Advertising Intensity and Welfare in an Equilibrium Search Model

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Abstract

We analyze an equilibrium search model in a duopoly setting with bilateral heterogeneities in production and search costs in which firms can advertise by announcing price and location. We study existence, stability, and comparative statics in such a setting, compare the market advertising level to the socially optimal level, and find conditions in which firms advertise more or less than the social optimum.

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1 Introduction

Imperfect price information is a fundamental aspect of any consumer search model. Avenues that can improve this information, such as advertising, therefore have a natural role as consumers can refine their knowledge of prices as they receive new information. In this paper, we study an equilibrium search model in a duopoly setting and introduce an advertising technology by which firms can inform consumers of their price. Our underlying market structure is similar to that of Carlson and McAfee (1983) and Bénabou (1993) with bilateral heterogeneities in production and search costs. The market consists of a continuum of consumers with individual search costs distributed along the unit interval, similar to Rob (1985), where all consumers enter the market with a free initial search and can choose to visit the other firm at some cost.[1] We ask, given that consumers engage in optimal search, will firms tend to over- or under-advertise relative to a planner? Our analysis provides good insight on the interaction between search and advertising in a duopoly setting and enhances our understanding of the welfare effects of advertising with search.

A priori, it is unclear whether the market advertising level generally exceeds that of a planner or vice versa. Since production costs and search cost expenditures are welfare losses, the planner advertises to save some cost of production or decrease search intensity. But acting purely as a profit maximizer, the firm advertises only to attract an excessively high search cost consumer, i.e., an inactive searcher. If we take as our measure of welfare the sum of consumer and producer surplus as well as search, advertising, and production costs, and consider inelastic demand so that the sum of consumer surplus and total revenue are fixed, then welfare depends completely on advertising, production, and search costs. In this

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[1]Our model is therefore a simplified version of Rob (1985), Bénabou (1993), and Robert and Stahl (1993). We assume a free first search so as to avoid keeping track of those consumers who elect not to buy. See Janssen, Moraga-González, and Wildenbeest (2005) for a relaxation of this assumption.

case, the planner sends buyers to the low price firm only if the decrease in search or production costs exceeds the cost of advertising, a tradeoff which the firm does not consider.

Note that our goal is not to establish equilibrium price dispersion under minimal conditions as in Reinganum (1979), Burdett and Judd (1983), Rob (1985), and Robert and Stahl (1993), among others. Indeed, with bilateral heterogeneities, price dispersion is more or less an automatic byproduct of the assumed market structure. Our goal in this paper is to develop an equilibrium search model that highlights the fundamental role of price advertising and, in doing so, provide definitive welfare results.

Given our duopoly setting, the model is fairly general. We allow for a relatively general search cost distribution, potentially downward-sloping demand, and an advertising function that can accommodate economies of scale. Under fairly mild assumptions, we prove existence in pure strategies and derive comparative statics with respect to production and advertising costs. Although these results can go either way, we show that the relevant dynamic stability conditions rule out counter-intuitive comparative statics.

We then turn to welfare issues, the main focus of the paper. The welfare standard we adopt is that of a social planner maximizing welfare, as previously discussed, subject to the first order conditions for price. We impose the latter constraint because the first best solution of a planner allowed to choose both prices and advertising intensity would be to essentially set the low cost firm's price to zero, making a useful comparison between the market and planner's advertising level impossible. The pricing constraint essentially forms a structural second best problem so that the planner and firm are on the same footing with respect to their advertising decisions.

Our analysis provides intuitive sufficient conditions such that market advertising intensity is above or below that of a planner. In general, firms over-advertise when the indifferent consumer's search cost is sufficiently low and under-advertise

3

when this search cost is sufficiently high relative to the cost of production. For symmetric search cost distributions, this implies that firms over-advertise when the majority of consumers do not search. We express this result in terms of production and advertising costs, the consumer's maximum willingness to pay, and advertising effectiveness—all of which relate to the tradeoff between advertising and search as a means to disseminate information to consumers. Intuitively, under-advertising results for two reasons. One, the firm only cares about attracting inactive searchers. Since this portion of the market decreases as the indifferent consumer's search cost increases, the firm has less incentive to advertise. Two, the planner's advertising decision is based partly on saving search costs. Since this savings increases with the indifferent consumer's search cost, the planner advertises more intensely. Similar intuition holds for over-advertising when the indifferent consumer's search cost is low. Since inactive searchers make up a relatively large portion of the market, the firm has more incentive to advertise. In addition, the search costs paid by consumers are generally lower, which decreases the planner's advertising incentive. The planner also understands that an increase in advertising converts some marginal consumers from active searchers—where they pay their search cost—to inactive searchers—where they may buy from the high cost firm. This implicit cost of sending a few small search cost consumers to the high cost firm further dulls the planner's advertising incentive.

Previous advertising and sequential search models include Butters (1977), Stegeman (1991), Robert and Stahl (1993), and Janssen and Non (2005). Robert and Stahl show that, without *ex ante* heterogeneities, there exists a unique equilibrium with price dispersion and derive comparative statics with regard to entry, search costs, and advertising costs. While their analysis thoroughly describes the strategic interaction of advertising and search in a general setting, they do not compare the competitive and socially optimal advertising levels. Janssen and Non develop a similar model for the special case of a duopoly and allow some small percentage of completely informed consumers, i.e., shoppers. They derive partly

4

contrasting results with Robert and Stahl and show that the inclusion of informed consumers has important implications for comparative statics—especially the limiting cases of zero search or advertising costs. But they also do not address the planner's advertising decision.

Although Butters' (1977) model with search does compare the competitive and socially optimal advertising levels, the advertising technology is such that any given advertisement reaches exactly one consumer, which excludes economies of scale. Buyers also do not adopt an optimal search process due to "certain unpalatable conclusions" and instead visit any given firm with some probability proportional to the firm's sales. As such, these welfare results are only based on optimal *firm* behavior. Finally, Butters does not impose the monopolistically competitive pricing constraint on the social planner's problem, making advertising comparisons problematic. He nonetheless finds that firms always over-advertise. Stegeman (1991) develops a similar model but allows for heterogeneous reservation prices. He derives equivalent results only if search costs are sufficiently small but generally finds that monopolistically competitive firms advertise *too little*.

Even without search, welfare results are not obvious. Dixit and Norman (1978) show that advertising is excessive, while Shapiro (1980) extends this work, showing that advertising is sometimes under-utilized.[2] Shapiro, however, only considers the monopoly case, and as Bagwell (2001) shows, Shapiro's model can be extended to several firms, which results in excessive advertising.

The remainder of the paper is organized as follows. In Section 2, we develop the model, prove existence and stability of equilibria, and derive comparative statics. Section 3 establishes the major welfare results and characterizes conditions in which the advertising firm over- or under-advertises. Proofs of all Propositions are deferred to the Appendix.

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[2]Butters (1976) and Bagwell (2001) provide good surveys of generally accepted results of the literature and of the ambiguous nature of advertising and its effect on equilibrium outcomes.

2 The Model

2.1 Model Setup

Consider a search model where consumers are identical except for their search costs. The market is normalized to one, and buyers are identified by their search cost $s \in [0,1]$. The distribution of search costs follows a cdf $Q(s)$, with pdf $q(s)$ and full support on $[0,1]$. We assume that $Q(0) = 0$, $q(s)$ is twice continuously differentiable, and that $q(s)$, $q'(s)$, and $q''(s)$ are bounded with $q_{max} = \max_{s \in [0,1]} q(s)$, $q_{min} = \min_{s \in [0,1]} q(s)$, and similarly for $q'(s)$ and $q''(s)$. We also make the standard assumption that consumers are perfectly informed as to the distribution of prices but are uncertain about which firms offer which price, as in Carlson and McAfee (1983) and Bénabou (1993). Given prices, individual demand arises from a quasi-linear utility function, with indirect utility $v(p) + y$. By Roy's Identity, each consumer purchases $d(p) = -v'(p)$ units at price p. Let p_{max} denote the consumer's maximum willingness to pay, and assume $d(p)$ is twice continuously differentiable with $d' < 0$ on $[0, p_{max})$. Buyers enter the market with a free initial search but must pay their search cost to visit another firm.

In a duopoly without advertising, half of the consumers randomly visit the high cost firm and half visit the low. Of the unlucky buyers reaching the high cost firm, only those consumers with sufficiently low search costs benefit from an additional search. The decision of such a consumer is based on

$$v(p_L) - s \geq v(p_H). \tag{2.1}$$

This yields the critical search value $\hat{s} = v(p_L) - v(p_H)$, which is the cost below which consumers search again to find the low cost firm and above which consumers purchase from whichever firm they randomly choose, i.e., for $s \geq \hat{s}$, consumers are inactive searchers. We refer to the buyer with $s = \hat{s}$ as the indifferent consumer.

There are two firms, each producing identical goods with heterogenous costs

of production. The low cost firm has marginal cost normalized to zero, while the high cost firm has constant marginal and average costs of $c > 0$. Both firms can advertise their price and location to a fraction of the market at some constant marginal cost $A > 0$. Note that, since the distribution of prices is known, any consumer receiving an advertisement is then perfectly informed of prices, in which case the high price firm never advertises. We denote the level of advertising by $x \in [0, 1]$, where the advertising firm is bound to charge the price advertised, e.g., for legal reasons. Given x, denote the proportion of uninformed consumers by $f(x)$, where $f(x)$ satisfies $f' < 0$, $f'' \geq 0$, $f(0) = 1$, and $f(1) = 0$. Therefore, given x, the proportion of informed consumers is $1 - f(x)$ drawn uniformly from $[0, 1]$, where each consumer is equally likely to observe an advertisement. Both firms take as given consumer behavior described above and play the subsequent game with prices and advertising as strategic variables. Figure 1 summarizes the setup thus far.

Figure 1: CONSUMER AND FIRM INTERACTION
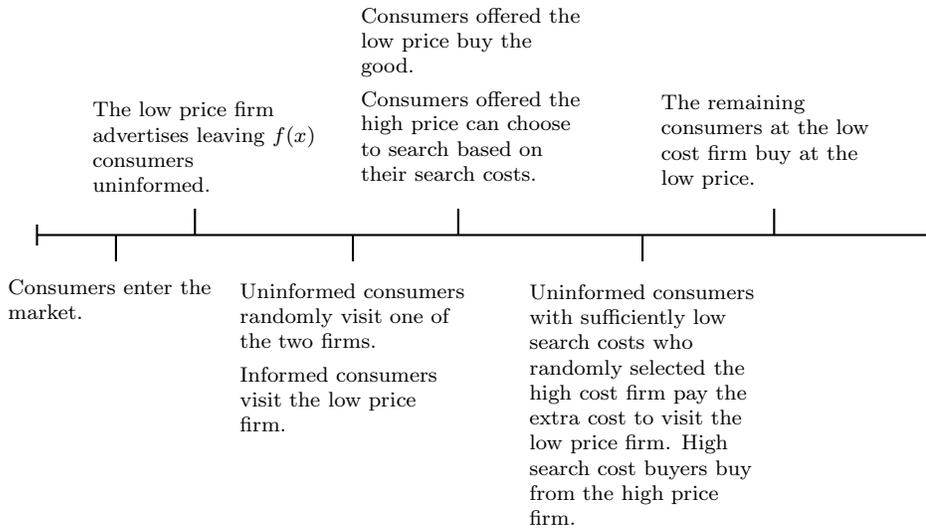
Although we formally address this issue in Section 2.2, assume for now that each firm prices according to cost so that the low cost firm is the low price firm.

In this case, the low and high price firms face the following demands:

$$q_L = d(p_L)\left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s}))\right];\tag{2.2}$$

$$q_H = \frac{1}{2}d(p_H)f(x)(1 - Q(\hat{s})).\tag{2.3}$$

In words, (2.3) comes from some proportion—determined by x—of consumers being informed of the low price via advertising, leaving $f(x)$ uninformed. Of these, half randomly select the high cost firm, and some portion $Q(\hat{s})$ are active searchers with sufficiently low search costs so that they never pay the high price. Equation (2.3) is therefore the probability that any given buyer purchases from the high price firm, where each buyer demands $d(p_H)$ units. The remaining consumers pay the low price and demand $d(p_L)$ units each, which yields (2.2).

2.2 Existence, Stability, and Comparative Statics

We assume the monopolist's problem

$$\max_{p\in[0,p_{max})} \Pi = d(p)(p - c)$$

has a unique solution, denoted p^*, where $\Pi_p > 0$ on $[0, p^*)$. Consumers receive sufficient indirect utility so that they always purchase at the monopoly price $(v(p^*) + y \geq p^*)$. We also assume

$$\Pi_p p > -d_p\Pi,\tag{2.4}$$

which essentially restricts the elasticity of demand. For notational convenience, denote the monopolist's first order condition evaluated at the high cost firm's price and cost by Π_p^H and similarly for the low cost firm.

In this paper, we want to focus on the natural equilibrium where the low cost firm is the low price firm. We therefore begin with an artificially restricted case

where the low cost firm must price below the high cost firm. As such, only the low cost firm advertises and faces the profit maximization problem

$$\max_{\substack{p_L \leq p_H \\ x \leq 1}} p_L q_L - Ax, \tag{2.5}$$

which yields the following first order conditions for price and advertising, respectively:

$$\frac{\partial \pi_L}{\partial p_L} = \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s}))\right]\Pi_p^L - \frac{1}{2}q(\hat{s})f(x)d(p_L)^2 p_L = 0; \tag{2.6}$$

$$\frac{\partial \pi_L}{\partial x} = -\frac{1}{2}f'(x)(1 - Q(\hat{s}))d(p_L)p_L - A = 0. \tag{2.7}$$

Similarly, we restrict the high cost firm to price above the low cost firm. The high cost firm therefore does not advertise and must solve

$$\max_{\substack{c \leq p_H \\ p_L \leq p_H < p_{max}}} q_H(p_H - c), \tag{2.8}$$

which yields

$$\frac{\partial \pi_H}{\partial p_H} = \frac{1}{2}f(x)(1 - Q(\hat{s}))\Pi_p^H - \frac{1}{2}f(x)q(\hat{s})d(p_H)^2(p_H - c) = 0. \tag{2.9}$$

Definition 1 *In a restricted game, the low and high cost firms solve (2.5) and (2.8), respectively. We define a **restricted Nash equilibrium** by the triplet (p_L^*, p_H^*, x^*) such that (p_L^*, p_H^*, x^*) is a Nash equilibrium of this restricted game.*

Having defined a restricted Nash equilibrium, we now show that such an equilibrium exists. We then find conditions such that restricted Nash equilibria and conventional Nash equilibria coincide.

Proposition 1 *The profit functions, π_L and π_H, are quasi-concave in firms' own actions, strictly concave in p_L and p_H (for a given x), and a restricted pure-*

9

strategy Nash equilibrium exists provided the hazard function satisfies

$$\frac{q'(\hat{s})}{q(\hat{s})} \in \left(-\frac{\Pi_p^H}{\Pi^H d}, \frac{\Pi_p^L}{\Pi^L d} \right). \tag{2.10}$$

Condition (2.10) is a standard hazard condition that imposes restrictions on the tails of the density. Certainly, the uniform distribution fits this requirement, but in general, any standard hill or bell-shaped density with $q'(s)$ relatively flat in the tails will suffice.

Given existence of a restricted Nash equilibrium, we now provide conditions such that the previous pricing restrictions are non-binding and the restricted Nash equilibrium is a Nash equilibrium in the conventional sense.

Proposition 2

(i) *For all $c > 0$, every restricted Nash equilibrium is a Nash equilibrium.*

(ii) *There exists some $\bar{c} > 0$ such that, for all $c > \bar{c}$, every Nash equilibrium involves the low cost firm pricing below the high cost firm.*

Part *(i)* simply says that both firms are content with pricing at $p_L^* \leq p_H^*$, while part *(ii)* ensures that, even if allowed to choose any price up to p_{max}, firms still choose prices consistent with the restrictions of equations (2.5) and (2.8).

We therefore have a duopoly game with heterogeneous consumers and firms in which the low cost firm prices below the high cost firm and can advertise to some fraction of consumers, where uninformed buyers can search for the lowest available price. A price dispersed equilibrium exists and consumers follow an optimal search rule, based on (2.1), so that both advertising and search effectively disseminate information between buyers and firms.

We now impose stability via a standard proportional marginal profitability adjustment rule.[3]

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[3]See equation (C.1) in Appendix C.

Proposition 3 *Given the hazard condition (2.10) and c such that $Q(c) < 1$, the conditions*

$$\frac{f'(x)}{f(x)} > -\frac{(1 - Q(c))}{(2p_{max} - c)^2 q_{max}}, \text{ and}$$
$$\frac{f''(x)}{f'(x)} < -\frac{3q_{max}}{1 - Q(c)},$$

are sufficient such that the triplet (p_L^*, p_H^*, x^*), where $p_L^*, p_H^* \in [0, p_{max})$ and $x^* \in [0, 1]$, is a locally stable Nash equilibrium.

Proposition 3 formalizes the role of the advertising function in determining stability, where we see that $f'(x)$ must be small relative to $f(x)$ and $f''(x)$ large relative to $f'(x)$, in absolute value. Intuitively, this is a standard contraction condition to ensure the effect of any given strategic variable on the marginal profitability of that variable exceeds the effect on the marginal profitability of all other variables.

Using these stability conditions, we can now discuss comparative statics.

Proposition 4 *Given stability and condition (2.10), the following relationships hold in equilibrium:*

(i) *p_L is increasing in c and decreasing in A;*

(ii) *p_H is increasing in c and decreasing in A;*

(iii) *x is decreasing in A and is non-monotonic in c, where there exists some c^* such that x is increasing in c for all $c < c^*$ and decreasing in c for all $c > c^*$;*

(iv) *price dispersion $p_H - p_L$ is increasing in both c and A; and*

(v) *an exogenous increase in x increases both p_H and p_L and decreases price dispersion.*

These results are fairly intuitive. Consider first the response to an increase in the cost of production, c. Naturally, the high cost firm must increase price. The low cost firm, now with more residual demand, also responds with a price increase but to a lesser degree. Therefore, p_H, p_L, as well as $p_H - p_L$ are all increasing in c.

11

With regard to advertising, low c implies low price dispersion, in which case many consumers are inactive searchers. Since there is only a small scope for price adjustments, the low cost firm advertises more as c increases. Conversely, a high cost of production implies high price dispersion so that price adjustments now play a larger role. Advertising is also less likely to reach an inactive searcher relative to when c is low. In this case, the low cost firm advertises less as c increases and relies more on price competition.

Now consider the response to an increase in the cost of advertising, A. The low price firm must decrease advertising intensity but also decreases p_L to maintain profits.[4] To avoid losing a large share of the market, the high cost firm also decreases price, but not by as much. Therefore p_H and p_L are both decreasing in A, while $p_H - p_L$ is increasing.

Condition *(v)* addresses the firms' responses in price to exogenous changes in advertising, i.e., the advertising decision of the planner. From the low cost firm's perspective, this essentially increases his residual demand at no cost. Similar to an increase in c, the low cost firm responds by increasing p_L. The high price firm adopts a similar strategy but to a lesser degree, which implies that $p_H - p_L$ is decreasing in x. This decrease in price dispersion subsequently decreases the proportion of consumers who engage in search. As we will see in Section 3, this tradeoff between advertising and search intensity has important welfare implications.

3   Welfare and Advertising Intensity

We now have a model in which advertising plays a purely informational role in announcing the true price and location of the low cost firm and thus implicitly doing so for the high cost firm. But as mentioned in Section 1, welfare effects are

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[4]It can be shown that, without stability, counter-intuitive comparative statics might result in which advertising intensity increases with the cost of advertising. See Chapter 4 of Vives (1999) for a thorough explanation.

unclear due to the inherent tension between the social planner and the advertising firm. To fully characterize when and how this tension might lead the firm to over- or under-advertise, we consider the basic pricing/advertising game proposed in Section 2 and study the firm's advertising level relative to the level chosen by a social planner. For simplification, we assume all consumers inelastically demand one unit up to some maximum price, which fixes consumer surplus and total revenue as a sum so that welfare depends totally on the transaction prices of advertising, production, and search costs.

Denote the welfare attributed to the low and high cost firms by

$$w_L = \bar{u} \left[1 - \frac{1}{2} f(x)(1 - Q(\hat{s})) \right] - \frac{1}{2} f(x) \int_0^{\hat{s}} sq(s)ds - Ax, \text{ and} \tag{3.1}$$

$$w_H = \frac{1}{2} \left(\bar{u} - c \right) f(x) \left(1 - Q(\hat{s}) \right), \tag{3.2}$$

respectively. In words, (3.1) comes from $\left[1 - \frac{1}{2} f(x)(1 - Q(\hat{s})) \right]$ consumers receiving utility $\bar{u}$ from purchasing the good, which the firm produces at zero cost. Further, $\frac{1}{2} f(x) \int_0^{\hat{s}} sq(s)ds$ represents those buyers who did not randomly select the low cost firm and who were not informed through advertising but who have sufficiently low search costs so that they pay to visit the other firm. This is a welfare loss as it is the accumulated cost paid by all consumers who search to reach the low price firm. The remaining term, Ax, is the cost of advertising, which decreases welfare by lessening producer surplus. Equation (3.2) is similar and differs due to no advertising, no extra search costs, and positive marginal costs of production.

From equations (3.1) and (3.2), we see that the planner advertises essentially for two reasons: one, so that consumers reach the low cost firm on their first attempt and do not pay additional search costs, and two, to save the cost of production incurred by the high cost firm. We also see that the planner has no interest in the specific profit level of either firm. The low cost firm, however, cares only about profit and is indifferent to whatever search costs its customers accrue.

13

We can now formally discuss the planner's problem and study existence. First note that, from Proposition 1, π_L and π_H are strictly concave in p_L and p_H, respectively. Therefore the first order conditions for price are necessary and sufficient for the constrained planner's problem. Assuming an interior solution, the social planner solves

$$\max_{x \in [0,1]} \bar{u} - Ax - \frac{1}{2}f(x)\int_0^{\hat{s}} sq(s)ds - \frac{1}{2}f(x)(1 - Q(\hat{s}))c, \qquad (3.3)$$

subject to

$$\frac{\partial \pi_L}{\partial p_L} = \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s}))\right] - \frac{1}{2}f(x)q(\hat{s})p_L = 0 \qquad (3.4)$$

$$\frac{\partial \pi_H}{\partial p_H} = -q(\hat{s})(p_H - c) + 1 - Q(\hat{s}) = 0, \qquad (3.5)$$

where $\hat{s} = p_H - p_L$.

By imposing the duopoly first order conditions, we focus on a structural second best where the planner chooses advertising at prices consistent with firm behavior.[5] We need to impose these first order conditions essentially because the low cost firm makes his advertising and pricing decisions simultaneously and must consider the strategic complementarities between the two. Allowing the planner to ignore this interaction therefore imposes one set of rules on the firm and a different set of rules on the planner. To resolve this issue and provide a fair comparison, we impose the price first order conditions on the planner's problem.

To solve the planner's problem, we solve the constraints implicitly for $\hat{s}(x)$ and substitute this into the planner's objective function, equation (3.3). Before we can explicitly compare the firm's and planner's advertising levels, however, we need to establish the existence of a unique socially optimal x.

Proposition 5 *Given stability and condition (2.10), there exists a unique* $\bar{x} \in$

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[5]See Vives (1999) Chapter 6 for a similar approach with product differentiation.

$[0, 1]$ *such that* $\bar{x}$ *maximizes* (3.3), *subject to the firms' first order conditions for price, provided*

$$\frac{q''(\hat{s})}{q(\hat{s})} \leq \frac{-2f''(x)}{f'(x)(2p_{max} - c)c} - \frac{2}{(p_{max} - c)c}. \tag{3.6}$$

Equation (3.6) simply restricts $q''(\hat{s})$ so that it is not "too" positive. This ensures that the planner's objective function, after substituting $\hat{s}(x)$, is globally concave if x.

Given uniqueness, we determine over- or under-advertising by imposing the first order condition for advertising from the low price firm, equation (2.7), on the planner's first order condition. The resulting sign indicates whether firms advertise excessively or vice versa. Specifically, denote the planner's objective function by $W(x)$ and the low price firm's first order condition for advertising by $g(x; p_L, p_H)$. Over-advertising therefore results for $\frac{dW(x)}{dx}\big|_{g(x;\cdot)=0} < 0$ and vice versa for under-advertising. We summarize conditions for each result in Proposition 6.

Proposition 6 *Denote the mean search cost consumer by μ, then given stability and conditions* (2.10) *and* (3.6),

(i) *there exists some $\bar{c}$, $\underline{A}$, and p_{max} such that the duopolistic advertising level always exceeds the socially optimal level for all $c \geq \bar{c}$, all $A \leq \underline{A}$, or all $p_{max} \leq \frac{2}{q_{max}} - \mu$;*

(ii) *for the specific advertising function denoted $f(t, x)$, where $f_t < 0$ and $f_x(t, 0)$ sufficiently large, there exists some $\bar{t}$ such that the duopolistic advertising level always exceeds the socially optimal level for all $t \geq \bar{t}$; and*

(iii) *there exists some cost combination $(\bar{A}, \underline{c})$ such that, for all $A \geq \bar{A}$ and $c \leq \underline{c}$, the duopolistic advertising level is always below that of a planner, provided $\frac{1-Q(c)}{q(c)} < c$.*

15

The comparison between the firm's and planner's advertising levels depends primarily on $\hat{s} - c$. Whenever this is a large negative number, the firm over-advertises, and as it approaches zero, the firm under-advertises. The planner cares about this difference because it represents an implicit cost associated with advertising. For instance, we know from Proposition 4 that price dispersion decreases in response to an exogenous increase in advertising. This means that, if the planner increases x, some marginal consumers near $\hat{s}$ will go from active searchers (where they always buy at the low price) to inactive searchers (where they might pay the high price). This implicit advertising cost increases as c increases relative to $\hat{s}$.

Part *(i)* therefore describes three cases. In two of the three (p_{max} or A sufficiently low), $\hat{s}$ is low both in absolute terms and relative to c. Here, the firm has a large incentive to advertise because inactive searchers make up a larger portion of the market, while the planner would rather advertise less because the implicit cost of advertising is high. The social benefit due to decreased search is also small because $\hat{s}$ is low. In the third case (c sufficiently high), $\hat{s}$ is only low relative to c but may be high in absolute terms. This is the case in which, although the firm may have only a small incentive to advertise, the implicit cost of sending relatively low search cost consumers from the low price firm to the high price firm is substantial.

To better understand how the third case might arise, consider $\hat{s}$ such that $Q(\hat{s}) \geq 1/2$ and $\hat{s} \leq (1/2)c$, and assume that the planner increases advertising by some small amount. Here, active searchers constitute the majority of the market, and the welfare loss associated with the cost of production is significantly larger than the welfare loss associated with search. An extra advertisement therefore most likely reaches an active searcher, which provides some welfare gain, but also sends some marginal active searcher to the high price firm. Since c is sufficiently high relative to $\hat{s}$, this advertising decision is most likely a net welfare loss, and the planner would regret adopting such a strategy. Note that this does not depend

on the actual size of $\hat{s}$, only the relative size. Another important feature is that advertising has a uniform effect on the market while changes in price dispersion have a direct effect on consumers near $\hat{s}$. If the planner could increase advertising and pinpoint exactly where the extra advertising went, this tradeoff determined by $\hat{s} - c$ would not be an issue.

Similar intuition holds for under-advertising in part *(iii)*. As A increases, both $\hat{s}$ and $\hat{s} - c$ increase. The firm therefore has lesser incentive to advertise due to high $\hat{s}$, and the implicit welfare cost determined by $\hat{s} - c$ is less relevant. Further, the potential welfare gain associated with decreased search costs is larger since $\hat{s}$ is high. Note that the requirement for $\frac{1 - Q(c)}{q(c)} < c$ is essentially a hazard rate condition that places an upper bound on the high cost firm's profit margin even for high price dispersion.

We are also interested in how the overall shape of the advertising function might affect welfare. Part *(ii)* formally describes an advertising function where, for given amounts of advertising, only a small share of the market remains uninformed. The intuition here is similar to that of low A or low p_{max} in part *(i)*—as the incentive to advertise increases, the firm takes takes excessive advantage.

Note that, for symmetric search cost distributions, the indifferent consumer having a low search cost is equivalent to a market composed primarily of inactive searchers. For such distributions, we conclude that firms over-advertise when inactive searchers compose the majority of the market. This does not hold for all distributions, however, as a highly skewed $q(\cdot)$ could imply a large proportion of consumers search while the indifferent consumer's search cost remains small. This also does not hold in reverse as we have already seen that, even if $\hat{s}$ is large in absolute terms, it might still be small relative to c, in which case the firm still over-advertises.

Also note that, since all functions are continuous, and since both under- or over-advertising can result, there must be some combination of distributions, functional form specifications, and cost parameters such that the interests of both the

firm and the planner align. Although this is a knife-edge situation, it is interesting in that the two firms, acting purely in self-interest, could reach the socially optimal outcome.

4 Conclusion

The imperfect nature of price information in search models provides a natural framework within which to study price advertising. Previous studies, however, have not offered definitive welfare results under optimal consumer and firm behavior. This is a nontrivial issue as the planner and firm have potentially conflicting definitions of the value of advertising. In this paper, we put enough structure on the market to explicitly compare optimal and market advertising levels. We do so in an equilibrium search setting and analyze a structural second best where we impose the firms' price first order conditions on the planner's problem. Our analysis explains well the relationship between the firm's and the planner's incentives to advertise.

We find that firms might under- or over-advertise relative to a planner and that the result depends on several factors—primarily the effectiveness and cost of advertising and the cost of production. We find that firms place significantly more weight on the informational role of advertising whenever the indifferent consumer's search cost is low and vice versa when this search cost is high relative to the cost of production.

In particular, we get both under- and over-advertising in a setting where advertising is purely informative and without focusing on many identical firms. We do so in the context of an equilibrium consumer search model where *(i)* advertising has an obvious role in forming and improving buyers' knowledge of prices and *(ii)* where advertising and search are imperfect substitutes for transmitting price information. Our results show that the welfare effects of advertising are not a strict byproduct of the type of advertising in question, the elasticity of demand,

or the nature of competition among firms.

A   Proof of Proposition 1

By assumption, all functions are continuous and strategy sets are compact intervals. Therefore, by the standard Nash-Debreu theorem, a restricted pure-strategy Nash equilibrium exists so long as profit functions are quasi-concave in own strategy variables. From (2.9),

$$\frac{\partial^2 \pi_H}{\partial p_H^2} = \frac{1}{2}f(x)(1 - Q(\hat{s}))\Pi_{pp}^H - \frac{1}{2}f(x)d(p_H)\left[q(\hat{s})\Pi_p^H + q'(\hat{s})d(p_H)\Pi^H\right]$$

$$- \frac{1}{2}f(x)q(\hat{s})\left[d(p_H)\Pi_p^H + d'(p_H)\Pi^H\right]. \quad \text{(A.1)}$$

By previous assumptions on the monopolist's problem, the hazard condition, and on the advertising function, we know that (A.1) is negative so that π_H is strictly concave. By these same conditions,

$$\frac{\partial^2 \pi_L}{\partial p_L \partial x} = -\frac{1}{2}f'(x)\left[(1 - Q(\hat{s}))\Pi_p^L + q(\hat{s})d(p_L)\Pi^L\right] > 0, \quad \text{(A.2)}$$

$$\frac{\partial^2 \pi_L}{\partial x^2} = -\frac{1}{2}f''(x)(1 - Q(\hat{s}))d(p_L)p_L \leq 0, \text{ and} \quad \text{(A.3)}$$

$$\frac{\partial^2 \pi_L}{\partial p_L^2} = \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s}))\right]\Pi_{pp}^L$$

$$- \frac{1}{2}f(x)d(p_L)\left[q(\hat{s})\Pi_p^L - q'(\hat{s})d(p_L)\Pi^L\right]$$

$$- \frac{1}{2}q(\hat{s})f(x)\left[d(p_L)\Pi_p^L + d'(p_L)\Pi^L\right] < 0. \quad \text{(A.4)}$$

Therefore, the determinant of the bordered Hessian for the low cost firm must be positive, which then implies that π_L is quasi-concave. ∎

B   Proof of Proposition 2

First Prove (i)

First note that, from $\Pi = d(p)(p - c)$, we know that for any common price $p_H = p_L = p$, $\Pi_p^H|_{p_H=p} = d'(p)(p - c) + d(p) > d'(p)p + d(p) = \Pi_p^L|_{p_L=p}$ for all $c > 0$. Now suppose there exists a restricted Nash equilibrium that is not a Nash equilibrium. In such a case, at least one player is not making a best response. Figure 2 represents a graphical example of such a situation, where at least one firm would like to deviate from the restricted pricing strategy for a given advertising intensity x.

Figure 2: Profit Functions
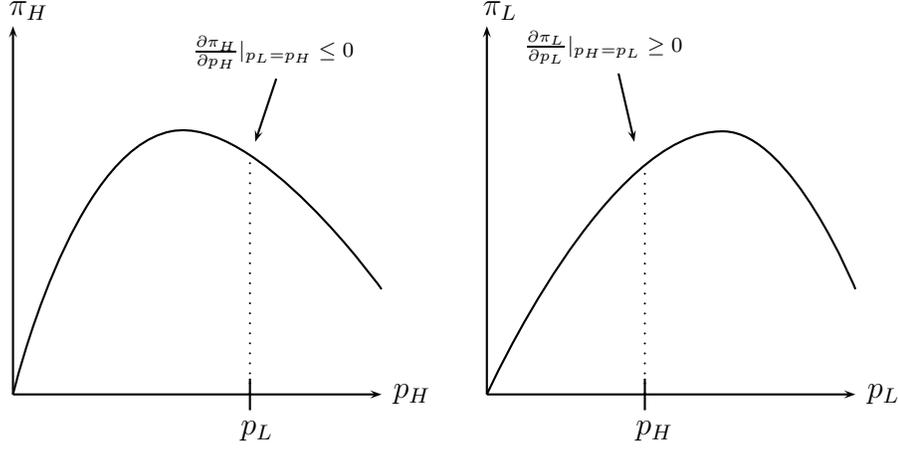
If the high cost firm is not making a best response, then $\frac{\partial \pi_H}{\partial p_H}|_{p_H=p_L} < 0$, while if the low cost firm is not making a best response, $\frac{\partial \pi_L}{\partial p_L}|_{p_L=p_H} > 0$. In either case, it must be that $\frac{\partial \pi_H}{\partial p_H}|_{p_H=p_L} \leq 0$ and $\frac{\partial \pi_L}{\partial p_L}|_{p_L=p_H} \geq 0$, where it follows that the restricted equilibrium must be at $p_H = p_L$, which implies that $x^* = 0$, $f(x^*) = 1$, and $Q(\hat{s}) = Q(0) = 0$. The resulting first order conditions are as follows (where $p_L = p_H = p$):

$$\frac{\partial \pi_H}{\partial p_H} = \frac{1}{2}\Pi_p^H|_{p_H=p} - \frac{1}{2}q(0)d(p)^2(p-c), \text{ and}$$

$$\frac{\partial \pi_L}{\partial p_L} = \frac{1}{2}\Pi_p^L|_{p_L=p} - \frac{1}{2}q(0)d(p)^2p.$$

Since $\Pi_p^H > \Pi_p^L$ from before, we see that

$$\frac{\partial \pi_H}{\partial p_H} > \frac{\partial \pi_L}{\partial p_L}$$

must hold for all $c > 0$. Without loss of generality, assume $\frac{\partial \pi_H}{\partial p_H} \leq 0$. Then it must be that

$$\frac{\partial \pi_L}{\partial p_L} < \frac{\partial \pi_H}{\partial p_H} \leq 0.$$

This cannot be a restricted equilibrium as $\frac{\partial \pi_L}{\partial p_L} < 0$, and the low cost firm wants to decrease price.

Now Prove (II)
From (I), we know that $p_H^* = p_L^*$ cannot hold in equilibrium, so we need only consider the case

21

where $p_H^* < p_L^*$. Denote each firm's monopoly price by $p_M(c) = \max(p-c)d(p)$, so that $p_M(0)$ is the monopoly price of the low cost firm. In any Nash equilibrium, $p_L^* < p_M(0) < p_{max}$. Then for $p_{max} > c > p_M(0)$, the high cost firm never prices below p_L. Accordingly, there exists some $\bar{c} > 0$, $\bar{c} < p_{max}$, such that, for all $c > \bar{c}$, every Nash equilibrium involves the low cost firm pricing below the high cost firm. ∎

C   Proof of Proposition 3

Assume that firms adjust their strategies according to

$$\frac{\mathrm{d}a_i}{\mathrm{d}t} = k_i \frac{\partial \pi_i(a_1, a_2, a_3)}{\partial a_i} \tag{C.1}$$

in a neighborhood of the equilibrium. In the usual way, we take a first-order Taylor approximation and, ignoring the constants k_i, we find

$$
\begin{bmatrix} \frac{\mathrm{d}p_L}{\mathrm{d}t} \\ \frac{\mathrm{d}p_H}{\mathrm{d}t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} \end{bmatrix}
=
\begin{bmatrix}
\frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial p_L^2} & \frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial p_L \partial p_H} & \frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial p_L \partial x} \\
\frac{\partial^2 \pi_H(p_L^*,p_H^*,x^*)}{\partial p_H \partial p_L} & \frac{\partial^2 \pi_H(p_L^*,p_H^*,x^*)}{\partial p_H^2} & \frac{\partial^2 \pi_H(p_L^*,p_H^*,x^*)}{\partial p_H \partial x} \\
\frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial x \partial p_L} & \frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial x \partial p_H} & \frac{\partial^2 \pi_L(p_L^*,p_H^*,x^*)}{\partial x^2}
\end{bmatrix}
\begin{bmatrix} p_L - p_L^* \\ p_H - p_H^* \\ x - x^* \end{bmatrix}.
$$

We need to show that the real parts of all eigenvalues are negative, which will ensure that our system is stable. A sufficient condition, therefore, is that our Hessian matrix has a dominant diagonal. By definition, any $n \times n$ matrix A has a dominant diagonal if there exists some $d_i > 0$, for $i = 1, 2, ..., n$, such that $d_i|\pi_{ii}| > \sum_{j \neq i} d_j|\pi_{ij}|$.

For convenience, denote the following

$$\lambda_{11} = (A.4),$$
$$\lambda_{12} = \frac{1}{2} f(x) d(p_H) \left[q(\hat{s}) \Pi_p^L - q'(\hat{s}) d(p_L) \Pi^L \right],$$
$$\lambda_{13} = (A.2),$$
$$\lambda_{21} = \frac{1}{2} f(x) d(p_L) \left[q(\hat{s}) \Pi_p^H + q'(\hat{s}) d(p_H) \Pi^H \right],$$
$$\lambda_{22} = (A.1),$$
$$\lambda_{23} = \frac{1}{2} f'(x) \left[(1 - Q(\hat{s})) \Pi_p^H - q(\hat{s}) d(p_H) \Pi^H \right],$$
$$\lambda_{31} = (A.2),$$
$$\lambda_{32} = \frac{1}{2} f'(x) q(\hat{s}) d(p_H) \Pi^H, \text{ and}$$
$$\lambda_{33} = (A.3).$$

Denote the matrix with the above elements by Λ. Sufficient conditions under elastic demand are complicated and omitted for space. It can be shown, however, that such conditions are maximized under inelastic demand. Accordingly, we consider unit inelastic demand to show that Λ has a dominant diagonal. After imposing the first order conditions for price and setting $d_1 = d_2 = d_3 = 1$, the following three conditions are sufficient for a dominant diagonal and thus

stability:

$$-\frac{1}{2}f(x)q(\hat{s}) - \frac{f'(x)}{f(x)} < 0,$$

$$-\frac{1}{2}f(x)q(\hat{s}) < 0, \text{ and}$$

$$-\frac{1}{2}f''(x)(1 - Q(\hat{s}))p_L - \frac{f'(x)}{f(x)} - \frac{1}{2}f'(x)q(\hat{s})p_L < 0.$$

These hold so long as

$$\frac{f'(x)}{f(x)} > -\frac{1}{2}f(x)q(\hat{s}), \text{ and} \tag{C.2}$$

$$\frac{f''(x)}{f'(x)} < -\frac{1}{1 - Q(\hat{s})}\left[\frac{2}{p_L f(x)} + q(\hat{s})\right]. \tag{C.3}$$

Using equilibrium conditions

$$p_L = \frac{2}{f(x)q(\hat{s})} - \frac{1 - Q(\hat{s})}{q(\hat{s})}, \text{ and}$$

$$p_H - c = \frac{1 - Q(\hat{s})}{q(\hat{s})},$$

we see that $\hat{s} = \frac{2}{q(\hat{s})}\left[1 - Q(\hat{s}) - \frac{1}{f(x)}\right] + c$, which implies that $\hat{s}$ is bounded above by c. We also see that $f(x)$ is bounded below by $\frac{2}{(2p_{max} - c)q_{max}}$ and that $q(\hat{s})$ is bounded below by $\frac{1 - Q(\hat{s})}{p_{max} - c}$. Therefore, assuming $Q(c) < 1$ provides an upper bound of $Q(\hat{s})$ and a lower bound on $1 - Q(\hat{s})$, and we can rewrite the above conditions as

$$\frac{f'(x)}{f(x)} > -\frac{(1 - Q(c))}{(2p_{max} - c)(p_{max} - c)q_{max}}, \text{ and} \tag{C.4}$$

$$\frac{f''(x)}{f'(x)} < -\frac{3q_{max}}{1 - Q(c)}. \tag{C.5}$$

Therefore, under conditions (C.4) and (C.5), Λ has a dominant diagonal, and the adjustment process defined by (C.1) is locally stable. Note that the expression for (C.4) given in the text is a slightly stronger sufficient condition. ∎

D   Proof of Proposition 4

Totally differentiating the system of first order conditions formed by (2.6), (2.9), and (2.7) with respect to p_L, p_H, x, A, and c provides the system of equations with which to derive comparative statics. Recalling Λ above, the differentiated system can then be written as follows:

$$\Lambda \begin{bmatrix} \mathrm{d}p_L \\ \mathrm{d}p_H \\ \mathrm{d}x \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}f(x)q(\hat{s})d(p_H)^2\mathrm{d}c \\ \mathrm{d}A \end{bmatrix}.$$

Just as in Appendix C, we impose the first order conditions for price, which greatly simplifies λ_{23} and λ_{13}. We also again consider the inelastic demand case for brevity, where it can be shown

that the determinant is maximized under this setting. This yields

$$|\Lambda| = -\frac{1}{8}f''(x)f(x)^2(1 - Q(\hat{s}))p_L\,(2q(\hat{s}) - q'(\hat{s})p_L)\,(2q(\hat{s}) + q'(\hat{s})(p_H - c))$$

$$+ \frac{1}{8}f''(x)f(x)^2(1 - Q(\hat{s}))p_L\,(q(\hat{s}) + q'(\hat{s})(p_H - c))\,(q(\hat{s}) - q'(\hat{s})p_L)$$

$$- \frac{f'(x)^2}{2f(x)}\left[f(x)q(\hat{s})p_L\,(q(\hat{s}) + q'(\hat{s})(p_H - c)) - (2q(\hat{s}) + q'(\hat{s})(p_H - c))\right].$$

Imposing stability conditions (C.2) and (C.3), it follows that $|\Lambda| < 0$, and applying Cramer's rule, we find

$$\frac{\mathrm{d}p_L}{\mathrm{d}c} = \frac{\frac{1}{2}f(x)q(\hat{s})\left[-\frac{1}{4}f(x)f''(x)(1 - Q(\hat{s}))p_L[q(\hat{s}) - q'(\hat{s})p_L] + \frac{f'(x)^2}{2f(x)}q(\hat{s})p_L\right]}{|\Lambda|} \geq 0,$$

$$\frac{\mathrm{d}p_L}{\mathrm{d}A} = -\frac{\frac{1}{2}f'(x)\left[2q(\hat{s}) + q'(\hat{s})(p_H - c)\right]}{|\Lambda|} \leq 0,$$

$$\frac{\mathrm{d}p_H}{\mathrm{d}c} = -\frac{\frac{1}{2}f(x)q(\hat{s})\left[\frac{1}{4}f''(x)f(x)(1 - Q(\hat{s}))p_L[2q(\hat{s}) - q'(\hat{s})p_L] - \frac{f'(x)^2}{f(x)^2}\right]}{|\Lambda|} \geq 0,$$

$$\frac{\mathrm{d}p_H}{\mathrm{d}A} = -\frac{\frac{1}{2}f'(x)[q(\hat{s}) + q'(\hat{s})(p_H - c)]}{|\Lambda|} \leq 0,$$

$$\frac{\mathrm{d}x}{\mathrm{d}c} = \frac{\frac{1}{4}f'(x)f(x)q(\hat{s})\left[-\frac{1}{2}f(x)q(\hat{s})p_L[2q(\hat{s}) - q'(\hat{s})p_L] + (q(\hat{s}) - q'(\hat{s})p_L)\right]}{|\Lambda|} \lessgtr 0,$$

$$\frac{\mathrm{d}x}{\mathrm{d}A} = \frac{\frac{1}{4}f(x)^2\left[(2q(\hat{s}) - q'(\hat{s})p_L)(2q(\hat{s}) + q'(\hat{s})(p_H - c))\right]}{|\Lambda|}$$

$$- \frac{\frac{1}{4}f(x)^2\left[(q(\hat{s}) - q'(\hat{s})p_L)(q(\hat{s}) + q'(\hat{s})(p_H - c))\right]}{|\Lambda|} \leq 0.$$

With respect to advertising, we treat x as exogenous and derive $\frac{\mathrm{d}\hat{s}}{\mathrm{d}x}$ in the usual way. Again looking at the inelastic demand case, totally differentiating (2.6) and (2.9) with respect to p_H, p_L, x, and c yields the following system

$$\begin{bmatrix} -q'(\hat{s})(p_H - c) - 2q(\hat{s}) & q'(\hat{s})(p_H - c) + q(\hat{s}) \\ \frac{1}{2}f(x)\,(q(\hat{s}) - q'(\hat{s})p_L) & -\frac{1}{2}f(x)\,(2q(\hat{s}) - q'(\hat{s})p_L) \end{bmatrix} \begin{bmatrix} \mathrm{d}p_H \\ \mathrm{d}p_L \end{bmatrix} = $$
$$\begin{bmatrix} -q(\hat{s})\mathrm{d}c \\ \frac{1}{2}f'(x)\,(1 - Q(\hat{s}) + q(\hat{s})p_L)\,\mathrm{d}x \end{bmatrix}.$$

For simplicity, define the following matrices:

$$\Omega = \begin{bmatrix} -q'(\hat{s})(p_H - c) - 2q(\hat{s}) & q'(\hat{s})(p_H - c) + q(\hat{s}) \\ \frac{1}{2}f(x)\,(q(\hat{s}) - q'(\hat{s})p_L) & -\frac{1}{2}f(x)\,(2q(\hat{s}) - q'(\hat{s})p_L) \end{bmatrix},$$

$$\Omega_{p_H} = \begin{bmatrix} -q(\hat{s})\mathrm{d}c & q'(\hat{s})(p_H - c) + q(\hat{s}) \\ \frac{1}{2}f'(x)\,(1 - Q(\hat{s}) + q'(\hat{s})p_L)\,\mathrm{d}x & -\frac{1}{2}f(x)\,(2q(\hat{s}) - q'(\hat{s})p_L) \end{bmatrix}, \text{ and}$$

$$\Omega_{p_L} = \begin{bmatrix} -q'(\hat{s})(p_H - c) - 2q(\hat{s}) & -q(\hat{s})\mathrm{d}c \\ \frac{1}{2}f(x)\,(q(\hat{s}) - q'(\hat{s})p_L) & \frac{1}{2}f'(x)\,(1 - Q(\hat{s}) + q'(\hat{s})p_L)\,\mathrm{d}x \end{bmatrix}.$$

From conditions (2.10) and (A.1), we know that

$$|\Omega| = \frac{1}{2}f(x)q(\hat{s})\left[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})\right] > 0.$$

24

Looking only at p_H, we see that

$$\mathrm{d}p_H = \frac{[2q(\hat{s}) - q'(\hat{s})p_L]\,\mathrm{d}c}{q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})}$$
$$- \frac{f'(x)\left[1 - Q(\hat{s}) + q(\hat{s})p_L\right]\left[q'(\hat{s})(p_H - c) + q(\hat{s})\right]\,\mathrm{d}x}{f(x)q(\hat{s})\left[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})\right]}.$$

Setting $\mathrm{d}c$ to zero, we find

$$\frac{\mathrm{d}p_H}{\mathrm{d}x} = \frac{-f'(x)\left[1 - Q(\hat{s}) + q(\hat{s})p_L\right]\left[q'(\hat{s})(p_H - c) + q(\hat{s})\right]}{f(x)q(\hat{s})\left[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})\right]}.$$

The same process for p_L yields

$$\frac{\mathrm{d}p_L}{\mathrm{d}x} = \frac{-f'(x)\left[1 - Q(\hat{s}) + q(\hat{s})p_L\right]\left[q'(\hat{s})(p_H - c) + 2q(\hat{s})\right]}{f(x)q(\hat{s})\left[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})\right]}.$$

Therefore, we know

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}x} = \frac{\mathrm{d}p_H}{\mathrm{d}x} - \frac{\mathrm{d}p_L}{\mathrm{d}x} = \frac{f'(x)\left[1 - Q(\hat{s}) + q(\hat{s})p_L\right]}{f(x)\left[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})\right]} < 0. \ \blacksquare$$

E   Proof of Proposition 5

Since π_L and π_H are concave in p_L and p_H respectively, the first order conditions for p_L and p_H are necessary and sufficient for a constrained optimum of the planner's problem. We can then solve the constraints implicitly for $\hat{s}(x)$ and plug this into the objective function. To ensure a unique optimum, we need only show that the resulting function is strictly concave.

Differentiating the welfare function and rearranging terms yields

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = -\frac{1}{2}f''(x)\left[(1 - Q(\hat{s}))c + \int_0^{\hat{s}} sq(s)ds\right]$$
$$- f'(x)q(\hat{s})(\hat{s} - c)\frac{\mathrm{d}\hat{s}}{\mathrm{d}x} - \frac{1}{2}f(x)q(\hat{s})\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}x}\right)^2$$
$$- \frac{1}{2}f(x)(\hat{s} - c)\left[q'(\hat{s})\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}x}\right)^2 + q(\hat{s})\frac{\mathrm{d}^2\hat{s}}{\mathrm{d}x^2}\right]. \tag{E.1}$$

After substituting $\frac{\mathrm{d}\hat{s}}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2\hat{s}}{\mathrm{d}x^2}$ and some tedious algebra, we see that (E.1) is always negative provided $\frac{q''(\hat{s})}{q(\hat{s})} \leq \frac{-f''(x)}{f'(x)(2p_{max} - c)c} - \frac{2}{(p_{max} - c)c}$, and there exists a unique socially optimal advertising level subject to the equilibrium duopoly price level. $\blacksquare$

F   Proof of Proposition 6

FIRST PROVE (I)
For convenience, denote $\phi = \int_0^{\hat{s}} sq(s)\mathrm{d}s + (1 - Q(\hat{s}))(c - p_L)$, then substituting the firm's first

order condition for advertising, (2.7), yields

$$\frac{\mathrm{d}W}{\mathrm{d}x}\Big|_{x^d} = -\frac{1}{2}f'(x)\phi - \frac{1}{2}f(x)q(\hat{s})(\hat{s}-c)\frac{\mathrm{d}\hat{s}}{\mathrm{d}x}$$

$$= -\frac{1}{2}f'(x)\phi - \frac{1}{2}f'(x)q(\hat{s})(\hat{s}-c)\frac{1-Q(\hat{s})+q(\hat{s})p_L}{q'(\hat{s})(p_H-p_L-c)+3q(\hat{s})}$$

$$= -\frac{1}{2}f'(x)\phi - q(\hat{s})\frac{f'(x)(\hat{s}-c)}{f(x)\left[q'(\hat{s})(p_H-p_L-c)+3q(\hat{s})\right]}$$

where the third equality comes from substituting (2.6). From this equation, we know the sign of $\frac{\mathrm{d}W}{\mathrm{d}x}\big|_{x^d}$ depends on ϕ. To see this, note that equilibrium first order conditions, (2.6) and (2.9), require

$$\hat{s} - c = p_H - p_L - c = \frac{2}{q(\hat{s})}\left[1 - Q(\hat{s}) - f^{-1}(x)\right]$$

which is nonpositive as $f(x) \in [0,1]$ and $Q(\hat{s}) \geq 0$. Also, from (2.10) we know $q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s}) > 0$. So the following results hold:

$$\frac{\mathrm{d}W}{\mathrm{d}x}\Big|_{x^d} > 0 \text{ iff } \phi > \frac{-2q(\hat{s})(\hat{s}-c)}{f(x)\left[q'(\hat{s})(\hat{s}-c)+3q(\hat{s})\right]}; \tag{F.1}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x}\Big|_{x^d} < 0 \text{ iff } \phi < \frac{-2q(\hat{s})(\hat{s}-c)}{f(x)\left[q'(\hat{s})(\hat{s}-c)+3q(\hat{s})\right]}. \tag{F.2}$$

First consider the upper bound of ϕ. From equations (2.6) and (2.9), we see that $c - p_L = p_H - \frac{2}{q(\hat{s})f(x)}$, which is bounded above by $p_{max} - \frac{2}{q_{max}}$. This implies that ϕ is bounded above by $\mu + p_{max} - \frac{2}{q_{max}}$, where μ is the mean of s (note that $\mu \geq \int_0^{\hat{s}} sq(s)\mathrm{d}s$). Therefore, for $p_{max} < \frac{2}{q_{max}} - \mu$, it follows that $\frac{\mathrm{d}W}{\mathrm{d}x}\big|_{x^d} < 0$.

We proceed by examining the comparative statics of $\hat{s}$ to changes in c and A as well the upper bound of ϕ to determine when equation (F.2) holds. First, we rewrite the upper bound of ϕ by noting that $\int_0^{\hat{s}} sq(s)\mathrm{d}s \leq \hat{s}Q(\hat{s})$, which implies that $\phi \leq c + Q(\hat{s})(\hat{s}-c)$. Therefore, over-advertising results for

$$c + Q(\hat{s})(\hat{s}-c) \leq -\frac{2(\hat{s}-c)}{f(x)\left[\frac{q'(\hat{s})}{q(\hat{s})}(\hat{s}-c)+3\right]}. \tag{F.3}$$

We use both equations (F.2) and (F.3) in the following. Note that any equilibrium requires $\hat{s} \leq 1$. Otherwise, the low price firm could increase price to $p_H - 1$ and still get the entire market. Since $\hat{s} - c$ is always negative in equilibrium, this implies that $\min\{1, c\} \geq \hat{s} \geq 0$. Now consider the left and right hand sides of equation (F.2) as $\hat{s}$ goes to its maximum, where we see that the left hand side is bounded above by c for $c \leq 1$ and bounded above by μ for $c > 1$, and the right hand side is equal to 0 for $c \leq 1$ and is positive for $c > 1$. For the lower bound of $\hat{s}$ ($\hat{s} = 0$), we see that $\phi = c - p_H < 0$ as $\hat{s} = 0 \Rightarrow p_L = p_H$. Since the right hand side is positive for $\hat{s} = 0$, it follows that equation (F.2) is always satisfied for $\hat{s} = 0$ and unsatisfied for $\hat{s} = c \leq 1$. Figure 3 describes these bounds graphically, where over-advertising is depicted by the range in which RHS is above LHS. We only consider graphically the case where $c \leq 1$, but a similar result holds for $c > 1$ as ϕ remains bounded above by μ.

Figure 3: OVER-ADVERTISING
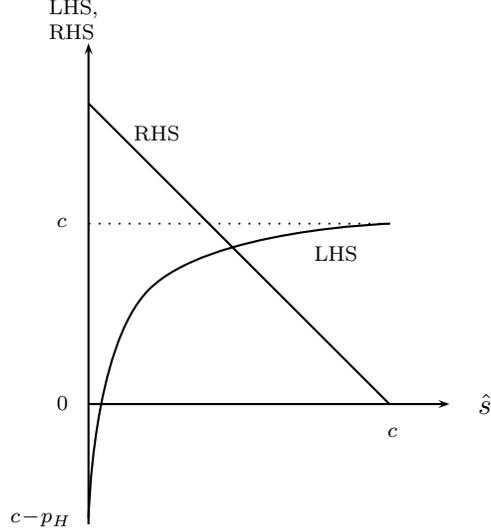
While the functions on the left and right hand sides both change as $\hat{s}$ and c change, the bounds remain fixed as in figure 3. Since all functions are continuous, it follows that there exists some $\underline{\hat{s}}$ such that over-advertising results for all $\hat{s} \leq \underline{\hat{s}}$. From Proposition 4, we see that $\hat{s}$ is increasing in A. Therefore, there exists some $\underline{A}$ such that, for all $A \leq \underline{A}$, the duopolistic advertising level exceeds the social optimum.

Finally, recall that $\hat{s} \to 1$ as c increases, in which case $\phi = \int_0^{\hat{s}} sq(s)\mathrm{d}s + (1 - Q(\hat{s}))(c - p_L) \to \mu$ since $Q(1) = 1$. Now, the right hand side of equation (F.2) is increasing in c while the left hand side does not change, so there exists some $\bar{c}$ such that over-advertising results for all $c \geq \bar{c}$.[6] This proves part *(i)*.

NOW PROVE (II)

Since the upper bound for ϕ is independent of x, and since the lower bound for

$$\frac{-2q(\hat{s})(\hat{s} - c)}{f(x)\left[q'(\hat{s})(\hat{s} - c) + 3q(\hat{s})\right]}$$

is increasing in x, it follows that over-advertising results for $f(x) \in [0, 1]$ sufficiently small. We formally characterize this by considering $f(t, x)$, $t > 0$, in which case $f_t < 0$ and t sufficiently large implies that $f(x)$ is small even for small levels of advertising. Finally, we restrict problems along the boundary by assuming that, as $x \to 0$, f_x becomes large. This proves part *(ii)*.

NOW PROVE (III)

Recall that over-advertising occurs iff $\phi > \frac{-2q(\hat{s})(\hat{s}-c)}{f(x)[q'(\hat{s})(\hat{s}-c)+3q(\hat{s})]}$. Following a similar process as with part *(i)*, $\hat{s} \to c$ as A increases, which implies that $p_H - c = p_L$. This also implies that $(1 - Q(\hat{s}))(c - p_L) \to (1 - Q(c))(c - (p_H - c))$.

Now consider $(1 - Q(c))(2c - p_H)$ as a lower bound for ϕ at $\hat{s} = c$. To ensure that this is positive, we need $\frac{1-Q(c)}{q(c)} < c$, in which case ϕ is positive at this lower bound and the right

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[6]We consider the more extreme argument of $\hat{s} \to 1$ so as to avoid the extra assumptions dealing with how the right hand side of equation (F.2) shifts relative the left hand side as c increases, although it can be show that under assumptions consistent with existence of a social optimum and stability, such a result holds.

hand side goes to zero. Therefore, provided c is relatively small so that $\hat{s}$ is bounded below 1, equation (F.1) will hold for some A sufficiently large since $\hat{s}$ is increasing in A. So, there exists some cost pair $(\bar{A}, \underline{c})$ such that for all $A \geq \bar{A}$ and $c \leq \underline{c}$ the duopolistic advertising level is below that of a planner. This proves part *(iii)*. ∎

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