SIXTH GRADERS' CONSTRUCTION OF QUANTITATIVE REASONING AS A FOUNDATION FOR ALGEBRAIC REASONING

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In a year-long constructivist teaching experiment with four 6th grade students, their quantitative reasoning with fractions was found to form an important basis for their construction of algebraic reasoning. Two of the four students constructed anticipatory schemes for solving problems that could be solved by an equation such as ax = b. In doing so, these students operated on the structure of their schemes. In solving similar problems, the other two students could not foresee the results of their schemes in thought—they had to carry out activity and then check afterwards to determine whether their activity had solved the problems. Operating on the structure of one's schemes is argued to be fundamentally algebraic, and a hypothesis is proposed that algebraic reasoning can be constructed as a reorganization of quantitative operations students use to construct fractional schemes.

In this paper, I demonstrate how 6th grade students' quantitative reasoning with fractions can form an important basis for their construction of multiplicative algebraic reasoning. In particular, during a year-long constructivist teaching experiment with four 6th grade students, the coordination of fractional schemes and whole-number multiplying schemes was seen as pivotal in the students' work toward solving basic linear equations of the form $ax = b$.

Since $ax = b$ is essentially a statement of division, considering its construction and solution requires understanding how students produce division, which entails understanding students' multiplying schemes and multiplicative reasoning. Furthermore, any statement of division inherently involves reasoning with fractions: While fractions may be implicit or disguised in solving equations like $4x = 28$, they soon become explicit in solving equations like $3x = 7$. So in contrast to recent studies on elementary school students' early algebraic reasoning that are largely additive in nature (e.g., Blanton & Kaput, 2005; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006), I was specifically interested in investigating students' construction of multiplicative algebraic reasoning.

Two of the four students in the teaching experiment constructed anticipatory schemes for solving problems that could be solved by an equation such as $ax = b$. The anticipatory nature of these schemes meant that the students did not have to carry out activity in a computer microworld and then check afterwards to determine whether their activity had solved the problem. Instead, these two students could foresee, in thought, aspects of the results of implementing their schemes (1). In doing so, these two students operated on the structure of their schemes. The larger purposes of this paper are (1) to argue that operating on the structure of one's schemes is fundamentally algebraic and (2) to propose a hypothesis that algebraic reasoning can be constructed as a reorganization of quantitative operations students use to construct fractional schemes.

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Quantitative and Algebraic Reasoning

Fractions as Quantities

I use the phrase quantitative reasoning with fractions to refer to the purposeful functioning of a person's fractional schemes and operations in the context of quantities and quantitative relationships. In my study, this approach to fractions meant that problem situations often involved quantities whose values were fractional amounts identified with standard units of measurement, such as 3/4 of a yard of ribbon. More generally, fractions were measures of quantities (often lengths) made in relation to an identified (but non-standard) unit quantity. So, conceiving of 4/5, for example, meant being able to draw 4/5 of a previously identified unit length. Taking 1/3 of 4/5 of a unit length invariably meant being able to draw this amount and determine the resulting length in relation to the unit length. This approach to fractions was facilitated by the use of JavaBars (Biddlecomb & Olive, 2000), a computer program in which students can draw rectangles (bars) of various dimensions and operate on those bars by partitioning them into parts, further partitioning parts, disembedding parts, iterating parts, etc.

Early Multiplicative Algebraic Reasoning

In my study, I did not want to take the equation ax = b as a given. Instead, I wanted to understand more about what was required for students to generate that equation out of their reasoning and what schemes students might construct to solve it, even in the case where a and b are fractions. This focus meant that I was interested in how students operated on both known and unknown quantities that stand in multiplicative relationship to each other: For example, if 3/5 of a length is 7 inches, how do students come to know that 3/5 multiplied by the unknown length is a length exactly 7 inches long? How do they come to know that 5/3 of the 7 inches produces the unknown length? From this point of view, in generating the equation ax = b, operating on both knowns and unknowns is involved. At the very least, conceiving of (3/5)x requires conceiving of 3/5 operating multiplicatively on an unknown quantity represented by x.

Some researchers have proposed explanations for students' difficulties in operating on unknowns when solving linear equations (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994), while others have contested such difficulties (e.g., Brizuela & Schliemann, 2004). Often researchers have not considered a operating multiplicatively on x as "operating on the unknown," opting instead to examine students' solutions of more complex equations with "unknowns on both sides," such as ax + b = cx + d. The reasons for exploring such equations have varied but sometimes have resulted in students' solutions of ax = b being dismissed as not yet algebraic—as merely involving the reversal of arithmetic operations in order to be solved. Yet students in Filloy and Rojano's study did not correctly solve equations of the form ax = b in the case of 102x = 51, which indicates that reversing one's thinking to solve ax = b is not trivial and can be considered a component of early algebraic reasoning (cf. Sfard and Linchevski, 1994).

Abstracting Conceptual Structures

More broadly, many researchers characterize algebraic reasoning as generalizing mathematical activity into structural ways of thinking (e.g., Carpenter, et al., 2003; Sfard & Linchevski, 1994). Consistent with this view, in my study a central distinction between students' quantitative and algebraic reasoning was the extent to which students had abstracted a conceptual structure. A concept, according to von Glasersfeld, is "a kind of place-holder or variable for some of the properties in the sensory complex we have abstracted from our experiences of particular things" (1991, p. 49), where the place-holder is often linguistic.

By abstracting a mathematical conceptual structure, I mean abstracting a “program of operations” from the experiences of using particular schemes that includes an awareness of how the schemes are composed (their structure) and an ability to operate with this awareness. So attributing a conceptual structure to a student means inferring the student is operating on or with the structure of his schemes. I use this notion of abstracting conceptual structures to account for differences in the schemes constructed by the four students in the study. In particular, students’ multiplicative structures were central explanatory constructs. I conceive of students’ multiplicative structure as the units coordinations that they have abstracted from the activity and results of their multiplying schemes and can take as given. Taking a units coordination as given means a student can project the coordination into a situation and operate further with it. Such units coordinations are said to be interiorized.

Methods of Inquiry

To investigate the purposes I have described, I conducted a year-long constructivist teaching experiment (Steffe & Thompson, 2000) in which I taught two pairs of sixth-grade students at a rural middle school in north Georgia from October 30, 2003 to May 12, 2004. The four students were invited to participate after individual selection interviews conducted during September and October of 2003. In the interviews I used fraction tasks to select students who were reasoning multiplicatively (see Hackenberg, 2005, for full details).

The pairs and I met twice weekly in 30-minute episodes for two to three weeks, followed by a week off. Most sessions included the use of JavaBars, and all sessions were videotaped with two cameras for on-going and retrospective analysis. One camera captured the interaction between the pair of students and myself, and the other camera recorded the students’ written or computer work. Two witness-researchers were present at all sessions to assist in videotaping and to provide other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and retrospective analysis of the videotapes.

In retrospective analysis, I aimed to construct second-order models (Steffe & Thompson, 2000) of the students’ ways and means of operating and changes in those ways of operating. A second-order model is a researcher’s constellation of constructs formulated to describe and account for another person’s activity. In my study, I used scheme theory (Piaget, 1968; von Glasersfeld, 1995) as a central tool toward this end, and thus I viewed mathematical learning as a process in which a person makes accommodations in her schemes and operations in response to perturbations (disturbances) brought about by her current schemes and operations in on-going interaction within her experiential reality.

Data Excerpts

All four students coordinated their fractional schemes with their whole-number multiplying schemes, at least to some extent. However, only two of the four students, Michael and Deborah, embedded their whole number multiplying schemes into their fractional schemes to construct powerful anticipatory schemes for solving problems that can be solved by basic linear equations of the form $ax = b$. I call such problems reversible multiplicative reasoning (RMR) problems.

For example, on February 18 Michael solved the following RMR problem by using his whole number multiplying scheme in service of his reversible fractional scheme:

The Candy Bar Problem. This collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?(2)

Michael quickly formed a goal to divide 7 inches into three equal parts. However, he had no immediate way of operating to achieve that goal—seven seemed to be a perturbing element. His activity in some previous episodes in January, in which he solved problems such as making a 2/2-bar into a 3/3-bar without erasing the half mark,(3) was an important basis for his elimination of this perturbation. That is, Michael partitioned each of the 1-inch bars into three equal parts and determined he had 21 parts, which he knew could be divided by 3. He then announced, prior to carrying out further activity, "I know how many there are now." He made the new collection by making five of the three equal parts of the 7 inches.

In this solution, Michael used the activity of his multiplying scheme to convert the composite unit of 7 inches into a composite unit consisting of number of parts (21) that he could split into three equal parts. I infer that he could operate in this way because he could view the 7 inches as a unit of seven units, into each of which he could insert more units (three units) to produce a number of units (21) that he could reorganize, in thought, as a unit of three units each containing seven units. So Michael inserted the coordination of units (the activity of his multiplying scheme) into the activity of his reversible fractional scheme—thereby operating with the structure of his multiplying scheme on the structure of this other scheme. This way of operating was novel for Michael, and during the rest of the study he used this new scheme, a reversible multiplying scheme with fractions, to solve other RMR problems like the Candy Bar Problem.

Problems like the Candy Bar Problem proved to be quite a challenge for Michael's partner Carlos. As a result, by mid-March I began to pose "basic" RMR problems where both the known quantity and the quantitative relationship were whole numbers:

Basic RMR Problem. That 2-foot candy bar is three times longer than your candy bar. Make your candy bar and tell how long it is. (No erasing the foot-mark on the bar.)

Carlos struggled to solve this problem over two teaching episodes, and even after several tries he wanted to erase the foot-mark and then split the unmarked bar into three equal parts. However, on his third try on March 29, he partitioned each of the two feet into three equal parts for a total of six parts, and pulled away two of the six pieces, explaining "I multiplied two by three and got six." So like Michael, Carlos used the activity of his multiplying scheme to convert a composite unit he could not split into three parts (the 2-unit bar) into a composite unit that he could split (the 6-unit bar). Yet because he had seen his partner operate in similar ways many times before, and because he may have been aiming for a particular visual image (he had created a bar that was 1/3 of 2 feet prior to this last solution), it is difficult to judge to what extent this was a significant or permanent modification for him.

Carlos did use this way of operating in subsequent episodes, at least when the goal was to make a 2-part bar into three equal parts. However, when working with bigger numbers, such as making a 5-part bar into four equal parts, his tendency was to partition each part of the 5-part bar into some number of small parts, select a number of those small parts that he estimated would constitute 1/4 of the whole bar, and iterate that selection four times to check. If he was not successful, he would adjust his estimate and try again. So he did not appear to make a modification in his activity at the same level of generality as his partner's modification. My main explanation for this difference between the boys was that Carlos had not yet interiorized the coordination of three levels of units (Hackenberg, 2005). That is, Carlos could use his multiplying scheme to determine that a 5-part bar with each part partitioned into 4 equal parts would produce a 20-part bar. But he could not take that structure as a given while mentally reorganizing the 20-part bar into a different number of units of units—i.e., as a unit of four units each containing five. I call his scheme for solving basic RMR problems an enactive reversible multiplying scheme (without fractions).

On March 29, Bridget also demonstrated that she could solve basic RMR problems by solving this problem:

Chocolate Bar Problem. Four yards of chocolate is three times what Phillip has. Can you make Phillip's chocolate bar? How long is his bar?

Bridget partitioned each of the four yards into three equal parts, disembedded one of those parts, and iterated it to produce a 4-part bar. In explanation she said, "you can't divide three into four, so you go, three times four is twelve. And twelve divided by three is four. You have to divide it by three because you said three times."

To test out the nature of Bridget's scheme for solving basic RMR problems, I posed a series of them in the next teaching episode on March 31. Bridget began to solve them by partitioning each unit of the known quantity into the number of parts given in the relationship between known and unknown. However, then she said things like "Hmm. I don't know what else to do," and "Why am I doing that?" Although she generally completed solutions of the problems, she did not appear to be certain about her activity unless it was confirmed in some way by me or by her partner Deborah (who seemed certain). During the episode I challenged Deborah to solve a basic RMR problem without enacting the physical coordinations in the microworld, and she did. When I then posed another such problem to Bridget, she exclaimed "I can't do that in my head!" Thus I can conclude that like Carlos, Bridget had constructed an enactive reversible multiplying scheme to solve basic RMR problems. She needed to carry out the activity of the scheme in the microworld in order to solve these problems, and thus a three-levels-of-units structure was something she could make in activity. Like Carlos, she did not seem to be able to take such a structure as given so as to reorganize the known quantity into a different units-of-units structure, in order to anticipate the result of the scheme.

Meanwhile, Deborah began to construct an anticipatory scheme for solving the most complex type of RMR problem she encountered. On May 12, using the context of a homemade racecar contest between two teams, I posed this problem to Deborah:

Race Car Problem. The Lizards' car goes 1/2 of a meter. That's 3/4 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?

Deborah partitioned her 1/2-meter bar into three equal parts, disembedded one of the parts, and iterated it to produce a 4-part bar which she called 4/6 of a meter. When I asked her what she had to take times the Lizards' distance to make the Cobras' distance, she promptly said, "three-fourths—I mean four-thirds." In explanation, she pointed to the Lizards' distance and stated that it was 3/4 of the Cobras' distance.

I then posed the same problem except the Lizards' car went 2/3 of a meter. Deborah partitioned each third of her 2/3-meter bar into six equal parts, disembedded one of the parts, and iterated it to produce a 16-part bar. In explanation, she said, "I knew each third is four pieces. So four times four, because you need four thirds for this one," pointing to the Cobras' car's distance. A short time later, she also referred to the 16-part bar as "four-fourths."

At this point in the teaching experiment, Deborah had constructed a reversible multiplying scheme with fractions similar to Michael's. However, as shown here, a characteristic of her solutions was her use of reciprocal relationships between the two quantities. In fact, she was the only student of the four to state these relationships swiftly and to use them in operating. Her initial justification for knowing that the Cobras' distance was 4/3 of the Lizard's distance indicates that she had constructed a multiplicative relationship that did not rely on reference to material parts. In other words, she did not explain that the Cobras' distance was 4/3 of the Lizards' distance because the Cobras' distance was four equal parts and the Lizard's three. Instead, she relied on the given multiplicative relationship, that the Lizards' distance was 3/4 of the Cobras' distance. Her explanation leads me to infer that she knew the following: If Bar A was 3/4 of Bar B, then Bar B was 4/3 of Bar A. When solving the second Race Car Problem and other variations, she used this kind of reasoning. Yet she did not compress her reasoning so much as to, for example, take 4/3 times the known distance to solve the Race Car Problems—i.e., she did not appear to view these problems as problems that could be solved by fraction multiplication.

Discussion

So, what was algebraic about Michael's and Deborah's ways of operating in contrast with their partners? Both Michael and Deborah operated on the structure of their reversible fractional schemes with the structure of their multiplying schemes. That is, they both seemed to view the known quantity as a unit of so many units where the goal was to divide it into some number of parts—the number being determined, in the case of a fractional relationship, by the number of units of fractional size that the known quantity was of the unknown quantity. Once they used the activity of their multiplying schemes to accomplish this goal, each part they had created was a unit fractional part of the unknown and could be used in iteration to make the unknown. This aspect of their scheme was one that they could foresee and one reason for calling their schemes anticipatory. In contrast, Carlos and Bridget needed to carry out activity materially and check the result in retrospect—the results were not "contained" in the implementation.

Thus anticipatory schemes have a general, structural quality to them. They seem general in the sense that they apply to a wide range of situations, and this generality is related to their structural nature. For example, to both Michael and Deborah I can attribute a conceptual structure for splitting any known whole-number quantity into any number of parts required to solve a particular RMR problem. In contrast, I cannot be sure how general Carlos's or Bridget's reversible multiplying schemes were, nor can I claim they were operating with an awareness of the structure of their schemes.

My main explanation for the difference between the constructions of anticipatory versus enactive schemes involves the students' multiplicative structures. Being able to take the coordination of three levels of units as given prior to activity meant that both Michael and Deborah could project two different units of units structures into the known quantity, and flexibly switch between them, while also keeping track of the "larger" unit structure between known and unknown. Not having interiorized three levels of units was a significant constraint in Carlos's and Bridget's activity. However, being able to coordinate three levels of units in activity seemed to facilitate their construction of enactive schemes.

Although powerful, this explanation does not account for the main difference between Michael and Deborah: the use of reciprocal relationships in solving RMR problems. My current explanation is that Deborah had abstracted fractions as programs of operations that included all the ways to make as well as "unmake" the fractions. So, for example, 1/4 meant not only to disembed one part from a bar partitioned into 4 parts; it also meant that 1/4 taken four times, or 4*1/4, would yield the original bar. Thus if one bar was 3/4 of another bar, it was 3 times 1/4 of the other bar, and the other bar was 4 times 1/4 of itself, which was 4 times 1/3 of the original bar. Abstracting fractional concepts seemed to allow her to flexibly switch between viewing either quantity as the unit to which the other quantity could be compared. The reason Deborah made this abstraction while Michael did not is not yet clear.

Of course, in one sense the students' activity was clearly not algebraic: No students operated on standard algebraic notation in lieu of carrying out their own mental operations. In addition, with the exception of Deborah's use of reciprocal relationships, the students did not explicitly operate on unknowns (or knowns) multiplicatively with fractions (they did do so with whole numbers). However, this research supports and extends the notion put forth by early algebra researchers that arithmetic (including "arithmetic" with fractions) has an algebraic character (Blanton & Kaput, 2005; Carpenter, et al., 2003; Carraher, et al., 2006).

In fact, although early algebra researchers have indicated that reconceiving of arithmetic can bring out its algebraic qualities, they have not yet proposed a reorganization hypothesis for algebraic reasoning similar to the hypothesis proposed by Steffe (2002) for students' construction of fractions. Steffe's hypothesis is that students' operations on discrete quantities are reorganized to produce operations on continuous quantities—that students use the operations they have used to construct whole numbers in the construction of fractions.

I propose that such a hypothesis might be fruitful for understanding how students can construct multiplicative algebraic reasoning out of their previous quantitative reasoning with fractions. The main question in proposing such a hypothesis is: What quantitative operations are reorganized? One possible response has its origins in the research presented here: If students use their interiorized coordinations of three levels of units to operate with multiple units structures at once, it could be that students further abstract these interiorized units coordinations so that they can "apply" not just to numerical units but to schemes themselves. That is, algebraic reasoning may involve taking schemes as units that can be inserted into other schemes in at least 2-level and perhaps 3-level structures. In turn, this units-coordinating activity likely requires abstracting this "mega-structure" of embedded schemes as a program of operations.

Conclusion

The usefulness of this research, and of the proposed reorganization hypothesis for algebraic reasoning, will be measured by the degree to which it helps researchers understand more about the nature and value of early algebraic activity for elementary and middle school students. Ultimately, research that investigates and refines this reorganization hypothesis will contribute to testing Kaput's (1998) proposal that algebraic activity integrated across K-12 schooling will lend coherence, depth, and power to K-12 students' mathematical educations.

Endnotes

1. Still, activity in a computer microworld often accompanied the implementation of their schemes.
2. In solving RMR problems, the students usually determined the length of the unknown. Due to space limitations in this paper, I will restrict myself to discussing only how they made the unknown quantity.
3. To solve this problem, Michael partitioned each of the halves into six equal parts

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