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Hyper-special valued lattice-ordered groups

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Abstract. A lattice-ordered group G is *hyper-special valued* if it lies in the largest torsion class which is contained in the class of special-valued lattice-ordered groups. This is precisely the class of lattice-ordered groups G such that for each $g \in G$, every ℓ-homomorphic image K of the principal convex ℓ-subgroup generated by g has the feature that each $0 < x \in K$ is the supremum of pairwise disjoint special elements. It is shown in this article that if G is hyper-special valued, then for each $g \in G$, the space of values $Y(g)$ of g is a compact scattered space. This property naturally gives meaning to the notion of an α-special value of g: this is a value which corresponds to an isolated point of the α-th remainder in the Cantor–Bendixson sequence of $Y(g)$. It is shown that, for each ordinal α, the set of α-special values of G forms a disjoint union of chains, which is at once an order ideal and a dual order ideal of the root system of all values of G. If G is projectable, then in addition the set of special values of G is also a disjoint union of chains which is an order ideal and a dual order ideal. An archimedean lattice-ordered group G with weak order unit $u > 0$, given its Yosida representation, such that $u \equiv 1$ is hyper-special valued if and only if (a) G is projectable, (b) the Yosida space Y is scattered, and (c) for each $g \in G$ the image of the function g has finitely many ∞'s as well as finitely many accumulations of 0.

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1 Introduction

The motivation for this work comes largely from the prominence of special valued lattice-ordered groups, as well as that of the classes of lattice-ordered groups which are closed under taking ℓ-homomorphic images. As is the custom in this area of mathematics, we abbreviate "lattice-ordered group" as *ℓ-group*. In tandem, we use *ℓ-subgroup* for a subgroup which is at once a sublattice; *ℓ-homomorphism* for a map which preserves the group and the lattice structure; and the term *convex ℓ-subgroup* refers to an ℓ-subgroup which is convex in the ordering of the group. For any other unexplained terms of this sort, and as the general reference on the theory of ℓ-groups, we refer the reader to [D95], but also occasionally to [BKW77]. Certain references from general topology will enter our discussion, and we shall cite the appropriate ones as needed.

We proceed to the main definition; for openers, no assumption is made about the ℓ-groups under discussion. The notation is additive, following the Conrad tradition.

Definition & Remarks 1.1. (a) In this discussion G stands for a ℓ-group. $\mathscr{C}(G)$ denotes the lattice of all convex ℓ-subgroups of G; it is well known that $\mathscr{C}(G)$ is an algebraic Brouwerian lattice ([D95], Proposition 7.10); in particular, it is a distributive lattice.

We say that $N \in \mathscr{C}(G)$ is a *value* if it is maximal with respect to not containing some $a \in G$ as an element. In the event, we call N a *value of a*. The set of all values of a is denoted $Y(a)$. Observe that $Y(a)$ is, by definition, a trivially ordered set relative to inclusion. When $|Y(a)| = 1$, we say that both the element a and its value are *special*.

It is well known that if N is a value of G then $a \wedge b = 0$ implies that $a \in N$ or $b \in N$. A convex ℓ-subgroup with this feature is called a *prime subgroup*. There are many ways to characterize prime subgroups of an ℓ-group; the reader is referred to Theorem 9.1 in [D95]. Let us summarize some of the most frequently used properties of values and prime subgroups.

(b) Basic properties of values and prime subgroups:

(i) The set $\mathrm{Spec}_\ell(G)$ of all prime subgroups of G forms a root system under inclusion; that is, no two incomparable prime subgroups contain a third. (Theorem 9.8, [D95]) In particular, this is true of the set $\mathrm{Val}(G)$ of all values of G.

Here is an application that comes up often in any discussion about values in an ℓ-group. If $M \in \mathrm{Val}(G)$ and $a \in G$ is not in M then there is a unique value of a which contains M.

(ii) For each value N of G, the set

$$\{M \in \mathscr{C}(G) : N \subset M\}$$

has a least element N^*. (Proposition 10.2, [D95]) If N is normal in N^*, then N^*/N is order-isomorphic to a subgroup of the additive group of real numbers, $\mathbb{R}$, in its natural ordering. (The latter is due to Hölder's Theorem; [D95], Theorem 24.16.)

(iii) Assuming a and b are positive, $a \wedge b = 0$ precisely when each $M \in Y(a)$ is incomparable to each $N \in Y(b)$. (Proposition 11.7, [D95])

(c) If N is a special value then the convex ℓ-subgroup N^* mentioned in (b)(ii) is referred to as the *cover* of N. If N is special then it is always normal in its cover. (Theorem 40.8, [D95]) To motivate, in part, the definition of special valued ℓ-groups, let us first briefly recall the more particular situation in which each $Y(a)$ is finite. The theorem we now quote is due to Conrad, and first appeared in [Co65]. The reader may also find this material in [D95], §46. We stress the "local" version of it in [BKW77], Theorem 6.4.1.

Suppose that G is an ℓ-group and $0 < a \in G$. Then the following are equivalent:

(i) $Y(a)$ is finite.

(ii) a can be expressed as a disjoint sum $a = a_1 + \cdots + a_n$ of special elements a_i.

(iii) Each value of a is special.

An ℓ-group G which satisfies the preceding equivalent conditions for each of its elements is said to be *finite valued.*

If $g = a + b$ in an ℓ-group and $|a| \wedge |b| = 0$ we say that a is a *component* of g. Now, an ℓ-group G is said to be *special valued* if each positive $g \in G$ is a supremum of special components. Theorem 45.6, [D95], states that

> G is special valued if and only if (i) each $g \neq 0$ has a special value and (ii) if $N \in \mathrm{Val}(G)$ is special and M is a value containing N, then M too is special.

In a special valued ℓ-group every value is normal in its cover (Proposition 45.4, [D95]). When this happens in an ℓ-group G it is said that G is *normal valued.*

Finally we note, anticipating our central focus of discussion, that the class of special valued ℓ-groups is not closed under formation of ℓ-homomorphic images, but if $\phi : G \to H$ is a surjective ℓ-homomorphism which preserves all suprema, and G is special valued, then H is also special valued. The reader is referred to Example 45.11 and Proposition 45.13 of [D95].

(d) As is common in commutative ring theory we may and do put a "hull-kernel" topology on $\mathrm{Spec}_\ell(G)$. The basic open sets of the topology are the sets of the form

$$h(a) = \{N \in \mathrm{Spec}_\ell(G) : a \notin N\},$$

for all $a \in G$. We shall be most interested in the subspace topology for $Y(g)$, where $g \in G$. We shall also abuse the notation just introduced and identify $h(a)$ $(a \in G)$ with the basic relatively open set $h(a) \cap Y(g)$ in $Y(g)$. It is an easy exercise to verify that $Y(g)$ is a compact Hausdorff space; we shall speak of $Y(g)$ as the *Yosida space of g.*

The connection between special values and things topological is subject of the following lemma, for later use.

Lemma 1.2. *Let G be an ℓ-group and $g > 0$ in G. An element N of $Y(g)$ is special precisely when it is an isolated point of $Y(g)$.*

Proof. If $N \in Y(g)$ is special, and a is a special element that witnesses that fact, then $h(a) = \{N\}$, whence N is isolated.

Conversely, suppose that N is an isolated point in $Y(g)$. Then there exists an $s > 0$ in G such that $h(s) = \{N\}$; without loss of generality, one can replace s with $s \wedge g$ and assume $s \leq g$. Now $Y(g)\backslash\{N\}$ is also clopen, hence compact, and from a routine open-cover-to-finite-subcover argument, one can find $0 < a \leq g$ such that $h(a) = Y(g)\backslash\{N\}$. Subtracting the meet, $a \wedge s$ we may also assume that a and s are disjoint. Now, if M is a value of s, then $g \notin M$, whence it follows that M is contained in a value $P \in Y(g)$. As $a \wedge s = 0$ we have that $a \in M \subseteq P$, and we conclude that $M = P = N$, proving that N is the only value of s. $\square$

We conclude this introduction with a brief overview of radical and torsion classes of ℓ-groups. We refer the reader to §36 of [D95]. Torsion classes as such were introduced in [M75] by Martinez. We also recommend the survey in [GH89], Chapter 6.

Definition & Remarks 1.3. In this discussion we use the phrase "class of ℓ-groups" to mean a (proper) class which is closed under formation of ℓ-isomorphic copies. Let $\mathcal{R}$ be class of ℓ-groups. It is called a *radical class* if it is closed under taking convex ℓ-subgroups, and whenever $C_i \in \mathcal{C}(G)$ $(i \in I)$, and each $C_i \in \mathcal{R}$, then $C \equiv \bigvee_{i \in I} C_i \in \mathcal{R}$. If $\mathcal{R}$ is also closed under formation of ℓ-homomorphic images we say that it is a *torsion class*. If $\mathcal{R}$ is a radical class, and G is an ℓ-group, denote by $\mathcal{R}(G)$ the join of all the convex ℓ-subgroups of G belonging to $\mathcal{R}$.

Examples of torsion classes abound in the theory; here are a few:

(i) The class $\mathcal{FV}$ of all finite valued ℓ-groups.

(ii) The class of all hyper-archimedean ℓ-groups. (Recall that G is *archimedean* if $0 \leq na \leq b$, for each natural number n implies that $a = 0$; G is *hyper-archimedean* if each ℓ-homomorphic image of G is archimedean.)

(iii) The class $\mathcal{O}$ of all cardinal sums of totally ordered groups. (The term "cardinal sum" refers to a direct sum of ℓ-groups with coordinatewise partial ordering.) Observe that if $G \in \mathcal{O}$ then the Yosida space of every element is a finite, discrete space.

As was observed in 1.1(c), the class of special valued ℓ-groups is a radical class, which is not a torsion class.

The following proposition sets the stage for our discussion the rest of the way. As is the custom, if $a \in G$, then $G(a)$ denotes the convex ℓ-subgroup of G generated by a. The term "ℓ-ideal" refers to a normal convex ℓ-subgroup.

Proposition 1.4. *Suppose that $\mathcal{R}$ is a radical class of ℓ-groups. We denote by* Hyp($\mathcal{R}$) *the class of all G for which $G(a)/L \in \mathcal{R}$, for each $a \in G$ and each ℓ-ideal L of $G(a)$. Then* Hyp($\mathcal{R}$) *is a torsion class, and, indeed, the largest torsion class contained in $\mathcal{R}$.*

Proof. That Hyp($\mathcal{R}$) is closed under formation of convex ℓ-subgroups is easy, and we leave it to the reader. Next, suppose that each $C_i \in \mathcal{C}(G)$ $(i \in I)$ belongs to Hyp($\mathcal{R}$), and let $C = \bigvee_i C_i$. The reader will readily appreciate that it suffices to verify that quotients $G(a)/L$ belong to $\mathcal{R}$, for each positive $a \in C$. So let $a \in C$; then one may write $a = a_1 + \cdots + a_n$, with $0 \leq a_k \in C_{i_k}$. If L is an ℓ-ideal of $G(a)$, then

$$G(a)/L = \bigvee_{k=1}^{n} (G(a_k) \vee L)/L,$$

and each $(G(a_k) \vee L)/L \in \mathcal{R}$, by assumption. This implies that $G(a)/L \in \mathcal{R}$.

Now suppose that $\phi : G \to H$ is a surjective ℓ-homomorphism, and $G \in \mathrm{Hyp}(\mathscr{R})$. Pick $b \in H$ and an ℓ-ideal L of $H(b)$. There is a preimage $a \in G$ of b under ϕ such that $H(b) = \phi(K + G(a))$, with $K = \ker(\phi)$. Noting that

$$H(b) = \phi(K + G(a)) \cong (K + G(a))/K \cong G(a)/K \cap G(a),$$

we may conclude that $H(b)/L$ is an ℓ-homomorphic image of $G(a)$, and it is therefore in $\mathscr{R}$. This proves that $H \in \mathrm{Hyp}(\mathscr{R})$, and that $\mathrm{Hyp}(\mathscr{R})$ is a torsion class.

If $G \in \mathrm{Hyp}(\mathscr{R})$ then, for each $a \in G, G(a) \in \mathscr{R}$, and hence $G \in \mathscr{R}$, because $\mathscr{R}$ is a radical class. That it is the largest torsion class contained in $\mathscr{R}$ is routine to check, and this is left to the reader. □

Remark 1.5. Suppose that $\mathscr{R}$ is a radical class in which every convex ℓ-subgroup is normal. For each $G \in \mathscr{R}$ and $a \in G$ each ℓ-ideal of $G(a)$ is an ℓ-ideal of G. Therefore, if each ℓ-homomorphic image of G lies in $\mathscr{R}$, it follows that $G \in \mathrm{Hyp}(\mathscr{R})$. In this setting then Proposition 1.4 may be stated this way:

$$G \in \mathrm{Hyp}(\mathscr{R}) \text{ if and only if each } \ell\text{-homomorphic image of } G \text{ lies in } \mathscr{R}.$$

This characterization is valid if $\mathscr{R}$ consists of abelian ℓ-groups. However, in Example 6.11 we shall see that it does not work in general.

Finally, and formally, here is our central definition. Consideration of $\mathrm{Hyp}(\mathscr{R})$-groups, for different radical classes $\mathscr{R}$ of ℓ-groups is left for another time and place.

Definition 1.6. An ℓ-group G is said to be *hyper-special valued* if it belongs to the class $\mathrm{Hyp}(\mathscr{SV})$, where $\mathscr{SV}$ stands for the class of all special valued ℓ-groups. We point out that if G is such that every ℓ-homomorphic image of it is special valued, then G need not be hyper-special valued; an example is given in §6, 6.11. In sum then, one really needs the full strength of the definition of $\mathrm{Hyp}(\mathscr{R})$ in Proposition 1.4 to get it to be a radical class.

2 Scattered spaces and higher grades of specialty

In a sense, what makes hyper-special valued ℓ-groups tick is that each Yosida space $Y(g)$ is scattered. Let us first lay the topological groundwork by giving a brief overview of scattered spaces. For the most part we discuss compact spaces. Our central reference for this is [Ko89], §10.

Definition & Remarks 2.1. Recall that a space X is *scattered* if every nonvoid closed subspace Y of X contains an isolated point of Y. A compact scattered space is necessarily zero-dimensional. (Recall that a space X is said to be *zero-dimensional* if there is a base for the open sets consisting of clopen sets.) Under this circumstance we are able to take advantage of Stone duality. The dual of a scattered compact space is a *superatomic* boolean algebra: *every nontrivial homomorphic image has an atom.*

Superatomic boolean algebras are characterized in [Ko89], Proposition 17.5. Dualizing the statement of that theorem we obtain, for compact space X, that the following are equivalent.

(a) X is scattered.

(b) Every nonvoid closed subspace Y of X contains a dense set of isolated points.

(c) Every continuous image of X is scattered.

It should also be clear that any closed subspace of a scattered space is scattered. A boolean algebra is superatomic precisely when it contains no infinite free boolean subalgebras. ([Ko89], Lemma 10.19) Dualizing, and recalling that the dual of a free algebra on κ free generators is the so-called Cantor space 2^κ of weight κ (with the product topology), we get that X, compact and zero-dimensional, is scattered if and only if the Cantor set 2^ω is not a continuous image of X. More is true, however: if X is compact and scattered then all its continuous real-valued functions are onto compact scattered subspaces of the real line, which by a result of W. Rudin—[Ru57]—means that all such images are countable. With a little more work one gets the following bit of folklore, which appears as Lemma 1.5 in [MM∞]:

> Let X be a compact space. Then X is scattered if and only if the unit interval I is not a continuous image of X.

More crucial to our purposes are the Cantor–Bendixson derivatives, associated to a scattered space. We briefly review the definition.

Definition & Remarks 2.2. Let X be a compact scattered space. We define a nest of closed subspaces, ordinally indexed. First, let $\mathrm{Is}(X)$ denote the set of all isolated points of X, and put $X_1 \equiv X \backslash \mathrm{Is}(X)$. Suppose that, for an ordinal β the X_α have been defined for all $\alpha < \beta$. If β is a limit ordinal let

$$X_\beta \equiv \bigcap_{\alpha<\beta} X_\alpha.$$

Otherwise, if $\beta = \gamma + 1$, let

$$X_\beta \equiv X_\gamma \backslash \mathrm{Is}(X_\gamma).$$

X_β is called the *β-th Cantor–Bendixson derivative* of X. It is shown in §17 of [Ko89] that the last nontrivial Cantor–Bendixson derivative X_γ is finite, whereas all the preceding ones are infinite. The ordinal γ is called the *length* of the sequence.

The starting point in this section is a result for a class which is slightly bigger than $\mathrm{Hyp}(\mathscr{SV})$. Let us lead in with a remark.

Remark 2.3. The class $\pi\mathscr{SV}$ of all ℓ-groups G in which each $0 < g \in G$ has a special component was introduced in [M92]. The members of $\pi\mathscr{SV}$ are said to be *pseudo-*

special valued. It should be evident that every special valued ℓ-group is pseudospecial valued. It was shown in [M92] that pseudospecial valued ℓ-groups need not be special valued. Before we get to Proposition 2.5, it is useful to record the following result, which seems to be folklore. If G is any ℓ-group and $g \in G$, then N_g denotes the intersection of all the values of g in $G(g)$. Note that $M \in \mathscr{C}(G(g))$ is a value of g in $G(g)$ if and only if it is a maximal convex ℓ-subgroup of $G(g)$. It is a consequence of the discussion in [BCD86] on above and below subgroups that N_g is always order-closed in $G(g)$; this will be used in Proposition 6.6. $G(g)$ is clearly normal in $G(g)$.

Proposition 2.4. *Any pseudospecial valued ℓ-group is normal valued.*

Proof. Suppose that G is pseudospecial valued and $g > 0$ in G. It suffices to show that each maximal convex ℓ-subgroup of $G(g)$ is normal in $G(g)$. Now observe that to say that G is pseudospecial valued is to say that for each $g > 0$ there is a maximal pairwise disjoint set in $G(g)$ consisting of special components of g. Putting it another way: G is pseudospecial valued if and only if N_g is the intersection of all the special maximal convex ℓ-subgroups of $G(g)$. On the other hand, as is well known (see 40.8, [D95]), each special value is normal in its cover. This implies that $G(g)/N_g$ is a subdirect product of subgroups of $\mathbb{R}$, and, in particular, abelian. Thus, if M is any maximal convex ℓ-subgroup of $G(g)$, then M is normal in $G(g)$ because M/N_g is obviously normal in $G(g)/N_g$. It follows that G is normal valued. $\square$

Now here is the proposition we are really after.

Proposition 2.5. $\mathrm{Hyp}(\pi\mathscr{S}\mathscr{V})$ *is the class of all ℓ-groups G such that $Y(g)$ is a scattered space, for each $0 < g \in G$.*

Proof. We use Lemma 1.2. Suppose first that $G \in \mathrm{Hyp}(\pi\mathscr{S}\mathscr{V})$; then in each ℓ-homomorphic image H of $G(g)$, each nonzero $h \in H$ has a special component. Now let K be a closed subspace of $Y(g)$ $(g \in G)$. Since $Y(g)$ carries the hull-kernel topology, and since $Y(g)$ is homeomorphic to the space of values of g in $G(g)$, where all values are normal (per Proposition 2.4), there is an ℓ-ideal L of $G(g)$ such that $K = \{N \in Y(g) : L \subseteq N\}$. Since the values of $L + g$ in $G(g)/L$ are the subgroups of the form M/L, where M is a value of g in $G(g)$, containing L, it follows that $Y(L + g)$ is homeomorphic to K. Since $L + g$ has a special component, Lemma 1.2 gives us that K has an isolated point, and therefore $Y(g)$ is scattered.

If, conversely, each $Y(g)$ is scattered, and A is an ℓ-ideal of $G(g)$, then, once again $Y(A + g)$ is homeomorphic to a closed subspace of $Y(g)$. Owing to Lemma 1.2, $A + g$ has a special value, which is of the form N/A, for suitable $N \in Y(g)$. There is an $s \in G$ such that N/A is the only value of $A + s$. Without loss of generality one may choose $s > 0$, and by taking a sufficiently large multiple of s, $A + (s \wedge g)$ is a component of $A + g$. $\square$

We wish to use Proposition 2.5 in two instances. The first requires no preliminaries.

Corollary 2.6. *If G is hyper-special valued then each $Y(g)$ is scattered.*

Remark 2.7. Proposition 2.5 characterizes a radical class in terms of the Yosida spaces of the elements of the ℓ-groups belonging to that class. This kind of characterization of radical classes will be explored in more depth in [DM∞], where several of the results of this paper are generalized.

There is another torsion class of ℓ-groups to which we should like to apply Proposition 2.5.

Definition & Remarks 2.8. (a) We have already brought up the class $\mathcal{O}$ of all cardinal sums of totally ordered groups (1.3). We enlarge that class, using the "completion" idea from [M75]. The class $\mathcal{O}^*$ consists of all ℓ-groups G for which there exists a well ordered chain of ℓ-ideals

$$\{0\} \subset L_1 \subset \cdots \subset L_\alpha \subset \cdots$$

for which $L_\beta = \bigcup_{\alpha \in \beta} L_\alpha$ for each limit ordinal $\beta, L_\delta = G$, for some ordinal δ, and for each $\alpha < \delta$, $L_{\alpha+1}/L_\alpha \in \mathcal{O}$. It is well known (see [M75], Theorem 1.6) that $\mathcal{O}^*$ is a torsion class; indeed, it is the least torsion class containing all the cardinal sums of totally ordered groups, which is closed under extensions. (Note: a radical class $\mathcal{R}$ is *closed under extensions* if, for each G and each ℓ-ideal L of G, $G \in \mathcal{R}$ whenever $L \in \mathcal{R}$ and $G/L \in \mathcal{R}$.)

Define transfinitely the sequence $\mathcal{O}^1(G) = \mathcal{O}(G)$, the $\mathcal{O}$-radical of G, and $\mathcal{O}^{\alpha+1}$ so that

$$\mathcal{O}^{\alpha+1}(G)/\mathcal{O}^\alpha(G) = \mathcal{O}(G/\mathcal{O}^\alpha(G)),$$

while $\mathcal{O}^\beta(G) = \bigcup_{\alpha<\beta} \mathcal{O}^\alpha(G)$, for each limit ordinal β. Then $G \in \mathcal{O}^*$ precisely when $G = \mathcal{O}^\alpha(G)$, for a suitable ordinal α.

(b) Recall (p. 98, [D95]) that in an ℓ-group $G, 0 < b \in G$ is *basic* if the set

$$\{s : 0 \le s \le b\}$$

is a chain. G is said to have a *basis* if there is a maximal pairwise disjoint set consisting of basic elements. It is well known that G has a basis if and only if each $g > 0$ exceeds a basic element (Theorem 19.11, [D95]). Now if $G \in \mathcal{O}^*$, then $\mathcal{O}(G)$ is (order) dense in G ([M75], Corollary 1.7.1), and it is then not hard to see that G has a basis. Since $\mathcal{O}^*$ is a torsion class, this means that every ℓ-homomorphic image of G also has a basis. Conversely, suppose that each ℓ-homomorphic image of the ℓ-group G has a basis. By observing that $\mathcal{O}(G)$ contains any basis of G, arguing transfinitely, one obtains that $G \in \mathcal{O}^*$. Let us summarize this; we emphasize that this result is not new. One can easily ascribe it to folklore.

$G \in \mathcal{O}^*$ if and only if each ℓ-homomorphic image of G has a basis.

Recall as well, for later use, that any basic element is special, and note that each $a \in \mathcal{O}(G)$ is finite valued.

The point of the remarks in 2.8 is to explain the upcoming corollary (Corollary 2.10) of Proposition 2.5. It will be convenient to precede it with some notational remarks.

Remarks 2.9. Let G be an ℓ-group. Recall that for positive a and b one writes $a \ll b$ if $na < b$ for each positive integer n. Pick $g \in G$; denote by N_g the intersection of all the values of g in $G(g)$. The reader should reflect that the values of g in $G(g)$ are precisely the maximal convex ℓ-subgroups of $G(g)$. Then it is clear that N_g is invariant under all the ℓ-automorphisms of $G(g)$, and therefore is normal in $G(g)$. It is also easy to calculate that, if G is normal valued, then, for each $g \in G$,

$$N_g = \{x \in G : |x| \ll g\}.$$

Recall that the class $\mathcal{N}$ of all normal valued ℓ-groups is (a torsion class which is) closed under extensions ([HoM79], or else [GH89], Corollary 6.4.10). Then observe that if $G \in \mathcal{O}$ then it is finite valued and, necessarily, normal valued (1.1(c)). Therefore, arguing transfinitely, each $\mathcal{O}^*$-group must be normal valued.

Corollary 2.10. *If $G \in \mathcal{O}^*$ then each $Y(g)$ is a scattered space.*

Proof. Suppose that $G \in \mathcal{O}^*$ and $0 < g \in G$. Since $Y(g)$ is homeomorphic to the Yosida space of g in $G(g)$, and $\mathcal{O}^*$ is a torsion class, we assume without loss of generality that $G = G(g)$. There is a largest ordinal α for which $\mathcal{O}^\alpha(G) \subseteq N_g$. In view of the comments in 2.9, $Y(g) \cong Y(\mathcal{O}^\alpha(G) + g)$, the latter computed in $G/\mathcal{O}^\alpha(G)$. We may then, without loss of generality, assume that there is some $0 < a \in \mathcal{O}(G) \backslash N_g$. From the remarks in 2.8(b), we may assume that a is basic, and, therefore, special. Applying Lemma 1.2, it follows that $Y(g)$ contains an isolated point.

One notes again that each closed subspace K of $Y(g)$ arises as the hull of an ℓ-ideal of G (see the proof of Proposition 2.5). Since $\mathcal{O}^*$ is a torsion class, this implies that $Y(g)$ is scattered. □

Remark 2.11. $\mathcal{O}^*$ and Hyp($\mathcal{SV}$) are related, but neither one contains the other. The proof of Corollary 2.10 gives that $\mathcal{O}^*$ consists of pseudospecial valued ℓ-groups. As to examples showing that $\mathcal{O}^*$-groups need not be hyper-special valued, and conversely, we refer the reader to §6, 6.8. Indeed, §6 should be consulted for counterexamples to a number of conjectures the reader might reasonably be expected to form.

Armed with Proposition 2.5 we have a reasonable context of discourse for the so-called higher grades of specialty.

Definition 2.12. Suppose that G is an ℓ-group and $g \in G$, with scattered Yosida space. A value N of g is said to be *α-special* provided N is an isolated point in the α-th Cantor–Bendixson derivative of $Y(g)$. As explained in 2.2, there is a largest ordinal

$\delta \equiv \delta(g)$ such that g has δ-special values, and it has finitely many of them. For each $\alpha < \delta, g$ has infinitely many α-special values. As in earlier arguments, it follows that there is an ℓ-ideal L and finitely many positive $g_1, \dots, g_k$ so that

$$L + g = \sum_{i=1}^{k} L + g_i,$$

disjointly, with each $L + g_i$ special in $G(g)/L$.

We hasten to add that, in general, the α-specialty of a value N is a relative term; there must be an element g witnessing that feature, and N could be β-special relative to another element, for some $\beta \neq \alpha$. The beauty of a hyper-special valued ℓ-group is that there the term "α-special" is unambiguous and universal.

For convenience, we shall sometimes refer to special values as 0-special values. If g has exactly one α-special value we will say that g itself is α-special. When such a g is a component of an $h \in G$ we call g an α-special component of h. Thus, summarizing, the *specialty* of a value N is an ordinal α, witnessing the fact that N is α-special with respect to a suitable $g \in G$.

3 The root system of values of a hyper-special valued ℓ-group

This section is devoted to the fine structure of the root system of values in a hyper-special valued ℓ-group, as determined by the various grades of specialty. We state the goals of this section in one main theorem (Theorem 3.1) at the outset. The proofs are mostly by transfinite induction. Our plan is to first establish each assertion of Theorem 3.1 for the ordinal 1, and then proceed to the induction.

Theorem 3.1. *Suppose that G is a hyper-special valued ℓ-group. Suppose that $\beta > 0$ is an ordinal. Then we have the following:*

(a) *Suppose that $g, h \in G$. If M is a β-special value of h and $g \notin M$, then the value N of g containing M is either special, or else a β-special value of g.*

(b) *Suppose that M, N and P are β-special values and $M \vee N \subseteq P$. Then either $M \subseteq N$ or $N \subseteq M$.*

(c) *If M is a β-special value and N is a value contained in M, then N too is β-special.*

For the rest of this section G will denote a hyper-special valued ℓ-group, unless the contrary is expressly indicated. We start on the proof of Theorem 3.1 with a general observation.

Proposition 3.2. *Suppose that $0 < g \in G$ and that M is an α-special value of g. Then g has an α-special component t of which M is the lone α-special value.*

Proof. We induct on α. There is no loss of generality in assuming that $G = G(g)$. The claim is well known for $\alpha = 0$. So suppose that for each $\lambda < \alpha$ and each λ-special value N of g, there is a λ-special component of g having N as its only λ-special value.

Let L denote the ℓ-ideal of G generated by all the λ-special components of g. Then M/L is special; so let $L + x$ be a special component of $L + g$. Without loss of generality one may assume that $0 < x \leq g$; let $y = g \quad x$. Since $x \wedge y \in L$ there are ordinals $\lambda_1, \ldots, \lambda_n$ and there is a λ_i-special component g_i of g such that $x \wedge y \leq g_1 + \cdots + g_n$. Let $\hat{g} = g_1 \vee \cdots \vee g_n$; observe that $\hat{g}$ too is a component of g. Then $t = x \wedge (g \quad \hat{g})$ is the component we are looking for; here is the reason why. First, one has disjoint sums

$$x = (x \wedge \hat{g}) + (x \wedge (g \quad \hat{g})) \quad \text{and} \quad y = (y \wedge \hat{g}) + (y \wedge (g \quad \hat{g})),$$

and since $x \wedge y \wedge \hat{g} = 0$, the following is also a disjoint sum:

$$g = u + t + (y \wedge (g \quad \hat{g})),$$

where $u \equiv a \wedge \hat{g} + b \wedge \hat{g}$. Thus, t is indeed a component of g. Now, as $t \leq x$ and x lies in any value P of g whose grade of specialty exceeds α, and also in any value $P \neq M$ of specialty α, t must have these properties as well.

To finish the argument it then suffices to prove two things: first, that M is a value of t; and second, that the specialty of M is still α relative to t. As to the first, note that

$$L + t = L + x \wedge (L + g \quad L + \hat{g}) = L + x,$$

since $\hat{g} \in L$. Thus $t \notin M$ which is enough to make M a value of t. As to the second point, consider the topology of the scattered space $Y(g)$: $h(t)$ is a clopen set in $Y(g)$; if it is $Y(g)$ then $t = g$ and there is nothing to prove. Otherwise, as $h(t)$ contains an isolated point of each of the Cantor–Bendixson derivatives which it intersects, we conclude that t has λ-special values for each $\lambda < \alpha$, and so M is an α-special value of t. □

Now that we have as many α-special components as we could want, we record an observation for later.

Lemma 3.3. *For each ordinal α such that $0 < g \in G$ has α-special values, g is the supremum of all its α-special components.*

Proof. If a is an α-special component of g and s is a special component, then $g \vee s$ is also an α-special component of g. Thus, each special component of g is bounded by an α-special component, which suffices to establish the claim, since G is special valued. □

The upcoming sequence of results establish the claims of Theorem 3.1 for 1-specialty.

Lemma 3.4. *Suppose that M is a 1-special value, and $g \notin M$. Then the value of g which contains M is either special or a 1-special value of g.*

Proof. Let P stand for the value of g containing M. Suppose P is not special. Pick $0 < h \in G$ 1-special, witnessed by M. We may replace h by $h \wedge g$ and then use Proposition 3.2 to produce a 1-special component of $h \wedge g$, whose lone 1-special value is M. Thus, without loss of generality, $h \leq g$. In the same vein, we may assume that $G = G(g)$.

Let L be the ℓ-ideal of G generated by all the special components of g. Observe that every special component of g is in M, whence $L \subseteq M$, and, in particular, $h \notin L$. On the other hand, since P is not special, every special component of h is also in L, and so it follows that M/L is a special value (of $L + h$), and then also P/L is special, because G/L is hyper-special valued. By definition, P is a 1-special value of g. □

We state a corollary which settles for 1-specialty, that the concept is, for hyper-special valued ℓ-groups, independent of the element that exhibits it.

Corollary 3.5. *If M is a 1-special value of h and also a value of g, then M is a 1-special value of g.*

The next lemma will be generalized later.

Lemma 3.6. *Suppose that $M \subseteq N$ are 1-special values, and $0 < g \leq h$ are 1-special elements witnessing this at M and N, respectively. Then g and h have all but finitely many of their special values in common.*

Proof. Suppose first that h has infinitely many special values which are incomparable to any special value of g. Let L be the ℓ-ideal of $G(h)$ generated by the special components of h which do contain a special value of g. In $G(h)/L$, $(M \vee L)/L$ is special. But then $L + h$ has infinitely many values, all special, which is impossible (see 1.1(c)). We conclude that h has at most finitely many special values which do not contain a special value of g.

To show that at most a finite number of special values of h contain one of g properly, let A be the ℓ-ideal of $G(h)$ generated by all the special components of g. Once again $(M \vee A)/A$ is special. But if there is an infinite number of special values of h properly containing one of g, then $A + h$ has infinitely many values, all special, again a contradiction. □

Lemma 3.7. *No 1-special value of G contains two incomparable 1-special values.*

Proof. Suppose that M_1, M_2 and N are 1-special values of G, and that $M_1 \| M_2$, while both are contained in N. Select positive g_1, g_2 and h, all 1-special, such that $g_1 \vee g_2 \leq h$ and having their unique 1-special value at M_1, M_2 and N, respectively. Without loss of generality, we may suppose that $g_1 \wedge g_2 = 0$. As none of these elements is special, each one has infinitely many special values, and therefore both g_1 and g_2 have infinitely many values which do not coincide with values of h. That contradicts Lemma 3.6. □

Lemma 3.8. *If M is a 1-special value of G and it contains the value N, then N too is 1-special.*

Proof. By way of contradiction, let us assume that the specialty of N is $\alpha > 1$. Suppose that $h > 0$ witnesses the 1-specialty of M, and that $g > 0$ does the same for the α-specialty of N.

If all the special values of g lie below a finite number of special values of h, then for a suitable sum h' of special components of h, there is a multiple nh' which exceeds all the special components of g and hence g itself since G is special valued. This is impossible, because $N \subseteq M$. Thus, there are infinitely many special components of h which contain a special component of g.

Next, since $\alpha > 1, g$ has infinitely many γ-special values, for each $\gamma < \alpha$. This is true, in particular, for $\gamma = 1$. Now at most one 1-special value of g can be contained in M, by Lemma 3.7, and so infinitely many 1-special components of g lie under special components of h. What is important about this is that there are infinitely many special values which contain values of g properly. Therefore, if we let L denote the ℓ-ideal of $G(h)$ generated by all the components of g of specialty $< \alpha$, we get that $L + g$ is special with value $(N \vee L)/L$, while $(M \vee L)/L$ is not special. This cannot happen since $G(h)/L$ is special valued. □

As an intermediate result let us summarize the information in Lemmas 3.4, 3.7 and 3.8:

Proposition 3.9. *The set* $\mathrm{Val}_1(G)$ *of all 1-special values of G is a disjoint union of chains which are order ideals in* $\mathrm{Val}(G)$. *Only special values may contain a 1-special value.*

Remark 3.10. We have verified Theorem 3.1 for the case $\beta = 1$. We now assume that $\beta > 1$ and that all three claims made there are true for all ordinals $1 \leq \lambda < \beta$. In the subsequent propositions we use this three-pronged hypothesis to verify one by one that the claims of Theorem 3.1 hold for β.

Proposition 3.11. *The claim in Theorem 3.1(a) holds for β.*

Proof. Suppose that M is a β-special value of G and the value N properly contains M. Assume that the specialty of N is $\gamma \geq 1$ (for a suitable element). Then the inductive hypothesis for Theorem 3.1(c) dictates that $\beta \leq \gamma$. Suppose, by way of contradiction, that $\beta < \gamma$. Without loss of generality we may assume that $g > h > 0$, with g is γ-special, witnessed by N, while h is β-special, exhibited by M. If $1 \leq \alpha < \beta$ and P is an α-special value of h, the value of g in which it is contained is special, or else α-special, by the induction hypothesis for Theorem 3.1(a). This implies that any value of g of specialty β or higher contains no values of h. As in earlier proofs, let L be the ℓ-ideal of $G(g)$ generated by all the α-special components of h, for all $\alpha < \beta$. In $G(g)/L$ then, $L + h$ is special with value $(M \vee L)/L$, while $(N \vee L)/L$ is not special. As before, this is a contradiction. We are forced to conclude that $\beta = \gamma$. □

Proposition 3.12. *The claim in Theorem 3.1(b) holds for β.*

Proof. We set things up: by way of contradiction, suppose there are β-special values M_1, M_2 and N, so that $M_1 \| M_2$ and both of these are contained in N. Pick positive g_1, g_2 and h, witnessing α-specialty at M_1, M_2 and N, respectively, and such that $g_1 \vee g_2 \leq h$. From here the proof consists of a number of claims.

Every value of $g_1 \wedge g_2$ has specialty $< \beta$.
For suppose that P is a value of $g_1 \wedge g_2$. It is contained in values Q_1 and Q_2 of g_1 and g_2, respectively. It should be clear that $M_i \neq Q_i$ for both $i = 1, 2$. Thus the specialty of Q_1 and Q_2 is $< \beta$. By the induction hypothesis of Theorem 3.1(c), it follows that the specialty of P too is $< \beta$.

Without loss of generality $g_1 \wedge g_2 = 0$.
If not already so, then as explained in 2.12 there is a largest ordinal $\delta < \beta$ such that every value of $g_1 \wedge g_2$ has specialty not exceeding δ, and $g_1 \wedge g_2$ has only a finite number of δ-special values. Thus, $g_1 \wedge g_2$ is a disjoint sum of δ-special components a_i $(i = 1, \ldots, n)$. It is then not hard to see that each a_i is a δ-special component of g_1 or g_2, so that subtracting a_i from the appropriate g_j $(j = 1, 2)$, one obtains a pair of elements with the same specifications as g_1 and g_2, now disjoint.

There are infinitely many values of h of specialty $< \beta$ which contain values of g_1 (necessarily of the same specialty, by the induction hypothesis regarding Theorem 3.1(c)), and likewise for g_2.
Suppose the contrary, that, say, all the values of g_1 of specialty $< \beta$ are contained in a finite number of values of h of specialty $< \beta$. By Lemma 3.3, this means that there is a component a, with no β-special values which exceeds g_1. This contradicts that $M_1 \subseteq N$, and the claim is established.

Arguing now as in the proof of Lemma 3.8, we are able to deduce that there are infinitely many values of h of specialty $< \beta$ which properly contain values of g_1 of equal specialty. A similar statement is true for g_2 as well. Now taking the ℓ-ideal L of $G(h)$ generated by all the α-components of g_1 for $\alpha < \beta$, then $L + g_1$ is special with value $(M_1 \vee L)/L$, while $(N \vee L)/L$ is not special. □

Proposition 3.13. *The claim in Theorem 3.1(c) holds for β.*

Proof. Suppose that N is a β-special value, which is exhibited by the β-special element $h > 0$. Suppose that the value M is contained in N, and has specialty α. By what we've done so far it follows that $\alpha \geq \beta$. Let us assume, by way of contradiction, that $\alpha > \beta$. Pick a positive g, α-special, which witnesses the α-specialty of M. If P is any value $h, P \neq N$, then any value of g contained in it must have the same specialty. This means that if $\beta \leq \delta \leq \alpha$, then every δ-special value of g must be contained in N. On the other hand (applying this to $\delta = \beta$), g has infinitely many β-special values, but only one may lie under N, because of Proposition 3.12. This is absurd, and so we must conclude that $\alpha = \beta$, thus proving this proposition. □

With Propositions 3.11, 3.12 and 3.13, Theorem 3.1 is now proved. □

For emphasis, we note the following formulation of Theorem 3.1.

Corollary 3.14. *For each ordinal* $\alpha \geq 1$, *let* $\mathrm{Val}_\alpha(G)$ *denote the set of all* α*-special values of G. Then* $\mathrm{Val}_\alpha(G)$ *is an order ideal and a disjoint union of chains, and only special values may contain a value in* $\mathrm{Val}_\alpha(G)$.

To conclude the section, let us record an observation which, by now, is obvious from Theorem 3.1(a).

Corollary 3.15. *Suppose that M is a* β*-special value of h and also a value of g. Then M is a* β*-special value of g.*

4 Projectable hyper-special valued ℓ-groups

In this section we consider the structure of projectable hyper-special valued ℓ-groups, and apply the main structure theorem (Theorem 4.2 below) to the Yosida representation of archimedean ℓ-groups with weak order unit.

Before proceeding, let us first review some issues relating to polars and projectable ℓ-groups.

Definition & Remarks 4.1. (a) Suppose that G is an arbitrary ℓ-group. For any subset X of G we denote

$$X^\perp = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}.$$

It is well known that $X^\perp \in \mathscr{C}(G)$, and if $A \in \mathscr{C}(G)$, then $A^\perp$ is the largest convex ℓ-subgroup of G such that $A \cap A^\perp = \{0\}$. We call $A \in \mathscr{C}(G)$ a *polar* of G if it is of the form $A = X^\perp$, for some $X \subseteq G$. Note that A is a polar if and only if $A = A^{\perp\perp}$. The set $\mathscr{P}(G)$ of all polars of G is a complete boolean algebra under inclusion, which is complete meet subsemilattice of $\mathscr{C}(G)$ (Theorem 13.7, [D95]). For each $g \in G$, $g^\perp$ stands for $\{g\}^\perp$, and $g^{\perp\perp}$ for $\{g\}^{\perp\perp}$.

(b) The ℓ-group G is *projectable* if $G = g^{\perp\perp} + g^\perp$, for each $g \in G$. It is well known that if G is projectable then the root system of prime subgroups of G is a disjoint union of chains; equivalently, each prime subgroup contains a unique minimal prime subgroup (Theorem 18.1, [D95]).

We characterize projectable hyper-special valued ℓ-groups as follows. For each ordinal α (including 0), recall that $\mathrm{Val}_\alpha(G)$ stands for the set of all α-special values of G.

Theorem 4.2. *Suppose that G is hyper-special valued. Then the following are equivalent.*

(a) *G has a basis, and for each basic element* $b > 0, G = b^{\perp\perp} + b^\perp$.

(b) $\mathrm{Val}_0(G)$, *the set of special values of G, is an order ideal which is a disjoint union of chains.*

(c) *G is projectable.*

In particular, if G is hyper-special valued and projectable then it is hyper-projectable.

Proof. (a) $\Rightarrow$ (b) Since $G = b^{\perp\perp} + b^{\perp}$ for each basic $b > 0$, and G has a basis, we get that every special element is basic: for if $s > 0$ is special but not basic, then we may choose $0 < a, b < s$ such that $a \wedge b = 0$, and, indeed, both a and b are basic. But then $s \notin a^{\perp\perp} + a^{\perp}$, a contradiction. This shows that $\mathrm{Val}_0(G)$ is a disjoint union of chains, and, with a little more thought, that no special value can contain a non-special value. We leave the verification to the reader. (Note: this implication has nothing to do with G being hyper-special valued.)

(b) $\Rightarrow$ (c) We take positive g and h and proceed to show that $g \in h^{\perp\perp} + h^{\perp}$. We do this by induction on the specialty of h. First, a few preliminary reductions. There is a largest ordinal α such that h has α-special values, and it has finitely many of those (2.12). So we may write h as a disjoint sum of its α-special components. If for each of those components $h', g \in h'^{\perp\perp} + h'^{\perp}$, then a routine bit of ℓ-group arithmetic yields that $g \in h^{\perp\perp} + h^{\perp}$. Thus, we may assume that h is α-special; let M_h be its single α-special value. Likewise, it is enough to assume that g is β-special, and we denote its β-special value by M_g. Finally, we may also suppose that $G = G(g)$.

Suppose h is special. If $g \wedge h = 0$ there is nothing to prove. Thus $g \wedge h > 0$. Then M_h is comparable to a value N of g, which by the assumption of (b) is unique and special. Let a be the special component of g with value N. It is then an easy exercise to check that $a \in h^{\perp\perp}$, while $g \;\; a$ is clearly disjoint to h. Thus $g \in h^{\perp\perp} + h^{\perp}$, and this proves the claim for $\alpha = 0$.

Now assume that $G = z^{\perp\perp} + z^{\perp}$ whenever every value of z has specialty $< \alpha$. The reader will readily convince himself that it suffices to assume that $h \leq g$. Thus also, $\alpha \leq \beta$, by Theorem 3.1. There is a value of g containing M_h, say P, and it is necessarily α-special. Pick an α-special component a of g with lone α-special value P. Let us consider $g \;\; a$; none of its α-special values is comparable to M_h. This means that $x \equiv h \wedge (g \;\; a)$ has no α-special values. By the inductive hypothesis, $g \;\; a \in x^{\perp\perp} + x^{\perp}$; that is,

$$g \;\; a = g_1 + g_2, \quad \text{with} \quad g_1 \in x^{\perp\perp} \subseteq h^{\perp\perp} \quad \text{and} \quad g_2 \wedge x = 0.$$

But then $g_2 \wedge h = 0$ as well, proving that $g \;\; a \in h^{\perp\perp} + h^{\perp}$.

The above calculation demonstrates that it suffices to consider the case in which $\alpha = \beta$, with $M_h \subseteq M_g$ (and $g = a$). Let L be the ℓ-ideal of G generated by all the components of h of specialty $\gamma < \alpha$. Then $L + h$ is special with value M_h/L, which means that M_g/L is special as well, since G is hyper-special valued. We choose a special component $L + s$ of $L + g$ with value M_g/L. Without loss of generality we may take $h \leq s \leq g$.

Suppose that Q is a value of s. If $L \subseteq Q$ then $Q = M_g$. If $L \nsubseteq Q$, then some γ-special component $\bar{h}$ of h ($\gamma < \alpha$) fails to be in Q. But then Q is contained in some value of h, which, in view of $h \leq s$, means that Q is a value of h. The upshot is that for each $\gamma < \alpha$, the γ-special values of h and s are identical. This implies that $s^{\perp} = h^{\perp}$, as the reader will readily verify. Next, consider $u = s \wedge (g \quad s); u \in L$, so that u has no α-special values. By the inductive hypothesis, $g \quad s \in u^{\perp\perp} + u^{\perp}$, which in turn implies that $g \quad s = c + d$, with

$$c \in u^{\perp\perp} \subseteq s^{\perp\perp} = h^{\perp\perp},$$

and $0 = d \wedge u = d \wedge s$, whence $d \in h^{\perp}$. Conclusion: $g \quad s \in h^{\perp\perp} + h^{\perp}$ and $s \in h^{\perp\perp}$, and hence $g \in h^{\perp\perp} + h^{\perp}$. This proves that (b) implies (c).

That (c) implies (a) should be clear, as, if G is projectable, then $\mathrm{Val}(G)$ is a disjoint union of chains. From this it readily follows that each special element is basic. This suffices for (a).

As to the final claim of the theorem, if G is projectable and hyper-special valued, and L is an ℓ-ideal of some $G(a)$, then $G(a)/L$ is again hyper-special valued, and (b) of the theorem is also satisfied. Thus, G is hyper-projectable. □

The following result is extracted from the proof of Theorem 4.2. It is technical and yet, somehow, fascinating. In our view, it encapsulates the thinking of the preceding proof. For a time it also seemed essential to proving Theorem 4.8, ahead in this section.

Lemma 4.3. *Suppose that G is a projectable, hyper-special valued ℓ-group and $0 < h \leq g$, both α-special, with unique α-special value $M_h \subseteq M_g$. Assume as well that $g^{\perp} = h^{\perp}$. Let L be the ℓ-ideal of $G(g)$ generated by all the components of h of specialty $\gamma < \alpha$. Then there exists $h \leq s \leq g$ such that*

(a) *$L + s$ is a special component of $L + g$ with value M_g/L;*

(b) *for each $\gamma < \alpha$, the γ-special value of h and s agree, and $s^{\perp} = h^{\perp}$;*

(c) *let $u = s \wedge (g \quad s)$; then $u \in L$, and u has no α-special values;*

(d) *$(g \quad s)^{\perp} = u^{\perp}$.*

Proof. (a), (b) and (c) were established in the foregoing proof. For (d), it suffices to show that $u^{\perp} \subseteq (g \quad s)^{\perp}$. Note that since $g^{\perp} = h^{\perp}$, we also have $g^{\perp} = s^{\perp}$. Thus, if $x \wedge u = 0$, then $x \wedge g \wedge (g \quad s) = 0$; that is, $x \wedge (g \quad s) = 0$. □

We immediately have the following corollary to Theorem 4.2.

Corollary 4.4. *If G is archimedean and hyper-special valued then it is hyper-projectable.*

Proof. Every special element of an archimedean ℓ-group is basic (Proposition 53.16, [D95]). Thus, if G is archimedean and special valued then it has a basis, and so (a) of Theorem 4.2 holds (Theorem 19.16, [D95]). The claim then follows from Theorem 4.2. □

Remark 4.5. In the class of hyper-projectable ℓ-groups some of the features of hyper-special valued ℓ-groups which were set forth in some detail in §3 can be recovered. This will discussed in [DM∞].

Example 4.6. The converse of Corollary 4.4 is false, even for archimedean ℓ-groups. Any hyper-archimedean ℓ-group is projectable, and therefore hyper-projectable. For example, consider the ℓ-group of all periodic real sequences. It is hyper-archimedean but has no special elements.

For the next observation we return to the class $\mathcal{O}^*$ of §2, for which we refer the reader to 2.8(a). A converse of Corollary 4.7 turns up in Theorem 5.4, later on in the article.

Corollary 4.7. *If G is projectable and hyper-special valued then it is also in* $\mathcal{O}^*$.

Proof. As spelled out by Theorem 4.2, every special element of such a group is basic, and, moreover, $\mathcal{O}(G)$ is precisely the subgroup generated by all the special elements. In $G/\mathcal{O}(G)$ the special elements are the cosets represented by 1-special elements of G. According to Theorem 3.1, these are all basic, whence $\mathcal{O}(G/\mathcal{O}(G))$ is the subgroup generated by the special elements of $G/\mathcal{O}(G)$. The reader may now easily complete the transfinite induction. $\square$

In projectable ℓ-groups, hyper-special valuedness implies more finely wrought structure. This is the subject of our next theorem.

Theorem 4.8. *Suppose that G is a projectable, hyper-special valued ℓ-group and* $0 < h \le g$, *both α-special, with respective α-special values $M_h \subseteq M_g$. Assume as well that $g^\perp = h^\perp$. Then, if $\alpha > 1$, g and h have all but finitely many of their γ-special values (all $\gamma < \alpha$) in common.*

Proof. First observe that, since $g^\perp = h^\perp$, each value of g of specialty $\gamma < \alpha$ must contain a value of h (which, necessarily, is of the same specialty.) This uses projectability, which implies that $Y(g)$ and $Y(h)$ are homeomorphic via the map that assigns to a value V of h the value of g which contains it.

Now suppose that there are infinitely many values of g which contain a value of h properly; they are all of specialty $\gamma < \alpha$. We may suppose that α is the least ordinal for which this happens. Else, let β be the first ordinal with the feature that there are infinitely many values of g of specialty less than β, containing a value of h properly. Choose a β-special component g_o of g, and let h_o be the projection of h on g_o. Then g_o and h_o satisfy the hypotheses of the theorem and we may proceed with them. Thus, we suppress the subscripts and assume α is least, as explained here.

It will aid our exposition to denote by T the set of ordinals γ with the property that some γ-special value of g properly contains a value of h. We claim that $\alpha = \bigvee T$. If not, let δ be this supremum. Let Z be the closure in $Y(g)$ of the set of all γ-special values of g, over $\gamma \in T$. Let L be the ℓ-ideal of $G(g)$ generated by the all the components of h of lesser specialty than α. Next, observe that $Y(L + g)$ is canonically homeomorphic to the closed subspace $K = Z \cup \{M_g\}$, and since we are supposing that $\delta < \alpha$, it follows that M_g is isolated in K. Since Z is scattered and δ indexes the last nontrivial member of its Cantor–Bendixson sequence, we have that there is a δ-special component g' of g, and by projection of h upon g', a δ-special component h' of h below it. Note now that g' and h' satisfy all the hypotheses of the theorem, with $\delta < \alpha$. This contradicts the minimality of α, and so our claim is established.

But now, returning to $G(g)/L$, we are able to conclude that M_g/L is not isolated in $Y(L + g)$, and hence not special, whereas M_h/L (beneath it) is special, thus contradicting the fact that $G(g)/L$ is special valued. From this the conclusion of the theorem follows. □

Theorem 4.8 has an important consequence for hyper-special valued archimedean ℓ-groups with a weak order unit, in terms of the Yosida representation of the group. We discuss that and other aspects of archimedean hyper-special valued ℓ-groups in the next section.

5 Archimedean hyper-special valued ℓ-groups

We begin by briefly recalling the elements of the Yosida representation. For amplification we recommend [HR77].

Definition & Remarks 5.1. Suppose that X is a compact Hausdorff space. We let $D(X)$ stand for the set of all continuous functions f on X with values in the extended real numbers, so that $f^{\ 1}(\mathbb{R})$ is a dense (open) set. Taking pointwise suprema and infima, $D(X)$ becomes a distributive lattice. However, only under special circumstances is $D(X)$ a group under pointwise addition. Observe though that, if $f \in D(X)$ then $\ f$ is as well. In any event, a subset $H \subseteq D(X)$ is *an ℓ-group in* $D(X)$ if H is an ℓ-group so that for each $f, g \in D(X), (f + g)(x) = f(x) + g(x)$, whenever $x \in f^{\ 1}(\mathbb{R}) \cap g^{\ 1}(\mathbb{R})$.

Suppose that G is an archimedean ℓ-group with a weak order unit $u > 0$. Set $Y = Y(u)$. It is convenient to follow the following convention: technically, a point $y \in Y$ is a value of u; in practice, however, we carry a bijection $y \mapsto V_y$, signifying that when we think of y as a point in a topological space in the context of a representation, we shall use the lower case y; when thinking of it as a value of u, we shall employ the subscripted V_y.

There is an ℓ-isomorphism $\phi : G \to G'$ onto an ℓ-group G' in $D(Y)$, such that $1 \in G'$ and $\phi(u) = 1$. Moreover, ϕ separates points, in the sense that if $g \neq 0$ in G, then $g(y) \neq 0$, for a suitable $y \in Y$. Finally, Y is canonical, in the following sense: if $\theta : G \to H$ is an ℓ-isomorphism onto an ℓ-group in $D(X)$, with X compact and Hausdorff, which separates points, then there is a homeomorphism $\tau : X \to Y$ such that for each $f \in G$,

$$\phi(f)(\tau(x)) = \theta(f)(x),$$

for each $x \in X$.

Recall that if Y is scattered and G is represented on Y as described here, then each $\phi(f)(Y)$ is countable (see 2.1).

And now let us record some preliminaries for Theorem 5.3 below.

Remark 5.2. Suppose that f is an extended real valued continuous function on a topological space X. Recall that $Z(f)$ stands for the *zeroset* of f, that is,

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The complement, $\mathrm{coz}(f)$, is the *cozeroset* of f. Now, we call $p \in Z(f)$ an *accumulated zero* of f if $p \in \mathrm{cl}(\mathrm{coz}(f))$. As one might guess, if g is continuous on X, with values in the extended real numbers, then $p \in X$ is an *infinity* of g if $g(p) = \pm\infty$.

The statement of Theorem 5.3 suppresses the ℓ-isomorphism in the Yosida representation.

Theorem 5.3. *Suppose that G is a hyper-special valued archimedean ℓ-group with designated weak order unit $u > 0$, and let $Y = Y(u)$. Then G is ℓ-isomorphic to an ℓ-group in $D(Y)$, such that each $f \in G$ has finitely many infinities and finitely many accumulated zeroes.*

Proof. Suppose that $f \in G$; then $p \in Y$ is an infinity of f if and only if p corresponds to a value V_p of u for which $f \notin V_p^*$. (Clearly such a V_p is not special.) Theorem 4.8 guarantees that there are at most a finite number of such points.

As to the accumulated zeroes, it suffices to consider $f > 0$: there is an $0 \le f' \in G$ such that $f \wedge f' = 0$ and $f + f'$ is a weak order unit. To see this, use the projectability of G and let f' be the projection of u disjoint from f. Now, Theorem 4.8 insures that in the Yosida representation (which identifies u with 1), $f + f'$ has finitely many zeroes, as those points p correspond to values such that $f + f' \in V_p$. They are necessarily accumulated zeroes of $f + f'$. We claim that every accumulated zero of f is one of these.

For suppose the contrary; that is, suppose $y \in Y$ is an accumulated zero of f, yet $f'(y) > 0$. Then for a suitable neighborhood U of y, $f'(U) > 0$, and so $f(U) = \{0\}$, which contradicts the assumption that $y \in \mathrm{cl}(\mathrm{coz}(f))$. □

Next, in this section, we turn to the converse of Corollary 4.7, promised earlier.

Theorem 5.4. *Suppose that G is a projectable ℓ-group with weak order unit $u > 0$, which is in $\mathcal{O}^*$, such that for each $g \in G$ all but finitely many values of g are also values of u. Then G is hyper-special valued.*

Proof. Observe at the outset that, by Corollary 2.10, every Yosida space is scattered.

We must show that, for each $g \in G$ and each ℓ-ideal L of $G(g)$, $G(g)/L$ is special valued. In this we may assume, without loss of generality, that $g > 0$. Next, observe that we may write g as

$$g = g_1 + \cdots + g_k, \quad \text{with} \quad g_i \wedge g_j = 0 \quad \text{for } i \neq j,$$

so that only one value of g_i differs from a value of u. By reduction we have

$$G(g)/L = \bigvee_{i=1}^{k} (G(g_i) \vee L)/L,$$

which makes it clear that it is sufficient to prove the claim for each component. Thus, we may assume that only one value of g is not a value of u.

Here is another reduction: project u on g; we write $u = u_1 + u_2$, with $u_1 \in g^{\perp\perp}$ and $u_2 \wedge g = 0$. We have that $G = u_1^{\perp\perp} \oplus u_2^{\perp\perp}$, and the same assumptions of the theorem apply to $u_1^{\perp\perp}$. As $g \in u_1^{\perp\perp}$, we may as well assume that g is a weak order unit too, and then $Y(g) = Y(u)$. For the rest of the argument then, $u = u_1$.

Suppose now, by way of contradiction, that $G(g)/L$ is not special valued. Then there exist $0 < x < y \in G(g)$ having values V_x and, respectively, V_y, such that $L \subseteq V_x \subset V_y$, with V_x/L special, but V_y/L not. Our standing assumption implies that $Y(y)\backslash Y(x)$ is finite. By taking suitable components, as in the preceding reductions, we may assume that $|Y(y)\backslash Y(x)| = 1$. Now, as V_x/L is special, it follows that each component of x of lower specialty than that of V_x lies in L. The same is then true of the components of lower specialty of y. But then V_y/L is special, a contradiction. □

Our final major result spells out when the necessary conditions in Theorem 5.3 suffice. To set up this theorem (Theorem 5.7), it will behoove us to review the $\mathcal{O}^*$-groups from the vantage point of the space of minimal prime subgroups. Several remarks are in order.

Definition & Remarks 5.5. In these comments G stands for an arbitrary ℓ-group.

(a) An elementary Zorn's Lemma argument shows that every prime subgroup contains a minimal prime subgroup (Theorem 9.6, [D95]). We let $\mathrm{Min}(G)$ denote the set of minimal prime subgroups of G. When we refer to the *minimal prime subgroup space* we mean $\mathrm{Min}(G)$ with the induced hull-kernel topology. $\mathrm{Min}(G)$ is always a zero-dimensional space. As a reference for this and other remarks made here about $\mathrm{Min}(G)$ we suggest [CM90]. According to Theorem 2.2 in [CM90], $\mathrm{Min}(G)$ is compact if and only if G is *complemented*; that is, to each $g \in G$ there is an $0 \leq h \in G$ such that $|g| \wedge h = 0$ and $|g| + h$ is a weak order unit. In this situation, h is called a *complement* of g.

(b) The reader will easily verify that a projectable ℓ-group with a weak order unit is complemented. If G has the feature that each prime subgroup contains a unique minimal prime subgroup, we say that G has *stranded primes*. Note that the class of ℓ-groups with stranded primes is a torsion class. Also, every projectable ℓ-group has stranded primes (Corollary 18.2, [D95]). Conversely, if G is complemented and has stranded primes, then it is projectable. This is easy to prove, and we leave it to the reader.

(c) We examine the topology of $\mathrm{Min}(G)$ a bit more closely. The closed sets are of the form

$$V(A) = \{P \in \mathrm{Min}(G) : A \subseteq P\},$$

and we may assume that A is an intersection of minimal prime subgroups. Recall that if N is a prime subgroup, then N is minimal precisely when $a \in N$ implies that $a^{\perp\perp} \subseteq N$ (Proposition 15.1, [D95]). A convex ℓ-subgroup with this feature is called a *z-subgroup*. (The reader is cautioned that some authors refer to such a convex ℓ-subgroup as a *d-ideal*.) It is clear that each $A \in \mathscr{C}(G)$ which is an intersection of minimal prime subgroups is a z-subgroup. When G is complemented the converse is true (§ 3, [CM90]).

Here is a proposition which characterizes projectable $\mathcal{O}^*$-groups in terms of the minimal prime subgroup space.

Proposition 5.6. *Suppose G is a projectable ℓ-group with weak unit. Then $G \in \mathcal{O}^*$ if and only if* $\mathrm{Min}(G)$ *is scattered.*

Proof. For use in this proof we introduce, for each ordinal α,

$$M_\alpha \equiv \{P \in \mathrm{Min}(G) : \mathcal{O}^\alpha(G) \subseteq P\}.$$

This is a closed subset of $\mathrm{Min}(G)$.

The proof depends on establishing the following claim:

$\mathcal{O}(G)$ is a z-subgroup and $G/\mathcal{O}(G)$ is again complemented.

For if $0 < a \in \mathcal{O}(G)$ then a is a sum of pairwise disjoint basic elements a_i $(1 \le i \le k)$. But then, as the reader will readily verify,

$$a^{\perp\perp} = \bigvee_{i=1}^{k} a_i^{\perp\perp},$$

and since each $a_i^{\perp\perp}$ is a totally ordered group (Theorem 19.1, [D95]), it follows that $a^{\perp\perp} \subseteq \mathcal{O}(G)$.

Next, as is established in [CM90], §3, if A is a z-subgroup of G and P is a prime subgroup, minimal with respect to containing A, then $P \in \mathrm{Min}(G)$. This means that $\mathrm{Min}(G/\mathcal{O}(G))$ is canonically homeomorphic with M_1. Being a closed subset of a compact space, it follows that $\mathrm{Min}(G/\mathcal{O}(G))$ is compact, which means that $G/\mathcal{O}(G)$ is complemented. This is obviously the first leg of an induction argument.

Continuing, suppose that $\beta > 1$ is an ordinal and that, for each $\alpha < \beta$, $\mathcal{O}^\alpha(G)$ is a z-subgroup. If β is a limit ordinal then $\mathcal{O}^\beta(G) = \bigcup_{\alpha<\beta} \mathcal{O}^\alpha(G)$, and any union of a chain of z-subgroups is a z-subgroup. Otherwise, if $\beta = \alpha + 1$, then one may prove that $G/\mathcal{O}^\alpha(G)$ is complemented, as shown in the preceding paragraph. This insures (see 5.5(b)) that $G/\mathcal{O}^\alpha(G)$ is projectable. Now

$$\mathcal{O}^\beta(G)/\mathcal{O}^\alpha(G) = \mathcal{O}(G/\mathcal{O}^\alpha(G)),$$

which is an intersection of minimal prime subgroups $P_i/\mathcal{O}^\alpha(G)$ $(i \in I)$ of $G/\mathcal{O}^\alpha(G)$. As $\mathcal{O}^\alpha(G)$ is a z-subgroup of G, each P_i is minimal among prime subgroups of G, proving that $\mathcal{O}^\beta(G)$ is a z-subgroup.

For each basic element $a, a^\perp$ is the unique minimal prime subgroup not containing a. Conversely, 2.6 of [CM90] tells us that an isolated point of $\mathrm{Min}(G)$ is the polar of a basic element. This makes it clear that M_1 is the first Cantor–Bendixson derivative of $\mathrm{Min}(G)$. Now suppose that it has been shown for all $\alpha < \beta$, with $\beta > 1$, that M_α is the α-th Cantor–Bendixson derivative. If β is a limit ordinal then it should be clear that $M_\beta = \bigcap_{\alpha<\beta} M_\alpha$, and so M_β is the β-th Cantor–Bendixson derivative, by definition. If β is a successor ordinal, say $\beta = \alpha + 1$, then the matter is settled by the fact that

$$\mathcal{O}^\beta(G)/\mathcal{O}^\alpha(G) = \mathcal{O}(G/\mathcal{O}^\alpha(G)),$$

and again by definition of the Cantor–Bendixson derivatives.

The proposition is now easily proved: $G \in \mathcal{O}^*$ precisely when $G = \mathcal{O}^\beta(G)$ for some β, which happens if and only if $M_\beta = \emptyset$. □

We have what we need for the following result.

Theorem 5.7. *Suppose that G is archimedean with weak order unit $u > 0$. If G is also projectable, $Y(u)$ is scattered, and, in the Yosida representation, each $g \in G$ has finitely many infinities and accumulated zeroes, then G is hyper-special valued.*

Proof. We identify G with its image under the Yosida representation on $Y = Y(u)$. The hypothesis about zeroes and infinities implies that, for each $g \in G$, there are only finitely many values of g which aren't values of u. All that's left to do is apply Proposition 5.6 to conclude that $G \in \mathcal{O}^*$, and then by Theorem 5.4 this theorem is proved.

To conclude that $G \in \mathcal{O}^*$ observe that, since G is projectable, $\mathrm{Min}(G)$ is homeomorphic to Y under the map which associates with a minimal prime subgroup P the value of u that lies above it. Thus $\mathrm{Min}(G)$ is scattered and Proposition 5.6 applies. □

The remainder of this section consists of a few observations about particular archimedean hyper-special valued ℓ-groups, and also about the torsion radical of $\mathrm{Hyp}(\mathcal{SV})$. Let us begin by applying Theorems 5.3 and 5.7 to integer valued continuous functions.

Definition & Remarks 5.8. We assume that X is a zero-dimensional space. $C(X, \mathbb{Z})$ denotes the ring of all continuous integer valued functions defined on X. It is also an ℓ-group under pointwise operations. It is well known that $C(X, \mathbb{Z})$ is always projectable. $\mathrm{Min}(C(X, \mathbb{Z}))$ is homeomorphic to the so-called Banaschewski compactification of X, usually denoted $\beta_0 X$. (See §4.7 of [PW89].) Like the Stone–Čech compactification, $\beta_0 X$ has a universal property, namely, that for each continuous map $f : X \to T$, where T is any compact zero-dimensional Hausdorff space there is a unique extension $\beta_0(f) : \beta_0 X \to T$ of f.

For X compact, $C(X, \mathbb{Z})$ is a *Specker group*; let us explain. An element $s > 0$ in an ℓ-group G is *singular* if $x \wedge (s \quad x) = 0$, for each $0 \leq x \leq s$. G is a *Specker* group if it is generated as an ℓ-group by its singular elements. If this is so, then it can be shown that G is, in fact, generated as a *group* by its singular elements (Proposition 55.11, [D95]). Note that each Specker ℓ-group is hyperarchimedean (Proposition 55.10, [D95]), and that the Specker ℓ-groups form a torsion class (Proposition 55.15, [D95]). Now if G is a Specker ℓ-group with a singular order unit u, then it can be shown that $G \cong C(\mathrm{Min}(G), \mathbb{Z})$, so that we may legitimately view the class of Specker ℓ-groups as the class of ℓ-groups $C(X, \mathbb{Z})$, with X compact and zero-dimensional.

In the context of this section we have the following. Recall that a topological space X is *pseudocompact* if each real valued continuous function from X is bounded.

Proposition 5.9. *If $C(X, \mathbb{Z})$ is hyper-special valued, then $\beta_0 X$ is a scattered space. This implies that X is pseudocompact and that $C(X, \mathbb{Z})$ is a Specker ℓ-group.*

Proof. By Corollary 4.7, $C(X, \mathbb{Z}) \in \mathcal{O}^*$, whence, from Proposition 5.6, $\mathrm{Min}(C(X, \mathbb{Z})) \cong \beta_0 X$ is scattered. Now a compact scattered space cannot contain a copy of $\beta\omega$, the Stone–Čech compactification of the discrete natural numbers. Thus, X must be pseudocompact, otherwise, by Corollary 1.21, [GJ76], X contains a C-embedded copy of ω, and $\beta_0 X$ therefore contains a copy of $\beta\omega$. (Also relevant here is 5G(4) of [PW89].) Finally, since X is pseudocompact, every $f \in C(X, \mathbb{Z})$ is bounded and, therefore, $C(X, \mathbb{Z})$ is generated by its characteristic functions. Conclusion: $C(X, \mathbb{Z})$ is Specker. □

For an arbitrary Hausdorff space X, we now investigate the analogue of Proposition 5.9 for $C(X)$, the ring of all continuous real valued functions on X. $C(X)$ is an archimedean ℓ-group with respect to the pointwise operations of sum, supremum and infimum. We begin with some preliminaries. Our general reference for rings of continuous functions is [GJ76]. All unexplained terminology may be found there.

Definition & Remarks 5.10. (a) As explained in Chapter 3 of [GJ76], when dealing with $C(X)$ one may always assume—as we do here—that the space X is *Tychonoff*: that is to say, Hausdorff, such that for each closed set K and each $p \notin K$ there is an $f \in C(X)$ such that $f(p) = 0$ and $f(K) = \{1\}$. Equivalently, X is Tychonoff if it is Hausdorff and the sets $\mathrm{coz}(f)$ $(f \in C(X))$ form a base for the open sets of the topology.

When viewing $C(X)$ as an archimedean ℓ-group without specific mention of a weak order unit, it will always be understood that the designated unit is the constant function 1. Its Yosida space is canonically homeomorphic to βX, the Stone–Čech compactification of X.

(b) Recall that a space X is *basically disconnected* if the closure of each cozeroset is open. It is well known that X is basically disconnected if and only if βX is basically disconnected (6M.1, [GJ76]). Also, as is well known, X is basically disconnected if and only if $C(X)$ is projectable.

X is an *F-space* if each cozeroset of X is C^*-embedded. It is well known that each basically disconnected space is an F-space ([GJ76], 14N.4), but the converse is false; e.g., $\beta\omega\backslash\omega$. Note as well that X is an F-space if and only if βX has this feature ([GJ76], Theorem 14.25). What we will need in this context is that each countable subset of an F-space is C^*-embedded ([GJ76], 14N.5). This implies that if X is a compact F-space contains a copy of $\beta\omega$. Thus, a compact F-space which is scattered is finite.

(c) Let us venture a little further afield. A space X is *cozero-complemented* if for each cozeroset U there is one V, such that $U \cap V = \varnothing$ and $U \cup V$ is dense. It is easy to see that X is cozero-complemented precisely when $C(X)$ is a complemented ℓ-group.

Recall that a continuous surjection $f : Y \to X$ between compact spaces is called *irreducible* if whenever $A \subseteq Y$ is closed and $f(A) = X$, then $A = Y$. It is well known that a continuous surjection $f : Y \to X$ induces an ℓ-embedding $C(f) : C(X) \to C(Y)$ by $C(f)(g) = g \cdot f$. Then f is irreducible if and only if $C(f)$ is a dense embedding of ℓ-groups. For a primer on irreducible maps between spaces we refer the reader to [H89].

For a given compact space X there is a minimum basically disconnected cover, in the following sense. There are a compact basically disconnected space BX and an irreducible surjection $b_X : BX \to X$ with the property that if $g : Y \to X$ is any irreducible surjection, with Y compact and basically disconnected, then there is an irreducible surjection $\hat{g} : Y \to BX$ such that $g = b \cdot \hat{g}$. For completeness we note that, if X is compact then $C(BX)$ is canonically ℓ-isomorphic to the Dedekind σ-completion of $C(X)$.

(d) Now, when X is compact and cozero-complemented then BX can be obtained as follows. As already noted (5.5(a)), $\mathrm{Min}(C(X))$ is compact, because $C(X)$ is complemented. Moreover, $\mathrm{Min}(C(X))$ is basically disconnected and the map $b_X : \mathrm{Min}(C(X)) \to X$, assigning to each minimal prime subgroup the maximal ideal containing it, is irreducible (up to a canonical homeomorphism), and b_X witnesses the fact that $\mathrm{Min}(C(X))$ is the basically disconnected cover of X. This can be pieced together from material in [HVW87] and [HM93]. Without apology we skip the details and rely on the interested reader to check the references.

Armed with the above remarks it is easy to say when $C(X)$ is hyper-special valued. The result may disappoint, but should not surprise.

Proposition 5.11. *For any Tychonoff space X, $C(X)$ is hyper-special valued if and only if X is finite.*

Proof. The sufficiency is clear, and so we move on to the necessity. If $C(X)$ is hyper-special valued, then, per Corollary 4.4, $C(X)$ is projectable, whence X and βX are basically disconnected. On the other hand, βX is scattered (Proposition 2.5), and so, by the remarks in 5.10(b), βX and X are finite. □

With the assumption of cozero-complementation we can pinpoint the hyper-special valued radical of $C(X)$.

Proposition 5.12. *Suppose that X is Tychonoff and cozero-complemented. Then the following coincidence of radicals occurs:*

$$\mathcal{O}(C(X)) = \mathrm{Hyp}(\mathcal{SV})(C(X)) = \mathcal{O}^*(C(X)) = \{f \in C(X) : |\mathrm{coz}(f)| < \infty\}.$$

Proof. For brevity let us denote $C = C(X)$. We begin by observing that, for any archimedean ℓ-group, special elements are basic, and that $f \in C$ is basic precisely when $|\mathrm{coz}(f)| = 1$; that is, f is zero except at an isolated point. This makes clear that

$$\mathcal{O}(C) = \{f \in C : |\mathrm{coz}(f)| < \infty\}.$$

Since $\mathcal{O} \subseteq \mathrm{Hyp}(\mathcal{SV})$, it suffices, to prove the claim, to show that

(i) for each $g \in \mathrm{Hyp}(\mathcal{SV})(C), \mathrm{coz}(g)$ is finite, and once this is done, that

(ii) $C/\mathcal{O}(C)$ has no basic elements.

Recall (5.10(d)) that $\mathrm{Min}(C)$ is compact and basically disconnected. Now, to show (i), if $g \in \mathrm{Hyp}(\mathcal{SV})(C)$, and $\mathrm{coz}(g)$ is infinite, then $C(g) \in \mathrm{Hyp}(\mathcal{SV})$, whence it is projectable and $Y(g)$, which is scattered, is also homeomorphic to a closed subspace of $\mathrm{Min}(C)$, a contradiction, according to the comments in 5.10(b).

As to (ii), recall that if $f_1, f_2, \ldots$ is any sequence of functions in C such that $|f_n| \le 1$ for each $n < \omega$, then, if we define, for each $x \in X$,

$$f(x) = \sum_{n=1}^{\infty} \frac{|f_n(x)|}{2^n},$$

we obtain $f \in C(X)$, and $f^{\perp\perp}$ is the polar generated by $f_1, f_2, \ldots$. Now suppose that $0 < g \in C$ and $\mathrm{coz}(g)$ is infinite. If $\mathcal{O}(C) + g$ is basic, then g has infinitely many special values, which means that there is a sequence $(x_n)_n$ of isolated points of X such that $g(x_n) > 0$, for each n. Let χ_n be the characteristic function of the singleton $\{x_n\}$. Now choose $f_o \in C$ so that $f_o^{\perp\perp}$ is the polar generated by all the χ_{2n+1}, and $f_e \in C$ so that $f_e^{\perp\perp}$ is the polar generated by all the χ_{2n}. Then, without loss of generality, we may also have $f_o, f_e \le g$, and it is clear that $f_o \wedge f_e = 0$, while neither one lies in $\mathcal{O}(C)$. This is a contradiction, showing that $\mathcal{O}(C) + g$ cannot be basic. This demonstrates (ii), and completes the proof of the proposition. □

Remark 5.13. Proposition 5.12 shows that the Hyp($\mathscr{S}\mathscr{V}$)-radical need not be *order-closed*; that is, closed under arbitrary suprema.

Remark 5.14. It would be nice to drop the requirement of cozero-complementation from Proposition 5.12, but we do not know if this can be done. There is precedent in a different context, of radicals in classical torsion theory of commutative rings (see [MM99], §4), which provides some evidence that this assumption may be needed.

We do not even know, under the assumption of Proposition 5.12, whether (ii) of the proof can be strengthened to say that $C/\mathscr{O}(C)$ contains no *special* elements.

And to conclude this section we make the following observation.

Remark 5.15. It may occur to the reader that a projectable hyper-special valued ℓ-group is, in some sense, "almost" archimedean. This is so, in the following sense. In any normal valued ℓ-group G, and for each $a \in G$, the factor $G(a)/N_a$ is archimedean, as has already been observed. If G is also hyper-special valued and projectable then $N_a \in \mathscr{O}$. We summarize:

If G is hyper-special valued and projectable, then for each $a \in G$, $G(a)$ is an extension of a $\mathscr{O}$-group by one which is archimedean and hyper-special valued.

6 Examples and comments

We round out the subject matter of this article with a number of observations and an array of examples. The latter, especially, ought to lay to rest a number of questions the reader might pose. Our exposition begins with some initial remarks on structural features of hyper-special valued ℓ-groups, and then compares a number of classes of ℓ-groups which have been mentioned in earlier sections to Hyp($\mathscr{S}\mathscr{V}$).

The first observation is a proposition concerning abelian ℓ-homomorphic images of a hyper-special valued ℓ-groups.

Proposition 6.1. *Suppose that G is a hyper-special valued ℓ-group. Then, for each $g, h \in G$, the commutator $[g, h]$ lies in the ℓ-ideal of $G(|g| \vee |h|)$ generated by the special components of $|g| \vee |h|$.*

Proof. Since G is normal valued, $|[g, h]| \ll |g| \vee |h|$ (see [D95], Theorem 41.1), and so every value of $[g, h]$ is properly contained in a value of $|g| \vee |h|$. Suppose, by way of contradiction, that $[g, h]$ has a 1-special value, say V, which is not contained in a special value of $|g| \vee |h|$. If W is the value of $|g| \vee |h|$ containing V then, according to Lemma 3.4, W is 1-special. But then $[g, h]$ and $|g| \vee |h|$ have 1-special components which have infinitely many special values in common (Lemma 3.6), which amounts to a contradiction. □

We record some immediate corollaries of Proposition 6.1.

Corollary 6.2. *Suppose that G is hyper-special valued. If* $\mathrm{Val}_0(G)$, *the set of special values of G, forms an order ideal, then* $G/\mathscr{F}\mathscr{V}(G)$ *is abelian.*

Corollary 6.3. *Suppose that G is hyper-special valued. If G has a unique maximal finite valued* ℓ*-subgroup, then* $G/\mathscr{F}\mathscr{V}(G)$ *is abelian.*

Corollary 6.4. *Suppose that G is hyper-special valued. If G is a subdirect product of totally ordered groups, then* $G/\mathscr{O}(G)$ *is abelian.*

Lemma 3.6 asserts that, in a hyper-special valued ℓ-group, two 1-special elements whose 1-special values are comparable must share all but finitely many of their special values. This might lead one to think that the lengths of chains of α-special values are "severely" restricted in a hyper-special valued ℓ-group. One could also surmise that the structure of such a chain might, likewise, be limited in complexity or density. The first example in this section suggests otherwise.

Example 6.5. *A hyper-special valued* ℓ*-subgroup of the group of all real sequences, possessing a chain of 1-special values which is order-isomorphic to an* η_1*-set.*

We denote the ℓ-group of all real sequences by $\mathbb{R}^\omega$, and the ℓ-subgroup of the finitely nonzero sequences by Σ. In the sequel we consider the family $\mathfrak{A}$ of all ℓ-subgroups of $\mathbb{R}^\omega$ which contain Σ as a prime convex ℓ-subgroup. $\mathfrak{A}$ is partially ordered by inclusion. A routine application of Zorn's Lemma shows that $\mathfrak{A}$ is inductive; thus, $\mathfrak{A}$ has a maximal element. We now proceed, in two steps, to show that if A is such a maximal element, then A fulfills the conditions stipulated here. That A is hyper-special valued is a consequence of Theorem 4.2.

We remind the reader that an η_1-set is a totally ordered set H such that for any two countable subsets U and V, with $U < V$, there is an $h \in H$ such that $U < h < V$. In the sequel G^d denotes the divisible hull of G.

(a) *If* $G \in \mathfrak{A}$ *then so is* G^d.

The proof is left as an exercise.

(b) *Suppose* $G \in \mathfrak{A}$ *and* $0 < b \in \mathbb{R}^\omega$, *such that for each* $g \in G$, *either* $\Sigma + |g| \ll \Sigma + b$ *or else* $\Sigma + b \ll \Sigma + |g|$. *Then* $G \oplus \mathbb{Z}b \in \mathfrak{A}$.

Proof. Suppose that $m \in \mathbb{Z}$, and $g \in G$, and consider $b_m = (mb + g) \vee 0$. Note that $\Sigma + b_m = [m(\Sigma + b) + (\Sigma + g)] \vee \Sigma$. If $gS + g \gg \Sigma + b$, then $\Sigma + b_m = \Sigma + mb + g$, while if $\Sigma \quad g \gg \Sigma + b$, then $\Sigma + b_m = \Sigma$. Likewise, if $\Sigma + |g| \ll \Sigma + b$, and m is positive, then $\Sigma + b_m = \Sigma + mb + g$, while if m is negative, $\Sigma + b_m = \Sigma$. All of this means, since $\Sigma \subseteq G$ that $b_m \in G \oplus \mathbb{Z}b$; that is to say, $G \oplus \mathbb{Z}b$ is an ℓ-subgroup of $\mathbb{R}^\omega$. It should be clear from the hypotheses that $G \oplus \mathbb{Z}b \in \mathfrak{A}$. □

(c) *If A is a maximal element in* $\mathfrak{A}$ *then* $\mathrm{Val}(A/\Sigma)$ *is an* η_1*-set.*

Proof. There are three parts to the argument.

(i) $\mathrm{Val}(A/\Sigma)$ *has no least and no greatest element*. If $\mathrm{Val}(A/\Sigma)$ has a greatest element, then there exists a positive $x \in A$ such that, for each $a \in A$, there is a positive integer k such that $\Sigma + |a| < \Sigma + kx$. Now define $b \in \mathbb{R}^{\omega}$ by $b(n) = nx(n)$. Then $\Sigma + b \gg \Sigma + x$, whence $\Sigma + b \gg \Sigma + |a|$, for each $a \in A$. But then, according to (b) above, $A \oplus \mathbb{Z}b \in \mathfrak{A}$, which contradicts the maximality of A. A similar argument shows that $\mathrm{Val}(A/\Sigma)$ cannot have a least element either.

(ii) $\mathrm{Val}(A/\Sigma)$ *has no cofinal or coinitial countable sets*. Suppose there is a countable cofinal subset. Then, owing to (i), there is a strictly increasing countable cofinal sequence $V_1, V_2, \ldots$. Pick real sequences $f_n > 0$ such that $\Sigma < \Sigma + f_n$ and V_n/Σ is the value of $\Sigma + f_n$. Without loss of generality we may assume that $0 < f_1 < f_2 < \cdots$ in A.

Now since $\Sigma + f_i \ll \Sigma + f_{i+1}$, we have for each i and each positive integer k an integer $n_{i,k}$ such that $f_{i+1}(n) \geq f_i(n)$, for each $n \geq n_{i,k}$. Next, let $b(n) = f_n(n)$; this defines $b \in \mathbb{R}^{\omega}$. Observe that, whenever $n > \max(i, n_{i,k}\}$,

$$b(n) \geq f_{i+1}(n) \geq kf_i(n),$$

and so $\Sigma + b \gg \Sigma + f_i$, for each i. This again implies that $\Sigma + b \gg \Sigma + |a|$, for each $a \in A$, which thanks to (b) once more leads to a contradiction. Thus, $\mathrm{Val}(A/\Sigma)$ has no countable cofinal subset, and, in a similar fashion, no countable coinitial subset either.

(iii) $\mathrm{Val}(A/\Sigma)$ *is an* η_1*-set*. Suppose $\mathcal{U}$ and $\mathcal{V}$ are countable subsets of $\mathrm{Val}(A/\Sigma)$, and that $\mathcal{U} < \mathcal{V}$. Note that (ii) above takes care of the case in which either $\mathcal{U}$ or $\mathcal{V}$ is empty. Enumerate $\mathcal{U}$ and $\mathcal{V}$:

$$\mathcal{U} = \{M_i/\Sigma : i < \omega\} \quad \text{and} \quad \mathcal{V} = \{N_j/\Sigma : j < \omega\}.$$

Choose $u_i > 0$ in A so that M_i/Σ is the value of $\Sigma + u_i$, and $v_j > 0$ in A so that N_j/Σ is the value of $\Sigma + v_j$. Consider

$$U' = \{\Sigma + qu_i : q \in \mathbb{Q}^+ \text{ and } i < \omega\} \quad \text{and} \quad V' = \{\Sigma + qv_j : q \in \mathbb{Q}^+ \text{ and } j < \omega\}.$$

Note that U' and V' are countable subsets of A/Σ and that $U' < V'$. Now U' has no largest element, which implies that there is a cofinal increasing sequence in U', $\Sigma + u_1' < \Sigma + u_2' < \cdots$. Similarly, V' has a coinitial decreasing sequence $\Sigma + v_1' > \Sigma + v_2' > \cdots$. Without loss of generality one may assume that $u_1' < u_2' < \cdots < v_2' < v_1'$ in A^+. Next, define $b \in \mathbb{R}^{\omega}$ by $b(n) = u_n'(n)$. Then, for all i and j,

$$\Sigma + u_i' < \Sigma + b < \Sigma + v_j'.$$

Observe as well that for each i and each positive integer k there is an i_o such that $\Sigma + u_{i_o}' > \Sigma + ku_i$, which permits us to conclude that $\Sigma + u_i \ll \Sigma + b$, for each i. By an analogous argument, $\sigma + b \ll \Sigma + v_j$, for each j.

Finally, suppose that there is no value strictly between $\mathscr{U}$ and $\mathscr{V}$. Then if $a \in A$ we must either have $\Sigma + |a| < \Sigma + u_i$, for suitable i, so that $\Sigma + |a| \ll \Sigma + b$, or else, similarly, $\Sigma + b \ll \Sigma + |a|$. Employing (b) once more, we conclude that $A \oplus \mathbb{Z}b \in \mathfrak{A}$, contradicting the maximality of A yet another time. Thus $\mathrm{Val}(A/\Sigma)$ is an η_1-set, as asserted.

The proof of (c) is complete. □

The following observation could have been made much earlier, but it doesn't really fit in any of the previous sections. Recall that an ℓ-group is *laterally (σ-)complete* if every (countable) disjoint set has a supremum. It is well known that an archimedean laterally σ-complete ℓ-group is projectable. (For a comprehensive discussion of archimedean laterally σ-complete ℓ-groups and related matters, we refer the reader to [HM96].) Projectability makes the argument in the proof which follows a bit crisper, but it is not needed.

Proposition 6.6. *Suppose that G is hyper-special valued and laterally σ-complete. Then G has a finite basis.*

Proof. Suppose that $b_1, b_2, \ldots$ is a countable pairwise disjoint set of special elements. Note that if no such infinite set exists then G is necessarily finite valued. It is well known that any finite valued laterally σ-complete ℓ-group has a finite basis.

Form $b = \bigvee_n b_n$ and $a = \bigvee_n nb_n$. It is evident that $a^{\perp\perp} = b^{\perp\perp}$. The reader will also quickly verify that $N_a = N_b$. Now pass to the archimedean ℓ-group $H = G(a)/N_a$; it is hyper-special valued as well. Next, observe that the special values of a in $G(a)$ are the values of the b_n $(n < \omega)$. Since N_a is order-closed in $G(a)$, we are able to conclude that the canonical ℓ-homomorphism $G(a) \to H$ preserves all suprema and infima ([D95], 21.2). It follows that H too is laterally σ-complete.

In H take $u = N_a + b$ for the designated unit, then notice that $Y(u) \cong Y(b)$. Owing to the results of the preceding paragraph and 2.3(d) of [HM96] we conclude that $Y(u)$ is a basically disconnected space, and a compactification of a countable discrete set N. Using the remarks in 5.10(b) it turns out that $Y(u)$ is, in fact the Stone–Čech compactification of N. It is then evident that $Y(u) \backslash N$ is infinite. Next, identifying u with the constant 1 in the Yosida representation, the remarks in the preceding paragraph imply that the identity function i is the coset of a in H, and that i has infinitely many infinities. This contradicts Theorem 5.3.

It follows that G has a finite basis. □

Next, we present an example showing that $\mathrm{Hyp}(\mathscr{SV})$ is not closed under formation of ℓ-subgroups.

Example 6.7. *A hyper-special valued archimedean ℓ-group having an ℓ-subgroup which is not hyper-special valued.*

G is the ℓ-subgroup of $\mathbb{Z}^{\omega}$ generated by Σ, the subgroup of finitely nonzero sequences, together with

$$e_1 = (1,0,1,0,\ldots), \quad e_2 = (0,1,0,1,\ldots), \quad x_1 = (1,0,3,0,5,\ldots) \quad \text{and}$$

$$x_2 = (0,2,0,4,0,6,\ldots).$$

Let A be the ℓ-subgroup of G generated by Σ, e_1, e_2 and $x_1 + x_2$. Note that G is an $\mathcal{O}^2$-group which is hyper-special valued. However, the 1-special value in A of $x_1 + x_2$ contains the two incomparable values of e_1 and e_2. Thus, A is not hyper-special valued.

At last, the example we promised in Remark 2.11.

Example 6.8. *An archimedean ℓ-group which is $\mathcal{O}^2$, special valued, but not hyper-special valued.*

Let G be the ℓ-subgroup of $\mathbb{R}^\omega$ generated by Σ, the finitely nonzero sequences, the constant 1 and b defined by $b(n) = 0$, if n is odd, and $b(n) = 1/n$, if n is even. G is $\mathcal{O}^2$, as G/Σ is a totally ordered group. Since G is archimedean with basis, it is also special valued. On the other hand, G has two 1-special elements, 1 and b, with comparable 1-special values, yet there are infinitely many special values of 1 which are not shared by b. This contradicts Lemma 3.6. Also, observe that G has scattered Yosida space, namely the one-point compactification of ω, and only one accumulated zero, at infinity. Thus, Theorem 5.7 fails here, the reason being that G is not projectable.

As we shall see, any ℓ-group in $\mathcal{O}^*$ is necessarily hyper-pseudospecial valued, but the converse is not true, even for archimedean ℓ-groups.

Example 6.9. *An ℓ-group in* Hyp($\pi\mathcal{S}\mathcal{V}$) *which is not $\mathcal{O}^*$.*

The ℓ-group we are about to describe is similar in spirit to Example 5.5 in [CD96]; the difference is that here we do not bother about making the example a group of integer valued sequences. We begin with some notational items. Let $q(n) = \frac{1}{n}$, for each $n < \omega$; for each positive integer n and each integer i, such that $1 \le i \le 2^n$, let $e_{n,i}$ be the sequence defined by

$$e_{n,i}(m) = \begin{cases} 1 & \text{if } m \equiv i \bmod 2^n \\ 0 & \text{otherwise} \end{cases}.$$

Now, G is the subgroup of real valued sequences generated by Σ, the finitely nonzero sequences, together with

$$q, q^2 e_{1,1}, q^2 e_{1,2}, \ldots, q^{2^n} e_{n,1}, q^{2^n} e_{n,2}, \ldots, q^{2^n} e_{n,2^n}, \ldots.$$

As is shown about Example 5.5, [CD96], G/Σ is a finite valued ℓ-group whose root system of values is the binary tree. It is easily seen that, mod Σ,

$$q^{2^n} e_{n,i} \gg q^{2^{n+1}} e_{n+1,i} \;\&\; q^{2^{n+1}} e_{n+1,i+2^n}.$$

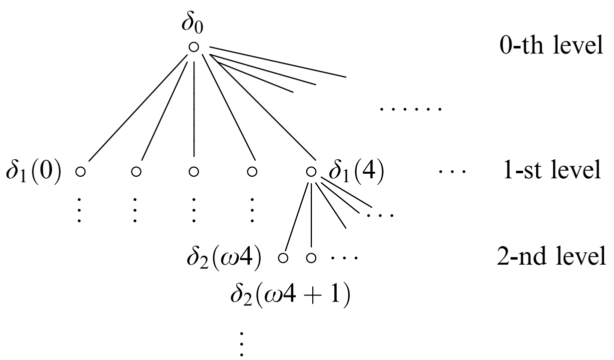
Thus, G is not in $\mathcal{O}^*$, as $\mathcal{O}(G/\Sigma) = \{0\}$.

G is countably valued—meaning that every element has countably many values; this is easily checked. Else, we let the reader look at Example 5.5 of [CD96]. As is demonstrated in [CD96], Proposition 2.3, the class of all countably valued ℓ-groups is a torsion class; since every countably valued ℓ-group is pseudospecial valued ([D95], Corollary 2.8) it follows that $G \in \mathrm{Hyp}(\pi\mathcal{S}\mathcal{V})$.

A hyper-special valued ℓ-group can fail to have basic elements. Indeed, we can do worse:

Example 6.10. *A hyper-special valued ℓ-group G such that $\mathcal{F}\mathcal{V}(G) = \{0\}$.*

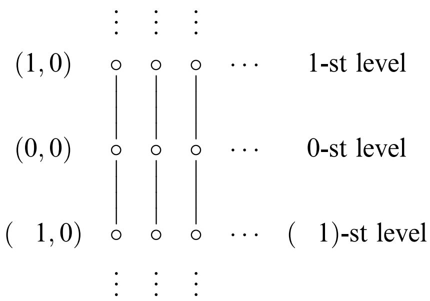
The root system Δ is the ω-ary tree pictured below. As in tree-theoretic language we shall refer to "levels" of the root system. The ordering (and notation) in Δ is defined recursively: δ_0 is the top element; each $\delta_1(n)$ in the 1-st level immediately preceeds δ_0. Recursively, the immediate predecessors of $\delta_n(\alpha)$—with $\alpha < \omega^n$—are the nodes in the $(n+1)$-st level, $\delta_{n+1}(\alpha\omega + k)$, for all $0 \le k < \omega$.



G is the ℓ-subgroup of $V(\Delta, \mathbb{R})$ generated by Σ, the finitely nonzero functions and, for each node $\delta_m(\alpha)$ at level m, the function $e_{\alpha,m}$ which is the characteristic function of $E_{\alpha,m}$, the intersection of the $(m+1)$-st level set with $\{\delta \in \Delta : \delta < \delta_m(\alpha)\}$. It is easily verified that the properties claimed for this group do, in fact, hold.

Example 6.11. *An ℓ-group G which is not hyper-special valued, yet every ℓ-homomorphic image is special valued.*

Consider the root system Δ which is the disjoint union of countably many copies of $\mathbb{Z}$ in its natural ordering, as pictured below.



Again we speak of levels, in the same sense as one does with trees. For each level n, the node in the k-th copy of $\mathbb{Z}$ is denoted by the pair (n,k); note that n ranges over $\mathbb{Z}$, while k is a nonnegative integer. The ordering in Δ is such that $(n,k) < (m,l)$ precisely when $k = l$ and $n < m$.

Now A stands for the ℓ-subgroup of $V(\Delta, \mathbb{R})$ generated by Σ, the subgroup of finitely nonzero functions, and the characteristic functions of each level set. Now form a splitting extension $G = A \times \mathbb{Z}$ of A by $\mathbb{Z}$ so that

$$((0, \quad m) + (a,0) + (0,m))(n,k) = (a,0)(n \quad m,k),$$

for each $m, n \in \mathbb{Z}$ and each positive k, and each $a \in A$; note that a is being identified with $(a,0)$. For the record, the ordering on G is lexicographic: (a,m) is positive if $m \geq 0$ and if $m = 0$ then $a \in A^+$. It should be clear that G is special valued.

Now suppose that L is an ℓ-ideal of G. We may suppose that $L \subset A \times \{0\}$: if not, then either $G = L$, in which case there is nothing to prove, or else $A \times \{0\} = L$, and then $G/L \cong \mathbb{Z}$, which is obviously special valued. From this point on there are a number of cases:

(a) *L is finite valued*. Then $L = \Sigma_{b \in B} \Sigma_b(\mathbb{Z}, \mathbb{R})$, where $B \subseteq \omega$.

  (i) *$\omega \backslash B$ is finite*. Then A/L is a finite cardinal sum of copies of $V(\mathbb{Z}, \mathbb{R})$, which is finite valued, and hence it is clear that G/L is special valued.

  (ii) *$\omega \backslash B$ is infinite*. Then

$$(A \times \{0\})/L \cong A,$$

  and again G/L is special valued.

(b) *L is not finite valued*. Then if L contains the characteristic function of one cofinite subset of a level set, it necessarily contains the characteristic function of the corresponding cofinite subset of *each* level set. In fact, there exist finitely many positive integers $m_1, m_2, \ldots, m_t$ such that

$$(a,0) \in L \text{ if and only if } a(n, m_i) = 0, \text{ for each } i = 1, 2, \ldots, t.$$

In this event, $(A \times \{0\})/L$ is once again a finite cardinal sum of copies of $V(\mathbb{Z}, \mathbb{R})$, and therefore special valued.

This shows that every ℓ-homomorphic image of G is special valued. On the other hand, $A \in \mathscr{C}(G)$ is not hyper-special valued, as Lemma 3.6 is violated.

Note, besides, that $G \in \mathrm{Loc}(\mathscr{O}^*)$; refer to the remarks in 6.12 just ahead.

To conclude the article, we present some comments concerning a number of radical classes of ℓ-groups which are closely related to the hyper-special valued ones.

Remarks 6.12. (a) We have discussed at length the relationship between $\mathrm{Hyp}(\mathscr{SV})$ and $\mathscr{O}^*$. Other classes that come to mind in this context are these:

(i) The torsion class $\mathscr{FV}$ of all finite valued ℓ-groups.

(ii) The radical class $\mathscr{SV}$ of all special valued ℓ-groups.

(iii) The radical class $\pi\mathscr{SV}$ of all pseudospecial valued ℓ-groups.

As to the pseudospecial valued ℓ-groups, we recall the notation of a *locally conditioned class*, as defined in [M92]. We say that the radical class is locally conditioned by the class $\mathscr{T}$ of ℓ-groups if $G \in \mathscr{R}$ precisely when $G(x)/N_x \in \mathscr{T}$, for all $x \in G$. When $\mathscr{R}$ is locally conditioned by $\mathscr{T}$ we write $\mathscr{R} = \mathrm{Loc}(\mathscr{T})$. The minimal requirements one puts on the class $\mathscr{T}$ needn't concern us here. We only point out that $\pi\mathscr{SV} = \mathrm{Loc}(\mathscr{B})$, where $\mathscr{B}$ is the class of ℓ-groups with a basis.

Incidentally, it is conjectured in [M92] that $\pi\mathscr{SV}$ is not a torsion class, and not even closed under formation of ℓ-homomorphic images which preserve all suprema. Darnel has an example confirming this; however, we do not feel that it should be included in this exposition.

(b) In the proposition that follows we indicate the known relationships between the classes discussed above. All implications are strict; a recounting of counterexamples showing the arrows do not reverse precedes the statement of this proposition.

Example 6.8 shows that $\mathrm{Hyp}(\mathscr{SV})$ is properly contained in $\mathscr{SV} \cap \mathrm{Loc}(\mathscr{O}^*)$, even in the archimedean context. Example 6.9 shows that the first arrow in (b) of Proposition 6.13 does not reverse; since the example is archimedean we have that the second implication in (c) is also strict. The example at the end of 3.1 in [M92] is in $\pi\mathscr{SV}$ and $\mathscr{O}^2$, but is not special valued; since it is in $\mathrm{Hyp}(\pi\mathscr{SV})$, by Proposition 6.13 below, it also demonstrates that $\mathrm{Hyp}(\pi\mathscr{SV})$ is properly contained in $\pi\mathscr{SV}$. In any case, referring to (c) of Proposition 6.13, the latter containment is proper even for archimedean ℓ-groups: $C(\omega)$ has a basis; however, modulo the basis subgroup, there are no special values whatsoever. For a suitably complicated root system Δ, $V(\Delta, \mathbb{R})$ is not $\mathscr{O}^*$; it is always special valued ([D95], Proposition 51.4).

Proposition 6.13. *The following implications are valid for arbitrary ℓ-groups:*

(a) $\mathrm{Hyp}(\mathscr{SV}) \Rightarrow \mathscr{SV} \cap \mathrm{Loc}(\mathscr{O}^*) \Rightarrow \mathscr{SV} \| \mathrm{Loc}(\mathscr{O}^*)$.

(b) $\mathrm{Loc}(\mathscr{O}^*) \Rightarrow \mathrm{Hyp}(\pi\mathscr{SV}) \Rightarrow \pi\mathscr{SV}$, and $\mathscr{SV} \Rightarrow \pi\mathscr{SV}$.

For archimedean ℓ-groups we have the implications

(c) $\mathrm{Hyp}(\mathscr{SV}) \Rightarrow \mathscr{O}^* \Rightarrow \mathrm{Hyp}(\pi\mathscr{SV}) \Rightarrow \mathscr{B} = \mathscr{SV} = \pi\mathscr{SV}$.

Proof. For the first inclusion, it is clear that every hyper-special valued ℓ-group is special valued; if G is hyper-special valued and $x \in G$, then $G(x)/N_x$ is hyper-special valued and archimedean. Using Corollary 4.7, we conclude that $G \in \mathrm{Loc}(\mathcal{O}^*)$.

The inclusion $\mathrm{Loc}(\mathcal{O}^*) \subseteq \mathrm{Hyp}(\pi\mathcal{S}\mathcal{V})$: Suppose that $G \in \mathrm{Loc}(\mathcal{O}^*)$ and $x \in G$ and L is an ℓ-ideal of $G(x)$. Since $\mathrm{Loc}(\mathcal{O}^*)$ is a torsion class (Proposition 2.9, [M92]) it follows that $H \equiv G(x)/L \in \mathrm{Loc}(\mathcal{O}^*)$. Thus, for each $L + g \in H, H(L+g)/N_{L+g} \in \mathcal{O}^*$, and so $H(L+g)/N_{L+g}$ has a basis. This shows that H is pseudospecial valued, whence $G \in \mathrm{Hyp}(\pi\mathcal{S}\mathcal{V})$.

Finally, recall that if $\mathcal{T}$ is a radical class and G is archimedean and in $\mathrm{Loc}(\mathcal{T})$, then, in fact, $G \in \mathcal{T}$, as $N_x = \{0\}$, for each $x \in G$. This establishes the claim for archimedean ℓ-groups. $\square$

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