pattern of the bifurcated solutions and phase transitions in physical spaces.

The study in the first two directions is related to application of a new bifurcation theory, called *attractor bifurcation*, which was developed by T. Ma and S. Wang [**24, 25**]. The third direction of the study is related to geometric study of fluid flows and includes structural stability theory.

The main ingredients of the theory include the attractor bifurcation theory based on the attractor bifurcation concept, new strategies for the Lyapunov-Schmidt and center manifold reductions, for saddle-node bifurcation and hysteresis and for transcritical bifurcation. The Center manifold and Lyapunov-Schmidt reductions of the original PDEs should be carried out to apply the general theory to specific problems.

There are a few important features of the attractor bifurcation and asymptotic stability is one of them. As an attractor itself, the bifurcated attractor has a basin of attraction and consequently is the right object to describe local transition. Moreover, detailed classification of the solutions in the bifurcated attractor enables us to study the asymptotic stability of the bifurcated attractor. It enables us to study the stability of different solutions in the bifurcated attractor as well, by providing a complete understanding of the transitions of the physical system as the system parameter changes. In addition, steady state bifurcation theories provide only part of information of the problem, particularly for the stability issue since fluid flows are normally time dependent. Therefore, the right notion of asymptotic stability should be best described by the attractor near, but excluding, the trivial state.

The geometric theory of 2D-incompressible flows was initiated by T. Ma and S. Wang to study the structure and its stability and transitions of 2D-incompressible fluid flows in physical space. This program of study consists of research in directions

1) the study of the structure and its transitions/evolutions of divergence free vector fields, and

2) the study of the structure and its transitions of velocity fields for 2D-incompressible fluid flows governed by the Navier-Stokes equations or the Euler equations.

The study of the first direction is more kinematic in nature and the results and methods developed can naturally be applied to other problems of mathematical physics involving divergence free vector fields. In fluid dynamics context, the study in the second direction involves specific connections between the solutions of the Navier-Stokes or the Euler equations and flow structure in physical space. In other words, this area of research links the kinematics to the dynamics of fluid flows.

A Large part of my work has been dedicated to the Rayleigh-Bénard convection with the infinite Prandtl number and the complex Ginzburg-Landau equation and are examined in the following five chapters.

Chapter 1 is an introduction to attractor bifurcation theory and the center manifold reduction which are main ingredients for the remaining chapters.

Chapter 2 is dedicated to bifurcation for complex Ginzburg-Landau equation. In the collaboration with T. Ma and S. Wang [**21**], we obtained the following results:

For the CGLE with the Dirichlet boundary condition, let λ_1 be the first eigenvalue of the elliptic operator $-\Delta$. Then we have

1. If $\rho \leq \varepsilon_0\lambda_1$, the trivial solution $u = 0$ is globally asymptotically stable. The global attractor of the CGLE consists exactly of the trivial steady state solution $u = 0$.

2. As ρ crosses $\varepsilon_0\lambda_1$, i.e., there exists an $\epsilon > 0$ such that for any $\varepsilon_0\lambda_1 < \rho < \varepsilon_0\lambda_1+\epsilon$, the CGLE bifurcates from the trivial solution to an attractor Σ_ρ.

In this case, the global attractor $\mathbf{A}_\rho$ is 2D, consisting of Σ_ρ, $u = 0$ and the connecting orbits between them.

For the CGLE equipped with the periodic boundary condition, similar results can be obtained. Moreover, bifurcation from any eigenvalue of the Laplacian can also be obtained as for the first eigenvalue. The complete structure of the global attractor for the bifurcations from the first eigenvalue is obtained, while no such information is available for bifurcations from the rest eigenvalues.

In Chapter 3, we study the bifurcation a bifurcation and stability for the generalized complex Ginzburg-Landau equation [**29**]. In this case We investigate two different modes of the CGLE. We study the first mode of the CGLE which has only cubic unstable nonlinear term and later we also study the second mode of the CGLE which has both cubic and quintic nonlinear terms. The solutions considered in the cubic CGLE bifurcate from the trivial steady state supercritically in some parameter range. However, for the cubic-quintic CGLE, solutions bifurcate from the trivial steady state subcritically. Due to the global attractor, we obtained a saddle node bifurcation point λ_c. We also study the steady state bifurcation of the CGLE and verify that the bifurcated invariant sets Σ_k contain steady state solutions.

Chapter 4 and 5 are devoted to the Rayleigh-Bénard convection which, in particular, has the infinite Prandtl number [**27, 28**]. In Chapter 4, We prove that the problem bifurcates from the trivial solution to an attractor Σ_R when the Rayleigh number R crosses the first critical Rayleigh number R_c. As a special case, we also prove another result which corresponds to the classical pitchfork bifurcation, that this bifurcated attractor Σ_R consists of only two stable steady states when the first eigenvalue R_1 is simple. In Chapter 5, we examines the bifurcation and structure of the bifurcated solutions of the two-dimensional infinite Prandtl number convection problem. We prove in this chapter that the bifurcated attractor Σ_R consists of only one cycle of steady state solutions and that it is homeomorphic to S^1. By thoroughly investigating the structure and transitions the solutions of the infinite Prandtl number convection problem in physical space, we confirm that the bifurcated solutions are indeed structurally stable. In turn, this will corroborate and justify the suggested results with the physical findings about the presence of the roll structure.

CHAPTER 1

Dynamic Bifurcation and Center Manifold Theory

1. Dynamic Bifurcation Theory

1.1. Attractor bifurcation. The purpose of this section is to introduce a new notion of bifurcation called *attractor bifurcation*, developed by T. Ma and W. Wang [**24, 25**].

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. Consider the following nonlinear evolution equations

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda \quad \text{a sectorial operator,} \\ A : H_1 \to H \quad \text{a linear homeomorphism,} \\ B_\lambda : H_1 \to H \text{ the parameterized linear compact operators.} \end{cases} \tag{1.3}$$

We can see that L_λ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t\geq 0}$ and then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_2 < \alpha_1$, and $H_0 = H$.

We now assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \to H$ for some $0 \leq \alpha < 1$ are a family of parameterized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}$, such that

$$G(u,\lambda) = o(\|u\|_{H_\alpha}), \quad \forall\ \lambda \in \mathbb{R}. \tag{1.4}$$

For the linear operator A we assume that there exists a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}$ and an eigenvector sequence $\{e_k, h_k\} \subset H_1$, i.e.,

$$\begin{cases} Az_k = \rho_k z_k,\ z_k = e_k + ih_k, \\ Re\rho_k \to +\infty \text{ as } k \to \infty, \\ |Im\rho_k/(Re\rho_k + a)| \leq C \text{ for some constants } a, C > 0, \end{cases} \tag{1.5}$$

such that $\{e_k, h_k\}$ is a basis of H.

For the compact operator $B_\lambda : H_1 \to H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ bounded}, \forall\ \lambda \in \mathbb{R}. \tag{1.6}$$

Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semigroup generated by (1.1) then the solution of (1.1) and (1.2) can be expressed as

$$u(t,u_0) = S_\lambda(t)u_0, \qquad t \geq 0.$$

Definition 1.1. *A set $\Sigma \subset H$ is called an invariant set of (1.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (1.1) is said to be an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$ we have*

$$\lim_{t\to\infty} \text{dist}_H(u(t,\varphi), \Sigma) = 0. \tag{1.7}$$

The largest open set U satisfying (1.7) is called the basin of attraction of Σ.

Definition 1.2. (1) *We say that the equation (1.1) bifurcates from $(u,\lambda) = (0,\lambda_0)$ to an invariant set Ω_λ, if there exists a sequence of invariant sets*

$\{\Omega_{\lambda_n}\}$ of (1.1), $0 \notin \Omega_{\lambda_n}$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda_0,$$

$$\lim_{n\to\infty} \max_{x\in\Omega_{\lambda_n}} |x| = 0.$$

(2) *If the invariant sets Ω_λ are attractors of (1.1), then the bifurcation is called attractor bifurcation.*

(3) *If Ω_λ are attractors and are homotopy equivalent to an m-dimensional sphere S^m, then the bifurcation is called S^m-attractor bifurcation.*

Let the eigenvalues (counting multiplicity) of L_λ be given by

$$\beta_k(\lambda) \in \mathbb{C} \quad (k \geq 1).$$

Suppose that

$$Re\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \qquad (1 \leq i \leq m), \tag{1.8}$$

$$Re\beta_j(\lambda_0) < 0 \qquad (m+1 \leq j). \tag{1.9}$$

Let the eigenspace of L_λ at λ_0 be

$$E_0 = \bigcup_{i=1}^{m} \left\{u \in H_1 \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0, k = 1, 2, \cdots \right\}.$$

The following dynamic bifurcation theorem for (1.1) was proved in [**24, 25**].

THEOREM 1.3 (Attractor Bifurcation). *Assume that (1.3)–(1.6) hold and $u = 0$ is a locally asymptotically stable equilibrium point of (1.1) at $\lambda = \lambda_0$. Then the following assertions hold.*

(1) *The equation (1.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to an attractor Σ_λ for $\lambda > \lambda_0$, with $m - 1 \leq \dim \Sigma_\lambda \leq m$, which is connected if $m > 1$.*

(2) *The attractor* Σ_λ *is a limit of a sequence of* m*-dimensional annulus* M_k *with* $M_{k+1} \subset M_k$*; in particular if* Σ_λ *is a finite simplicial complex, then* Σ_λ *has the homotopy type of* S^{m-1}*.*

(3) *For any* $u_\lambda \in \Sigma_\lambda$*,* u_λ *can be expressed as*

$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad v_\lambda \in E_0.$$

(4) *If* $u = 0$ *is globally stable for (1.1) at* $\lambda = \lambda_0$*, then for any bounded open set* $U \subset H$ *with* $0 \in U$ *there is an* $\varepsilon > 0$ *such that as* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$*, the attractor* Σ_λ *bifurcated from* $(0, \lambda_0)$ *attracts* U/Γ *in* H*, where* Γ *is the stable manifold of* $u = 0$ *with codimension* m*.*

The basic idea of the Theorem 1.3 can be demonstrated as follows, see [**25**] for details.

Near $\lambda = \lambda_0$, the flows of (1.1) in a small neighborhood of $u = 0$ is squeezed into an m–dimensional center manifold which can be expressed by a $h = h(x, \lambda)$, where $x \in \mathbb{R}^m$. The flow in the center manifold has the same topological structure as the flow in $\mathbb{R}^m$ of the equations

$$\frac{dx}{dt} = \mathcal{L}_\lambda x + PG(x, h(x, \lambda)), \tag{1.10}$$

where $\mathcal{L}_\lambda = L_\lambda|_{E_0}$, E_0 is the subspace of $\mathbb{R}^m$ on which all eigenvalues of $\mathcal{L}_\lambda$ possess nonnegative real part at $\lambda = \lambda_0$ and P is the projection.

It suffices then to consider the dynamic bifurcation for (1.10). When $\lambda < \lambda_0$, near $x = 0$ the linear term $\mathcal{L}_\lambda x$ determines the dynamical behavior of (1.10). Therefore the flow converges to $x = 0$. When $\lambda = \lambda_0$, assume that $x = 0$ is locally asymptotically stable for (1.10). Then when $\lambda > \lambda_0$ is sufficiently small, (1.10) can be rewritten as

$$\frac{dx}{dt} = PG(x, h(x, 0)) + \mathcal{L}_\lambda x + K(x, \lambda), \tag{1.11}$$

where $K(x, \lambda) = PG(x, h(x, \lambda)) - PG(x, h(x, 0)) \to 0$ as $\lambda \to 0$. Hence the flow of (1.11) is a superposition of the flow of

$$\frac{dx}{dt} = PG(x, h(x, 0)) \tag{1.12}$$

and the flow of

$$\frac{dx}{dt} = \mathcal{L}_\lambda x + K(x, \lambda) \ \ (\lambda > \lambda_0). \tag{1.13}$$

The flow of (1.13) is outward from $x = 0$, and the flow of (1.12) is inward to $x = 0$. Near $x = 0$, the linear term dominates the flow and for x far away from $x = 0$, the nonlinear term $PG(x, h(x, 0))$ dominates the flow structure. Thus the inward flow and outward flow squeeze an attractor near $x = 0$.

In the case of where $m = 2$, the bifurcated attractor can be further classified. Consider a two-dimensional system as follows:

$$\frac{dx}{dt} = \beta(\lambda)x - G(x, \lambda), \tag{1.14}$$

for $x \in \mathbb{R}^2$. Here $\beta(\lambda)$ is a continuous function of λ satisfying (1.8) and $G(x, \lambda)$ is defined as

$$G(v, \lambda) = G_k(v, \lambda) + o(|v|^k), \tag{1.15}$$

where $G_k(v, \lambda)$ is a k-multilinear operator($k \geq 2$)and satisfies the inequality

$$C_1|x|^{k+1} \leq < G_k(x, \lambda), x >_H \leq C_2|x|^{k+1}, \tag{1.16}$$

for some constants $0 < C_1 < C_2$ and $k = 2m + 1$, $m \geq 1$.

THEOREM 1.4. *(T. Ma and S. Wang, [**25**]) Under the condition (1.16), the system (1.14) bifurcates from $(x, \lambda) = (0, \lambda_0)$ to an attractor Σ_λ for $\lambda > \lambda_0$, which is homeomorphic to S^1. Moreover, one and only one of the following is true.*

(1) Σ_λ *is a periodic orbit,*

(2) Σ_λ *consists of infinitely many singular points, or*

(3) Σ_λ *contains at most* $2(k+1) = 4(m+1)$ *singular points, and has* $4N+n(N+n \geq 1)$ *singular points,* $2N$ *of which are saddle points,* $2N$ *of which are stable node points (possibly degenerate), and* n *of which have index zero.*

1.2. Asymptotic Stability. To apply the attractor bifurcation theorems, it is crucial to verify the asymptotic stability of the critical states.

Let the linear operator L_λ in (1.1) be symmetric and all the eigenvalues $\{\beta_k\} \subset \mathbb{R}$ of L_λ at $\lambda = \lambda_0$ satisfy

$$\begin{aligned} &\beta_i = 0 \quad \text{if} \quad 1 \leq i \leq m, \\ &\beta_j < 0 \quad \text{if} \quad m+1 \leq j. \end{aligned}$$

Set

$$\begin{aligned} E_0 &= \{u \in H_1 \mid L_{\lambda_0} u = 0\}, \\ E_1 &= E_0^{\perp} = \{u \in H_1 \mid \langle u, v\rangle_H = 0 \quad \forall\, v \in E_0\}. \end{aligned}$$

THEOREM 1.5. *(T. Ma and S. Wang* [**23, 25**]*) If* $G_{\lambda_0} : H_1 \to H$ *satisfies the following orthogonal condition:*

$$\langle G_{\lambda_0} u, u\rangle_H = 0, \qquad \forall\, u \in H_1.$$

Then one and only one of the following two assertions holds:

(1) *There exists a sequence of invariant sets* $\{\Gamma_n\} \subset E_0$ *of (1.1) at* $\lambda = \lambda_0$ *such that*

$$0 \notin \Gamma_n, \qquad \lim_{n\to\infty} \text{dist}(\Gamma_n, 0) = 0.$$

(2) *The trivial steady state solution* $u = 0$ *for (1.1) at* $\lambda = \lambda_0$ *is locally asymptotically stable under the* H*-norm.*

Furthermore, if (1.1) has no invariant sets in E_0 *except the trivial one* $\{0\}$*, then* $u = 0$ *is globally asymptotically stable.*

2. Center Manifold Reduction

2.1. Center manifold theorem. The purpose of this section is to recall some results of the center manifold theory under general settings, i.e., general function spaces and general functional equations. It is a powerful tool for the reduction method and for the dynamic bifurcation of abstract nonlinear evolution equations developed in [**25**].

We assume that the spaces H_1 and H can be decomposed into

$$\begin{cases} H_1 = E_1^\lambda \oplus E_2^\lambda, \quad \dim E_1^\lambda < \infty, \quad \text{near } \lambda_0 \in \mathbb{R}^1, \\ H = \widetilde{E}_1^\lambda \oplus \widetilde{E}_2^\lambda, \quad \widetilde{E}_1^\lambda = E_1^\lambda, \quad \widetilde{E}_2^\lambda = \text{closure of } E_2^\lambda \text{ in } H, \end{cases} \tag{1.17}$$

where E_1^λ and E_2^λ are two invariant subspaces of L_λ, i.e., L_λ can be decomposed into $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$ such that for any λ near λ_0,

$$\begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \to \widetilde{E}_1^\lambda, \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \to \widetilde{E}_2^\lambda, \end{cases} \tag{1.18}$$

where all eigenvalues of $\mathcal{L}_2^\lambda$ possess negative real parts, and all eigenvalues of $\mathcal{L}_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$.

Thus, for λ near λ_0, (1.1) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, y, \lambda), \\ \dfrac{dy}{dt} = \mathcal{L}_2^\lambda y + G_2(x, y, \lambda), \end{cases} \tag{1.19}$$

where $u = x + y \in H_1$, $x \in E_1^\lambda$, $y \in E_2^\lambda$, $G_i(x, y, \lambda) = P_i G(u, \lambda)$, and $P_i : H \to \widetilde{E}_i$ are canonical projections. Furthermore, we let

$$E_2^\lambda(\alpha) = E_2^\lambda \cap H_\alpha,$$

with α given by (1.4).

THEOREM 1.6. *(Center Manifold Theorem* [**15**]*) Assume (1.4)–(1.6) and (1.17). Then there exists a neighborhood of* λ_0 *given by* $|\lambda - \lambda_0| < \delta$ *for some* $\delta > 0$, *a* *neighborhood* $B_\lambda \subset E_1^\lambda$ *of* $x = 0$*, and a* C^1 *function* $\Phi(\cdot, \lambda) : B_\lambda \to E_2^\lambda(\alpha)$*, depending continuously on* λ*, such that*

(1) $\Phi(0, \lambda) = 0$, $D_x\Phi(0, \lambda) = 0$.

(2) *The set*

$$M_\lambda = \left\{(x, y) \in H_1 \mid x \in B_\lambda,\ y = \Phi(x, \lambda) \in E_2^\lambda(\alpha)\right\},$$

called the center manifold, is locally invariant for (1.1), i.e., for any $v_0 \in M_\lambda$,

$$v_\lambda(t, v_0) \in M_\lambda, \quad \forall\ 0 \leq t < t_{v_0},$$

for some $t_{v_0} > 0$*, where* $v_\lambda(t, v_0)$ *is the solution of (1.1).*

(3) *If* $(x_\lambda(t), y_\lambda(t))$ *is a solution of (1.19), then there are* $\beta_\lambda > 0$ *and* $k_\lambda > 0$ *with* k_λ *depending on* $(x_\lambda(0), y_\lambda(0))$*, such that*

$$\|y_\lambda(t) - \Phi(x_\lambda(t), \lambda)\|_H \leq k_\lambda e^{-\beta_\lambda t}.$$

By the help of the Center Manifold Theorem, we obtain the following bifurcation equation reduced to the finite dimensional system

$$\frac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x + \Phi(x), \lambda),$$

for $x \in B_\lambda \subset E_1^\lambda$.

2.2. Approximation of center manifold function. This subsection will be devoted to develop an approximation formula for the center manifold function Φ which plays crucial role in the bifurcation and stability study.

Let the nonlinear operator $G(v, \lambda)$ be defined as in (1.15), then we have the following theorem.

THEOREM 1.7. *(T.Ma and S.Wang,* [**25**]*) Under the conditions in Theorem 1.6 and (1.15), we have the following center manifold function approximation:*

$$\Phi(v, \lambda) = (-\mathcal{L}_2^\lambda)^{-1} P_2 G_k(v, \lambda) + O(|Re\beta(\lambda)| \cdot ||v||^k) + o(||v||^k),$$

where $\beta(\lambda) = (\beta_1(\lambda), \cdots, \beta_m(\lambda))$ *are the eigenvalues of* $\mathcal{L}_1^\lambda$.

In particular, if L_λ is symmetric and $G(\cdot, \lambda)$ is bilinear, then the eigenvalues of L_λ are real and the eigenvectors form an orthogonal basis of H. Suppose that the eigenvalues (counting multiplicity) and the eigenvectors of L_λ are

$$\beta_1(\lambda) = \beta_2(\lambda) = \cdots = \beta_m(\lambda) > \beta_{m+1}(\lambda) \geq \cdots,$$

$$e_1(x, \lambda), e_2(x, \lambda), \cdots, e_m(x, \lambda), e_{m+1}(x, \lambda), \cdots,$$

and $E_1^\lambda = span\{e_1, e_2, \cdots, e_m\}$, $E_2^\lambda = (E_1^\lambda)^\perp$ and $x = \sum_{j=1}^m x_j e_j$.

With the help of Theorem 1.6, the bifurcation equation of (1.1) can be reduced as

$$\frac{dx_k}{dt} = \beta_1 x_k + \frac{1}{< e_k, e_k >_H} < G(x + \Phi(x), \lambda), e_k >_H \tag{1.20}$$

for $k = 1, 2, \cdots, m$. It is known that the center manifold function

$$\Phi(x, \lambda) = \sum_{n=m+1}^{\infty} \Phi_n(x, \lambda) e_n$$

satisfies the following definition equation

$$\Phi_n(x, \lambda) = \int_{-\infty}^0 e^{-\beta_n \tau} \rho_\varepsilon < G(z(\tau, x) + \Phi, \lambda), e_n >_H d\tau, \tag{1.21}$$

where ρ_ε is a C^∞ cut-off function and

$$z(t, x) = \sum_{i=1}^m a_i(t, x) e_i$$

satisfies

$$\begin{cases} \dfrac{dz_i}{dt} = \beta_1 z_1 + \dfrac{\rho_\varepsilon(z)}{< e_i, e_i >_H} < G(z + \Phi, \lambda), e_i >_H, \\ z_i(0) = x_i. \end{cases} \tag{1.22}$$

Hence we have

$$z_i(t, x) = x_i e^{\beta_1 t} + o(|x|). \tag{1.23}$$

Replacing (1.21) with (1.23), we get

$$\Phi_n(x, \lambda) = \sum_{i,\ell=1}^m \left(\int_{-\infty}^0 e^{(2\beta_1 - \beta_n)\tau} d\tau \right) < G(e_i, e_\ell, \lambda), e_n >_H + o(|x|^2). \tag{1.24}$$

By (1.20) and (1.24), we conclude that

$$\frac{dx_k}{dt} = \beta_1 x_k + \sum_{i,j=1}^{m} b^k_{ij} x_i x_j + \sum_{i,j,\ell=1}^{m} a^k_{ij\ell} x_i x_j x_\ell + o(|x|^3), \tag{1.25}$$

where

$$b^k_{ij} = \frac{1}{< e_k, e_k >_H} < G(e_i, e_j, \lambda), e_k >_H$$

and

$$a^k_{ij\ell} = \sum_{n=m+1}^{\infty} \frac{< G(e_i, e_\ell, \lambda), e_n >_H < G(e_j, e_n, \lambda) + G(e_n, e_j, \lambda), e_k >_H}{(2\beta_1 - \beta_n) < e_n, e_n >_H < e_k, e_k >_H},$$

for $k = 1, 2, \cdots, m$.

CHAPTER 2

Bifurcatin for the Complex Ginzburg-Landau Equation

1. Introduction

The complex Ginzburg–Landau equation(CGLE) is a widely studied patrial differential equation with applications in many areas of science. If the dissipative terms in the CGLE are neglected, the equation is reduced to the nonlinear Schrödinger equation. It has become a model problem for the study of nonlinear evolution equations with chaotic spatio-temporal dynamics. In this chapter, we consider the bifurcation of attractors and invariant sets of the CGLE, which reads

$$\frac{\partial u}{\partial t} - (\alpha + i\beta)\triangle u + (\sigma + i\rho)|u|^2 u - \lambda u = 0, \tag{2.1}$$

where the unknown function $u : \Omega \times [0, \infty) \to \mathbb{C}$ is a complex-valued function and $\Omega \subset \mathbb{R}^n$ is an open, bounded, and smooth domain in $\mathbb{R}^n$ $(1 \leq n \leq 3)$. The parameters $\alpha, \beta, \sigma, \rho,$ and λ are real numbers and

$$\alpha > 0, \quad \sigma > 0. \tag{2.2}$$

The initial condition for (2.1) is given by

$$u(x, 0) = \phi + i\psi. \tag{2.3}$$

Also, (2.1) is supplemented with either the Dirichlet boundary condition,

$$u|_{\partial\Omega} = 0, \tag{2.4}$$

or the periodic boundary condition,

$$\Omega = (0, 2\pi)^n \text{ and } u \text{ is } \Omega\text{-periodic}. \tag{2.5}$$

This equation is a canonical model for weakly-nonlinear, dissipative systems and for this reason, it arises in a variety of settings, including nonlinear optics, fluid dynamics, chemical physics, mathematical biology, condensed matter physics, and statistical mechanics. In fluid dynamics the CGLE is found, for example, in the study of Poiseuille flow, the nonlinear growth of convection rolls in the Rayleigh–Beńard problem and Taylor–Couette flow. In this case, the bifurcation parameter ρ plays the role of a Reynolds number. The equation also arises in the study of chemical systems governed by reaction-diffusion equations. The CGLE plays the role of a simplified set of fluid dynamic equations in the following sense. The Navier-Stokes equations can be characterized as an infinite dimensional dynamical system whose behavior is often dominated by long wavelength instabilities with short wavelength dissipation (due to viscosity) and a nonlinear mode coupling (the convection term) that provides an energy transport from the long to the short wavelengths. A useful bifurcation parameter for the Navier-Stokes equations, the Reynolds number, is the ratio of the product of a long wavelength velocity and length scale and the short wavelength dissipation. Quiescent low Reynolds number flows are characterized by a low ratio of driving to damping, while the often chaotic or turbulent high Reynolds number flows correspond to a high ratio. The same qualitative features are present in the CGLE.

There are extensive studies from the mathematical point of view for the CGLE, and we refer in particular to [**3, 10, 2, 17, 19, 20, 36**] and the references therein for studies related to the global attractors, inertial manifolds, and soft and hard turbulences described by the CGLE.

For the CGLE, bifurcation is obtained with respect to the parameter λ, and the main results obtained can be summarized as follows.

First, for the CGLE with the Dirichlet boundary condition, let λ_1 be the first eigenvalue of the elliptic operator $-\triangle$. Our main results in this case include the following.

1. If $\lambda \leq \alpha\lambda_1$, the trivial solution $u = 0$ is globally asymptotically stable. The global attractor of the CGLE consists exactly of the trivial steady state solution $u = 0$.

2. As λ crosses $\alpha\lambda_1$, i.e., there exists an $\epsilon > 0$ such that for any $\alpha\lambda_1 < \lambda < \alpha\lambda_1 + \epsilon$, the CGLE bifurcates from the trivial solution an attractor Σ_λ. The bifurcated attractor Σ_λ attracts the open set $L^2(\Omega, \mathbb{C})/\Gamma$, where Γ is the stable manifold of $u = 0$ having codimension two in $L^2(\Omega, \mathbb{C})$.

More detailed structure of this bifurcated attractor can be classified as follows (see Figures 2.1 and 2.2.)

(a) If $|\beta| + |\rho| \neq 0$, then the bifurcated attractor consists of exactly one stable limiting cycle, i.e., $\Sigma_\lambda = S^1$, which is asymptotically stable. The global attractor $\mathcal{A}_\lambda$ is a two-dimensional (2D) disk consisting of the stable limiting cycle $\Sigma_\lambda = S^1$, the (unstable) trivial steady state solution $u = 0$, and orbits connecting $\Sigma_\lambda = S^1$ and $u = 0$.

In particular, if $\beta \neq 0$, then the bifurcation is a Hopf bifurcation to a stable limiting cycle.

(b) If $\beta = \rho = 0$, then the bifurcated attractor Σ_λ has dimension between 1 and 2 and is a limit of a sequence of $2D$ annulus M_k with $M_{k+1} \subset M_k$, i.e., $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$.

Again in this case, the global attractor $\mathcal{A}_\lambda$ is 2D, consisting of Σ_λ, $u = 0$ and the connecting orbits between them. [1]

[1] Using a different method, we can in fact prove that Σ_λ is also homeomorphic to S^1, which shall be reported elsewhere.

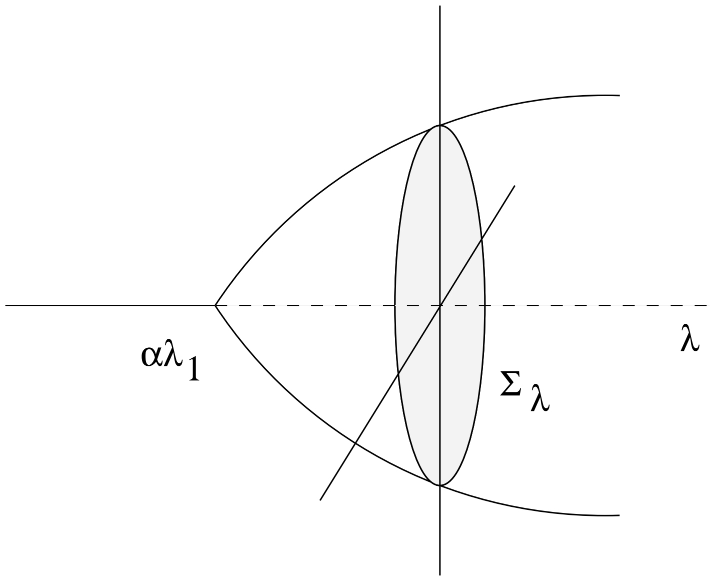

Figure 2.1. Bifurcation diagram for the CGLE with the Dirichlet boundary condition: (1) bifurcation appears at $\lambda = \alpha\lambda_1$, (2) bifurcated attractor $\Sigma_\lambda = S^1$ is the boundary of the shaded region, and (3) the global attractor $\mathcal{A}_\lambda$ is the 2D disk, shown as the shaded region. Here the dotted line stands for the unstable trivial solution $u = 0$.

Second, for the CGLE equipped with the periodic boundary condition, similar results can be obtained as well. In particular, in the case where $|\beta|+|\rho| \neq 0$, we prove that the bifurcated attractor Σ_λ is a sphere S^1, containing no steady state solutions, and the global attractor $\mathcal{A}_\lambda$ is a 2-dimensional ball consisting of the trivial steady state $u = 0$, Σ_λ, and the orbits connecting them.

Finally, bifurcation from *any* eigenvalue of the Laplacian can also be obtained as for the first eigenvalue. It is worth mentioning that the complete structure of the global attractor for the bifurcations from the first eigenvalue is obtained, while no such information is available for bifurcations from the rest eigenvalues.

Important work on lower and upper bounds of the global attractor of the GL equation, together with their physical mechanisms, was done in the 1980's in [**3, 10,**

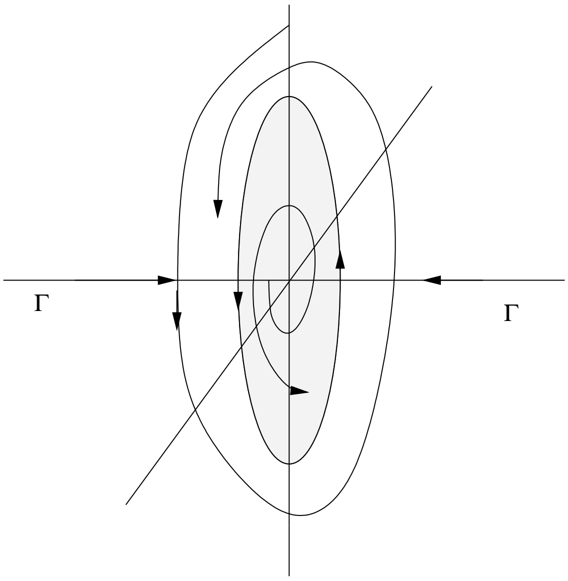

FIGURE 2.2. Phase space structure in $L^2(\Omega, \mathbb{C}) \times \{\lambda\}$ in the case where $|\beta| + |\rho| \neq 0$. Here the bifurcated attractor $\Sigma_\lambda = S^1$ is a stable limiting cycle.

2]. As mentioned earlier, the main objective of this chapter is to study bifurcation and transitions from the trivial solution. Hence we focus only on the local attractor near the trivial solution, which is part of the global attractor. Of course, near the first eigenvalue, complete information for both the global attractor and the bifurcated attractor is obtained in this chapter. For λ near other eigenvalues, the results here demonstrate only the transitions of the trivial solution and provide some partial information on the low bounds of the global attractor. As far as the dimension of the global attractor is concerned, our results are consistent with the work in [**3, 10, 2**].

This chapter is organized as follows. We study the CGLE with Dirichlet and periodic boundary condition in the first two sections, respectively. Bifurcation for any other eigenvalues will be investigated in Section 4.

2. Bifurcation of the CGLE with Dirichlet Boundary Condition

As mentioned in the introduction, we study in this chapter attractor bifurcation of the CGLE under either the Dirichlet or the periodic boundary conditions.

We start with the CGLE with the Dirichlet boundary condition. Let

$$H^k(\Omega, \mathbb{C}) = \{u_1 + iu_2 \,|\, u_j \in H^k(\Omega), j = 1, 2\},$$
$$H_0^1(\Omega, \mathbb{C}) = \{u \in H^1(\Omega, \mathbb{C}) \mid u|_{\partial\Omega} = 0\},$$

where $H^k(\Omega)$ is the usual real-valued Sobolev space.

Let λ_1 be the first eigenvalue of $-\triangle$ with the Dirichlet boundary condition (2.4). Then we have the following main bifurcation theorem for the CGLE with the Dirichlet boundary condition.

Theorem 2.1. (1) *If $\lambda \leq \alpha\lambda_1$, $u = 0$ is a globally asymptotically stable equilibrium point of (2.1) with (2.4).*

(2) *As λ crosses $\alpha\lambda_1$, i.e., for any $\alpha\lambda_1 < \lambda < \alpha\lambda_1 + \epsilon$ for some $\epsilon > 0$, the problem (2.1) with (2.4) bifurcates from $(u, \lambda) = (0, \alpha\lambda_1)$ to an attractor Σ_λ.*

(3) *The bifurcated attractor Σ_λ has dimension between 1 and 2 and is a limit of a sequence of $2D$ annulus M_k with $M_{k+1} \subset M_k$; i.e. $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$.*

(4) *If $\beta \neq 0$, then the bifurcation is a Hopf bifurcation, i.e., $\Sigma_\lambda = S^1$, which is asymptotically stable (limiting cycle).*

(5) *If $\beta = 0$ and $\rho \neq 0$, then the bifurcated attractor Σ_λ is a periodic orbit, which is a limiting cycle.*

(6) *Moreover, for each $\alpha\lambda_1 < \lambda < \alpha\lambda_1 + \epsilon$, the bifurcated attractor Σ_λ attracts the open set $L^2(\Omega, \mathbb{C})/\Gamma$, where Γ is the stable manifold of $u = 0$ having codimension two in $L^2(\Omega, \mathbb{C})$.*

Proof. We proceed in several steps as follows.

STEP 1. Let $u = u_1 + iu_2$. The CGLE (2.1) with (2.3) can be equivalently written as follows:

$$\begin{cases} \frac{\partial u_1}{\partial t} = \alpha \triangle u_1 - \beta \triangle u_2 + \lambda u_1 - \sigma |u|^2 u_1 + \rho |u|^2 u_2, \\ \frac{\partial u_2}{\partial t} = \beta \triangle u_1 + \alpha \triangle u_2 + \lambda u_2 - \sigma |u|^2 u_2 - \rho |u|^2 u_1, \\ u_1(x,0) = \phi(x), \quad u_2(x,0) = \psi(x). \end{cases} \tag{2.6}$$

We shall apply Theorems 1.3 and 1.6 to prove this theorem. Let

$$H_1 = H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C}), \quad H = L^2(\Omega, \mathbb{C}).$$

The mappings $L_\lambda = -A + B_\lambda$ and $G : H_1 \to H$ are defined as

$$\begin{aligned} -Au &= \begin{pmatrix} \alpha \triangle u_1 - \beta \triangle u_2 \\ \beta \triangle u_1 + \alpha \triangle u_2 \end{pmatrix}, \\ B_\lambda u &= \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ Gu &= \begin{pmatrix} -\sigma |u|^2 u_1 + \rho |u|^2 u_2 \\ -\sigma |u|^2 u_2 - \rho |u|^2 u_1 \end{pmatrix}. \end{aligned}$$

It is known that $H_{1/2} = H_0^1(\Omega, \mathbb{C})$. By the Sobolev embedding theorems and $1 \leq n \leq 3$, the mapping $G : H_{1/2} \to H$ is C^∞. The condition (1.4) is fulfilled.

Let $\{\lambda_k\} \subset \mathbb{R}$ and $\{e_k\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be the eigenvalues and eigenvectors of $-\triangle$ with the Dirichlet boundary condition (2.4)

$$\begin{cases} -\triangle e_k = \lambda_k e_k, \\ e_k|_{\partial\Omega} = 0. \end{cases}$$

We know that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and $\{e_k\}$ is an orthogonal basis of $L^2(\Omega)$.

It is easy to see that the eigenvalues of A are given by

$$\alpha\lambda_k \pm i\beta\lambda_k,\ \ k = 1, 2, \cdots$$

with the corresponding eigenvectors

$$z_k = e_k + ie_k,$$

and $\{e_k, ie_j | 1 \leq k, j < \infty\}$ is an orthogonal basis of H. Thus the conditions (1.5) and (1.6) are valid for A and B_λ. The eigenvalues of $L_\lambda = -A + B_\lambda$ are as follows:

$$(\lambda - \alpha\lambda_k) \pm i\beta\lambda_k, \qquad k = 1, 2, \cdots . \tag{2.7}$$

In addition, the spaces H and H_1 can be decomposed into the form

$$\begin{aligned} H_1 &= E_1 \oplus E_2 \quad \text{and} \quad H = E_1 \oplus \widetilde{E}_2, \\ E_1 &= \{x_1 e_1 + iy_1 e_1 \,|\, x_1, y_1 \in \mathbb{R}\}, \\ E_2 &= \left\{ \sum_{k=2}^{\infty} (x_k + iy_k) e_k \,|\, \sum_{k=2}^{\infty} \lambda_k^2 (x_k^2 + y_k^2) < \infty \right\}, \\ \widetilde{E}_2 &= \left\{ \sum_{k=2}^{\infty} (x_k + iy_k) e_k \,|\, \sum_{k=2}^{\infty} (x_k^2 + y_k^2) < \infty \right\}, \end{aligned}$$

and the operator L_λ is decomposed into

$$\begin{cases} L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda, \\ \mathcal{L}_1^\lambda = L_\lambda|_{E_1} \,:\, E_1 \to E_1,\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2} \,:\, E_2 \to \widetilde{E}_2. \end{cases}$$

Thus the conditions (1.17) and (1.18) are satisfied.

By the center manifold theorem, the attractor bifurcation of (2.1) with (2.4) is equivalent to that of the bifurcation equations

$$\begin{cases} \dfrac{dx_1}{dt} = (\lambda - \alpha\lambda_1)x_1 + \beta\lambda_1 y_1 + P_1 G_1(x_1 + iy_1 + h), \\ \dfrac{dy_1}{dt} = -\beta\lambda_1 x_1 + (\lambda - \alpha\lambda_1)y_1 + P_1 G_2(x_1 + iy_1 + h), \end{cases} \tag{2.8}$$

where $h = h_1 + ih_2$ is the center manifold function satisfying

$$h(x_1, y_1) = o(|x_1| + |y_1|),$$

and $P_1G_i(u)\,(i=1,2)$ are given by

$$
\begin{aligned}
&P_1G_1(u)=\int_\Omega[-\sigma|u|^2u_1+\rho|u|^2u_2]e_1dx,\\
&P_1G_2(u)=\int_\Omega[-\sigma|u|^2u_2-\rho|u|^2u_1]e_1dx,\\
&u=u_1+iu_2=\sum_{k=1}^{\infty}(x_k+iy_k)e_k.
\end{aligned}
\tag{2.9}
$$

STEP 2. Now we show that $u=0$ is a globally asymptotically stable equilibrium of (2.1) for $\lambda\le\alpha\lambda_1$. In fact, from (2.6) we can derive that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2dx&=\int_\Omega(-\alpha|\nabla u|^2+\lambda|u|^2-\sigma|u|^4)dx\\
&\le-\int_\Omega[(\alpha\lambda_1-\lambda)|u|^2+\sigma|u|^4]dx
\end{aligned}
$$

which implies that $u=0$ is globally stable.

STEP 3. We know that (2.1) has a global attractor; see [**37**]. Obviously, for the eigenvalues (2.7) of L_λ, the conditions (1.8) and (1.9) for $\lambda_0=\alpha\lambda_1$ are satisfied. Therefore, by Theorem 1.3, (2.1) bifurcates from $(u,\lambda)=(0,\alpha\lambda_1)$ to an attractor Σ_λ which attracts H/Γ.

STEP 4. We now prove that $\Sigma_\lambda=S^1$.

Obviously, when $\beta\neq 0$ the bifurcation is the typical Hopf bifurcation. Therefore, we have to consider only the case where $\beta=0$. In this case, the bifurcation equations (2.8) read

$$
\begin{cases}
\dfrac{dx}{dt}=\varepsilon x+P_1G_1(xe_1+h_1+iye_1+ih_2),\\
\dfrac{dy}{dt}=\varepsilon y+P_1G_2(xe_1+h_1+iye_1+ih_2),
\end{cases}
\tag{2.10}
$$

where $\varepsilon = \lambda - \alpha\lambda_1 > 0$ sufficiently small. From (2.9) we have

$$\begin{aligned} P_1G_1(x,y) &= \int_\Omega [-\sigma u_1^3 + \rho u_2^3 - \sigma u_2^2 u_1 + \rho u_1^2 u_2] e_1 dx \\ &= (\text{by } h(x,y) = o(|x|+|y|)) \\ &= a(-\sigma x^3 + \rho y^3 - \sigma y^2 x + \rho x^2 y) + o(|x|^3 + |y|^3), \\ P_1G_2(x,y) &= a(-\sigma y^3 - \rho x^3 - \sigma x^2 y - \rho y^2 x) + o(|x|^3 + |y|^3), \end{aligned}$$

where $u_1 = xe_1 + h_1(x,y),\ u_2 = ye_1 + h_2(x,y)$, and

$$a = \int_\Omega e_1^4(x) dx > 0.$$

Thus, the bifurcation equations (2.10) lead to

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon x - a(\sigma x^3 - \rho y^3 + \sigma y^2 x - \rho x^2 y) + o(|x|^3 + |y|^3), \\ \dfrac{dy}{dt} = \varepsilon y - a(\sigma y^3 + \rho x^3 + \sigma x^2 y + \rho y^2 x) + o(|x|^3 + |y|^3). \end{cases} \tag{2.11}$$

We can see that the attractor Σ_λ has no nonzero singular point; i.e., the singular point $u = 0$ of (2.1) with (2.4) is unique provided $|\rho| + |\beta| \neq 0$, because from (2.6) we have

$$\int_\Omega \left[u_2 \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_2}{\partial t} \right] dx = \int_\Omega [\beta |\nabla u|^2 + \rho |u|^4] dx.$$

By Theorem 1.3, Σ_λ has the homotopy type of S^1; hence Σ_λ contains at least one periodic orbit provided $\rho \neq 0$.

Take the polar coordinate system

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then (2.11) becomes

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{\varepsilon - a\sigma r^2 + o(r^2)}{a\rho r}, \\ r(0) = r_0. \end{cases} \tag{2.12}$$

From (2.12) it follows that

$$\frac{a\rho}{2}(r^2(2\pi) - r^2(0)) = \int_0^{2\pi} [\varepsilon - a\sigma r^2 + o(r^2)]d\theta.$$

Because $r^2 = r^2(\theta, r_0)$ is C^∞ on $r_0 \geq 0$, we have the Taylor expansion

$$r^2(\theta, r_0) = r_0^2 + R(\theta) \cdot o(|r_0|^2), \quad R(0) = 0.$$

Hence we get

$$\frac{a\rho}{2}(r^2(2\pi) - r^2(0)) = 2\pi\varepsilon - 2\pi a\sigma r_0^2 + o(|r_0|^2).$$

Obviously the initial values $r_0 > 0$ in (2.12) satisfying

$$2\pi\varepsilon - 2\pi a\sigma r_0^2 + o(|r_0|^2) = 0 \tag{2.13}$$

are corresponding to the periodic orbits of (2.11). It is easy to see that the solution $r_0^2 > 0$ of (2.13) near $r_0 = 0$ is unique. Thus we deduce that Σ_λ is a periodic orbit provided $\rho \neq 0$.

The proof is complete. □

3. Bifurcation of the CGLE with Periodic Boundary Condition

For the CGLE with periodic boundary condition, the first eigenspace is larger than that in the Dirichlet boundary condition case, and to proceed, we need the following function spaces:

$$H^k_{per}(\Omega, \mathbb{C}) = \{u \in H^k(\Omega, \mathbb{C}) \,|\, u \text{ satisfy (2.5)}\}.$$

Then the main result in this section is the following theorem.

THEOREM 2.2. *For the CGLE (2.1) with the periodic boundary condition (2.5), we have the following assertions.*

(1) (a) *As $\lambda > \alpha$, the problem (2.1) with (2.5) bifurcates from $(u, \lambda) = (0, \alpha)$ to an invariant set Σ_λ. Σ_λ has dimension between $4n - 1$ and $4n$ and*

is a limit of a sequence of $4n$ *annulus* M_k *with* $M_{k+1} \subset M_k$*; i.e.,* $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$.

(b) *If* $|\rho| + |\beta| \neq 0$*, then* Σ_λ *contains no steady state solutions of (2.1) with (2.5).*

(2) (a) *As* $\lambda \leq 0$*,* $u = 0$ *is globally asymptotically stable.*

(b) *As* $\lambda > 0$ *the problem (2.1) with (2.5) bifurcates from* $(u, \lambda) = (0, 0)$ *an attractor* $\Sigma_\lambda \subset L^2(\Omega, \mathbb{C})$*. The bifurcated attractor* Σ_λ *has dimension between* 1 *and* 2*, and is a limit of a sequence of* $2D$ *annulus* M_k *with* $M_{k+1} \subset M_k$*, i.e.* $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$.

(c) *If* $\rho \neq 0$ *then* Σ_λ *is a periodic orbit.*

(d) Σ_λ *attracts* $L^2(\Omega, \mathbb{C})/\Gamma$*, where* Γ *is the stable manifold of* $u = 0$ *with codimension two in* $L^2(\Omega, \mathbb{C})$.

Proof. Let $H_1 = H^2_{per}(\Omega, \mathbb{C})$, $H = L^2_{per}(\Omega, \mathbb{C})$ and the mappings L_λ and $G : H_1 \to H$ be as defined in the previous section. Similar to the proof of Theorem 2.1, L_λ and G satisfy the conditions in Theorem 1.6.

We know that the eigenvalue problem

$$\begin{cases} -\triangle e_k = \lambda_k e_k, \\ e_k(x + 2k\pi) = e_k(x) \end{cases} \tag{2.14}$$

has an eigenvalue sequence

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and an eigenvector sequence $\{e_k\}$ which constitutes a common orthogonal basis of H_1 and H. The second eigenvalue $\lambda_1 = 1$ has multiplicity $2n$, i.e., $\lambda_1 = \cdots = \lambda_{2n}$, with the first eigenvectors

$$\sin x_j, \quad \cos x_j \qquad (x = (x_1, \cdots, x_n) \in \Omega = (0, 2\pi)^n).$$

Eigenvalues of L_λ are as in (2.7), and the second eigenvalue $\Lambda_1 = (\lambda-\alpha) \pm i\beta$ has multiplicity $4n$. For simplicity, let

$$e_{2j-1} = \sin x_j, \quad e_{2j} = \cos x_j, \quad (j = 1, \cdots, n).$$

Then the spaces H and H_1 can be decomposed into the following form

$$\begin{aligned} H &= E_1 \oplus \widetilde{E}_2, \\ E_1 &= \left\{ \sum_{j=1}^{2n} (z_{1j} + i z_{2j}) e_j \mid z_{1j}, z_{2j} \in \mathbb{R} \right\}, \\ \widetilde{E}_2 &= E_1^{\perp}. \end{aligned}$$

Then the bifurcation equations of (2.1) with (2.5) are given by

$$\begin{cases} \dfrac{dZ_1}{dt} = (\lambda-\alpha) Z_1 + \beta Z_2 + PG_1(u), \\ \dfrac{dZ_2}{dt} = -\beta Z_1 + (\lambda - \alpha) Z_2 + PG_2(u), \end{cases} \tag{2.15}$$

where $u = u_1 + iu_2$ and

$$\begin{aligned} (u_1, u_2) &= (Z_1 + h_1(Z_1, Z_2), Z_2 + h_2(Z_1, Z_2)), \\ (Z_1, Z_2) &= \sum_{j=1}^{2n} (z_{1j}, z_{2j}) e_j. \end{aligned}$$

Here $h = h_1 + ih_2 : E_1 \to \widetilde{E}_2$ is the center manifold function satisfying

$$h(Z_1, Z_2) = o(|Z_1| + |Z_2|) \tag{2.16}$$

and

$$\begin{aligned} PG_1(u) &= \sum_{j=1}^{2n} e_j \int_\Omega [-\sigma |u|^2 u_1 + \rho |u|^2 u_2] e_j dx, \\ PG_2(u) &= \sum_{j=1}^{2n} e_j \int_\Omega [-\sigma |u|^2 u_1 - \rho |u|^2 u_2] e_j dx. \end{aligned}$$

By Theorem 1.3, we infer from (2.16) that the problem (2.1) and (2.5) bifurcates from $(u, \lambda) = (0, \alpha)$ an invariant set Σ_λ.

The proof is complete. □

REMARK 2.3. In fact, the invariant set Σ_λ of (2.1) with (2.5) is a sphere S^{4n-1}; namely, Σ_λ is homeomorphic to a sphere S^{4n-1}. The topological structure of an attractor of vector fields should be stable provided some nondegenerate conditions hold.

4. Bifurcation of invariant sphere S^m

More generally, for the CGLE we have the bifurcation theorem of invariant sphere S^m $(m \geq 1)$ at any eigenvalue.

THEOREM 2.4. *Let λ_k be an eigenvalue of $-\triangle$ with the boundary condition (2.4), or (2.5), which has multiplicity $m \geq 1$. Then, as $\lambda > \alpha\lambda_k$, the problem (2.1) with (2.4), or (2.1) with (2.5), bifurcates from $(u, \lambda) = (0, \alpha\lambda_k)$ to an invariant set Σ_λ. This invariant set Σ_λ has dimension between $2m - 1$ and $2m$ and is a limit of a sequence of $2m$ dimensional annulus M_k with $M_{k+1} \subset M_k$; i.e., $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$. If $|\beta| + |\rho| \neq 0$, then there is no singular point in Σ_λ.*

PROOF. We denote the eigenvectors of $-\triangle$ corresponding to λ_k by

$$\{e_1^*, \cdots, e_m^*\}.$$

Thus, the space H_1 and H defined in Theorem 2.1 or Theorem 2.2 can be decomposed into

$$\begin{aligned}
&H_1 = E_m \oplus E_m^{\perp}, \quad H = \widetilde{E}_m \oplus \ \widetilde{E}_m^{\perp}, \\
&E_m = \text{span}\{e_i^* + ie_j^* \mid 1 \leq i, j \leq m\}, \\
&E_m^{\perp} = \{u \in H_1 \mid \langle u, v\rangle_{H_1} = 0 \ \forall v \in E_m\}, \\
&\widetilde{E}_m = E_m, \quad \widetilde{E}_m^{\perp} = \{u \in H \mid \langle u, v\rangle_H = 0 \ \forall v \in \widetilde{E}_m\}.
\end{aligned}$$

By the center manifold theorem, the bifurcation equations of (2.1) with (2.4), or (2.1) with (2.5), at $\lambda = \lambda_k$ are equivalent to

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = \alpha \triangle v_1 - \beta \triangle v_2 + \lambda v_1 + PG_1(v + h(v)), \\ \dfrac{\partial v_2}{\partial t} = \beta \triangle v_1 + \alpha \triangle v_2 + \lambda v_2 + PG_2(v + h(v)), \end{cases} \tag{2.17}$$

where λ is near λ_k, $v = v_1 + iv_2 \in E_m$, and $h : E_m \to E_m^{\perp}$ is the center manifold function, $G = (G_1, G_2) : H_1 \to H$ defined as in Theorem 2.1 or Theorem 2.2, and $P : H \to \widetilde{E}_m$ is the projection.

The equations (2.17) are a system of ordinary differential equations with order $2m$:

$$\begin{cases} \dfrac{dZ_1}{dt} = (\lambda - \alpha)Z_1 + \beta Z_2 + [-\sigma|Z|^2 Z_1 + \rho|Z|^2 Z_2] + o(|Z|^3), \\ \dfrac{dZ_2}{dt} = -\beta Z_1 + (\lambda - \alpha)Z_2 + [-\sigma|Z|^2 Z_2 - \rho|Z|^2 Z_1] + o(|Z|^3), \end{cases} \tag{2.18}$$

where $Z = Z_1 + iZ_2$. The eigenvalues of the linear part are still $(\lambda - \alpha\lambda_k) \pm i\beta\lambda_k$, with multiplicity $2m$.

By Theorem 1.3 it suffices to prove that $v = 0$ is asymptotically stable for (2.18) at $\lambda = \alpha\lambda_k$. For $\lambda = \alpha\lambda_k$, we infer from (2.18) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx &= \int_\Omega G(v + h(v))v dx \\ &= (\text{by } h(v) = o(|v|) \\ &= \int_\Omega G(v)v dx + o(|v|^4) \\ &= -\sigma \int_\Omega |v|^4 dx + o(|v|^4), \end{aligned}$$

which implies that $v = 0$ is asymptotically stable for (2.18) at $\lambda = \alpha\lambda_k$.

The proof is complete. □

CHAPTER 3

Bifurcation for the generalized Complex Ginzburg-Landau Equation

1. Introduction

In this chapter we work with the generalized Complex Ginzburg-Landau equation(CGLE) which reads

$$\frac{\partial u}{\partial t} = \rho u + (\varepsilon_0 + i\rho_0)\Delta u + (\varepsilon_1 + i\rho_1)|u|^2 u - (\varepsilon_2 + i\rho_2)|u|^4 u, \tag{3.1}$$

where the unknown function $u : \Omega \times [0, \infty) \to \mathbb{C}$ is a complex-valued function and $\Omega \subset \mathbb{R}^n$ is an open, bounded, and smooth domain in $\mathbb{R}^n$ $(1 \leq n \leq 3)$. The parameters ε_i and $\rho_i (i = 0, 1, 2)$ are real numbers and ρ is the system parameter. This is the equation we shall consider, supplemented with either the Dirichlet boundary condition,

$$u|_{\partial\Omega} = 0, \tag{3.2}$$

or the space periodic boundary condition,

$$\Omega = (0, 2\pi)^n \text{ and } u \text{ is } \Omega - \text{periodic}, \tag{3.3}$$

and suitable initial data

$$u(x, 0) = \phi + i\psi. \tag{3.4}$$

There are extensive studies for the CGLE and we refer, in particular, to [**1, 2, 5, 6, 7, 21, 32**] and the references therein for studies related to the global attractors, inertial manifolds, soft and hard turbulences, coherent structure, kink and solitons, stabilities and bifurcation theories described by the CGLE. If we take a look at the equation (3.1), then we can realize that we have choices to set three coefficients ε_i and

ρ_i [**32**]. When $\varepsilon_1 < 0$, the equation has a supercritical bifurcation at some critical values, and one often takes $\varepsilon_2 = \rho_2 = 0$ and $\varepsilon_0 = -\varepsilon_1 = 1$ since ε_2 is not necessary for stability (We will briefly mention about it at the end of the chapter). Therefore, the equation has two parameters ρ_0 and ρ_1. When $\varepsilon_1 > 0$, the equation has a subcritical bifurcation, and we must retain $\varepsilon_2 > 0$ which allows us to avoid an expected blowup of the solutions. It is usual to take $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$ and the equation has the three parameters ρ_0, ρ_1 and ρ_2.

The first supercritical case was studied in Chapter 2. It was shown that the equation bifurcates supercritically as ρ crosses some critical numbers, for both Dirichlet and periodic boundary conditions. In this chapter, we study the CGLE which has an unstable cubic nonlinear term. The results can be summarized as follows.

First, we consider the CGLE in the case of $\varepsilon_1 = 1 > 0$ and $\varepsilon_2 = \rho_2 = 0$ (the so-called *cubic* CGLE) with (3.3):

$$\frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2 u, \tag{3.5}$$

supplemented with a parameter range (ρ_0, ρ_1). If ρ_0 and ρ_1 stay in the region of $1 - \frac{\rho_1}{\rho_0} < 0$, the solutions of (3.5) bifurcate from $u = 0$ "supercritically" in spite of the fact that the sign of the real part of the nonlinear term signals a subcritical bifurcation(See Figure 3.1). Second, we consider the CGLE in the case of nonzero cubic and quintic terms $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ (the so-called $cubic - quintic$ CGLE) with either (3.2) or (3.3):

$$\frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2 u - (1 + i\rho_2)|u|^4 u. \tag{3.6}$$

If we let λ_1 be the first eigenvalue of $-\Delta$ under (3.2), then the equation bifurcates subcritically from $u = 0$ for $\rho < \lambda_1$. Moreover, with the help of the global attractor near $\rho = 0$, we obtain a saddle node bifurcation point $(u, \rho) = (u^*, \lambda_c)$ $(0 < \lambda_c < \lambda_1)$ from which two branches of solutions Σ_1 and Σ_2 bifurcate. We can easily see that

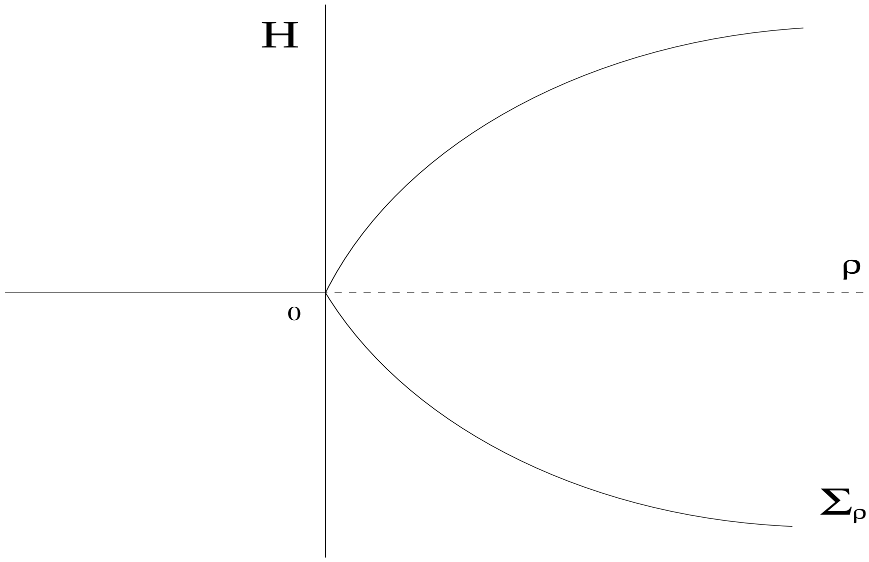

FIGURE 3.1. Supercritical bifurcation diagram for the cubic CGLE. Solid lines denote stable states and dashed lines unstable states. For $\rho > 0$, the problem bifurcates supercritically.

Σ_1 is a repeller and convergent to $(0, \lambda_1)$ as $\rho \to \lambda_1-$ and Σ_2 is an attractor and extends to $\rho > \lambda_1$. We can also prove that these two bifurcated invariant sets are S^1. Considering the periodic boundary condition (3.3), we obtain the same result as the Dirichlet boundary case, for $\rho < 0$(See Figure 3.2).

It is worth mentioning that the above two results can be verified using a new notion of bifurcation theory which was developed in [**25**] and the results coincide with those in [**1, 5, 7, 32**].

Chapter 3 is organized as follows. We will study the cubic CGLE in Section 2 which will be followed by the cubic-quintic CGLE in Section 3 and 4. Section 4 will be devoted to see the existence of a global attractor of the CGLE.

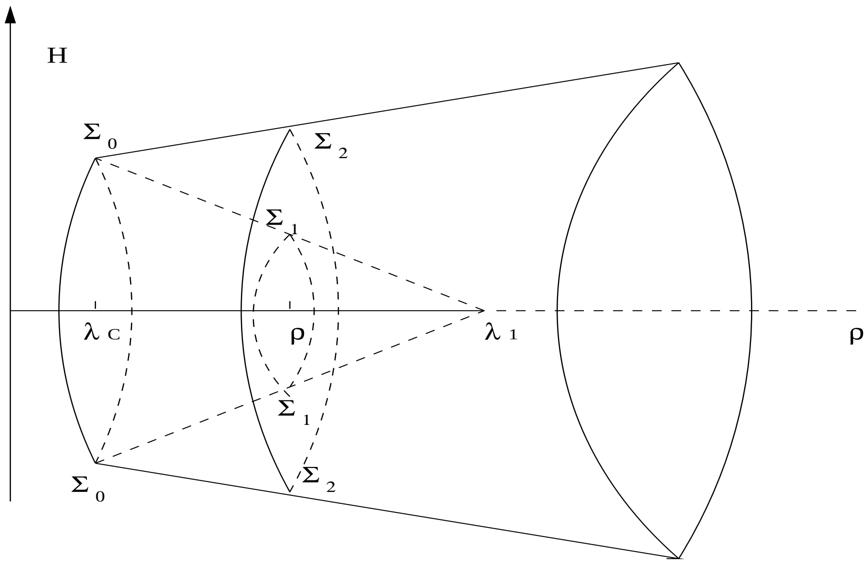

FIGURE 3.2. Subcritical bifurcation diagram for cubic-quintic CGLE. Solid lines denote stable states and dashed lines unstable states. For $\lambda_c < \rho < \lambda_1$, Σ_1 represents the inner circle $(e^{i\theta}u_1^\rho, \rho)$ and Σ_2 represents the outer circle $(e^{i\theta}u_2^\rho, \rho)$, both of which are bifurcated from nonzero point $(u^*, \lambda_c) = (e^{i\theta}u^*, \lambda_c)$.

2. Bifurcation of Cubic CGLE with Periodic Boundary Condition

In this section, we study the cubic CGLE under the periodic boundary condition. We need the following function spaces:

$$H = L^2_{per}(\Omega) = \{u \in L^2(\Omega) \,|\, u \text{ is } \Omega - \text{periodic}\},$$

$$H_1 = H^2_{per}(\Omega) = \{u \in H^2(\Omega) \,|\, u \text{ is } \Omega - \text{periodic}\}.$$

We now have the following supercritical bifurcation theorem for the cubic CGLE.

THEOREM 3.1. *If two coefficients ρ_0 and ρ_1 satisfy*

$$1 - \frac{\rho_1}{\rho_0} < 0,$$

then the following assertions hold.

(1) *$u = 0$ is globally asymptotically stable for $\rho \leq 0$.*

(2) *The problem (3.5) bifurcates from* $(u, \rho) = (0, 0)$ *to an attractor* Σ_ρ *as* ρ *crosses* 0.

PROOF. Let $u = u_1 + iu_2$ then (3.5) can be equivalently written as

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 - \rho_0 \Delta u_2 + \rho u_1 + |u|^2 u_1 - \rho_1 |u|^2 u_2, \\ \dfrac{\partial u_2}{\partial t} = \rho_0 \Delta u_1 + \Delta u_2 + \rho u_2 + |u|^2 u_2 + \rho_1 |u|^2 u_1, \\ u_1(x,0) = \phi(x), \quad u_2(x,0) = \psi(x). \end{cases} \tag{3.7}$$

The mappings $L_\rho = -A + B_\rho$ and $G : H_1 \to H$ are defined as

$$\begin{aligned} -Au &= \begin{pmatrix} \Delta u_1 - \rho_0 \Delta u_2 \\ \rho_0 \Delta u_1 + \Delta u_2 \end{pmatrix}, \\ B_\rho u &= \rho \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ Gu &= \begin{pmatrix} |u|^2 u_1 - \rho_1 |u|^2 u_2 \\ |u|^2 u_2 + \rho_1 |u|^2 u_1 \end{pmatrix}. \end{aligned}$$

It is known that $H_{1/2} = H^1_{per}$, therefore $G : H_{1/2} \to H$ is C^∞.

We know that the eigenvalue problem

$$\begin{cases} -\Delta e_k = \lambda_k e_k, \\ e_k(x + 2j\pi) = e_k(x) \end{cases} \tag{3.8}$$

has eigenvalues given by

$$\lambda_k = |k|^2 = k_1^2 + \cdots + k_n^2, \; k = (k_1, \cdots, k_n) \text{ and } k_i = 0, 1, \cdots \; (1 \le i \le n)$$

and the eigenvectors corresponding to λ_k are given by

$$\sin(k_1 x_1 + \cdots + k_n x_n) \text{ and } \cos(k_1 x_1 + \cdots + k_n x_n).$$

It is easy to see that the eigenvalues of $L_\rho = -A + B_\rho$ are given by

$$\beta_k = (\rho - |k|^2) + i|k|^2 \rho_0$$

and the eigenvectors are

$$\cos kx \pm i\cos kx, \quad \cos kx \pm i\sin kx,$$

$$\sin kx \pm i\cos kx, \quad \sin kx \pm i\sin kx,$$

where $kx = k_1x_1 + \cdots + k_nx_n$.

For the first eigenvalue $\beta_0 = \rho$ of the operator $L_\rho : H_1 \to H$, we can see that it satisfies (1.8) and (1.9) for $\lambda_0 = 0$ and β_0 has multiplicity two.

From equation (3.7), we have

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2} = \rho|u|^2_{L^2} - |\nabla u|^2_{L^2} + |u|^4_{L^4} \tag{3.9}$$

and

$$\rho_0|\nabla u|^2_{L^2} = \rho_1|u|^4_{L^4}. \tag{3.10}$$

Replacing (3.9) by (3.10), we have

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2} = \rho|u|^2_{L^2} + (1 - \frac{\rho_1}{\rho_0})|u|^4_{L^4}. \tag{3.11}$$

Since $1 - \frac{\rho_1}{\rho_0} < 0$, we have a following inequality

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2} \leq \rho|u|^2_{L^2} + (1 - \frac{\rho_1}{\rho_0})|\Omega|^{-1}|u|^4_{L^2}.$$

Therefore, $u = 0$ is globally asymptotically stable for $\rho \leq 0$. Thanks to Theorem 1.3 we can conclude that the problem (3.5) has a supercritical bifurcation from the trivial solution.

The proof is complete. □

3. Bifurcation of Cubic-Quintic CGLE with Dirichlet Boundary Condition

In the previous section we achieved a supercritical bifurcation in some parameter range. Outside of this range we need, at the least, a quintic term to saturate the explosive instability provided by the nonlinear cubic term. In this section, we explore attractor bifurcations of the cubic-quintic CGLE under the Dirichlet boundary condition.

Let λ_1 be the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition (3.2). Then we have the following bifurcation theorem for the cubic-quintic CGLE.

THEOREM 3.2. *For the Dirichlet boundary condition we have the following assertions*

(1) $u = 0$ *is locally asymptotically stable for* $\rho < \lambda_1$ *and unstable for* $\rho \geq \lambda_1$.

(2) *The problem (3.6) bifurcates from* $(u, \rho) = (0, \lambda_1)$ *to an invariant set* Σ_k *and has no bifurcation on* $\rho > \lambda_1$.

(3) *There exists a saddle node bifurcation point* $0 < \lambda_c < \lambda_1$ *of the problem (3.6).*

(4) *At* $\rho = \lambda_c$*, there is an invariant set* $\Sigma_0 = \Sigma_{\lambda_c}$ *with* $0 \notin \Sigma_0$.

(5) *For* $\rho < \lambda_c$*, there is no invariant set near* Σ_0.

(6) *For* $\lambda_c < \rho < \lambda_1$*, there are two branches of invariant sets* Σ_1 *and* Σ_2*, and* Σ_2 *extends to* $\rho \geq \lambda_1$ *and near* λ_1 *as well. Moreover, we have*
   (a) Σ_1 *is a repeller and* Σ_2 *is an attractor with* $0 \notin \Sigma_2$.
   (b) $\Sigma_i = S^1$ *is a cycle consisting of steady states* $(i = 1, 2)$*. In particular, the bifurcation is a Hopf bifurcation provided* $\rho_0 \neq 0$.

PROOF. We divide the proof into several steps.

STEP 1. Let $H_1 = H^2(\Omega) \cap H_0^1(\Omega)$, $H = L^2(\Omega)$ and $u = u_1 + iu_2$. Then (3.6) with (3.4) can be equivalently written as

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = \Delta u_1 - \rho_0 \Delta u_2 + \rho u_1 + |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2, \\
\dfrac{\partial u_2}{\partial t} = \rho_0 \Delta u_1 + \Delta u_2 + \rho u_2 + |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1, \\
u_1(x, 0) = \phi(x), \quad u_2(x, 0) = \psi(x).
\end{cases}
\tag{3.12}
$$

The mappings $L_\rho = -A + B_\rho$ and $G : H_1 \to H$ are defined as

$$\begin{aligned} -Au &= \begin{pmatrix} \Delta u_1 - \rho_0 \Delta u_2 \\ \rho_0 \Delta u_1 + \Delta u_2 \end{pmatrix}, \\ B_\rho u &= \rho \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ Gu &= \begin{pmatrix} |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2 \\ |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1 \end{pmatrix}. \end{aligned}$$

It is known that $H_{1/2} = H_0^1(\Omega)$, therefore $G : H_{1/2} \to H$ is C^∞.

Let $\{\lambda_k\} \subset \mathbb{R}$ and $\{e_k\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be the eigenvalues and eigenvectors of $-\Delta$ with the Dirichlet boundary condition (3.2)

$$\begin{cases} -\Delta e_k = \lambda_k e_k, \\ e_k|_{\partial\Omega} = 0. \end{cases}$$

Note that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and $\{e_k\}$ forms an orthogonal basis of $L^2(\Omega)$.

Since the eigenvalues of A are given by

$$\lambda_k \pm i\rho_0 \lambda_k, \ k = 1, 2, \cdots$$

with the corresponding eigenvectors

$$z_k = e_k + ie_k,$$

and $\{e_k, ie_j | 1 \leq k, j < \infty\}$ is an orthogonal basis of $L^2(\Omega)$, the operator L_ρ is a sectorial operator. Also note that the eigenvalues of $L_\rho = -A + B_\rho$ are given by

$$(\rho - \lambda_k) \pm i\rho_0 \lambda_k, \qquad k = 1, 2, \cdots. \tag{3.13}$$

From (3.13), we have the principles of the exchange of stabilities :

$$Re((\rho-\lambda_1)\pm i\rho_0\lambda_1)\begin{cases} <0 & \text{if } \rho<\lambda_1, \\ =0 & \text{if } \rho=\lambda_1, \\ >0 & \text{if } \rho>\lambda_1, \end{cases}$$

$$Re((\rho-\lambda_k)\pm i\rho_0\lambda_k)=\rho-\lambda_k<0 \quad \text{for } \rho=\lambda_1\ (k\geq 2).$$

Step 2. We now find bifurcation equations of (3.6) reduced to the center manifold. The space H and H_1 can be decomposed into the form

$$\begin{aligned} H_1 &= E_1\oplus E_2 \quad \text{and} \quad H=E_1\oplus \widetilde{E}_2, \\ E_1 &= \{(x_1+iy_1)e_1 \,|\, x_1,y_1\in\mathbb{R}\}, \\ E_2 &= \left\{\sum_{k=2}^{\infty}(x_k+iy_k)e_k \,|\, \sum_{k=2}^{\infty}\lambda_k^2(x_k^2+y_k^2)<\infty\right\}, \\ \widetilde{E}_2 &= \left\{\sum_{k=2}^{\infty}(x_k+iy_k)e_k \,|\, \sum_{k=2}^{\infty}(x_k^2+y_k^2)<\infty\right\}, \end{aligned}$$

and the operator L_ρ is decomposed into

$$\begin{cases} L_\rho=\mathcal{L}_1^\rho\oplus\mathcal{L}_2^\rho, \\ \mathcal{L}_1^\rho=L_\rho|_{E_1} : E_1\to E_1,\ \mathcal{L}_2^\rho=L_\rho|_{E_2} : E_2\to\widetilde{E}_2. \end{cases}$$

It fulfills the conditions of Theorem 1.6. Therefore there exists a center manifold function $\Phi=\Phi_1+i\Phi_2 : E_1\longrightarrow\widetilde{E}_2$ satisfying

$$\Phi(x_1,y_1)=o(|x_1|,|y_1|).$$

Let $u=v_1+v_2\in E_1\oplus E_2$ then we have have the bifurcation equations of (3.6)

$$\frac{dv_1}{dt}=\mathcal{L}_1^\rho v_1+G(v_1+\Phi(v_1)),$$

and it is equivalent to

$$\begin{cases} \dfrac{dx_1}{dt}=(\rho-\lambda_1)x_1+\rho_0\lambda_1y_1+P_1G(v_1+\Phi(v_1)), \\ \dfrac{dy_1}{dt}=-\rho_0\lambda_1x_1+(\rho-\lambda_1)y_1+P_2G(v_1+\Phi(v_1)), \end{cases} \tag{3.14}$$

where

$$P_1G(u) = \int_\Omega \left[|u|^2u_1 - \rho_1|u|^2u_2 - |u|^4u_1 + \rho_2|u|^4u_2\right] e_1dx,$$
$$P_1G(u) = \int_\Omega \left[|u|^2u_2 + \rho_1|u|^2u_1 - |u|^4u_2 - \rho_2|u|^4u_1\right] e_1dx.$$

Since

$$u_1 = x_1e_1 + \Phi_1(x_1, y_1), \;\; u_2 = y_1e_1 + \Phi_2(x_1, y_1)$$

and

$$\Phi(x_1, y_1) = o(|x_1|, |y_1|),$$

we have

$$\begin{aligned} P_1G(x_1, y_1) = & \int_\Omega \left[(x_1^2 + y_1^2)x_1 - \rho_1(x_1^2 + y_1^2)y_1\right] e_1^4dx \\ & + \int_\Omega \left[-(x_1^2 + y_1^2)^2x_1 + \rho_2(x_1^2 + y_1^2)^2y_1\right] e_1^6dx \\ & + o(|x_1|^3, |y_1|^3) \\ & = \tau(x_1 - \rho_1y_1)(x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3). \end{aligned}$$

where $\tau = \int_\Omega e_1^4dx > 0$. Therefore, (3.14) is rewritten as

$$\begin{cases} \dfrac{dx_1}{dt} = (\rho - \lambda_1)x_1 + \rho_0\lambda_1y_1 + \tau(x_1 - \rho_1y_1)(x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3), \\ \dfrac{dy_1}{dt} = -\rho_0\lambda_1x_1 + (\rho - \lambda_1)y_1 + \tau(y_1 + \rho_1x_1)(x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3). \end{cases} \tag{3.15}$$

Let

$$v(x_1, y_1) = -\tau \begin{pmatrix} (x_1 - \rho_1y_1)(x_1^2 + y_1^2) \\ (y_1 + \rho_1x_1)(x_1^2 + y_1^2) \end{pmatrix}.$$

Then, we have

$$< v(x_1, y_1), (x_1, y_1) >= -\tau(x_1^2 + y_1^2)^2 < 0. \tag{3.16}$$

Step 3. Now, we consider the time-reversed semigroup $S_\rho(-t)$ generated by (3.15). We can see that $S_\rho(-t)$ has the same dynamic properties as the equations

$$\begin{cases} \dfrac{dx_1}{dt} = -(\rho - \lambda_1)x_1 - \rho_0\lambda_1y_1 - \tau(x_1 - \rho_1y_1)(x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3), \\ \dfrac{dy_1}{dt} = \rho_0\lambda_1x_1 - (\rho - \lambda_1)y_1 - \tau(y_1 + \rho_1y_1)(x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3). \end{cases} \tag{3.17}$$

and it is equivalent to

$$\frac{dz_1}{dt} = -(\rho - \lambda_1)z_1 + i\rho_0\lambda_1 z_1 - \tau(1+i\rho_1)|z_1|^2 z_1 + o(|z_1|^3).$$

Therefore we have

$$\frac{1}{2}\frac{d}{dt}|z_1|^2 = -(\rho - \lambda_1)|z_1|^2 - \tau|z_1|^4 + o(|z_1|^4). \tag{3.18}$$

Let $\widetilde{S_\rho}(t)$ be a semigroup which is generated by (3.17). Then from (3.18) we can see that $u = 0$ is the locally asymptotically stable solution of (3.17) for $\rho \geq \lambda_1$. Therefore, $u = 0$ is the locally asymptotically stable solution of (3.15) for $\rho < \lambda_1$ and unstable solution for $\rho \geq \lambda_1$. Hence, for the semigroup $S_\rho(t) = \widetilde{S_\rho}(-t)$ generated by (3.15) bifurcates from $(u, \rho) = (0, \lambda_1)$ to an attractor Σ_1 for $\rho < \lambda_1$ and it is obvious that it is a repeller for the time-reversed case.

If $\rho_0 \neq 0$, the bifurcation is the typical Hopf bifurcation since λ_1 is simple so that it $\Sigma_1 = S^1$. Moreover, if $\rho_0 = 0$ then we can infer (3.16) and Theorem 1.4 that the bifurcated invariant set Σ_1 is homeomorphic to S^1.

Step 4. From equation (3.18) and the Hölder's inequality, we have

$$\frac{1}{2}\frac{d}{dt}||z_1||^2_{L^2} \leq -(\rho - \lambda_1)||z_1||^2_{L^2} - \frac{\tau}{|\Omega|}||z_1||^4_{L^2},$$

which implies that $u = 0$ is the globally asymptotically stable steady state solution of (3.12) near $\rho = 0$. By the existence of global attractor of (3.6), we can see that there exists a $\lambda_c\,(0 < \lambda_c < \lambda_1)$, such that if $\rho < \lambda_c$, the equation has no nonzero singular points and it generates, at least, a cycle Σ_0 of singular point at $\rho = \lambda_c$. Moreover, if $\rho > \lambda_c$, it bifurcates from Σ_0 to two cycles Σ_1 and Σ_2, which consist of singular points, such that Σ_1 is a repeller and Σ_2 is an attractor with $0 \notin \Sigma_2$ at $\rho = \lambda_1$. We can describe the structure of them in detail as follows:

For $\rho > \lambda_c$, the equations (3.12) bifurcate from $(u, \rho) = (u^*, \lambda_c)\,(u^* \neq 0)$ to two singular points $u_1^\rho = e^{i\theta} u_1^\rho$ and $u_2^\rho = e^{i\theta} u_2^\rho$ in H satisfying

$$\lim_{\rho \to \lambda_c + 0} u_i^\rho = u^*, \ i = 1, 2,$$

$$\lim_{\rho \to \lambda_1 - 0} u_1^\rho = 0,$$

$$u_2^\rho \neq 0 \text{ at } \rho = \lambda_1,$$

and u_2^ρ are in an attractor Σ_2 for $\rho > \lambda_c$, as shown in Figure 3.2.

Step 5. Now we want to consider the steady state bifurcation. If $\rho_0 \neq 0$ then the eigenvalues (3.13) of L_ρ at $\rho = \lambda_1$ are nonzero so that $L_\rho : H_1 \to H$ is a linear homeomorphism at $\rho = \lambda_1$, which means that (3.12) has no steady state bifurcation. Thus we assume $\rho_0 = 0$ which is a necessary condition for steady state bifurcation. Since the first eigenvalue of the linear operator $L_\rho = \rho u + (1 + \rho_0)\Delta u$ in the steady state equation

$$\rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2 u - (1 + i\rho_2)|u|^4 u = 0$$

is simple when we restrict our concerns only to real valued function space, classical Krasnoselskii theorem leads us to the steady state bifurcation for $\lambda_c < \rho < \lambda_1$. Since cubic-quintic CGLE is invariant under the gauge transformation

$$\psi \to \psi e^{i\theta}, \ \theta \in \mathbb{R}^1,$$

the set S_ρ of steady state solutions of the cubic-quintic CGLE appears as a cycle S^1. Since $\Sigma_\rho \subset \Sigma_k = S^1$, the bifurcated attractor $\Sigma_k\,(k = 1, 2)$ consists of only steady states.

The proof is complete. □

4. Bifurcation of Cubic-Quintic CGLE with Periodic Boundary Condition

For the cubic-quintic CGLE with the periodic boundary condition, we need the same function space and operators described in Section 2.

Referring to eigenvalue problem (3.8), we have the first eigenvalue $\beta_0 = 0$ of the operator L_ρ at $\rho = 0$. Let e_0 be the eigenvector corresponding to β_0 and

$$E_1 = \{z_0 e_0 \,|\, z_0 \in \mathbb{C}\} \text{ and } E_2 = E_1^{\perp}.$$

Let $u(= u_1 + iu_2) = v_1 + v_2 \in E_1 \oplus E_2$ then

$$u = z_0 e_0 + \sum_{k\geq 1} z_k e_k = (x_0 + iy_0)e_0 + \sum_{k\geq 1}(x_k + iy_k)e_k$$

where $\{e_k \,|\, e_k$ are eigenvectors of $\beta_k,\ k = 1, 2, \cdots\}$. Let

$$x = x_0 e_0,\ y = y_0 e_0$$

then by the center manifold theorem, there exists a center manifold function $\Phi = \Phi_1 + i\Phi_2 : E_1 \to \widetilde{E}_2$ satisfying

$$\Phi(x, y) = o(|x|, |y|)$$

and

$$u_1 = x + \Phi_1(x, y),\ u_2 = y + \Phi_2(x, y).$$

Then the bifurcation equations of (3.6) are given by

$$\frac{dv_1}{dt} = \mathcal{L}_1^\rho v_1 + G(v_1 + \Phi(v_1))$$

and it is equivalent to

$$\begin{cases} \dfrac{dx_0}{dt} = \rho x_0 + P_1 G(v_1 + \Phi(v_1)), \\ \dfrac{dy_0}{dt} = \rho y_0 + P_2 G(v_1 + \Phi(v_1)), \end{cases}$$

where

$$P_1 G(u) = \int_\Omega \left[|u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2\right] e_0 dx,$$
$$P_1 G(u) = \int_\Omega \left[|u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1\right] e_0 dx.$$

In the same fashion as in the proof of Theorem 3.2, we can obtain the following bifurcation theorem.

Theorem 3.3. *For the periodic boundary condition, we have the following assertions.*

1. $u = 0$ *is locally asymptotically stable for* $\rho < 0$ *and unstable for* $\rho \geq 0$.
2. *The problem (3.6) bifurcates from* $(u, \rho) = (0, 0)$ *to an invariant set* Σ_k *and has no bifurcation on* $\rho > \lambda_1$.
3. *There exists a saddle node bifurcation point* $\lambda_c < 0$ *of the problem (3.6).*
4. *For* $\lambda_c < \rho < 0$, *there are two branches of invariant sets* Σ_1 *and* Σ_2 *and* Σ_2 *extends to* $\rho \geq \lambda_1$ *and near* λ_1 *as well. Moreover, we have*
   (a) Σ_1 *is a repeller and* Σ_2 *is an attractor with* $0 \notin \Sigma_i$ *for* $i = 1, 2$.
   (b) $\Sigma_i = S^1$ *is a cycle consisting of steady states (for* $i = 1, 2$*). In particular, the bifurcation is a Hopf bifurcation provided* $\rho_0 \neq 0$.

Remark 3.4. It is known that the saddle node bifurcation point λ_c can be obtained for the periodic boundary condition, and it is given as $\lambda_c = -\frac{1}{4}$, see [**7, 32**] for details.

Remark 3.5. We can replace the stable quintic nonlinear term with an arbitrary odd degree stable term, i.e., we can replace $|u|^4 u$ by $|u|^{2\sigma} u$ $(\sigma \geq 2)$.

Remark 3.6. If both cubic and quintic terms are stable, the results are the same as in [**21**]. Therefore, we only need to take an unstable cubic and a stable quintic term.

5. The Existence of a Global Attractor of CGLE

In this subsection, we shall describe the existence of a global attractor of the cubic-quintic CGLE which was used in the proof of Theorem 3.2. We shall follow the same functional settings, boundary conditions and eigenvalues as defined in previous sections.

For the equation under consideration, we will prove the existence of absorbing sets in $L^2(\Omega)$ and $H^2(\Omega) \cap H_0^1(\Omega)$ under the Dirichlet boundary condition, and proving the existence of absorbing sets amounts to proving a priori estimates.

To prove the existence of an absorbing set in $L^2(\Omega)$, we multiply (3.6) by complex conjugate u^* of u and integrate over Ω, then we obtain

$$\frac{d}{dt}|u|_{L^2}^2 + 2|\nabla u|_{L^2}^2 + \int_\Omega 2|u|^6 - 2|u|^4 - 2\rho|u|^2 \, dx = 0.$$

Since

$$\frac{1}{2}|u|^6 - C_1 \leq |u|^6 - |u|^4 - \rho|u|^2, \quad \text{for some } C_1 > 0,$$

we have

$$\frac{d}{dt}|u|_{L^2}^2 + 2|\nabla u|_{L^2}^2 + \int_\Omega |u|^6 dx \leq 2C_1|\Omega|,$$

where $|\Omega|$ is the measure of Ω. Due to the Poincaré inequality, we have

$$\frac{d}{dt}|u|_{L^2}^2 + 2\lambda_1|u|_{L^2}^2 \leq C_2, \quad C_2 = 2C_1|\Omega|.$$

Using the classical Gronwall lemma, we see that

$$|u(t)|_{L^2}^2 \leq |u_0|_{L^2}^2 \exp(-2\lambda_1 t) + \frac{C_2}{2\lambda_1}(1 - \exp(-2\lambda_1 t)).$$

Thus

$$\limsup_{t \to +\infty} |u(t)|_{L^2} \leq \eta_0, \quad \eta_0^2 = \frac{C_2}{2\lambda}.$$

There exists an absorbing set $\mathcal{B}_0$ in $L^2(\Omega)$, namely, any ball of $L^2(\Omega)$ centered at 0 of radius $\eta_0' > \eta_0$. If $\mathcal{B}$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0, R)$ of $L^2(\Omega)$, centered at 0 of radius R, then $S(t)\mathcal{B} \subset B(0, \eta_0')$ for $t \geq t_0(\mathcal{B}, \eta_0')$, where

$$t_0 = \frac{1}{2\lambda_1} \ln \frac{R^2}{(\eta_0')^2 - \eta_0^2}.$$

We now prove the existence of an absorbing set in $H^2(\Omega) \cap H_0^1(\Omega)$ and the uniform compactness of $S(t)$. Multiplying (3.6) by $-\Delta u^*$ and integrating over Ω again, we obtain

$$-\int_\Omega \frac{\partial u}{\partial t}\Delta u^* dx = -(1+i\rho_0)\int_\Omega |\Delta u|^2 dx - \rho\int_\Omega u\Delta u^* dx$$
$$-(1+i\rho_1)\int_\Omega |u|^2 u\Delta u^* dx + (1+i\rho_2)\int_\Omega |u|^4 u\Delta u^* dx.$$

We have, using the boundary condition and the Green formula,

$$-\int_\Omega \frac{\partial u}{\partial t}\Delta u^* dx = \frac{1}{2}\frac{d}{dt}|\nabla u|^2_{L^2},$$

$$-\int_\Omega \left[\rho u + (1+i\rho_1)|u|^2 u - (1+i\rho_2)|u|^4 u\right]\Delta u^* dx$$
$$= \int_\Omega \left[\rho u + (1+i\rho_1)|u|^2 u - (1+i\rho_2)|u|^4 u\right]' |\nabla u|^2 dx.$$

Hence

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2_{L^2} = -|\Delta u|^2_{L^2} + \int_\Omega \left[\rho u + |u|^2 u - |u|^4 u\right]' |\nabla u|^2 dx$$
$$= -|\Delta u|^2_{L^2} + \int_\Omega \left[\rho u + \frac{3}{2}|u|^2 - 3|u|^4\right] |\nabla u|^2 dx.$$

We also infer from general results on the Dirichlet problem in Ω on $H^2(\Omega)\cap H^1_0(\Omega)$, and therefore,

$$|\nabla u|^2_{L^2} \le \frac{1}{\lambda_1}|\Delta u|^2_{L^2}.$$

Due to

$$2\rho + 3|u|^2 - 6|u|^4 \le -\frac{1}{2}|u|^4 + C_3 \quad \text{for some } C_3 > 0,$$
$$\frac{d}{dt}|\nabla u|^2_{L^2} \le \frac{C_3}{2}|\nabla u|^2_{L^2} \equiv C_4|\nabla u|^2_{L^2}.$$

If $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$, then the Gronwall lemma shows that

$$|\nabla u(t)|^2_{L^2} \le |\nabla u_0|^2_{L^2}\exp(C_4 t),\ \forall t > 0.$$

The same arguments as in $L^2(\Omega)$ can be considered and we can conclude that the cubic-quintic CGLE possesses a maximal attractor $\mathcal{A}$ which is bounded in $H^2(\Omega)\cap H^1_0(\Omega)$, compact and connected in $L^2(\Omega)$. Its basin of attraction is the whole space $L^2(\Omega)$, and $\mathcal{A}$ attracts the bounded sets of $L^2(\Omega)$.

Remark 3.7. For the periodic boundary condition, we achieve the same result using Hölder's and Young inequalities instead of the Poincaré inequality.

CHAPTER 4

Bifurcation for the Infinite Prandtl Number Convection

1. Introduction

Rayleigh-Bénard convection, that is, a buoyancy-driven convection in a fluid layer heated from below and cooled from above, is one of the prime examples of bifurcating high-dimensional systems. It has long been a subject of intense theoretical and experimental study and has been applied to many different areas of study such as meteorology, geophysics, and astrophysics. In this chapter, we consider a three-dimensional (3D) convection problem. The governing equations are the following Boussinesq equations:

$$\frac{1}{P_r}\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nabla p - \Delta u - RTk = 0, \tag{4.1}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \Delta T = 0, \tag{4.2}$$

$$\nabla \cdot u = 0, \tag{4.3}$$

where $u = (u_1, u_2, u_3)$ is the velocity field, p is the pressure function, T is the temperature field and $k = (0, 0, 1)$ is the unit vector in x_3-direction.

In the Boussinesq equations, we have two important numbers: the Rayleigh number,

$$R = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa},$$

which measures the ratio of overall buoyancy force to the damping coefficients, and the Prandtl number,

$$P_r = \frac{\nu}{\kappa},$$

which measures the relative importance of kinematic viscosity over thermal diffusivity. Here, ν and κ are the kinematic viscosity and thermal diffusive coefficients respectively, α is the thermal expansion coefficient of the fluid, g is the gravitational constant, h is the distance between two plates confining the fluid and $T_2 - T_1$ is the temperature difference between the bottom and top plates.

Due to the fact that mathematical information is very limited, the complicated equations have been simplified. For example, the infinite Prandtl number limit of the Boussinesq equations has been used as the standard model for the convection of earth's mantle, where it is argued that P_r could be of the order 10^{24}, as well as for many gasses under high pressure. A Broader rationale for investigating the infinite Prandtl number convection is based on the observation of both the linear and weakly nonlinear theories; that fluids with $P_r > 1$ convect in a similar fashion. Moreover, the infinite Prandtl number model of convection can also be justified as the limit of the Boussinesq approximation to the Rayleigh-Bénard convection as the Prandtl number approaches infinity [**38, 39**].

In the limit of the infinite Prandtl number, the inertial terms in the momentum equation can be dropped, thus we are left with a linear dependence of the velocity field on temperature:

$$\nabla p - \Delta u - RTk = 0, \tag{4.4}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \Delta T = 0, \tag{4.5}$$

$$\nabla \cdot u = 0. \tag{4.6}$$

Since the velocity field is linearly dependent on the temperature field in the infinite Prandtl number convection, the velocity field has a behavior that is much more regular than the finite Prandtl number model. Thus this is a distinct advantage of studying this model. In particular, by investigating the structure of the attractor generated only by T(much more convenient), it is possible to reconstruct the structure of the attractor in terms of (u, T), which has the same topological structure as is obtained in terms of only T.

Extensive mathematical studies have been conducted for the Rayleigh-Bénard convection since Rayleigh's work. In particular, for the Rayleigh-Bénard convection with the finite Prandtl numbers, readers are referred to Chandrasekhar [**4**], to Drazin and Reid [**11**] for linear theories; to Foias, Manley and Temam [**13**] for the existence and physical bounds of attractors; to Rabinowitz [**31**] for the existence of rectangular solutions; and to Ma and Wang [**23, 25**] for attractor bifurcation. For the case regarding the infinite Prandtl number, the readers are referred to Constantin and Doering [**8, 9**] for the upper bounds of the minimal conduction value; to Schnaubelt and Busse [**35**] for two-dimensional convection rolls; to Keken [**18**] and to Yanagisawa and Tamagishi [**40**] for mantle and spherical shell convection studies, respectively; to X. Wang [**38, 39**] for the justification of the infinite Prandtl number convection as the infinite Prandtl number limit of the Boussinesq equations; and to Park [**27**] for the existence of attractor bifurcation.

For the Rayleigh-Bénard convection with the infinite Prandtl number, bifurcation is obtained with respect to the parameter R and the Rayleigh number. The main results obtained can be summarized as follows.

First, we show that as the Rayleigh number R crosses the first critical value R_c, the Boussinesq equations bifurcate from the trivial solution to an attractor Σ_R, with dimension between $m-1$ and m. Here the first critical Rayleigh number R_c is defined to be the first eigenvalue of the linear eigenvalue problem, and m is the multiplicity of R_c.

Second, as an attractor, the bifurcated attractor Σ_R has asymptotic stability in the sense that it attracts all solutions with initial data in the phase space outside of the stable manifold of the trivial solution, with codimension m.

Third, if, the first eigenvalue R_c is simple, the system particularly bifurcates exactly two steady state solutions and each of them is a minimal attractor.

In [**23**], Ma and Wang showed the existence of bifurcation of the Rayleigh-Bénard convection in the case of finite Prandtl number P_r. In the infinite Prandtl number case, the following two features make this problem worthy of study.

(i) The first equation of the Boussinesq equations does not involve the time derivative term, thus the solution u, the velocity field, becomes diagnostic and can be solved in terms of the temperature function T. This allows us to approach this problem from different point of view. That is, instead of considering the bifurcation problem of equations (4.4)–(4.6) in $(u, T) \in L^2(\Omega)^3 \times L^2(\Omega)$, we consider

$$\frac{\partial T}{\partial t} + (u(T) \cdot \nabla)T - \Delta T = 0, \tag{4.7}$$

$$\nabla \cdot u = 0 \tag{4.8}$$

in only $T \in L^2(\Omega)$. In Theorem 4.2, which will follow in Section 3, the Boussinesq equation bifurcates from $T = 0$ to an attractor Σ_R.

(ii) Since the velocity field depends on the temperature field T, we just investigate the structure of the attractor generated by only T. Moreover, since we can get the velocity field $u(T)$ associated with the temperature field T, the structure of the velocity field is reconstructed by T in terms of the eigenvectors. This tells us that we can reconstruct the attractor in terms of (u, T), which has the same topological structure as that which is achieved in terms of only T.

This chapter is organized as follows. In Section 2, we will summarize the the Boussinesq equations and their mathematical setting and in Section 3, we will prove the existence of bifurcation.

2. Bénard Problem and It's Mathematical Setting

In this chapter we deal with the Boussinesq equations on the non-dimensional domain $\Omega = D \times (0,1) \subset \mathbb{R}^3$, where $D \subset \mathbb{R}^2$ is an open set and impose the boundary conditions at the plates which are

$$u = 0, \quad T = 0, \quad \text{at} \quad x_3 = 1,$$

$$u = 0, \quad T = 1, \quad \text{at} \quad x_3 = 0.$$

(See the figure 4.1)

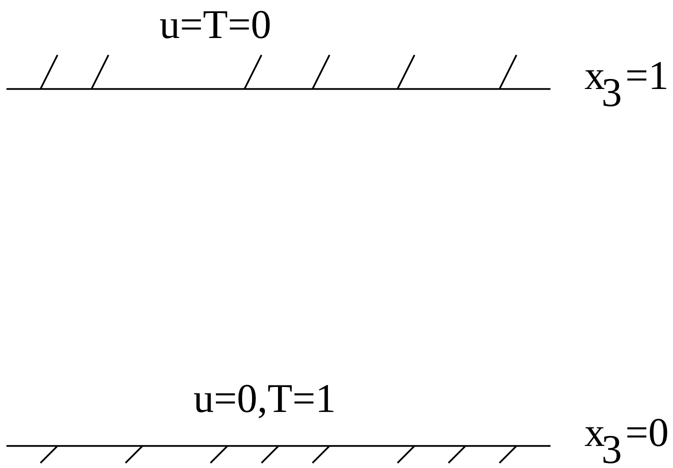

FIGURE 4.1. Flow between two plates heated from below

The basic linear profiles of (4.4)–(4.6) are steady state solutions given by

$$u = 0,$$

$$T = 1 - x_3,$$

$$p = p_0 + R(x_3 - \frac{x_3^2}{2})k.$$

Let q be the difference between p and the steady state solutions, and let θ be the difference between T and the steady states solutions, i.e, $p = p_0 + R(x_3 - \frac{x_3^2}{2})k + q$ and $T = (1 - x_3) + \theta$. Then the equations for the perturbation of these trivial solutions are derived as

$$\nabla q - \Delta u - R\theta k = 0,$$
$$\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \Delta\theta - u_3 = 0,$$
$$\nabla \cdot u = 0,$$

and the boundary conditions changed into

$$u = 0, \quad \theta = T, \quad \text{at} \quad x_3 = 1,$$
$$u = 0, \quad \theta = T - 1, \quad \text{at} \quad x_3 = 0$$

Let $T = \sqrt{R}\sqrt{\theta}$ and $p = q$. Then we have

$$\nabla p - \Delta u - \sqrt{R}Tk = 0, \tag{4.9}$$
$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \Delta T - \sqrt{R}u_3 = 0, \tag{4.10}$$
$$\nabla \cdot u = 0, \tag{4.11}$$
$$u = 0,\ T = 0 \ \text{ at } x_3 = 0, 1. \tag{4.12}$$

The Boussinesq equations (4.9)–(4.11) are basic equations to study the infinite Prandtl number convection problem in this chapter. For an initial value problem we also provide the initial value of (u, T) as

$$(u, T) = (u_0, T_0) \qquad \text{at } t = 0. \tag{4.13}$$

On the lateral boundary $\partial D \times [0, 1]$, we use the periodic condition;

$$\begin{cases} u = 0, \quad T = 0 \qquad \text{at } x_3 = 0, 1, \\ (u, T)(x, t) = (u, T)(x_1 + k_1L_1, x_2 + k_2L_2, x_3, t), \ \text{ for } x \in \Omega. \end{cases} \tag{4.14}$$

Since (4.9) together with (4.11) is a Stokes equation, we have the solution $u = u(T)$ under the above boundary condition. By replacing u and its third component u_3 in (4.10) by solution which depends on T, we would consider only the following equation.

$$\frac{\partial T}{\partial t} + (u(T) \cdot \nabla)T - \Delta T - \sqrt{R} u_3(T) = 0, \tag{4.15}$$

where $u(T)$ and $u_3(T)$ satisfy (4.9) together. The Boussinesq equations (4.15) is a crucial equation to study the Rayleigh-Bénard problem in this paper.

For an initial value problem we also provide the initial value of T as

$$T = T_0 \qquad \text{at } t = 0. \tag{4.16}$$

In addition, we can consider boundary condition such as

$$\begin{cases} T = 0 & \text{at } x_3 = 0, 1, \\ T(x,t) = T(x_1 + k_1 L_1, x_2 + k_2 L_2, x_3, t), & \text{for } x \in \Omega \end{cases} \tag{4.17}$$

instead of boundary condition (4.14).

We recall here the functional setting of equation (4.15) with initial and boundary conditions (4.16) and (4.17). Let

$$H = L^2(\Omega) \quad \text{and} \quad H_1 = H_0^1(\Omega) \cap H^2(\Omega)$$

where $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$, which vanish at $x_3 = 0, 1$ and are periodic in the x_i-directions ($i = 1, 2$).

Thanks to existence results, we can define a semi-group

$$S(t) : (u_0, T_0) \to (u(t), T(t)),$$

which enjoys the semi-group properties.

Remark 4.1. We can use other boundary conditions as follows;

(1) Dirichlet boundary condition:

$$u = 0, \quad T = 0 \ (\text{or } \frac{\partial T}{\partial n} = 0),$$

(2) Free boundary condition:

$$T = 0, \quad u_n = 0, \quad \frac{\partial u_\tau}{\partial n} = 0,$$

where n and τ are the unit normal and tangent vectors on $\partial D \times [0, 1]$ respectively, and $u_n = u \cdot n$, $u_\tau = u \cdot \tau$.

3. Bifurcation for Infinite Prandtl Number Convection

First, we want to find the critical Rayleigh number which plays the role of the critical value in the abstract theory. To do so, we consider the linearized equations of (4.15) with the boundary conditions (4.17):

$$-\Delta T - \sqrt{R} u_3(T) = 0 \tag{4.18}$$

where $u_3(T)$ satisfies equation (4.9)

Since this eigenvalue problem for the Rayleigh number R is symmetric, all eigenvalues R_k with multiplicities m_k of (4.18) with (4.17) are real numbers, and

$$0 < R_1 < \cdots < R_k < R_{k+1} < \cdots \tag{4.19}$$

in the sense that

$$\begin{cases} -\Delta u + \nabla p - \lambda T k = 0, \\ -\Delta T - \lambda u_3 = 0, \\ \nabla \cdot u = 0 \end{cases}$$

is equivalent to

$$\int_\Omega \left[\nabla u \cdot \nabla \widetilde{u} + \nabla T \cdot \nabla \widetilde{T} \right] dx = \lambda \int_\Omega \left[T \widetilde{u_3} + u_3 \widetilde{T} \right] dx,$$

where $\lambda = \sqrt{R}$.

The first eigenvalue R_1, denoted by $R_1 = R_c$, is called the critical Rayleigh number. Let the multiplicity of R_c be $m_1 = m$ ($m \geq 1$) and the first eigenvectors

$T_1 \cdots, T_m$ of (4.18) be orthonormal:

$$\langle T_i, T_j \rangle_H = \int_\Omega T_i \cdot T_j dx = \delta_{ij}.$$

Then E_0, the first eigenspace of (4.18) with (4.17) is

$$E_0 = \left\{ \sum_{k=1}^{m} \alpha_k T_k \mid \alpha_k \in \mathbb{R},\ 1 \leq k \leq m \right\}. \tag{4.20}$$

Now we are ready to introduce the main result.

THEOREM 4.2. *For the Bénard problem (4.15) with (4.17), we have the following assertions:*

(1) *If $R \leq R_c$, the steady state $T = 0$ is a globally asymptotically stable equilibrium point of the equation.*
(2) *The equation bifurcates from $(T, R) = (0, R_c)$ to an attractor Σ_R for $R > R_c$, with $m - 1 \leq \dim \Sigma_R \leq m$, which is connected when $m > 0$.*
(3) *For any $T \in \Sigma_R$, the associated velocity field $u = u(T)$ which is achieved from given T, can be expressed as*

$$u = \sum_{k=1}^{m} \alpha_k e_k + o\left(\sum_{k=1}^{m} \alpha_k e_k \right), \tag{4.21}$$

where e_k are eigenvectors of (4.15) corresponding to each T_k.
(4) *The attractor Σ_R has the homotopy type of an $(m-1)$-dimensional sphere S^{m-1} provided Σ_R is a finite simplicial complex.*
(5) *For any bounded open set $U \subset L^2(\Omega)$ with $0 \in U$ there is an $\varepsilon > 0$ such that as $R_c < R < R_c + \varepsilon$, the attractor Σ_R attracts U/Γ in $L^2(\Omega)$, where Γ is the stable manifold of $T = 0$ with co-dimension m.*

PROOF. STEP 1. In the first step we will prove that $T = 0$ is a globally asymptotically stable equilibrium point of (4.15) at $\lambda_1 = \sqrt{R_c}$.

We know that the Boussinesq equation (4.15) has a bounded absorbing set in H. Assume that (4.15) has an invariant set $B \subset E_0$, with $B \neq \{0\}$ at $\lambda_1 = \sqrt{R_c}$. If restricted in B, (4.15) can be rewritten as

$$\frac{\partial T}{\partial t} + (u(T) \cdot \nabla)T = 0. \tag{4.22}$$

Let $T \in B$ be the solutions of (4.22). Then $\alpha T(\alpha t) \in \alpha B \subset E_0$ are also solutions of (4.22), since

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha T(\alpha t)) + [u(\alpha T(\alpha t) \cdot \nabla](\alpha T(\alpha t)) &= \alpha^2 \frac{\partial T(s)}{\partial s} + [u(\alpha T(s)) \cdot \nabla]\,(\alpha T(s)) \\ &= \alpha^2 \left[\frac{\partial T(s)}{\partial s} + (u(T(s)) \cdot \nabla)T(s)\right] = 0,\end{aligned}$$

where $s := \alpha t$, that is, for any $\alpha \in \mathbb{R}$, the set $\alpha B \subset E_0$ is an invariant set of (4.22). But it is an unbounded invariant set, which contradicts the existence of a bounded absorbing set since all invariant sets have the same bound in H as the absorbing set. Hence the invariant set B can only consist of $T = 0$. By Theorem 1.5, $T = 0$ at $\lambda = \sqrt{R_c}$ is globally asymptotically stable.

Step 2. Let H and H_1 be defined as in Section 2. Define $L_\lambda = -A + B_\lambda : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{cases} A(T) = -\Delta T, \\ B_\lambda(T) = \lambda u_3(T), \\ G(T) = -(u(T) \cdot \nabla)T, \end{cases} \tag{4.23}$$

where $\lambda = \sqrt{R}$. Then

$$\begin{aligned}\langle A(T), \widetilde{T}\rangle_H &= \int_\Omega -\Delta T \cdot \widetilde{T} dx \\ &= \int_\Omega T \cdot (-\Delta \widetilde{T}) dx \\ &= \langle T, A(\widetilde{T})\rangle_H.\end{aligned}$$

For arbitrary T and $\widetilde{T}$, we have two equations:

$$\nabla p - \Delta u(T) - \lambda T k = 0$$

and

$$\nabla p - \Delta u(\widetilde{T}) - \lambda \widetilde{T} k = 0,$$

which are true for T and $\widetilde{T}$, respectively. Moreover, these equations become

$$\int_\Omega \nabla p \cdot u(\widetilde{T}) dx - \int_\Omega \Delta u(T) \cdot u(\widetilde{T}) dx - \lambda \int_\Omega T \cdot u_3(\widetilde{T}) dx = 0$$

and

$$\int_\Omega \nabla p \cdot u(T) dx - \int_\Omega \Delta u(\widetilde{T}) \cdot u(T) dx - \lambda \int_\Omega \widetilde{T} \cdot u_3(T) dx = 0.$$

Due to the incompressibility of u, we have

$$\begin{aligned}\langle B_\lambda(T), \widetilde{T} \rangle_H &= \lambda \int_\Omega u_3(T) \cdot \widetilde{T} dx \\ &= \lambda \int_\Omega T \cdot u_3(\widetilde{T}) dx \\ &= \langle T, B_\lambda(\widetilde{T}) \rangle_H.\end{aligned}$$

Thus, A, B_λ are symmetric operators and so is $L_\lambda = -A + B_\lambda$. Moreover, G is orthogonal since

$$\begin{aligned}\langle G(T), T \rangle_H &= \int_\Omega -(u(T) \cdot \nabla) T \cdot T dx \\ &= -\int_\Omega \operatorname{div}\, u(T) \cdot T^2 dx = 0.\end{aligned}$$

By (4.19) we have (1.5). Moreover (1.4) and (1.6) also hold.

Due to the above arguments, we have an operator equation which is equivalent to the Boussinesq equation (4.15):

$$\frac{dT}{dt} = L_\lambda T + G(T). \tag{4.24}$$

The eigenvalue problem of the linear equation

$$L_\lambda T = \beta(\lambda) T \tag{4.25}$$

is equivalent to

$$-\Delta T - \lambda u_3(T) + \beta(\lambda) T = 0.$$

Because $u(T)$ satisfies

$$-\Delta u(T) + \nabla p - \lambda T k = 0,$$

we can put two equations together as follows:

$$\begin{cases} -\Delta u(T) + \nabla p - \lambda T k = 0, \\ -\Delta T - \lambda u_3(T) + \beta(\lambda) T = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{4.26}$$

Since L_λ is a symmetric completely continuous field, the eigenvalue $\beta_k(\lambda)\,(k = 1, 2, \cdots)$ of (4.26) are real numbers satisfying

$$\begin{cases} \beta_1(\lambda) \geq \beta_2(\lambda) \geq \cdots \geq \beta_k(\lambda) \geq \cdots, \\ \beta_k(\lambda) \to -\infty \,\text{as}\, k \to \infty. \end{cases} \tag{4.27}$$

The first eigenvalue $\beta_1(\lambda)$ of (4.26) has the Rayleigh's formula:

$$-\beta_1(\lambda) = \min_{T \in H_1(\Omega)} \frac{\int_\Omega \left[|\nabla T|^2 - \lambda T u_3(T)\right] dx}{\int_\Omega T^2 dx}.$$

For the critical Rayleigh number $\lambda_1 = \sqrt{R_c}$, (4.18) and (4.19) lead us to

$$\int_\Omega \left[|\nabla u(T)|^2 + |\nabla T|^2\right] dx \begin{cases} > 2\int_\Omega \lambda T u_3(T) dx & \text{if } \lambda < \lambda_1, \\ = 2\int_\Omega \lambda T u_3(T) dx & \text{if } \lambda = \lambda_1, \\ < 2\int_\Omega \lambda T u_3(T) dx & \text{if } \lambda > \lambda_1. \end{cases}$$

From the first equation of (4.26), we have

$$\int_\Omega |\nabla u(T)|^2 dx = \int_\Omega \lambda T u_3(T) dx,$$

which gives us

$$\int_\Omega \left[|\nabla T|^2 - \lambda T e_3(T)\right] dx = \int_\Omega \left[|\nabla T|^2 - |\nabla e(T)|^2\right] dx \begin{cases} > 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ < 0 & \text{if } \lambda > \lambda_1 \end{cases}$$

for the first eigenvectors $(e(T), T) \in H_0^1(\Omega)^3 \times H_1$. Therefore,

$$\beta_1(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ > 0 & \text{if } \lambda > \lambda_1, \end{cases}$$

and $\beta_j(\lambda_1) < 0$ $(2 \leq j < \infty)$.

Thus the condition (1.8) and (1.9) are obtained. By Theorem 1.3, we complete the proof. □

REMARK 4.3. In Theorem 4.2 and its proof, we can say that all statements are true for (u, T) instead of T alone, i.e., $(u, T) = 0$ is a globally asymptotically stable solution of (4.9)–(4.11) and the equations (4.9)–(4.11) bifurcate from $(u, T) = (0, R_c)$ to an attractor $\Sigma_R \subset L^2(\Omega)^3 \times L^2(\Omega)$, since we have $u(T)$ associated with a given T which satisfies equation (4.9), and $u = 0$ when $T = 0$.

Now we have another result which corresponds to the classical pitchfork bifurcation. But the below theorem gives us the stability of bifurcated steady states and that is another advantage of the attractor bifurcation theory.

THEOREM 4.4. *Let Σ_R be the bifurcated attractor of (4.15) with (4.17) achieved in Theorem 4.2. If the first eigenvalue R_1 is simple, then Σ_R consists of exactly two points, $\phi_1, \phi_2 \in H_1$ given by*

$$\phi_1 = \alpha T_1 + o(|\alpha|), \quad \phi_2 = -\alpha T_1 + o(|\alpha|),$$

for some $\alpha \neq 0$. Moreover, for any bounded open set $U \subset H$ with $0 \in U$, if $R > R_c$ and $R - R_c$ is small enough, then U can be decomposed into two open sets U_1 and U_2 such that

(1) $\overline{U} = \overline{U}_1 + \overline{U}_2$, $U_1 \cap U_2 = \emptyset$ *and* $0 \in \partial U_1 \cap \partial U_2$,

(2) $\phi_i \in U_i$ $(i = 1, 2)$, *and*

(3) *for any* $\phi_0 \in U_i$ $(i = 1, 2)$, $\lim_{t\to\infty} S_\lambda(t)\phi_0 = \phi_i$, *where* $S_\lambda(t)\phi_0$ *is the solution of (4.15) with (4.17) and initial data* $\phi_0 = T_0$.

Proof. Since $L_\lambda + G : H_1 \to H$ defined in the proof of Theorem 4.2 is a symmetric completely continuous field, H_1 can be decomposed into

$$H_1 = E_1^\lambda \oplus E_2^\lambda,$$
$$E_1^\lambda = \{xT_1(\lambda) : x \in \mathbb{R}\},$$
$$E_2^\lambda = \{y \in H_1 : \langle y, T_1\rangle_H = 0\}.$$

Let $P_i : H_1 \to E_i^\lambda$ be the canonical projection. For $\phi \in H$, ϕ can be expressed by $\phi = xT_1 + y$, for some $x \in \mathbb{R}$, $y \in E_2^\lambda$. Then

$$\begin{aligned} 0 &= P_1(L_\lambda(\phi) + G(\phi)) \\ &= P_1(xL_\lambda(T_1) + L_\lambda(y)) + P_1(G(\phi)) \\ &= xL_\lambda(T_1) + \langle G(\phi), T_1\rangle_H \\ &= x\beta_1(\lambda) + \langle G(\phi), T_1\rangle_H, \end{aligned}$$

and

$$\begin{aligned} 0 &= (I - P_1)(L_\lambda(\phi) + G(\phi)) \\ &= (I - P_1)(xL_\lambda(T_1) + L_\lambda(y)) + (I - P_1)(G(\phi)) \\ &= L_\lambda(y) + (I - P_1)G(\phi) \\ &= L_\lambda(y) + P_2G(\phi). \end{aligned}$$

So, the equation $L_\lambda(\phi) + G(\phi) = 0$ can be decomposed into

$$\beta_1(\lambda)x + \langle G(\phi), T_1(\lambda)\rangle_H = 0, \tag{4.28}$$

$$L_\lambda(y) + P_2G(\phi) = 0. \tag{4.29}$$

Since the eigenvalues $\beta_j(\lambda)$ in (4.27) satisfy that $\beta_j(\lambda_1) \neq 0$ for $j \geq 2$, and $\lambda_1 = \sqrt{R_c}$, the restriction

$$L_\lambda \,|_{E_2^\lambda} : E_2^\lambda \longrightarrow E_2^\lambda$$

is invertible. The decomposed equation (4.29) together with the implicit function theorem imply that y is a function of x:

$$y = y(x, \lambda), \tag{4.30}$$

which satisfies (4.29). Since $G(\phi) = G(xT_1 + y)$ is an analytic function of ϕ, y is also analytic in x and so is the function $f(x, \lambda) := \langle G(xT_1 + y(x, \lambda)), T_1(\lambda)\rangle_H$. Hence there exist $\alpha(\lambda) \neq 0$ in $\mathbb{R}$ and $k \geq 2$ such that $\beta_1(\lambda)x + f(x, \lambda)$ has the Taylor expansion as follows:

$$\beta_1(\lambda)x + f(x, \lambda) = \beta_1(\lambda)x + \alpha(\lambda)x^k + o(|x|^k). \tag{4.31}$$

By Theorem 4.2, if $\lambda \leq \lambda_1$ and $\lambda_1 - \lambda$ is small, (4.28) and (4.29) have no non-zero solutions, which implies that $\alpha(\lambda_1) < 0$ and k=odd. Thus, (4.31) has exactly two solutions

$$x_\pm = \pm \left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/(k-1)} + o\left(\left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/(k-1)}\right),$$

for $\lambda > \lambda_1$ and $\lambda - \lambda_1$ small.

By Theorem 4.2 again, (4.15) bifurcates from $(0, \lambda_1)$ to an attractor Σ_R with $\dim \Sigma_R \leq 1$. It is clear that Σ_R consists of exactly two singular points $\phi_\lambda := x_\pm T_1 + o(|x_\pm|)$ if $\lambda > \lambda_1$ and $\lambda - \lambda_1$ is small. By the stable manifold theorem, there is a stable manifold with codimension 1 of (4.15) at $\phi = 0$ dividing the open set U into two parts: U_1 and U_2, such that $\phi_i \in U_i$ and ϕ_i attracts U_i. This completes the proof. $\square$

We have the following remarks.

Remark 4.5. By the same arguments in Remark 4.3, Theorem 4.4 hold true for $\Sigma_R \subset L^2(\Omega)^3 \times L^2(\Omega)$. That is, the bifurcated attractor Σ_R consists of exactly two points $\phi_1, \phi_2 \in H_0^1(\Omega)^3 \times H_1(\Omega)$ given by

$$\phi_1 = \alpha(e_1(T_1), T_1) + o(|\alpha|), \quad \phi_2 = -\alpha(e_1(T_1), T_1) + o(|\alpha|),$$

where $e_1(T_1)$ is an eigenvector of (4.15) corresponding to T_1.

Remark 4.6. Both theorems hold for Boussinesq equations (4.15) with different combinations of boundary conditions as described in Remark 4.3.

Remark 4.7. Both theorems hold for two-dimensional (2D) Boussinesq equations as well.

CHAPTER 5

Structure of Bifurcated solutions for 2-D Infinite Prandtl number convection

1. Introduction

In this chapter, we consider two-dimensional (2D) Boussinesq equations and the following two problems will be addressed:

(1) the classification of the structure of Σ_R,
(2) the transition of the bifurcated solutions in physical space.

For the first result, we use an approximation formula for the center manifold function given in Theorem 1.7 [**25**]. Of course, the key ingredient of the analysis is to derive the reduction equation for our problem. For the second result, it is essential to study the topological structure of the divergence-free vector fields for 2D-incompressible flows governed by the Navier-Stokes equations or Euler equations. Such a result would involve a specific connection between solutions of the equations and flow structure in physical space, i.e., this area of research links the kinematics to the dynamics of fluid flows.

As a result, we have a bifurcation from the trivial state to an attractor Σ_R as R crosses the critical Rayleigh number R_c, the first eigenvalue of the eigenvalue problem of linearized equations [**27**]. Next, we prove that Σ_R is homeomorphic to S^1 and that Σ_R consists of steady state solutions for $R_c < R < R_c + \varepsilon$ for some $\varepsilon > 0$. These results can be obtained by analyzing eigenvectors in terms of the temperature field T and later on, the velocity field $u(T)$ associated with the temperature T can be reconstructed by the eigenvectors of T. Thanks to the geometric theory of 2D-incompressible flows [**26**], the structure and its transitions between the convection states in physical space is analyzed. Through this, we prove that the associated velocity field is structurally stable and show what the asymptotic structure looks like. This leads us, in particular, to a rigorous justification of the roll structure.

We take free–free boundary conditions on the top and bottom plates

$$T = 0, \quad u_2 = 0, \quad \frac{\partial u_1}{\partial x_2} = 0 \quad \text{at} \quad x_2 = 0, 1 \tag{5.1}$$

to study the infinite Prandtl number convection. However, the free–free boundary condition (which was first considered by Rayleigh) can be realized only under very artificial conditions, such as by having the liquid layer floating on top of a somewhat heavier liquid. From the point of view of realizability in the laboratory for precise quantitative experiments, the Dirichlet boundary condition (rigid-rigid) is, of course, of the greatest interest. Nevertheless, the case is of theoretical interest because it allows for an explicit solution.

The chapter is organized as follows. After introducing a geometric theory for the structural analysis of 2D divergence-free vector field in Section 2, topological structure of the bifurcated solutions and the structural stability and the asymptotic structure of the bifurcated solutions will be discussed in Section 3 and 4.

2. Geometric Theory of Incompressible Flows

In this section, we shall recall a geometric theory for the structural analysis of 2D divergence-free vector field governed by the Navier-Stokes equations or the Euler equations with the free boundary condition; see [**26**].

Let $C^r(\Omega, \mathbb{R}^2)$ be the space of all $C^r (r \geq 1)$ vector fields on Ω, which are periodic in x_1 direction with periods L, and let

$$\begin{aligned}
D^r(\Omega, \mathbb{R}^2) &= \{v \in C^r(\Omega, \mathbb{R}^2) \mid \nabla \cdot v = 0, v_2 = 0 \text{ at } x_2 = 0, 1\},\\
B_1^r(\Omega, \mathbb{R}^2) &= \{v \in D^r(\Omega, \mathbb{R}^2) \mid v_2 = \frac{\partial v_1}{\partial x_2} = 0 \text{ at } x_2 = 0, 1\},\\
B_2^r(\Omega, \mathbb{R}^2) &= \{v \in B_1^r(\Omega, \mathbb{R}^2) \mid \int_\Omega u dx = 0\}.
\end{aligned}$$

Definition 5.1. *Two vector fields $u, v \in C^r(\Omega, \mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\varphi : \Omega \to \Omega$, which takes the orbits of u to that of v and preserves their orientations.*

Definition 5.2. *Let $X = D^r(\Omega, \mathbb{R}^2)$. A vector field $v \in X$ is called structurally stable in X if there exists a neighborhood $U \subset X$ of v such that for any $u \in U$, u and v are topologically equivalent.*

Next we recall some basic facts and definitions on divergence-free vector fields. Let $v \in D^r(\Omega, \mathbb{R}^2)$.

(1) A point $p \in \Omega$ is called a singular point of v if $v(p) = 0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate.
(2) An interior non-degenerate singular point of v can be either a center or a saddle, and a non-degenerate boundary singularity must be a saddle.
(3) Saddles of v must be connected to saddles. An interior saddle p is called self-connected if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number "8".
(4) v is structurally stable near each non-degenerate singular point of v.

(5) If $v \in D^r(\Omega, \mathbb{R}^2)$ be regular, then the topological set of orbits of v consists of finite connected components of circle cells, circle bands, and saddle connections.

When $u \in B_1^r(\Omega, \mathbb{R}^2)$ $(r \geq 2)$, different singularity concept for points on the boundary was introduced in [**26**]. We recall it as follows.

(1) A point $p \in \partial\Omega$ is called a ∂-regular point of u if $\partial u_\tau(p)/\partial n \neq 0$; otherwise, $p \in \partial\Omega$ is called a ∂-singular point of u.

(2) A ∂-singular point $p \in \partial\Omega$ of u is called non-degenerate if

$$det \begin{pmatrix} \frac{\partial^2 u_\tau(p)}{\partial\tau\partial n} & \frac{\partial^2 u_\tau(p)}{\partial n^2} \\ \frac{\partial^2 u_n(p)}{\partial\tau\partial n} & \frac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0.$$

A non-degenerate ∂-singular point of u is also called a ∂-saddle point of u.

(3) A vector $u \in B_1^r(\Omega, \mathbb{R}^2)$ $(r \geq 2)$ is called D-regular if u is regular in the interior of Ω, and all ∂-singular points of u on $\partial\Omega$ are non-degenerate.

The following lemma and theorems were proved by T. Ma and S. Wang and they provide necessary and sufficient conditions for structural stability of divergence-free vector fields.

Lemma 5.3. *(Connection Lemma, [**26**]) Let $u, v \in D^r(\Omega, \mathbb{R}^2)$ with v sufficiently small, and $L \subset \Omega$ be an extended orbit of u starting at p. Then an extended orbit γ of $u + v$ starting at p passes through a point $q \in L$ if and only if*

$$\int_{L[p,q]} v \times d\ell = 0,$$

where $L[p, q]$ is the curve segment on L from p and q.

Theorem 5.4. [**26**] *Let $u \in B_1^r(\Omega, \mathbb{R}^2)$ $(r \geq 2)$(resp. $u \in B_2^r(\Omega, \mathbb{R}^2)$). Then u is structurally stable in $B_1^r(\Omega, \mathbb{R}^2)$ (resp. $B_2^r(\Omega, \mathbb{R}^2)$) if and only if*

(1) *u is regular.*

(2) *All interior saddle points of u are self-connected, i.e., each interior saddle point is connected only to itself.*

(3) *Each boundary saddle point of u is connected to boundary saddle points on the same connected component of $\partial\Omega$ (resp. each boundary saddle point of u is connected to boundary saddle points not necessarily on the same connected component).*

Since $((\alpha, 0), 0)$ is a solution of (4.9)–(4.11) for any constant α under the free-free boundary condition, we have to take velocity field u from the following function spaces:

$$\widetilde{H} = \{u \in L^2(\Omega)^2 \mid \nabla \cdot u = 0, \int_\Omega u dx = 0, \ u_1 \text{ is periodic in } x_1 \text{ direction}$$
$$\text{and } u_2 = 0 \text{ at } x_2 = 0, 1\},$$
$$\widetilde{H}_1 = \{u \in \widetilde{H} \cap H^2(\Omega)^2 \mid u \text{ satisfies (4.17) and (5.1)}\}.$$

3. Structure of Bifurcated Solutions for the infinite Prandtl number convection

In this section, we shall state and prove the first results which are related to the classification of the structure of Σ_R.

THEOREM 5.5. *For the problem (4.15) with (4.17) and (5.1), we have the following assertions: the equation bifurcates from the trivial solution $(T, R) = (0, R_c)$ to an attractor Σ_R for $R > R_c$ such that*

(1) *Σ_R consists of exactly one cycle of steady state solutions.*

(2) *Σ_R is homeomorphic to S^1.*

PROOF. STEP 1.We divide the proof into several steps. In the first step, we shall consider the eigenvalue problem of the linearized equation of (4.15) and shall find the eigenvectors and the critical Rayleigh number R_c.

Let's consider the eigenvalue problem of the linear equation,

$$L_\lambda T = \beta(\lambda)T. \tag{5.2}$$

It is equivalent to

$$-\Delta T - \lambda u_2(T) + \beta(\lambda)T = 0,$$

where $\lambda = \sqrt{R}$. Because $u(T)$ satisfies

$$-\Delta u(T) + \nabla p - \lambda T k = 0,$$

we can put two equations together as follows:

$$\begin{cases} -\Delta u(T) + \nabla p - \lambda T k = 0, \\ -\Delta T - \lambda u_2(T) + \beta(\lambda)T = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{5.3}$$

For the free-free boundary conditions, the following separation of variables is appropriate: for $T \in H_1$,

$$\begin{aligned}(u,T) &= (u_1,u_2,T) \\ &= \left(-\sin\frac{2k\pi}{L}x_1 H'(x_2), \frac{2k\pi}{L}\cos\frac{2k\pi}{L}x_1 H(x_2), \cos\frac{2k\pi}{L}x_1\Theta(x_2)\right),\end{aligned}$$

or,

$$\begin{aligned}(u,T) &= (u_1,u_2,T) \\ &= \left(\cos\frac{2k\pi}{L}x_1 H'(x_2), \frac{2k\pi}{L}\sin\frac{2k\pi}{L}x_1 H(x_2), \sin\frac{2k\pi}{L}x_1\Theta(x_2)\right).\end{aligned}$$

Then, from (5.3) we can derive a system of ODEs

$$\begin{cases} (D^2 - a_k^2)^2 H = \lambda a_k \Theta, \\ (D^2 - a_k^2)\Theta = -\lambda a_k H + \beta(\lambda)\Theta, \end{cases} \tag{5.4}$$

supplemented with the boundary conditions

$$H = D^2 H = \Theta = 0 \quad \text{at} \quad x_2 = 0, 1, \tag{5.5}$$

where $D^2 = \frac{d^2}{dx_2^2}$ and $a_k = \frac{2k\pi}{L}$.

It is clear that for each $k \geq 0$, the solutions of (5.4) with (5.5) are given by

$$\begin{aligned} H_j(x_2) &= \sin j\pi x_2, \\ \lambda_{kj} &= \frac{[(j\pi)^2 + a_k^2]^{3/2}}{a_k}, \\ \Theta_{kj}(x_2) &= \sqrt{(j\pi)^2 + a_k^2}\sin j\pi x_2, \end{aligned}$$

and

$$\beta_{k,2j-1}(\lambda) = \beta_{k,2j}(\lambda) = \frac{\lambda a_k}{\sqrt{(j\pi)^2 + a_k^2}} - [(j\pi)^2 + a_k^2]$$

for $j = 1, 2, \cdots$.

From the above arguments, it can be seen that the following sequences are the set of eigenvectors of (5.3):

(5.6)

$$\begin{aligned} &(u_{k,2j-1}, T_{k,2j-1}) \\ &= \left(-(j\pi)^2 \sin\frac{2k\pi}{L}x_1 \cos j\pi x_2, \frac{2k\pi}{L}\cos\frac{2k\pi}{L}x_1 \sin j\pi x_2, c_{kj}\cos\frac{2k\pi}{L}x_1 \sin j\pi x_2\right), \end{aligned}$$

(5.7)

$$\begin{aligned} &(u_{k,2j}, T_{k,2j}) \\ &= \left((j\pi)^2 \cos\frac{2k\pi}{L}x_1 \cos j\pi x_2, \frac{2k\pi}{L}\sin\frac{2k\pi}{L}x_1 \sin j\pi x_2, c_{kj}\sin\frac{2k\pi}{L}x_1 \sin j\pi x_2\right), \end{aligned}$$

where $k \geq 0$, $j \geq 1$ and $c_{kj} = \sqrt{(j\pi)^2 + (\frac{2k\pi}{L})^2}$. Here, $L = L_c$ leads $\lambda_{11}(L)$ to be the critical Rayleigh number.

Therefore, we get

$$\beta_{11}(\lambda) = \beta_{12}(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_{11}, \\ = 0 & \text{if } \lambda = \lambda_{11}, \\ > 0 & \text{if } \lambda > \lambda_{11}, \end{cases}$$

$$\beta_{kj}(\lambda_{11}) < 0 \qquad \text{if } \quad (k, j) \neq (1, 1),$$

and the first eigenvectors of (5.3) corresponding to β_{11} are

$$(u_{11}, T_{11}) \quad \text{and} \quad (u_{12}, T_{12}).$$

STEP 2.We shall show that the bifurcated attractor of (4.15) and (4.17) contains a singularity cycle. In order to show that, we note that each steady state solution of (4.15) generates one S^1 in H. We can easily see that the above claim is true from the following arguments: since the solutions of (4.15) are translation invariant,

$$T(x,t) \to T(x_1+\theta, x_2, t), \quad \text{for any } \theta \in \mathbb{R},$$

and if T_0 is a steady state solution of (4.15), the set

$$\Gamma = \{T_0(x_1+\theta, x_2) | \theta \in \mathbb{R}\}$$

represents S^1 in H_1.

Applying the classical Krasnoselskii theorem, we can see that the rest of Step 2 holds true. To this end, let's consider the odd function spaces

$$H^{odd} = \{T \in H \mid T \text{ is an odd function in the } x_1 \text{ direction}\},$$
$$H_1^{odd} = H_1 \cap H^{odd}.$$

Then H^{odd} and H_1^{odd} are invariant for $L_\lambda + G : H_1^{odd} \to H^{odd}$ and a set

$$\{T_{k,2j-1} | k \geq 0,\ j \geq 1\}$$

forms a basis of H_1^{odd}.

Since the first eigenvalue $\beta_{11}(\lambda)$ is simple in H^{odd}, $L_\lambda + G$ bifurcates from the trivial solution to a steady state solution in H_1^{odd}, i.e., (4.15) bifurcates from $(T, R) = (0, R_c)$ to a steady state solution. From the first argument in the Step 2, we can conclude that the bifurcated attractor contains a singularity cycle of (4.15).

STEP 3. In the third step, we shall investigate the topological structure of Σ_R. Let $E_1^\lambda = E_0 = span\{T_{11}, T_{12}\}$ and $E_2^\lambda = E_0^\perp$. For $T \in H$, we have the Furier expansion

$$T = \sum_{k\geq 0, j\geq 1} y_{kj} T_{kj}.$$

Then the reduction equations of (4.24) are as follows:

$$\begin{cases} \frac{dy_{11}}{dt} = \beta_{11}(\lambda) y_{11} + \frac{1}{<T_{11},T_{11}>_H} < G_2(T,T), T_{11} >_H, \\ \frac{dy_{12}}{dt} = \beta_{11}(\lambda) y_{12} + \frac{1}{<T_{12},T_{12}>_H} < G_2(T,T), T_{12} >_H, \end{cases} \tag{5.8}$$

where G_2 is the bilinear operator of G such that

$$G_2(T,T) = -(u(T)\cdot\nabla)T,$$
$$< G_2(T_1, T_2), T_3 >_H = -\int_\Omega (u(T_1)\cdot\nabla T_2) T_3 dx.$$

Let Φ be the center manifold function. Then

$$T = y_{11}T_{11} + y_{12}T_{12} + \Phi,$$

and (5.8) becomes

$$\begin{cases} \frac{dy_{11}}{dt} = \beta_{11}(\lambda) y_{11} + \frac{1}{<T_{11},T_{11}>_H} < G_2(& y_{11}T_{11} + y_{12}T_{12} + \Phi(y), \\ & y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{11} >_H, \\ \frac{dy_{12}}{dt} = \beta_{11}(\lambda) y_{12} + \frac{1}{<T_{12},T_{12}>_H} < G_2(& y_{11}T_{11} + y_{12}T_{12} + \Phi(y), \\ & y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{12} >_H, \end{cases} \tag{5.9}$$

where $y = y_{11}T_{11} + y_{12}T_{12}$.

Note that for T_{11} and T_{12},

$$< G_2(T_{1\ell}, T_{1m}), T_{1n} >_H = 0 \quad \text{for} \quad \ell, m, n = 1 \text{ or } 2,$$
$$< G_2(\Phi, T_{1i}), T_{1i} >_H = 0 \qquad \text{for} \quad i = 1, 2$$
$$< G_2(T_{1\ell}, T_{1m}), T_{1n} >_H = - < G_2(T_{1\ell}, T_{1n}), T_{1m} >_H .$$

Therefore,

$$\begin{aligned}
&< G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{11} >_H \\
&= - < G_2(T_{11}, T_{11}), \Phi(y) >_H y_{11} + < G_2(T_{12}, \Phi(y)), T_{11} >_H y_{12} \\
&+ < G_2(\Phi(y), T_{12}), T_{11} >_H y_{12} + < G_2(\Phi(y), \Phi(y)), T_{11} >_H,
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
&< G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{12} >_H \\
&= - < G_2(T_{12}, T_{12}), \Phi(y) >_H y_{12} + < G_2(T_{11}, \Phi(y)), T_{12} >_H y_{11} \\
&+ < G_2(\Phi(y), T_{11}), T_{12} >_H y_{11} + < G_2(\Phi(y), \Phi(y)), T_{12} >_H .
\end{aligned} \tag{5.11}$$

It is easy to check the following inner products:

$$< G_2(T_{11}, T_{11}), T_{kj} >_H \begin{cases} \neq 0 & \text{if} \quad k = 0, 2, \text{ and } j = 3, \\ = 0 & \text{otherwise,} \end{cases}$$

$$< G_2(T_{12}, T_{12}), T_{kj} >_H \begin{cases} \neq 0 & \text{if} \quad k = 0, 2, \text{ and } j = 3, \\ = 0 & \text{otherwise,} \end{cases}$$

$$< G_2(T_{12}, T_{kj}), T_{11} >_H \begin{cases} \neq 0 & \text{if} \quad k = 0, 2, \text{ and } j = 4, \\ = 0 & \text{otherwise,} \end{cases}$$

$$< G_2(T_{11}, T_{kj}), T_{12} >_H \begin{cases} \neq 0 & \text{if} \quad k = 0, 2, \text{ and } j = 4, \\ = 0 & \text{otherwise,} \end{cases}$$

$$< G_2(T_{kj}, T_{12}), T_{11} >_H \begin{cases} \neq 0 & \text{if} \quad k = 0, 2, \text{ and } j = 4, \\ = 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.7, it is seen that the center manifold function Φ, near $\lambda = \lambda_{11}$, can be expressed as

$$\Phi(y) = \sum_{(k,j) \neq (1,1),(1,2)} \Phi_{kj}(y) T_{kj}, \tag{5.12}$$

where

$$\Phi_{kj}(y) = -\frac{1}{\beta_{kj}(\lambda)} \sum_{1 \le p,q \le 2} a_{pq}^{kj} y_{1p} y_{1q},$$

$$a_{pq}^{kj} =< G_2(T_{1p}, T_{1q}), T_{kj} >_H .$$

Due to the above calculations, (5.10) and (5.11) can now be written as

$$\begin{aligned} & < G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{11} >_H \\ & = -\left[< G_2(T_{11}, T_{11}), T_{03} >_H \Phi_{03} + < G_2(T_{11}, T_{11}), T_{23} >_H \Phi_{23}\right] y_{11} \\ & + \left[< G_2(T_{12}, T_{04}), T_{11} >_H \Phi_{04} + < G_2(T_{12}, T_{24}), T_{11} >_H \Phi_{24}\right] y_{12} \\ & + \left[< G_2(T_{04}, T_{12}), T_{11} >_H \Phi_{04} + < G_2(T_{24}, T_{12}), T_{11} >_H \Phi_{24}\right] y_{12} \\ & + < G_2(\Phi(y), \Phi(y)), T_{11} >_H, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} & < G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{12} >_H \\ & = -\left[< G_2(T_{11}, T_{04}), T_{12} >_H \Phi_{04} + < G_2(T_{11}, T_{24}), T_{12} >_H \Phi_{24}\right] y_{11} \\ & + \left[< G_2(T_{04}, T_{11}), T_{12} >_H \Phi_{04} + < G_2(T_{24}, T_{11}), T_{12} >_H \Phi_{24}\right] y_{11} \\ & - \left[< G_2(T_{12}, T_{12}), T_{03} >_H \Phi_{03} + < G_2(T_{12}, T_{12}), T_{23} >_H \Phi_{23}\right] y_{12} \\ & + < G_2(\Phi(y), \Phi(y)), T_{12} >_H . \end{aligned} \tag{5.14}$$

By (5.12), near $\lambda = \lambda_{11}$ we have

$$\begin{aligned} & \Phi_{03} = -\frac{1}{\beta_{03}(\lambda)} \left[\pi^3 \sqrt{\pi^2 + \left(\frac{2\pi}{L}\right)^2} (y_{11}^2 + y_{12}^2)\right], \\ & \Phi_{04} = \Phi_{23} = \Phi_{24} = 0. \end{aligned} \tag{5.15}$$

Replacing (5.13) and (5.14) by (5.15), we obtain

$$\begin{aligned} & < G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{11} >_H \\ & = \frac{1}{\beta_{03}(\lambda)} \pi^6 \left[\pi^2 + \left(\frac{2\pi}{L}\right)^2\right] y_{11} \left[y_{11}^2 + y_{12}^2\right] + < G_2(\Phi, \Phi), T_{11} >_H, \end{aligned}$$

and

$$
\begin{aligned}
&< G_2(y_{11}T_{11} + y_{12}T_{12} + \Phi(y), y_{11}T_{11} + y_{12}T_{12} + \Phi(y)), T_{12} >_H \\
&= \frac{1}{\beta_{03}(\lambda)}\pi^6 \left[\pi^2 + \left(\frac{2\pi}{L}\right)^2\right] y_{12}\left[y_{11}^2 + y_{12}^2\right] + < G_2(\Phi, \Phi), T_{12} >_H .
\end{aligned}
$$

Therefore, the reduction equations (5.9) are transformed into

$$
\begin{cases}
\frac{dy_{11}}{dt} = \beta_{11}(\lambda)y_{11} + \frac{\alpha}{<T_{11},T_{11}>_H} y_{11}(y_{11}^2 + y_{12}^2) + < G_2(\Phi, \Phi), T_{11} >_H, \\
\frac{dy_{12}}{dt} = \beta_{11}(\lambda)y_{12} + \frac{\alpha}{<T_{12},T_{12}>_H} y_{12}(y_{11}^2 + y_{12}^2) + < G_2(\Phi, \Phi), T_{12} >_H,
\end{cases} \tag{5.16}
$$

where

$$
\alpha = \frac{1}{\beta_{03}(\lambda)}\pi^6 \left[\pi^2 + \left(\frac{2\pi}{L}\right)^2\right] < 0 \quad \text{for} \quad \lambda_{11} < \lambda < \lambda_{11} + \varepsilon.
$$

We know that the center manifold function involves only higher order terms

$$
\Phi(y) = O(|y_{11}|^2, |y_{12}|^2),
$$

which implies

$$
< G_2(\Phi, \Phi), T_{1i} >_H = o(|y_{11}|^3, |y_{12}|^3), \quad (i = 1, 2).
$$

Since the following equations,

$$
\begin{cases}
\frac{dy_{11}}{dt} = \beta_{11}(\lambda)y_{11} + \frac{\alpha}{<T_{11},T_{11}>_H} y_{11}(y_{11}^2 + y_{12}^2) + o(|y_{11}|^3, |y_{12}|^3), \\
\frac{dy_{12}}{dt} = \beta_{11}(\lambda)y_{12} + \frac{\alpha}{<T_{12},T_{12}>_H} y_{12}(y_{11}^2 + y_{12}^2) + o(|y_{11}|^3, |y_{12}|^3),
\end{cases} \tag{5.17}
$$

satisfy conditions of Theorem 1.4, we can conclude that Σ_R is homeomorphic toS^1.

This completes the proof.

□

Remark 5.6. It is known that under free-free boundary condition, the critical Rayleigh number can be calculated analytically as $\frac{27\pi^4}{4} = 657.5$, which is independent of the Prandtl number of the fluid; see [**4, 11**] for details.

4. Structural Stability for the Infinite Prandtl Number Convection

In this section, we shall study the structural stability and asymptotic structure of the bifurcated solutions of the infinite Prandtl number convection in physical space. We begin with a simple but strongly necessary lemma, which links the structure of the solutions in terms of (u, T) and that of the solutions in terms of only T. This occurs only in the infinite Prandtl number convection not finite Prandtl number convection.

LEMMA 5.7. *Let $T \in \Sigma_R$, $T = \Sigma_{k\geq 0, j\geq 1} y_{kj} T_{kj}$ and $u = u(T)$ be the associated velocity field which is obtained from T. Then u can be expressed as*

$$u = \sum_{k=1}^{2} \alpha_{1k} u_{1k} + o(\sum_{k=1}^{2} \alpha_{1k} u_{1k}),$$

where u_{1k} are eigenvectors of (4.15) corresponding to T_{1k}. Moreover,

$$\alpha_{1k} = \tau y_{1k}, \text{ for some } \tau \in \mathbb{R}.$$

PROOF. We infer from Theorem 4.2 that for any $T \in \Sigma_R$, the associated velocity field $u = (u_1, u_2)$ can be expressed as

$$u = \sum_{k=1}^{2} \alpha_{1k} u_{1k} + o(\sum_{k=1}^{2} \alpha_{1k} u_{1k}) \text{ for some } \alpha_{1k},$$

where u_{1k} are eigenvectors of (4.15) corresponding to T_{1k}. Thus it suffices to prove that α_{1k} are proportional to y_{1k}.

From the first equation in (5.3), we have

$$\Delta(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) - \lambda \frac{\partial T}{\partial x_1} = 0.$$

With the sequences of the eigenvectors (5.16) and (5.17), we can rewrite the above equality as follows:

$$\left[\left(\pi^3 (\frac{2\pi}{L})^2 + (\frac{2\pi}{L})^4 - \pi^5 - (\frac{2\pi}{L})^2 \pi^2 \right) \alpha_{11} - \lambda y_{11} c_{11} (\frac{2\pi}{L}) \right] \sin \frac{2\pi}{L} x_1 \sin \pi x_2 = 0,$$
$$\left[\left(\pi^3 (\frac{2\pi}{L})^2 + (\frac{2\pi}{L})^4 - \pi^5 - (\frac{2\pi}{L})^2 \pi^2 \right) \alpha_{12} - \lambda y_{12} c_{11} (\frac{2\pi}{L}) \right] \cos \frac{2\pi}{L} x_1 \sin \pi x_2 = 0,$$

where c_{11} is as defined in (5.16) and (5.17). Therefore, we arrive at

$$\alpha_{1k} = \tau y_{1k} \text{ for some } \tau \in \mathbb{R} \ (k = 1, 2).$$

□

With the help of Lemma 5.7, we can finally state and prove structural stability theorem.

THEOREM 5.8. *For any $T_0 \in H \backslash \Gamma$, there exists a time $t_0 \geq 0$ such that for any $t \geq t_0$, the associated vector field $u(t, T) = u(t, T, \Psi_0)$ is structurally stable and is topologically equivalent to the structure as shown in Figure 5.1, where $\Psi = (u(t, \Psi_0), T(t, \Psi_0))$ is the solution of (4.9)–(4.11), (4.17) and (5.1), $\Psi_0 = (u_0(T_0), T_0) \in (\widetilde{H} \times H) \backslash \widetilde{\Gamma}$ and Γ(resp. $\widetilde{\Gamma}$) is the stable manifold of $T = 0$(resp. of $(u, T) = 0$) with co-dimension 2 in H(resp. in $\widetilde{H} \times H$).*

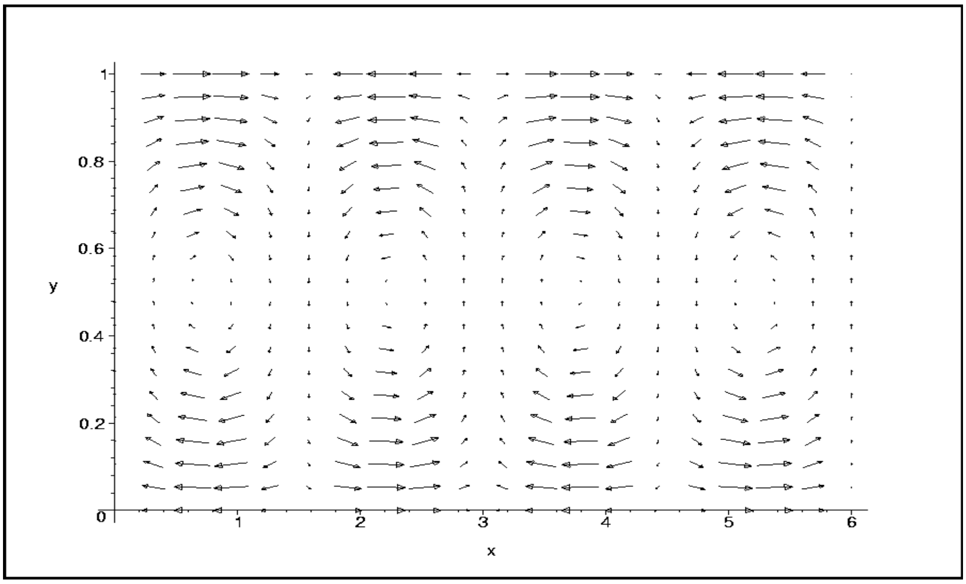

FIGURE 5.1. The roll structure of solutions of Benard convection with infinite Prandtl number.

PROOF. By [**13**], we know that the solution $(u(t,\psi_0), T(t,\Psi_0))$ is analytic in time and all the H^m norms of $u(t,\psi_0)$, $T(t,\Psi_0)$ remain uniformly bounded in time for $t \geq \delta > 0$, $m \geq 0$.

From Lemma 5.7 and equation (5.17) at the end of the proof of Theorem 5.5, we have

$$\begin{cases} u_1 = \alpha_{11}(-\pi^2 \sin \frac{2\pi}{L} x_1 \cos \pi x_2) + \alpha_{12}(\pi^2 \cos \frac{2\pi}{L} x_1 \cos \pi x_2) + v_1(\alpha_{11}, \alpha_{12}, \beta_{11}), \\ u_2 = \alpha_{11}(\frac{2\pi}{L} \cos \frac{2\pi}{L} x_1 \sin \pi x_2) + \alpha_{12}(\frac{2\pi}{L} \sin \frac{2\pi}{L} x_1 \sin \pi x_2) + v_2(\alpha_{11}, \alpha_{12}, \beta_{11}). \end{cases}$$

where $\alpha_{11} = \tau y_{11}$, $\alpha_{12} = \tau y_{12}$ and τ is the constant obtained in the proof of Lemma 5.7.

Now, note that the above expansions were done in terms of the eigenvectors. Thus, they must hold not only on H, but also on higher dimensional Sobolev spaces H^m. By simple modifications, the following velocity fields are obtained:

$$\begin{cases} u_1 = \pi^2 r \sin \frac{2\pi}{L}(\theta - x_1) \cos \pi x_2 + v_1(\alpha_{11}, \alpha_{12}, \beta_{11}), \\ u_2 = \frac{2\pi}{L} r \cos \frac{2\pi}{L}(\theta + x_1) \sin \pi x_2 + v_2(y_{11}, y_{12}, \beta_{11}), \end{cases} \tag{5.18}$$

for some $0 \leq \theta \leq 2\pi$. Here,

$$\begin{aligned} & r^2 = \alpha_{11}^2 + \alpha_{12}^2 = \tau^2 \frac{\beta_{11}}{\alpha} < T_{11}, T_{11} >_H \quad \text{for } \lambda > \lambda_{11}, \\ & v_i(\alpha_{11}, \alpha_{12}, \beta_{11}) = o(r) \ \ (i = 1, 2). \end{aligned}$$

Since $\int_\Omega u dx = 0$, we have

$$\int_\Omega v_i dx = 0 \ \ (i = 1, 2) \tag{5.19}$$

for the velocity field u given above. By the connection Lemma, it follows that

$$u_0 = (-\pi^2 r \sin \frac{2\pi}{L} x_1 \cos \pi x_2, \frac{2\pi}{L} r \cos \frac{2\pi}{L} x_1 \sin \pi x_2) \tag{5.20}$$

is topologically equivalent to $u = (u_1, u_2)$. Now it suffices to prove that u_0 is structurally stable.

For any singular point

$$(x_1, x_2) = (\frac{kL}{2}, \frac{1}{2}) \text{ or } (\frac{(2k+1)L}{4}, \frac{1}{2}), \quad k = 1, 2, \cdots,$$

of u_0 in Ω,

$$\det Du_0(x_1, x_2) \neq 0.$$

Therefore, u_0 is structurally stable and we can easily show that the topological structure is as shown in Figure 5.1. It provides us the structural stability of u and its structure as well.

This completes the proof.

□

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