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Endogenous Mechanisms and Nash Equilibrium in Competitive Contracting Games

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Abstract

We model strategic competition in a market with asymmetric information as a noncooperative game in which each firm competes for the business of a buyer of unknown type by offering the buyer a catalog of products and prices. The timing in our model is Stackelberg: in the first stage, given the distribution of buyer types known to all firms and the deducible, type-dependent best responses of the agent, firms simultaneously and noncooperatively choose their catalog offers. In the second stage the buyer, knowing his type, chooses a single firm and product-price pair from that firm's catalog. By backward induction, this Stackelberg game with asymmetric information reduces to a game over catalogs with payoff indeterminacies. In particular, due to ties within catalogs and/or across catalogs, corresponding to any catalog profile offered by firms there may be multiple possible expected firm payoffs, all consistent with the rational optimizing behavior of the agent for each of his types. The resolution of these indeterminacies depends on the tie-breaking mechanism which emerges in the market. Because each tie-breaking mechanism induces a particular game over catalogs, a reasonable candidate would be a tie-breaking mechanism which supports a Nash equilibrium in the corresponding catalog game. We call such a mechanism an *endogenous Nash mechanism*. The fundamental question we address in this paper is, does there exist an endogenous Nash mechanism - and therefore, does there exist a Nash equilibrium for the catalog game? We show under fairly mild conditions on primitives that catalog games naturally possess tie-breaking mechanisms which support Nash equilibria.

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1 Introduction

Firms often compete by offering potential buyers catalogs of products and prices. Examples of catalog competition abound. Fidelity Investments Inc. competes in the mutual funds market by offering investors a catalog of funds with differing risks and fee structures (e.g., equity funds, bond funds, index funds). California competes with other states to attract businesses by offering a catalog of amenities and taxes. Intel competes with other high tech firms to attract top engineers by offering a catalog of compensation and benefit packages. Cell phone companies compete by offering a plethora of calling plans. And the list could go on. A common feature of all of these examples is the presence of asymmetric information. Firms do not know all of the relevant characteristics of potential buyers. By offering a catalog each firm is better able to screen potential buyers by allowing buyers to sort themselves. Moreover, by offering a well-chosen catalog of products and prices a firm may be able to deter a competitor from defecting to a new catalog or to prevent such a defection from eroding the firm's customer base.

In this paper we model strategic competition in a market with asymmetric information as a noncooperative game in which each firm competes for the business of a buyer of unknown type by offering the buyer a catalog of products and prices. Thus each firm's strategy space is a set of catalogs. The buyer's type parameter, known to the buyer at the time the buyer chooses a firm and a contract, is known only up to a distribution by firms at the time firms choose their catalogs.[1] The timing in our model is Stackelberg: in the first stage, given the distribution of buyer types known to all firms and the deducible, type-dependent best responses of the agent, firms simultaneously and noncooperatively choose their catalog offers. In the second stage the buyer, knowing his type, chooses a single firm and product-price pair from that firm's catalog. By backward induction, this Stackelberg game with asymmetric information reduces to a game over catalogs with expected payoff indeterminacies. In particular, due to ties within catalogs and/or across catalogs, corresponding to any catalog profile offered by firms there may be multiple possible expected firm payoffs, all consistent with the rational optimizing behavior of the agent for each of his types. The resolution of these indeterminacies depends on the tie-breaking mechanism which emerges in the market. Because each tie-breaking mechanism induces a particular game over catalogs, a reasonable candidate for tie-breaking mechanism would be a mechanism which supports a Nash equilibrium in the corresponding catalog game. We call such a mechanism an *endogenous Nash mechanism*. The fundamental question we address in this paper is, does there exist an endogenous Nash mechanism?

[1] We can also interpret the model as one in which several firms compete in a market populated by many buyers where each buyer is identified (or indexed) by a random type parameter and where firms at the time they choose their catalogs only know the distribution of the type parameter in the population of buyers.

We make two contributions. First, we show that if the private information of the agent is sufficiently granular, that is, if the probability space of agent types is atomless (for example, if the set of agent types is a Borel subset of R^L and the distribution of agent types is given by a density with respect to Lebesgue measure), then there exists an endogenous Nash mechanism. Second, we show that even without the atomless condition, there exists a randomized endogenous Nash mechanism. Thus we show under fairly mild conditions on primitives that competitive contracting games - that is, catalog games - naturally possess tie-breaking mechanisms which support Nash equilibria.

The primitives of our model consist of a probability space of agent types, a compact metric space of contracts, an agent payoff function continuous in contracts and measurable in types, and firm profit functions also continuous in contracts and measurable in agent types (i.e., both agent and firm payoff functions are Caratheodory). While each firm's profit function is continuous in contracts, each firm's expected profit function over catalog profiles, induced by the prevailing tie-breaking mechanism and the optimizing behavior of the agent, is discontinuous. To deal with discontinuities in expected profit we follow the seminal approach introduced by Simon and Zame (1990) and formulate the game over set-valued expected catalog profits as a game with an endogenous sharing rule. In particular, we begin by identifying a (measurable) selection from the expected catalog payoff correspondence such that the induced game with firm expected profit functions over catalogs specified via this selection (i.e., via this sharing rule) has a Nash equilibrium in mixed catalog strategies. Next, using a beautiful measurable selection result due to Mertens (1987) and the classical Filippov's Implicit Function Theorem (see Himmelberg (1975)), we show that if the probability space of agent types is atomless, then this payoff sharing rule can be implemented by a mechanism mapping from agent types and catalog profiles into firm-contract pairs with the property that for each given catalog profile the resulting direct mechanism (a mapping from agent types into firm-contract pairs) is rational and incentive compatible. Thus we conclude that the catalog game induced by this endogenous mechanism has a Nash equilibrium in mixed catalog strategies, and thus we conclude that this endogenous mechanism is Nash. In the absence of the atomless condition, we show that this payoff sharing rule (i.e., our selection from the expected payoff correspondence) can be implemented by a randomized mechanism. Thus we conclude that in the absence of the atomless condition the catalog game induced by this randomized endogenous mechanism has a Nash equilibrium in mixed catalog strategies - and thus that this randomized endogenous mechanism is Nash.

Related Literature

There has been much work on catalog (or menu) models of competitive contracting games. Most notably for the analysis we carry out here Page (1999), Peters (2001), Martimort and Stole (2002), and Page and Monteiro (2003) have shown for various versions of the competitive contracting game (or common agency game) that attention can be restricted without loss of generality to games played over contract catalogs rather than over indirect contracting mechanisms (see, for example, Theorem 2 in Page and Monteiro (2003); also see Martimort (2006)).

While there has been much work on the relationship between catalog games and competitive contracting games over mechanisms (direct and indirect), far less work has been done on the Nash existence problem. In Page and Monteiro (2003) firm profit functions are taken to be upper semicontinuous in contracts (rather than continuous in contracts as is assumed here) and a Nash equilibrium in mixed catalog strategies is established under the assumption that each firm behaves as if all ties will be broken in the firm's favor. Under this *optimistic tie-breaking rule* each firm's induced expected profit function over catalogs is upper semicontinuous. The existence of a Nash equilibrium in mixed strategies is then established via Reny (1999) by identifying conditions sufficient to guarantee that the mixed catalog game is payoff secure.[2] In particular, Page and Monteiro (2003) first establish, under some additional conditions on primitives, that no matter what the tie-breaking rule (or mechanism), catalog games are *uniformly payoff secure*.[3] Next they show that if the catalog game is uniformly payoff secure, then no matter what the tie-breaking rule the mixed extension of the game is payoff secure.[4] Together these two results imply that the question of existence of Nash equilibrium in discontinuous mixed catalog games reduces (via Reny (1999)) to a search for tie-breaking mechanisms which guarantee that the induced expected profit functions over catalogs are at least reciprocally upper semicontinuous - a property guaranteed by the upper semicontinuity of expected profits under the optimistic tie-breaking rule. In Monteiro and Page (2006b), we introduce a more realistic tie-breaking rule - the *efficiency rule* - is introduced. Under the efficiency rule contracts are awarded to firms which value it most (i.e., breaks ties in favor of firms which stand to make the highest profit). Monteiro and Page (2006b) show that under the efficiency rule firm expected profit functions are reciprocally upper semicontinuous (i.e., that under the efficiency rule the mixed

[2] It follows from Reny (1999) that a payoff secure game with upper semicontinuous (or at least reciprocally upper semicontinuous) payoffs over compact strategy sets has a Nash equilibrium.

Following Reny, a game is said to be payoff secure if for every joint strategy, $x = (x_i, x_{-i})$, each agent i has a strategy x_i^* that virtually guarantees the payoff he receives at x even if other agents play strategies, $x_{-i}^{'}$, slightly different from x_{-i}.

Informally, payoffs are reciprocally upper semicontinuous if, whenever some agent's payoff jumps down, some other agent's payoff jumps up (see Simon (1987), Dasgupta and Maskin (1986), and Reny (1999)). Reciprocal upper semicontinuity is implied by upper semicontinuity (but the converse does not hold in general).

[3] A catalog game is uniformly payoff secure if for every firm j, every catalog C_j^*, and every $\varepsilon > 0$, there exists a catalog $\widehat{C}_j$ such that for any catalogs C_{-j} offered by other firms there is a neighborhood of C_{-j} such that

$$\Pi_j(\widehat{C}_j, C'_{-j}) \geq \Pi_j(C_j^*, C_{-j}) - \varepsilon,$$

for all deviations $C_{-j}^{'}$ in that neighborhood of C_{-j}. Thus, the catalog game is uniformly payoff secure if for each firm j and each starting catalog strategy C_j^* there is a defensive catalog strategy, $\widehat{C}_j$, that firm j can move to in order to secure an expected payoff of at least

$$\Pi_j(C_j^*, C_{-j}) - \varepsilon$$

given any starting strategy profile, C_{-j}, of other firms and any deviation by other firms to strategies, C'_{-j}, in a neighborhood of C_{-j}. A formal definition of uniform payoff security is given in section 3.1.

[4] This connection between the uniformly payoff security of a game and the payoff security of its mixed extension is extended to arbitrary compact games in Monteiro and Page (2006a).

catalog game is reciprocally upper semicontinuous) - and therefore show that under the efficiency rule a uniformly payoff secure catalog game with payoffs only upper semicontinuous in contracts has a Nash equilibrium in mixed strategies. In this paper, we take a completely new approach based on Simon and Zame (1990). Under our new approach *the tie-breaking rule is determined endogenously.* In particular, we show that if profit functions are continuous in contracts (even though the induced expected profit functions over catalogs are still only upper semicontinuous), then the Nash problem reduces (via Simon and Zame (1990)) to an implementation problem - that is, to a problem of finding an endogenous mechanism which implements the payoff sharing rule with respect to which there is a Nash equilibrium in mixed strategies (...a payoff sharing rule whose existence is guaranteed by Simon and Zame (1990)). Our contribution viewed from this perspective is to show that this implementation problem has a solution - that is that there exists an endogenous Nash mechanism.

The work most closely related to our analysis here is that of Carmona and Fajardo (2006). Their strategy for solving the Nash problem differs from our strategy. In particular, they model the common agency game as an extensive game and then try to extend Simon and Zame to show the existence of a sequential equilibrium. Here, we show that the common agency game - because it is a Stackelberg screening game - reduces by straight forward backward induction to a game (i.e., a catalog game) already contained in the class of games covered by Simon and Zame. Hence under our approach, no extension of Simon and Zame is required and, as noted above, the Nash existence problem reduces to an implementation problem. For the case of simultaneous move games with incomplete information and communication (i.e., for the non-Stackelberg case where the type of reduction via backward induction carried out here is *not* possible), Jackson, Simon, Swinkels, and Zame (2002) extend Simon and Zame (1990) and prove the existence of a Nash equilibrium.

We shall proceed as follows: In Section 2 we present the basic ingredients of our model including the agent's choice problem under catalog competition and the notion of a tie-breaking mechanism. In section 3, we consider catalog games induced by tie-breaking mechanisms and we introduce the notion an endogenous Nash mechanism. Also in Section 3 we state our main results. Finally, in Section 4 we provide proofs of our main results. Throughout we illustrate our model via an example of competitive executive compensation.

2 The Catalog Model of Competitive Contracting with a Privately Informed Agent

We construct a model in which m firms, indexed by i *and* j ($= 1, 2, \ldots, m$), compete for the business of a single, privately informed agent via the catalogs of contracts firms offer to the agent. The agent represents for each of his types a buyer type in the market. In our model, firms move first, simultaneously choosing their catalogs. The agent moves second, choosing for each of his types a firm with which to do business and a contract from that firm's catalog. The game we shall ultimately consider provides a model of how firms choose their catalogs in the first stage given

the distribution of agent types (i.e., the distribution of buyers) and the optimizing behavior of buyers in the second stage. We begin by considering the second stage, that is, by considering the agent's problem.

2.1 The Agent's Problem

2.1.1 Agent Types and Contracts

We shall assume that

(A-1) the set of agent types is given by a probability space, $(T, \Im, \mu)$, where T is a set of agent types, $\Im$ is a σ -field in T, and μ is a probability measure defined on $\Im$.

Let K be a set representing all possible contracts firms can offer the agent and for each firm $i = 1, 2, \ldots, m$ let $K_i \subseteq K$ be the set of all possible contracts firm i can offer to the agent. Elements of K will be denoted by f.

We shall assume that,

(A-2) (i) K is a compact metric space, containing an element 0 which we shall agree denotes "no contracting," and (ii) K_i is a nonempty closed subset of K containing 0.

Example 1 *(Executive Compensation Contracts) Consider a situation in which several firms compete for the services of an executive (an agent) of unknown type. To begin, let $(Z, B(Z), \eta)$ be a probability space of states-of-nature where Z is a metric space equipped with the Borel σ-field $B(Z)$ and η is a probability measure know to all firms and the agent. Suppose that the set of all possible compensation contracts is given by a convex set K of $B(Z)$-measurable functions defined on the state space Z taking values in some closed bounded interval, $[L, H]$, where $L \leq 0 < H$.*[5] *Under compensation contract $f = f(\cdot) \in K$, the executive is paid an amount $f(z) \in [L, H]$ in state $z \in Z$. Assume now that the set of compensation contracts K has the following properties:*

(i) K contains a contract,

$$0(\cdot) : Z \to [L, H],$$

such that

$$0(z) = 0 \text{ for all } z \in Z.$$

(ii) K is sequentially compact for the topology of pointwise convergence on Z, that is, for any sequence of contracts $\{f^n\}_n$ in K there is a subsequence $\{f^{n_k}\}_k$ in K and a contract $f' \in K$ such that

$$f^{n_k}(z) \to f'(z) \text{ for all } z \in Z.$$

[5] See Aliprantis and Border (1999) for a definition of $B(Z)$-measurability.

(iii) K contains no redundant contracts, that is, if the compensation levels under contracts f and f' in K differ in some state of nature $z' \in Z$, then the payoffs under contracts f and f' differ on a set of states having positive probability. Stated formally, K contains no redundant funds if for any pair of contracts f and f' in K

$$f(z') \neq f'(z') \text{ for some } z' \in Z, \text{ implies that}$$
$$\eta\left\{z \in Z : f(z) \neq f'(z)\right\} > 0.$$

The uniform boundedness of K (by L and H) together with conditions (ii) and (iii) imply that the set of contracts K is compact and metrizable for the topology of pointwise convergence (see Proposition 1 in Tulcea (1973)). In particular, under (ii), (iii), and uniform boundedness,

$$d_\eta(f, f') := \int_Z \left|f(z) - f'(z)\right| d\eta(z),$$

defines a metric on K which generates the topology of pointwise convergence[6] *and makes K a compact metric space. Taking as the "no-contracting" contract the function $0(\cdot)$, the set K satisfies (A-2)(i). If, in addition, we assume (a) that each firm $i = 1, 2, \ldots, m$ has available only a d_η-closed subset of compensation contracts $K_i \subseteq K$ that can be offered to the executive, and (b) that each firm-specific contract set K_i contains the contract $0(\cdot)$, then assumptions (A-2) are satisfied.*

In order to allow for the possibility that some types of the agent may wish to abstain from contracting altogether, we assume that there is a fictitious firm $i = 0$ with feasible set of contracts $K_0 := \{0\}$.

Letting $I = \{0, 1, 2, \ldots, m\}$, define the set

$$\mathbb{K} := \left\{(i, f) \in I \times K : f \in K_i\right\}. \tag{1}$$

A firm-contract pair $(i, f) \in \mathbb{K}$ indicates that the agent has chosen sales contract $f \in K_i$ from firm i, while $(i, f) = (0, 0) \in \mathbb{K}$ indicates that the agent has chosen to abstain from contracting altogether. Note that the set $\mathbb{K}$ is a closed subset of the compact set $I \times K$.[7] Thus, $\mathbb{K}$ is a compact set.

2.1.2 Catalogs of Contracts

For each firm $i = 1, 2, \ldots, m$, let C_i be a nonempty, closed subset of K_i. We can think of the subset C_i as representing a *catalog* of contracts that the ith firm might offer

[6] Thus, for any sequence of contracts $\{f^n\}_n \subseteq K$ and contract $f' \in K$, $d_\eta(f^n, f') \to 0$ if and only if

$$f^n(z) \to f'(z) \text{ for all } z \in Z.$$

[7] Equip I with the discrete metric $d_I(\cdot, \cdot)$ given by

$$d_I(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

to the agent. For $i = 0, 1, 2, \ldots, m$, let $P_f(K_i)$ denote the collection of all possible catalogs, that is, the collection of *all nonempty, closed subsets of* K_i.[8] Since K_i is a compact metric space, the collection of catalogs, $P_f(K_i)$, equipped with the Hausdorff metric h is automatically compact (see Aliprantis and Border (1999) for the definition of the Hausdorff metric and a discussion).

If firms compete via catalogs, then their strategy choices can be summarized via a catalog profile,

$$(C_1, \ldots, C_m)\,. \tag{2}$$

Here, the ith component of the m-tuple $(C_1, \ldots, C_m)$ is the catalog offered by the ith firm to the agent. Let

$$\mathbf{P} := P_f(K_1) \times \cdots \times P_f(K_m)$$

denote the space of all catalog profiles. If $\mathbf{P}$ is equipped with the metric $h_{\mathbf{P}}$ given by

$$h_{\mathbf{P}}\left((C_1, \ldots, C_m), (C'_1, \ldots, C'_m)\right) := \max\{h(C_i, C'_i) : i = 1, 2, \ldots, m\}, \tag{3}$$

then the space of catalog profiles $(\mathbf{P}, h_{\mathbf{P}})$ is a compact metric space.

2.1.3 The Agent's Choice Problem Under Catalog Competition

We shall assume that

(A-3) the agent's utility function,

$$v(\cdot, \cdot, \cdot, \cdot) : T \times I \times K \times \mathbf{P} \to R, \tag{4}$$

is such that, (i) for each $t \in T$, $v(t, \cdot, \cdot, \cdot)$ is continuous and for each $(i, f, C) \in I \times K \times \mathbf{P}$, $v(\cdot, i, f, C)$ is $\Im$-measurable and (ii) $v(t, i, 0, C) \leq v(t, 0, 0, C)$ for all $t \in T$ and $i = 1, 2, \ldots, m$.

Note that we allow the agent's utility to depend not only on the contract f but also on brand name i (i.e., the name of the firm with which the agent contracts) and the catalogs offered by all firms. However, by (A-3)(ii) if the agent is to derive any utility from a firm's brand name beyond the reservation level, $v(t, 0, 0, C)$, then the agent must enter into a contract with the firm. Allowing utility to depend on brand names *does not* rule out the possibility that some (or all) types of the agent are completely indifferent to brand names.

Example 2 *(The Agent's Utility Function for the Executive Compensation Example)*
As in example 1, let K be a convex set of state-contingent compensation contracts compact metrizable for the topology of pointwise convergence satisfying assumptions (A-2). We shall assume that if the agent is of type $t \in T$ and chooses to work for firm $i \in I$, then the states of nature upon which compensation is based are generated

[8] Note that since $K_0 = \{0\}$, $P_f(K_0)$ consists of one nonempty, closed subset, namely the set $\{0\}$.

according to a conditional probability measure $\zeta(\cdot|t,i)$ satisfying the property that for each event $E \in B(Z)$ and each firm $i \in I$

$$\zeta(E|\cdot,i) : T \to [0,1]$$

$\Im$-measurable. The conditional probability measure $\zeta(\cdot|t,i)$ thus represents the stochastic production technology available to firm i under the management of a type t executive. Let

$$u(\cdot,\cdot,\cdot,\cdot) : T \times I \times [L,H] \times \mathbf{P} \to R$$

be a function such that (i) for each $t \in T$, $u(t,\cdot,\cdot,\cdot)$ is continuous on $I \times [L,H] \times \mathbf{P}$, (ii) for each $(i,c,C) \in I \times [L,H] \times \mathbf{P}$, $u(\cdot,i,c,C)$ is $\Im$-measurable, and (iii) for each $C \in \mathbf{P}$ and $i \in 1,2,\ldots,m$, $u(t,i,0,C) \leq u(t,0,0,C)$ on T. Finally, let the type t agent's (expected) utility over firm-contract-catalog profile 3-tuples be given by

$$v(t,i,f,C) := \int_Z u(t,i,f(z),C)\zeta(dz|t,i).$$

Specified in this way, the agent's utility function satisfies assumptions (A-3). Note that we allow the type t agent's expected utility over firm-contract pairs to be influenced by the compensation catalogs offered by all firms.

Given catalog profile $C = (C_1,\ldots,C_m)$ the agent's choice set is given by

$$\Gamma(C) = \Gamma(C_1,\ldots,C_m) := \{(i,f) \in \mathbb{K} : f \in C_i\}, \tag{5}$$

and the agent's choice problem is given by

$$\max\{v(t,i,f,C) : (i,f) \in \Gamma(C)\}. \tag{6}$$

Under assumptions (A-1)-(A-3), for each t the agent's choice problem has a solution. Let

$$v^*(t,C) := \max\{v(t,i,f,C) : (i,f) \in \Gamma(C)\} \tag{7}$$

and

$$\Phi(t,C) := \{(i,x,p) \in \Gamma(C) : v(t,i,f,C) = v^*(t,C)\}. \tag{8}$$

The set-valued mapping $C \to \Phi(t,C)$ is a type t agent's best response mapping. For each catalog profile $C \in \mathbf{P}$, $\Phi(t,C)$ is a nonempty closed subset of $\mathbb{K}$. The following Proposition summarizes the continuity and measurability properties of the mappings, Γ and Φ, and the optimal utility function, v^*.

Proposition *(Continuity and measurability properties): Suppose assumptions (A-1)-(A-3) hold. Then the following statements are true. (a) The choice correspondence $\Gamma(\cdot)$ is $h_{\mathbf{P}}$-continuous on the space of catalog profiles $\mathbf{P}$ (i.e., is continuous with respect to the metric $h_{\mathbf{P}}$), (b) The function $v^*(\cdot,\cdot)$ is $h_{\mathbf{P}}$-continuous on $\mathbf{P}$ for each $t \in T$, and is $\Im$-measurable on T for each $C = (C_1,\ldots,C_m) \in \mathbf{P}$. (c) For each*

$t \in T$, $\Phi(t,\cdot)$ is $h_{\mathbf{P}}$ -upper semicontinuous on $\mathbf{P}$ and $\Phi(\cdot,\cdot)$ is $\Im \times B(\mathbf{P})$ -measurable on $T \times \mathbf{P}$.[9]

The proof of the Proposition above follows from Propositions 4.1 and 4.2 in Page (1992).

2.1.4 Tie-Breaking Mechanisms

We shall denote by Σ_Φ the set of all $\Im \times B(\mathbf{P})$-measurable selections from the best response mapping,

$$(t, C) \to \Phi(t, C),$$

that is, the set of $\Im \times B(\mathbf{P})$-measurable functions

$$(t, C) \to (i(t, C), f(t, C))$$

such that

$$(i(t, C), f(t, C)) \in \Phi(t, C) \text{ for all } (t, C) \in T \times \mathbf{P}.$$

We shall refer to each selection $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ as a tie-breaking mechanism. Given tie-breaking mechanism $(i(\cdot,\cdot), f(\cdot,\cdot))$ and catalog profile $C \in \mathbf{P}$ offered by firms, the corresponding mapping $t \to (i(t, C), f(t, C))$ is a direct contracting mechanism. Moreover, because

$$v(t, i(t, C), f(t, C), C) = v^*(t, C)$$

$$\text{for all } (t, C) \in T \times \mathbf{P} \text{ and for all } (i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi,$$

each tie-breaking mechanism $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ and each catalog profile $C \in \mathbf{P}$ induces a rational and incentive compatible direct mechanism $(i(\cdot, C), f(\cdot, C))$. This mechanism is rational because for all agent types t in T

$$v(t, i(t, C), f(t, C), C) \geq v(t, 0, 0, C), \tag{9}$$

and it is incentive compatible because for all agent types t and t' in T

$$v(t, i(t, C), f(t, C), C) \geq v(t, i(t', C), f(t', C), C). \tag{10}$$

Under tie-breaking mechanism $(i(\cdot,\cdot), f(\cdot,\cdot))$, given catalog profile $C \in \mathbf{P}$ offered by firms, it is intended that a type $t \in T$ agent do business with firm $i(t, C)$ and enter into contract $f(t, C) \in C_{i(t,C)}$. Moreover, because $(i(\cdot, C), f(\cdot, C))$ is rational and incentive compatible, it is reasonable to assume that a type t agent will do as intended by the mechanism (i.e., will obey the mechanism).

By the Kuratowski - Ryll-Nardzewski Selection Theorem (see Aliprantis and Border (1999), p. 567), the set of tie-breaking mechanisms Σ_Φ is nonempty.

[9] Here $B(\mathbf{P})$ denotes the Borel σ-field in the compact metric space $(\mathbf{P}, h_{\mathbf{P}})$. Moreover,

$$B(\mathbf{P}) = B(P_f(K_1)) \times \cdots \times B(P_f(K_m)),$$

where $B(P_f(K_j))$ denotes the Borel σ-field in the compact metric space of catalogs $(P_f(K_j), h)$ (see Aliprantis and Border (1999) Theorem 4.43, p. 146).

2.2 Firms

2.2.1 The Firm's Profit Function

We shall assume that

(A-4) the jth firm's profit is given by the function,

$$\pi_j(\cdot,\cdot,\cdot,\cdot) : T \times I \times K \times \mathbf{P} \to R,$$

where (i) for each $t \in T$, $\pi_j(t,\cdot,\cdot,\cdot)$ is continuous and for each $(i,f,C) \in I \times K \times \mathbf{P}$, $\pi_j(\cdot,i,f,C)$ is $\Im$-measurable, and (ii) there exists a μ-integrable function $\xi_j(\cdot) : T \to R$ such that for each for each $(i,f,C) \in I \times K \times \mathbf{P}$, $|\pi_j(t,i,f,C)| \leq \xi_j(t)$ for all $t \in T$.

Example 3 *(Firm Profit Functions for the Executive Compensation Example)*
Continuing our executive compensation example, for $j = 1,2,\ldots,m$, let firm j's profit be given by

$$\pi_j(t,i,f,C) = \left(\int_Z (r_j(z) - f(z))\zeta(dz|t,j)\right) I_j(i) - h_j(C_j, C_{-j})$$

where

$$I_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and where $r_j(\cdot) : Z \to R$ is firm j's bounded, $B(Z)$-measurable, state-contingent gross payoff function and $h_j(C_j, C_{-j})$ is the fixed cost to firm j of offering compensation catalog C_j given the catalogs C_{-j} offered by other firms. If we assume that the fixed cost functions are continuous in catalog profiles (with respect to the metric $h_{\mathbf{P}}$), then company profit functions satisfy assumptions (A-4).

3 Endogenous Nash Mechanisms and Equilibrium in Catalog Games

Each firm's expected profit is determined by three factors: (i) the catalogs offered by all firms, (ii) the optimizing behavior of the agent for each of his types (i.e., optimizing behavior of buyers in the market), and (iii) the tie-breaking mechanism which prevails. *But which tie-breaking mechanisms are reasonable? Can such a mechanism be determined endogenously*. Because each tie-breaking mechanism induces a game over catalogs (i.e., a catalog game) to be played by firms in their strategic competition for the privately informed agent, a reasonable candidate would be a tie-breaking mechanism which supports a Nash equilibrium in the corresponding catalog game. We call such a mechanism a *Nash mechanism*. The fundamental question we wish to address now is, does there exist a Nash mechanism?

3.1 Catalog Games Induced by Tie-Breaking Mechanisms

Consider the collection of catalog games indexed by the tie-breaking mechanisms, given by

$$\left((P_f(K_j))_{j=1}^m, (\Pi_j(i(\cdot,\cdot), f(\cdot,\cdot),\cdot))_{j=1}^m\right)_{(i(\cdot,\cdot),f(\cdot,\cdot))\in\Sigma_\Phi}. \tag{11}$$

with firm payoff functions $\Pi_j(i(\cdot,\cdot), f(\cdot,\cdot),\cdot)$ given by

$$C \to \Pi_j((i(\cdot,C), f(\cdot,C)), C) := \int_T \pi_j(t,(i(t,C), f(t,C)), C)d\mu(t).$$

Next let $\Delta(P_f(K_j))$ denote the jth firm's space of mixed catalog strategies. Note that because $P_f(K_j)$ is a compact metric space (with the Hausdorff metric), $\Delta(P_f(K_j))$ is compact and metrizable for the topology of weak convergence of probability measures.

Given tie-breaking mechanism $(i'(\cdot,\cdot), f'(\cdot,\cdot)) \in \Sigma_\Phi$ with induced catalog game

$$\left((P_f(K_j))_{j=1}^m, (\Pi_j(i'(\cdot,\cdot), f'(\cdot,\cdot),\cdot))_{j=1}^m\right),$$

the corresponding mixed catalog game is given by

$$\left((\Delta(P_f(K_j)))_{j=1}^m, \left(Q_j(\cdot,\cdot|(i'(\cdot,\cdot), f'(\cdot,\cdot))\right)_{j=1}^m\right),$$

where for mixed strategy profile $(\lambda_1,\ldots,\lambda_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$,

$$\begin{aligned}
&Q_j\left(\lambda_1,\ldots,\lambda_m|(i'(\cdot,\cdot), f'(\cdot,\cdot))\right)\\
&:= \textstyle\int_{\mathbf{P}} \Pi_j((i'(\cdot,C), f'(\cdot,C)), C)\lambda_1(dC_1)\cdot\cdots\cdot\lambda_m(dC_m)\\
&= \textstyle\int_{\mathbf{P}}\int_T \pi_j(t,(i'(t,C), f'(t,C)), C)d\mu(t)\lambda_1(dC_1)\cdot\cdots\cdot\lambda_m(dC_m)
\end{aligned}$$

and where recall $\mathbf{P} := P_f(K_1) \times \cdots \times P_f(K_m)$.

Definition 1 *(Endogenous Nash Mechanisms and Nash Equilibrium)*

A tie-breaking mechanism $(i^*(\cdot,\cdot), f^*(\cdot,\cdot)) \in \Sigma_\Phi$ is an endogenous Nash mechanism if the induced mixed catalog game

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot,\cdot|(i^*(\cdot,\cdot), f^*(\cdot,\cdot)))_{j=1}^m\right)$$

has a Nash equilibrium, that is, if there exists a mixed catalog strategy profile

$$(\lambda_1^*,\ldots,\lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)) := \Delta(\mathbf{P})$$

such that for all $j = 1,2,\ldots,m$

$$Q_j^*\left(\lambda_j^*, \lambda_{-j}^*|(i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right) \geq Q_j^*\left(\lambda_j, \lambda_{-j}^*|(i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right)$$

for all $\lambda_j \in \Delta(P_f(K_j))$.

If $(i^*(\cdot,\cdot), f^*(\cdot,\cdot)) \in \Sigma_\Phi$ is an endogenous Nash mechanism with Nash equilibrium $(\lambda_1^*, \ldots, \lambda_m^*,) \in \Delta(\mathbf{P})$, then for all $j = 1, \ldots, m$,

$$\left.\begin{array}{l} \max_{C_j \in P_f(K_j)} Q_j^* \left(C_j, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right) \\ = \max_{\lambda_j \in \Delta(P_f(K_j))} Q_j^* \left(\lambda_j, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right) \\ = Q_j^* \left(\lambda_j^*, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right), \end{array}\right\} \tag{12}$$

where

$$Q_j^* \left(C_j, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot)\right) := \int_{\mathbf{P}_{-j}} \Pi_j((i^*(\cdot,C), f^*(\cdot,C)), C) \lambda_{-j}^*(dC_{-j}).$$

Thus, for each $j = 1, \ldots, m$,

$$\lambda_j^* \left(\arg \max_{C_j \in P_f(K_j)} Q_j^* \left(C_j, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right)\right) = 1. \tag{13}$$

From the perspective of the jth firm, if the mixed catalog strategy, λ_j^*, is the jth firm's part of a Nash equilibrium with respect to tie-breaking mechanism $(i^*(\cdot,\cdot), f^*(\cdot,\cdot))$, then firm j will choose with probability 1 a catalog that maximizes

$$Q_j^* \left(C_j, \lambda_{-j}^* | (i^*(\cdot,\cdot), f^*(\cdot,\cdot))\right)$$

over the jth firms feasible set of catalogs, $P_f(K_j)$ - and thus will choose an optimal catalog against the mixed strategies, λ_{-j}^*, played by other firms given tie-breaking mechanism $(i^*(\cdot,\cdot), f^*(\cdot,\cdot))$.[10]

3.2 The Existence of Endogenous Nash Mechanisms

Our main result is the following:

Theorem 1 *(Existence of Endogenous Nash Mechanisms)*

Suppose assumptions (A-1)-(A-4) hold. If the probability space of agent types is atomless, then there exists an endogenous Nash mechanism $(i^*(\cdot,\cdot), f^*(\cdot,\cdot)) \in \Sigma_\Phi$. Thus, the induced mixed catalog game,

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot,\cdot | (i^*(\cdot,\cdot), f^*(\cdot,\cdot)))_{j=1}^m\right),$$

has a Nash equilibrium.

[10] Here,

$$\begin{array}{c} (\lambda_j, \lambda_{-j}^*) = (\lambda_1^*, \ldots, \lambda_{j-1}^*, \lambda_j, \lambda_{j+1}^*, \ldots, \lambda_m^*), \\ C_{-j} = (C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_m) \\ \mathbf{P}_{-j} = (P_f(K_1) \times \cdots \times P_f(K_{j-1}) \times P_f(K_{j+1}), \ldots, P_f(K_m)), \\ \text{and} \\ \lambda_{-j}(dC_{-j}) = \prod_{i \neq j} \lambda_i(dC_i). \end{array}$$

A subset of agent types $E \in \Im$ is an *atom* in the probability space $(T, \Im, \mu)$ if $\mu(E) > 0$ and for all $F \in \Im$ such that $F \subseteq E$, either $\mu(F) = 0$ or $\mu(E - F) = 0$. If $(T, \Im, \mu)$ contains no atoms, then it is said to be nonatomic or atomless. The atomless assumption will be satisfied if, for example, T is given by

$$T := [a_1, b_1] \times \cdots \times [a_L, b_L],$$

equipped with the Borel product σ -field and if the probability measure μ is given by a density, that is, if for all Borel subsets E of T

$$\mu(E) = \int_E h(t) d\sigma(t),$$

where $h(\cdot) : T \to R_+$ is a probability density function and σ is the Lebesgue product measure on T.

Without the atomless assumption we can show that there exists a *randomized* endogenous Nash mechanism. A randomized endogenous mechanism, denoted by $(\alpha_k(\cdot), i_k(\cdot,\cdot), f_k(\cdot,\cdot))_{k=0}^m$, consists of $m+1$, $B(\mathbf{P})$-measurable, nonnegative functions

$$\alpha_0(\cdot), \alpha_1(\cdot), \ldots, \alpha_m(\cdot)$$

such that for all $C \in \mathbf{P}$, $\sum_{k=0}^m \alpha_k(C) = 1$, and $m+1$ tie-breaking mechanisms

$$((i_0(\cdot,\cdot), f_0(\cdot,\cdot)), \ldots, (i_m(\cdot,\cdot), f_m(\cdot,\cdot)))$$

such that $(i_k(\cdot,\cdot), f_k(\cdot,\cdot)) \in \Sigma_\Phi$ for all $k = 0, 1, \ldots, m$. The number $\alpha_k(C) \in [0,1]$ is the probability that direct mechanism $(i_k(\cdot, C), f_k(\cdot, C))$ is selected under randomization $(\alpha_0(C), \alpha_1(C), \ldots, \alpha_m(C))$ given catalog profile C. Because

$$v(t, i_k(t,C), f_k(t,C), C) = v^*(t,C)$$

$$\text{for all } (t,C) \in T \times \mathbf{P} \text{ and for all } (i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi,$$

the agent has no incentive not to follow the selected direct mechanism and because each direct mechanism $(i_k(\cdot, C), f_k(\cdot, C))$ is rational and incentive compatible the agent has no incentive not to choose the firm-contract pair intended for his type under the selected mechanism.

A randomized endogenous mechanism, $(\alpha_k(\cdot), i_k(\cdot,\cdot), f_k(\cdot,\cdot))_{k=0}^m$ is Nash if the induced mixed catalog game

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot,\cdot|(\alpha_k(\cdot), i_k(\cdot,\cdot), f_k(\cdot,\cdot))_{k=0}^m)_{j=1}^m \right),$$

has a Nash equilibrium. Here

$$Q_j\left(\lambda_1, \ldots, \lambda_m | (\alpha_k(\cdot), i_k(\cdot,\cdot), f_k(\cdot,\cdot))_{k=0}^m\right)$$

$$:= \int_{\mathbf{P}} \left(\sum_{k=0}^m \alpha_k(C) \Pi_j((i_k(\cdot,C), f_k(\cdot,C)), C)\right) \lambda_1(dC_1) \cdot \cdots \cdot \lambda_m(dC_m)$$

$$= \int_{\mathbf{P}} \int_T \left(\sum_{k=0}^m \alpha_k(C) \pi_j(t, (i_k(t,C), f_k(t,C)), C)\right) d\mu(t) \lambda_1(dC_1) \cdot \cdots \cdot \lambda_m(dC_m).$$

We now have our main result on randomized endogenous Nash mechanisms.

Theorem 2 *(Existence of Randomized Endogenous Nash Mechanisms)*

Suppose (A-1)-(A-4) hold. Then there exists a randomized endogenous Nash mechanism $(\alpha_k^*(\cdot), i_k^*(\cdot,\cdot), f_k^*(\cdot,\cdot))_{k=0}^m$. *Thus, the induced mixed catalog game,*

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot,\cdot|(\alpha_k^*(\cdot), i_k^*(\cdot,\cdot), f_k^*(\cdot,\cdot))_{k=0}^m)_{j=1}^m\right),$$

has a Nash equilibrium.

4 Proofs

4.1 Preliminary Lemmas

We begin by defining the following set-valued mapping which we will call the profit possibilities mapping:

$$\Psi(t,C)$$

$$:= \{\pi \in R^m : \pi = (\pi_1(t,(i,f),C), \ldots, \pi_m(t,(i,f),C)) \text{ for some } (i,f) \in \Phi(t,C)\}.$$

For each agent type t and catalog profile $C \in \mathbf{P}$ offered by firms, $\Psi(t,C)$ is the set of all profit possibilities for firms under some optimal firm-contract choice by the agent. This correspondence has nonempty, compact values and by Theorem 6.5 in Himmelberg (1975) is $\Im \times B(\mathbf{P})$-measurable.

Lemma 1 *(upper semicontinuity of* $\Psi(t,\cdot)$*)*

Under assumptions (A-1)-(A-4), the mapping $\Psi(t,\cdot)$ *is upper semicontinous on* $\mathbf{P}$.

Proof. It suffices to show that $\Psi(t,\cdot)$ has a closed graph. But this follows from the continuity of profit function $\pi_1(t,(\cdot,\cdot),\cdot)$ and the upper semicontinuity of the best response mapping $\Phi(t,\cdot)$ on $\mathbf{P}$. ■

Let Σ_Ψ denote the set of all $\Im \times B(\mathbf{P})$-measurable selections from the profit possibilities mapping $\Psi(\cdot,\cdot)$ and let Σ_Φ denote the set of all $\Im \times B(\mathbf{P})$-measurable selections from the best response mapping $\Phi(\cdot,\cdot)$. By Filippov's Implicit Function Theorem (see Himmelberg (1975) Theorem 7.1), for all $\psi(\cdot,\cdot) \in \Sigma_\Psi$ there exists $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ such that for all $(t,C) \in T \times \mathbf{P}$,

$$\psi(t,C) = (\pi_1(t,i(t,C),f(t,C),C), \ldots, \pi_m(t,i(t,C),f(t,C),C)).$$

Next consider the integral of profit possibilities mapping $\Psi(\cdot,\cdot)$ given by

$$C \to \int_T \Psi(t,C)d\mu(t) := \left\{\int_T \psi(t,C)d\mu(t) : \psi(\cdot,\cdot) \in \Sigma_\Psi\right\}.$$

Lemma 2 *(upper semicontinuity of* $\int_T \Psi(t,\cdot)d\mu(t)$*)*

Under assumptions (A-1)-(A-4), the mapping $C \to \int_T \Psi(t,C)d\mu(t)$ *is upper semicontinous on* $\mathbf{P}$ *with nonempty, compact values in* R^m.

Proof. It suffices to show that $\int_T \Psi(t,\cdot)d\mu(t)$ has a closed graph. Let $\{(C^n, z^n)\}_n$ be a sequence in $grph\left\{\int_T \Psi(t,\cdot)d\mu(t)\right\} \subseteq \mathbf{P} \times \mathbb{R}^m$ converging to $(C', z') \in \mathbf{P} \times \mathbb{R}^m$. Thus for each n

$$z^n \in \int_T \Psi(t, C^n)d\mu(t).$$

Corresponding to the sequence $\{(C^n, z^n)\}_n$ let $\{\psi_n(\cdot,\cdot)\}_n$ be a sequence in Σ_Ψ such that for each n

$$z^n = \int_T \psi_n(t, C^n)d\mu(t).$$

Thus, we have

$$z' = \lim_n z^n = \lim_n \int_T \psi_n(t, C^n)d\mu(t)$$

By Fatou's Lemma in several dimensions (Artstein (1979)), there exists a μ-integrable function $z'(\cdot)$ such that

$$z'(t) \in Ls\left\{\psi_n(t, C^n)\right\} \text{ for all } t \in T\backslash N,\ \mu(N) = 0,$$

and

$$z' = \lim_n \int_T z_n(t)d\mu(t) = \int_T z'(t)d\mu(t).$$

Now consider the minimization problem

$$\min\left\{\left\|z'(t) - \pi(t,(i,f),C)\right\| : (i,f) \in \Phi(t,C)\right\},$$

where

$$\pi(t,(i,f),C) = (\pi_1(t,(i,f),C), \ldots, \pi_m(t,(i,f),C)).$$

Let

$$\delta(t,C) := \min\left\{\left\|z'(t) - \pi(t,(i,f),C)\right\| : (i,f) \in \Phi(t,C)\right\} \text{ for all } (t,C) \in T \times \mathbf{P}.$$

By Schal (1974), there exists a $\Im \times B(\mathbf{P})$-measurable selection $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ such that

$$\left\|z'(t) - \pi(t, i(t,C), f(t,C), C)\right\| = \delta(t,C) \text{ for all } (t,C) \in T \times \mathbf{P}.$$

The proof will be complete if we can show that $\delta(t, C') = 0$ for all $t \in T\backslash N$.

First, let $t \in T\backslash N$ be given and without loss of generality assume that

$$\psi_n(t, C^n) = (\pi_1(t,(i_t^n, f_t^n), C^n), \ldots, \pi_m(t,(i_t^n, f_t^n), C^n)) \to z'(t),$$

where for each n, $(i_t^n, f_t^n) \in \Phi(t, C^n)$. By the compactness of $I \times K$ we can assume without loss of generality that $(i_t^n, f_t^n) \to (i_t', f_t') \in I \times K$. We already have $C^n \to C'$ and thus by the continuity of $\pi_j(t,\cdot,\cdot,\cdot)$ on $I \times K \times \mathbf{P}$,

$$(\pi_1(t,(i_t^n, f_t^n), C^n), \ldots, \pi_m(t,(i_t^n, f_t^n), C^n))$$

$$\to (\pi_1(t,(i_t', f_t'), C'), \ldots, \pi_m(t,(i_t', f_t'), C')) = \pi(t,(i_t', f_t'), C) = z'(t).$$

Moreover, because

$$(i_t^n, f_t^n) \in \Phi(t, C^n) \text{ for all } n$$
$$\text{and}$$
$$C^n \to C' \in \mathbf{P},$$

by the upper semicontinuity of $\Phi(t, \cdot)$, we have $(i_t', f_t') \in \Phi(t, C')$. Thus,

$$\left\| z'(t) - \pi(t, i(t, C), f(t, C), C) \right\| = \left\| \pi(t, (i_t', f_t'), C) - \pi(t, i(t, C), f(t, C), C) \right\| = 0$$

because $(i_t', f_t') \in \Phi(t, C')$ and $(i(t, C), f(t, C)) \in \Phi(t, C')$

$$\text{minimizes } \left\| \pi(t, (i_t', f_t'), C') - \pi(t, (i, f), C') \right\| \text{ over } (i, f) \in \Phi(t, C').$$

Thus, $\delta(t, C') = 0$ for all $t \in T \backslash N$. ■

Let

$$F(C) := \int_T \Psi(t, C) d\mu(t)$$
$$\text{and}$$
$$G(C) := coF(C),$$

where *co* denotes convex hull. If the probability space of agent types, $(T, \Im, \mu)$, is atomless, then $F(C)$ is convex for all catalog profiles $C \in \mathbf{P}$ and we have

$$G(C) := coF(C) = F(C).$$

We shall refer to the upper semicontinuous correspondence

$$C \to G(C)$$

from catalog profiles $\mathbf{P}$ into the collection of nonempty, closed convex subsets of R^m as the *expected profit possibilities correspondence* (see Aliprantis and Border (1999), Theorem 16.36).

Now consider the game

$$\Gamma_\Psi := (P_f(K_1), \ldots, P_f(K_m), G(\cdot)).$$

In the terminology of Simon and Zame (1990), the game Γ_Ψ is a game with an endogenous sharing rule. A sharing rule is a $B(\mathbf{P})$-measurable function

$$g(\cdot) : \mathbf{P} \to \mathbf{R}^m$$
$$\text{such that}$$
$$g(C) \in G(C) \text{ for all catalog profiles } C \in \mathbf{P}.$$

Thus the collection of all endogenous sharing rules for the game Γ_Ψ is given by the set of all measurable selections from the expected profit possibilities correspondence. Let Σ_G denote the collection of all endogenous sharing rules and let Σ_F denote the collection of all $B(\mathbf{P})$-measurable selections from $C \to F(C)$. We shall refer to each selection $g(\cdot) \in \Sigma_F$ as an expected profit function. By Theorem 8.2 in Wagner (1977),

each endogenous sharing $g(\cdot) \in \Sigma_G$, has a measurable Caratheodory representation with respect to expected profit functions, that is, there exists

$$\begin{gathered} m+1 \text{ expected profit functions } g_0(\cdot), g_1(\cdot), \ldots, g_m(\cdot), \\ \text{and } m+1,\ B(\mathbf{P})\text{-measurable nonnegative functions} \\ \alpha_0(\cdot), \alpha_1(\cdot), \ldots, \alpha_m(\cdot) \\ \text{such that for all } C \in \mathbf{P} \\ \textstyle\sum_{k=0}^m \alpha_k(C) = 1 \text{ and } g(C) = \sum_{k=0}^m \alpha_k(C) g_k(C) \in R^m. \end{gathered}$$

Definition 2 *(A Solution for the Game Γ_Ψ)*

A solution for the game Γ_Ψ is a sharing rule $g^*(\cdot) = (g_1^*(\cdot), \ldots, g_m^*(\cdot)) \in \Sigma_G$ and a mixed catalog strategy profile $\lambda^*(\cdot) = (\lambda_1^*(\cdot), \ldots, \lambda_m^*(\cdot)) \in \Delta(\mathbf{P})$ such that λ^* is a Nash equilibrium of the game $(\Delta(P_f(K_j)), g_j(\cdot))_{j=1}^m$, that is, such that for the mixed catalog strategy profile λ^*,

$$\int_{\mathbf{P}} g_j^*(C_j, C_{-j}) \lambda_j^*(dC_j) \lambda_{-j}^*(dC_{-j}) \geq \int_{\mathbf{P}} g_j^*(C_j, C_{-j}) \lambda_j(dC_j) \lambda_{-j}^*(dC_{-j}),$$

for all firms $j = 1, 2, \ldots, m$ and for all mixed catalog strategies $\lambda_j(\cdot) \in \Delta(P_f(K_j))$.

Lemma 3 *(Γ_Ψ has a solution, Simon-Zame (1990))*

Under assumptions (A-1)-(A-4), Γ_Ψ has a solution.

Let $(g^*(\cdot), \lambda^*(\cdot))$ be a solution to the game Γ_Ψ and suppose $g^*(\cdot)$ has Caratheodory representation

$$g^*(C) = \sum_{k=0}^m \alpha_k^*(C) g_k^*(C),$$

where $\{g_0^*(\cdot), g_1^*(\cdot), \ldots, g_m^*(\cdot)\} \subset \Sigma_F$ and $\{\alpha_0^*(\cdot), \alpha_1^*(\cdot), \ldots, \alpha_m^*(\cdot)\}$ are $B(\mathbf{P})$-measurable nonnegative functions such that for all $C \in \mathbf{P}$, $\sum_{k=0}^m \alpha_k^*(C) = 1$. We may think of the number $\alpha_k^*(C) \in [0, 1]$ as the probability that expected profit possibilities vector $g_k^*(C) \in R^m$ will prevail under catalog profile C. Question: can each R^m-valued, expected profit function $g_k^*(\cdot)$ in Σ_F be implemented by a tie-breaking mechanism in Σ_Φ? That is, does there exist a tie-breaking mechanism $(i_k^*(\cdot, \cdot), f_k^*(\cdot, \cdot)) \in \Sigma_\Phi$ such that for each firm $j = 1, 2, \ldots, m$ and each function $g_{jk}^*(\cdot)$, $k = 0, 1, \ldots, m$ composing the jth firm's part of the Caratheodory representation we have

$$g_{jk}^*(C) = \int_T \pi_j(t, (i_k^*(t, C), f_k^*(t, C)), C) d\mu(t) \text{ for all } C \in \mathbf{P}.$$

Lemma 4 *(Each expected profit function $g(\cdot) \in \Sigma_F$ is implementable via a tie-breaking mechanism)*

Suppose assumptions (A-1)-(A-4) hold. Then the following statements are equivalent:

1. $g(\cdot) = (g_1(\cdot), \ldots, g_m(\cdot)) \in \Sigma_F$.

2. *There exists a tie-breaking mechanism,* $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ *such that for each* $j = 1, 2, \ldots, m$

$$g_j(C) = \int_T \pi_j(t, (i(t,C), f(t,C)), C) d\mu(t) \text{ for all } C \in \mathbf{P}.$$

Proof. It is easy to see that (2) $\Rightarrow$ (1), thus in order to complete the proof we must show that (1) $\Rightarrow$ (2). The key ingredient in establishing this implication is the measurable 'Measurable Choice' theorem due to Mertens (1987). Let $g(\cdot) = (g_1(\cdot), \ldots, g_m(\cdot)) \in \Sigma_F$. Thus, for all $C \in \mathbf{P}$,

$$(C, g(C)) \in graph(F).$$

By Mertens (1987) there exists a measurable function

$$\zeta : graph(F) \times T \to R^m$$

such that

$$g(C) = \textstyle\int_T \zeta(C, g(C), t) d\mu(t) \text{ for all } C \in \mathbf{P}$$
$$\text{and}$$
$$\zeta(C, g(C), t) \in \Psi(t, C) \text{ for all } (t, C) \in T \times \mathbf{P}$$

Thus, $\zeta(\cdot, g(\cdot), \cdot) \in \Sigma_\Psi$, and thus by Filippov's Implicit Function Theorem (see Himmelberg (1975) Theorem 7.1), there exists $(i(\cdot,\cdot), f(\cdot,\cdot)) \in \Sigma_\Phi$ such that for all $(t, C) \in T \times \mathbf{P}$,

$$\zeta(C, g(C), t) = (\pi_1(t, i(t,C), f(t,C), C), \ldots, \pi_m(t, i(t,C), f(t,C), C)).$$

■

4.2 Proof of Theorem 2

Consider the game with an endogenous sharing rule

$$\Gamma_\Psi := (P_f(K_1), \ldots, P_f(K_m), G(\cdot))$$
$$\text{where}$$
$$G(\cdot) = coF(C) = co \textstyle\int_T \Psi(t, C) d\mu(t).$$

By Lemma 3 this game has a solution $(g^*(\cdot), \lambda^*(\cdot))$. Let

$$g^*(C) = \sum_{k=0}^{m} \alpha_k^*(C) g_k^*(C)$$

be a Caratheodory representation of endogenous sharing rule $g^*(\cdot)$ with respect to expected profit functions $\{g_0^*(\cdot), g_1^*(\cdot), \ldots, g_m^*(\cdot)\}$ and catalog dependent randomization $\{\alpha_0^*(\cdot), \alpha_1^*(\cdot), \ldots, \alpha_m^*(\cdot)\}$. By Lemma 4 each R^m-valued expected profit function $g_k^*(\cdot)$, $k = 0, 1, \ldots, m$, can be implemented via an endogenous mechanism $(i_k^*(\cdot,\cdot), f_k^*(\cdot,\cdot)) \in \Sigma_\Phi$ such that for each firm $j = 1, 2, \ldots, m$

$$g_{jk}^*(C) = \int_T \pi_j(t, (i_k^*(t,C), f_k^*(t,C)), C) d\mu(t) \text{ for all } C \in \mathbf{P}.$$

We have for all $C \in \mathbf{P}$ and $k = 0, 1, \ldots, m$,

$g_k^*(C)$

$= (g_{1k}^*(C), \ldots, g_{mk}^*(C))$

$= \left(\int_T \pi_1(t, (i_k^*(t, C), f_k^*(t, C)), C) d\mu(t), \ldots, \int_T \pi_m(t, (i_k^*(t, C), f_k^*(t, C)), C) d\mu(t)\right).$

Thus,

$$g^*(C) = (\textstyle\sum_{k=0}^m \alpha_k^*(C) g_{1k}^*(C), \ldots, \sum_{k=0}^m \alpha_k^*(C) g_{mk}^*(C))$$

where for each for each firm $j = 1, \ldots, m$ and each $k = 0, 1, \ldots, m$

$$\textstyle\sum_{k=0}^m \alpha_k^*(C) g_{jk}^*(C) = \int_T \left(\sum_{k=0}^m \alpha_k^*(C) \pi_j(t, (i_k^*(t, C), f_k^*(t, C)), C)\right) d\mu(t).$$

Because $(g^*(\cdot), \lambda^*(\cdot)) = (\sum_{k=0}^m \alpha_k^*(\cdot) g_k^*(\cdot), \lambda^*(\cdot))$ is a solution to the game $\Gamma_\Psi := (P_f(K_1), \ldots, P_f(K_m), G(\cdot))$, we conclude that the mixed catalog game

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot, \cdot | (\alpha_k^*(\cdot), i_k^*(\cdot, \cdot), f_k^*(\cdot, \cdot))_{k=0}^m)_{j=1}^m\right)$$

determined by the randomized endogenous mechanism $(\alpha_k^*(\cdot), i_k^*(\cdot, \cdot), f_k^*(\cdot, \cdot))_{k=0}^m$ has a Nash equilibrium,

$$\lambda^*(\cdot) = (\lambda_1^*(\cdot), \ldots, \lambda_m^*(\cdot)),$$

and thus we conclude that

$$(\alpha_k^*(\cdot), i_k^*(\cdot, \cdot), f_k^*(\cdot, \cdot))_{k=0}^m$$

is a randomized endogenous Nash mechanism. ■

4.3 Proof of Theorem 1

Consider the game with an endogenous sharing rule

$$\Gamma_\Psi := (P_f(K_1), \ldots, P_f(K_m), G(\cdot))$$
where again
$$G(\cdot) = coF(C) = co \int_T \Psi(t, C) d\mu(t).$$

Because the probability space of agent types is atomless, in fact we have,

$$G(\cdot) = coF(C) = F(C) = \int_T \Psi(t, C) d\mu(t).$$

Let $(g^*(\cdot), \lambda^*(\cdot))$ be a solution (see Lemma 3). By Lemma 4, the R^m-valued endogenous sharing rule $g^*(\cdot) \in \Sigma_F$ can be implemented via an endogenous mechanism $(i^*(\cdot, \cdot), f^*(\cdot, \cdot)) \in \Sigma_\Phi$ such that for all $C \in \mathbf{P}$

$g^*(C)$

$= (g_1^*(C), \ldots, g_m^*(C))$

$= \left(\int_T \pi_1(t, (i^*(t, C), f^*(t, C)), C) d\mu(t), \ldots, \int_T \pi_m(t, (i^*(t, C), f^*(t, C)), C) d\mu(t)\right).$

Because $(g^*(\cdot), \lambda^*(\cdot))$ is a solution to the game $\Gamma_\Psi := (P_f(K_1), \ldots, P_f(K_m), G(\cdot))$, where due to the atomless condition $G(\cdot) = F(\cdot)$, we conclude that the mixed catalog game

$$\left((\Delta(P_f(K_j)))_{j=1}^m, (Q_j(\cdot,\cdot|(i^*(\cdot,\cdot), f^*(\cdot,\cdot)))_{j=1}^m\right)$$

determined by the endogenous mechanism $(i^*(\cdot,\cdot), f^*(\cdot,\cdot)) \in \Sigma_\Phi$ has a Nash equilibrium,

$$\lambda^*(\cdot) = (\lambda_1^*(\cdot), \ldots, \lambda_m^*(\cdot)),$$

and thus we conclude that $(i^*(\cdot,\cdot), f^*(\cdot,\cdot))$ is an endogenous Nash mechanism. ■

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