



Riemann–Finsler geometry and Lorentz-violating kinematics

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ABSTRACT

Effective field theories with explicit Lorentz violation are intimately linked to Riemann–Finsler geometry. The quadratic single-fermion restriction of the Standard-Model Extension provides a rich source of pseudo-Riemann–Finsler spacetimes and Riemann–Finsler spaces. An example is presented that is constructed from a 1-form coefficient and has Finsler structure complementary to the Randers structure.

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1. Introduction

The study of Riemann–Finsler geometry, which has roots in Riemann's 1854 *Habilitationsvortrag* and Finsler's 1918 dissertation [1,2], is now an established mathematical field with a variety of physical applications. A well-known example intimately linked to physics is Randers geometry [3], in which the Riemann metric at each point is augmented by a contribution from a 1-form. For instance, a pseudo-Randers metric on $(3+1)$ -dimensional spacetime can be identified with the effective metric experienced by a relativistic charged massive particle minimally coupled to a background electromagnetic 1-form potential.

The present work concerns the relationship between a large class of Riemann–Finsler geometries and theories with explicit Lorentz violation. Tiny Lorentz violation offers a promising prospect for experimental detection of new physics from the Planck scale and could arise in an underlying unified theory such as strings [4]. At attainable energies, effective field theory provides a useful tool for describing observable signals of Lorentz and CPT violation [5,6], with explicit Lorentz violation characterized by background coefficients. However, explicit Lorentz violation is generically incompatible with the Bianchi identities of pseudo-Riemann geometry [7] and so presents an obstacle to recovering the usual geometry of General Relativity. This problem can be avoided via spontaneous Lorentz breaking as, for example, in cardinal gravity [8]. An alternative might be to subsume the usual Riemann geometry into a more general geometrical structure. Here, this is taken to be Riemann–Finsler geometry, and one method is provided to connect it with Lorentz-violating effective field theories.

The notion of distance in Riemann–Finsler spaces and pseudo-Riemann–Finsler spacetimes is controlled by additional quantities beyond the Riemann metric (for textbook treatments see, e.g., Refs. [9–13]). Intuitively, the role of these quantities can be played by the background coefficients for explicit Lorentz violation.

The comprehensive realistic effective field theory with Lorentz violation that incorporates both the Standard Model and General Relativity is known as the Standard-Model Extension (SME) [7,14]. To relate the SME to pseudo-Riemann–Finsler spacetimes and Riemann–Finsler spaces, this work adopts as a starting point the single-fermion renormalizable restriction of the SME in Minkowski spacetime, which is a comparatively simple quantum field theory with explicit Lorentz violation. In the presence of fermion self-interactions, a connection between SME coefficients and pseudo-Riemann–Finsler geometry has been proposed by Bogoslovsky [15]. Here, attention is focused on a free SME fermion, which has a wave packet propagating with a dispersion relation modified by Lorentz violation. The dispersion relation is a quartic in the plane-wave 4-momentum p_μ and is exactly known [16]. Some of its properties have been discussed by Lehnert [17] and by Altschul and Colladay [18]. It can be associated with an action for a relativistic point particle [19], which encodes in a classical description part of the key physical content of the free quantum field theory while avoiding some of the complications associated with spin.

The basic observation underlying the present work is that the SME-based Lorentz-violating classical lagrangian plays the role of a pseudo-Finsler structure, which leads to some interesting geometrical consequences. Pseudo-Finsler structures play a role in the context of modified particle dispersion relations [20]. They are also relevant for modified photon dispersion relations [21], for which the general photon dispersion relation arising from operators of arbitrary dimension is known [22]. A substantial recent literature

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links Lorentz violation with pseudo-Riemann–Finsler geometries in the contexts of spacetime, gravity, and field theory [23–32]. Note, however, that at present no compelling experimental evidence exists for Lorentz violation in nature [33], although SME-based models provide simple explanations for certain unconfirmed experimental results including anomalous neutrino oscillations [34] and anomalous meson oscillations [35].

Interpreting the classical SME lagrangian as a pseudo-Finsler structure implies that the Lorentz-violating trajectories of the relativistic particles are governed by pseudo-Riemann–Finsler metrics and hence define pseudo-Riemann–Finsler geometries in $(3+1)$ dimensions. Corresponding Riemann–Finsler geometries can be generated by Wick rotation or restriction to spatial submanifolds. This construction can be extended to arbitrary dimensional curved spaces in a straightforward way. Following a general discussion of these ideas, this work presents some basic results for a particular Riemann–Finsler space that is constructed using a Riemann metric and a 1-form but differs from Randers space for dimensions $n \geq 3$. Its Finsler structure is complementary to the Randers structure in a certain sense described below.

The conventions used here are as follows. Coordinates for $(n+1)$ -dimensional pseudo-Riemann–Finsler spacetime are denoted x^μ , $\mu = 0, 1, \dots, n$. The velocities u^μ along a curve with path parameter λ are $u^\mu = dx^\mu/d\lambda$, and a pseudo-Finsler structure is denoted as $L = L(x, u)$. The Minkowski metric in $(n+1)$ dimensions is defined to have positive signature for $n > 2$. Index raisings or lowerings and contractions are performed with the pseudo-Riemann metric $r_{\mu\nu}$ and its inverse $r^{\mu\nu}$; for example, $u_\mu = r_{\mu\nu}u^\nu$ and $u^2 = u^\mu r_{\mu\nu}u^\nu$. The pseudo-Riemann–Finsler metric is $g_{\mu\nu}$ with inverse $g^{\mu\nu}$. To match conventions in mathematics (see, e.g., Ref. [9]), coordinates for n -dimensional Riemann–Finsler space are denoted x^j , $j = 1, \dots, n$, the velocities are $y^j = dx^j/d\lambda$, and a Finsler structure is denoted as $F = F(x, y)$. The Riemann metric is r_{jk} with inverse r^{jk} , while the Riemann–Finsler metric is g_{jk} with inverse g^{jk} . Index raisings or lowerings and contractions are performed with the Riemann metric; for example, $y_j = r_{jk}y^k$, $y^2 = y^j r_{jk}y^k$. The norm $\|y\|$ of y^j is $\|y\| = \sqrt{y^2}$. Partial derivatives with respect to y^j are denoted by subscripts; for example, $\partial F/\partial y^j = F_{y^j}$.

2. SME-based Finsler structures

The Lagrange density for the renormalizable single-fermion restriction of the minimal SME in $(3+1)$ -dimensional Minkowski spacetime includes all operators quadratic in the fermion field having mass dimensions three and four [14]. Each Lorentz-violating operator is contracted with a controlling coefficient, so the physics is coordinate independent. The coefficients of mass dimension one are conventionally denoted as a_μ , b_μ , $H_{\mu\nu}$, while the dimensionless ones are $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ , f_μ , and $g_{\lambda\mu\nu}$. A constructive procedure has recently been given for generating the classical relativistic point-particle lagrangian L from which the SME plane-wave dispersion relation can be derived [19]. The complete action is involved, but various special cases are tractable and some limits of L have been explicitly obtained. One example of relevance in what follows is the limiting situation of vanishing coefficients $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ , f_μ , $g_{\lambda\mu\nu}$, and $H_{\mu\nu}$, for which the particle lagrangian is

$$L_{ab} = -m\sqrt{-u^2} - a \cdot u \mp \sqrt{(b \cdot u)^2 - b^2 u^2}. \quad (1)$$

The two possible signs for the last term reflect the presence of two particle spin projections in the quantum field theory.

This work extends the full L and its limits to include minimal coupling to a background gravitational field given by a pseudo-Riemann metric $r_{\mu\nu}$ in $(n+1)$ dimensions and to allow position

dependence of all coefficients. For example, this extension affects the contractions in Eq. (1) and permits L_{ab} to describe the motion of a relativistic particle on an $(n+1)$ -dimensional curved spacetime manifold in the presence of varying background coefficients $a_\mu(x)$ and $b_\mu(x)$. The extended L could be obtained via a suitable Foldy–Wouthuysen transformation [36] of the gravitationally coupled Dirac equation. Position dependence of SME coefficients appears naturally in the gravity context [7,37–39], and Seifert has shown this can result from topologically nontrivial field configurations [40]. Note that comparatively large a_μ coefficients could have escaped experimental detection to date [41]. In what follows the fermion mass m is set to unity, $m = 1$, for simplicity.

The classical relativistic lagrangian can be viewed as a function $L = L(x, u)$ on the tangent bundle TM of the background spacetime manifold M . The Lorentz violation is assumed sufficiently small that nonzero values of L have only one sign, fixed by the mass term. The function L is smoothly differentiable everywhere except along a subset $S = S_0 + S_1$ of TM that includes the usual slit S_0 with $u^\mu = 0$ and possibly also an extension S_1 . The requirement of curve-reparametrization invariance imposes positive homogeneity of L of degree one in u^μ : $L(x, \kappa u) = \kappa L(x, u)$ for $\kappa > 0$. The Lorentz violation is also assumed sufficiently small that the nonsingular pseudo-Riemann metric dominates the background fields, so the effective metric $g_{\mu\nu} := -\partial^2(L^2/2)/\partial u^\mu \partial u^\nu$ felt by the relativistic particle is nonsingular. Inspection reveals that the above results are the defining properties of a local pseudo-Riemann–Finsler spacetime with pseudo-Finsler structure L defined on $TM \setminus S$ (for a textbook discussion see, e.g., Ref. [12]). This pseudo-Riemann–Finsler spacetime therefore underlies the motion of a relativistic classical particle experiencing general Lorentz violation.

The explicit forms of L and of many of its limits are involved and unknown in detail, so the corresponding pseudo-Finsler geometries may be challenging to explore. However, a variety of special pseudo-Riemann–Finsler spacetimes can be obtained by taking limits in which certain coefficients vanish. One simple example is the structure $L_a := L_{ab}|_{b \rightarrow 0}$ obtained as the limit $b_\mu \rightarrow 0$ of Eq. (1), which takes the familiar pseudo-Randers form. The ‘face’ limit L_{acef} of L with coefficients a_μ , $c_{\mu\nu}$, e_μ , and f_μ also yields a pseudo-Randers structure.

An interesting class of comparatively simple limits of L consists of ‘bipartite’ pseudo-Finsler structures involving two terms and taking the generic form

$$L_S = -\sqrt{-u^2} \mp \sqrt{-u^\mu s_{\mu\nu} u^\nu}, \quad (2)$$

where the symmetric quantity $s_{\mu\nu}$ satisfies $u^\mu s_{\mu\nu} u^\nu \leq 0$. Several of the more tractable limits of L fall into this class. For example, the b structure $L_b := L_{ab}|_{a \rightarrow 0}$ is bipartite, with $s_{\mu\nu} = b^2 r_{\mu\nu} - b_\mu b_\nu$. The example with $H_{\mu\nu}$ given in Eq. (15) of Ref. [19] is also a bipartite structure L_H . The choice $s_{\mu\nu} = -a_\mu a_\nu$ yields the two structures $L_{|a|} = -\sqrt{-u^2} \mp |a \cdot u|$ jointly spanning L_a , so L_{ab} has a tripartite form in this sense. It is likely that other bipartite limits of L remain to be discovered. Note that the quantity $s_{\mu\nu}$ is reminiscent of a secondary metric but may lack an inverse. For instance, $s_{\mu\nu}$ for the b structure L_b is noninvertible because it has a zero eigenvalue for the eigenvector b^μ . As a result, the subset S for this example consists of the extended slit $u^\mu = \kappa b^\mu$ for real κ . In the more general case, S_1 includes all u^μ that are nonzero eigenvectors of $s_{\mu\nu}$ with zero eigenvalues.

When $S = S_0$, which holds for the face structure and other pseudo-Randers limits of L , global pseudo-Finsler spacetimes arise. However, when S_1 is nonempty, the geometry is only local. An interesting open question is whether it is possible to resolve the geometry at S_1 to yield global pseudo-Finsler spacetimes. The fourth-order polynomial dispersion relation for the wave-

packet 4-momentum p_μ can be viewed as an algebraic variety $\mathcal{R}(p_\mu)$. The structure L is constructed using \mathcal{R} , the requirement of homogeneity, and the intrinsic derivatives of L defining the 4-velocity u^μ [19]. This construction generates five equations that combine to yield a polynomial $\mathcal{P}(L)$, which has physical roots yielding the local pseudo-Finsler structures L of interest and spurious roots corresponding to the set S_1 . The latter arise from the singularities of \mathcal{R} , which according to the implicit function theorem are determined by the p_μ derivatives of \mathcal{R} . Resolving the geometry at S_1 therefore corresponds to resolving the singularities of the variety \mathcal{R} .

At the level of quantum field theory, the singularities of \mathcal{R} reflect degeneracy of the wave-packet energies, which can be resolved using spin. For example, in the Lorentz-invariant case the two spin projections for the particle modes are degenerate for all momenta because the variety $\mathcal{R} = (p^2 + 1)^2$ is singular everywhere, but considering only one spin projection at a time yields the nonsingular variety $\mathcal{R} = p^2 + 1$ instead. This spin-based resolution also underlies the global nature of the pseudo-Randers face geometry. When any of b_μ , $d_{\mu\nu}$, $g_{\lambda\mu\nu}$, or $H_{\mu\nu}$ is nonzero, \mathcal{R} is generically a nontrivial quartic. Singularities occur for a subset of momenta at which the two spin projections are degenerate in energy, and these generate the set S_1 in TM . It is therefore plausible that the pseudo-Finsler geometry at S_1 for the general structure L could be resolved by the introduction of a spin variable. Note that a resolution for the corresponding variety \mathcal{R} is guaranteed by, for example, the Hironaka theorem [42]. The geometry at S_1 might therefore alternatively be resolved using a standard technique for singularities of algebraic varieties such as blowing up. The above comments suggest the existence of a global pseudo-Finsler geometry associated with the general structure L is a reasonable conjecture, but its proof remains open.

An interesting class of SME-based Finsler structures can be obtained from the pseudo-Finsler ones by restriction to the spatial submanifold or by Wick rotation. The full Finsler structure $F(x, y)$ obtained in this way retains much of the complexity of $L(x, u)$, but some limits are amenable to explicit investigation. One comparatively simple example arises by converting to n euclidean dimensions the pseudo-Finsler ab structure L_{ab} given in Eq. (1), yielding the Finsler ab structure

$$F_{ab} = \sqrt{y^2} + a \cdot y \pm \sqrt{b^2 y^2 - (b \cdot y)^2}. \quad (3)$$

Note that $F_a := F_{ab}|_{b \rightarrow 0}$ generates the usual Randers geometry. The Finsler face structure F_{acef} corresponding to L_{acef} also generates a Randers space.

Applied to the bipartite pseudo-Finsler structure (2), the above procedure generates a bipartite Finsler structure F_s given by

$$F_s = \sqrt{y^2} \pm \sqrt{y^j s_{jk} y^k}, \quad (4)$$

where the contractions now involve a positive-definite Riemann metric r_{jk} . For the lower sign choice, the nonnegativity of F_s implies s_{jk} must be bounded and yields the constraint $\det(1 - r^{-1}s) > 0$. This corresponds to the assumption that the Lorentz violation is perturbative. Among bipartite examples are the Finsler b structure $F_b := F_{ab}|_{a \rightarrow 0}$ and the H structure F_H obtained by restricting the pseudo-Finsler structure L_H for the coefficient $H_{\mu\nu}$ to spatial components. With the positive sign in Eq. (4) and invertible s_{jk} , F_s reduces to the two-metric y -global Finsler structure mentioned by Antonelli, Ingarden, and Matsumoto in the context of photon birefringence in uniaxial crystals (see Eq. (4.2.29) of Ref. [43]). In the present fermion context, however, $\det s \geq 0$ can vanish and hence s_{jk} may have no inverse, implying a nonempty slit extension S_1 . Indeed, the Finsler structures F and its limits associated with L are

typically y -local, although global pseudo-Randers structures yield global Randers structures. It is plausible that the putative y -global completions of pseudo-Finsler structures discussed above would also yield y -global Riemann–Finsler submanifolds.

3. The b structure

As an explicit example of an SME-based Riemann–Finsler geometry, consider the b structure $F_b := F_{ab}|_{a \rightarrow 0}$ obtained as a limit of Eq. (4). In fact this specifies two Finsler structures, one for each choice of the \pm sign, originating in the two spin degrees of freedom in the SME. For notational simplicity, it is convenient to write the ab structure $F_{ab}(x, y)$ as

$$F_{ab} = \rho + \alpha + \beta, \quad (5)$$

where

$$\rho := \sqrt{y^2}, \quad \alpha := a \cdot y, \quad \beta := \pm \sqrt{b^2 y^2 - (b \cdot y)^2}. \quad (6)$$

For a Riemann space with metric r_{jk} , the Finsler structure is $F_r = \rho$, for Randers a space it is $F_a = \rho + \alpha$, and for b space it is $F_b = \rho + \beta$. The dependence on x^j arises through $r_{jk}(x)$, $a_j(x)$, and $b_j(x)$. Constancy of the metric and coefficients would imply that the canonical momentum is conserved and that the Riemann–Finsler space is locally Minkowski, which parallels the treatment of Ref. [19]. The notational pairings (r, ρ) , (a, α) , (b, β) here match the standard literature on Lorentz violation; the conventional mathematics notation is recovered by the replacements $(r, \rho) \rightarrow (a, \alpha)$, $(a, \alpha) \rightarrow (b, \beta)$, $(b, \beta) \rightarrow (\star, \star)$.

A first observation is that the b structure F_b offers a kind of complement to the Randers a structure F_a . Given a nonzero 1-form a_j , F_a can be constructed by adding to F_r the parallel projection of the velocity y^j along a_j ,

$$F_a = \rho + \alpha = \sqrt{y^2} + \|a\| y_{\parallel}, \quad (7)$$

where $y_{\parallel} = a \cdot y / \|a\|$ is the a -normalized parallel projection. By splitting the Randers structure into two pieces, the last term can be written $\alpha = \pm \|a\| \sqrt{y_{\parallel}^2}$. However, given another nonzero 1-form b_j , a complementary structure can be obtained by combining F_r with the perpendicular projection of the velocity y^j along b_j instead. This gives the b structure F_b ,

$$F_b = \rho + \beta = \sqrt{y^2} \pm \|b\| \sqrt{y_{\perp}^2}, \quad (8)$$

where $y_{\perp}^j = y^j - (b \cdot y) b^j / b^2$ is the b -normalized perpendicular projection. One natural formulation of this perpendicular projection uses the Gram determinant or gramian. Given two vectors b^j , y^j and the Riemann metric r_{jk} , the gramian $\text{gram}(b, u)$ is given by $\text{gram}(b, u) = b^2 y^2 - (b \cdot y)^2$, so the b structure can be written

$$F_b = \sqrt{y^2} \pm \sqrt{\text{gram}(b, y)}. \quad (9)$$

In euclidean space, the gramian of two vectors represents the square of the area of the parallelogram formed by the vectors. The Cauchy–Schwarz inequality implies the gramian is a nonnegative quantity, $\text{gram}(b, y) \geq 0$, confirming that the square-root term in Eq. (9) is real, as required.

For low dimensions, the b structure F_b generates known geometries. When $n = 1$ the gramian vanishes, $\text{gram}(b, y) = 0$, so the Riemann–Finsler space reduces to a Riemann curve. When $n = 2$, the parallel and perpendicular projections y_{\parallel}^j and y_{\perp}^j span vector spaces of the same dimension. This enables the introduction of a vector v^j via the identification $y_{\perp}^j \rightarrow v_{\parallel}^j$, $y_{\parallel}^j \rightarrow v_{\perp}^j$, which maps

$\rho(y) \rightarrow \rho(v)$ and $\beta = \pm \|b\| \sqrt{y_\perp^2} \rightarrow \pm \|b\| \sqrt{v_\perp^2}$. The b structure with its two signs therefore maps to the two pieces of a Randers structure. An equivalent way to see this result is to identify the corresponding Randers 1-form a_j with the dual of b_j , $a_j = \epsilon_{jk} b^k$, which is perpendicular to b_j . The contribution to F_b from the perpendicular projection to b_j is equivalent to a contribution to F_b from the parallel projection to the dual $\epsilon_{jk} b^k$, so for $n = 2$ the b structure generates a Randers geometry. However, the duality equivalence is unavailable in higher dimensions, so for $n \geq 3$ the b space is expected to be neither a Riemann nor a Randers geometry. This result is proved in the next section by direct construction of the Matsumoto torsion.

To be a Finsler structure, F_b must satisfy certain basic criteria [9]. One is nonnegativity on TM . For the positive sign in Eq. (8), F_b is always nonnegative because $\rho \geq 0$ and $\beta \geq 0$. For the negative sign, $F_b \geq 0$ iff $\|b\| < 1$. This can be checked as follows. If b_j is zero then $F_b = \rho$, which is nonnegative. If y^j is zero then $F_b = 0$, which is also nonnegative. If both b_j and y^j are nonzero, define the nonzero real angle $\cos \theta = (b \cdot y) / (\|b\| \|y\|)$. Then $F_b = \|y\| (1 \pm \|b\| |\sin \theta|)$. So if $F_{b-} > 0$ then $\|b\| < 1$ because $0 \leq |\sin \theta| \leq 1$. Also, if $\|b\| < 1$ then $F_{b-} > 0$ for the same reason. The nonnegativity of F_b is therefore assured for both signs in F_b when $\|b\| < 1$. This condition is assumed in what follows.

Another criterion for a Finsler structure is C^∞ regularity. Since the Riemann metric is positive definite, the component ρ of F_b is C^∞ on the usual slit bundle $TM \setminus S_0$ for which $y^j \neq 0$. In contrast, the component β vanishes on the slit extension S_1 for which $y_\perp^j = 0$ and $y^j \neq 0$, so on $TM \setminus S_0$ only C^0 continuity of F_b is assured in the general case. However, β is positive definite outside the set $S = S_0 + S_1$ for which $\text{gram}(b, y) = 0$. This implies that F_b is C^∞ on $TM \setminus S$. Where necessary, the restriction of F_b to $TM \setminus S$ is assumed in what follows. As discussed in the previous section, when S_1 is nonempty this restriction implies the geometry associated with F_b is typically singular on S and hence is y local. Exceptions are the case $n = 1$, which generates a Riemann curve and is y global, and the case $n = 2$, which can be mapped to a y -global Randers geometry as described above. The singularities at $\text{gram}(b, y) = 0$, $y^j \neq 0$ originate in those at $\text{gram}(b, u) = 0$, $u^j \neq 0$ arising from the pseudo-Finsler structure L_b . In turn, these are associated with singularities of the algebraic variety \mathcal{R} mentioned in the previous section. Some calculation shows the latter appear at $\text{gram}(b, p) = 0$, $p^\mu \neq 0$, where the dispersion relation has solutions with degenerate energies for spin projections satisfying $p_\mu = \pm \sqrt{(1 + m^2/b^2)} b_\mu$ for timelike b_μ . Colladay, McDonald, and Mullins have exhibited the dispersion relation as intersecting pairs of deformed spheres [44]. In projection, the degenerate energies appear as cusps on the energy-momentum plot [16]. Resolving these singularities and generating the corresponding y -global Riemann–Finsler geometries for F_b is an interesting open problem.

The two remaining criteria for F_b to be a Finsler structure are positive homogeneity of degree one in y^j , $F_b(x, \kappa y) = \kappa F_b(x, y)$ for $\kappa > 0$, and positive definiteness of the symmetric Finsler metric $g_{jk} := (F_b^2/2)_{y^j y^k}$ associated with F_b . The former holds by inspection, but to demonstrate the latter some explicit results are useful.

A short calculation shows g_{jk} can be expressed compactly as

$$g_{jk} = \frac{F_b}{\rho} \frac{B}{\beta} r_{jk} - \rho \beta \kappa_j \kappa_k - \frac{F_b}{\beta} b_j b_k, \tag{10}$$

where $B := \beta + b^2 \rho$ and where κ_j represents the convenient combination

$$\kappa_j := \frac{\rho y_j}{\rho} - \frac{\beta y_j}{\beta} \tag{11}$$

involving the y^j derivatives of ρ and β . The latter are $\rho_{y_j} = y_j/\rho$ and $\beta_{y_j} = s_{jk} y^k/\beta$, where $s_{jk} = b^2 r_{jk} - b_j b_k$ for the b structure. One way to investigate positive definiteness of g_{jk} is via the determinant $\det g$. For $n = 1$ the determinant is $\det g = \det r$, matching expectations for a Riemann curve. For arbitrary $n \geq 2$, some calculation gives the pleasantly simple formula

$$\det g = \left(\frac{B}{\beta}\right)^{n-2} \left(\frac{F_b}{\rho}\right)^{n+1} \det r. \tag{12}$$

For $n = 2$, the first factor reduces to the identity and the remaining factors match the well-known determinant of the Randers metric, as might be expected from the $n = 2$ mapping between the a and b structures. Also, in the limit $\|b\| \rightarrow 0$ the formula produces $\det g = \det r$, as required.

Given the result (12), a standard argument [9] verifies positive definiteness of g_{jk} . Introducing $F_{\epsilon b} = \rho + \epsilon \beta$, it follows from (12) that $\det g_\epsilon$ is positive and so $g_{\epsilon jk}$ has no vanishing eigenvalues. At $\epsilon = 0$ the eigenvalues of $g_{\epsilon jk}$ are those of r_{jk} and hence are all positive, while as ϵ increases to 1 no eigenvalue can change sign because none vanishes. This ensures positive definiteness and also invertibility of g_{jk} .

4. Some properties of b space

For any $n > 1$, the Finsler b space with metric (10) cannot be a Riemann geometry. One way to see this is to construct the Cartan torsion $C_{jkl} := (g_{jk})_{y^l}/2$, which measures the non-euclidean nature of a Finsler structure viewed as a Minkowski norm on any tangent space TM_x . For b space, the Cartan torsion takes the simple form

$$C_{jkl} = -\frac{1}{2} \rho \beta \sum_{(jkl)} \kappa_j \kappa_{kl}, \tag{13}$$

where the sum is over cyclic permutations of j, k, l . Here, κ_{jk} is the combination

$$\kappa_{jk} := \frac{\rho_{y_j y_k}}{\rho} - \frac{\beta_{y_j y_k}}{\beta} \tag{14}$$

of the second y^j derivatives of ρ and β , which are $\rho_{y_j y_k} = (r_{jk} - \rho_{y_j} \rho_{y_k})/\rho$ and $\beta_{y_j y_k} = (s_{jk} - \beta_{y_j} \beta_{y_k})/\beta$. Note that $\beta_{y_j y_k}$ vanishes for $n = 2$. Since C_{jkl} is nonzero, Diecke's theorem [45] implies that F_b is non-euclidean as a Minkowski norm, so b space cannot be a Riemann geometry. The mean Cartan torsion $I_j := (\ln(\det g))_{y^j}/2$ is found to be

$$I_j = -\frac{1}{2} \left[(n+1) \frac{\beta}{F_b} - (n-2) \frac{b^2 \rho}{B} \right] \kappa_j, \tag{15}$$

which is also nonvanishing for $n > 1$.

For any $n > 2$, the b space also differs from Randers space. This can be seen by calculating the Matsumoto torsion M_{jkl} , which separates Randers and non-Randers metrics when $n > 2$. This torsion is defined as $M_{jkl} := C_{jkl} - \frac{1}{(n+1)} \sum_{(jkl)} I_j h_{kl}$, where the angular metric h_{jk} is $h_{jk} := g_{jk} - F_{y_j} F_{y_k}$. For F_b , the Matsumoto torsion can be written as

$$M_{jkl} = -\frac{1}{2} F_b \sum_{(jkl)} \kappa_j \left[\frac{(n-2)}{(n+1)} \frac{b^2 \rho}{B} (\rho_{kl} + \beta_{kl}) - \beta_{kl} \right]. \tag{16}$$

Since this is nonzero for $n > 2$, the Matsumoto–Hōjō theorem [46] shows that the b structure F_b cannot correspond to a Randers structure for $n > 2$, despite being constructed from a 1-form b_j and despite its comparative simplicity and calculability.

One way to explore features of a Riemann–Finsler space is to study its geodesics (for a textbook treatment see, e.g., Ref. [47]). The Finsler geodesics for b space are solutions of the equation

$$F_b \frac{d}{d\lambda} \left(\frac{1}{F_b} \frac{dx^j}{d\lambda} \right) + G^j = 0, \tag{17}$$

where the spray coefficients $G^j := g^{jm} \Gamma_{mkl} y^k y^l$ are defined in terms of the Christoffel symbol Γ_{jkl} for the Riemann–Finsler metric g_{jk} ,

$$\Gamma_{jkl} := \frac{1}{2} (\partial_{x_k} g_{jl} + \partial_{x_l} g_{jk} - \partial_{x_j} g_{kl}). \tag{18}$$

The geodesics solving Eq. (17) are valid for any choice of diffeomorphism gauge or, equivalently, for any choice of geodesic speed.

The spray coefficients G^j for b space can be calculated explicitly by first deriving $G_j := \Gamma_{jkl} y^k y^l$ and then contracting with the inverse Riemann–Finsler metric to get $G^j := g^{jk} G_k$. Some calculation reveals the compact result

$$G_j = \rho F_b \tilde{\gamma}_{j\bullet\bullet} + \rho^2 (\partial_{\bullet} \beta - \beta \tilde{\gamma}_{\bullet\bullet\bullet}) \kappa_j + \frac{\rho^2 F_b}{\beta} \hat{\gamma}_{j\bullet\bullet}. \tag{19}$$

Here, a lower index m contracted with $r^{mk} \rho_{y_k}$ is denoted by a bullet \bullet , with contractions understood to be external to any derivatives. Also, the Christoffel symbol $\tilde{\gamma}_{jkl}$ for the Riemann metric r_{jk} takes the usual form

$$\tilde{\gamma}_{jkl} := \frac{1}{2} (\partial_{x_k} r_{jl} + \partial_{x_l} r_{jk} - \partial_{x_j} r_{kl}), \tag{20}$$

while the symbol $\hat{\gamma}_{jkl}$ is defined analogously as

$$\hat{\gamma}_{jkl} := \frac{1}{2} (\partial_{x_k} s_{jl} + \partial_{x_l} s_{jk} - \partial_{x_j} s_{kl}) \tag{21}$$

using the form of s_{jk} for F_b .

To proceed, the inverse Riemann–Finsler metric is required. This can be determined to be

$$g^{jk} = \frac{\rho}{F_b} \left(r^{jk} + \frac{(b \cdot y)^2 \rho}{B \beta^2} \lambda^j \lambda^k - \frac{\rho}{B} b^{jk} \right), \tag{22}$$

where

$$\lambda_j := \frac{(b \cdot y)}{F_b} \rho_{y_j} - b_j. \tag{23}$$

Contracting with G_j gives the spray coefficients G^j as

$$G^j = \rho^2 \tilde{\gamma}^j_{\bullet\bullet} + \frac{\rho^3}{B \beta^3} [\beta^3 \hat{\gamma}^j_{\bullet\bullet} + \rho^2 \beta \hat{\gamma}_{\bullet\bullet\bullet} b^j - \rho y_{\circ} (\hat{\gamma}_{\bullet\bullet\bullet} + \beta \hat{\gamma}_{\bullet\bullet\bullet}) \lambda^j], \tag{24}$$

where a lower index m contracted with b^m is denoted by an open circle \circ . This result implies that the geodesic equation on b space can be viewed as the usual Riemann geodesic equation corrected by terms involving the symbol $\hat{\gamma}_{jkl}$.

The expression (24) for the spray coefficients leads to some insights about b space. Suppose the 1-form b_j is parallel with respect to the Riemann metric r_{jk} , $\tilde{D}_j b_k = 0$. Then, the Finsler geodesics reduce to the standard Riemann geodesics for the metric r_{jk} . This can be demonstrated via the explicit formula

$$r_{jk} \hat{\gamma}^k_{\bullet\bullet} = 2\rho y_j \tilde{D}_{\bullet} b_{\circ} - \tilde{D}_j b_{\circ} - b_j \tilde{D}_{\bullet} b_{\bullet} - b_{\bullet} \tilde{D}_{\bullet} b_j + b_{\bullet} \tilde{D}_j b_{\bullet}, \tag{25}$$

where \tilde{D}_j is the Riemann covariant derivative and contractions are understood to be external to derivatives, as before. It follows that if $\tilde{D}_j b_k = 0$ then $\hat{\gamma}^j_{\bullet\bullet} = 0$ and so also $\hat{\gamma}_{\bullet\bullet\bullet} = \hat{\gamma}_{\circ\bullet\bullet} = 0$. The Finsler spray coefficients (24) therefore become $G^j = \rho^2 \tilde{\gamma}^j_{\bullet\bullet} = \tilde{\gamma}^j_{kl} y^k y^l$, which are the usual Riemann spray coefficients for the metric r_{jk} . For constant Finsler speed or, equivalently, the gauge choice $F_b = 1$

fixing the curve parameter λ to a definite time $\lambda = t$, the geodesics then become solutions of the usual Riemann geodesic equation $\ddot{x}^j + \tilde{\gamma}^j_{kl} \dot{x}^k \dot{x}^l = 0$.

Remarkably, this result shows an r -parallel b_j coefficient has no effect on the motion. Intuitively, local conditions along the geodesics appear uniform, so local geodesic observations cannot unambiguously detect nonzero b_j . This suggests a suitable transformation or coordinate redefinition could be found to remove a parallel b_j from F_b , in analogy to the removal of certain unphysical coefficients in suitable limits of the SME [7,14,19,22,41,48,49]. For example, at least one component of the Randers coefficient a_{μ} can be removed by a phase redefinition of the fermion [7]. At the relativistic quantum level, the b_{μ} coefficients cannot generically be removed due to the entanglement of the spin components, which is absent at the classical level away from the set S . However, for constant b_{μ} in Minkowski spacetime, a chiral phase transformation can eliminate b_{μ} in the massless limit [14], and Lehnert has exhibited a nonlocal field redefinition that simultaneously removes b_{μ} from both spin components [49].

The expression (24) for the spray coefficients permits in principle the direct derivation of various geometric quantities characterizing b space, including the nonlinear connection $N^j_k := (G^j)_{y_j} / 2$, the Berwald connection ${}^B \Gamma^j_{kl} := (G^j)_{y_j y_k} / 2$, and the Berwald h - v curvature ${}^B P_k{}^j{}_{lm} := -F_b (G^j)_{y_k y_l y_m} / 2$. The Cartan, Chern (Rund), and Hashiguchi connections and the various associated curvatures and torsions can also in principle be obtained. However, the explicit formulae appear lengthy and are omitted here.

One result of interest pertaining to Berwald curvature is that any b space having b_j parallel with respect to r_{jk} is a Berwald space. Since $\tilde{D}_j b_k = 0$ implies $\hat{\gamma}^j_{\bullet\bullet} = 0$ and hence $G^j = \tilde{\gamma}^j_{kl} y^k y^l$, and since $\tilde{\gamma}^j_{kl}$ is independent of y^j , three y derivatives of G^j vanish. The Berwald h - v curvature is therefore zero, and so any r -parallel b space is a Berwald space. The converse statement that any Berwald b space is necessarily an r -parallel space appears plausible but is left open here.

Note that the analogous results for Randers space, which in present terminology state that any a space is a Berwald space iff it is an r -parallel space, are well established [50–53]. It is natural to conjecture that any SME-based Riemann–Finsler space is a Berwald space iff it has r -parallel coefficients for Lorentz violation. This attractive conjecture is amenable to direct investigation in various special cases, while a general proof is likely to offer valuable insights.

Another open challenge is to identify physical interpretations of SME-based pseudo-Riemann–Finsler and Riemann–Finsler structures, including the b structure. Examples for the a structure are well known. As mentioned in the introduction, the dynamics of a relativistic charged particle moving in an electromagnetic potential is governed by a pseudo-Randers a structure L_a , while the Randers a structure F_a has applications in several physical contexts including Zermelo navigation, optical metrics, and magnetic flow (see, e.g., Refs. [54–58]). Physical applications of the b structure would also be interesting from both physical and mathematical perspectives. By construction, the SME-based pseudo-Riemann–Finsler b structure L_b controls the motion of a relativistic particle in the presence of Lorentz violation involving the b_{μ} coefficients. However, identifying an application of the Riemann–Finsler b structure F_b appears challenging.

Some insight can be obtained by converting the variational problem associated with F_b into a form with similarities to the Randers structure F_a . This can be accomplished by introducing two additional coordinate variables, a 2-form $\Sigma_{jk} = -\Sigma_{kj}$ and a scalar κ , and defining

$$(F_b)_{\Sigma\kappa} := \rho + b^j \Sigma_{jk} y^k + \kappa \rho \left(\frac{1}{2} \text{tr} \Sigma^2 + 1 \right). \quad (26)$$

Note that the factor of ρ in the last term is included to maintain explicit homogeneity of degree one in y^j but has no essential effect on the argument to follow. Note also that the conjugate velocities for Σ_{jk} and κ are absent from $(F_b)_{\Sigma\kappa}$, so an effective metric defined in the enlarged space with coordinates $(x^j, \Sigma^{jk}, \kappa)$ would have zero eigenvalues.

In the variational problem (26), the scalar κ plays the role of a Lagrange multiplier, enforcing the norm constraint $\Sigma_{jk} \Sigma^{jk} = 2$. Variation with respect to the 2-form Σ_{jk} imposes the condition $b_j y_k - y_j b_k = 2\kappa \rho \Sigma_{jk}$. These equations can be solved to yield $\kappa = \pm \beta / 2\rho$ and $\Sigma_{jk} = \pm (b_j y_k - b_k y_j) / \beta$, which in turn can be used to show that the canonical momentum p_j associated with x^j in F_b coincides on shell with that in $(F_b)_{\Sigma\kappa}$, i.e., $p_j := \partial F_b / \partial y^j = \partial (F_b)_{\Sigma\kappa} / \partial y^j$. It follows that $(F_b)_{\Sigma\kappa}$ and F_b have the same geodesics. The two signs in F_b correspond to $\kappa > 0$ and $\kappa < 0$ in $(F_b)_{\Sigma\kappa}$. Also, the condition $\kappa = 0$ corresponds to $\beta = 0$ and hence for nonzero y^j defines the set S_1 of singularities in $TM \setminus S_0$. A similar construction works for the pseudo-Riemann–Finsler structure L_b , where the 2-form $\Sigma_{\mu\nu}$ takes the attributes of the usual spin 2-tensor.

The expression (26) reveals that for the b structure the combination $b^k \Sigma_{kj}$ plays a role analogous in certain respects to that of the Randers a_j coefficient. Since Σ_{jk} is a dynamical variable, this suggests b space can be viewed in terms of a Randers space with a dynamical coefficient a_j . Shen [54] has shown that the usual Randers geodesics can be identified with solutions to the Zermelo problem of navigation control in an external wind related to the coefficient a_j (for a detailed exposition, see the treatment by Bao and Robles [56]). The dynamical coefficient $b^k \Sigma_{kj}$ therefore suggests a related interpretation for b space in which the effect of the external flow b_j is adjustable, in analogy to the change of effective wind direction arising from the combination of a boat's sail and keel. A direct application to the Zermelo problem falls short because in the Randers case the external flow is related not only to the Zermelo wind but also to the Riemann metric of the navigation space, whereas the term ρ in the expression (26) is independent of Σ_{jk} . However, an interpretation of F_b along these lines may be achievable for a system described by the more general theory of optimal control.

Another approach is to seek a physical system in which the notion of distance is intrinsically quartic rather than quadratic. In the optical-metric interpretation, for example, the Randers structure F_a generates geodesics matching the spatial trajectories of null geodesics in a stationary spacetime, which are determined by a quadratic spacetime interval $ds^2 = 0$ [57]. In contrast, geodesics of the b structure F_b match geodesics defined by a null quartic interval $ds^4 = 0$ in a certain class of spacetimes. The quartic nature of b space is directly reflected in its close ties to the motion of a massive Dirac fermion, which for nonzero b_μ generically has four distinct modes corresponding to the two spin degrees of freedom for particles and antiparticles, whereas the spin-independent Randers case involves only two distinct modes.

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