MATHEMATICAL CARING RELATIONS AS A FRAMEWORK FOR SUPPORTING RESEARCH AND LEARNING

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Mathematical caring relations (MCRs), a framework for conceptualizing student-teacher interaction, was used in a year-long constructivist teaching experiment with 4 6th grade students. MCRs supported (1) the extension of previous research on how students construct improper fractions and (2) the learning of students and their teacher (the researcher). Establishing a MCR entails aiming for mathematical learning while attending to affective responses of both student and teacher. Although all students entered the experiment with the splitting operation deemed necessary for constructing improper fractions (Steffe, 2002), during the experiment 2 students did not construct improper fractions. One of these students is the focus of this paper. The current hypothesis is that splitting does not automatically engender the coordination of 3 levels of units that seems necessary to construct improper fractions. Analyzing MCRs in research is seen to facilitate interactions that can lead to learning and to validate the experiential difficulties of learning.

Purpose

Mathematical caring relations (MCRs) is a framework for conceptualizing student-teacher interaction that conjoins cognitive and affective realms (Hackenberg, 2005b). This paper explores how establishing MCRs in a year-long, constructivist teaching experiment with four sixth grade students supported the extension of previous research on how students construct improper fractions (Steffe, 2002). The paper addresses two central questions: What does describing and analyzing affective responses contribute to research on mathematics learning? What learning occurred (and did not occur) for one of the students and her teacher (the researcher and author) in the process of establishing a MCR?

Background

Interaction and Learning

Social interaction is basic to mathematics learning: Interactions among teachers and students form a major, though not comprehensive (cf. Confrey, 1995), impetus for the learning of both teachers and students. I contend that any social interaction can trigger depletion and stimulation for the people involved. I define *depletion* as a feeling of being taxed, usually accompanied by a decrease in energy or a diminishment of overall well-being. I define *stimulation* as a feeling of being excited or awake, usually accompanied by a boost in energy or a stronger sense of aliveness. In any social interaction these two feelings may be negligible and the dominant feeling may be "evenness" or neutrality. But in some social interactions the two feelings fluctuate more obviously or one greatly outweighs the other. In social interactions focused on learning, sustaining some level of depletion is often necessary for subsequent feelings of stimulation. However, feelings of depletion may dominate for a variety of reasons.

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Caring Relations

Sustained depletion may contribute to comments from both students and teachers that the other party doesn't care. Students who experience sustained depletion may say that their teachers don't care about students' lives or ways of thinking, which implies a lack of awareness about or valuation of them. These students don't feel *cared for* by their teachers. In turn, teachers may say that these students don't care about learning, a particular subject area like mathematics, or school. These teachers don't feel cared for *as teachers* because they don't experience their students' responsiveness to and engagement with the activities they orchestrate and the questions they pose. These feelings may not correspond with the other party's intentions—teachers may care very much about their students, and students may be interested in learning. But teachers and students often fail to develop *caring relations* (Noddings, 2002). Such failures can interrupt or stunt learning for both students and teachers.

In Noddings' (2002) notion of caring relations, a teacher (a carer) orchestrates experiences in which a student (a cared-for) feels that her or his need for care in student-teacher interactions is satisfied. For Noddings, establishing care as a teacher means being engrossed in students' ideas and worlds while working cooperatively with students to realize and expand their ideas and worlds. Receiving care as a student means experiencing renewed interest or activity, an increase in commitment or energy, or even a "glow of well-being" (p. 28). A caring relation is reciprocal because evidence of the reception of the teacher's care is what the teacher needs most to feel "cared for" by the student and to continue to care. Thus caring is not simply a feeling, or a virtue. Caring is an orientation to co-create and participate in social interactions that are responsive to the cognitive and affective states and needs of both teachers and students.

I conceive of establishing care in *mathematical* interaction as inseparable from engendering mathematical learning (Hackenberg, 2005b). Mathematics teachers may act as carers in general, but they start to act as *mathematical* carers when they hold their work of orchestrating mathematical learning for their students together with an orientation to monitor and balance feelings of stimulation and depletion that may accompany student-teacher interactions. From a student's perspective, participating in a MCR involves being open to the teacher's interventions in the student's mathematical activity and pursuing questions and ideas of interest.

Theoretical Framework

Mathematical Learning

Because MCRs cannot be established without aiming for mathematical learning, situating MCRs in a model of learning is essential. Following Piaget (1970) and von Glasersfeld (1995), I define mathematical learning as a process in which a person makes relatively permanent modifications or reorganizations in her or his ways of operating in response to perturbations (disturbances) brought about by her or his current ways of operating. The outcome of mathematical learning is a new way of operating adapted from a previous way of operating. An outcome is more or less permanent if, from the learner's point of view (but not necessarily at the level of awareness), it continues to be useful to the learner in on-going interaction. If the new way of operating solves situations not previously solved and can serve in further learning, then it can be considered more powerful than a previous way of operating.

Schemes and operations. I use the phrase "way of operating" to refer to a range of repeatable activity in which a person engages, such as a student regularly telling the teacher that his head hurts when a problem seems difficult. In contrast, I use the words "scheme" and "operation" more specifically. An operation is a mental action, such as repeating an item in imagination to create a plurality of items (an iterating operation). Operations are the components of schemes, goal-directed ways of operating that consist of a situation, an activity, and a result (Piaget, 1970; von Glasersfeld, 1995). To initiate the activity of a scheme, a situation must be perceived by a person as similar in some way to previous situations in which the person used the scheme. This perception or recognition is the result of assimilation, the basis for construction—and modification—of schemes. The perceived situation then triggers the activity of the scheme, which may be mental or physical or both. The person generally anticipates that the result of the scheme, an outgrowth of the activity, will be expected or satisfying in some way.

Accommodations. Modification or reorganization of a scheme—i.e., an accommodation—may occur when a person's current schemes produce an unexpected result: The person does not achieve her intended goal. This "disturbed" state of affairs is one example of a perturbation and is often accompanied by a sense of disappointment or surprise. As von Glasersfeld (1995) emphasizes, a person's "unobservable expectations" (p. 66) are instrumental in initiating a perturbation because what is crucial is the degree to which the unexpected result "matters" to the person at an intentional or unintentional level. This aspect of perturbations means they are not always consciously conflictive: An unexpected result may remain largely unnoticed by a person and yet have some impact on a person's subsequent activity—perhaps in a vague sense of unease or a somewhat heightened interest. Thus even perturbations that are mostly outside of immediate awareness involve an affective aspect. As a person (consciously or unconsciously) eliminates perturbations, or equilibrates, the perturbation has the potential to trigger an accommodation.

Affective aspects of perturbations. The affective aspect of a perturbation is a major point of connection between mathematical learning and mathematical caring. That is, experiencing a perturbation can be accompanied by feelings of both depletion and stimulation. Feelings of depletion may occur if a person senses that she or he does not know what to do to eliminate the perturbation, or that such activity will be particularly onerous. If a feeling of depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity either immediately or in the future. Perturbations can also provide stimulation in the form of a challenge, particularly if a person senses that she or he can meet that challenge, or that such activity itself will be enjoyable. If a feeling of stimulation is sufficient, the student's interest in or curiosity about a situation may prolong mathematical activity and open new opportunities for learning. If, over time, feelings of stimulation outweigh feelings of depletion, the student may feel mathematically cared for.

Enacting Mathematical Caring Relations

Teacher's activity. In establishing MCRs, teachers attempt to pose situations in which students may construct more powerful schemes while experiencing a balance between stimulation and depletion. Three activities help teachers to enact MCRs with their students. First, teachers pose situations that harmonize with students' mathematical ways of operating and affective responses to mathematical activity. Second, teachers pose situations that challenge students—that open opportunities for them to make accommodations and thereby expand their mathematical ways of operating. Third, teachers track students' affective responses to mathematical activity as indications of whether they feel mathematically cared for, and then teachers make adjustments in the situations that they pose to reinitiate harmonizing with and challenging students' ways of operating. In this process, teachers also monitor their own feelings of stimulation and depletion.

Student's activity. When a student's response to a teacher's mathematical care includes consideration of a new situation the teacher has posed, or a sense of interest or aliveness, the student participates in a MCR with the teacher. Students who do so are likely to feel that they are being listened to, that their ideas are valued, and, perhaps, that they are understood. As a result, these students may experience stimulation—may feel energized or stronger in some way. Students may also feel some depletion—for example, uncertainty or confusion—in response to provocations posed by the teacher. The provocations can be stimulating if students find that it is possible—or even satisfying—to resolve them (although some may not be resolved quickly!). Such experiences may help students sustain or increase their engagement in mathematical activity engendered by the teacher or by themselves.

Connections between engendering caring and learning. Decentering from one's own ways of operating as a teacher is required to orchestrate learning for students and to initiate MCRs with them. In decentering, teachers practice close listening to and observing of students' ways of operating. Teachers test out activities and pose problems that they conjecture students can solve with their current schemes and operations. The teacher's main goal in listening and observing is not to confirm her or his own mathematical thinking but to make images of and conjectures about the students' mathematical thinking. In effect, the teacher is trying to learn mathematical ways of operating from students. In this sense the teacher works to harmonize with students' ways of operating, a central aspect of establishing mathematical care.

Yet to engender student learning, teachers also must act. Based on their images of and conjectures about students' ways of operating, teachers pose situations to provoke perturbations that students may eliminate by making accommodations in their current schemes. Such situations can expand students' mathematical realities in ways that students have likely not envisioned. Doing so can be stimulating for students and accomplishes the aspect of challenging them in establishing mathematical care. As teachers continue to observe and reflect on the consequences of these interactions with students, teachers may feel stimulated by their sense that they are communicating with their students and providing mathematical care.

Methods of Inquiry

In a constructivist teaching experiment (Steffe & Thompson, 2000), researchers seek to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching. Since researchers' mathematical knowledge may be insufficient to understand students' ways of operating, researchers aim to learn mathematics from students. Researchers also engage in on-going conceptual analysis of how students *might* operate in mathematical situations. Based on learning from students and conceptual analysis, researchers make conjectures and test them through posing tasks in teaching episodes. Teaching practices include presenting students with problem situations, analyzing students' responses, and determining new situations that might allow students to make accommodations. Thus this methodology is compatible with establishing MCRs because researchers harmonize with students' current schemes and open opportunities for students to construct new schemes.

Four sixth grade students from a rural middle school in Georgia participated in my teaching experiment from October of 2003 to May of 2004 (Hackenberg, 2005a). They were invited to participate after demonstrating during unrecorded selection interviews that they reasoned multiplicatively. I taught them biweekly in pairs for two to three weeks, followed by a week off.

Each teaching episode occurred during school hours, lasted 30 minutes, often involved the use of a computer software program called JavaBars (Biddlecomb & Olive, 2000), and was videotaped with two cameras. Two witness-researchers assisted in videotaping, offered feedback on the teaching activities, and provided triangulation of perspectives in data analysis. Intensive retrospective analysis of the videotapes began in the summer of 2004.

The Splitting Operation

In their selection interviews, all four of the participating students demonstrated that they had constructed the splitting operation (Steffe, 2002). This fundamental multiplicative operation is involved in solving a problem like the following: "Here's a picture of my stick, which is five times longer than yours. Can you make your stick?" Students who can solve this problem engage in partitioning and iterating nearly simultaneously. Students need to posit their bar, which stands in relationship to the given bar but is also separate from it: Their bar can be iterated five times to make the given bar, and at the same time their bar is formed from partitioning the given bar into five equal parts. So students who solve the problem are aware of at least one multiplicative relationship between their bar and the given bar, namely that their bar taken five times produces the given bar. They may not necessarily be aware that 1/5 of the given bar produces their bar, depending on the state of their fraction language and fraction schemes. Solving a splitting problem is more complex than solving the problem of making 1/5 of a candy bar. In this latter problem, making 1/5 is a result of a student's fraction scheme, whereas Steffe has commented that the concept of 1/5 is *input* to the splitting operation, even if the student does not name the part as 1/5 of a bar (Leslie P. Steffe, personal communication, February 11, 2005).

Steffe has posited that a student's construction of the splitting operation opens the way for the student to construct improper fractions. He states, "Upon the emergence of the splitting operation, I regard the partitive fractional scheme as an *iterative fractional scheme*" (2002, p. 299, italics in the original). Steffe places so much importance on splitting for the construction of improper fractions (and an iterative fractional scheme) because splitting involves positing a bar that is independent from a given bar and yet stands in relation to it. Students who have constructed only *partitive* fractional schemes can disembed 1/5 of a bar from a whole bar but cannot iterate it beyond the whole (i.e., 6/5 of a bar makes no sense to these students). Students who have constructed an *iterative* fractional scheme can disembed unit fractions and iterate them any number of times so that, for example, 17/5 means one-fifth iterated 17 times and simultaneously means three groups of five-fifths and two more one-fifths. In this way, 17/5 is a unit of 17 units, any of which can be iterated five times to make the whole (5/5), another unit embedded within the 17/5. Thus 17/5 can be seen as a structure involving three levels of units.

Results and Discussion

Bridget's Lacuna

Although both Deborah and Bridget had constructed a splitting operation at the start of the teaching experiment, they had not yet constructed an iterative fractional scheme. But on November 11th, Deborah made a bar 2/15 longer than a unit bar and called the result 17/15 of the unit bar. I infer that for her 17/15 was 1/15 iterated 17 times and simultaneously 2/15 more than a whole. My hypothesis, which I support by the discussion to follow, is that she operated with three levels of units: She viewed 17/15 as a unit of seventeen units, any of which could be iterated fifteen times to make the whole, 15/15, another unit. Seventeen-fifteenths was distinct from this whole (versus part of it) and yet still stood in relation to it. So I can attribute an iterative fractional scheme to Deborah at this point.

However, Bridget called the bar 17/17. She later agreed with Deborah's assessment of the bar and could often operate with "small" improper fractions (fractions just more than one) as Deborah did. But "large" improper fractions (fractions quite a bit greater than one) continued to be problematic for Bridget. For example, on February 9th we were imagining a long peppermint stick of length *P*. I asked them to imagine how they'd use *P* to make another stick that was 15/4 of *P*. Deborah said to make *P* three times and add on 3/4 of *P*. Bridget said, "you've got to keep going, you've got to go 14 more of the original one." My hypothesis is that Bridget was *not* operating with three levels of units: For Bridget, 15/4 was a unit of 15 units, but she did not form units of four of those units as equal to *P*.

Even at the end of the experiment, Bridget demonstrated this lacuna. On May 3rd I posed the following "apple" problem: "An apple costs 75 cents; how much does five-thirds of an apple cost?" Bridget exclaimed, with some annoyance in her voice, "Oh my gosh!" Then, while Deborah swiftly solved the problem, Bridget said, "Two and one-third?" For Deborah, five-thirds was one-third five times and she could take it as a given in operating (i.e., 5/3 of an apple costs five times \$.25 or \$1.25). Bridget did not seem to be able to operate similarly at that time, and appeared glum that Deborah could solve the problem so quickly.

Establishing a MCR with Bridget

Because early on November 11th I perceived that Bridget's activity with fractions larger than one was different from Deborah's activity, I posed this problem to the girls in that episode: "This cake is magic in that the cake fills right back in when you take out a piece. Can you make a cake that's 17/5 of that cake?" Such problems are designed to free students from feeling confined to the material of whole—i.e., the magic refilling of the cake allows them to use material that goes beyond the whole. So in posing a magic cake problem to Bridget I was harmonizing with my inference that she had a partitive fractional scheme but not yet an iterative fractional scheme. However, she was able to complete these problems with ease by disembedding and iterating a unit fraction beyond the whole, yet without eliminating the lacuna in her activity with fractions larger than one (as I have described in the previous section). So posing such problems did not sufficiently challenge her to make an accommodation in her fraction scheme and thereby expand her mathematical world, a central aspect of establishing a MCR.

In fact, throughout the experiment Bridget often made small, and sometimes large, improper fractions. But she did not seem to be able to take small or large improper fractions as a given in solving a problem. So improper fractions seemed to remain "odd" or not quite "legitimate" numbers for her. However, because she tended to be able to interpret Deborah's responses to problems involving improper fractions, I did not always take seriously enough the evidence that fractions larger than one bothered her and that not having constructed improper fractions may have prevented her from making accommodations similar to Deborah's during the experiment.

For example, in December the girls worked on problems like this one: "Make a 3/3-bar. Make that into a 9/8-bar without erasing the thirds marks." I infer that Deborah made an accommodation in her iterative fractional and multiplying schemes in order to solve this problem, but Bridget remained stumped by it. At the time I was aware of the depletion Bridget seemed to experience as we worked on these problems, but I was not sure what to do so that it might be alleviated through her operative activity. In retrospect, I can see that asking Bridget to make a 3/3-bar into an 8/8-bar (or something "easier" like a 4/4-bar) might have been one way to better mathematically care for her, provoking the coordination of two different fractions within the same bar while eliminating the "bothersome" improper fraction.

Similarly, in early May I posed a series of problems like the "apple" problem. Bridget appeared unable to operate and seemed emotionally shut down. I decentered enough to understand that she was experiencing significant depletion, and I endeavored to break down the problems so that she might act more independently, as well as feel more autonomous and in control. But because my interventions relied heavily on my own ways of structuring the problems, my suggestions were not very effective for her and indicated that I had not decentered enough cognitively. Thus my suggestions did not alleviate her depletion very well, and the longer she remained in a depleted state, the more depletion I felt!

My interactions with Bridget in early May triggered a perturbation for me that set off a rather fervent search for better ways to communicate with her mathematically. Out of that search came a gradual reestablishment of our MCR over the final three episodes of the teaching experiment. I began to take more seriously that large improper fractions were cognitively and affectively problematic for her. So I carefully planned a sequence of tasks for her in subsequent May episodes that avoided taking improper fractions as given. Bridget responded quite positively to the sequence, both cognitively and affectively (Hackenberg, 2005a). Thus I can conclude that we reestablished a MCR. But I cannot conclude that, at that time, Bridget made an accommodation in her partitive fractional scheme in order to construct an iterative fractional scheme.

Hypothesis Revision

While the splitting operation still seems to be instrumental in the construction of an iterative fractional scheme, it does not appear to be sufficient for it. Thus my current, revised hypothesis is that students can construct the splitting operation, in which units of the whole become iterable *within the whole*, without also constructing three levels of units that are necessary for constructing improper fractions. I conjecture that constructing a splitting operation *with fractional parts* is what allowed Bridget to iterate *beyond the whole* even though she did not yet operate with three levels of units. That is, Bridget could make the whole bar given 2/5 of the bar. So for her 2/5 was 1/5 two times, and she could use her splitting operation on 2/5 by partitioning it into two equal parts, either of which could be iterated five times to make the whole bar. Thus she could also iterate 1/5 to make, say, 17/5 of the whole bar, but 17/5 lost its relationship to the whole. Since another student in the teaching experiment operated similarly to Bridget, it is possible that many students may construct this way of operating with fractions larger than one.

Conclusions

Although Bridget did not learn to make improper fractions during the experiment, the depletion she demonstrated triggered significant learning for me, the researcher. My revision of the hypothesis about what is necessary for students to construct improper fractions opens the way to establishing better MCRs with students who operate like Bridget. Bringing forth the construction of three levels of units appears to be crucial, but such learning is not easy to engender! So, understanding that students like Bridget cannot yet operate with three levels of units may allow teachers to pose tasks that harmonize better with these students' current schemes—and therefore do not bring about the sustained depletion that Bridget seemed to experience. Thus the contribution of attending to affective responses in orchestrating and researching mathematical learning appears to involve (a) opening possibilities for making and refining interventions so that both teachers and students may learn more, and (b) validating the difficulties that both teachers and students may experience in eliminating perturbations.

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