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A KHINTCHINE DECOMPOSITION FOR FREE PROBABILITY

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Let μ be a probability measure on the real line. In this paper we prove that there exists a decomposition $\mu = \mu_0 \boxplus \mu_1 \boxplus \cdots \boxplus \mu_n \boxplus \cdots$ such that μ_0 is infinitely divisible, and μ_i is indecomposable for $i \geq 1$. Additionally, we prove that the family of all $\boxplus$-divisors of a measure μ is compact up to translation. Analogous results are also proven in the case of multiplicative convolution.

1. Introduction. In classical probability theory, it has long been known that the set of all convolution divisors of a random variable is compact up to translation. That is, given a family of decompositions $\mu = \mu_{1,i} * \mu_{2,i}$ with $i \in I$, the families $\{\mu_{j,i}\}_{i\in I, j=1,2}$ can be translated to form sequentially compact families $\{\hat{\mu}_{i,j}\}_{i\in I, j=1,2}$ so that $\mu = \hat{\mu}_{1,i} * \hat{\mu}_{2,i}$ for all $i \in I$. The proof of this result is a simple application of Lévy's lemma; see Chapter 5 in [15] for a full account of the classical case. This compactness lemma serves as the cornerstone for the proof of the following classical result of Khintchine.

THEOREM 1.1. *Let μ be a probability measure. Then there exist measures μ_i with $i = 0, 1, 2, \ldots$ such that μ_0 is $*$-infinitely divisible, μ_i is indecomposable for $i = 1, 2, \ldots$ and $\mu = \mu_0 * \mu_1 * \mu_2 * \cdots$. This decomposition is not unique.*

The equation $\mu = \mu_0 * \mu_1 * \mu_2 * \cdots$ is in the sense that in the weak* topology we have that $\lim_{n\uparrow\infty} \mu_0 * \mu_1 * \cdots * \mu_n = \mu$. This type of equality will be used throughout the paper without further comment.

In free probability theory, the corresponding compactness and decomposition theorems have hitherto been absent from the literature. Partial results of the corresponding compactness theorem are near trivialities. Indeed, consider a W^* probability space (A, τ) and a random variable $X \in A$ with mean 0 and finite variance. Let $X = X_1 + X_2$ be a decomposition with the X_i's freely independent and of mean 0. Then the equation $\tau(X^2) = \tau(X_1^2) + \tau(X_2^2)$ would imply the necessary tightness result when applied to families of decompositions.

It is the first aim of this paper to prove the corresponding tightness results in the fullest possible generality. That is, we make no assumptions as to the finiteness of moments. It is the second aim of this paper to prove versions of Theorem 1.1 for additive and multiplicative free convolution.

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This paper is organized as follows: in Section 2 we give the background and terminology of additive free convolution; in Section 3 we state and prove a number of compactness results for families of decompositions with respect to additive free convolution; in Section 4 we prove the existence of the Khintchine decomposition with respect to additive free convolution; Sections 5, 6 and 7 are the respective analogs of Sections 2, 3 and 4, but with regard to multiplicative free convolution for measures supported on the positive real numbers; in Section 8 we give the background and terminology for multiplicative free convolution of measures supported on the unit circle; in Section 9 we prove the existence of the Khintchine decomposition for measures supported on the unit circle; in Section 10 we provide applications of our compactness results.

2. Background and terminology for additive-free convolution. We refer to [21] for a full account of the basics of free probability theory.

Let (A,τ) be a W^* probability space. We say that a family of unital subalgebras $\{A_i\}_{i\in I}$ are *freely independent* if $\tau(x_{i_1}x_{i_2}\cdots x_{i_n})=0$ for $x_{i_j}\in A_{i_j}$ whenever $i_j\neq i_{j+1}$ for $j=1,\dots,n-1$ and $\tau(x_{i_k})=0$ for $k=1,\dots,n$. We say that random variables $x,y\in A$ are freely independent if the unital algebras that they generate in A satisfy the above definition.

Assume that $A\subset B(H)$. We say that a not necessarily bounded operator x is affiliated with A (in symbols, $x\eta A$) if the spectral projections of x are elements in A. Equivalently, $x\eta A$ if for every $y\in A'$ (the commutant of A), we have that $yx\subset xy$. This expanded class of random variables allows us to study measures with unbounded support.

Let $x\eta A$ be a self-adjoint random variable with distribution μ, a probability measure supported on $\mathbb{R}$. We associate to μ its Cauchy transform,

$$G_\mu(z)=\int_{\mathbb{R}}\frac{d\mu(t)}{z-t}=\tau\big((z-x)^{-1}\big).$$

Observe that $zG_\mu(z)\to 1$ as $z\to\infty$ nontangentially. It follows that G_μ is univalent on a set of the form $\Gamma_{\alpha,\beta}=\{z\in\mathbb{C}^+:\Im(z)>\alpha,\Im(z)>\beta\Re(z)\}$ for sufficiently large $\alpha,\beta>0$. Throughout this paper we shall refer to a set of this type as a *Stolz angle*. The set $G_\mu(\Gamma_{\alpha,\beta})$ contains a set of the form $\Lambda_{\alpha',\beta'}=\{z\in\mathbb{C}^-:0<\Im(z)\le\alpha',\beta'\Re(z)<\Im(z)\}$ on which we have a well-defined left inverse, G_μ^{-1}. The function $R_\mu(z)=G_\mu^{-1}(z)-1/z$ is called the R-transform of μ. First proved in [18], the following equality is fundamental in free probability theory:

$$R_{\mu\boxplus\nu}(z)=R_\mu(z)+R_\nu(z).$$

In what follows, it will be more convenient to consider the following functions:

$$F_\mu(z)=\frac{1}{G_\mu(z)},$$

$$\varphi_\mu(z)=F_\mu^{-1}(z)-z=R_\mu(1/z).$$

These functions are referred to as the F and Voiculescu transform, respectively. They have the following properties which are proven to various degrees of generality in [10, 18] and [16]:

(1) $|F_\mu(z) - z| = o(|z|)$ uniformly as $|z| \to \infty$ in $\Gamma_{\alpha,\beta}$ for all $\alpha, \beta > 0$.
(2) $\Im(F_{\mu(z)}) \geq \Im(z)$ for all $z \in \mathbb{C}^+$.
(3) F_μ has a well-defined left inverse on $\Gamma_{\alpha,\beta}$ for some $\alpha, \beta > 0$ (hence, the Voiculescu transform is defined on this set).
(4) There exist $\alpha, \beta > 0$ such that $\varphi_{\mu \boxplus \nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$ when $z \in \Gamma_{\alpha,\beta}$.
(5) $F_{\mu \boxplus \delta_c}(z) = F_\mu(z - c)$ and $\varphi_{\mu \boxplus \delta_c} = c + \varphi_\mu(z)$ for $c \in \mathbb{R}$.

Given a decomposition $\mu = \mu_1 \boxplus \mu_2$, it was shown in [20] and [13] that there exist analytic subordination functions $\omega_i : \mathbb{C}^+ \to \mathbb{C}^+$ such that:

(1) $F_\mu(z) = F_{\mu_i}(\omega_i(z))$ for $z \in \mathbb{C}^+$ and $i = 1, 2$.
(2) $\lim_{y \uparrow +\infty} \frac{\omega_i(iy)}{iy} = 1$ for $i = 1, 2$.
(3) $\omega_1(z) + \omega_2(z) = z + F_\mu(z)$.

Observe that ω_i and F_μ satisfy the same asymptotic properties in (2) above. A classical result, due to Nevanlinna (whose full account can be found in [1], Volume 2, page 7), implies that these functions have the following representation:

$$\omega_i(z) = r_i + z + \int_{-\infty}^{\infty} \frac{1 + tz}{z - t} \, d\sigma_i(t),$$

$$F_\mu(z) = r + z + \int_{-\infty}^{\infty} \frac{1 + tz}{z - t} \, d\sigma(t),$$

where $r, r_i \in \mathbb{R}$ and σ, σ_i are positive, finite measures which are uniquely determined by ω_i and F_μ. Observe that property (3) above and uniqueness imply that $r_1 + r_2 = r$ and $\sigma_1 + \sigma_2 = \sigma$.

We denote by $\mathfrak{F}_\mu(t) = \mu((-\infty, t])$ the cumulative distribution function of μ. This function is used to define two metrics on the space of probability measures, namely the Kolmogorov and Lévy metric, d_∞ and d, respectively. These are defined as follows:

$$d_\infty(\mu, \nu) = \sup_{t \in \mathbb{R}} |\mathfrak{F}_\mu(t) - \mathfrak{F}_\nu(t)|,$$

$$d(\mu, \nu) = \inf\{\varepsilon > 0 : \mathfrak{F}_\mu(t - \varepsilon) - \varepsilon \leq \mathfrak{F}_\nu(t) \leq \mathfrak{F}_\mu(t + \varepsilon) + \varepsilon\}.$$

The Lévy metric induces the weak topology on the space of probability measures on the line, while the Kolmogorov metric induces a stronger topology, which we call the Kolmogorov topology. We have the the following facts, first proven in [10], which will be used throughout, often without reference:

LEMMA 2.1. *Let μ_n and ν_n converge to probability measures μ and ν, respectively, in the weak* (resp., Kolmogorov) topology. Then $\mu_n \boxplus \nu_n$ converges to $\mu \boxplus \nu$ in the weak* (resp., Kolmogorov) topology.*

The proof of this lemma relies on the following inequalities which will be used in what follows:

$$d(\mu \boxplus \nu, \mu' \boxplus \nu') \leq d(\mu, \mu') + d(\nu, \nu'),$$
$$d_\infty(\mu \boxplus \nu, \mu' \boxplus \nu') \leq d_\infty(\mu, \mu') + d_\infty(\nu, \nu').$$

The next two lemmas were first proven in Section 5 of [10].

LEMMA 2.2. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a tight sequence of measures. Then there exists a Stolz angle $\Gamma_{\alpha,\beta}$ such that the functions $|F_{\mu_n}(z) - z| = o(z)$ uniformly as $|z| \to \infty$ in this set. In particular, the functions $F_{\mu_n}^{-1}$ exist on a common domain for all n.*

LEMMA 2.3. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$. The following assertions are equivalent*:

(1) *The sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges in the weak* topology to a probability measure μ.*

(2) *There exist $\alpha, \beta > 0$ such that the functions $\{\varphi_{\mu_n}\}_{n\in\mathbb{N}}$ are defined and converge uniformly on compact subsets of $\Gamma_{\alpha,\beta}$ to a function φ and $\varphi_{\mu_n}(z) = o(z)$ uniformly in n as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.*

Moreover, if (1) *and* (2) *are satisfied, we have that $\varphi = \varphi_\mu$ in $\Gamma_{\alpha,\beta}$.*

DEFINITION 2.4. A probability measure μ on the real line is said to be $\boxplus$-infinitely divisible if for every $n \in \mathbb{N}$ there exists a measure $\mu_{1/n}$ such that $\mu = \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$, where the measure on the right is the n-fold free convolution.

In dealing with infinitely divisible measures, the following characterization, first proven in [8], will prove invaluable.

THEOREM 2.5. *Let $\{\mu_{i,j}\}_{i\in\mathbb{N}, j=1,\ldots,k_i}$ be an array of Borel probability measures on $\mathbb{R}$ and $\{c_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers. Assume that $\lim_{i\to\infty} \max_{j=1,\ldots,k_i} \mu_{i,j}(\{t : |t| > \varepsilon\}) = 0$ for all $\varepsilon > 0$ and that the measures $\delta_{c_i} \boxplus \mu_{i,1} \boxplus \cdots \boxplus \mu_{i,k_i}$ converge to a probability measure μ in the weak* topology. Then μ is $\boxplus$-infinitely divisible.*

DEFINITION 2.6. Let μ be a probability measure. A decomposition $\mu = \nu \boxplus \rho$ is said to be nontrivial if neither ν nor ρ is a Dirac mass. We say that a measure μ is indecomposable if it has no nontrivial decomposition.

Such measures were studied extensively in [2, 3] and [12]. We close with a theorem, first proven in [10] and [3] from which we derive a corollary that will play a key role in the proof of Theorem 4.4.

THEOREM 2.7. *Let μ and ν be two Borel probability measures on $\mathbb{R}$, neither of them a Dirac mass. Then*:

(1) *The point $a \in \mathbb{R}$ is an atom of the measure $\mu \boxplus \nu$ if and only if there exist points $b, c \in \mathbb{R}$ such that $a = b + c$ and $\mu(\{b\}) + \nu(\{c\}) > 1$. Moreover, $(\mu \boxplus \nu)(\{a\}) = \mu(\{b\}) + \nu(\{c\}) - 1$.*

(2) *The absolutely continuous part of $\mu \boxplus \nu$ is always nonzero, and its density is analytic wherever positive and finite. More precisely, there exists an open set $U \subseteq \mathbb{R}$ so that the density function $f(x) = \frac{d(\mu\boxplus\nu)^{ac}(x)}{dx}$ with respect to Lebesgue measure is locally analytic on the set U and $(\mu \boxplus \nu)^{ac}(\mathbb{R}) = \int_U f(x)\,dx$.*

(3) *The singular continuous part of $\mu \boxplus \nu$ is zero.*

COROLLARY 2.8. *Let μ and ν be as above. There exists a point $s \in \mathbb{R}$ such that the cumulative distribution function $\mathfrak{F}_{\mu\boxplus\nu}$ is continuous and increasing in a neighborhood of s.*

PROOF. First observe that (1) implies that $\mu \boxplus \nu$ has only finitely many point masses. To see this, assume that $a = a_1 + a_2$ and $b = b_1 + b_2$ are point masses of $\mu \boxplus \nu$ where

$$(\mu \boxplus \nu)(\{a\}) = \mu(\{a_1\}) + \nu(\{a_2\}) - 1,$$
$$(\mu \boxplus \nu)(\{b\}) = \mu(\{b_1\}) + \nu(\{b_2\}) - 1.$$

Further assume that $a_1 \neq b_1$. This implies that

$$\mu(\{a_1\}) + \mu(\{b_1\}) \leq 1.$$

Combined with the previous equalities, this implies that $1 < \nu(\{a_2\}) + \nu(\{b_2\})$ so that $a_2 = b_2$. This implies that, under these assumptions, there are at most $(1 - \nu(\{b_2\}))^{-1}$ point masses of $\mu \boxplus \nu$.

Note that the nonatomic part of $\mu \boxplus \nu$ has mass strictly greater than 0. To see this, let $\{x_i\}_{i=1}^n$ be the set of point masses of $\mu \boxplus \nu$. Let y and $\{z_i\}_{i=1}^n$ satisfy $y + z_i = x_i$ and $\nu(y) + \mu(z_i) - 1 = (\mu \boxplus \nu)(x_i)$ for $i = 1, 2, \ldots, n$ where these points arise as in the previous paragraph. Summing over both sides of the equation and recalling that $\nu(y) < 1$, we have that

$$\sum_{i=1}^{n} (\mu \boxplus \nu)(x_i) = n\nu(y) - n + \sum_{i=1}^{n} \mu(z_i) < n - n + \mu(\mathbb{R}) = 1.$$

Thus, for U as in the previous theorem, pick an open subset $V \subseteq U$ that contains no point masses. This set satisfies our claim. $\square$

3. Compactness results for additive free convolution. We begin our investigation with a technical lemma.

LEMMA 3.1. *Let μ be a probability measure on $\mathbb{R}$. Let Ω denote a Stolz angle on which F_μ^{-1} is defined. If $\mu = \mu_1 \boxplus \mu_2$ is any decomposition of μ, then φ_{μ_1} and φ_{μ_2} have analytic extensions to Ω. These extensions satisfy $\Im(\varphi_{\mu_1}(z))$ and $\Im(\varphi_{\mu_2}(z)) \le 0$ for all $z \in \Omega$.*

PROOF. By assumption, φ_μ exists and is analytic on all of Ω and, since F_μ increases the imaginary part, $\varphi_\mu(z) \le 0$ for all $z \in \Omega$.

Turning our attention to μ_1, consider the subordination function ω satisfying $F_\mu(z) = F_{\mu_1}(\omega(z))$ for $z \in \mathbb{C}^+$. Recall that

$$\lim_{y\uparrow\infty} \frac{F_\mu(iy)}{iy} = \lim_{y\uparrow\infty} \frac{F_{\mu_1}(iy)}{iy} = \lim_{y\uparrow\infty} \frac{\omega(iy)}{iy} = 1.$$

These facts imply that on a sufficiently small Stolz angle, all three functions are invertible, and we have the following:

$$\omega \circ F_\mu^{-1} = F_{\mu_1}^{-1}.$$

Since the left-hand side is defined on Ω, the right-hand side must also extend to Ω. This implies that this implies that the Voiculesu transform of μ_1 extends to Ω and, by abuse of notation, we continue to call this extension φ_{μ_1}

With regard to the negativity of the imaginary part of our analytic extension, note that on a large enough Stolz angle, F_{μ_1} acts as a left inverse for $F_{\mu_1}^{-1}$ and $\omega \circ F_\mu^{-1} = F_{\mu_1}^{-1}$. Thus, $F_{\mu_1}(\omega(F_\mu^{-1}(z))) = z$. As the left-hand side of the equation is defined and analytic for all $z \in \Omega$, by analytic continuation, the same equality holds for all $z \in \Omega$. Thus

$$\varphi_{\mu_1}(z) = \omega(F_\mu^{-1}(z)) - z = \omega(F_\mu^{-1}(z)) - F_{\mu_1}(\omega(F_\mu^{-1}(z)))$$

for all $z \in \Omega$. As F_{μ_1} increases the imaginary part, our result holds. □

With this preliminary result out of the way, we now begin proving tightness results. The *diameter* of a subset $\sigma \subset \mathbb{R}$ is defined in the usual way: $\operatorname{diam}(\sigma) = \sup_{x,y\in\sigma} |x - y|$. The *support* of a measure μ [in symbols, $\operatorname{supp}(\mu)$] is the complement of the largest open μ-null set.

THEOREM 3.2. *Let μ be a probability measure with compact support, and consider a decomposition $\mu = \mu_1 \boxplus \mu_2$. Then* $\operatorname{diam}(\operatorname{supp}(\mu_i)) \le \operatorname{diam}(\operatorname{supp}(\mu))$ *with equality if and only if one of the μ_i is a Dirac mass.*

PROOF. Consider the subordination functions ω_i satisfying $F_\mu(z) = F_{\mu_i}(\omega_i(z))$ with the following Nevanlinna representations:

$$\omega_i(z) = r_i + z + \int \frac{1+zt}{t-z}\, d\sigma_i(t),$$

$$F_\mu(z) = r + z + \int \frac{1+zt}{t-z}\, d\sigma(t).$$

Let $\alpha, \beta \in \mathbb{R}$ satisfy $\operatorname{supp}(\mu) \subseteq [\alpha, \beta]$, where the interval on the right-hand side is the smallest for which this containment holds. Observe that G_μ has a nonzero real analytic continuation across $(-\infty, \alpha)$ so that the same must hold for F_μ. This implies that $\sigma((-\infty, \alpha)) = 0$. Since $\sigma = \sigma_1 + \sigma_2$, we also have that $\sigma_i(-\infty, \alpha) = 0$ so that, by the Schwarz reflection principle, ω_i admits analytic continuation across $(-\infty, \alpha)$. Furthermore, ω_i is increasing on $(-\infty, \alpha)$ so that $F_{\mu_i} = F_\mu \circ \omega_i^{-1}$ has an analytic continuation to $\omega_i(-\infty, \alpha)$. This tells us that $\operatorname{supp}(\mu_i) \subset \mathbb{R} \setminus \omega_1((-\infty, \alpha))$.

Now, observe that $\omega_i(x) - x \to r_i - m_i$ as $x \to \pm\infty$, where m_i is the first moment of σ_i. Differentiating the Nevanlinna representation of ω_i, it is clear that $\omega_i'(x) \geq 1$ for $x < \alpha$. Thus

$$\begin{aligned}\omega_i(\alpha - \varepsilon) &= \int_x^{\alpha-\varepsilon} \omega_i'(t)\, dt + \omega_i(x) \geq \int_x^{\alpha-\varepsilon} dt + x + \bigl(\omega_i(x) - x\bigr) \\ &\to \alpha - \varepsilon + r_i - m_i.\end{aligned}$$

It follows that $(-\infty, \alpha + r_i - m_i) \subseteq \omega_i((-\infty, \alpha))$. Similarly, $(\beta + r_i - m_i, \infty) \subseteq \omega_i(\beta, \infty)$. These two observations imply that $\operatorname{supp}(\mu_i) \subseteq [\alpha + r_i - m_1, \beta + r_i - m_i]$. Hence, we have that $\operatorname{diam}(\operatorname{supp}(\mu_i)) \leq \operatorname{diam}(\operatorname{supp}(\mu))$.

With regard the equality claim, observe that our measure $\sigma_1 = 0$ implies that μ_1 is a translation of μ. This implies that μ_2 is a Dirac mass. Thus, by assuming that neither μ_1 nor μ_2 is a Dirac mass, we have that $\sigma_i \neq 0$ for $i = 1, 2$. This implies that $\omega_i'(t) > 1$ for $t < \alpha$. It follows that $\operatorname{supp}(\mu_i) \subsetneq [\alpha + r_i - m_1, \beta + r_i - m_i]$, and our claim follows. $\square$

In what follows, for $O \subset \mathbb{R}$, we let $\operatorname{conv}(O)$ be the smallest interval containing the set O.

LEMMA 3.3. *Let μ_1 and μ_2 be probability measures with compact support. Then* $\operatorname{supp}(\mu_1 \boxplus \mu_2) \subseteq \operatorname{conv}(\operatorname{supp}(\mu_1) + \operatorname{supp}(\mu_2))$.

PROOF. Let x_1 and x_2 be freely independent random variables in a W^* probability space (A, τ) with respective distributions μ_1 and μ_2. Let $c_i = \inf\{t \in \sigma(x_i)\}$ and $d_i = \sup\{t \in \sigma(x_i)\}$. It is precisely the content of Theorem 4.16 in [10] that $x_1 - c_1 I + x_2 - c_2 I$ is a positive random variable. Thus, its spectrum is contained in the positive real numbers. Since the spectrum of a self-adjoint operator contains the support of its distribution, we have that the distribution of $x_1 - c_1 I + x_2 - c_2 I$ is supported in the positive reals. Similarly, the distribution of $x_1 - d_1 I + x_2 - d_2 I$ is supported in the negative reals. Thus, $\operatorname{supp}(\mu_1 \boxplus \mu_2) \subseteq [c_1 + c_2, d_1 + d_2]$, which is equivalent to our claim. $\square$

We now extend the above theorem to measures with unbounded support. For a measure μ, recall that $\mathfrak{F}_\mu$ denotes its cumulative distribution function. We shall let $\Omega_\varepsilon(\mu) = \{t \in \mathbb{R} : \varepsilon < \mathfrak{F}_\mu(t) < 1 - \varepsilon\}$.

THEOREM 3.4. *Let $\mu = \mu_1 \boxplus \mu_2$. For $\varepsilon > 0$ we have that $\overline{\Omega_\varepsilon(\mu)} \subseteq \overline{\Omega_{\varepsilon/2}(\mu_1)} + \overline{\Omega_{\varepsilon/2}(\mu_2)}$.*

PROOF. Let a_i and b_i denote the left and right endpoints of $\Omega_{\varepsilon/2}(\mu_i)$. Consider the probability measures $\mu_{i,\varepsilon/2}$ defined as follows:

$$\mu_{i,\varepsilon/2}(\sigma) = \mu_i\big(\sigma \cap \Omega_{\varepsilon/2}(\mu_i)\big) + \left(\frac{1 - \mu_i(\Omega_{\varepsilon/2}(\mu_i))}{2}\right)\delta_{a_i}(\sigma)$$
$$+ \left(\frac{1 - \mu_i(\Omega_{\varepsilon/2}(\mu_i))}{2}\right)\delta_{b_i}(\sigma).$$

Observe that $d_\infty(\mu_i, \mu_{i,\varepsilon/2}) \le \varepsilon/2$ where d_∞ denotes the Kolmogorov metric. Further observe that $\operatorname{supp}(\mu_{i,\varepsilon/2}) = \Omega_{\varepsilon/2}(\mu_i)$. It follows that

$$d_\infty(\mu, \mu_{1,\varepsilon/2} \boxplus \mu_{2,\varepsilon/2}) \le d_\infty(\mu_1, \mu_{1,\varepsilon/2}) + d_\infty(\mu_2, \mu_{2,\varepsilon/2}) \le \varepsilon.$$

Observe that $\mathfrak{F}_\mu(t) \in (\varepsilon, 1 - \varepsilon)$ implies that $\mathfrak{F}_{\mu_{1,\varepsilon/2} \boxplus \mu_{2,\varepsilon/2}}(t) \in (0, 1)$. Thus, $\Omega_\varepsilon(\mu) \subseteq \operatorname{supp}(\mu_{1,\varepsilon/2} \boxplus \mu_{2,\varepsilon/2})$. By Lemma 3.3, we have that $\overline{\operatorname{supp}(\mu_{1,\varepsilon/2} \boxplus \mu_{2,\varepsilon/2})} \subset \operatorname{conv}(\operatorname{supp}(\mu_{1,\varepsilon/2}) + \operatorname{supp}(\mu_{2,\varepsilon/2})) = \operatorname{conv}(\overline{\Omega_{\varepsilon/2}(\mu_1)} + \overline{\Omega_{\varepsilon/2}(\mu_2)})$. $\square$

We close with the main result of the section. Observe that this theorem lacks the quantitative information found in Theorem 3.4. The hope was to extend Theorem 3.2 in a similar manner, but such an approach proved elusive. We have found no negative results in this direction so we conjecture that $\Omega_\varepsilon(\mu_i) \subseteq x + \Omega_{\varepsilon/2}(\mu)$ for some $x \in \mathbb{R}$. However, the theorem below provides us with tightness and will suffice for the applications that follow.

Let ν be a measure satisfying $0 < \nu(\mathbb{R}) \le 1$. We extend the definition of the Cauchy and F-transform by letting $G_\nu(z) = \int_{\mathbb{R}} (z - t)^{-1}\, d\nu(t)$ and $F_{\nu(z)} = 1/G_{\nu(z)}$. Observe that for $\lambda = \nu(\mathbb{R})$, the measure $\hat{\nu} = \lambda^{-1}\nu$ is in fact a probability measure. This provides us with the following inequality which we shall exploit in what follows.

$$\Im F_{\nu(z)} = \lambda^{-1} \Im F_{\hat{\nu}}(z) \ge \lambda^{-1} \Im(z).$$

THEOREM 3.5. *Let $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ for all $k \in \mathbb{N}$. Then there exist translations $\{\hat{\mu}_{i,k}\}$ so that $\mu = \hat{\mu}_{1,k} \boxplus \hat{\mu}_{2,k}$ and the family of measures $\{\hat{\mu}_{i,k}\}$ is tight for $i = 1, 2$.*

Before embarking on the proof, we remark that there are two ways for tightness to fail. The first is to take an otherwise tight sequence of measures and translate their support to $\pm\infty$. The second is if the mass of your measures becomes more spread out. Since our theorem assumes away the former case, the idea of the proof is to show that the latter cannot happen. We quantify the latter case as follows: a sequence of measures $\{\mu_k\}_{k\in\mathbb{N}}$ cannot be translated to tightness if and only if there exists a $\gamma \in [0, 1)$ such that $\liminf_k \sup_{t\in\mathbb{R}}(\mu_k(t-a, t+a)) < \gamma$ for all $a \in \mathbb{R}^+$.

PROOF OF THEOREM 3.5. Assume that $\{\mu_{1,k}\}_{k\in\mathbb{N}}$ is tight which is equivalent to sequential precompactness in the weak* topology. As we established in Lemma 3.1, F_μ^{-1}, $F_{\mu_{1,k}}^{-1}$ and $F_{\mu_{2,k}}^{-1}$ extend to a common domain for all k, which we shall denote by Ω in what follows, on which they satisfy $F_\mu^{-1}(z) - F_{\mu_{1,k}}^{-1}(z) + z = F_{\mu_{2,k}}^{-1}(z)$. Recall that, according to Lemma 2.3, weak convergence is equivalent to the uniform convergence of the functions $F_{\mu_{1,k}}^{-1}$ on compact subsets of a Stolz angle $\Gamma_{\alpha,\beta}$ to a function F satisfying $F(iy)/iy \to 1$ as $y \to \infty$. The equation above implies that $F_{\mu_{2,k}}^{-1}$ is similarly behaved on $\Gamma_{\alpha,\beta}$ so that $\{\mu_{2,k}\}$ is also weakly convergent along this subsequence. Thus $\{\mu_{1,k}\}$ is tight, which implies the same for $\{\mu_{2,k}\}$.

With that in mind, we may assume, for the sake of contradiction, that the family $\{\mu_{1,k}\}$ cannot be translated to form a tight family of measures along any subsequence. This implies that there exists a $\gamma \in (0, 1)$ such that $\liminf_k \sup_{x\in\mathbb{R}}(\mu_{1,k}(-a + x, a + x)) < \gamma < 1$ for all $a \in \mathbb{R}^+$. Passing to subsequences and possibly renumbering our measures, we may assume that $\sup_{x\in\mathbb{R}}(\mu_{1,k}(x-k, x+k)) < \gamma$.

Now, pick $\varepsilon > 0$ such that $(1-\varepsilon) > \gamma$. Let $w = ib$ where $b \in \mathbb{R}^+$ is chosen so that $w \in \Omega$ and $|F_\mu^{-1}(w) - w| \le \varepsilon|w| = \varepsilon b$. Observe that

$$F_\mu^{-1}(w) = F_{\mu_{1,k}}^{-1}(w) + F_{\mu_{2,k}}^{-1}(w) - w$$

implies that

$$\Im F_{\mu_{1,k}}^{-1}(w) + \Im F_{\mu_{2,k}}^{-1}(w) \ge b(2-\varepsilon).$$

In Lemma 3.1, we showed that $F_{\mu_{i,k}}^{-1}$ decreases the imaginary part so that

$$\Im F_{\mu_{1,k}}^{-1}(ib) \ge b(1-\varepsilon).$$

Further observe that analytic continuation implies that $F_{\mu_{1,k}}(F_{\mu_{1,k}}^{-1}(z)) = z$ for all $z \in \Omega$ so that, in particular, $ib = F_{\mu_{1,k}}(F_{\mu_{1,k}}^{-1}(ib))$.

Now, let $z_k = F_{\mu_{1,k}}^{-1}(ib)$, and denote by t_k the real part of this number (the real part can vary as wildly as you would like but we will show that this is not a problem). We decompose $\mu_{1,k}$ so that $\mu_{1,k} = \nu_{1,k} + \rho_{1,k}$ where $\nu_{1,k}(\mathbb{R}) = \lambda_k < \gamma$ and

$\rho_{1,k}([t_k - k, t_k + k]) = 0$. A decomposition with these properties exists because of the fact that $\sup_{x\in\mathbb{R}}(\mu_{1,k}(x-k, x+k)) < \gamma$. We will use the last of the above properties to show that $|F_{\mu_{1,k}}(z_k) - F_{\nu_{1,k}}(z_k)| \to 0$. We will then use the fact that $F_{\nu_{1,k}}$ increases the imaginary part in proportion to λ_k^{-1} to derive a contradiction.

Observe that

$$F_{\mu_{1,k}}(z_k) = \frac{1}{G_{\nu_{1,k}}(z_k) + G_{\rho_{1,k}}(z_k)}$$

and that

$$|G_{\rho_{1,k}}(z_k)| = \left|\int_{\mathbb{R}\setminus(t_k-k,t_k+k)} \frac{1}{z_k - t}\, d\rho_{1,k}(t)\right| \to 0$$

as $k \to \infty$. This second fact is clear since $\rho_{1,k}$ is a subprobability measure, and, since $\Re(z_k) = t_k$, the above integrand converges to 0 uniformly on the domain of integration as $k \uparrow \infty$. Now, if $\liminf_k |G_{\nu_{1,k}}(z_k)| = 0$, then $\limsup_k |F_{\mu_{1,k}}(z_k)| = \infty$ which would contradict the fact that $F_{\mu_{1,k}}(z_k) \equiv ib$. Thus, we may assume that $|G_{\nu_{1,k}}(z_k)| \geq c > 0$. This implies that $\lambda_k > 0$.

Consider the quantity

$$|F_{\mu_{1,k}}(z_k) - F_{\nu_{1,k}}(z_k)| = \bigl|\bigl(\bigl(G_{\nu_{1,k}}(z_k) - G_{\mu_{1,k}}(z_k)\bigr)\bigr)(G_{\mu_{1,k}}(z_k)G_{\nu_{1,k}}(z_k))^{-1}\bigr|.$$

Observe that the numerator of the right-hand side goes to zero since $G_{\mu_{1,k}} - G_{\nu_{1,k}} = G_{\rho_{1,k}}$, and the denominator is bounded away from zero since $|G_{\nu_{1,k}}(z_k)| \geq c > 0$ and $|G_{\mu_{1,k}}(z_k)| \equiv b^{-1} > 0$. Thus, $|F_{\mu_{1,k}}(z_k) - F_{\nu_{1,k}}(z_k)| \to 0$ as $k \uparrow \infty$.

Recalling the remarks preceding this theorem, we consider the probability measure $\hat{\nu}_{1,k} = \lambda_k^{-1}\nu_{1,k}$ so that $F_{\nu_{1,k}}(z_k) = \lambda_k^{-1}F_{\hat{\nu}_{1,k}}(z_k)$. We then have

$$\begin{aligned} b = \Im F_{\mu_{1,k}}(z_k) &= \lim_{k\uparrow\infty} \Im F_{\mu_{1,k}}(z_k) = \lim_{k\uparrow\infty} \lambda_k^{-1}\Im F_{\hat{\nu}_{1,k}}(z_k) \\ &\geq \lim_{k\uparrow\infty} \lambda_k^{-1}\Im(z_k) \geq \gamma^{-1}\bigl(b(1-\varepsilon)\bigr) > b. \end{aligned}$$

This contradiction completes our proof. $\square$

We end with a few remarks and corollaries. We single out the following fact from last theorem for easy reference.

COROLLARY 3.6. *Let $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ be a family of decompositions. Assume that $\{\mu_{1,k}\}_{k\in\mathbb{N}}$ is tight. Then $\{\mu_{2,k}\}_{k\in\mathbb{N}}$ is tight.*

As we stated before the proof of Theorem 3.5, a family of measures can fail to be tight either by being translated to $\pm\infty$ or by becoming more spread out. For $t \in (0,1)$, we shall say that a measure μ is *t-centered* if $\mathfrak{F}_\mu(s) < t$ for $s < 0$ and $\mathfrak{F}_\mu(s) \geq t$ for $s \geq 0$. Right continuity of the distribution function implies that a measure μ has a unique t-centered translation. Observe that when $t = 1/2$,

t-centered is simply the more familiar median 0. Now, if we assume that we have a family of decompositions $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ where $\{\mu_{1,k}\}_{k\in\mathbb{N}}$ are assumed to be t-centered, then the supports of these measures are not being sent to ∞. By Theorem 3.5, we have the following corollary.

COROLLARY 3.7. *Let $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ where $\{\mu_{1,k}\}$ are t-centered where t is allowed to range over a compact subset of $(0, 1)$. Then $\{\mu_{i,k}\}_{k\in\mathbb{N}}$ forms a tight family.*

The following variation will prove useful in what follows.

COROLLARY 3.8. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a tight sequence of measures. Assume that to each member of this family we associate a family of decompositions $\mu_n = \nu_{n,k} \boxplus \rho_{n,k}$ for $k \in \mathbb{N}$. Then we may translate our measures so as to form tight families $\{\hat{\nu}_{n,k}\}_{n,k\in\mathbb{N}}$ and $\{\hat{\rho}_{n,k}\}_{n,k\in\mathbb{N}}$ with the property that $\mu_n = \hat{\nu}_{n,k} \boxplus \hat{\rho}_{n,k}$ for all $n, k \in \mathbb{N}$.*

PROOF. We assume that each $\nu_{n,k}$ has median 0.

Assume that $\{\nu_{n(i),k(i)}\}_{i\in\mathbb{N}}$ has no subconvergent sequence. Let μ be a cluster point of $\{\mu_{n(i)}\}_{i\in\mathbb{N}}$. By Lemmas 2.2 and 3.1, we have that there exists a truncated cone $\Gamma_{\alpha,\beta}$ so that for i large enough, $F_{\mu}^{-1}, F_{\mu_{n(i)}}^{-1}, F_{\nu_{n(i),k(i)}}^{-1}$ and $F_{\rho_{n(i),k(i)}}^{-1}$ are all defined and satisfy $F_{\nu_{n(i),k(i)}}^{-1}(z) + F_{\rho_{n(i),k(i)}}^{-1}(z) - z = F_{\mu_{n(i)}}^{-1}(z) \to F_{\mu}^{-1}(z)$ uniformly over compact subsets of $\Gamma_{\alpha,\beta}$.

Now, since we have centered our measures $\nu_{n(i),k(i)}$ by assuming median 0, the lack of a convergent subsequence amounts to assuming that

$$\liminf_{i}\Big(\sup_{t\in\mathbb{R}} \nu_{n(i),k(i)}([t-a, t+a])\Big) \to \gamma < 1$$

for all $a \in \mathbb{R}^+$. At this point, one need only observe that every step of the proof of Theorem 3.5 holds under the weaker assumption that $F_{\nu_{n(i),k(i)}}^{-1}(z) + F_{\rho_{n(i),k(i)}}^{-1}(z) - z \to F_{\mu}^{-1}(z)$ as opposed to assuming outright equality. This completes our proof. $\square$

4. A Khintichine decomposition for additive free convolution.

LEMMA 4.1. *Let $\{\mu_i\}_{i\in I}$ be a tight family of probability measures. Then, for every $C > 0$, there exists a Stolz angle $\Gamma_{\alpha,\beta}$ such that $|\varphi'_{\mu_i}(z)| \le C|z|$ for all $z \in \Gamma_{\alpha,\beta}$ and $n \in \mathbb{N}$.*

PROOF. It was shown in the proof of Theorem 5.2 in [10] that, given a tight family of measures $\{\mu_i\}_{i\in I}$, there exists an $\alpha > 0$ such that $F_{\mu_i} = z(1 + o(1))$ uniformly as $|z| \uparrow \infty$ for $z \in \Gamma_{\alpha,0}$. Thus, for fixed $C > 0$, we may find a β large enough so that $|\varphi_{\mu_i}(z)| = |F_{\mu_i}^{-1}(z) - z| \le C|z|$ for $z \in \Gamma_{\alpha,\beta}$, $n \in \mathbb{N}$. By Cauchy's formula,

$$|\varphi'_{\mu_i}(z)| = (2\pi)^{-1}\left|\int_{|\zeta|=1} \frac{\varphi_{\mu_i}(z)}{(\zeta - z)^2}\, d\zeta\right| \le C|z|. \qquad \square$$

We will now define the functional that will be the main tool in the proof of our main theorems. Let μ be a probability measure. Let M_0 be the set of all median 0 probability measures ν satisfying $\mu = \nu \boxplus \rho$ for some probability measure ρ. It is a consequence of Corollary 3.7 that this is a tight family of measures. Let $\Gamma_{\alpha,\beta}$ be a Stolz angle on which F_μ^{-1} is defined and for which Lemma 4.1 is satisfied with regard to M_0. Consider the set $\Gamma' = \{z \in \mathbb{C}^+ : \alpha + 1 > \Im(z) > \alpha, \Im(z) > \beta\Re(z)\} \subset \Gamma_{\alpha,\beta}$, and let $M_{\Gamma'}$ be the set of probability measures ν such that φ_ν has analytic extension to Γ' such that $\Im\varphi_\nu(z) \le 0$ for all $z \in \Gamma'$. For $\nu \in M_{\Gamma'}$, let $\Lambda(\nu) := -\int_{\Gamma'} \Im\varphi_{\nu(z)}\, dA(z)$ where A denotes the area measure.

Observe that, by Lemma 4.2, for any decomposition $\mu = \rho \boxplus \nu$ we have that $\rho, \nu \in M_{\Gamma'}$. Furthermore, we claim the following properties for our functional Λ:

(1) Λ is weakly continuous.
(2) $\Lambda(\nu \boxplus \rho) = \Lambda(\nu) + \Lambda(\rho)$ for all $\nu, \rho \in M_{\Gamma'}$.
(3) $0 \le \Lambda(\nu) < \infty$ for all $\nu \in M_{\Gamma'}$. $\Lambda(\nu) = 0$ if and only if ν is a Dirac mass.
(4) $\Lambda(\nu \boxplus \delta_t) = \Lambda(\nu)$ for all $t \in \mathbb{R}$ and $\nu \in M_{\Gamma'}$.

The only fact that requires argument is that $\Lambda(\nu) = 0$ if and only if ν is a Dirac mass. One direction is clear since the Voiculescu transform of a Dirac mass is simply a real constant. Furthermore, since $-\Im(\varphi_\nu(z)) \ge 0$ for all $z \in \Gamma'$, $\Lambda(\nu) = 0$ implies that $-\Im(\varphi_\nu(z)) \equiv 0$ for $z \in \Gamma'$. Analytic continuation implies that φ_ν is a real constant which implies that ν is a Dirac mass.

THEOREM 4.2. *Let μ be a probability measure with the property that for every nontrivial decomposition $\mu = \mu_1 \boxplus \mu_2$, neither μ_1 nor μ_2 is indecomposable. Then μ is infinitely divisible.*

PROOF. We first note that for every $\varepsilon > 0$, there exists a decomposition $\mu = \mu_1 \boxplus \mu_2$ such that $0 < \Lambda(\mu_1) < \varepsilon$. Assume otherwise and let $\alpha > 0$ be the infimum of Λ over all nontrivial decompositions of μ. By Theorem 3.5, there exists a sequence of decompositions $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ so that the families $\{\mu_{i,k}\}_{k=1}^\infty$ are tight and so that $\Lambda(\mu_{1,k}) \to \alpha$. Taking weak cluster points μ_1 and μ_2, by weak continuity of both Λ and $\boxplus$ we have that $\mu = \mu_1 \boxplus \mu_2$ and $\Lambda(\mu_1) = \alpha$. By assumption, μ_1 has a nontrivial decomposition $\mu_1 = \nu_0 \boxplus \nu_1$. Since neither component is a Dirac mass, we have that $\alpha > \Lambda(\nu_i) > 0$ so that the decomposition $\mu = \nu_0 \boxplus (\nu_1 \boxplus \mu_2)$ violates minimality of α.

We now claim that for every $t \in (0, \Lambda(\mu))$ there exists a decomposition $\mu = \mu_1 \boxplus \mu_2$ such that $\Lambda(\mu_1) = t$. To see this, let α be the supremum of all values of

$\Lambda(\mu_1)$ that are $\le t$. The previous paragraph implies that $\alpha > 0$. We again take a sequence $\mu = \mu_{1,k} \boxplus \mu_{2,k}$ so that $\Lambda(\mu_{1,k}) \uparrow \alpha$ so that the cluster points μ_i satisfy $\mu = \mu_1 \boxplus \mu_2$ and $\Lambda(\mu_1) = \alpha$. If $\alpha < t$, by the above argument, we can break a chunk of size less than $t - \alpha$ from μ_2 so as to attain a contradiction. Thus, Λ takes values on all of $(0, \Lambda(\mu))$ as it ranges over divisors of μ.

By induction, for every $n \in \mathbb{N}$ we can find a decomposition $\mu = \mu_{n,1} \boxplus \cdots \boxplus \mu_{n,n} \boxplus \delta_{x_n}$ such that $\Lambda(\mu_{n,i}) = \Lambda(\mu)/n$ and $\mu_{n,i}$ has median 0 for all $i = 1, \dots, n$. The real number x_n is the shift constant that necessarily arises when centering these measures. We now claim that the array $\{\mu_{n,j}\}_{n\in\mathbb{N}, j=1,\dots,n}$ converges to δ_0 uniformly as $n \uparrow \infty$.

Observe that Corollary 3.7 implies that our array is tight. Let ν be any cluster point, and let $\{\mu_{k_n,j_n}\}_{n\in\mathbb{N}}$ be a subsequence converging to ν. By Lemma 2.3, $\varphi_{\{\mu_{k_n,jn}\}}(z) \to \varphi_\nu(z)$ uniformly on compact subsets of a Stolz angle $\Gamma^* \subseteq \Gamma$. Now, observe that Γ' and Γ^* may be disjoint. However, there exist $a, b \in \mathbb{R}$ such that $ia \in \Gamma'$ and $ib \in \Gamma^*$.

Observe that $\varphi_{\mu_{k_n,jn}}$ is a normal family on $\Gamma' \cup i[a, b]$, which implies precompactness. By analytic continuation, any cluster point must agree with φ_ν on $i[a, b] \cap \Gamma^*$. This implies that φ_ν has analytic continuation to Γ' that satisfies $\varphi_\nu(z) = \lim_{n\uparrow\infty} \varphi_{\mu_{k_n,jn}}(z)$ for $z \in \Gamma'$. Now, observe that the fact that $\Lambda(\mu_{k_n,j_n}) \to 0$ implies that $-\int_{\Gamma'} \varphi_{\mu_{k_n,jn}}(z)\, dA(z) \to 0$. By Lemma 4.1, we have a bound on the derivatives of these functions so that, recalling that the imaginary parts of these functions are negative, $\Im\varphi_{\mu_{k_n,jn}}(z) \to 0$ for $z \in \Gamma'$. This implies that $\Im\varphi_\nu(z) = 0$ for $z \in \Gamma'$. Thus, ν is a dirac mass and our median 0 assumption implies that $\nu = \delta_0$.

Thus, our array is tight and every subsequence converges to δ_0. This implies that our array converges to δ_0 uniformly over n. By Theorem 2.5, μ is infinitely divisible. $\square$

LEMMA 4.3. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of t-centered measures that converge weakly to μ. Assume that for $s \in \mathbb{R}$ such that $\mathfrak{F}_\mu(s) = t$, we have that $\mathfrak{F}_\mu$ is continuous and strictly increasing in a neighborhood s. Then $s = 0$, or, in other words, μ is t-centered.*

PROOF. Choose $\varepsilon > 0$ such that $\mathfrak{F}_\mu$ is continuous on $(s - 2\varepsilon, s + 2\varepsilon)$. Let $0 < \varepsilon' < \varepsilon$ and observe that utilizing the Lévy metric and our assumption of weak convergence, we have the following inequality for n large enough, independent of ε:

$$\mathfrak{F}_\mu(s - \varepsilon - \varepsilon') - \varepsilon' \le \mathfrak{F}_{\mu_n}(s - \varepsilon) \le \mathfrak{F}_\mu(s - \varepsilon + \varepsilon') + \varepsilon'.$$

By continuity of $\mathfrak{F}_\mu$ at these points, it follows that $\mathfrak{F}_{\mu_n}(s - \varepsilon) \to \mathfrak{F}_\mu(s - \varepsilon)$. Similarly $\mathfrak{F}_{\mu_n}(s + \varepsilon) \to \mathfrak{F}_\mu(s + \varepsilon)$. Thus, for n large enough, we have that $\mathfrak{F}_{\mu_n}(s - \varepsilon) < t$ and $\mathfrak{F}_{\mu_n}(s - \varepsilon) > t$. This implies that $0 \in (s - \varepsilon, s + \varepsilon)$. As ε was arbitrary, this implies that $s = 0$. $\square$

It is clear from the statement of the previous lemma that it will be used in conjunction with Corollary 2.8. Indeed, it is precisely the content of this corollary that measures with nontrivial decompositions satisfy the hypotheses in Lemma 4.3, which will play a small but key role in the proof of the following theorem.

THEOREM 4.4. *Let μ be a probability measure. Then there exist measures μ_i with $i = 0, 1, 2, \dots$ such that μ_0 is $\boxplus$-infinitely divisible, μ_i is indecomposable for $i = 1, 2, \dots$ and $\mu = \mu_0 \boxplus \mu_1 \boxplus \mu_2 \boxplus \cdots$. This decomposition is not unique.*

PROOF. If μ is infinitely divisible, then we are done. If not, by Theorem 4.2, μ has nontrivial divisors. Otherwise, let $\alpha_0 = \sup\{\Lambda(\rho)\}$ where the supremum is taken over all indecomposable probability measures ρ satisfying $\mu = \nu \boxplus \rho$ for some probability measure ν. Let μ_1 be chosen so that $\mu = \mu_{0,1} \boxplus \mu_1$, $\Lambda(\mu_1) > \alpha_0/2$ and μ_1 is indecomposable. By translating our measures, μ_1 is assumed to be t-centered for a t to be chosen later [for the real number s such that $\mu_1 \boxplus \delta_s$ is t-centered, we need only consider the decomposition $\mu = (\mu_{0,1} \boxplus \delta_{-s}) \boxplus (\mu_1 \boxplus \delta_s)$ and all of the relevant properties will be satisfied].

At the nth stage of this process, we let $\alpha_{n-1} = \sup\{\Lambda(\rho)\}$ where the supremum is taken over all indecomposable probability measures ρ satisfying $\mu_{0,n-1} = \nu \boxplus \rho$ for some measure ν (unless $\mu_{0,n-1}$ is infinitely divisible, at which point we are done). We then let μ_n be chosen such that $\mu_{0,n-1} = \mu_{0,n} \boxplus \mu_n$, $\Lambda(\mu_n) > \alpha_n/2$ and μ_n is indecomposable. By translating $\mu_{0,n}$ and μ_n, we may further assume that $\mu_1 \boxplus \cdots \boxplus \mu_n$ is t-centered. If at any point $\alpha_n = 0$, then by Theorem 4.2, we are done. We therefore assume that $\alpha_n > 0$ for all $n \in \mathbb{N}$.

In what follows, we utilize the following notation:

$$\nu_n = \mu_1 \boxplus \cdots \boxplus \mu_n,$$

$$\nu_{n,m} = \mu_{m+1} \boxplus \cdots \boxplus \mu_n,$$

$$\nu_{\infty,m} = \lim_{n\uparrow\infty} \mu_{m+1} \boxplus \cdots \boxplus \mu_n,$$

where we will show at a latter point that the latter actually converges.

Note that Corollary 3.8 implies that $\{\nu_{n,m}\}_{n,m\in\mathbb{N}}$ is a tight family. It follows that $\{\nu_n\}_{n\in\mathbb{N}}$ is also tight. We now claim that this sequence of measures is actually convergent for an appropriate choice of t in the sense of t-centeredness.

Proceeding with our claim, observe that $\Lambda(\mu) = \Lambda(\mu_{0,n}) + \Lambda(\nu_n) = \Lambda(\mu_{0,n}) + \Lambda(\nu_m) + \Lambda(\nu_{n,m})$ for all $m < n \in \mathbb{N}$. Observe that $\Lambda(\mu_{0,n})$ is bounded and decreasing so necessarily converges. This implies that $\Lambda(\nu_{n,m})$ represents the tail of a convergent series and must therefore go to 0 uniformly as $m \uparrow \infty$ (note that this implies that $\alpha_n \to 0$).

Let $\hat{\nu}_{n,m}$ be the translation of $\nu_{n,m}$ with median 0 and observe that $\Lambda(\nu_{n,m}) = -\int_{\Gamma'} \Im\varphi_{\nu_{n,m}}(z)\,dA(z) = -\int_{\Gamma'} \Im\varphi_{\hat{\nu}_{n,m}}(z)\,dA(z) = \Lambda(\hat{\nu}_{n,m})$. By Lemma 4.1, $\varphi'_{\hat{\nu}_{n,m}}$ is bounded on Γ'. Since $\Lambda(\hat{\nu}_{n,m}) \to 0$ as $m \uparrow \infty$, we have that $-\Im\varphi_{\hat{\nu}_{n,m}}(z) \to 0$

uniformly over Γ' as $m \uparrow \infty$. By Lemma 2.3, any cluster point $\hat{\nu}$ of $\{\hat{\nu}_{n,m}\}_{n,m\in\mathbb{N}}$ must satisfy $\varphi_\nu(z) = 0$ for $z \in \Gamma'$. This implies that $\hat{\nu}$ is a Dirac mass. Thus, any cluster point ν of $\{\nu_{n,m}\}$ as $m \uparrow \infty$ must also be a Dirac mass.

Thus, the set of cluster points of $\{\nu_n\}_{n\in\mathbb{N}}$ is of the form $\{\rho \boxplus \delta_r\}_{r\in K}$ where K is a compact subset of $\mathbb{R}$. Since we are assuming that $\alpha_n > 0$ for all $n \in \mathbb{N}$, we have that $\rho = \mu_1 \boxplus \nu$ where ν is some nontrivial cluster point of $\{\nu_{n,1}\}_{n\in\mathbb{N}}$. In particular, ρ has a nontrivial decomposition so that, by Corollary 2.8, there exist points $s \in \mathbb{R}$ and $t \in (0, 1)$ such that $\mathfrak{F}_\rho(s) = t$ and $\mathfrak{F}_\rho$ is continuous and increasing in a neighborhood of s. We therefore assume that $\{\nu_n\}_{n\in\mathbb{N}}$ are t-centered (we may do this retroactively since this only translates our measures ν_n and does not affect the fact that they cluster to translations of ρ). By Lemma 4.3, all cluster points of $\{\nu_n\}_{n\in\mathbb{N}}$ must be t-centered so that, by uniqueness of this property, our sequence converges to a single measure.

Now, observe that these facts together imply that $\{\nu_{n,m}\}_{n,m}$ must converge to the Dirac mass at 0 as $m \uparrow \infty$. This further implies that $\nu_{\infty,m}$ is the limit of a convergent sequence. We next claim that if μ_0 is any cluster point of $\{\mu_{0,n}\}_{n\in\mathbb{N}}$, then $\mu = \lim_{n\uparrow\infty} \mu_0 \boxplus \nu_n$.

To see this, let i_n be a subsequence along which $\mu_{0,n}$ converges to μ_0. Observe that $\lim_{n\uparrow\infty} \mu_0 \boxplus \nu_n = \lim_{n\uparrow\infty} \mu_{0,i_n} \boxplus \nu_n = \lim_{n\uparrow\infty} \mu_{0,i_n} \boxplus \nu_{i_n} \boxplus \nu_{n,i_n} = \lim_{n\uparrow\infty} \mu \boxplus \nu_{n,i_n}$. As $n \to \infty$, the right-hand side converges to $\mu \boxplus \delta_0 = \mu$, proving our claim.

We have shown that $\mu = \lim_{n\uparrow\infty} \mu_0 \boxplus \nu_n$ so that our theorem will be proven once we show that μ_0 is infinitely divisible. Toward this end, we claim that $\mu_{k,0} = \mu_0 \boxplus \nu_{\infty,k+1}$ for all $k \in \mathbb{N}$. To see this, observe that the right-hand side is equal to $\mu_0 \boxplus \nu_{\infty,k+1} = \lim_{n\uparrow\infty} \mu_{0,i_n} \boxplus \nu_{\infty,k+1} = \lim_{n\uparrow\infty} \mu_{0,i_n} \boxplus \nu_{i_n,k+1} \boxplus \nu_{\infty,i_n} = \lim_{n\uparrow\infty} \mu_{0,k} \boxplus \nu_{\infty,i_n} \to \mu_{0,k}$ as $n \to \infty$. This proves our claim.

Now, assume that μ_0 has a decomposition $\mu_0 = \rho \boxplus \nu$ where ν is indecomposable. Assume that $\Lambda(\nu) > 0$. Pick n large enough so that $\alpha_n < \Lambda(\nu)$ and recall that $\mu_{n,0} = \mu_0 \boxplus \nu_{\infty,n+1}$. The left-hand side has no indecomposable divisor whose Λ value is larger than α_n. This contradiction implies that μ_0 has no indecomposable divisors so that, by Theorem 4.2, our theorem holds. $\square$

The failure of uniqueness will be addressed in Section 10.

5. Background and terminology for the multiplicative convolution of measures supported on the positive real line. Let x, y be positive random variables in (A, τ) with respective distributions μ and ν. We denote by $\mu \boxtimes \nu$ the distribution of the random variable xy. Since τ is a trace, the distribution of xy is the same as that of $y^{1/2}xy^{1/2}$, so that $\boxtimes$ preserves the property that the distribution is a measure supported on the positive real numbers.

Let $M_{\mathbb{R}^+}$ denote the set of probability measures supported on $\mathbb{R}^+$. Observe that, with exception of δ_0, all such measures have nonzero first moment and we assume throughout that we are not dealing with this measure. Consider the following function:

$$\psi_\mu(z) = \int_0^\infty \frac{zt}{1-zt}\, d\mu(t)$$

for $z \in \mathbb{C} \setminus \mathbb{R}^+$. As seen in [19] and [10], $\psi_\mu|_{i\mathbb{C}^+}$ is univalent and maps into an open neighborhood about the interval $(\mu(\{0\}) - 1, 0)$. It is also true that $\psi_\mu(i\mathbb{C}^+) \cap \mathbb{R} = (\mu(\{0\}) - 1, 0)$.

Let $\Omega_\mu = \psi_\mu(i\mathbb{C}^+)$ and let $\chi_\mu : \Omega_\mu \to i\mathbb{C}^+$ denote the inverse function. We refer to the S-transform as the following function:

$$S_\mu(z) = \frac{(1+z)\chi_\mu(z)}{z}.$$

These functions have the following properties which will be used, often without reference, in what follows:

(1) $S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z)$ for all z in their common domain.
(2) $S_\mu(z) > 0$ and $S'_\mu(z) \leq 0$ for $z \in (\mu(\{0\}) - 1, 0)$.
(3) $(\mu_1 \boxtimes \mu_2)(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$.
(4) $\chi'_\mu(z) > 0$ for all $z \in (\mu(\{0\}) - 1, 0)$.
(5) $\chi_{\mu \boxtimes \delta_c}(z) = \chi_\mu(z)/c$ and $S_{\mu \boxtimes \delta_c}(z) = S_\mu(z)/c$.

Observe that (3) above implies a multiplicative version of Lemma 3.1. That is, for any nontrivial decomposition $\mu = \mu_1 \boxtimes \mu_2$, (3) implies that real part of the domain of χ_μ is contained in the real part of the domain of χ_{μ_i} for each $i = 1, 2$. We will use this fact without reference throughout.

The following results on convergence and tightness were first proven in full generality in [10].

LEMMA 5.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be sequence of probability measures on $\mathbb{R}^+$. Assume the these sequences weakly converge to μ and ν, respectively. Then $\{\mu_n \boxtimes \nu_n\}_{n\in\mathbb{N}}$ converges to $\mu \boxtimes \nu$ in the weak* topology.*

LEMMA 5.2. *Let M be a set of probability measures on $\mathbb{R}^+$. The following conditions are equivalent*:

(1) *M is tight, and the weak* closure of M does not contain δ_0.*
(2) *There exists an $\alpha > 0$ such that*:
   (a) *$-\alpha$ belongs to the domain of χ_μ for all $\mu \in M$.*
   (b) $\sup\{|\chi_\mu(-\alpha)| : \mu \in M\} < \infty$.
   (c) $\inf\{|\chi_\mu(-\beta)| : \mu \in M|\} > 0$ *for all* $\beta \in (0, \alpha)$.
(3) *There exists an $\alpha > 0$ such that*:
   (a) *$-\alpha$ belongs to the domain of S_μ for all $\mu \in M$.*
   (b) $\sup\{|S_\mu(-\alpha)| : \mu \in M\} < \infty$.
   (c) $\inf\{|S_\mu(-\beta) : \mu \in M|\} > 0$ *for all* $\beta \in (0, \alpha)$.

LEMMA 5.3. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $\mathbb{R}^+$ such that δ_0 is not in the weak* closure of our sequence. The following are equivalent*:

(1) *The sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges to a measure μ in the weak* topology.*
(2) *There exist positive numbers $\beta < \alpha$ such that the sequence $\{\chi_{\mu_n}\}$ converges uniformly on the interval $(-\alpha, -\beta)$ to a function χ.*
(3) *There exist positive numbers $\beta < \alpha$ such that the sequence $\{S_{\mu_n}\}$ converges uniformly on the interval $(-\alpha, -\beta)$ to a function S.*

Moreover, if (1) *and* (2) *are satisfied, we have $\chi = \chi_\mu$ in $(-\alpha, -\beta)$.*

In a manner analogous to the additive case, we have the following subordination result for multiplicative convolution. This was first proven in full generality in [13] and is proven by different means in [6].

THEOREM 5.4. *Let μ be a probability measure on $\mathbb{R}^+$ with decomposition $\mu = \mu_1 \boxtimes \mu_2$. There exist analytic subordination functions $\omega_i : \mathbb{C}\setminus\mathbb{R}^+ \to \mathbb{C}\setminus\mathbb{R}^+$ for $i = 1, 2$, such that*:

(1) $\omega_i(0-) = 0$.
(2) *for every $\lambda \in \mathbb{C}^+$ we have that $\omega_i(\bar{\lambda}) = \overline{\omega_i(\lambda)}$, $\omega_i(\lambda) \in \mathbb{C}^+$ and*

$$\arg(\omega_j(\lambda)) \geq \arg(\lambda).$$

(3) $\psi_\mu(\lambda) = \psi_{\mu_i}(\omega_i(\lambda))$ *for all $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$.*
(4) $\omega_1(\lambda)\omega_2(\lambda) = \lambda\psi_\mu(\lambda)$.

Consider next the following result which may be found in [5].

THEOREM 5.5. *Let $\eta : \Omega \to \mathbb{C}\setminus\{0\}$ be an analytic function such that $\eta(\bar{z}) = \overline{\eta(z)}$ for all $z \in \Omega$. The following are equivalent*:

(1) *There exists a probability measure $\mu \neq \delta_0$ on $[0, \infty)$ such that $\eta = \psi_\mu/(1 + \psi_\mu)$.*
(2) $\eta(0-) = 0$ *and* $\arg(\eta(z)) \in [\arg(z), \pi)$ *for all $z \in \mathbb{C}^+$.*

These two theorems may be combined to give us the following corollary. We have no direct reference for this fact but can be sure that it is well known and are recording it only for the reader's convenience.

COROLLARY 5.6. *Let ω_i be a subordination function arising from the decomposition $\mu = \mu_1 \boxtimes \mu_2$ as above. Then*

$$\omega_i(z) = \frac{\psi_\nu(z)}{1 + \psi_\nu(z)}$$

for a probability measure ν with the property that $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\mu)$.

PROOF. The existence of such a representation is a direct consequence of the previous theorems. It remains to prove to the assertion about the support of ν.

In the proof of Theorem 6.1 in the next section, we will show that ω_i will have analytic continuation and is real on $\mathbb{R} \setminus (\mathrm{supp}(\mu)^{-1})$ where $\mathrm{supp}(\mu)^{-1} = \{t^{-1} : t \in \mathrm{supp}(\mu)\}$. This implies that $\Im\psi_\nu(t + i\varepsilon) \to 0$ as $\varepsilon \to 0$ for $t \notin (\mathrm{supp}(\mu)^{-1})$. Since $G_\nu(1/z) = z(\psi_\nu(z) + 1)$, this implies that $t^{-1} \notin \mathrm{supp}(\nu)$. Our claim follows. $\square$

This final result was first proven in [7] and will be used in proving a multiplicative version of Theorem 4.2.

THEOREM 5.7. *Consider* $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *and an array* $\{\mu_{n,j}\}_{n\in\mathbb{N},j=1,2,\ldots,k_n}$ *of probability measures on* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} \min_{1\le j\le k_n} \mu_{n,j}\big((1-\varepsilon, 1+\varepsilon)\big) = 1$$

for every $\varepsilon > 0$. *If the measures* $\delta_{c_n} \boxtimes \mu_{n,1} \boxtimes \cdots \boxtimes \mu_{n,k_n}$ *have a weak limit* μ *which is a probability measure*, *then* μ *is infinitely divisible*.

Observe that the assumptions in this theorem may be weakened so that we need only assume that $\mu_{n,j}(\{0\}) = 0$ for all $n \in \mathbb{N}$ and $j = 1, 2, \ldots, k_n$. Indeed, every element in such an array can be approximated arbitrarily well by a measure supported on $(0,\infty)$. It is under this weakened assumption that we will later invoke this theorem.

6. Compactness results for measures supported on the positive real half-line.

We define $\mathrm{logdiam}(\mu) := \sup_{x,y\in\mathrm{supp}(\mu)}(|\log(x) - \log(y)|)$ to be the *logarithmic diameter* of the measure μ.

THEOREM 6.1. *Let* μ *be a compactly supported probability measure on* $\mathbb{R}^+$. *Then for any decomposition* $\mu = \mu_1 \boxtimes \mu_2$ *we have that* $\mathrm{logdiam}(\mu_i) \le \mathrm{logdiam}(\mu)$. *If* $\mu(\{0\}) = 0$, *then equality occurs if and only if one of the* μ_i *is a Dirac mass*.

PROOF. If $\{0\}$ is contained in the support of μ, the theorem is trivial. Thus, we assume that $[\alpha, \beta] = \mathrm{conv}(\mathrm{supp}(\mu))$ and $[\alpha_1, \beta_1] = \mathrm{conv}(\mathrm{supp}(\mu_1))$ with α, $\alpha_1 > 0$. Observe that ψ_μ has analytic extension to $\mathbb{R} \setminus [\beta^{-1}, \alpha^{-1}]$. We claim that the subordination function ω_1 does also.

To see this, note that $\psi_{\mu_1}(\omega_1(te^{i\theta})) = \psi_\mu(te^{i\theta}) = (G_\mu(1/te^{i\theta})/te^{i\theta}) + 1$ for $t \in \mathbb{R} \setminus [\beta^{-1}, \alpha^{-1}]$. Since $1/t$ is not contained in the support of μ, the Stieltjes inversion formula tells us that the imaginary part of the right-hand side goes to zero as θ goes to 0. Since ψ_{μ_1} increases argument, the imaginary part of $\omega_1(te^{i\theta})$ must go to zero. The Schwarz reflection principle implies that ω_1 extends analytically across t.

As we saw in Corollary 5.6, we have that $\omega_1(z) = \psi_{\nu(z)}/(1+\psi_{\nu(z)})$ for ν supported on $[\alpha,\beta]$. Thus, $\omega_1'(z) = (\int t(1-zt)^{-2}\,d\nu(t))/(\int(1-zt)^{-1}\,d\nu(t))^2$ so that $\lim_{\lambda\uparrow\infty}\omega_1'(\lambda) = (\int t^{-1}\,d\nu(t))^{-1}$. We call this limit $\omega_1'(\infty)$.

We now claim that $\lambda\omega_1'(\infty) - \omega_1(\lambda) \to C < 0$ as $\lambda\uparrow\infty$. Indeed,

$$
\begin{aligned}
&\omega_1(\lambda) - \lambda\omega_1'(\infty)\\
&\quad = \frac{\int_\alpha^\beta t\lambda/(1-t\lambda)\,d\nu(t)}{\int_\alpha^\beta 1/(1-t\lambda)\,d\nu(t)} - \frac{\lambda}{\int_\alpha^\beta t^{-1}\,d\nu(t)}\\
&\quad = \frac{\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta t\lambda/(1-t\lambda)\,d\nu(t) - \lambda\int_\alpha^\beta 1/(1-t\lambda)\,d\nu(t)}{\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta 1/(1-t\lambda)\,d\nu(t)}\\
&\quad = \lambda\frac{\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta t/(\lambda^{-1}-t)\,d\nu(t) - \int_\alpha^\beta 1/(\lambda^{-1}-t)\,d\nu(t)}{\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta 1/(\lambda^{-1}-t)\,d\nu(t)}\\
&\quad = \lambda\biggl(\int_\alpha^\beta t^{-1}\,d\nu(t)\biggl(1+\int_\alpha^\beta \frac{t}{\lambda^{-1}-t}\,d\nu(t)\biggr)\\
&\qquad\quad - \biggl(\int_\alpha^\beta t^{-1}\,d\nu(t) + \int_\alpha^\beta \frac{1}{\lambda^{-1}-t}\,d\nu(t)\biggr)\biggr)\\
&\qquad\quad \times\biggl(\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta \frac{1}{\lambda^{-1}-t}\,d\nu(t)\biggr)^{-1}\\
&\quad = \frac{\int_\alpha^\beta t^{-1}\,d\nu(t)\int_\alpha^\beta 1/(\lambda^{-1}-t)\,d\nu(t) - \int_\alpha^\beta 1/(t(\lambda^{-1}-t))\,d\nu(t)}{\int_\alpha^\beta t^{-1}\int_\alpha^\beta 1/(\lambda^{-1}-t)\,d\nu(t)}\\
&\quad \to \frac{-(\int_\alpha^\beta t^{-1}\,d\nu(t))^2 + \int_\alpha^\beta t^{-2}\,d\nu(t)}{-(\int_\alpha^\beta t^{-1})^2} = C
\end{aligned}
$$

as $\lambda\uparrow\infty$. Note that $f(t) = t^2$ is a strictly convex function on $[\alpha,\beta]$. Assuming that ν is not a Dirac mass, it follows from Jensen's inequality that C is a strictly negative number (we may assume that ν is not a Dirac mass since this would imply that μ_1 is a Dirac mass and our theorem is trivially true in this case).

Now, by Cauchy–Schwarz, we have that $|\omega_1'(z)| \geq \omega_1'(\infty)$ for all $z \in \mathbb{C}^+ \setminus [\beta^{-1},\alpha^{-1}]$. Indeed, we have that

$$
|\omega_1'(z)| = \left|\frac{\int t/(1-zt)^2\,d\nu(t)}{(\int 1/(1-zt)\,d\nu(t))^2}\right| = \frac{\|\sqrt{t}/(z-t)\|_2^2}{|\langle 1/\sqrt{t}, \sqrt{t}/(z-t)\rangle|^2} \geq \omega_1'(\infty).
$$

Thus,

$$
\omega_1(\alpha^{-1}+\varepsilon) = \omega_1(\lambda) - \int_{\alpha^{-1}+\varepsilon}^{\lambda}\omega_1'(t)\,dt \leq \omega_1(\lambda) - \lambda\omega_1'(\infty) + (\alpha^{-1}+\varepsilon)\omega_1'(\infty),
$$

which converges to $(\alpha^{-1}+\varepsilon)\omega_1'(\infty) + C$ as $\lambda\uparrow\infty$.

To complete our claim, note that

$$\omega_1(\beta^{-1}-\varepsilon)=\omega_1(0)+\int_0^{\beta^{-1}-\varepsilon}\omega'(t)\,dt\le\omega_1'(\infty)(\beta^{-1}-\varepsilon)$$

since $\omega_1(0)=0$. Thus, $\mathbb{R}^+\setminus[\omega_1'(\infty)\beta^{-1},\omega_1'(\infty)\alpha^{-1}+C]\subseteq\omega_1(\mathbb{R}^+\setminus[\beta^{-1},\alpha^{-1}])$. Since ψ_{μ_1} can be continued analytically to the right-hand set, we have that it also has analytic continuation to $\mathbb{R}^+\setminus[\omega_1'(\infty)\beta^{-1},\omega_1'(\infty)\alpha^{-1}+C]$. This implies that the support of μ_1 is contained in $([\omega_1'(\infty)\beta^{-1},\omega_1'(\infty)\alpha^{-1}+C])^{-1}\subseteq\omega_1'(\infty)^{-1}[\alpha,\beta]$ with equality if and only if one of the μ_i is a Dirac mass. The theorem follows. $\square$

THEOREM 6.2. *Let μ be a probability measure with* $\operatorname{supp}(\mu)\subset\mathbb{R}^+$ *different from δ_0. Let $\mu=\mu_{1,k}\boxtimes\mu_{2,k}$ be a family of decompositions. There exists a sequence $\{\lambda_k\}\subset\mathbb{R}^+$ so that the families $\{\mu_{1,k}\circ D_{\lambda_k}\}_{k\in\mathbb{N}}$ and $\{\mu_{2,k}\circ D_{\lambda_k^{-1}}\}_{k\in\mathbb{N}}$ are tight. Furthermore, δ_0 is not in the weak closure of either of these families of measures.*

PROOF. Let $1-\mu(\{0\})=-\alpha<0$. Recall that ψ_μ maps the negative half line injectively onto $(-\alpha,0)$. Also recall that, for each k, ψ_{μ_k} maps the negative half line injectively onto $(1-\mu_{i,k}(\{0\}),0)$ and that $\mu_{i,k}(\{0\})\le\mu(\{0\})$. Thus, for each k, there exists a unique real number λ_k so that $\psi_{\mu_{1,k}\circ D_{\lambda_k}}(-1)=-\alpha/2$. Denote the new measure by $\nu_{1,k}$. Dilate $\mu_{2,k}$ by $D_{\lambda_k^{-1}}$, and denote the new measure by $\nu_{2,k}$. Observe that $\mu=\nu_{1,k}\boxtimes\nu_{2,k}$ for all $k\in\mathbb{N}$.

Now, observe that $-\alpha/2$ is contained in the domain of $\chi_{\nu_{1,k}}$ and that $|\chi_{\nu_{1,k}}(-\alpha/2)|=1$ for all $k\in\mathbb{N}$. By Lemma 5.2, if we can show that $\inf_{k\in\mathbb{N}}|\chi_{\nu_{1,k}}(-\beta)|>0$ for all $\beta\in(0,\alpha/2)$, then $\{\nu_{1,k}\}$ is tight.

Consider the following equation for $t\in(0,\alpha/2)$:

$$\frac{-t+1}{-t}\chi_{\nu_{1,k}}(-t)\chi_{\nu_{2,k}}(-t)=\chi_\mu(-t). \tag{6.1}$$

Assume that for $\beta\in(0,\alpha/2)$, we have that $\inf_{k\in\mathbb{N}}(\chi_{\nu_{1,k}}(-\beta))=0$. Our assumption that $\mu\ne\delta_0$ implies that $\chi_\mu(-\beta)>0$. Manipulating (6.1), this implies that $\{\chi_{\nu_{2,k}}(-\beta)\}$ are unbounded over k and negative. As $\chi'_{\nu_{2,k}}(t)>0$, this implies that $\{\chi_{\nu_{2,k}}(-\alpha/2)\}$ are unbounded over k. However, (6.1) and the assumption that $\chi_{\nu_{1,k}}(-\alpha/2)\equiv-1$ results in contradiction. By Lemma 5.2, $\{\nu_{1,k}\}$ is a tight family.

It is easily seen that $\{\nu_{2,k}\}$ is also a tight family. Indeed, $\chi_{\nu_{1,k}}(-\alpha/2)\equiv-1$ implies that

$$\chi_{\nu_{2,k}}(-\alpha/2)\equiv\frac{\alpha/2}{1-\alpha/2}\chi_\mu(-\alpha/2).$$

Thus, the first two criteria of Lemma 5.2 are satisfied, and the last follows from the fact that for fixed $\beta\in(0,\alpha/2)$,

$$|\chi_{\nu_{2,k}}(-\beta)|=\frac{\beta\chi_\mu(-\beta)}{(1-\beta)\chi_{\nu_{1,k}}(-\beta)}\ge\frac{-\beta\chi_\mu(-\beta)}{(1-\beta)}>0.$$

$\square$

7. A Khintchine decomposition for multiplicative free convolution with measures supported on the positive half line.

THEOREM 7.1. *Let μ be a probability measure with the property that, for any nontrivial decomposition $\mu = \mu_1 \boxtimes \mu_2$, neither μ_1 nor μ_2 is indecomposable. Then μ is $\boxtimes$-infinitely divisible.*

PROOF. Let $\alpha = 1 - \mu(\{0\})$. We will show later that $\alpha = 1$. Recall that S_μ, S_{μ_1} and S_{μ_2} are all defined on an open neighborhood of $(-\alpha, 0)$ for any decomposition $\mu = \mu_1 \boxtimes \mu_2$. We assume without loss of generality that $S_\mu(-\beta) = 1$ for some $\beta \in (0, \alpha)$ [indeed, pick any β in this interval, and then consider $\mu \boxtimes \delta_c$ where $c = (-\beta\chi_\mu(-\beta))/(1-\beta)$].

We denote by M_β the set of all probability measures $\nu \in M_{\mathbb{R}^+}$ such that $S_\nu(-\beta) = 1$ and $\mu = \nu \boxtimes \rho$ for a probability measure $\rho \in M_{\mathbb{R}^+}$. Observe that $S_\mu(-\beta) = S_\nu(-\beta)S_\rho(-\beta)$ implies that $\rho \in M_\beta$. Further note that for any decomposition $\mu = \nu' \boxtimes \rho'$ there exists a real number c such that $\nu' \boxtimes \delta_c$, $\rho \boxtimes \delta_{c^{-1}} \in M_\beta$. Last, it is the content of Theorem 6.2 that M_β is weak* compact.

Fix $\gamma \in (0, \beta)$. We claim that given any $\varepsilon > 0$, there exists an element $\nu \in M_\beta$ such that $1 > S_\nu(-\gamma) > 1 - \varepsilon$. To show this, assume instead that there is a $\delta > 0$ so that $1 - \delta$ is the supremum of $S_\nu(-\gamma)$ ranging over all nontrivial elements in M_β. By compactness, we may pass to a cluster point, and assume that we have a decomposition $\mu = \mu_1 \boxtimes \mu_2$ where $S_{\mu_1}(-\gamma)$ takes on this supremum. Now, by assumption, we have a nontrivial decomposition $\mu_1 = \nu_0 \boxtimes \nu_1$ where $S_{\nu_i}(-\beta) = 1$ for $i = 0, 1$. Since $S'_{\nu_i} \leq 0$, this implies that both $S_{\nu_i}(-\gamma) < 1$ (we would have equality if and only if ν_i were a Dirac mass, which we have assumed away). As their product satisfies $S_{\nu_0}(-\gamma)S_{\nu_1}(-\gamma) = S_{\mu_1}(-\gamma) = 1 - \delta$, we have that $S_{\nu_i}(-\gamma) > 1 - \delta$ for $i = 1, 2$. Thus the decomposition $\nu_0 \boxtimes (\nu_1 \boxtimes \mu_2)$ violates the above supremum.

We next claim that $S_\nu(-\gamma)$ takes on all values of the interval $[S_\mu(-\gamma), 1]$ as we range over elements in M_β. Clearly our compactness result implies that the range of the $S_\nu(-\gamma)$ is closed. We assume, for the sake of contradiction, that there exist real numbers $\delta > 0$ and $\lambda > S_\mu(-\gamma)$ such that $S_\nu(-\gamma)$ does not take on any values in the interval $(\lambda - \delta, \lambda)$ for $\nu \in M_\beta$ and that this interval is maximal in this regard. Passing to cluster points, we assume that $S_{\mu_1}(-\gamma) = \lambda$ for a decomposition $\mu = \mu_1 \boxtimes \mu_2$. Now, pick a nontrivial decomposition $\mu_2 = \nu_0 \boxtimes \nu_1$ so that $S_{\nu_0}(-\gamma)$ is close enough to 1 so that $\lambda S_{\nu_0}(-\gamma) \in (\lambda - \delta, \lambda)$. Transferring this mass, we obtain our contradiction.

By induction, there exists a decomposition $\mu = \mu_{n,1} \boxtimes \cdots \boxtimes \mu_{n,n}$ such that $S_{\mu_{n,i}}(-\beta) = 1$ and $S_{\mu_{n,i}}(-\gamma) = \sqrt[n]{S_\mu(-\gamma)}$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. Observe that this implies that $S_{\mu_{n,i}}(-t) \to 1$ uniformly for $t \in (\gamma, \beta)$ and $n \in \mathbb{N}$ ($S_{\mu_{n,i}}$ is nonincreasing on this interval). By Lemma 5.3 this implies that any subsequence of our array $\{\mu_{n,i}\}_{n\in\mathbb{N}, i=1,2,\ldots,n}$ converges to δ_1. Compactness implies that our array converges to δ_1 uniformly over n. Last, note that this implies that our measures satisfy $\mu_{n,i}(\{0\}) = 0$. Indeed, observe that $\max_{i=1,2,\ldots,n} \mu_{n,i}(\{0\}) \to 0$ since we have uniform weak convergence to δ_1. Since $\mu(\{0\}) = \max_{i=1,2,\ldots,n} \mu_{n,i}(\{0\})$ we must have no mass at 0 for μ or for any element in our array.

Thus, we may now invoke Theorem 5.7 which implies that our measure μ is $\boxtimes$-infinitely divisible. $\square$

THEOREM 7.2. *Let $\mu \in M_{\mathbb{R}_+}$ different from δ_0. Then there exist measures μ_i with $i = 0, 1, 2, \ldots$ such that μ_0 is $\boxtimes$-infinitely divisible, μ_i is $\boxtimes$-indecomposable for $i = 1, 2, \ldots$ and $\mu = \mu_0 \boxtimes \mu_1 \boxtimes \mu_2 \boxtimes \cdots$. This decomposition is not unique.*

PROOF. We again assume without loss of generality that $S_\mu(-\beta) = 1$ for some $\beta \in (0, 1 - \mu(\{0\}))$. In what follows, all decompositions will be taken from elements in M_β.

Pick $\gamma \in (0, \beta)$. Let $\alpha = S_\mu(-\gamma) \le 1$ (with equality if and only if $\mu = \delta_1$, in which case the theorem is trivially true). Now, let $\alpha_0 = \inf\{S_\nu(-\gamma)\}$ where the infimum is taken over all indecomposable $\nu \in M_\beta$. If $\alpha_0 = 1$, then, by Theorem 7.1, our theorem holds. If not, let $\mu = \mu_{0,1} \boxtimes \mu_1$ with $\mu_1 \in M_\beta$ indecomposable satisfying $S_{\mu_1}(-\gamma) > \sqrt{\alpha_0}$.

At the nth stage of this process, we start with a decomposition $\mu = \mu_{0,n-1} \boxtimes \mu_1 \boxtimes \mu_{n-1}$ where all divisors are elements of M_β and μ_i is indecomposable for $i = 1, 2, \ldots, n-1$. We let $\alpha_{n-1} = \inf\{S_\nu(-\gamma)\}$ where the infimum is taken over all indecomposable $\nu \in M_\beta$ such that $\mu_{0,n-1} = \nu \boxtimes \rho$ for some $\rho \in M_\beta$ (observe that $\mu_{0,n-1}, \nu \in M_\beta$ implies that $\rho \in M_\beta$). If at any point $\alpha_n = 1$ then, by Theorem 7.1, we are done. Thus, we assume that $\alpha_n < 1$ for all $n \in \mathbb{N}$. Let $\mu_{0,n-1} = \mu_{0,n} \boxtimes \mu_n$ where $\mu_n \in M_\beta$ is indecomposable and satisfies $S_{\mu_n}(-\gamma) > \sqrt{\alpha_n}$. At this point, we have a decomposition $\mu = \mu_{0,n} \boxtimes \mu_1 \boxtimes \cdots \boxtimes \mu_n$ satisfying $\mu_{0,n}, \mu_i \in M_\beta$, μ_i is indecomposable and $S_{\mu_i}(-\gamma) > \sqrt{\alpha_i}$ for all $i = 1, 2, \ldots, n-1$.

In what follows, we will use the following notation for $n > m$:

$$\nu_n = \mu_1 \boxtimes \cdots \boxtimes \mu_n,$$

$$\nu_{n,m} = \mu_{m+1} \boxtimes \cdots \boxtimes \mu_n,$$

$$\nu_{\infty,m} = \lim_{n \uparrow \infty} \mu_{m+1} \boxtimes \cdots \boxtimes \mu_n.$$

We will show later that this last element actually converges to a measure in M_β.

Now, observe that $\{\nu_{n,m}\}_{m<n\in\mathbb{N}}$ is a tight family since it is a subset of M_β. We claim that $\nu_{n,m} \to \delta_1$ uniformly in the weak* topology as $m \uparrow \infty$. Indeed, observe that $S_{\mu_{0,n}}(-\gamma)$ is increasing and bounded by 1 which implies convergence. Furthermore,

$$S_\mu(-\gamma) = S_{\mu_{0,n}}(-\gamma) * S_{\nu_n}(-\gamma) = S_{\mu_{0,n}}(-\gamma) * S_{\nu_m}(-\gamma) * S_{\nu_{n,m}}(-\gamma)$$

so that $S_{\nu_{n,m}}(-\gamma)$ represents the tail of a convergent product. This implies that $S_{\nu_{n,m}}(-\gamma) \to 1$ uniformly over $n \in \mathbb{N}$ as $m \uparrow \infty$ (observe that this also implies that $\alpha_n \uparrow 1$). By Lemma 5.3, any convergent subsequence must converge to δ_1. By tightness, we must have uniform convergence to δ_1 as $m \uparrow \infty$.

Now, let μ_0 be a cluster point of $\mu_{0,n}$. We claim that $\mu_0 \boxtimes \nu_n \to \mu$ in the weak* topology. Indeed, let i_k be a subsequence on which μ_{0,i_k} converges to μ_0 and let f map $\mathbb{N}$ onto this subsequence by letting $f(n) = i_k$ where $i_k \le n < i_{k+1}$. We then have that $\lim_{n\uparrow\infty} \mu_0 \boxtimes \nu_n = \lim_{n\uparrow\infty} \mu_{0,f(n)} \boxtimes \nu_n = \lim_{n\uparrow\infty} \mu_{0,f(n)} \boxtimes \nu_{f(n)} \boxtimes \nu_{n,f(n)} = \lim_{n\uparrow\infty} \mu \boxtimes \nu_{n,f(n)}$. As we saw in the previous paragraph, the right-hand side converges to μ.

It remains to show that μ_0 is infinitely divisible. As in Theorem 4.4, we will show that $\mu_0 \boxtimes \nu_{\infty,n} = \mu_{n,0}$. Indeed, note that $\mu_0 \boxtimes \nu_{\infty,n} = \lim_{k\uparrow\infty} \mu_{0,i_k} \boxtimes \nu_{\infty,n} = \lim_{k\uparrow\infty} \mu_{0,i_k} \boxtimes \nu_{i_k,n} \boxtimes \nu_{\infty,i_k} = \lim_{k\uparrow\infty} \mu_{n,0} \boxtimes \nu_{\infty,i_k} = \mu_{n,0}$, proving our claim (the second to last equality follows from the fact that, by construction, $\mu_{0,n} \boxtimes \nu_{n,m} = \mu_{0,m}$ for all $m < n \in \mathbb{N}$).

To complete the proof, assume that $\mu_0 = \nu \boxtimes \rho$ where ν is indecomposable and satisfies $S_\nu(-\gamma) < 1$. Pick n such that $\alpha_n < S_\nu(-\gamma)$. As $\mu_{0,n} = \mu_0 \boxtimes \nu_{\infty,n}$ and the left-hand side has no indecomposable divisors satisfying the above inequality, we have a contradiction. Thus, μ_0 has no nontrivial divisors so that, by Theorem 7.1, our theorem holds. $\square$

8. Background and terminology for measures supported on the unit circle. Let $M_{\mathbb{T}}$ be the set of all Borel probability measures supported on the unit circle. Let M_* be the set of all Borel probability measures on $\mathbb{C}$ with nonzero first moment. For a measure $\mu \in M_* \cap M_{\mathbb{T}}$, we have the following definition:

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{zt}{1 - zt}\, d\mu(t).$$

Observe that $\psi_\mu(0) = 0$ and $\psi'_\mu(0) = \int_C t\, d\mu_t$ so that our assumption of nonzero first moment implies that $\psi_\mu^{-1} = \chi_\mu$ is defined and analytic in the neighborhood of 0. We again define $S_\mu(z) = (1+z)\chi_{\mu(z)}/z$. Observe that $S_\mu(0) = 1/\psi'_\mu(0)$ so that S_μ is also defined and analytic in a neighborhood of 0. Further note that

$$|\psi'_\mu(0)| = \left|\int_{\mathbb{T}} \zeta\, d\mu(\zeta)\right| \le \int_{\mathbb{T}} |\zeta|\, d\mu(\zeta) = 1,$$

which implies that $|S_\mu(0)| \ge 1$ for $\mu \in M_* \cap M_{\mathbb{T}}$.

We now record the following lemmas and theorems for use in proving our main results. These were first proven in [7, 9] and [19].

LEMMA 8.1. *Let $\mu \in M_* \cap M_{\mathbb{T}}$ satisfy $|S_\mu(\{0\})| = 1$. Then $\mu = \delta_\alpha$ for some $\alpha \in \mathbb{T}$.*

LEMMA 8.2. *Let $\mu_i \in M_* \cap M_{\mathbb{T}}$ be such that $S_{\mu_i}(z)$ converge uniformly in some neighborhood of 0 to a function $S(z)$. Then there exists $\mu \in M_* \cap M_{\mathbb{T}}$ such that $S = S_\mu$.*

THEOREM 8.3. *Consider $\mu \in M_* \cap M_{\mathbb{T}}$, and let $\mu_i \in M_{\mathbb{T}}$ for $i \in \mathbb{N}$. If μ_i converge to μ in the weak* topology, the $\mu_i \in M_* \cap M_{\mathbb{T}}$ eventually and the functions S_{μ_i} converge to S_μ uniformly in some neighborhood of zero. Conversely, if $\mu_i \in M_* \cap M_{\mathbb{T}}$ and S_{μ_i} converge to S_μ uniformly in some neighborhood of zero, then the measures μ_i converge to μ in the weak* topology.*

THEOREM 8.4. *Let $c_n \in \mathbb{T}$ be a sequence of numbers and $\{\mu_{n,j}\}_{n\in\mathbb{N}, j=1,\dots,k_n}$ be and array of probability measures in $M_{\mathbb{T}}$ such that*

$$\lim_{n\uparrow\infty} \max_{j=1,\dots,k_n} \mu_{n,j}(\{z : |z-1| < \varepsilon\}) = 1$$

for every $\varepsilon > 0$. If the measures $\delta_{c_n} \boxtimes \mu_{n,1} \boxtimes \cdots \boxtimes \mu_{n,k_n}$ have a weak limit μ, then μ is $\boxtimes$-infinitely divisible.

9. Main results for measures supported on the unit circle. The last case considered are measures $\mu \in M_{\mathbb{T}} \cap M_*$ where $M_{\mathbb{T}}$ are those probability measures supported on the complex circle and M_* are those probability measures with nonzero first moment. Observe that our decompositions will be supported on the unit circle so that a family of decompositions $\mu = \mu_{1,k} \boxtimes \mu_{2,k}$ are trivially tight.

THEOREM 9.1. *Let $\mu \in M_{\mathbb{T}} \cap M_*$ have the property that, for any nontrivial decomposition $\mu = \nu \boxtimes \omega$ with $\nu, \omega \in M_{\mathbb{T}} \cap M_*$, neither ν nor ω is indecomposable. Then μ is $\boxtimes$-infinitely divisible.*

PROOF. Let $\Lambda : M_{\mathbb{T}} \to \mathbb{C}$ be defined by $\Lambda(\nu) = S_\nu(0)$. Observe that $|\Lambda(\mu)| \geq 1$ with equality if and only if μ is a Dirac mass situated on the circle. We may then assume that $|\Lambda(\mu)| = 1 + \alpha > 1$. In a manner analogous to Theorems 4.2 and 7.1, for every $\alpha > \varepsilon > 0$, there exists a nontrivial decomposition $\mu = \nu \boxtimes \omega$ such that $|\Lambda(\nu)| < 1 + \varepsilon$. Through a similar maximality argument, one can show that for every $n \in \mathbb{N}$ there exists a decomposition $\mu = \mu_{n,1} \boxtimes \cdots \boxtimes \mu_{n,n}$ such that $|\Lambda(\mu_{n,i})| = \sqrt[n]{|\Lambda(\mu)|}$ for all $i = 1, 2, \dots, n$. We forgo the proof due to extreme similarity to the first two cases.

Now, observe that $\Lambda(\mu_{n,i} \boxtimes \delta_c) = \Lambda(\mu_{n,i})/c$ for $c \in \mathbb{T}$. Thus, we may assume that $\mu = \delta_{c_n} \boxtimes \mu_{n,1} \boxtimes \cdots \boxtimes \mu_{n,n}$ for all $n \in \mathbb{N}$ where we additionally assume that $\Lambda(\mu_{n,i}) = \sqrt[n]{|\Lambda(\mu)|}$.

Note that $\{\mu_{n,j}\}_{n\in\mathbb{N}, j=1,2,\dots,n}$ forms a tight array since all of our measures are compactly supported. Further observe that, by Theorem 8.3 any cluster point ν of this array satisfies $\Lambda(\nu) = 1$. By Lemma 8.1, this implies that $\nu = \delta_1$. Tightness implies that our array converges to δ_1 uniformly over n. By Theorem 8.4, this implies $\boxtimes$-infinite divisibility. $\square$

We close with our Khinthine decomposition for measures in $M_{\mathbb{T}}$. Several steps of the proof are indistinguishable from Theorem 7.2 so are not presented in full detail.

THEOREM 9.2. *Let $\mu \in M_{\mathbb{T}} \cap M_*$ be a probability measure. There exists a decomposition $\mu = \mu_0 \boxtimes \mu_1 \boxtimes \mu_2 \boxtimes \cdots$ such that $\mu_i \in M_{\mathbb{T}} \cap M_*$ for all $i = 0, 1, 2, \ldots,$ μ_0 is infinitely divisible, and μ_i is indecomposable for $i = 1, 2, \ldots.$ Such a decomposition need not be unique.*

PROOF. In a manner entirely analogous with the previous cases, for all $n \in \mathbb{N}$, we construct a decomposition

$$\mu = \mu_{0,n} \boxtimes \mu_1 \boxtimes \cdots \boxtimes \mu_n$$

with the following properties:

(1) The measure $\mu_i \in M_{\mathbb{T}}$ is indecomposable for all $i \in \mathbb{N}$.

(2) Let $\alpha_{i-1} = \sup |\Lambda(\nu)|$ where the supremum is taken over all indecomposable measures $\nu \in M_{\mathbb{T}}$ satisfying $\mu_{0,1} = \nu \boxtimes \rho$ for some $\rho \in M_{\mathbb{T}}$. We have that $1 \leq \Lambda(\mu_i) < \sqrt{\alpha_i}$ [in particular, we may assume that $\Lambda(\mu_i)$ is real].

We again define ν_n, $\nu_{n,m}$ and $\nu_{\infty,m}$ as in the proof of Theorem 7.2. That is,

$$\nu_n = \mu_1 \boxtimes \cdots \boxtimes \mu_n,$$

$$\nu_{n,m} = \mu_{m+1} \boxtimes \cdots \boxtimes \mu_n,$$

$$\nu_{\infty,m} = \lim_{n \uparrow \infty} \mu_{m+1} \boxtimes \cdots \boxtimes \mu_n.$$

Observe that tightness is trivial in this case since $M_{\mathbb{T}}$ is compact. We then have that $\Lambda(\mu) = \Lambda(\mu_{0,n}) * \Lambda(\nu_n) = \Lambda(\mu_{0,n}) * \Lambda(\nu_m) * \Lambda(\nu_{n,m})$. Since $\Lambda_{\mu_{0,n}}$ is decreasing and bounded as $n \uparrow \infty$, this is a convergent sequence. This implies that $\nu_{n,m}$ represents the tail of a convergent product so that it goes to 0 as $m \uparrow \infty$ (this implies that $\alpha_i \to 1$). Thus, $\{\nu_{n,m}\}_{m<n\in\mathbb{N}}$ is tight and any cluster point ν of a subsequence with unbounded m must satisfy $\Lambda(\nu) = 1$. By Lemma 8.1, $\nu = \delta_1$. This implies that $\nu_{n,m} \to \delta_1$ uniformly as $m \uparrow \infty$.

Once again, we let μ_0 be a cluster point of $\{\mu_{0,n}\}_{n\in\mathbb{N}}$. In the same manner as in Theorem 7.2, we have that $\mu_0 \boxtimes \nu_n \to \mu$ as $n \uparrow \infty$.

The theorem is proved when we can show that μ_0 is infinitely divisible. It is again true that $\mu_0 \boxtimes \nu_{\infty:m} = \mu_{0,m-1}$ with no deviation from the previous proof. Our result then follows by the same line of reasoning as Theorem 7.2. $\square$

10. Applications. We begin by extending the class of $\boxplus$-indecomposable measures.

THEOREM 10.1. *Let μ be a measure with the property that the left and right endpoints of the support of μ are Dirac masses. Then μ is indecomposable.*

PROOF. Assume that $\mu = \mu_1 \boxplus \mu_2$ and that the support of μ has respective left and right endpoints a and b. Recall that Theorem 2.7 states that

$$\mu(\{a\}) = \mu_1(\{a_1\}) + \mu_2(\{a_2\}) - 1,$$

$$\mu(\{b\}) = \mu_1(\{b_1\}) + \mu_2(\{b_2\}) - 1$$

for masses $a_i, b_i \in \operatorname{supp}(\mu_i)$, and that these points satisfy $a = a_1 + a_2$ and $b = b_1 + b_2$. Now, if $a_1 \neq b_1$, then $\mu_1(\{a_1\}) + \mu_1(\{b_1\}) \leq 1$. Thus

$$\begin{aligned} 0 < \mu(\{a\}) + \mu(\{b\}) &= \mu_1(\{a_1\}) + \mu_2(\{a_2\}) + \mu_1(\{b_1\}) + \mu_2(\{b_2\}) - 2 \\ &\leq \mu_2(a_2) + \mu_2(\{b_2\}) - 1. \end{aligned}$$

Thus, $\mu_2(a_2) + \mu_2(\{b_2\}) > 1$ so that $a_2 = b_2$. Translating our measures, we may assume that $a_2 = b_2 = 0$. Thus, $a_1 = a$ and $b_1 = b$. This implies that $\operatorname{diam}(\operatorname{supp}(\mu_1)) \geq \operatorname{diam}(\operatorname{supp}(\mu))$. By Theorem 3.2, it follows that $\mu_2 = \delta_0$ so that μ is indecomposable. $\square$

Now, given a measure μ, it was proven by Nica and Speicher in [17] that we may associate to μ a semigroup of measures $\{\mu_t\}_{t\geq 1}$ so that $\mu_1 = \mu$ and $\mu_{s+t} = \mu_s \boxplus \mu_t$. In particular, $\mu_n = \mu \boxplus \cdots \boxplus \mu$, the n-fold free convolution. When μ is infinitely divisible, this family may be extended to $t \in \mathbb{R}^+$.

It was shown in [4] that for $\mu = (\delta_1 + \delta_{-1})/2$, we have that μ_t is a sum of two atoms concentrated at $\pm t$ and an absolutely continuous measure concentrated on $[-2\sqrt{t-1}, 2\sqrt{t-1}]$. This implies the following corollary to our theorem.

COROLLARY 10.2. *For $\mu = (\delta_1 + \delta_{-1})/2$, the elements of the family of measures $\{\mu_t\}_{t\in[1,2)}$ are indecomposable.*

Observe that this family of examples also dashes any hope of uniqueness for our Khintchine decomposition. Indeed, for μ and $\{\mu_t\}_{t\geq 1}$ as in the previous example we have that, for $s = 2 + \varepsilon$, $\mu_s = \mu_t \boxplus \mu_{s-t}$ for all $t \in (1, 1 + \varepsilon)$. This is an uncountable family of distinct decompositions of μ_s into a sum of indecomposable elements.

Note that the even the infinitely divisible divisor in the Khintchine composition cannot be determined uniquely. Indeed denote by μ the semicircle distribution with mean 0 and variance 1, an infinitely divisible measure. It was shown in [11] that there is a nontrivial decomposition $\mu = \nu \boxplus \rho$ where neither ν nor ρ is infinitely divisible. Taking the Khintchine decompositions for each ν and ρ and combining the respective infinitely divisible divisors, we obtain a decomposition $\mu = \mu_0 \boxplus \mu_1 \boxplus \mu_2 \boxplus \cdots$ such that μ_0 infinitely divisible, μ_i indecomposable for $i \geq 1$ and μ_1 nontrivial. This implies that $\mu \neq \mu_0$.

Last, it has come to the author's attention that these results have been addressed independently in [14]. They rightly point out the following improvement on Theorems 4.2 and 4.4. Namely, the class of measures that satisfy the hypotheses of Theorem 4.2 are precisely the Dirac measures. For a simple justification of this fact, note that we have shown that such measures are necessarily infinitely divisible. It was shown in [10] that infinitely divisible measures may be decomposed into the free convolution of a semicircular measure and a free Poisson measure. Free Poisson measures have indecomposable divisors, almost by definition. As was shown in [11], semicircular measures also have indecomposable divisors. These facts taken together imply the above statement so that Theorem 4.4 may be improved into a purely prime decomposition, with no infinitely divisible component.

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