observations of only one of the samples and calculate the values of R_i in succession feeding them thereupon into the adder. Thus an apparatus with restricted memory could process quite effectively arbitrarily long sequences of pairs of observations.

4. In conclusion, the author wishes to express his gratitude to E. M. Gabidullin and D. M. Chibisov for the great number of suggestions during the writing of the paper.

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ON THE ASYMPTOTIC NORMALITY OF ONE SAMPLE RANK ORDER TEST STATISTICS*

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Summary

The asymptotic normality of a class of one sample rank order test statistics is established. This class includes among other test statistics the well-known normal scores test of symmetry developed by Fraser [2] and the Wilcoxon paired comparison test [8].

1. Introduction

Chernoff and Savage [1] established the asymptotic normality of a class of two sample non-parametric test statistics which includes among other test statistics the well-known Fisher–Yates normal scores statistic and the Wilcoxon statistic. With the aid of this theorem they also established that the test for translation based on the normal scores statistic is at least as efficient as the t-test. Their theorem was extended by the present authors [6], [7] in order to derive optimum tests for the c-sample univariate as well as multivariate problems. Recently, Govindarajulu, Le Cam and Raghavachari [3] established the results of Chernoff and Savage [1] and Puri [6] by relaxing some of the sufficient conditions required by them in proving the asymptotic normality. The purpose of the present note is to provide a formal proof of the asymptotic normality of a class of one sample Chernoff–Savage type test statistics which include among other tests the well-known normal scores test of symmetry developed by Fraser [2] and the Wilcoxon [8] paired comparison test.

Let $X_1, \cdots, X_n$ be n independent and identically distributed random variables having an absolutely continuous distribution function (cdf) $F(x)$, defined on the real line $-\infty < x < \infty$. Let $Z_{n,\alpha}$ be equal to 1 if the α-th smallest observation among the values of $|X_i|, i = 1, \cdots, n$, is from a positive X and otherwise let $Z_{n,\alpha}$ be equal to zero, $\alpha = 1, \cdots, n$. By virtue of the assumed continuity of $F(x)$, the possibilities of ties among the observations $|X_i|, i = 1, \cdots, n$, and of some

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of them being exactly equal to zero may be ignored, in probability. We shall be concerned with the Chernoff–Savage [1] type of one sample rank order statistics which may be expressed as

(1) $$T_n = \frac{1}{n}\sum_{\alpha=1}^{n} Z_{n,\alpha}E_{n,\alpha},$$

where $E_{n,\alpha} = J_n(\alpha/(n+1))$, $\alpha = 1, \cdots, n$, are functions of the ranks $\alpha(= 1, \cdots, n)$ and are explicitly known.

The statistics given by (1) include as special cases a number of well-known test statistics. For example, if $E_{n,\alpha}$ is the expected value of the α-th order statistic in a sample of size n from the rectangular distribution (0, 1), then the statistic T_n reduces to the one sample Wilcoxon [8] signed rank statistic. If $E_{n,\alpha}$ is the expected value of the α-th order statistic in a sample of size n from the chi-distribution with one degree of freedom, then the statistic T_n coincides with Fraser's [2] one sample normal scores statistic.

The function $J_n(u)$ is defined only at $u = \alpha/(n+1)$, $\alpha = 1, \cdots, n$, but we may extend its domain of definition to (0, 1) by letting it have constant values over

$$\left[\frac{\alpha}{n+1}, \frac{\alpha+1}{n+1}\right), \qquad \alpha = 1, \cdots, n.$$

Further, we define

(2) $$F_n(x) = \frac{1}{n} \text{ (Number of } X_\alpha \leqq x),$$

(3) $$H_n(x) = \frac{1}{n} \text{ (Number of } |X_\alpha| \leqq x) = F_n(x) - F_n(-x-0),$$

(4) $$H(x) = F(x) - F(-x).$$

Then, T_n defined in (1), can equivalently be written as

(5) $$T_n = \int_0^\infty J_n\left[\frac{n}{n+1}H_n(x)\right] dF_n(x).$$

2. The Main Theorem

Theorem. *If*

(I) $\lim_{n\to\infty} J_n(u) = J(u)$ *exists for* $0 < u < 1$ *and is not a constant,*

(II) $$\int_0^\infty \left\{J_n\left[\frac{n}{n+1}H_n(x)\right] - J\left[\frac{n}{n+1}H_n(x)\right]\right\} dF_n(x) = o_p(n^{-1/2}),$$

and

(III) $J(u)$ *is absolutely continuous for* $0 < u < 1$ *and*

$$|J^{(i)}(u)| = \left|\frac{d^i}{du^i}J(u)\right| \leqq K\{u(1-u)\}^{-1/2-i+\delta} \quad \textit{for} \quad i = 0, 1,$$

where K is some constant and $\delta > 0$, then

(6) $$\lim_{n\to\infty} \mathbf{P}\{n^{1/2}[T_n - \sigma_n]/\mu_n \leqq x\} = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

where

(7) $$\mu_n = \int_0^\infty J[H(x)]\, dF(x),$$

$$
\begin{aligned}
\sigma_n^2 = & \int_0^\infty J^2[H(x)]\,dF(x) - \left[\int_0^\infty J[H(x)]\,dF(x)\right]^2 \\
& + 2\Bigg[\iint_{0<x<y<\infty} H(x)[1 - H(y)]J'[H(x)]J'[H(y)]\,dF(x)\,dF(y) \\
& - \iint_{0<x<y<\infty} H(x)J'[H(x)]J[H(y)]\,dF(x)\,dF(y) \\
& + \iint_{0<x<y<\infty} J[H(x)][1 - H(y)]J'[H(y)]\,dF(x)\,dF(y)\Bigg],
\end{aligned} \tag{8}
$$

provided $\sigma_n \neq 0$.

PROOF. We write T_n as

$$T_n = \int_0^\infty J_n\left[\frac{n}{n+1}H_n(x)\right]dF_n(x) = B_{1n} + B_{2n} + \sum_{i=1}^4 C_{in}, \tag{9}$$

where

$$B_{1n} = \int_0^\infty J[H(x)]d[F_n(x)], \tag{10}$$

$$B_{2n} = \int_0^\infty [H_n(x) - H(x)]J'[H(x)]\,dF(x), \tag{11}$$

$$C_{2n} = \frac{-1}{n+1}\int_0^\infty H_n(x)J'[H(x)]\,dF_n(x), \tag{12}$$

$$C_{2n} = \int_0^\infty [H_n(x) - H(x)]J'[H(x)]d[F_n(x) - F(x)], \tag{13}$$

$$C_{3n} = \int_0^\infty \left\{J\left[\frac{n}{n+1}H_n(x)\right] - J[H(x)] - \left[\frac{n}{n+1}H_n(x) - H(x)\right]J'[H(x)]\right\}dF_n(x), \tag{14}$$

$$C_{4n} = \int_0^\infty \left\{J_n\left[\frac{n}{n+1}H_n(x)\right] - J\left[\frac{n}{n+1}H_n(x)\right]\right\}dF_n(x). \tag{15}$$

The term μ_n is finite by condition (III), and in the appendix we show that C_{in}, $i = 1, 2, 3, 4$, are all $o_p(N^{-1/2})$. Thus we require only to show that $n^{1/2}(B_{1n} + B_{2n} - \mu_n)$ has asymptotically a normal distribution with zero mean and variance σ_n^2, defined by (8). Now,

$$B_{1n} + B_{2n} = \frac{1}{n}\sum_{\alpha=1}^n B(X_\alpha), \tag{16}$$

$$B(X_\alpha) = J[H(|X_\alpha|)]c(X_\alpha) + \int_0^\infty [c(x - |X_\alpha|) - H(x)]J'[H(x)]\,dF(x), \tag{17}$$

where

$$c(u) = \begin{cases} 1, & u \geqq 0, \\ 0, & u < 0. \end{cases}$$

By an application of Fubini's theorem [5], it is easily seen that

$$\mathbf{E}_F\{B(X)\} = \mu_n \quad \text{and} \quad \mathrm{Var}_F\{B(X)\} = \sigma_n^2, \tag{18}$$

defined by (8). Also by condition III, $\mathbf{E}_F|B(x)|^{2+\sigma} < \infty$ for some $\delta > 0$. Since $B_{1n} + B_{2n}$ defined by (16) is the average of n independent and identically distributed random variables having mean μ_n and finite variance, the asymptotic normality of $n^{1/2}(B_{1n} + B_{2n} - \mu_n)$ follows from the central limit theorem.

Corollary. *If* (i) *the assumptions of the above theorem hold, and* (ii) $F(x)$ *is replaced by a sequence of cdfs* $\{F^{(n)}(x)\}$, *where* $F^{(n)}(x) = F(x + \theta/\sqrt{n})$, θ *being real and finite, and* $F(x)$ *is symmetric about* $x = 0$, *then*

$$\lim_{n\to\infty} \sigma_n^2 = \tfrac{1}{4}\int_0^1 J^2(x)\,dx. \tag{19}$$

The proof is an immediate consequence of the above theorem and is therefore omitted.

REMARK 1. Let J be the inverse of the chi-distribution with one degree of freedom. Then the statistic T_n reduces to Fraser's [2] normal scores statistic. The asymptotic Pitman efficiency of this statistic relative to the Student's t-statistic is

$$\sigma^2\left(\int_{-\infty}^{\infty} \frac{f^2(x)\,dx}{\varphi[\Phi^{-1}\{F(x)\}]}\right)^2,$$

where σ^2 is the variance of F, f is the density of F, and φ is the density of the standard cumulative normal distribution function Φ. This efficiency is the same as that of the corresponding tests in the two-sample or c-sample location problem and is known to be at least one for all F (see, e.g., Kendall and Stuart [4], or Puri [6]).

REMARK 2. Let J be the inverse of the rectangular distribution over (0, 1). Then the statistic T_n reduces to the Wilcoxon signed rank statistic. The asymptotic Pitman efficiency of this statistic relative to the Student's t-statistic is $12\sigma^2\left(\int_{-\infty}^{\infty} f^2(x)\,dx\right)^2$. This efficiency is the same as that of the corresponding tests in the two-sample or c-sample location problem and is known to be at least 0.864 for all F (see, e.g., Kendall and Stuart [4]).

3. Appendix. Treatment of Higher Order Terms

The proof that C_{1n} is $o_p(n^{-1/2})$ is essentially the same as in Chernoff and Savage [1], and that $C_{4n} = o_p(n^{-1/2})$ follows from the assumption II. We shall show that $C_{in} = o_p(n^{-1/2})$. Let a_n be a value of x such that $1 - H(a_n) = k_\varepsilon/n$; $S_{n,\varepsilon} = \{x : 0 \leqq x \leqq a_n\}$, where k_ε is so chosen that

$$\mathbf{P}\{|X_\alpha| \in S_{n,\varepsilon} \quad \text{for} \quad \alpha = 1, \cdots, n\} \geqq 1 - \varepsilon/2, \tag{20}$$

ε being arbitrarily small. Denoting $\bar{S}_{n,\varepsilon} = \{x : x > a_n\}$, we have

$$n^{1/2}C_{2n} = \int_{S_{n,\varepsilon}} + \int_{\bar{S}_{n,\varepsilon}} n^{1/2}[H_n(x) - H(x)]J'(H(x))d(F_n(x) - F(x)). \tag{21}$$

From (20), we have with probability greater than (or equal to) $1 - \varepsilon/2$, $H_n(x) = F_n(x) = 1$ for all $x \in \bar{S}_{n,\varepsilon}$. Hence

$$\begin{aligned} \left|\int_{\bar{S}_{n,\varepsilon}} n^{1/2}[H_n - H]J'(H)d[F_n - F]\right| &= \left|n^{1/2}\int_{a_n}^{\infty} [1 - H]J'(H)d[1 - F]\right| \\ &\leqq n^{1/2}\int_{a_n}^{\infty} [1 - H]\,|J'(H)|\,d[1 - H], \end{aligned} \tag{22}$$

since by (4),

$$H(x) \leqq F(x) \Rightarrow 1 - F(x) \leqq 1 - H(x) \quad \text{and} \quad dH(x) \leqq dF(x). \tag{23}$$

Thus, using assumption III, (22) and (23), it readily follows that (22) is bounded by

$$Kn^{1/2}[u^{1/2+\delta}]_0^{k_\varepsilon/n} = Kn^{1/2}[k_\varepsilon/n]^{1/2+\delta} = O(n^{-\delta}) = o(1). \tag{24}$$

Hence

$$n^{1/2}C_{2n} = n^{1/2}\int_{S_{n,\varepsilon}} [H_n(x) - H(x)]J'(H(x))\,d[F_n - F] + o_p(1). \tag{25}$$

Now $H(x)$ and $H_n(x)$ being the true and empirical cdf (respectively) of $|X\alpha|$'s it follows precisely as in [3] that

$$\mathbf{P}\{|n^{1/2}[H_n - H]| \leqq c(\varepsilon)[H(x)(1 - H(x))]^{(1-\delta)/2}\} \geqq 1 - \varepsilon/2, \tag{26}$$

where $c(\varepsilon)$ is a finite constant. Thus, with probability greater than or equal to $1 - \varepsilon/2$,

$$n^{1/2}|[H_n - H]J'(H)| \leqq K[H(1 - H)]^{-1+\delta/2}. \tag{27}$$

Also

$$\int [H(1 - H)]^{-1+\delta/2}\, dH < \infty, \tag{28}$$

and by Khintchine's law of large numbers, we have

$$\int [H(1 - H)]^{-1+\delta/2}\, dH_n \overset{\text{a.s.}}{\to} \int [H(1 - H)]^{-1+\delta/2}\, dH < \infty. \tag{29}$$

Thus noting that $0 < dF_n < dH_n$; $0 < dF < dH$ we get from (28), (29) and the dominated convergence theorem that

$$\int_0^\infty [H(1 - H)]^{-1+\delta/2} d[F_n - F] \overset{\text{a.s.}}{\to} 0. \tag{30}$$

Hence, from (25), (27), and (30), we obtain $C_{2n} = o_p(n^{-1/2})$. Finally, noting again that $0 < dF_n < dH_n$, the proof of $C_{3n} = o_p(n^{-1/2})$ follows as an immediate corollary to the corresponding proof for C_{3n} in the two-sample case considered in Govindarajulu, LeCam and Raghavachari [3].

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A REMARK ON WALD'S SEQUENTIAL PROBABILITY RATIO TEST

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(*Translated by B. Seckler*)

It is well-known that Wald's probability ratio test, which is used to test two simple hypotheses $H_0:\theta = \theta_0$ against $H_1:\theta = \theta_1$, is executed as follows: Two positive quantities A and B are chosen ($A > B$). If

$$\log B < \sum_{i=1}^{m} z_i < \log A, \tag{1}$$

where $z_i = \log[f(x_i, \theta_1)/f(x_i, \theta_0)]$ and $f(x, \theta)$ is the probability density of the random variable x, the experiment is continued and the next observation x_{m+1} is made; if $\sum_{i=1}^m z_i \geqq \log A$, hypothesis H_0 is rejected (hypothesis H_1 is accepted). Finally, if $\sum_{i=1}^m z_i \leqq \log B$, hypothesis H_0 is accepted.