Theory Probab. Appl. Translated from Russian Journal
Vol. 41, No. 3

NORMAL APPROXIMATION OF U-STATISTICS IN HILBERT SPACE*

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(*Translated by V. V. Sazonov*)

Abstract. Let $\{U_n\}$, $n = 1, 2, \ldots$, be Hilbert space H-valued U-statistics with kernel $\Phi(\cdot\,,\cdot)$, corresponding to a sequence of observations (random variables) $X_1, X_2, \ldots$. The rate of convergence on balls in the central limit theorem for $\{U_n\}$ is investigated. The obtained estimate is of order $n^{-1/2}$ and depends explicitly on $\mathbf{E}\|\Phi(X_1, X_2)\|^3$ and on the trace and the first nine eigenvalues of the covariance operator of $\mathbf{E}(\Phi(X_1, X_2)|X_1)$.

Key words. U-statistic, Hilbert space, central limit theorem, normal (Gaussian) approximation, rate of convergence

PII. S0040585X97975198

1. Introduction. Let $X_1, \ldots, X_n$ be independent random variables with values in a measurable space $(X, \mathcal{X})$ having on it the same distribution $\mathbf{P}$. Let H be a real separable Hilbert space with inner product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$. Consider the UH-statistic

$$U_n = \binom{n}{2}^{-1} \sum_{1 \leqq i < j \leqq n} \Phi(X_i, X_j)$$

with a symmetric kernel Φ: $X^2 \to H$ such that $\mathbf{E}\Phi(X_1, X_2) = 0$, $\mathbf{E}\|\Phi(X_1, X_2)\| < \infty$.

By the Hoeffding decomposition,

$$U_n = 2n^{-1} \sum_{j=1}^{n} g_1(X_j) + 2n^{-1}(n-1)^{-1} \sum_{1 \leqq i < j \leqq n} g_2(X_i, X_j), \tag{1}$$

where

$$g_1(x) = \int_X \Phi(x, y)\,\mathbf{P}(dy) = \mathbf{E}\big(\Phi(X_1, X_2) \mid X_1 = x\big),$$
$$g_2(x, y) = \Phi(x, y) - g_1(x) - g_1(y).$$

We assume that $\sigma^2 = \mathbf{E}\|g_1(X_1)\|^2 > 0$ and observe that the definitions of $g_1(x)$ and $g_2(x)$ imply

$$\int_X g_2(x, y)\,\mathbf{P}(dy) = 0 \quad \text{(a.s.)}. \tag{2}$$

*Received by the editors May 17, 1994. This research was supported by the Office of Naval Research Contract N 00014-91-J-1020. The results of this paper were partially obtained during the visit of the last author to the Department of Management, Hong Kong University of Science and Technology, winter 1993–1994.

http://www.siam.org/journals/tvp/41-3/97519.html

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Assuming also that $\sigma < \infty$ we can define the covariance operator V of $g_1(X_1)$ in the usual way:

$$(Vx,\, y) = \int_X \bigl(x,\, g_1(z)\bigr)\bigl(y,\, g_1(z)\bigr)\,\mathbf{P}(dz), \quad x,\, y \in H.$$

Let Y be a Gaussian H-valued random variable with mean zero and covariance operator $\sigma^{-2}V$. Denote by $\sigma_1^2 \geqq \sigma_2^2 \geqq \cdots$ the eigenvalues of V. Finally, for any $r \geqq 0$, $a \in H$, let $B_r(a) = \{x \in H\colon\ \|x - a\| < r\}$ and $B_r = B_r(0)$.

By the central limit theorem for UH-statistics, if

$$\mathbf{E}\bigl\|g_1(X_1)\bigr\|^2 < \infty, \qquad \mathbf{E}\bigl\|g_2(X_1, X_2)\bigr\|^{4/3} < \infty,$$

then (see [1], [2], [5])

$$\Delta_n(a) = \sup_r \Bigl|\mathbf{P}\bigl\{\|2^{-1}n^{1/2}\sigma^{-1}U_n - a\| < r\bigr\} - \mathbf{P}\bigl\{\|Y - a\| < r\bigr\}\Bigr| \longrightarrow 0, \quad n \to \infty.$$

The rate of convergence of $\Delta_n(0)$ to zero with respect to n was estimated in [1], [2], [5], [12], where under the additional moment conditions

$$\mathbf{E}\bigl\|g_1(X_1)\bigr\|^3 < \infty, \qquad \mathbf{E}\bigl\|g_2(X_1, X_2)\bigr\|^2 < \infty$$

it was shown that

$$\Delta_n(0) = \rho_n(V)\, n^{-1/2}$$

with $\rho_n(V) \to 0$ as $n \to \infty$. To prove this, it was observed that the problem of estimating the probability for a value of the UH-statistic to be in the ball B_r may be treated as the problem of estimating the rate of convergence in the central limit theorem in Hilbert space for balls with a nonzero random center which tends to zero with probability 1 as $n \to \infty$. However, it was assumed in [1], [2], [5], [12] that the number of nonzero eigenvalues of the covariance operator V is sufficiently large. The explicit dependence of $\rho_n(V)$ on V in [1], [2], [5], [12] was not determined. Estimates of order $O(n^{-1/3})$ for any $a \in H$ with explicit dependence on the second and third moments were obtained in [14].

Note that in the one-dimensional case the study of the problem considered in the present paper has a rather long history. It was initiated in 1973 by Grams and Serfling [11] and later a number of gradual improvements of their result was made (for details see [13] and [10]). In [10] it was also shown that the estimate obtained by Korolyuk and Borovskikh in [4] is essentially unimprovable.

In the present paper we study the dependence of $\Delta_n(a)$ on n, the covariance operator V and $\beta = \mathbf{E}\|\Phi(X_1, X_2)\|^3$. When $\beta < \infty$ an estimate of order $n^{-1/2}$ is obtained, the dependence of which on Φ involves only β, the first nine eigenvalues of V and $\sigma^2 = \operatorname{tr} V$. The order of dependence of our estimate on n is precise, but its dependence on σ_j^2, σ_j^2 and β apparently admits improvement. To prove the estimate we employ the methods developed in [3] and [12].

2. The theorem.

THEOREM. *There exists an absolute constant c such that, for all $n \geqq 2$ and $a \in H$,*

$$\Delta_n(a) \leqq c\,\bigl(1 + \|a\|^3\bigr)\sigma^4 \quad \prod_{j=1}^{9} \sigma_j^{-1}\Biggr)\sigma_9^{-1}\beta^2 n^{-1/2}. \tag{3}$$

In what follows we will assume that $\sigma = 1$. The general case is reduced to this one if we replace Φ by $\sigma^{-1}\Phi$. We can assume also that $\beta < \infty$ and $\sigma_9 > 0$ since otherwise (3) would be obvious.

In the proof of the theorem the upper estimates for $\Delta_n(a)$ are calculated more precisely; except for n and V, they depend only on the moments

$$\mathbf{E}\|g_1\|^3 = \mathbf{E}\|g_1(X_1)\|^3, \qquad \mathbf{E}\|g_2\|^3 = \mathbf{E}\|g_2(X_1, X_2)\|^3,$$

which are related to β by the inequalities

$$\mathbf{E}\|g_1\|^3 \leqq \beta, \qquad \mathbf{E}\|g_2\|^3 \leqq 27\beta. \tag{4}$$

We formulate the theorem using only β for the sake of simplicity.

3. Proof of the theorem. Below $c, c_1, c_2, \ldots$ denote absolute constants; the same symbol may stand for different constants. The proof is divided into several lemmas.

LEMMA 1. *We have*

$$\Delta_n(a) \leqq \Delta_{n1}(a) + n\,\mathbf{P}\big\{\|g_1(X_1)\| > n^{1/2}\big\}, \tag{5}$$

where

$$\Delta_{n1}(a) = \sup_r \Big|\mathbf{P}\big\{\|\overline{S}_n - a + V_n\| < r\big\} - \mathbf{P}\big\{\|Y - a\| < r\big\}\Big|,$$

$$\overline{S}_n = n^{-1/2} \sum_{j=1}^{n} g_1(X_j)\, I\Big(\big\|g_1(X_j)\big\| \leqq n^{1/2}\Big),$$

$I(A)$ is the indicator function of the set A and

$$V_n = (n-1)^{-1} n^{-1/2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} g_2(X_i, X_j).$$

Proof. According to (1),

$$2^{-1} n^{1/2} U_n - a = S_n - a + V_n,$$

where

$$S_n = n^{-1/2} \sum_{j=1}^{n} g_1(X_j)$$

and we obviously have

$$\Delta_n(a) \leqq \Delta_{n1}(a) + \sup_r \Big|\mathbf{P}\big\{\|S_n - a + V_n\| < r\big\} - \mathbf{P}\big\{\|\overline{S}_n - a + V_n\| < r\big\}\Big|.$$

Now applying Lemma 1 of [17] with $f(x) = I(\|x\| < r)$, $u_1 = \infty$, $u_2 = n^{1/2}$ and $Z = -a + V_n$, we see that the second summand in the right-hand side of this inequality is not greater than $n\mathbf{P}\{\|g_1(X_1)\| > n^{1/2}\}$ and the lemma follows.

Let $k = n - [n^{2/3}]$, where $[n^{2/3}]$ is the integral part of $n^{2/3}$ and denote

$$\alpha_k = \sum_{1 \leqq i<j \leqq k} g_2(X_i, X_j), \qquad \beta_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} g_2(X_i, X_j),$$

$$\delta_k = \sum_{k+1 \leqq i<j \leqq n} g_2(X_i, X_j), \qquad \overline{S}_n^k = n^{-1/2} \sum_{j=1}^{k} \overline{g}_1(X_j),$$

$$\overline{g}_1(X_j) = g_1(X_j)\, I\Big(\|g_1(X_j)\| \leqq n^{1/2}\Big), \quad j = 1, \ldots, n,$$

$$Z_n = 2\varepsilon_n\big(\overline{S}_n^k - a,\, \alpha_k\big), \quad \varepsilon_n = (n-1)^{-1} n^{-1/2}.$$

Lemma 2. *For $\Delta_{n1}(a)$ we have*

$$\Delta_{n1}(a) \leqq \Delta_{n2}(a) + c_1\big(1 + \|a\|^3\big)\, \big(\beta^2 + \sigma_1^{-1}\sigma_2^{-1}\big)\, n^{-1/2}, \tag{6}$$

where

$$\Delta_{n2}(a) = \sup_r \Big|\mathbf{P}\big\{\|\overline{S}_n - a\|^2 + Z_n < r\big\} - \mathbf{P}\big\{\|Y - a\|^2 < r\big\}\Big|.$$

Proof. First we observe that

$$\begin{aligned} \big\|\overline{S}_n - a + V_n\big\|^2 &= (\overline{S}_n - a + V_n,\, \overline{S}_n - a + V_n) \\ &= \big\|\overline{S}_n - a\big\|^2 + 2(\overline{S}_n - a,\, V_n) + \|V_n\|^2. \end{aligned} \tag{7}$$

In our notation,

$$\|V_n\|^2 = (n-1)^{-2} n^{-1} \left\| \sum_{j=2}^{n} \sum_{i=1}^{j-1} g_2(X_i, X_j) \right\|^2,$$

$$2\,(\overline{S}_n - a,\, V_n) = Z_n + 2\varepsilon_n\big(\overline{S}_n^k - a,\, \beta_k + \delta_k\big) + 2\,\big(\overline{S}_n - \overline{S}_n^k,\, V_n\big),$$

and

$$\big(\overline{S}_n^k - a,\, \beta_k + \delta_k\big) = \big(\overline{S}_n^k,\, \beta_k\big) - \big(a,\, \beta_k\big) + \big(\overline{S}_n^k,\, \delta_k\big) - \big(a,\, \delta_k\big),$$

$$\big(\overline{S}_n - \overline{S}_n^k,\, V_n\big) = \varepsilon_n\big(\overline{S}_n - \overline{S}_n^k,\, \alpha_k\big) + \varepsilon_n\big(\overline{S}_n - \overline{S}_n^k,\, \beta_k\big) + \varepsilon_n\big(\overline{S}_n - \overline{S}_n^k,\, \delta_k\big).$$

In what follows we will need the following inequalities: for any $p \geqq 2$,

$$\mathbf{E}\big\|\overline{S}_n^k\big\|^p \leqq c_p, \qquad \big\|\mathbf{E}\,\overline{g}_1(X_1)\big\| = \left\| \int_{\|g_1(x)\| > n^{1/2}} g_1(x)\, \mathbf{P}(dx) \right\| \leqq n^{-1/2} \tag{8}$$

(see [9], [16]) and

$$\begin{aligned} \mathbf{E}\|\alpha_k\|^p &\leqq c_p \mathbf{E}\|g_2\|^p k^p, \\ \mathbf{E}\|\beta_k\|^p &\leqq c_p \mathbf{E}\|g_2\|^p k^{p/2} (n-k)^{p/2}, \\ \mathbf{E}\|\delta_k\|^p &\leqq c_p \mathbf{E}\|g_2\|^p (n-k)^p \end{aligned} \tag{9}$$

(see [5]).

The Chebychev inequality and (9) imply

$$\mathbf{P}\big\{\|V_n\|^2 \geqq n^{-1/2}\big\} \leqq n^{1/2} \mathbf{E}\,\|V_n\|^2 \leqq c_1 \mathbf{E}\,\|g_2\|^2 n^{-1/2}.$$

Furthermore,

$$\mathbf{P}\Big\{\big|(\overline{S}_n^k, \varepsilon_n\beta_k)\big| > n^{-1/2}\Big\} \leqq \Pi_1 + \Pi_2, \tag{10}$$

where

$$\Pi_1 = \mathbf{P}\Bigg\{2k\varepsilon_n\Bigg|\ \ \mathbf{E}\overline{g}_1(X_1),\ \sum_{i=1}^k h(X_i)\Bigg)\Bigg| > 1\Bigg\},$$
$$\Pi_2 = \mathbf{P}\Bigg\{2\varepsilon_n\Bigg|\ \ \sum_{i=1}^k \overline{g}(X_i),\ \sum_{i=1}^k h(X_i)\Bigg)\Bigg| > 1\Bigg\},$$

$$\overline{g}(X_i) = \overline{g}_1(X_i) - \mathbf{E}\overline{g}_1(X_1), \qquad h(X_i) = \sum_{j=k+1}^n g_2(X_i, X_j), \quad i = 1, \dots, k.$$

For Π_1 by the Chebychev inequality and (8), (9)

$$\Pi_1 \leqq (2k\varepsilon_n)^3 \big\|\mathbf{E}\, g_1(X_1)\big\|^3 \mathbf{E}\Bigg\|\sum_{i=1}^k h(X_i)\Bigg\|^3 \leqq c_1 \mathbf{E}\|g_2\|^3 n^{-1/2}.$$

To estimate Π_2 we first observe that

$$\sum_{i=1}^k \overline{g}(X_i),\ \sum_{i=1}^k h(X_i)\Bigg) = \sum_{i=1}^4 A_i,$$

where

$$A_1 = k\,\mathbf{E}_1\big(\overline{g}(X_1),\, h(X_1)\big),$$
$$A_2 = \sum_{i=1}^k \Big(\big(\overline{g}(X_i),\, h(X_i)\big) - \mathbf{E}_1\big(\overline{g}(X_1),\, h(X_1)\big)\Big),$$
$$A_3 = \sum_{i_1=2}^k \sum_{i_2=1}^{i_1-1} \big(\overline{g}(X_{i_1}),\, h(X_{i_2})\big), \qquad A_4 = \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \big(\overline{g}(X_{i_1}),\, h(X_{i_2})\big)$$

and, obviously,

$$\mathbf{E}_k|A_1|^3 \leqq k^3\Big|\mathbf{E}_1\big(\overline{g}(X_1),\, h(X_1)\big)\Big|^3$$

by [8]

$$\mathbf{E}_k|A_2|^3 \leqq c_1 k^{3/2}\mathbf{E}_1\Big|\big(\overline{g}(X_1),\, h(X_1)\big)\Big|^3$$

and by [9]

$$\mathbf{E}_k|A_3|^3 \leqq c_1 k^3 \mathbf{E}_2\Big|\big(\overline{g}(X_1),\, h(X_2)\big)\Big|^3, \qquad \mathbf{E}_k|A_4|^3 \leqq c_1 k^3 \mathbf{E}_2\Big|\big(\overline{g}(X_1),\, h(X_2)\big)\Big|^3,$$

$\mathbf{E}_k(\cdot)$ denotes the expectation with respect to $X_1, \dots, X_k$ with fixed $X_{k+1}, \dots, X_n$, $k = 1, \dots, n$.

Moreover, since $\mathbf{E}\|g_1(X_1)\|^2 = 1$, we have

$$\Big|\mathbf{E}\big(\overline{g}(X_1),\, h(X_1)\big)\Big|^3 \leqq \mathbf{E}_1\big\|h(X_1)\big\|^3.$$

And, obviously,

$$\mathbf{E}_1\Big|\big(\overline{g}(X_1),\,h(X_1)\big)\Big|^3 \leqq 8n^{3/2}\mathbf{E}_1\big\|h(X_1)\big\|^3,$$
$$\mathbf{E}_2\Big|\big(\overline{g}(X_1),\,h(X_2)\big)\Big|^3 \leqq c_1\mathbf{E}_1\big\|\overline{g}(X_1)\big\|^3\mathbf{E}_1\big\|h(X_1)\big\|^3$$

and by [8]

$$\overline{\mathbf{E}}_k\big\|h(X_1)\big\|^3 \leqq c_1(n-k)^{3/2}\overline{\mathbf{E}}_k\big\|g_2(X_1, X_{k+1})\big\|^3,$$

$\overline{\mathbf{E}}_k(\cdot)$ denotes the expectation with respect to $X_{k+1},\dots,X_n$ with fixed $X_1,\dots,X_k$, $k=1,\dots,n$.

The above implies

$$\Pi_2 \leqq c_1\mathbf{E}\|g_1\|^3\mathbf{E}\|g_2\|^3 n^{-1/2}$$

and, hence we obtain an estimate for the probability in the left side of (10).

Similarly, using (8), (9) we get

$$\begin{aligned}
&\mathbf{P}\Big\{\big|c\,(\overline{S}_n^k,\,\varepsilon_n\delta_k)\big| > n^{-1/2}\Big\} \leqq \mathbf{P}\Big\{c\|(\overline{S}_n^k\| > n^{1/6}\Big\} + \mathbf{P}\Big\{n^{1/6}\varepsilon_n\|\delta_k\| > n^{-1/2}\Big\}\\
&\quad\leqq c^3\mathbf{E}\|\overline{S}_n^k\|^3 n^{-1/2} + n^{1/2}n^{3/2}\varepsilon_n^3\mathbf{E}\|\delta_k\|^3 \leqq c_1 n^{-1/2} + c_2\mathbf{E}\|g_2\|^3 n^{-1/2},\\
&\mathbf{P}\Big\{\big|c\,(a,\,\varepsilon_n\beta_k)\big| > n^{-1/2}\Big\} \leqq c^3\|a\|^3\varepsilon_n^3 n^{3/2}\mathbf{E}\|\beta_k\|^3 \leqq c_1\mathbf{E}\|a\|^3\mathbf{E}\|g_2\|^3 n^{-1/2},\\
&\mathbf{P}\Big\{\big|c\,(a,\,\varepsilon_n\delta_k)\big| > n^{-1/2}\Big\} \leqq c^3\|a\|^3\varepsilon_n^3 n^{3/2}\mathbf{E}\|\delta_k\|^3 \leqq c_1\mathbf{E}\|a\|^3\mathbf{E}\|g_2\|^3 n^{-1/2},\\
&\mathbf{P}\Big\{\big|c\,(\overline{S}_n - \overline{S}_n^k,\,\varepsilon_n\alpha_k)\big| > n^{-1/2}\Big\} \leqq c^3 n^{3/2}\mathbf{E}\|\overline{S}_n - \overline{S}_n^k\|^3\varepsilon_n^3\mathbf{E}\|\alpha_k\|^3\\
&\quad\leqq c_1\mathbf{E}\|g_1\|^3\mathbf{E}\|g_2\|^3 n^{-1/2},\\
&\mathbf{P}\Big\{c\big|(\overline{S}_n - \overline{S}_n^k,\,\varepsilon_n\beta_k)\big| > n^{-1/2}\Big\} \leqq \mathbf{P}\big\{c\|\overline{S}_n - \overline{S}_n^k\| > 1\big\} + \mathbf{P}\big\{\varepsilon_n\|\beta_k\| > n^{-1/2}\big\}\\
&\quad\leqq c^3\mathbf{E}\big\|\overline{S}_n - \overline{S}_n^k\big\|^3 + n^{3/2}\varepsilon_n^3\mathbf{E}\|\beta_k\|^3 \leqq c_1\mathbf{E}\|g_1\|^3 n^{-1/2} + c_2\mathbf{E}\|g_2\|^3 n^{-1/2},\\
&\mathbf{P}\Big\{\big|c\,(\overline{S}_n - \overline{S}_n^k,\,\varepsilon_n\delta_k)\big| > n^{-1/2}\Big\} \leqq \mathbf{P}\big\{c\|\overline{S}_n - \overline{S}_n^k\| > 1\big\} + \mathbf{P}\big\{\varepsilon_n\|\delta_k\| > n^{-1/2}\big\}\\
&\quad\leqq c^3\mathbf{E}\big\|\overline{S}_n - \overline{S}_n^k\big\|^3 + n^{3/2}\varepsilon_n^3\mathbf{E}\|\delta_k\|^3 \leqq c_1\mathbf{E}\|g_1\|^3 n^{-1/2} + c_2\mathbf{E}\|g_2\|^3 n^{-1/2}.
\end{aligned}$$

Denote now

$$Y_n = 2\varepsilon_n\big(\overline{S}_n^k - a,\,\beta_k + \delta_k\big) + 2\big(\overline{S}_n - \overline{S}_n^k,\,V_n\big) + \|V_n\|^2.$$

Then, according to (7),

$$\big\|\overline{S}_n - a + V_n\big\|^2 = \big\|\overline{S}_n - a\big\|^2 + Z_n + Y_n$$

and the estimates above imply

$$\mathbf{P}\big\{|Y_n| > n^{-1/2}\big\} \leqq c_1\big(1 + \|a\|^3\big)\,\mathbf{E}\|g_1\|^3\mathbf{E}\|g_2\|^3 n^{-1/2}. \tag{11}$$

Since, for all $r \geqq 0$,

$$\begin{aligned}
&\mathbf{P}\big\{\|\overline{S}_n - a\|^2 + Z_n < r - n^{-1/2}\big\} - \mathbf{P}\big\{|Y_n| > n^{-1/2}\big\}\\
&\quad\leqq \mathbf{P}\big\{\|\overline{S}_n - a\|^2 + Z_n + Y_n < r\big\}\\
&\quad\leqq \mathbf{P}\big\{\|\overline{S}_n - a\|^2 + Z_n < r + n^{-1/2}\big\} + \mathbf{P}\big\{|Y_n| > n^{-1/2}\big\},
\end{aligned}$$

we have

(12) $$\Delta_{n1}(a) \leqq \Delta_{n2}(a) + \mathbf{P}\{|Y_n| > n^{-1/2}\} + \sup_r \mathbf{P}\{r - n^{-1/2} \leqq \|Y - a\|^2 \leqq r + n^{-1/2}\}.$$

In [6] it is proved that

(13) $$\sup_r \mathbf{P}\{r - n^{-1/2} \leqq \|Y - a\|^2 \leqq r + n^{-1/2}\} \leqq c\,\sigma_1^{-1}\sigma_2^{-1} n^{-1/2}.$$

Estimates (11)–(13) imply (6).

Denote now

$$\varphi_n(t) = \mathbf{E}\exp\big(it\,\|\overline{S}_n - a\|^2 + it\,Z_n\big),$$
$$\varphi(t) = \mathbf{E}\exp\big(it\,\|Y - a\|^2\big),$$
$$\psi_n(t) = \mathbf{E}\exp\big(it\,\|\overline{S}_n - a\|^2\big).$$

The density of the random variable $\|Y - a\|^2$ is not greater than $c(\sigma_1\sigma_2)^{-1}$ (see [6]). Thus for $\Delta_{n2}(a)$ in (6), applying Esséen's inequality (see, e.g. [7]) for any $T > 0$ we have

(14) $$\Delta_{n2}(a) \leqq c_1 \int_{-T}^{T} \big|\varphi_n(t) - \varphi(t)\big|\,|t|^{-1} dt + c_2(\sigma_1\sigma_2)^{-1} T^{-1}.$$

We choose in (14)

(15) $$T = \frac{c\,n^{1/2}}{L}, \quad L = 3\sigma_9^{-2}\mathbf{E}\,\|g_1\|^3,\ \sigma_9 > 0,$$

where c is a small enough absolute constant and then apply Lemma 12 of [3]. Note that with L as in (15) the following conditions of Lemma 12 in [3]

(16) $$\frac{2}{3} \leqq L^2 \leqq \frac{n}{2e}, \qquad \mathbf{P}(B_L) \geqq \frac{1}{2}$$

are fulfilled (for a proof see [3], [9], [16]). When (16) is satisfied the distribution of $\overline{g}_1 = g_1(X_1)\,I(\|g_1(X_1)\| \leqq n^{1/2})$ may be represented as $(Q + Q_1)/2$, where

$$Q(A) = \frac{\mathbf{P}(A \cap B_L)}{\mathbf{P}(B_L)}, \quad A \in H,$$

and Q_1 is a probability measure (see [15]).

Let V_1 be the covariance operator of Q and $\sigma_{11}^2 \geqq \sigma_{12}^2 \geqq \cdots$ be its eigenvalues. By our assumptions, $\operatorname{tr} V = 1$. Hence (see [3]) $\operatorname{tr} V_1 \leqq 2$ and

(17) $$\frac{2\sigma_j^2}{3} \leqq \sigma_{1j}^2 \leqq 2\sigma_j^2, \quad j = 1, \dots, 9.$$

To estimate the integral in (14) we will need the following generalization of Lemma 11 in [3] (see also [9, Lemma 4]).

Lemma 3. *Denote*

$$Z = n^{-1/2} \sum_{j=1}^{l+m} \overline{g}_1(X_j), \qquad Z_1(f) = \sum_{j=1}^{l+m} f(X_j),$$

where $\overline{g}_1 = g_1(X_1)\, I(\|g_1(X_1)\| \leqq n^{1/2})$ *and* $f\colon\ X \to \mathbf{R}$ *is a measurable function, l, m are positive integers, $l \leqq m$, $l + m \leqq n$. Then, for any $A > 0$, even $k \geqq 0$, integer $r \geqq 0$, $k_j \geqq 0$, $j = 1, \dots, r$, and any $x, x_j \in H$, $j = 1, \dots, r$, for $|t| \leqq c(A)\, nL^{-1}(l\log(l/L^2))^{-1/2}$ if $l > L^2$, and for any t if $l \leqq L^2$*

$$\left|\mathbf{E}\exp\left(it\,\|Z+x\|^2 + it\,Z_1(F)\right)\|Z\|^k \prod_{j=1}^{r}(x_j, Z)^{k_j}\right|$$

$$\leqq c_1 \prod_{j=1}^{r}\|x_j\|^{k_j} \quad \exp(-c_2 l) + c_3\left(\frac{L^2}{l}\right)^A + h^{1/2}\left(\frac{c_4 T_1^2 lm}{n^2},\, V_1\right)\Bigg), \tag{18}$$

where c_1, c_2 are functions of $\overline{k} = k + k_1 + \dots + k_r$; c_3, c_4 are functions of A and $\overline{k}$

$$T_1 = \min\left\{|t|,\ nL^{-1}\left(m\log\left(\frac{m}{L^2}\right)\right)^{-1/2}\right\}, \qquad h(s,\, V_1) = \prod_{j=1}^{\infty}\left(1 + 2s\sigma_{1j}^4\right)^{-1/2}.$$

Proof. This lemma reduces to Lemma 11 in [3] if $Z_1(f) = 0$. The proof of the present lemma is essentially the same as the proof of Lemma 11 in [3].

Let us estimate the integral in (14). Obviously,

$$\int_{-T}^{T}\left|\varphi_n(t) - \varphi(t)\right|\, |t|^{-1} dt \leqq I_1 + I_2, \tag{19}$$

where

$$I_1 = \int_{-T}^{T}\left|\ \ _n(t) - \varphi(t)\right|\, |t|^{-1} dt, \qquad I_2 = \int_{-T}^{T}\left|\varphi_n(t) - \psi_n(t)\right|\, |t|^{-1} dt.$$

By Lemma 12 in [3]

$$I_1 \leqq c_1 U, \tag{20}$$

where

$$\begin{aligned} U = {} & c_7(V)\Big(\mathbf{E}\big(\|g_1\|^2 + (a,\, g_1)^2\big)\, I\big(\|g_1\| > n^{1/2}\big)\Big) \\ & + \Big(\mathbf{E}\big(\|g_1\|^3 + \left|(a,\, g_1)\right|^3\big)\, I\big(\|g_1\| \leqq n^{1/2}\big) + (Va,\, a)^{3/2}\Big)\, n^{-1/2} \\ & + c_3(V)\,\sigma_9^{-2}\mathbf{E}\|g_1\|^3 n^{-1/2}, \quad c_s(V) = \prod_{j=1}^{s}\sigma_j^{-1}, \end{aligned}$$

and for U we have

$$\begin{aligned} U & \leqq \left(1 + \|a\|^3\right) c_7(V)\,\mathbf{E}\,\|g_1\|^3 n^{-1/2} + c_3(V)\,\sigma_9^{-2}\mathbf{E}\,\|g_1\|^3 n^{-1/2} \\ & \leqq 2\left(1 + \|a\|^3\right)\beta\, c_9(V)\,\sigma_9^{-1} n^{-1/2}. \end{aligned}$$

Consider now I_2 in (19). Obviously,

$$\varphi_n(t) - \psi_n(t) = \mathbf{E}\exp\left\{it\,\|\overline{S}_n - a\|^2\right\}\left(e^{it\,Z_n} - 1\right).$$

Next we will use the representation

$$e^z - 1 = \sum_{s=1}^{r-1}\frac{z^s}{s!} + z^r \int_0^1 \frac{(1-u)^{r-1}e^{uz}}{(r-1)!}\, du, \tag{21}$$

valid for all complex z and positive integers r, with $r = 3$, $z = it\, Z_n$. Thus we have

$$\varphi_n(t) - \psi_n(t) = it\, J_1(t) - \frac{t^2}{2}\, J_2(t) + \frac{(it)^3}{2}\, J_3(t), \tag{22}$$

where

$$J_1(t) = \mathbf{E}\Big(Z_n \exp\big\{it\, \|\overline{S}_n - a\|^2\big\}\Big), \qquad J_2(t) = \mathbf{E}\Big(Z_n^2 \exp\big\{it\, \|\overline{S}_n - a\|^2\big\}\Big),$$
$$J_3(t) = \int_0^1 du\, (1-u)^2 \mathbf{E}\Big(Z_n^3 \exp\big\{it\, \|\overline{S}_n - a\|^2 + itu\, Z_n\big\}\Big).$$

LEMMA 4. *If* $|t| \leqq T = cn^{1/2}L^{-1}$, *then*

$$\begin{aligned}\big|J_3(t)\big| \leqq c_1 \quad & L^{3/2}n^{-1/2} + \prod_{j=1}^{7}\left(1 + \frac{c_2\sigma_j^4 t^2}{n^{2/3}}\right)^{-1/4}\Bigg) \\ & \times \big(\mathbf{E}\, \|g_1\|^3 + \|a\|^3\big)\, \mathbf{E}\, \|g_2\|^3 n^{-3/2}. \end{aligned} \tag{23}$$

Proof. The random variable Z_n does not depend on $X_{k+1}, \dots, X_n$. Hence,

$$\left|\mathbf{E}\Big(Z_n^3 \exp\big\{it\, \|\overline{S}_n - a\|^2 + it\, uZ_n\big\}\Big)\right| \leqq \mathbf{E}_k|Z_n|^3 \left|\overline{\mathbf{E}}_k \exp\Big\{it\, \big\|\overline{S}_n - a\big\|^2\Big\}\right|,$$

where as above $\overline{\mathbf{E}}_k(\cdot)$ denotes expectation with respect to $X_{k+1}, \dots, X_n$ with fixed $X_1, \dots, X_k$. Representing $\overline{S}_n = (\overline{S}_n - \overline{S}_n^k) + \overline{S}_n^k$ we apply Lemma 3 with $Z = \overline{S}_n - \overline{S}_n^k$, $1 \sim n^{2/3}/4$, $m \sim 3n^{2/3}/4$, $A = \frac{3}{4}$. Noting that here $T_1 = |t|$, we obtain

$$\begin{aligned} & \left|\overline{\mathbf{E}}_k \exp\big\{it\, \|\overline{S}_n - a\|^2\big\}\right| \\ & \leqq c_1 \quad \exp(-c_2 n^{2/3}) + c_3 L^{3/2} n^{-1/2} + \prod_{j=1}^{7}\left(1 + \frac{c_4\sigma_j^4 t^2}{n^{2/3}}\right)^{-1/4}\Bigg). \end{aligned} \tag{24}$$

To estimate $\mathbf{E}|Z_n|^3$ we first write

$$\mathbf{E}\, |Z_n|^3 \leqq 32\, \varepsilon_n^3 \Big(\mathbf{E}\big|(\overline{S}_n^k,\, \alpha_k)\big|^3 + \|a\|^3 \mathbf{E}\, \|\alpha_k\|^3\Big). \tag{25}$$

Then we represent the scalar product $(\overline{S}_n^k, \alpha_k)$ in the form

$$\begin{aligned} (\overline{S}_n^k,\, \alpha_k) = n^{-1/2} \quad & \sum_{j=1}^{k} \overline{g}(X_j), \sum_{1 \leqq i_1 < i_2 \leqq k} g_2\big(X_{i_1}, X_{i_2}\big)\Bigg) \\ & + n^{-1/2} k \Bigg(\mathbf{E}\overline{g}_1(X_1), \sum_{1 \leqq i_1 < i_2 \leqq k} g_2\big(X_{i_1}, X_{i_2}\big)\Bigg). \end{aligned} \tag{26}$$

Using (8)–(9) we have

$$\begin{aligned} & \mathbf{E}\left| \quad \mathbf{E}\overline{g}_1(X_1), \sum_{1 \leqq i_1 < i_2 \leqq k} g_2\big(X_{i_1}, X_{i_2}\big)\Bigg)\right|^3 \\ & \leqq \big\|\mathbf{E}\overline{g}_1(X_1)\big\|^3 \mathbf{E}\left\|\sum_{1 \leqq i_1 < i_2 \leqq k} g_2\big(X_{i_1}, X_{i_2}\big)\right\|^3 \leqq c_1 n^{-3/2} k^3 \mathbf{E}\|g_2\|^3. \end{aligned}$$

To estimate the first term in the right side of (26) we will use the representation

$$\sum_{j=1}^{k}\bar{g}_1(X_j),\quad \sum_{1\leqq i_1<i_2\leqq k} g_2\big(X_{i_1},X_{i_2}\big)\bigg)$$
$$=\sum_{j=1}^{k}\sum_{1\leqq i_1<i_2\leqq k}\big(\bar{g}_1(X_j),\,g_2\big(X_{i_1},X_{i_2}\big)\big)=\sum_{i=1}^{5}G_i,$$

where

$$G_1=\sum_{j=3}^{k}\sum_{i_2=2}^{j-1}\sum_{i_1=1}^{i_2-1}\big(\bar{g}(X_j),\,g_2(X_{i_1},X_{i_2})\big),$$
$$G_2=\sum_{j=2}^{k}\sum_{i_1=1}^{j-1}\big(\bar{g}(X_j),\,g_2(X_{i_1},X_j)\big),$$
$$G_3=\sum_{j=2}^{k-1}\sum_{i_1=1}^{j-1}\sum_{i_2=j+1}^{k}\big(\bar{g}(X_j),\,g_2(X_{i_1},X_{i_2})\big),$$
$$G_4=\sum_{j=1}^{k}\sum_{i_2=j+1}^{k}\big(\bar{g}(X_j),\,g_2(X_j,X_{i_2})\big),$$
$$G_5=\sum_{j=1}^{k}\sum_{i_2=j+2}^{k}\sum_{i_1=j+1}^{i_2-1}\big(\bar{g}(X_j),\,g_2(X_{i_1},X_{i_2})\big).$$

By (9) we have

$$\begin{aligned}&\mathbf{E}|G_i|^3\leqq c_1\mathbf{E}\|g_1\|^3\mathbf{E}\|g_2\|^3n^{9/2}, && i=1,\,3,\,5,\\ &\mathbf{E}|G_i|^3\leqq c_2\mathbf{E}\|g_2\|^3n^{9/2}, && i=2,\,4.\end{aligned}\tag{27}$$

Combining (26) and (27) we obtain

$$\mathbf{E}|Z_n|^3\leqq c_1\big(\mathbf{E}\|g_1\|^3+\|a\|^3\big)\,\mathbf{E}\|g_2\|^3n^{-3/2}.$$

Together with (24) this implies (23).

For $m\sim n$ partition the interval $|t|\leqq T=cn^{1/2}L^{-1}$ into the two subsets

$$|t|\leqq T(m)=nL^{-1}\left(m\log\frac{m}{L^2}\right)^{-1/2}\tag{28}$$

and

$$T(m)\leqq|t|\leqq T.\tag{29}$$

Lemma 5. *For t satisfying* (28)

$$\begin{aligned}\big|J_1(t)\big|\leqq{}& c_1\big(1+\|a\|^3\big)\,\big(\mathbf{E}\|g_1\|^3\big)\,\big(\mathbf{E}\|g_2\|^3\big)^{1/3}\\ &\times\quad L^3n^{-1/2}+\big(|t|+t^2\big)\prod_{j=1}^{7}\big(1+c_2\sigma_j^4t^2\big)^{-1/4}\\ &+|t|^3\prod_{j=1}^{9}\big(1+c_2\sigma_j^4t^2\big)^{-1/4}\bigg)\,n^{-1/2},\end{aligned}\tag{30}$$

and for t satisfying (29)

(31) $$\left|J_1(t)\right| \leqq c_1\big(1+\|a\|^3\big)\left(\mathbf{E}\|g_1\|^3\right)\left(\mathbf{E}\|g_2\|^3\right)^{1/3} c_9(V)\, L^2 n^{-1}.$$

Proof. Let us show first that

(32) $$J_1(t) = 4n^{-1/2}\varepsilon_n\binom{k}{2} h_1(t) + 2(k-2)\, n^{-1/2}\varepsilon_n\binom{k}{2} h_2(t) - 2\varepsilon_n\binom{k}{2} h_3(t),$$

where

$$h_1(t) = \mathbf{E}\Big(\big(\overline{g}_1(X_1),\, g_2(X_1,X_2)\big)\exp\Big\{it\,\big\|\overline{g}_{1n}(X_1)+\overline{g}_{1n}(X_2)+\overline{S}_{n2}\big\|^2\Big\}\Big),$$

$$h_2(t) = \mathbf{E}\Big(\big(\overline{g}_1(X_1),\, g_2(X_2,X_3)\big) \times \exp\Big\{it\,\big\|\overline{g}_{1n}(X_1)+\overline{g}_{1n}(X_2)+\overline{g}_{1n}(X_3)+\overline{S}_{n3}\big\|^2\Big\}\Big),$$

$$h_3(t) = \mathbf{E}\Big(\big(a,\, g_2(X_1,X_2)\big)\exp\Big\{it\,\big\|\overline{g}_{1n}(X_1)+\overline{g}_{1n}(X_2)+\overline{S}_{n2}\big\|^2\Big\}\Big),$$

$$\overline{g}_{1n}(X_j) = \overline{g}_1(X_j)\, n^{-1/2}, \qquad \overline{S}_{nr} = n^{-1/2}\sum_{j=r+1}^{n}\overline{g}_1(X_j) - a.$$

Indeed, according to the definition,

$$J_1(t) = 2n^{-1/2}\varepsilon_n\sum_{i=1}^{5} H_i(t) - 2\varepsilon_n H_6,$$

$$H_1(t) = \sum_{j=3}^{k}\sum_{i_2=2}^{j-1}\sum_{i_1=1}^{i_2-1}\mathbf{E}\Big(\big(\overline{g}_1(X_j),\, g_2(X_{i_1},X_{i_2})\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{3} h_2(t),$$

$$H_2(t) = \sum_{j=2}^{k}\sum_{i_2=1}^{j-1}\mathbf{E}\Big(\big(\overline{g}_1(X_j),\, g_2(X_{i_2},X_j)\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{2} h_1(t),$$

$$H_3(t) = \sum_{j=2}^{k-1}\sum_{i_1=1}^{j-1}\sum_{i_2=j+1}^{k}\mathbf{E}\Big(\big(\overline{g}_1(X_j),\, g_2(X_{i_1},X_{i_2})\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{3} h_2(t),$$

$$H_4(t) = \sum_{j=1}^{k-1}\sum_{i_1=j+1}^{k}\mathbf{E}\Big(\big(\overline{g}_1(X_j),\, g_2(X_{i_1},X_j)\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{2} h_1(t),$$

$$H_5(t) = \sum_{j=1}^{k-2}\sum_{i_2=j+2}^{k}\sum_{i_1=j+1}^{i_2-1}\mathbf{E}\Big(\big(\overline{g}_1(X_j),\, g_2(X_{i_1},X_{i_2})\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{3} h_2(t),$$

$$H_6(t) = \sum_{i_2=2}^{k}\sum_{i_1=1}^{i_2-1}\mathbf{E}\Big(\big(a,\, g_2(X_{i_1},X_{i_2})\big)\exp\Big\{it\,\big\|\overline{S}_n-a\big\|^2\Big\}\Big) = \binom{k}{2} h_3(t).$$

These representations imply (32). Next we estimate functions $h_i(t)$ in (32). Obviously,

$$\big\|\overline{g}_{1n}(X_1)+\overline{g}_{1n}(X_2)+\overline{S}_{n2}\big\|^2 = \|\overline{S}_{n2}\|^2 + W_1 + W_2,$$

where

$$W_1 = \left\|\overline{g}_{1n}(X_1) + \overline{g}_{1n}(X_2)\right\|^2, \qquad W_2 = 2\left(\overline{g}_{1n}(X_1) + \overline{g}_{1n}(X_2),\, \overline{S}_{n2}\right).$$

By (21) with $r = 1$ and $z = \mathrm{it}(W_1 + W_2)$, taking into consideration (2) we have

$$\begin{aligned} h_1(t) = it \int_0^1 du\, \mathbf{E}\Big(&\left(\overline{g}_1(X_1),\, g_2(X_1, X_2)\right) W_1 \exp\{it\, uW_1\} \\ &\times \overline{\mathbf{E}}_2 \exp\left\{ it \left\|\overline{S}_{n2}\right\|^2 + itu\, W_2 \right\} \Big) \\ + it \int_0^1 du\, \mathbf{E}\Big(&\left(\overline{g}_1(X_1),\, g_2(X_1, X_2)\right) \exp\{itu\, W_1\} \\ &\times \overline{\mathbf{E}}_2 \Big(\exp\left\{ it \left\|\overline{S}_{n2}\right\|^2 + itu\, W_2 \right\} W_2 \Big) \Big). \end{aligned}$$

The expectation $\overline{\mathbf{E}}_2$ with respect to $X_3, \dots, X_n$ we will estimate using Lemma 3. To estimate $h_1(t)$ in the interval (28) we take in Lemma 3 $l \sim n/2$, $m = n - l$, $A = 1$. Then $T_1 = |t|$ and

$$\begin{aligned} &\left| \overline{\mathbf{E}}_2 \exp\left\{ it \|\overline{S}_{n2}\|^2 + itu\, W_2 \right\} \right| \\ (33) \qquad &\leqq c_1 \quad \exp(-c_2 n) + c_3 L^2 n^{-1} + \prod_{j=1}^{5} \left(1 + c_2 \sigma_j^4 t^2\right)^{-1/4} \Big), \\ &\left| \overline{\mathbf{E}}_2 \Big(\exp\left\{ it \|\overline{S}_{n2}\|^2 + itu\, W_2 \right\} W_2 \Big) \right| \leqq c_1 \left(1 + \|a\|\right) \left\|\overline{g}_{1n}(X_1) + g_{1n}(X_2)\right\| \\ (34) \qquad &\times \quad \exp(-c_2 n) + c_3 L^2 n^{-1} + \prod_{j=1}^{5} \left(1 + c_2 \sigma_j^4 t^2\right)^{-1/4} \Big). \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbf{E}\Big(\left| \left(\overline{g}_1(X_1),\, g_2(X_1, X_2)\right) \right| \left\|\overline{g}_{1n}(X_1) + \overline{g}_{1n}(X_2)\right\| \Big) \\ &\quad \leqq n^{-1/2} \Big(\mathbf{E}\Big(\left\|g_1(X_1)\right\|^2 \left\|g_2(X_1, X_2)\right\| \Big) + \mathbf{E}\Big(\left\|g_1(X_1)\right\| \left\|g_1(X_2)\right\| \left\|g_2(X_1, X_2)\right\| \Big) \Big) \\ &\quad \leqq 2n^{-1/2} \left(\mathbf{E}\|g_1\|^3\right)^{2/3} \left(\mathbf{E}\|g_2\|^3\right)^{1/3}, \\ &\mathbf{E}\Big(\left| \left(\overline{g}_1(X_1),\, g_2(X_1, X_2)\right) \right| \left\|\overline{g}_{1n}(X_1) + \overline{g}_{1n}(X_2)\right\|^2 \Big) \\ &\quad \leqq 2n^{-1/2} \left(\mathbf{E}\|g_1\|^3\right)^{2/3} \left(\mathbf{E}\|g_2\|^3\right)^{1/3}. \end{aligned}$$

Thus, by (33) and (34), in the interval (28)

$$\begin{aligned} &\left|h_1(t)\right| \leqq c_1 \left(1 + \|a\|\right) \left(\mathbf{E}\|g_1\|^3\right)^{2/3} \left(\mathbf{E}\|g_2\|^3\right)^{1/3} \\ (35) \qquad &\qquad \times \quad Ln^{-1/2} + |t| \prod_{j=1}^{7} \left(1 + c_2 \sigma_j^4 t^2\right)^{-1/4} \Big) n^{-1/2}. \end{aligned}$$

To estimate $h_1(t)$ for t satisfying (29) we take in Lemma 3, $l \sim n/\log n$, $m = n-l$, $A = \frac{5}{4}$. Then $T_1 = T(m)$ and similar reasoning gives for t satisfying (29)

$$\begin{aligned}\bigl|h_1(t)\bigr| &\leqq c_1|t|\,\bigl(1+\|a\|\bigr)\,\bigl(\mathbf{E}\|g_1\|^3\bigr)^{2/3}\bigl(\mathbf{E}\|g_2\|^3\bigr)^{1/3}\\ &\qquad\times\quad \exp\biggl(-\frac{c_2 n}{\log n}\biggr) + c_3 L^{5/2} n^{-5/4}(\log n)^{5/4}\\ &\qquad + \prod_{j=1}^{5}\biggl(1+\frac{c_4\sigma_j^4 n}{L^2(\log n)\log(m/L^2)}\biggr)^{-1/4}\biggr) n^{-1/2}\\ &\leqq c_5\bigl(1+\|a\|\bigr)\,\bigl(\mathbf{E}\|g_1\|^3\bigr)^{2/3}\bigl(\mathbf{E}\|g_2\|^3\bigr)^{1/3} c_5(V)\, L^{3/2} n^{-1}. \end{aligned} \tag{36}$$

Consider now $h_2(t)$ in (32). Denote

$$\xi = \sum_{j=1}^{3}\xi_j, \qquad \bar\xi = \sum_{j=1}^{3}\mathbf{E}\,\overline{g}_{1n}(X_j), \quad \xi_j = \overline{g}_{1n}(X_j) - \mathbf{E}\,\overline{g}_{1n}(X_j),\ j=1,\,2,\,3.$$

Obviously,

$$\begin{aligned}\bigl\|\overline{g}_{1n}(X_1)+\overline{g}_{1n}(X_2)+\overline{g}_{1n}(X_3)+\overline{S}_{n3}\bigr\| &= \|\xi+\overline{S}_{n3}+\bar\xi\|^2\\ = \bigl\|\overline{S}_{n3}+\bar\xi\bigr\|^2 + W_3+W_4+W_5,\end{aligned}$$

where

$$W_3 = \sum_{j=1}^{3}\|\xi_j\|^2, \qquad W_4 = 2\sum_{j=1}^{3}(\xi_i,\,\overline{S}_{n3}+\bar\xi), \qquad W_5 = 2\sum_{1\leqq i<j\leqq 3}(\xi_i,\xi_j).$$

Applying (21) with $r=2$ and $z = it\,W_5$ we can write

$$h_2(t) = h_{21}(t)+h_{22}(t)+h_{23}(t), \tag{37}$$

where

$$\begin{aligned} h_{21}(t) &= \mathbf{E}\Bigl(\bigl(\overline{g}_1(X_1),\,g_2(X_2,X_3)\bigr)\,\exp\Bigl\{it\,\bigl\|\overline{S}_{n3}+\bar\xi\bigr\|^2 + it\,(W_3+W_4)\bigr)\Bigr\}\Bigr),\\ h_{22}(t) &= it\,\mathbf{E}\biggl(\bigl(\overline{g}_1(X_1),\,g_2(X_2,X_3)\bigr)\,W_5\,\exp\Bigl\{it\,\bigl\|\overline{S}_{n3}+\bar\xi\bigr\|^2 + it\,(W_3+W_4)\Bigr\}\biggr),\\ h_{23}(t) &= -t^2\int_0^1 du\,(1-u)\,\mathbf{E}\biggl(\bigl(\overline{g}_1(X_1),\,g_2(X_2,X_3)\bigr)\,W_5^2\\ &\qquad\times\exp\Bigl\{it\,\bigl\|\overline{S}_{n3}+\bar\xi\bigr\|^2 + it\,(W_3+W_4+uW_5)\Bigr\}\biggr).\end{aligned}$$

Furthermore, observe that

$$\bigl|h_{23}(t)\bigr| \leqq t^2\mathbf{E}\biggl(\bigl|\bigl(\overline{g}_1(X_1),\,g_2(X_2,X_3)\bigr)\bigr|\,W_5^2\Bigl|\overline{\mathbf{E}}_3\exp\Bigl\{it\,\bigl\|\overline{S}_{n3}+\bar\xi\bigr\|^2 + it\,W_4\Bigr\}\Bigr|\biggr).$$

In the interval (28) by Lemma 3 with $l\sim n/2$, $m=n-l$, $A=\frac{3}{2}$

$$\begin{aligned}&\Bigl|\overline{\mathbf{E}}_3\exp\bigl\{it\,\|\overline{S}_{n3}+\bar\xi\|^2 + it\,W_4\bigr\}\Bigr|\\ &\quad\leqq c_1\quad \exp(-c_2 n) + c_3 L^3 n^{-3/2} + \prod_{j=1}^{7}\bigl(1+c_4\sigma_j^4 t^2\bigr)^{-1/4}\biggr),\end{aligned} \tag{38}$$

and for t satisfying (29) Lemma 3 with $l \sim n/\log n$, $m = n - l$, $A = \frac{7}{4}$ gives

$$\begin{aligned} &\Big|\overline{\mathbf{E}}_3 \exp\left\{it\,\|\overline{S}_{n3} + \bar{\xi}\|^2 + it\,W_4\right\}\Big| \\ &\quad \leqq c_1 \quad \exp\left(-\frac{c_2 n}{\log n}\right) + c_3 L^{7/2} n^{-7/4} (\log n)^{7/4} \\ &\qquad + \prod_{j=1}^{7}\left(1 + \frac{c_4\sigma_j^4 n}{L^2(\log n)\,\log(m/L^2)}\right)^{-1/4}\Bigg). \end{aligned} \tag{39}$$

Moreover,

$$\mathbf{E}\Big(\big|\left(\bar{g}_1(X_1),\, g_2(X_2,X_3)\right)\big|\, W_5^2\Big) \leqq c\left(\mathbf{E}\big\|g_1(X_1)\big\|^3\right)^{2/3}\left(\mathbf{E}\|g_2\|^3\right)^{1/3} n^{-3/2}. \tag{40}$$

Combining (38)–(40) we obtain an estimate for $h_{23}(t)$.

By (21) with $r = 1$, $z = it\,(W_3 + W_4)$ we have

$$h_{22}(t) = F_1(t) + F_2(t), \tag{41}$$

where

$$\begin{aligned} F_1(t) &= it\,\mathbf{E}\Big(\left(\mathbf{E}\bar{g}_1(X_1),\, g_2(X_2,X_3)\right) W_5 \exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + it\,(W_3 + W_4)\Big\}\Big), \\ F_2(t) &= -t^2 \int_0^1 du\,\mathbf{E}\Big(\left(\bar{g}_1(X_1) - \mathbf{E}\bar{g}_1(X_1),\, g_2(X_2,X_3)\right) W_5 (W_3 + W_4) \\ &\qquad \times \exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + itu\,(W_3 + W_4)\Big\}\Big). \end{aligned}$$

Here

$$\big|F_1(t)\big| \leqq |t|\,\mathbf{E}\Big(\Big|\left(\mathbf{E}\bar{g}_1(X_1),\, g_2(X_2,X_3)\right) W_5\Big|\,\Big|\overline{\mathbf{E}}_3 \exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + it\,W_4\Big\}\Big|\Big),$$

and

$$\mathbf{E}\Big(\Big|\left(\mathbf{E}\bar{g}_1(X_1),\, g_2(X_2,X_3)\right)\Big|\, W_5\Big) \leqq 24\left(\mathbf{E}\big\|g_2\|^2\right)^{1/2} n^{-3/2}.$$

Thus for t satisfying (28) by Lemma 3 with $l \sim n/2$, $m = n - l$, $A = 1$

$$\begin{aligned} &\big|F_1(t)\big| \leqq c_1 |t|\left(\mathbf{E}\|g_2\|^2\right)^{1/2} \\ &\qquad \times c_1 \quad \exp(-c_2 n) + c_3 L^2 n^{-1} + \prod_{j=1}^{5}\left(1 + c_4\sigma_j^4 t^2\right)^{-1/4}\Bigg) n^{-3/2}, \end{aligned} \tag{42}$$

and for t satisfying (29) by Lemma with $l \sim n/\log n$, $m = n - l$ and $A = \frac{5}{4}$,

$$\begin{aligned} &\big|F_1(t)\big| \leqq c_1 |t|\left(\mathbf{E}\|g_2\|^2\right)^{1/2} c_1\Bigg(\exp\left(-\frac{c_2 n}{\log n}\right) + c_3 L^{5/2} n^{-5/4} (\log n)^{5/4} \\ &\qquad + \prod_{j=1}^{5}\left(1 + \frac{c_4\sigma_j^4 n}{L^2(\log n)\log(m/L^2)}\right)^{-1/4}\Bigg) n^{-3/2}. \end{aligned} \tag{43}$$

To estimate $f_2(t)$ in (41) we first observe, that

$$\big|F_2(t)\big| \leqq t^2 \int_0^1 du\, \mathbf{E}\Bigg(\Big|\big(\overline{g}_1(X_1) - \mathbf{E}\overline{g}_1(X_1),\, g_2(X_2, X_3)\big)\, W_5\Big| \times \Big|\overline{\mathbf{E}}_3(W_3 + W_4)\, \exp\Big\{it\,\big\|S_{n3} + \bar{\xi}\big\|^2 + itu\, W_4\Big\}\Big|\Bigg),$$

and

$$\mathbf{E}\Bigg(\Big|\big(\overline{g}_1(X_1) - \mathbf{E}\overline{g}_1(X_1),\, g_2(X_2, X_3)\big)\, W_5\Big|\Big[W_3 + \sum_{j=1}^{3}\|\xi_j\|\,\big(1 + \|a\|\big)\Big]\Bigg) \leqq c\,\mathbf{E}\|g_1\|^3\big(\mathbf{E}\|g_2\|^3\big)^{1/3}\,\big(1 + \|a\|\big)\, n^{-3/2}.$$

Applying now Lemma 3 for t satisfying (28) with $l \sim n\log n$, $m = n - l$ and $A = \frac{3}{2}$, we have

$$\big|F_2(t)\big| \leqq c_1 t^2\,\big(1 + \|a\|\big)\,\mathbf{E}\|g_1\|^3\big(\mathbf{E}\|g_2\|^3\big)^{1/3} \times \Bigg(\exp(-c_2 n)\, c_3 L^3 n^{-3/2} + \prod_{j=1}^{7}\big(1 + c_4\sigma_j^4 t^2\big)^{-1/4}\Bigg)\, n^{-3/2}, \tag{44}$$

and for t satisfy (29), using Lemma 3 with $l \sim n\log n$, $m = n - l$ and $A = \frac{7}{4}$, we have

$$\big|F_2(t)\big| \leqq c_1 t^2\,\big(1 + \|a\|\big)\,\mathbf{E}\|g_1\|^3\big(\mathbf{E}\|g_2\|^3\big)^{1/3} \times \Bigg(\exp\Big(-\frac{c_2 n}{\log n}\Big) + c_3 L^{7/2} n^{-7/4}(\log n)^{7/4} + \prod_{j=1}^{7}\Big(1 + \frac{c_4\sigma_j^4 n}{L^2(\log n)\,\log(m/L^2)}\Big)^{-1/4}\Bigg)\, n^{-3/2}. \tag{45}$$

In (37) it remains only to estimate $h_{21}(t)$. We use the representation

$$h_{21}(t) = F_3(t) + F_4(t), \tag{46}$$

where

$$F_3(t) = \mathbf{E}\Big(\big(\mathbf{E}\overline{g}_1(X_1),\, g_2(X_2, X_3)\big)\,\exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + it\,(W_3 + W_4)\Big\}\Big),$$

$$F_4(t) = \mathbf{E}\Big(\big(\overline{g}_1(X_1) - \mathbf{E}\overline{g}_1(X_1),\, g_2(X_2, X_3)\big) \times W_5\,\exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + it\,(W_3 + W_4)\Big\}\Big).$$

In view of (2) we can also write

$$F_3(t) = \mathbf{E}\,\Bigg(\big(\mathbf{E}\overline{g}_1(X_1),\, g_2(X_2, X_3)\big)\,\exp\Big\{it\,\big\|\overline{S}_{n3} + \bar{\xi}\big\|^2 + it\,\|\xi_1\|^2 + 2it\,(\xi_1, \overline{S}_{n3} + \bar{\xi})\Big\} \times \prod_{j=2}^{3}\Big(\exp\Big\{it\,\|\xi_j\|^2 + 2it\,(\xi_j, \overline{S}_{n3} + \bar{\xi})\Big\} - 1\Big)\Bigg), \tag{47}$$

$$F_4(t) = \mathbf{E}\ \ \big(\bar g_1(X_1) - \mathbf{E}\bar g_1(X_1),\, g_2(X_2, X_3)\big)\ \exp\Big\{it\,\big\|S_{n3} + \bar\xi\big\|^2\Big\}$$

$$\times \prod_{j=1}^{3}\bigg(\exp\Big\{it\,\|\xi_j\|^2 + 2it\,(\xi_j, \overline{S}_{n3} + \bar\xi)\Big\} - 1\bigg)\bigg). \tag{48}$$

Furthermore, applying (21) with $r = 1$ in (47) and (48), we have

$$F_3(t) = -t^2\mathbf{E}\ \ \big(\mathbf{E}\bar g_1(X_1),\, g_2(X_2, X_3)\big)$$

$$\times \exp\Big\{it\,\big\|\overline{S}_{n3} + \bar\xi\big\|^2 + it\,\|\xi_1\|^2 + 2it\,(\xi_1, \overline{S}_{n3} + \bar\xi)\Big\}$$

$$\times \prod_{j=2}^{3}\Big(\big\|\xi_j\big\|^2 + 2(\xi_j, \overline{S}_{n3} + \bar\xi)\Big)$$

$$\times \int_0^1 du\ \exp\Big\{itu\,\big\|\xi_j\big\|^2 + 2itu\,(\xi_j, \overline{S}_{n3} + \bar\xi)\Big\}\bigg), \tag{49}$$

$$F_4(t) = (it)^2\mathbf{E}\ \ \big(\bar g_1(X_1) - \mathbf{E}\bar g_1(X_1),\, g_2(X_2, X_3)\big)\ \exp\Big\{it\,\big\|\overline{S}_{n3} + \bar\xi\big\|^2\Big\}$$

$$\times \prod_{j=1}^{3}\Big(\big\|\xi_j\big\|^2 + 2(\xi_j, \overline{S}_{n3} + \bar\xi)\Big)$$

$$\times \int_0^1 du\ \exp\Big\{itu\,\big\|\xi_j\big\|^2 + 2itu\,(\xi_j, \overline{S}_{n3} + \bar\xi)\Big\}\bigg). \tag{50}$$

Denote

$$b_j = \|\xi_j\|^2 + 2(\xi_j, \bar\xi) - 2(\xi_j, a).$$

Then

$$\prod_{j=2}^{3}\Big(\big\|\xi_j\big\|^2 + 2(\xi_j, \overline{S}_{n3} + \bar\xi)\Big) = \prod_{j=2}^{3}\Big(b_j + 2(\xi_j, S_{n3} + a)\Big)$$

$$= b_2 b_3 + 2b_2(\xi_3, \overline{S}_{n3} + a) + 2b_3(\xi_2, \overline{S}_{n3} + a) + 4(\xi_2, \overline{S}_{n3} + a)\,(\xi_3, \overline{S}_{n3} + a), \tag{51}$$

where according to the notation introduced above

$$\overline{S}_{n3} + a = n^{-1/2}\big(\bar g_1(X_4) + \cdots + \bar g_1(X_n)\big).$$

Note that

$$\begin{gathered} \|\xi_j\| \leqq 2, \qquad \|\xi_j\| \leqq n^{-1/2}\Big(1 + \big\|g_1(X_j)\big\|\Big), \\ |b_j| \leqq 4\big(1 + \|a\|\big)\,\Big(1 + \big\|g_1(X_j)\big\|\Big)\,n^{-1/2}. \end{gathered} \tag{52}$$

Substituting now (51) in (49) we obtain four expectations. Then we apply Lemma 3 to each of them, taking into account (52) and the estimate $\|\mathbf{E}\bar g_1(X_1)\| \leqq n^{-1/2}$. For t satisfying (28) we take in Lemma 3 $l \sim n/2$, $m = n - l$, $A = \frac{3}{2}$ and obtain

$$\big|F_3(t)\big| \leqq c_1 t^2\,\big(1 + \|a\|\big)^2\big(\mathbf{E}\|g_2\|^2\big)^{1/2}$$

$$\times \quad \exp(-c_2 n) + c_3 L^3 n^{-3/2} + \prod_{j=1}^{7} \left(1 + c_4 \sigma_j^4 t^2\right)^{-1/4} \Bigg) n^{-3/2}, \tag{53}$$

and for t satisfying (29) we take $l \sim n/\log n$, $m = n - l$ and $A = \frac{7}{4}$ and obtain

$$\begin{aligned} \left|F_3(t)\right| &\leqq c_1 t^2 \left(1 + \|a\|\right)^2 \left(\mathbf{E}\|g_2\|^2\right)^{1/2} \\ &\quad \times \quad \exp\left(-\frac{c_2 n}{\log n}\right) + c_3 L^{7/2} n^{-7/4} (\log n)^{7/4} \\ &\quad + \prod_{j=1}^{7} \left(1 + \frac{c_4 \sigma_j^4 n}{L^2 (\log n) \log(n/L^2)}\right)^{-1/4} \Bigg) n^{-3/2}. \end{aligned} \tag{54}$$

To estimate $f_4(t)$ in (50) we use the following formula which is analogous to (51):

$$\begin{aligned} \prod_{j=1}^{3} \left(\left\|\xi_j\right\|^2 + 2(\xi_j, \overline{S}_{n3} + \bar{\xi})\right) &= b_1 b_2 b_3 + 2 b_1 b_2 (\xi_3, \overline{S}_{n3} + a) \\ &+ 2 b_1 b_3 (\xi_2, \overline{S}_{n3} + a) + 2 b_2 b_3 (\xi_1, \overline{S}_{n3} + a) + 4 b_1 (\xi_2, \overline{S}_{n3} + a)\,(\xi_3, \overline{S}_{n3} + a) \\ &+ 4 b_2 (\xi_1, \overline{S}_{n3} + a)\,(\xi_3, \overline{S}_{n3} + a) + 4 b_3 (\xi_1, \overline{S}_{n3} + a)\,(\xi_2, \overline{S}_{n3} + a) \\ &+ 8 (\xi_1, \overline{S}_{n3} + a)\,(\xi_2, \overline{S}_{n3} + a)\,(\xi_3, \overline{S}_{n3} + a). \end{aligned}$$

Substituting it in (50) we obtain eight similar expectations. To estimate these expectations we again apply Lemma 3. For t satisfying (28) we take in it $l \sim n/2$, $m = n - l$, $A = 2$ and obtain,

$$\begin{aligned} \left|F_4(t)\right| &\leqq c_1 |t|^3 \left(1 + \|a\|\right)^3 \left(\mathbf{E}\|g_2\|^2\right)^{1/2} \\ &\quad \times \quad \exp(-c_2 n) + c_3 L^4 n^{-2} + \prod_{j=1}^{9} \left(1 + c_4 \sigma_j^4 t^2\right)^{-1/4} \Bigg) n^{-3/2}, \end{aligned} \tag{55}$$

and for t satisfying (29) we take $l \sim n/\log n$, $m = n - l$ and $A = \frac{9}{4}$ and obtain

$$\begin{aligned} \left|F_4(t)\right| &\leqq c_1 |t|^3 \left(1 + \|a\|\right)^3 \left(\mathbf{E}\|g_2\|^2\right)^{1/2} \\ &\quad \times \quad \exp\left(-\frac{c_2 n}{\log n}\right) + c_3 L^{9/2} n^{-9/4} (\log n)^{9/4} \\ &\quad + \prod_{j=1}^{9} \left(1 + \frac{c_4 \sigma_j^4 n}{L^2 (\log n) \log(m/L^2)}\right)^{-1/4} \Bigg) n^{-3/2}. \end{aligned} \tag{56}$$

Estimates (37)–(38), (41)–(42), (53), and (55) together for t satisfying (28) imply

$$\begin{aligned} \left|h_2(t)\right| &\leqq c_1 \left(1 + \|a\|\right)^3 \left(\mathbf{E}\|g_1\|^3\right) \left(\mathbf{E}\|g_2\|^3\right)^{1/3} \\ &\quad \times \quad L n^{-1/2} + \left(|t| + t^2\right) \prod_{j=1}^{7} \left(1 + c_4 \sigma_j^4 t^2\right)^{-1/4} \\ &\quad + |t|^3 \prod_{j=1}^{9} \left(1 + c_4 \sigma_j^4 t^2\right)^{-1/4} \Bigg) n^{-3/2}, \end{aligned} \tag{57}$$

and (37), (39), (41), (43), (54), and (56) for t satisfying (29) imply

$$\big|h_2(t)\big| \leqq c_1\big(1+\|a\|\big)^3\big(\mathbf{E}\|g_1\|^3\big)\,\big(\mathbf{E}\|g_2\|^3\big)^{1/3} c_9(V)\,L^{3/2}n^{-2}. \tag{58}$$

It remains to estimate $h_3(t)$ in (32). Applying (21) with $r=1$ and $z = it\,(W_1 + W_2)$, $z = itu\,(W_1+W_2)$ twice, we have

$$\begin{aligned} h_3(t) = it\int_0^1 du\,\mathbf{E}\Big(\big(a,\, g_2(X_1,X_2)\big)\,W_1\exp\{itu\,W_1\}\,\overline{\mathbf{E}}_2\exp\Big\{it\,\big\|\overline{S}_{n2}\big\|^2 + itu\,W_2\Big\}\Big) \\ - t^2\int_0^1 dv\int_0^1 du\,u\mathbf{E}\Big(\big(a,\, g_2(X_1,X_2)\big)\,\exp\{ituv\,W_1\} \\ \times\overline{\mathbf{E}}_2\Big(\exp\Big\{it\,\big\|\overline{S}_{n2}\big\|^2 + ituv\,W_2\Big\}(W_1+W_2)\Big)\Big). \end{aligned}$$

Arguing in the same way as in proving (35) and (36) we obtain for t satisfying (28)

$$\begin{aligned} \big|h_3(t)\big| &\leqq c_1\big(1+\|a\|\big)^3\big(\mathbf{E}\|g_1\|^3\big)^{2/3}\big(\mathbf{E}\|g_2\|^3\big)^{1/3} \\ &\times\left[Ln^{-1/2} + \big(|t|+t^2\big)\prod_{j=1}^{7}\big(1+c_2\sigma_j^4t^2\big)^{-1/4}\right]n^{-1}, \end{aligned} \tag{59}$$

and for t satisfying (29)

$$\big|h_3(t)\big| \leqq c_1\big(1+\|a\|^3\big)\big(\mathbf{E}\|g_1\|^3\big)^{2/3}\big(\mathbf{E}\|g_2\|^3\big)^{1/3}c_7(V)\,L^{3/2}n^{-3/2}. \tag{60}$$

Now (32), (35), (57), (59) imply (30) and (32), (36), (58), (60) imply (31).

Lemma 6. *If t belongs to the interval* (28), *then*

$$\begin{aligned} \big|J_2(t)\big| &\leqq c_1\big(1+\|a\|^2\big)\,\big(\mathbf{E}\|g_1\|^3\big)^{2/3} \\ &\times\quad L^2n^{-1/2} + n^{1/6}\prod_{j=1}^{9}\left(1+\frac{c_2\sigma_j^4t^2}{n^{1/3}}\right)^{-1/6}\Bigg)\,n^{-1}, \end{aligned} \tag{61}$$

and if t satisfies (29)

$$\big|J_2(t)\big| \leqq c_1\big(1+\|a\|^2\big)\,\big(\mathbf{E}\|g_1\|^3\big)^{2/3}c_9(V)\,L^3n^{-3/2}. \tag{62}$$

Proof. Noting that

$$J_2(t) = \mathbf{E}\left(Z_n^2\overline{\mathbf{E}}\,\exp\left\{it\,\|\overline{S}_n - a\|^2\right\}\right)$$

we have

$$\begin{aligned} \big|J_2(t)\big| &\leqq 4\varepsilon_n^2\mathbf{E}\Big(\big\|\overline{S}_n^k\big\|^2\|\alpha_k\|^2\big|\overline{\mathbf{E}}_k\exp\big\{it\,\big\|\overline{S}_n - a\big\|^2\big\}\big|\Big) \\ &+ 4\varepsilon_n^2\|a\|^2\mathbf{E}\Big(\|\alpha_k\|^2\big|\overline{\mathbf{E}}_k\exp\big\{it\,\big\|\overline{S}_n - a\big\|^2\big\}\big|\Big). \end{aligned} \tag{63}$$

By Hölder's inequality

$$\begin{aligned} &\mathbf{E}\Big(\big\|\overline{S}_n^k\big\|^2\|\alpha_k\|^2\big|\overline{\mathbf{E}}_k\exp\big\{it\,\big\|\overline{S}_n - a\big\|^2\big\}\big|\Big) \\ &\leqq \big(\mathbf{E}\|\alpha_k\|^3\big)^{2/3}\Big(\mathbf{E}\Big(\big\|\overline{S}_n^k\big\|^6\Big|\overline{\mathbf{E}}_k\exp\big\{it\,\big\|\overline{S}_n - a\big\|^2\big\}\Big|^3\Big)\Big)^{1/3}. \end{aligned} \tag{64}$$

Since for any $x > 0$ and $p > 0$

$$\|\overline{S}_n^k\|^6 \leqq x^6 + x^{-p}\|\overline{S}_n^k\|^{p+6},$$

we have in (64)

$$\mathbf{E}\Big(\|\overline{S}_n^k\|^6\Big|\overline{\mathbf{E}}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big\}\Big|^3\Big)$$
$$\leqq x^6\mathbf{E}\Big|\overline{\mathbf{E}}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big\}\Big|^2 + x^{-p}\mathbf{E}\,\|\overline{S}_n^k\|^{p+6}. \tag{65}$$

To estimate the second expectation in (63) we also use Hölder's inequality and obtain

$$\mathbf{E}\Big(\|\alpha_k\|^2\Big|\overline{\mathbf{E}}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big\}\Big|\Big) \leqq \mathbf{E}\big(\|\alpha_k\|^3\big)^{2/3}\Big(\mathbf{E}\big|\mathbf{E}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big\}\big|^3\Big)^{1/3}$$
$$\leqq \mathbf{E}\big(\|\alpha_k\|^3\big)^{2/3}\Big(\mathbf{E}\big|\mathbf{E}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big\}\big|^2\Big)^{1/3}. \tag{66}$$

Substituting (64) in (63) and using the elementary inequality $(\alpha+\beta)^{1/3} \leqq \alpha^{1/3} + \beta^{1/3}$ we get

$$\big|J_2(t)\big| \leqq 4\varepsilon_n^2\mathbf{E}\big(\|\alpha_k\|^3\big)^{2/3}$$
$$\times\Big[x^{-p/3}\big(\mathbf{E}\|\overline{S}_n^k\|^{p+6}\big)^{1/3} + \big(x^2 + \|a\|^2\big)\Big(\mathbf{E}\big|\mathbf{E}_k \exp\big\{it\,\|\overline{S}_n - a\|^2\big|^2\Big)^{1/3}\Big]. \tag{67}$$

Applying the same argument as in the proof of Lemma 10 in [3] with $l \sim n^{2/3}$, $m = n - l$, $A = \frac{3}{2}$ for $|t| \leqq cn^{1/2}L^{-1}$, we have

$$\mathbf{E}\Big|\overline{\mathbf{E}}_k \exp\big\{it\,\|\overline{S}_n - a\|\big\}\Big|^2 \leqq c_1\quad L^3n^{-1} + \prod_{j=1}^{9}\Big(1 + \frac{c_2\sigma_j^4T_1^2}{n^{1/3}}\Big)^{-1/2}\Big),$$

where T_1 is defined in Lemma 3. Furthermore, taking in (67) $p = 36$, $x = n^{1/12}$ and using (8) and (9) we obtain the estimate

$$\big|J_2(t)\big| = c_1\big(1 + \|a\|^2\big)\big(\mathbf{E}\|g_2\|^3\big)^{2/3}$$
$$\times\quad L^2n^{-1/2} + n^{1/6}\prod_{j=1}^{9}\Big(1 + \frac{c_2\sigma_j^4T_1^2}{n^{1/3}}\Big)^{-1/6}\Big)n^{-1},$$

which implies (61) and (62).

LEMMA 7. *I_2 in (19) satisfies the inequality*

$$I_2 \leqq c_1\big(1 + \|a\|^3\big)\,\beta^2\sigma_9^{-1}c_9(V)\,n^{-1/2}. \tag{68}$$

Proof. (22) implies

$$\big|\varphi_n(t) - \psi_n(t)\big|\,|t|^{-1} \leqq \big|J_1(t)\big| + |t|\,\big|J_2(t)\big| + t^2\big|J_3(t)\big|. \tag{69}$$

By Lemma 4

$$\int_{-T}^{T} t^2\big|J_3(t)\big|\,dt \leqq c_1\big(1 + \|a\|^3\big)\,\mathbf{E}\|g_1\|^3\mathbf{E}\|g_2\|^3c_7(V)\,n^{-1/2}. \tag{70}$$

By Lemma 5

$$\int_{-T}^{T} \left|J_1(t)\right| dt \leqq c_1\left(1+\|a\|^3\right) \left(\mathbf{E}\|g_1\|^3\right)\left(\mathbf{E}\|g_2\|^3\right)^{1/3} c_9(V)\, L^{1/2} n^{-1/2}. \tag{71}$$

By Lemma 6

$$\int_{-T}^{T} |t| \left|J_2(t)\right| dt \leqq c_1\left(1+\|a\|^2\right) \left(\mathbf{E}\|g_1\|^3\right)^{2/3} c_9(V)\, L n^{-1/2}. \tag{72}$$

Noting now that the moments in (70)–(72) satisfy (4) and integrating both sides in (69), we obtain (68).

From (14), (19)–(20), and (68) it follows that

$$\Delta_{n2}(a) \leqq c_1\left(1+\|a\|^3\right) \beta^2 \sigma_9^{-1} c_9(V)\, n^{-1/2}. \tag{73}$$

Estimates (5), (6), and (73) imply (3) and the theorem is proved.

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